On Non-Vanishing And Linear Independence Results

By

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DECLARATION

I declare that the thesis entitled **"On Non-Vanishing And Linear Independence Results"** submitted by me for the degree of **Doctor of Philosophy in Mathematics** is the record of academic work carried out by me under the guidance of Professor Purusottam Rath and this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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CERTIFICATE

I certify that the thesis entitled **"On Non-Vanishing And Linear Independence Results"** submitted for the degree of **Doctor of Philosophy in Mathematics** by Abhishek T Bharadwaj is the record of research work carried out by him under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent his independent work in a very substantial measure.

Chennai Mathematical Institute
Date: July 7, 2020.

Professor Purusottam Rath
Thesis Supervisor.

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Index of Symbols

Notation	Description
$\mathbb{N}$	The set of natural numbers $\{1, 2, 3, \ldots, \}$
$\mathbb{Z}$	The set of integers $\{\ldots, -1, 0, 1, \ldots\}$
$\mathbb{Q}$	The set of rational numbers
$\overline{\mathbb{Q}}$	The set of algebraic numbers
$\mathcal{P}$	The set of prime numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{Z}/n\mathbb{Z}$	The residue class of integers modulo n.
$\mathbb{F}_p$	The field $\mathbb{Z}/p\mathbb{Z}$ for a prime p.
$\left(\frac{\cdot}{p}\right)$	Legendre symbol $\bmod p$.
(a, b)	The gcd of two integers a and b.
$\mathbb{Z}_p$	The p adic completion of $\mathbb{Z}$.
$\mathbb{Q}_p$	The p adic completion of $\mathbb{Q}$.
$\mathbb{C}_p$	The p adic completion of the algebraic closure of $\overline{\mathbb{Q}}_p$.
Ω	a finite set of primes.
ζ_p	The primitive root of unity $e^{2\pi i/p}$.
φ	The Euler totient function.
μ	The Möbius function.
$\pi(x)$	The prime numbers in $\mathbb{N}$ less than or equal to x.

Some notations associated to arithmetic functions

Let f and g be two arithmetical functions $\mathbb{N} \mapsto \mathbb{C}$.

1. We say $f(x) = O(g(x))$ if there exists an absolute constant C such that $|f(n)| \leq Cg(n)$ for all $n \in \mathbb{N}$.

2. We say $f(x) = o(g(x))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$.

3. We say $f(x) >> g(x)$ if there exists an absolute constant C such that $f(n) \geq Cg(n)$ for all $n \in \mathbb{N}$.

4. For a finite set S, We say $f(x) \asymp_S g(x)$ if there exist absolute constants $c_1(S)$ and $c_2(S)$ (i.e. constants depending on the set S) such that $c_1(S)g(n) \leq f(n) \leq c_2(S)g(n)$ for all large natural numbers n.

Contents

Abstract

For a periodic function $f : \mathbb{N} \to \overline{\mathbb{Q}}$ of period N, the infinite sum $L(1, f)$ exists if and only if $\sum_{a=1}^{N} f(a) = 0$. By using Baker's theorem of linear forms in logarithms of non-zero algebraic numbers, it is known that this value is either zero or transcendental. However, this theorem cannot be applied individually for understanding the transcendence of the Euler constants $\gamma(a, N)$. The theme of studying these constants not individually, but as a family was initiated by Murty and Saradha [47] and implemented to a large family of these constants by Gun, Murty and Saha [28]. Here, we study the linear relations spanned by these constants and improve the dimension estimates mentioned in [28]. In the p-adic setup, these constants were introduced by Diamond [22] and transcendence results for a few cases were obtained in [15, 48]. We prove more general theorems akin to the ones proved in [28, 47].

It is a conjecture of Baker [6] that the special values $\{L(1, \chi) : \chi \bmod q, \chi \neq 1\}$ are linearly independent over $\mathbb{Q}$, and this result is known when $(q, \varphi(q)) = 1$. For a number field K and a prime p, we study the K linear relations of the vector space spanned by these special values. A similar study is conducted for $L(k, f)$ with k and f satisfying a 'parity' condition.

We study a conjecture of Erdös about the non vanishing of $L(1, f)$ with conditions on the values of periodic function f of period q. This conjecture is proved for $q \equiv 3 \bmod 4$ by Murty and Saradha [49], and a few more cases were covered by Chatterjee and Murty in [17]. We produce infinitely many examples of natural numbers q not covered earlier in the literature with an additional restriction on f to be an odd function.

List of publications associated to this thesis

1. A. T. Bharadwaj. A note on the arithmetic Chowla-Milnor space. *J. Ramanujan Math. Soc.*, 34(4):411–416, 2019.
2. A. T. Bharadwaj and P. Rath. On a question of Alan Baker over arbitrary number fields. *Mathematika*, 66(1):103–111, 2020.
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Chapter 1

Introduction

The complex numbers which satisfy a non-zero polynomial equation with integer coefficients are called algebraic numbers. These numbers (henceforth denoted by $\overline{\mathbb{Q}}$) form a subfield of $\mathbb{C}$ with usual addition and multiplication. However, there are complex numbers which do not satisfy any non-zero polynomial equation with integer coefficients. Such numbers are called as transcendental numbers. The explicit construction of such a number was given by Liouville in 1844 where he proved that the number $\sum_{n=1}^{\infty} 10^{-n!}$ is transcendental. Later, Cantor through his famous diagonal argument proved that the set of transcendental numbers is uncountable. He also proved that algebraic numbers are countable.

A typical method in proving the transcendence of a number α involves proving the $\mathbb{Q}$ linear independence of the numbers $\{\alpha^i\}_{i=0}^{N}$ as N varies over the natural numbers $\mathbb{N}$. We assume that the number α is algebraic and proceed to construct a sequence of non-zero polynomials $P_n(x) \in \mathbb{Z}[x]$ of degree n such that $|P_n(\alpha)|$ is non-zero, 'integer' valued and a decreasing function of n. Hence we obtain a contradiction for large n, as there are no integers between zero and one. Using this principle, Hermite in 1873 proved that e is transcendental. To prove the transcendence of π, Lindemann had to consider the fictitious conjugates and use a similar technique not only for the element π, but also for its conjugates. The use of Euler's identity

$$e^{i\pi} + 1 = 0$$

was critical to his proof. Finally, Weierstrass proved the $\overline{\mathbb{Q}}$-linear independence of e^{α_i} as α_i varies over the set of distinct non-zero algebraic numbers in 1885. A consequence of this theorem is the following :

$$\text{“ If } \alpha \in \overline{\mathbb{Q}} \setminus \{0,1\}, \text{ then } \log\alpha \notin \overline{\mathbb{Q}}.\text{”}$$

In the above, we can take any branch of logarithm log provided $\log\alpha \neq 0$.

In his famous 1900 address to the International congress of Mathematicians in Paris, Hilbert posed 23 problems in mathematics of great importance. The seventh one is a question on transcendence which we quote below.

"... *Indeed, we expect transcendental functions to assume, in general, transcendental values for even algebraic arguments; and, although it is well known that there exist integral transcendental functions which even have rational values for all algebraic arguments, we shall still consider it highly probable that the exponential function $e^{\pi i z}$, for example, which evidently has algebraic values for all rational arguments z, will on the other hand always take transcendental values for irrational algebraic values of the argument z. ... I consider the proof of this theorem very difficult; as also the proof that the expression α^β, for an algebraic base α and an irrational algebraic exponent β, e.g., the number $2^{\sqrt{2}}$ or $e^\pi = i^{-2i}$ always represents a transcendental or at least an irrational number.*"

The proof for the transcendence of $2^{\sqrt{2}}$ was given by Kuzĭmin [37] in 1930 . The transcendence of α^β was subsequently proved by Gelfond [25] and Schneider independently around 1934. In particular, the result states the following : For any non-zero algebraic numbers α, β we have

$$\frac{\log\alpha}{\log\beta} \in \overline{\mathbb{Q}} \Leftrightarrow \frac{\log\alpha}{\log\beta} \in \mathbb{Q}.$$

In other words, the $\overline{\mathbb{Q}}$-linear dependence of logarithms of two algebraic numbers arises from the fact that they are multiplicatively dependent, that is there exists two non-zero natural numbers m, n such that $\alpha^m = \beta^n$. This observation is the starting point of one of the crucial results in transcendental number theory. In his groundbreaking work, Baker [4] had proved the n-dimensional analogue. We quote the theorem below :

Theorem 1. Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers. If $\log\alpha_1, \ldots, \log\alpha_n$ are linearly independent over $\mathbb{Q}$, then $1, \log\alpha_1, \ldots, \log\alpha_n$ are linearly independent over $\overline{\mathbb{Q}}$.

As mentioned by Serre [58], if we denote $V = \overline{\mathbb{Q}} \left\langle \log \alpha \; : \; \alpha \in \overline{\mathbb{Q}}^* \right\rangle$, the above theorem means that the map

$$V \underset{\mathbb{Q}}{\otimes} \overline{\mathbb{Q}} \to \mathbb{C},$$

is injective and well behaved under the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Baker [4] had also provided a lower bound for the sum whenever it is non-zero. Baker subsequently used this theorem to solve the class number one problem of finding all the imaginary quadratic fields K with class number one. Our inclination towards this theorem rests in the following consequence :

Corollary 1. *Any $\overline{\mathbb{Q}}$-linear form of logarithm of non-zero algebraic numbers is either zero or transcendental.*

In this thesis, our focus is on the non-vanishing and the transcendence of the following infinite series

$$L(k, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^k}. \tag{1.1}$$

for a periodic function f of period N and a positive integer k. The earliest known examples of the transcendence are undoubtedly the special values of the Riemann Zeta function. We shall see how the preceding theorems play a role in this case.

1.0.1 Special Values of the Riemann Zeta functions

The function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \; \Re s > 1$$

was introduced by Euler for real variable $s > 1$ and generalised to complex values by Riemann. In 1734, Euler proved that

$$\zeta(2k) \in \pi^{2k}\mathbb{Q}^*.$$

This formula was proved by taking the logarithmic differentiation of the following identity :

$$\sin \pi t = \pi t \prod_{n=1}^{\infty} (1 - \frac{t^2}{n^2}).$$

The same method can be generalised for finding special values of the Dirichlet L function at 'certain' integers. Let χ be a Dirichlet character of conductor N with

$\chi(-1) = (-1)^k$. Then, we have

$$L(k, \chi) \in \overline{\mathbb{Q}}^* \pi^k. \tag{1.2}$$

The non-vanishing of these values can be observed owing to the Euler product expression of $L(k, \chi)$. In fact, in the domain of absolute convergence (that is whenever $k > 1$), we have

$$L(k, \chi) = \prod_p (1 - \frac{\chi(p)}{p^k})^{-1},$$

where p runs over all the prime numbers. Hence, $L(k, \chi) \neq 0$ whenever $k > 1$. While the non-vanishing is evident, one should note that this does not give any information about the irrationality (even in the case of the odd values of the Riemann Zeta function). In fact, Apery [1] in 1978 proved that $\zeta(3)$ is irrational. Further progress for the irrationality of special values at odd integers are by Ball-Rivoal [7], Zudilin [63], and Fischler [24].

If we are outside the domain of absolute convergence, it is no longer immediate that $L(1, \chi)$ is non-vanishing. In fact, it is a theorem of Dirichlet that $L(1, \chi)$ is not zero whenever χ is not a principal Dirichlet character. This fact implies the infinitude of primes in arithmetic progression.

The explicit expression for $L(1, \chi)$ (see Appendix A.1.2) shows that these numbers are transcendental. More generally, the same is true for a periodic arithmetic function f taking algebraic values under some conditions. For a periodic arithmetic function f of period N, the infinite sum $L(1, f)$ (whenever it converges) is intimately connected to linear form of logarithm of non-zero algebraic numbers as shown below :

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = -\sum_{a=1}^{N-1} \widehat{f}(a) \log(1 - \zeta_N^a). \tag{1.3}$$

In the above, $\zeta_N := e^{2\pi i/N}$ and $\widehat{f}(a) := \frac{1}{N} \sum_{b=1}^{N} f(a) \zeta_N^{-ba}$ denotes the Fourier transform of the function f. Hence by Theorem 1, the above sum is either zero or transcendental. More often than not, the non-vanishing of the above infinite sum is a hard question. For special values of the L function associated to the Dirichlet characters χ, this is a stark contrast where it is easy to prove the non-vanishing for $L(k, \chi)$ for integers k greater than 1 whereas the transcendence of the same is a much deeper issue and not completely resolved.

We mention a few words about the p-adic analogue. In 1964, Kubota and Leopoldt [36] had defined the p-adic L function $L_p(s, \chi)$ for an even Dirichlet character χ. To prove the non-vanishing of the regulator of the p-adic zeta function, the non-vanishing of $L_p(1, \chi)$ was necessary. It was observed that this value can be written as a $\overline{\mathbb{Q}}$-linear form of logarithm of algebraic numbers. Finally, a p-adic analogue of Theorem 1 was proved by Brumer [14]. We also remark that the non-vanishing of $\zeta_p(k)$ for all integers k is still unknown. Here $\zeta_p(s)$ denotes the p-adic analogue of the Riemann zeta function.

This thesis is centered around three parts. The definitions of some terms are given in the subsequent sections.

1. Obtain an optimal estimate for the $\overline{\mathbb{Q}}$ span of the generalized Euler Briggs constants, and associated results in the p-adic setup.

2. Study the linear relations and dimension estimates of the space spanned by the special values of the Dirichlet characters $L(1, \chi)$ over arbitrary number fields as χ varies over a prime modulus p. We also study similar questions for $k > 1$.

3. Provide some conditions to ensure the non-vanishing of $L(1, f)$ when f is an odd function (i.e. $f(-n) = -f(n)$ for all natural numbers n).

We now proceed to mention the conjectures and our main results regarding the transcendence of the infinite series (1.1).

1.1 The Euler constant γ

In 17^{th} century, Euler defined and studied the properties of the following function :

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where he considered s to be a real variable greater than 1. The above sum converges absolutely when $s > 1$, and diverges at $s = 1$. He introduced γ as the limit

$$\gamma := \lim_{x \to \infty} \Big(\sum_{n \le x} \frac{1}{n} - \log x \Big).$$

This mysterious constant appears in various contexts and its irrationality is elusive till date. For instance we have :

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1).$$

In the above $\zeta(s)$ denotes the Riemann zeta function and for $\Re s > 1$ it is given by the infinite series $\sum_{n\geq 1} \frac{1}{n^s}$. We mention a strong conjecture below asserting its transcendence for which we require the following definition.

Definition 1. *A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of algebraic functions with algebraic coefficients over domains in $\mathbb{R}^n$ given by polynomial inequalities with algebraic coefficients.*

The classical examples of periods are algebraic numbers and their logarithms. A beautiful exposition to this topic can be found in [60]. We now state a conjecture of Kontsevich and Zagier [35].

Conjecture 1. *γ is not a period.*

This constant lies in an extended ring containing the period known as exponential period. We quote the expression of γ involving the exponential integral (see [38]).

$$\gamma = \int_0^1 \int_x^1 \frac{e^{-x}}{y} dy dx - \int_1^\infty \int_1^x \frac{e^{-x}}{y} dy dx. \tag{1.4}$$

So far there is no progress concerning Conjecture 1. Rather than studying the irrationality/transcendence of γ in isolation, some authors studied the question by considering a family of constants "associated to gamma". One such family was defined by Briggs [13].

Definition 2. *Let $q \geq 2$ and $1 \leq a \leq q$ be integers. We define*

$$\gamma(a,q) := \lim_{x\to\infty} \Big(\sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \frac{1}{n} - \frac{1}{q}\log(x) \Big). \tag{1.5}$$

The arithmetic properties of these constants were studied by Lehmer [40]. These constants are naturally connected to the infinite series $\sum_{n=1}^{\infty} f(n)/n$ for a periodic function f whenever the sum converges. In fact,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{a=1}^{q} f(a)\gamma(a,q). \tag{1.6}$$

Thus under some conditions, these constants are connected to linear forms in logarithms of algebraic numbers. The authors [49] proved the following theorem on the transcendence of families associated to the Briggs constants.

Theorem 2 (Murty and Saradha). At most one number in the infinite list of numbers

$$\gamma(a,q),\ 1 \le a < q,\ q \ge 2,$$

is an algebraic number. Further if γ is algebraic, then only the number $\gamma(1,2) = \gamma/4$ from the above list is algebraic.

We mention another generalisation of the Euler's constant γ defined by Diamond and Ford [21].Here they sieve out a finite set of primes Ω from the summand. More precisely, for a finite set of Primes Ω, they define :

$$\gamma(\Omega) = \lim_{x\to\infty} \Big(\sum_{\substack{n\le x \\ (n,P_\Omega)=1}} \frac{1}{n} - \delta_\Omega \log x \Big).$$

Here,

$$P_\Omega := \prod_{p\in\Omega} p \quad \text{and} \quad \delta_\Omega := \prod_{p\in\Omega} \Big(1 - \frac{1}{p}\Big).$$

By convention, $P_\Omega = 1$, $\delta_\Omega = 1$ when $\Omega = \emptyset$. Akin to Theorem 2, there are results on transcendence of families associated to this constant.

$$H_p(s,a,F) = \frac{<a>^{1-s}}{F(s-1)} \int_{\mathbb{Z}_p} <1 + \frac{F}{a}t>^{1-s} dt. \tag{1.7}$$

The "arithmetic progression" analogue of this constant was introduced by Gun, Saha and Sinha [29] and the authors proved some properties of these constants similar to that of [40]. We record the definition here.

Definition 3. *Let Ω be a finite set of primes. Let $q \ge 1$ be such that $(q, P_\Omega) = 1$ and let $1 \le a \le q$. The generalised Euler constant $\gamma(\Omega,a,q)$ is defined as :*

$$\gamma(\Omega,a,q) = \lim_{x\to\infty} \Big(\sum_{\substack{n\le x \\ n\equiv a \bmod q \\ (n,P_\Omega)=1}} \frac{1}{n} - \frac{\delta_\Omega}{q} \log x \Big)$$

Furthermore, the authors [28] proved the following linear independence result strengthening the earlier results.

Theorem 3 (Gun Murty and Saha). Consider the $\mathbb{Q}$ vector space

$$V_{\mathbb{Q},N} := \mathbb{Q} < \gamma(\Omega, m, n) \,:\, 1 \le m \le n \le N,\ (m,n) = 1 = (n, P_\Omega) > .$$

Then for sufficiently large N, we have $\dim_{\mathbb{Q}} V_{\mathbb{Q},N} \gg_\Omega N$.

1.1.1 Our contribution

Contrary to the goal of Theorem 3, we wanted to investigate $\mathbb{Q}$ linear relations of the generalised Euler-Briggs constants. We define the following vector space over $\mathbb{Q}$.

Definition 4. *Let Ω be a finite set of primes. For any integer N co-prime to P_Ω, let $W^0_{\Omega,N}$ denote the following $\mathbb{Q}$ vector space:*

$$W^0_{\Omega,N} := \Big\{ \sum_{\substack{a=1 \\ (a,N)=1}}^{N} \alpha_a \gamma(\Omega, a, N) \,:\, \alpha_a \in \mathbb{Q},\ \sum_{\substack{a=1 \\ (a,N)=1}}^{N} \alpha_a = 0 \Big\}.$$

We prove the following theorem highlighting the existence of relations between these constants under 'comparable levels'.

Theorem 4. For any N, N' co-prime to P_Ω such that $N \mid N'$, we have $W^0_{\Omega,N} \subset W^0_{\Omega,N'}$.

It was observed in [28] that $\dim_{\mathbb{Q}} V_{\mathbb{Q},N} = O(N^2)$. Our next theorem improves the earlier known bound mentioned in Theorem 3.

Theorem 5. With the same notations as in Theorem 3, we have

$$\dim_{\overline{\mathbb{Q}}} V_{\overline{\mathbb{Q}},N} \asymp_\Omega N^2.$$

A portion of the above work is done in [12]. We proved similar theorems in the p-adic setup. In the p-adic setup, these constants were introduced by Diamond [22]. Given the technicality involved in the definitions, we discuss the same in the later chapters. This work is done in [11].

1.2 The Chowla-Chowla and Milnor Conjecture

In 1982, P. Chowla and S.Chowla [19] made the following conjecture.

Conjecture 2. *Let p be any prime and f be any rational valued function with period p. Then $L(2, f) \neq 0$ except in the case when*

$$f(1) = f(2) - \cdots = f(p-1) = \frac{f(p)}{(1-p^2)}.$$

The above conjecture can be re-interpreted as follows : If f is a non-zero rational valued function with prime period p, then

$$L(2, f) = 0 \Leftrightarrow L(s, f) = \big(1 - \frac{p^2}{p^s}\big)\zeta(s).$$

In 1983, John Milnor [43] interpreted this conjecture in terms of the values of Hurwitz zeta function.

Conjecture 3. *Let $q > 1, k > 1$ be two integers. Then the following $\varphi(q)$ real numbers :*

$$\zeta(k, a/q) \text{ with } 1 \leq a < q, \ (a, q) = 1$$

are linearly independent over $\mathbb{Q}$.

Using Plancherel's formula, one can indeed write $L(k, f)$ as a linear form of polylogarithm of algebraic numbers.

Definition 5. *Let z be a complex number in the closed unit disk and k be an integer greater than one. The $k-th$ polylogarithm function $Li_k(z)$ is defined by :*

$$Li_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

This function satisfies the following integral representation :

$$Li_{k+1}(z) = \int_0^z \frac{Li_k(t)}{t}\, dt \tag{1.8}$$

With the above definition, for a periodic function f of period N, we can write

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^k} = \sum_{a=1}^{N} \widehat{f}(a) Li_k(\zeta_N^a).$$

Using this function, Gun, Murty and Rath [30] formulated the Polylog conjecture which is an analogue to Baker's theory on linear forms in logarithms.

Conjecture 4. *Suppose that* $\alpha_1, \ldots, \alpha_n$ *are algebraic numbers with* $|\alpha_i| \leq 1$ *for all* $1 \leq i \leq n$ *such that* $Li_k(\alpha_1), \ldots Li_k(\alpha_n)$ *are linearly independent over* $\mathbb{Q}$*. Then they are linearly independent over* $\overline{\mathbb{Q}}$*.*

Note that one can construct a $\mathbb{Q}$-linear relation between the values of the polylogarithm function with the help of the distribution identity and the orthogonality relations of the characters.

$$\sum_{m=0}^{N-1} Li_k(z\, e^{2\pi i m/N}) = N^{1-k}\, Li_k(z^N) \tag{1.9}$$

Assuming Conjecture 4, the authors were able to establish Conjecture 3. In order to understand the conjecture, the authors defined the following $\mathbb{Q}$ vector space in the set of complex numbers.

Definition 6. *For integers* $k > 1, q \geq 2$*, define the Chowla-Milnor space* $V_k(q)$ *by :*

$$V_k(q) := \mathbb{Q}\left\langle \zeta(k, \frac{a}{q}) \;:\; 1 \leq a < q,\ (a,q) = 1 \right\rangle,$$

Conjecture 3 states that the dimension of $V_k(q)$ as a $\mathbb{Q}$ vector space is $\varphi(q)$. Unconditionally, the following lower bound was obtained :

Theorem 6 (Gun, Murty and Rath). Let $k > 1$ and $q > 2$ be two integers. Then

$$\dim V_k(q) \geq \frac{\varphi(q)}{2}.$$

This bound was attained by identifying a canonical subspace of $V_k(q)$ consisting of elements which are algebraic multiple of π^k. We mention the subspace in consideration below.

Definition 7. *Let* $k > 1, q > 2$ *be positive integers. The arithmetic space* $V_{ar}(\mathbb{Q})$ *is the* $\mathbb{Q}$ *linear space defined by*

$$V_{ar}(\mathbb{Q}) = \mathbb{Q}\left\langle \zeta(k, \frac{a}{q}) + (-1)^k \zeta(k, 1 - \frac{a}{q}) \;:\; (a,q) = 1, \quad 1 \leq a < \frac{q}{2} \right\rangle.$$

The elements of the subspace are linked with the derivatives of the cotangents $\cot \pi z$ at rational values of z. We mention a lemma by Okada [52] about the linear independence of co-tangents.

Lemma 1 (Okada). *Let k and q be positive integers with $k > 0$ and $q > 2$. Let T be a set of $\varphi(q)/2$ representations mod q such that the union $T \cup (-T)$ constitutes a complete set of co-prime residue classes $\bmod q$. Let K be a number field such that $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then the set of real numbers*

$$\frac{d^{k-1}}{dz^{k-1}} \cot(\pi z)|_{z=a/q}, a \in T$$

is linearly independent over K.

An elegant proof of the above lemma can be found in [26].

1.2.1 Our contribution

We study the precise dimension estimates of the arithmetic space $V_{ar}(K)$ over number fields K. This work is done in [8]. The main theorem is the following :

Theorem 7. Let K be a number field contained in $\mathbb{Q}(\zeta_q)$. Then we have,

$$dim_K V_{ar}(K) = \begin{cases} \frac{\varphi(q)}{2[K:\mathbb{Q}]} & \text{if } K \text{ is totally real} \\ \frac{\varphi(q)}{[K:\mathbb{Q}]} & \text{if } K \text{ is totally imaginary} \end{cases} . \quad (1.10)$$

1.3 Baker's question over number fields

During a number theory conference in 1969, Chowla posed a question to generalise the non-vanishing of the value $L(1, \chi)$ for a Dirichlet character χ to non-vanishing of $L(1, f)$ whenever f is a rational valued periodic function of prime period p.

Question 1. *Does there exist a rational valued arithmetic function f, periodic with prime period p such that $L(1, f) = 0$ whenever it converges?*

A complete solution to this question was provided by Baker, Birch and Wirsing [5] in a more general setup by appealing to Baker's theorem and the non-vanishing of $L(1, \chi)$. We quote the theorem below :

Theorem 8 (Baker, Birch and Wirsing). Let f be a non-zero arithmetic function of period q satisfying $f(n) = 0$ whenever $1 < (n, q) < q$, and that the cyclotomic polynomial $\Phi_q(x)$ is irreducible over $\mathbb{Q}(f(1), \ldots, f(q))$. Then $L(1, f) \neq 0$ whenever the sum converges.

This theorem has its connections to the following :

- Linear independence of Briggs Constants over the field of rational numbers.
- Linear independence of $L(1, \chi)$ as χ varies over the non-trivial characters over rational numbers.

The ideas elaborating on the first point stem from [47]. We quote a couple of results from the same paper before proceeding further. We recall that two number fields L and K are linearly disjoint if the tensor product $K \otimes_{\mathbb{Q}} L \mapsto KL$ is an injection. When either of K or L is a Galois number field then this implies $K \cap L = \mathbb{Q}$.

Conjecture 5. *Let K be an algebraic number field linearly disjoint with $\mathbb{Q}(\zeta_q)$. Then the numbers $\Gamma'/\Gamma(a/q)$ where $1 \leq a \leq q$ with $(a, q) = 1$ are linearly independent over the field K.*

The authors also proved the following theorem with the help of Theorem 8.

Theorem 9 (Murty and Saradha). Either Conjecture 5 is true or γ is a period.

Therefore, the Conjectures 1 and 5 are “compatible" with each other. Naturally, Conjecture 5 also holds true when the values $\Gamma'/\Gamma(a/q)$ are replaced by the Briggs constants $\gamma(a, q)$ due to the following relation :

$$\gamma(r, N) = -\frac{1}{N}\left(\frac{\Gamma'}{\Gamma}\left(\frac{r}{N}\right) + \log N\right),$$

This concludes the impact of Theorem 8 on the Euler constants.

Moving on to the linear independence of $L(1, \chi)$, we start with the following corollary of Theorem 8.

Corollary 2. *Let $(q, \varphi(q)) = 1$. Then the numbers $\{L(1, \chi)\}$ as χ varies over the non-trivial characters* $\bmod q$ *are linearly independent over* $\mathbb{Q}$.

The question of Baker [6, Pg. 48] asks whether the condition $(q, \varphi(q)) = 1$ is required for the conclusion of the statement. Unconditionally, the following was observed by many authors [5, 45, 53].

Corollary 3. *For a fixed $q > 1$, the elements $L(1, \chi)$ where χ runs over all the non-trivial even characters modulo q are linearly independent over $\overline{\mathbb{Q}}$.*

Naturally, the above corollary cannot be encompassed by Theorem 8 and one would require a system of $\varphi(q)/2 - 1$ multiplicatively independent elements in $\mathbb{Q}(\zeta_q)$ to conclude the above corollary by appealing to Baker's theorem. Concerning $L(1, \chi)$ where χ is an odd primitive character $\bmod N$, we note that the explicit value is an algebraic multiple of π(see Appendix A.1.2). Thus, in this case the theorem of Baker is not helpful for addressing the linear independence of $L(1, \chi)$ for an odd Dirichlet character of arbitrary modulus.

1.3.1 Our Contribution

We study the question of Baker over arbitrary number fields. As observed in the earlier paragraph, this involves analysing the linear relations between the values of $L(1, \chi)$ as χ ranges over the odd Dirichlet characters $\bmod p$ where p is a prime. We quote the two major results in [9].

Theorem 10. Let p be a prime greater than 7 and K be a number field such that $\mathbb{Q}(\zeta_{p-1}) \subseteq K \subseteq \mathbb{Q}(\zeta_{p(p-1)})$. Let $[K : \mathbb{Q}(\zeta_{p-1})] = d$. Then

$$\dim_K K \langle L(1, \chi) \,:\, \chi \text{ non-trivial Dirichlet character } \bmod p \rangle = \frac{p-1}{2} + \frac{p-1}{2d}\delta_d - 1,$$

where $\delta_d = 1$ if d is odd and 2 if d is even.

Similar computations have been done when K contains the cyclotomic field $\mathbb{Q}(\zeta_p)$. We also define the following counting function over primes.

Definition 8. *For each prime p, let $B(p)$ be the set of integers*

$$B(p) := \left\{ n \;\middle|\; \frac{p-1}{2} < n < p-2 \textit{ and } \dim\ V(K, p) = n \textit{ for some number field } K \right\},$$

and $b(p)$ denotes the cardinality of $B(p)$.

We prove the following limsup estimate using Theorem 10.

Theorem 11. There exists a constant $c > 0$ such that

$$b(p) > \exp\left(\frac{c \log p}{\log\log p}\right)$$

for infinitely many primes p. In particular, for any integer $N > 1$ we have

$$\limsup_{p\to\infty} \frac{b(p)}{(\log p)^N} = \infty.$$

1.4 Erdős Conjecture

In a written communication to Livingston [42], Erdős had mentioned the following conjecture which we quote verbatim.

Conjecture 6. *If q is a positive integer and f is a number theoretic function with period q and $f(n) \in \{-1, 1\}$ when $n = 1, 2, \ldots q-1$ and $f(n) = 0$ whenever $n \equiv 0 \bmod q$ then* $\sum \frac{f(n)}{n} \neq 0$.

This conjecture can be thought as the non-multiplicative analogue of the non-vanishing of $L(1,\chi)$ where χ is a quadratic character mod$q$. This conjecture is vacuously true when q is an even number as we will have $\sum_{n=1}^{q} f(n) \equiv 1 \bmod 2$ and therefore the infinite sum $\sum_{n\geq 1} \frac{f(n)}{n}$ does not converge. When the period of the arithmetic function f is a prime, then by Theorem 8, the Conjecture follows. Further progress towards this conjecture rests on criteria of the vanishing of the sum $L(1, f)$ and this was initiated by Okada [51]. We mention the two results highlighting the current progress that has been made towards this conjecture. Throughout we say that the Conjecture 6 is true mod$N$ if it holds for arithmetic functions f of period N satisfying the hypothesis.

This conjecture has been solved by Murty and Saradha [49] for periodic functions of period N in the equivalence class 3 mod 4. The crux of the proof relies on proving the non-vanishing of the coefficient of π (see (1.11)) while expressing $L(1, f)$ as a linear form of logarithm of algebraic numbers. We state the theorem specifying the algebraic coefficient of π.

Theorem 12 (Murty and Saradha). Let $f : \mathbb{Z}/q\mathbb{Z} \to \overline{\mathbb{Q}}$ be an algebraic valued function which is not identically zero, and ζ_q a primitive q-th root of unity. Further suppose that $\sum_{a=1}^{q} f(a) = 0$. If

$$\frac{f(q)}{2q} + \frac{1}{q}\sum_{b=1}^{q-1} \frac{f(b)}{1-\zeta_q^b} \neq 0, \tag{1.11}$$

then $L(1, f)$ is transcendental.

The above method is not applicable when the period N is congruent to 1 mod 4 as we can construct even functions of period N satisfying the hypothesis of the conjecture.

On the other hand, there are also analytic approaches to understanding this conjecture. The following result relies on a criteria by Saradha and Tijdeman [57] on the vanishing of $L(1, f)$. This proposition was used by Chatterjee and Murty [17] wherein they work with a density theoretic approach to study Conjecture 6. The key proposition in obtaining the density theoretic result is the following :

Proposition 1 (Chatterjee and Murty). *If Conjecture 6 is false for a periodic function f of period q with q odd, then*

$$1 \leq \sum_{\substack{d|q \\ d\geq 3 \\ d\neq q}} \frac{1}{\varphi(d)}. \tag{1.12}$$

1.4.1 Our Contribution

We say that an arithmetic function f is Erdősian if it satisfies the hypothesis of Conjecture 6. Our goal is to produce examples of natural numbers q which satisfy (1.12) for which the above conjecture is true. However, we prove that the co-efficient of π mentioned in (1.11) is non-zero, and hence our method cannot be applied to even functions. We mention an elementary proposition giving a sufficient criteria for the non-vanishing of $L(1, f)$ when f is an odd function. We work with arithmetic functions f of square-free period N. A portion of this work is done in [10].

Proposition 2. *Let N be square-free and $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}$ be an odd function. For each $d|N$, $d \neq 1$, we set*

$$\alpha_d := \sum_{\substack{i=1 \\ (i,\frac{N}{d})=1}}^{[N/2d]} f(di)\frac{1+\zeta_N^{di}}{1-\zeta_N^{di}}.$$

If $\alpha_{N/p}$ is not an algebraic integer for some prime p dividing N, then $L(1, f) \neq 0$.

From the above proposition, we conclude that any odd Erdősian function f of squarefree period N which has either 3 or 5 as a divisor satisfies Conjecture 6. We also prove the following stronger result under additional assumptions on N.

Theorem 13. Let N be an integer satisfying the following conditions

1. $(N, \varphi(N)) = 1$.
2. There exists a prime $p \equiv 1 \bmod 4$ dividing N satisfying the following condition:

   If $q \mid N$ is a prime and q is a square $\bmod p$.

If $L(1, f) = 0$, for an integer valued odd periodic function f of period N, then the following conditions are satisfied :

$$\sum_{a=1}^{(p-1)/2} f(\frac{N}{p}a)a^{-1} \equiv 0 \bmod p \quad \text{and} \quad \sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right) f(\frac{N}{p}a)a^{-1} \equiv 0 \bmod p. \tag{1.13}$$

Here $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol $\bmod p$.

We prove the following corollary when $17 \mid N$.

Corollary 4. *Let N be an integer divisible by 17 satisfying the following conditions :*

1. $(N, \varphi(N)) = 1$.
2. *If $q \mid N$ is a prime, then q is a square* $\bmod 17$.

Then, for all Erdősian functions f such that the arithmetic function $g(a) := f(Na/17)$ is not even , we have $L(1, f) \neq 0$. Hence, we obtain infinitely many natural numbers q satisfying (1.12) and Conjecture 6 under some additional restrictions.

1.5 Organisation of the thesis

We end with the brief description of the chapters mentioned in the thesis. In Chapter 2, we list out the preliminaries that are required for the further chapters.

In Chapter 3, we highlight some $\mathbb{Q}$-linear relations spanned by the generalised Euler-Briggs constants as highlighted in Section 1.1. We also improve the lower bound of $V_{\mathbb{Q},N}$ mentioned in Theorem 3.

In Chapter 4, we study the p-adic counterpart of the Euler constants. More precisely, we obtain the analogue of the theorem of Baker, Birch and Wirsing in this setup, define the p-adic generalised Euler Briggs constants. We also prove certain properties analogous to the Archimedean setup.

In Chapter 5, we study the arithmetic Chowla Milnor space over number fields. As mentioned in Section 1.2, for a fixed integer k and q greater than one, there is a canonical subspace of $V_k(\mathbb{Q})$ whose elements are algebraic multiples of π^k. We study the same space over arbitrary number fields.

In Chapter 6, we study the linear independence of $L(1,\chi)$ over arbitrary number fields. Here, we restrict our attention to the characters χ of prime modulus. As we observed in Section 1.3, we need to consider the vector space spanned by the special values $L(1,\chi)$ as χ ranges over the odd characters. We also give some limsup estimates as we vary over p.

In Chapter 7, we study the Conjecture of Erdős for odd arithmetic functions f of period q satisfying the condition $(q, \varphi(q)) = 1$. More precisely, we list an infinite family of numbers q congruent to 1 mod 4 (albeit zero density) satisfying (1.12) for which the conjecture is true. For a prime p, we also study the $\mathfrak{p}$-adic valuations of the elements of $\mathbb{Z}[\zeta_p]$ as an ideal in $\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})]$ for the prime ideal $\mathfrak{p}$ in $\mathbb{Z}[\zeta_p]$ above p.

Chapter 2

Preliminaries

The goal of this chapter is to introduce the notions and the results required for the subsequent chapters. We divide this section into three parts, namely the algebraic and the analytic aspects, and towards the end, we will touch upon the p-adic notions required for the subsequent chapters.

2.1 The analytic aspects

As mentioned in the introduction, our goal is to understand the non-vanishing of the Dirichlet series

$$L(s, f) = \sum_{n \geq 1} \frac{f(n)}{n^s},$$

where $f : \mathbb{N} \to \overline{\mathbb{Q}}$ is periodic mod$q$ and s takes positive integer values. We first discuss about the non-vanishing of $L(s, f)$ when $s = 1$. We remark here that $L(s, f)$ cannot be evaluated at $s = 1$ for all periodic arithmetic functions f. We use the notation $L(1, f)$ to denote $\sum_{n \geq 1} \frac{f(n)}{n}$ whenever the sum converges.

2.1.1 Convergence of $L(s, f)$ at $s = 1$

Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function of period q. The function $L(s, f)$ converges absolutely and uniformly on compact subsets of the complex plane for $\Re(s) > 1$ and hence is a holomorphic function in this domain. To derive the necessary and sufficient condition for the convergence of $L(1, f)$, we use Abel's summation formula

[2, Theorem 4.2, Pg 77]

$$\sum_{n\leq x}\frac{f(n)}{n}=\frac{F(x)}{x}+\int_1^x\frac{F(t)}{t^2}dt. \tag{2.1}$$

where

$$F(x)=\sum_{n\leq x}f(n)=\frac{x}{q}\sum_{a=1}^{q}f(a)+O(q).$$

Substituting this expression in (2.1) we get,

$$\sum_{n\leq x}\frac{f(n)}{n}=\sum_{a=1}^{q}f(a)(\frac{1}{q}+\log x)+O(q). \tag{2.2}$$

Hence as $x\to\infty$, we get the following theorem (See [46] for the proof of the second part).

Theorem 14. The limit of $L(s,f)$ at $s=1$ exists if and only if $\sum_{a=1}^{q}f(a)=0$. When the function f satisfies this condition, the evaluation at $s=1$ is given by

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}=\sum_{a=1}^{q}f(a)\gamma(a,q)=-\sum_{a=1}^{N-1}\widehat{f}(a)\log(1-\zeta_N^a). \tag{2.3}$$

2.1.2 Linnik's theorem

By Dirichlet's theorem for primes in arithmetic progression, we know that the set $\{a+nq|n\geq 0\}$ has infinitely many primes whenever a and q are co-prime.

Question 2. *Fix two co-prime positive integers a and q. What is the smallest natural number n(depending on q) such that a + nq is a prime number?*

The theorem of Linnik [41] (also see [55]) addresses the above question :

Theorem 15 (Linnik). Let a,n be two positive integers with $(a,n)=1, n\geq 2$. Let $p(a,n)$ denote the least prime p such that $p\equiv a \mod (n)$. There exist absolute positive constants C and L such that

$$p(a,n)<Cn^L.$$

The constant L is known as *"Linnik's constant"*. Various mathematicians have determined admissible values for Linnik's constant. The best known bound for this is due to Xylouris [62] who proves that $L\leq 5.18$.

2.1.3 Estimating partial sums using convolution

We quote a result mentioned in [54] which helps us in computing partial sums via understanding weighted partial sum of a certain convolution.

Theorem 16 (Popken). Let f be an arithmetic function and α be another arithmetic function such that $f = \alpha * \iota$ ($f_1 * f_2$ denotes Dirichlet convolutions of functions f_1 and f_2). Here ι denotes the identity function $\iota(n) = n$. Then

$$\sum_{n \leq x} f(n) = \frac{1}{2} \sum_{d=1}^{x} \alpha(d) \left\{ \left\lfloor \frac{x}{d} \right\rfloor \left(\left\lfloor \frac{x}{d} \right\rfloor + 1 \right) \right\}.$$

This completes the analytic requisites. We proceed to the algebraic aspects.

2.2 Algebraic aspects

We begin by mentioning a few results about Dirichlet characters

2.2.1 Dirichlet Characters

A character χ of a finite abelian group G is a homomorphism $G \to \mathbb{C}^*$. When $G \cong (\mathbb{Z}/N\mathbb{Z})^*$, we call it a Dirichlet Character. The group of all the Dirichlet characters corresponding to G is denoted by $\widehat{G}$. It is a standard fact that $G \cong \widehat{G}$ when G is a finite abelian group. We also have an isomorphism $G \to \widehat{\widehat{G}}$ given by $x \to F_x$ with $F_x(\chi) = \chi(x)$ where $\chi \in \widehat{G}$.

Lemma 2. *Let $\chi \in \widehat{G}$. We have the following identity :*

$$\sum_{a \in G} \chi(a) = \begin{cases} |G| & \text{if } \chi \equiv Id, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1. *The same holds true when G is replaced by $\widehat{G}$. Indeed if we write the sum, we have*

$$\sum_{\chi \in \widehat{G}} \Psi_a(\chi) = 0 \text{ if } a \neq 1, \qquad \Psi_a \in \widehat{\widehat{G}}.$$

But we have $\Psi_a(\chi) = \chi(a)$ by definition. As a consequence we also obtain the orthogonality relations.

Let χ be a Dirichlet character $\bmod N$. For any natural number N' such that $N \mid N'$, we can construct a Dirichlet character χ' mod N' by the following : If $\pi_{NN'}$ denotes the surjection from $(\mathbb{Z}/N'\mathbb{Z})^* \to (\mathbb{Z}/N\mathbb{Z})^*$, then $\chi' : (\mathbb{Z}/N'\mathbb{Z})^* \to \mathbb{C}^*$ is given by $\chi' \equiv \chi \circ \pi_{NN'}$.

Given a Dirichlet character χ, the smallest N for which χ does not factor through $(\mathbb{Z}/N\mathbb{Z})^*$ is called the conductor of χ. If χ is a character $\bmod N$, the characters χ are extended to the whole of integers $\mathbb{Z}$, in a natural manner :

$$\chi(n) = \begin{cases} \chi(n \bmod N) & \text{if } (n, N) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

A character χ of $(\mathbb{Z}/N\mathbb{Z})^*$ is said to be primitive if the conductor of χ is N. For a fixed N, we denote the number of primitive characters of conductor N to be $\Psi'(N)$. We derive an explicit formula for $\Psi'(N)$ below. Noting that a character $\chi \bmod N$ arises from a primitive character χ' mod d for some divisor d of N, we get

$$\varphi(N) = \sum_{d|N} \Psi'(d). \tag{2.4}$$

Hence by the Möbius inversion formula, we note that

$$\Psi'(N) = \sum_{d|N} \mu(d)\varphi(\frac{N}{d}). \tag{2.5}$$

We record a few observations (also see [23]) concerning the values of $\Psi'(N)$ from the above formula.

$$\Psi'(p) = p - 2 \text{ if } p \text{ is prime} \tag{2.6}$$

$$\Psi'(p^k) = p^{k-2}(p-1)^2 \text{ if } k \geq 2 \tag{2.7}$$

$$\Psi'(mn) = \Psi'(m)\Psi'(n) \text{ if } (m, n) = 1 \tag{2.8}$$

We can also derive the number of primitive even characters $\bmod N$.

Proposition 3. *Let $\Psi^e(n)$ denote the number of primitive even characters $\bmod n$. Then we have,*

$$\Psi^e(n) = \frac{\Psi'(n) + \mu(n) + \delta_2(n)\mu(n/2)}{2},$$

with $\delta_2(n) = 1$ if n is even and 0 otherwise.

Proof. For $n = 1, 2$, we note that there is one even character, namely the trivial one. For n greater than 2, we have :

$$\frac{\varphi(n)}{2} = \sum_{d|n} \Psi^e(d).$$

We obtain the above by counting the number of even characters $\bmod n$ (denoted by $g(n)$ for this proposition). Therefore, by the Möbius inversion formula, we have :

$$\Psi^e(n) = \sum_{d|n} \mu(\frac{n}{d}) g(d) = \frac{1}{2} \sum_{d|n} \mu(\frac{n}{d}) \varphi(d) + \frac{\mu(n)}{2} \mu(n) + \delta_2(n) \frac{\mu(n/2)}{2}.$$

We obtain the proposition by (2.5). □

We mention the following theorem of Artin [3]:

Theorem 17. Let $\chi_1, \dots \chi_n$ be distinct group homomorphisms from an arbitrary group G to $\mathbb{C}^*$. They are linearly independent over $\mathbb{C}$, that is, if

$$\sum_{i=1}^{n} c_i \chi_i \equiv 0,$$

then $c_i = 0$ for all i.

2.2.2 Elementary properties of certain families of arithmetic functions

Throughout, we consider periodic arithmetic functions $f : \mathbb{N} \to \overline{\mathbb{Q}}$ of period q (that is $f(n+q) = f(n)$ for all $n \in \mathbb{N}$) such that $\sum_{a=1}^{q} f(a) = 0$. These arithmetic functions can naturally be extended to the whole of integers $\mathbb{Z}$ by setting $f(n+q) = f(n)$ for all $n \in \mathbb{Z}$. We say that an arithmetic function f of period q is even if $f(-n) = f(n)$ for all $n \in \mathbb{Z}$. Likewise, we say that a periodic function f of period q is odd if $f(-n) = -f(n)$ for all $n \in \mathbb{Z}$. We recall that a periodic arithmetic function f of period q is said to be of Dirichlet type if it is supported on the residue classes of $(\mathbb{Z}/q\mathbb{Z})^*$. For such an algebraic valued function f, by orthogonality relations, we can write :

$$f(n) = \sum_{\chi \neq \chi_0} a_\chi \chi(n),$$

where χ runs over the character group of $(\mathbb{Z}/q\mathbb{Z})^*$, χ_0 denotes the principal character $\bmod q$ and $a_\chi \in \overline{\mathbb{Q}}$. Consequently,

$$L(1,f) = \sum_{\chi \neq \chi_0} a_\chi L(1,\chi). \tag{2.9}$$

If f is even (resp. odd) function of period N, then the sum varies over the even (resp. odd) characters of $(Z/NZ)^*$.

2.2.3 Cyclotomic requisites

We set $\zeta_n = e^{2\pi i/n}$ and $\Phi_n(X)$ as the minimal polynomial of ζ_n over $\mathbb{Q}$. We have the following identity.

Lemma 3. *Let $n = p^k m$ with $(m,p) = 1$ and $k \geq 1$.*

$$\Phi_n(x) = \frac{\Phi_m(x^{p^k})}{\Phi_m(x^{p^{k-1}})}. \tag{2.10}$$

Proof. We claim that $\frac{\Phi_m(x^p)}{\Phi_m(x)}$ is a polynomial, whenever $(m,p) = 1$. This is true as the map $\sigma_p : \mathbb{Q}(\zeta_m) \to \mathbb{Q}(\zeta_m)$ given by $\zeta_m \to \zeta_m^p$ is an automorphism of $\mathbb{Q}(\zeta_m)$. Therefore, any root of $\Phi_m(x)$ is a root of $\Phi_m(x^p)$. To prove the equality of the polynomials $\Phi_n(x)$ and $\frac{\Phi_m(x^{p^k})}{\Phi_m(x^{p^{k-1}})}$, we note that both the polynomials are monic, have same degree, and have the same set of zeros. □

Lemma 4. *Let $n \not\equiv 2 \bmod 4$ and p be a prime divisor of n. Then*

$$Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}(\zeta_n) = \begin{cases} -\zeta_{n/p} & \text{if } p^2 \nmid n \\ 0 & \text{otherwise} \end{cases}. \tag{2.11}$$

Proof. We consider the first case when $p \nmid (n/p)$. Let a and b be integers such that $1/n = a/(n/p) + b/p$. Since $\zeta_n = e^{2\pi i/n}$, we have $\zeta_n = \zeta_p^a \zeta_{n/p}^b$. Then,

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}(\zeta_n) = \zeta_{n/p}^b \mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}(\zeta_p)^a = \zeta_{n/p}^b \sum_{i=1}^{p-1} \zeta_p^i = -\zeta_{n/p}.$$

To conclude the last step, we use the following identity :

$$\sum_{i=0}^{p-1} \zeta_p^i = 0. \tag{2.12}$$

To prove the second case, we first observe that, if $n = p^k m$ with $p \nmid m$ and $k \geq 2$, then

$$\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})) = \{\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \mid a \equiv 1 \bmod p^{k-1}\},$$

where $\sigma_a : \mathbb{Q}(\zeta_n) \to \mathbb{Q}(\zeta_n)$ is the automorphism sending $\zeta_n \to \zeta_n^a$. Therefore,

$$\begin{aligned}
\mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}(\zeta_n) &= \zeta_m \mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}(\zeta_{p^k}). \\
&= \zeta_m \sum_{a=0}^{p-1} \zeta_{p^k}^{1+ap^{k-1}}. \\
&= \zeta_m \zeta_{p^k} \sum_{a=0}^{p-1} \zeta_{p^k}^{ap^{k-1}} = 0,
\end{aligned}$$

where we conclude the last step again by (2.12). □

We mention a few preliminary observations on the norms and traces which will be used in Chapter 7.

Proposition 4. *We have the following :*

1. *If n is a power of a prime p, we have $N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(1-\zeta_n) = p$. Similarly if n has at least two odd prime factors, we have $N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(1-\zeta_n) = 1$.*
2. *Let n be a positive integer. We have $Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n) = \mu(n)$, where μ denotes the Möbius function.*

Proof. To prove the first part of the proposition, it suffices to evaluate $\Phi_n(1)$ as $N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(1-\zeta_n) = \prod_{\substack{i=1 \\ (i,n)=1}}^{n} (1-\zeta_n^i) = \Phi_n(1)$. When n is a prime power say p^k, then the cyclotomic polynomial $\Phi_n(x)$ is given by

$$\Phi_n(x) = \frac{x^{p^k}-1}{x^{p^{k-1}}-1} = \Phi_n(x) = \sum_{i=0}^{p-1} x^{p^{k-1}i}.$$

From the above expression, we conclude $\Phi_n(1) = p$. To prove the second part, we use Lemma 3 and induction. Let n have 2 odd prime factors say $p^i q^j$. Then by substituting $x = 1$ in (2.10), we obtain :

$$\Phi_n(1) = \frac{\Phi_{q^j}(1)}{\Phi_{q^j}(1)} = 1.$$

Thus the statement is true whenever n has two prime factors. Assume that $\Phi_m(1) = 1$ whenever m has at k odd prime factors. We prove the lemma when $n = p^l m$ with $p \nmid m$ that is when n has $k+1$ odd prime factors. By substituting n and $x = 1$ in (2.10), we have $\Phi_n(1) = 1$ from the induction hypothesis.

The proof of second part of the proposition consists of 2 cases, when $p^2 \mid n$ and when n is odd and squarefree. If $p^2 \mid n$ for some prime p, then by Lemma 4, we have

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n) = \mathrm{Tr}_{\mathbb{Q}(\zeta_{n/p})/\mathbb{Q}} \circ \mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}(\zeta_n) = 0.$$

If n is odd and squarefree, then we note that

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n) = \mathrm{Tr}_{\mathbb{Q}(\zeta_{n/p})/\mathbb{Q}} \circ \mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}(\zeta_n) = -\mathrm{Tr}_{\mathbb{Q}(\zeta_{n/p})/\mathbb{Q}}(\zeta_{n/p}).$$

Iterating the above $\omega(n) - 1$ times,

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}\zeta_n = (-1)^{\omega(n)}$$

□

Lemma 5. *Let m, n be two natural numbers. Then we have $\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{(m,n)})$.*

For a proof, see [61].

2.2.4 Dedekind determinant

Definition 9. *Let G be a finite abelian group of order n and $F : G \to \mathbb{C}$ be any complex valued function on G. The determinant of the $n \times n$ matrix $[F(xy^{-1})]_{x,y \in G}$ is called the Dedekind determinant.*

With the same notations as above we have,

Proposition 5. *Let $F : G \to \mathbb{C}$ Then,*

$$|[F(xy^{-1})]_{x,y\in G}| = \prod_{\chi \in \widehat{G}} \Big(\sum_{x \in G} \chi(x) F(x) \Big).$$

Here, we recall that $\widehat{G}$ denotes the group of Dirichlet characters on G for a square matrix A, $|A|$ denotes the determinant of A.

A proof of the above proposition is in [46].

2.3 The p-adic analogies

Throughout, p is an odd prime number. Let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ under the p-adic metric and $\mathbb{Z}_p$ denote the unit ball in $\mathbb{Q}_p$. Let us denote the completion of $\overline{\mathbb{Q}}_p$ under the p-adic metric as $\mathbb{C}_p$. We note that $\mathbb{C}_p$ is non-Archimedean, algebraically closed, and the absolute value is the normalised absolute value i.e $|p|_p = \frac{1}{p}$. We denote $\mathfrak{O}_p$ as the unit ball $\mathbb{C}_p$ (under the absolute value $|.|_p$), and U_p as the units of $\mathfrak{O}_p$. Let us also denote

$$\mathbb{Q}_p[[X]] = \{\sum_{i\geq 0}^{\infty} a_i X^i | a_i \in \mathbb{Q}_p\}.$$
$$\mathbb{Q}_p\{X\} = \{\sum_{i\geq 0}^{\infty} a_i X^i | a_i \to 0\}.$$

We note that $\mathbb{Q}_p[[X]]$ is an integral domain and its field of fractions is given by $\mathbb{Q}_p((X))$. Also, note that $\mathbb{Q}_p\{X\}$ is a ring under usual addition and multiplication. We note that for $f \in \mathbb{Q}_p\{X\}$ $f(a) \in \mathbb{C}_p$ whenever $|a|_p \leq 1$. This can be seen by using the standard fact in p-adic analysis, that a sequence of partial sums $\sum_{n=1}^{k} a_n$ converges if and only if $a_n \to 0$.

For an open subset U of $\mathbb{C}_p$, we say that a function $f : U \to \mathbb{C}_p$ is locally holomorphic if at each point a in the domain U, there exists an open subset $U_a \subset U$ containing a and $g_a \in \mathbb{Q}_p((X))$ (depending on a) such that $f(x) = g_a(x)$ for all $x \in U_a$.

2.3.1 The p-adic logarithm

Consider $x \in p\mathfrak{O}_p$. We can define the p-adic logarithm as follows

$$\log_p(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

and we note that the sum converges. The above logarithm can be extended analytically to whole of $\mathbb{C}_p^*$, and is denoted by $\log_p(x)$. The analytic extension is in such a way that

1. $\log_p(ab) = \log_p(a) + \log_p(b) \qquad \forall a, b \in \mathbb{C}_p^*$.
2. $\log_p(p) = 0$.

If $t \in p\mathbb{Z}_p$, then $(1+t)^{p^k} \in 1 + p^k\mathbb{Z}_p$ for all natural numbers k. Therefore, for $x = \sum_{i=0}^{\infty} a_i p^i \in \mathbb{Z}_p$, we define

$$(1+t)^x := \lim_{k\to\infty} \prod_{i=0}^{k} (1+t)^{a_i p^i}.$$

The existence of the limit is ensured by the fact that $(1+t)^{p^j} \to 1$ as $j \to \infty$. We now state the following proposition :

Proposition 6. *Let $t \in p\mathbb{Z}_p$ and let $f : \mathbb{Z}_p \to \mathbb{C}_p$ be given by $f(x) = (1+t)^x$. Then the derivative of the function f at the origin is $\log_p(1+t)$.*

Remark 2. *Note that the derivative mentioned above is the strict differentiation as defined in [56, Pg. 218], but as mentioned in Pg. 238 ibid., for restricted power series $f \in \mathbb{Q}_p\{X\}$ one may also take the derivative with respect to X and evaluate it at a point $a \in Z_p$.*

2.3.2 p-adic Dirichlet L functions

In this section, we shall look at the analogue of Dirichlet L functions over $\mathbb{C}_p$. For a fixed positve integer $s > 1$, the partial sums

$$S(k) := \sum_{n=1}^{k} \frac{f(n)}{n^s},$$

do not converge in the p-adic setup as $|\frac{f(n)}{n^s}|_p$ do not converge to zero as n tends to infinity. Hence in order to construct the analogue, Kubota and Leopoldt [36] used the analytic extension of the Hurwitz zeta function, and since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, they constructed an analytic function $H_p(s, a, F)$ with values in $\mathbb{C}_p$ interpolating at certain points of $\mathbb{Z}$.

Before proceeding further, following Cohen [20], we define the operator $< . >$: $\mathbb{Q}_p^* \to U_1$, where U_1 denotes the group of 1 units in $\mathbb{Z}_p$ that is

$$U_1 = \{x \in \mathbb{Z}_p | |x-1|_p < 1\}.$$

Definition 10. *The map* $< . >$: $\mathbb{Z}_p^* \to U_1$ *is defined as*

$$< x > := \frac{x}{\omega(x)}.$$

Here, $\omega : \mathbb{Z}_p^* \to \mu_\infty$ *is the Teichmúller character, that is* $\omega(x)$ *is the unique* $(p-1)^{st}$ *root of unity such that* $\omega(x) \equiv x \bmod p$. *The map* $< . >$ *is extended to* $\mathbb{Q}_p^*$ *by setting* $< x > := < x/p^{v_p(x)} >$.

We mention the following theorem from Washington [61].

Theorem 18. Suppose $p \mid F$ and $p \nmid a$. There exists a unique p-adic meromorphic function $H_p(s,a,F)$ on

$$\{s \in \mathbb{C}_p \mid |s|_p < qp^{-1/p-1} > 1\}$$

such that

$$H_p(1-n,a,F) = \omega^{-n}(a) H(1-n,a,F),$$

for all natural numbers $n \geq 1$. In particular when $n \equiv 0 \mod (p-1)$, we have

$$H_p(1-n,a,F) = H(1-n,a,F)$$

The function H_p is analytic except for a simple pole at $s=1$ with residue $\frac{1}{F}$.

The function $H_p(s,a,F)$ is given by

$$H_p(s,a,F) := \frac{1}{s-1}\frac{1}{F} < a >^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{k} (B_j)(\frac{F}{a})^j. \tag{2.13}$$

We would like to interpret $H_p(s,a,F)$ as an "integral".

2.3.2.1 The Volkenborn Integral

Definition 11. *For a function $f \in \mathbb{Q}_p\{X\}$, the Volkenborn integral is defined as follows :*

$$\int_{\mathbb{Z}_p} f(t)dt := \lim_{r\to\infty} \frac{1}{p^r} \sum_{0\leq n<p^r} f(n). \tag{2.14}$$

Moreover, if we define

$$F(x) := \int_{\mathbb{Z}_p} f(x+t)dt$$

Then we have

$$\frac{d}{dx}F(x) = \int_{\mathbb{Z}_p} \frac{\partial}{\partial x} f(x+t)dt. \tag{2.15}$$

For natural numbers a, F such that $p \mid F$, $p \nmid a$, we can re-write $H_p(s, a, F)$ as

$$H_p(s,a,F) = \frac{< a >^{1-s}}{F(s-1)} \int_{\mathbb{Z}_p} < 1 + \frac{F}{a}t >^{1-s} dt. \tag{2.16}$$

The above integral is the same as the infinite series mentioned in the introduction by the example and proposition mentioned in [56, Pg 173,270].

2.3.3 Analogies in Transcendental Number Theory

An element $y \in \mathbb{C}_p$ is said to be transcendental if y is not in the algebraic closure of $\mathbb{Q}$ (considered as a subset of $\mathbb{C}_p$). By abuse of notation, we also denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}_p$ as $\overline{\mathbb{Q}}$. Having defined the p-adic logarithm, we proceed to state Brumer's theorem [14] which is the analogue of Baker's theorem on linear forms in logarithms.

Theorem 19. [Brumer] Let $\alpha_1, \ldots, \alpha_n$ be the elements of U_1 which are algebraic over $\mathbb{Q}$ and whose p-adic logarithms are linearly independent over $\mathbb{Q}$. These logarithms are linearly independent over the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}_p$.

While it is true that the logarithms are linearly independent over $\overline{\mathbb{Q}}$, it is not clear whether the linear combination of logarithm is transcendental. The following is the consequence of the theorem of Kauffmann [33] as noted in [48].

Corollary 5. *[Murty and Saradha] Let K be a number field. There exists a constant $c > 0$ depending on K such that the following holds : Suppose $\alpha_1, \ldots \alpha_m$ are multiplicatively* *independent algebraic numbers in K satisfying* $|\alpha_i - 1|_p \leq p^{-c}$ *for* $1 \leq i \leq m$ *and* $\beta_1, \ldots \beta_m \in K$. *Then the linear form*

$$\beta_1 \log_p \alpha_1 + \cdots + \beta_m \log_p \alpha_m$$

is either zero or transcendental.

Remark 3. *Taking the Iwasawa logarithm, we can remove the condition on c. Indeed, this is true as we can write* $\sum_{i=1}^{n} \beta_i \log_p \alpha_i = \sum_{i=1}^{n} \frac{\beta_i}{n} \log_p \alpha_i^n$. *Therefore, choosing n such that* $|\alpha_i^n - 1|_p < p^{-c}$, *we can say that* $\sum_{i=1}^{n} \beta_i \log_p \alpha_i$ *is either zero or transcendental. The existence of n is guaranteed by the following fact : Let* $K/\mathbb{Q}_p$ *be a finite field extension with ring of integers* O_K *and prime ideal* $\mathfrak{p}$. *Then the quotient group* $(1 + \mathfrak{p}O_K)/1 + \mathfrak{p}^m O_K$ *is finite for all natural numbers m (also see [39, Pg 47]). This was also worked out in [15].*

We end this section with the definition of the p-adic Euler constants as defined by Diamond [22], which is required for Chapter 4.

2.3.4 p-adic Euler constants

Definition 12. *Let* $f \geq 1$ *and* $0 \leq r \leq f - 1$. *When* $v_p(r/f) < 0$, *we define*

$$\gamma_p(r, f) = -\lim_{k \to \infty} \frac{1}{fp^k} \sum_{\substack{m=0 \\ m \equiv r \bmod f}}^{fp^k - 1} log_p(m). \tag{2.17}$$

When $v_p(r/f) \geq 0$, *we write* $f = p^k f^*$, *with* $(p, f^*) = 1$ *and let* $\phi = \varphi(f^*)$. *We then define*

$$\gamma_p(r, f) = \frac{p^\phi}{p^\phi - 1} \sum_{n \in N(r,f)} \gamma_p(r + nf, p^\phi f),$$

where

$$N(r, f) = \{n \mid 0 \leq n < p^\phi, nf + r \not\equiv 0 \bmod p^{\phi + k}\}.$$

We also define $\gamma_p = \gamma_p(0, 1)$.

The convergence of the limit in the above definition relies on the following theorem of Diamond [22, Theorem 2], which we state in its simplified version. This theorem is also required to show the existence of limit in definition of the p-adic generalised Euler-Briggs constants.

Theorem 20 (Diamond). Let a, b, M be rational integers with $a \geq 0$, $b, M \geq 1$. Let R be an open set in $\mathbb{C}_p$ with $a + b\mathbb{Z}_p \subseteq R$. Let B be a Banach space over $\mathbb{C}_p$ and let $f : R \to B$ is locally holomorphic. We define

$$S(k,b) = \frac{1}{bp^k} \sum_{\substack{n=0 \\ n\equiv a \bmod M}}^{Mbp^k-1} f(n).$$

Then the value $L := \lim_{k\to\infty} S(k,b)$ exists and is independent of the choice of b.

The required properties of these constants are mentioned in Appendix A.2. In the real setup, the Euler constants are naturally related with the Hurwitz zeta function that is if $H(s,a,F)$ denotes the Hurwitz zeta series

$$H(s,a,F) = \sum_{n\geq 0} \frac{1}{(a+nF)^s} \qquad \text{where } \Re s > 1,$$

then we have

$$\frac{d}{ds}(s-1)H(s,a,F)|_{s=1} = \gamma(a,F).$$

We believe that a similar analogy was made by Diamond while defining $\gamma_p(a,F)$ whenever $p \mid F$, $p \nmid a$.

Proposition 7. *Let F be a natural number greater than one, with $p \mid F$ and $a < F$ such that $p \nmid a$. We have*

$$\gamma_p(a,F) = \frac{d}{ds}(s-1)H_p(s,a,F)|_{s=1}.$$

Proof. Since $< \frac{F}{a}t + 1 >^s \in \mathbb{Q}_p\{t\}$, we can evaluate

$$\begin{aligned}
\frac{d}{ds}(s-1)H_p(s,a,F) &= \frac{d}{ds}\frac{< a >^{1-s}}{F} \int_{\mathbb{Z}_p} < 1 + \frac{F}{a}t >^{1-s} dt. \\
&= \frac{1}{F}\Big(-\log_p < a >< a >^{1-s} \int_{\mathbb{Z}_p} < 1 + \frac{F}{a}t >^{1-s} dt \\
&+ < a >^{1-s} \int_{\mathbb{Z}_p} \frac{\partial}{\partial s} < 1 + \frac{F}{a}t >^{1-s} dt\Big).
\end{aligned}$$

Evaluating the derivative at $s = 1$, we have

$$\frac{d}{ds}(s-1)H_p(s,a,F)|_{s=1} = \frac{-1}{F}(\log_p < a > + \int_{\mathbb{Z}_p} \log_p < 1 + \frac{F}{a}t > dt) = \gamma_p(a,F),$$

the last equality from Definitions 11 and 12. $\square$

We end this section with the following non-vanishing remark.

Remark 4. *Morita [44] had also defined γ_p earlier as the logarithmic derivative of the p-adic Gamma function. It is noted by Diamond that $\gamma_p(0,1)$ is the same as γ_p upto a rational factor $p/p-1$. Unlike γ, it is not clear from the definition of γ_p that the value is non-zero. Whenever $(p-1)! \not\equiv -1 \mod p^2$, it is highlighted in [20] that the p-adic valuation is 0. The definition of the p-adic gamma function was crucial in the proof. A complete solution to this "non-vanishing" question for other cases is not known to the best of our knowledge.*

Chapter 3

Generalised Euler-Briggs constants -Classical Setup

3.1 Introduction

The relation between the Euler constants $\gamma(a, N)$ (see Definition 2) was investigated by Lehmer [40], and these results were extended in the framework of generalized Euler-Briggs constants $\gamma(\Omega, a, N)$ (see Definition 3) by Gun, Saha and Sinha [29]. The question concerning the transcendence of these constants is open. However, with the help of Theorems 1 and 14, the authors [47] proved results pertaining to transcendence of Euler constants as a family and this was subsequently strengthened in [28].

In this chapter, we investigate the rational linear relations spanned by the generalised Euler-Briggs constants by appealing to Baker's Theorem of linear forms in logarithms of algebraic numbers. This method was already employed in [28, 29]. Here, we improve their results. A portion of this chapter is from [12].

Throughout, Ω will denote a finite set of primes numbers and N will be a natural number greater than 1 co-prime to P_Ω (recall that $P_\Omega = \prod_{p \in \Omega} p$). The breakup of this chapter will be as follows. In Section 3.2, we give explicit relations between a canonical space spanned by the generalised Euler constants $\gamma(\Omega, a, N)$ and the constants $\gamma(a, N)$. This serves as a reduction lemma thereby providing the ease in constructing arithmetical functions required for consequent theorems. In Section 3.3, we improve the dimension estimates of the vector space spanned by Euler constants mentioned in Theorem 3. We require the notations mentioned in Section 2.2.2. We begin with a few notations and remarks.

3.1.1 Notations and Remarks

For a natural number N greater than 1, we work with a fixed number field K such that $K \cap \mathbb{Q}(\zeta_N) = \mathbb{Q}$. For the ease of notations, henceforth, we will omit the dependency on K. Let $F(N)$ denote the set of periodic functions $f : \mathbb{N} \to K$ of period N satisfying $\sum_{a=1}^{N} f(a) = 0$.

Definition 13. *Let $N \geq 1$ be a natural number and $I \subseteq \{1, 2, \ldots, N\}$. We define the $\mathbb{Q}$-vector space $U_{N,I}$ as follows :*

$$U_{N,I} = \{L(1, f) \mid f \in F(N) \text{ and } f(a) = 0 \text{ if } a \notin I\}.$$

We denote $U_{N,\{1,\ldots,N\}}$ as U_N. For $(b, N) = 1$, we define an operator $T_{b,N} : U_N \to U_N$ as follows :

$$T_{b,N}(L(1, f)) := L(1, f_b),$$

where $f_b : \mathbb{N} \to \overline{\mathbb{Q}}$ denotes the arithmetic function $f_b(n) := f(bn)$. Note that if $f \in F(N)$, then $f_b \in F(N)$. From [5, Lemma 4], we note that $L(1, f_b) = 0$ whenever $L(1, f) = 0$ as we can construct an automorphism $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ mapping ζ_N^b to ζ_N and fixing the field $\mathbb{Q}(f(1), \ldots, f(N))$. Moreover $T_{b,N}(L(1, f + g)) = L(1, f_b) + L(1, g_b)$ for $f, g \in F(N)$. This shows that $T_{b,N}$ is linear. Further note that, $T_{b+N,N} = T_{b,N}$ and for $(bc, N) = 1$, $T_{bc,N} = T_{b,N} \circ T_{c,N}$. In particular the set of linear operators,

$$G := \{T_{b,N} | b \in \mathbb{N}, (b, N) = 1\},$$

form a group under composition. Naturally, $G \cong (\mathbb{Z}/N\mathbb{Z})^*$. In this section, we shall consider the action of the group ring $\mathbb{Q}[G]$ on U_N to understand the space spanned by the Euler constants. We remind the reader that the group ring $\mathbb{Q}[G]$ is a commutative ring as G is an abelian group. We end with the following definition.

Definition 14. *Let Ω be a finite set of primes. For any integer N co-prime to P_Ω, and a fixed number field K, let $W_{\Omega,K}(N)$ denote the following K-vector space :*

$$W_{\Omega,K}(N) := \Big\{ \sum_{\substack{a=1\\(a,N)=1}}^{N} \alpha_a \gamma(\Omega, a, N) \mid \alpha_a \in K, \sum_{\substack{a=1\\(a,N)=1}}^{N} \alpha_a = 0 \Big\}. \tag{3.1}$$

We remind the reader that $W_{\Omega,\mathbb{Q}}(N) = W^0_{\Omega,N}$ (See Definition 4). We define $W_{\Omega,\overline{\mathbb{Q}}}(N) = \cup_K W_{\Omega,K}(N)$, where the union is taken over all the number fields K.

Let $y \in W_{\emptyset,\mathbb{Q}}(N)$ i.e. $y = \sum_{\substack{a=1\\(a,N)=1}}^{N} \alpha_a \gamma(a, N)$ such that $\sum_{\substack{a=1\\(a,N)=1}}^{N} \alpha_a = 0$. To this element, we associate a rational arithmetic function f of period N as follows :

$$f(n) = \begin{cases} \alpha_a & \text{if } n \equiv a \bmod N, (n, N) = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Note that $\sum_{n=1}^{N} f(n) = 0$, and therefore by Theorem 14, $L(1, f)$ is defined and is equal to y.

3.2 Some linear relations

The goal of this section is to prove the following theorem :

Theorem 4. For any N, N' co-prime to P_Ω such that $N \mid N'$, we have $W^0_{\Omega,N} \subset W^0_{\Omega,N'}$.

We would require the following lemma.

Lemma 6. *Let Ω be a finite set of primes, $N \geq 1$ be a natural number co-prime to P_Ω. Let $I \subset \{1, \ldots, N\}$ and assume that I satisfies the following condition:*

$$\{pi \bmod N \mid i \in I\} = \{i \bmod N \mid i \in I\} \text{ for all } p \in \Omega. \tag{3.3}$$

Then $$U_{N,I} = \Big\{ \sum_{a=1}^{N} f(a) \gamma(\Omega, a, N) \mid f \in F(N) \text{ and } f(a) = 0 \text{ if } a \notin I \Big\}. \tag{3.4}$$

Proof. Let $f \in F(N)$. By Appendix A.2.2, we have

$$\sum_{\substack{n=1\\(n,P_\Omega)=1}}^{\infty} \frac{f(n)}{n} = \sum_{a=1}^{N} f(a)\gamma(\Omega, a, N). \tag{3.5}$$

We derive another expression for the infinite sum mentioned in (3.5).

$$\begin{aligned}\sum_{\substack{n=1\\(n,P_\Omega)=1}}^{\infty} \frac{f(n)}{n} &= \sum_{d|P_\Omega} \frac{\mu(d)}{d} \sum_{n=1}^{\infty} \frac{f(dn)}{n} &= \Big(\sum_{d|P_\Omega} \frac{\mu(d)}{d} T_{d,N}\Big)(L(1,f)). \\ &= \Big(\prod_{p\in\Omega}(1-\frac{T_{p,N}}{p})\Big)L(1,f). \end{aligned} \tag{3.6}$$

Since any prime $p \in \Omega$ is co-prime to N, we can choose k(depending on p) such that $p^k \equiv 1 \bmod N$. Note that

$$\Big(\sum_{i=0}^{k-1} \frac{T_{p^i,N}}{p^i}\Big)\Big(1-\frac{T_{p,N}}{p}\Big) = \Big(1-\frac{T_{p^k,N}}{p^k}\Big) = \Big(1-\frac{1}{p^k}\Big). \tag{3.7}$$

In particular, the linear map $(1-\frac{T_{p,N}}{p})$ is invertible.

To prove the lemma, we first claim that $(1-\frac{T_{p,N}}{p})U_{N,I} = U_{N,I}$ whenever $p \in \Omega$. To see this, it is enough to show that $T_{p,N}U_{N,I} = U_{N,I}$ after which we can deduce the result from (3.7). Note that if $y \in U_{N,I}$, then $y = L(1,f)$ for some $f \in F(N)$ and $f(a) = 0$ for $a \notin I$. Therefore, $T_{p,N}(y) = L(1,f_p)$, where f_p is the arithmetic function given by $f_p(n) = f(pn)$. From Appendix A.2.1, we have

$$T_{p,N}(y) = \sum_{a=1}^{N} f(pa)\gamma(a,N). \tag{3.8}$$

The condition $f(a) = 0$ if $a \notin I$, in this case will translate to

$$T_{p,N}(y) = \sum_{a\in p^{-1}I} f_p(a)\gamma(a,N), \tag{3.9}$$

where $p^{-1}I$ denotes the set of integers $\{y_i \mid 1 \le y_i \le N \text{ and } py_i \equiv i \bmod N \text{ for } i \in I\}$. From (3.3), we note that $p^{-1}I = I$. Thus we prove the claim.

Since the same holds true for all $p \in \Omega$, we have

$$\prod_{p\in\Omega}(1 - \frac{T_{p,N}}{p})U_{N,I} = U_{N,I}. \tag{3.10}$$

Any element y in the left hand side is the infinite series as mentioned in (3.6) with the arithmetic function f an element of $F(N)$ such that $f(a) = 0$ whenever $a \notin I$ as $L(1, f) \in U_{N,I}$ and therefore by (3.5), we prove the lemma. □

Corollary 6. *Let $f : \mathbb{N} \to \overline{\mathbb{Q}}$ be an arithmetic function of period q co-prime to P_Ω, satisfying the conditions of Theorem 8 and assume that $\sum_{a=1}^{q} f(a) = 0$. Then $\sum_{a=1}^{q} f(a)\gamma(\Omega, a, q) \neq 0$ unless $f \equiv 0$.*

Proof. From (3.7) and (3.6), if $\sum_{a=1}^{q} f(a)\gamma(\Omega, a, q) = 0$, then $L(1, f) = 0$. Therefore from Theorem 8, $f \equiv 0$. □

Remark 5. *For any q co-prime to P_Ω, we note that $I = \{a|1 \le a \le q, (a, q) = 1\} \cup \{q\}$ satisfies the condition mentioned in Lemma 6. Hence, it should be noted that the proof of Lemma works verbatim when the set $U_{N,I}$ is replaced by $W_{\emptyset,\mathbb{Q}}(N)$ In this case, we will have*

$$W_{\emptyset,\mathbb{Q}}(N) = W_{\Omega,\mathbb{Q}}(N). \tag{3.11}$$

From (3.6), we can conclude that whenever $(N, P_\Omega) = 1$ and $f \in U_N$, we can write

$$\sum_{\substack{n=1\\(n,P_\Omega)=1}}^{\infty} \frac{f(n)}{n} = L(1, T(f)), \tag{3.12}$$

where T is a specific linear transform on U_N.

We proceed to the proof of Theorem 4. To show the inclusion, $W_{\Omega,\mathbb{Q}}(N) \subset W_{\Omega,\mathbb{Q}}(N')$, given an element $y \in W_{\Omega,\mathbb{Q}}(N)$, we shall construct a rational arithmetic function g(depending on y) of period N' such that $L(1, g) = 0$.

Proof of Theorem 4. Since P_Ω is co-prime to N, by Remark 5, $W_{\Omega,\mathbb{Q}}(N) = W_{\emptyset,\mathbb{Q}}(N)$. Therefore, it is enough to prove $W_{\emptyset,\mathbb{Q}}(N) \subset W_{\emptyset,\mathbb{Q}}(Np)$ for any prime p not in Ω.

Let $y \in W_{\emptyset,\mathbb{Q}}(N)$. Then $y = L(1, f)$ as mentioned in (3.2). We now consider the arithmetic function g of period Np whose Dirichlet series is given by

$$L(s, g) = (1 - \frac{p}{p^s})L(s, f). \tag{3.13}$$

Explicitly, we can write the arithmetic function g as follows :

$$g(n) = \begin{cases} f(n) & \text{if } (p,n) = 1 \\ f(n) - pf(n/p) & \text{if } p \mid n \end{cases}.$$

Since $f(n) = 0$, whenever $(n, N) > 1$, we have $g(n) = 0$ whenever $(n, N) \neq 1, p$. We also note that $\sum_{a=1}^{pN} g(a) = 0$ and from (3.13), $L(1, g) = 0$. Therefore,

$$\begin{aligned} \sum_{\substack{a=1 \\ (a,Np)=1}}^{pN} g(a)\gamma(a, Np) &= - \sum_{\substack{a=1 \\ (a,Np)=p}}^{pN} g(a)\gamma(a, Np). \\ &= - \sum_{\substack{a=1 \\ (a,N)=1}}^{N} (f(pa) - pf(a))\gamma(pa, Np). \end{aligned}$$

From Appendix A.2.1, we obtain

$$\sum_{\substack{a=1 \\ (a,Np)=1}}^{pN} g(a)\gamma(a, Np) = - \sum_{\substack{a=1 \\ (a,N)=1}}^{N} (f(pa) - pf(a))(\frac{1}{p}\gamma(a, N) - \frac{1}{N}\log(p)).$$

With the above equation, we proceed to the two cases, namely when $p \mid N$ and when $p \nmid N$.

1. When $p \mid N$. By definition, $f(pn) = 0$ for all $n \in \mathbb{N}$. Therefore

$$\sum_{\substack{a=1 \\ (a,Np)=1}}^{pN} g(a)\gamma(a, Np) = L(1, f) = y. \tag{3.14}$$

2. When $p \nmid N$. Since $\sum_{a=1}^{N} f(a) = 0$, we have

$$\sum_{\substack{a=1 \\ (a,Np)=1}}^{pN} g(a)\gamma(a, Np) = (1 - \frac{1}{p}T_{p,N})L(1, f). \tag{3.15}$$

Now for each i such that $0 \leq i \leq ord_N(p) - 1$ (Here $ord_N(p)$ denotes the order of p in $(\mathbb{Z}/N\mathbb{Z})^*$), we define a rational valued periodic arithmetic function g_i

of period Np satisfying the following :

$$L(s, g_i) = (1 - \frac{p}{p^s})L(s, f^i),$$

where $f^i(n) := f(p^i n)$. As highlighted in (3.15),

$$\sum_{\substack{a=1 \\ (a,Np)=1}}^{pN} g_i(a)\gamma(a, Np) = (1 - \frac{1}{p}T_{p,N})L(1, f^i) = (T^i_{p,N} - \frac{1}{p}T^{i+1}_{p,N})L(1, f). \quad (3.16)$$

We have,

$$\sum_{i=0}^{ord_N(p)-1} \frac{1}{p^i}(T^i_{p,N} - \frac{1}{p}T^{i+1}_{p,N}) = (1 - \frac{1}{p^{ord_N(p)}})T_{1,N}. \quad (3.17)$$

Therefore,

$$y = L(1, f) = \frac{p^{ord_N(p)}}{p^{ord_N(p)} - 1} \sum_{\substack{a=1 \\ (a,Np)=1}}^{pN} \sum_{i=0}^{ord_N(p)-1} \frac{g_i(a)}{p^i}\gamma(a, pN). \quad (3.18)$$

This concludes the proof of the theorem as $y \in W_{\emptyset,pN}$.

□

3.3 Improving the dimension estimates

The goal of this section is to prove the following theorem :

Theorem 5. With the same notations as in Theorem 3, we have

$$\dim_{\overline{\mathbb{Q}}} V_{\overline{\mathbb{Q}},N} \asymp_\Omega N^2.$$

We first compare the methods used in Theorems 3 and 5. In the proof of Theorem 3, the authors proved that there exists a positive integer N_0 such that for all u co-prime to N_0, the elements of $\Gamma_{\Omega,u}$, consisting of the generalised Euler-Briggs constants $\gamma(\Omega, a, u)$, $1 \le a \le u$, $(a, u) = 1$ are linearly independent over $\mathbb{Q}$. This is primarily done by the theorem of Baker, Birch and Wirsing (Theorem 8) and a linear independence result on $L(1,\chi)$ for even characters χ (see [28, Theorem 12]) over co-prime periods. To complete the proof, the authors appeal to Bertand's postulate.

Our proof revolves around the linear independence result on $L(1,\chi)$ as χ ranges over all the non-principal primitive even characters. We deviate from the 'coprimality argument' mentioned in [28, Theorem 12]. More precisely, for a fixed integer N greater than one co-prime to P_Ω, we introduce the following subspace of $W_{\Omega,\overline{\mathbb{Q}}}(N)$.

$$W_{\Omega,\overline{\mathbb{Q}}}(N)^e := \{y + T_{-1,N}(y) \mid y \in W_{\Omega,\overline{\mathbb{Q}}}(N)\} \tag{3.19}$$

We remind the reader that if $y \in W_{\Omega,\overline{\mathbb{Q}}}(N)^e$, then $y = L(1,f)$ for some even arithmetic function f of period N as mentioned in (3.2). Moreover,

$$W_{\Omega,\overline{\mathbb{Q}}}(N)^e = \overline{\mathbb{Q}} < L(1,\chi) \mid \chi \text{ non-trivial even character } \bmod N > . \tag{3.20}$$

From here, we can conclude

$$\dim_{\overline{\mathbb{Q}}} V_{\overline{\mathbb{Q}},N} \gg_\Omega \sum_{k \leq N} \Psi^e(k), \tag{3.21}$$

where $\Psi^e(k)$ denotes the number of even primitive characters $\bmod k$. We can replace the arithmetic function Ψ^e by Ψ' (the function counting the number of primitive characters) in the above sum. Computing this estimate with the help of standard analytic number theory tools proves the result. We now prove the following lemma which computes the estimate when k is squarefree. We require the squarefree condition for simplifying the computations.

Lemma 7. *For $\epsilon > 0$, we have*

$$H(N) := \sum_{\substack{n \leq N \\ n \text{ squarefree}}} \Psi'(n) = cN^2 + o(N^{3/2+\epsilon}).$$

Proof. If we write $\Psi'\mu^2 = \alpha * \iota$ (Recall $*$ denotes the Dirichlet convolution of arithmetic functions and ι denotes the identity function), with $\alpha : \mathbb{N} \to \mathbb{C}$, then we have

$$L(s, \Psi'\mu^2) = L(s,\alpha)\zeta(s-1).$$

Therefore, $L(s,\alpha)$ is given by the following Dirichlet series.

$$L(s,\alpha) = \prod_p (1 + \frac{p-2}{p^s}) \prod_p (1 - \frac{p}{p^s}) = \prod_p (1 - \frac{2}{p^s} - \frac{p^2-2p}{p^{2s}}),$$

where p runs over all the primes. Consequently,

$$\alpha(n) = 0 \text{ if } p^3 \mid n \text{ for some prime } p.$$

We follow the method mentioned in [32]. From Theorem 16, we have :

$$H(N) = \frac{N^2}{2}\sum_{n=1}^{\infty}\frac{\alpha(n)}{n^2} + O(N\sum_{n=1}^{N}\frac{\alpha(n)}{n}) + O(N^2\sum_{n\geq N}\frac{\alpha(n)}{n^2}) \tag{3.22}$$

We claim that $\sum_{n\leq N}\frac{|\alpha(n)|}{n} = O(N^{1/2+\epsilon})$ for any ϵ greater than zero. To prove the same, we write any natural number $n = d^2 r$ with d and r squarefree. We note that

$$\sum_{n\leq N}\frac{|\alpha(n)|}{n} = \sum_{\substack{d\leq\sqrt{N}\\ d \text{ squarefree}}}\frac{|\alpha(d^2)|}{d^2}\sum_{\substack{r\leq N/d^2\\ r \text{ squarefree}}}\frac{|\alpha(r)|}{r}. \tag{3.23}$$

Noting that $|\alpha(p^2)| < p^2$ for primes p and therefore $|\alpha(n^2)| < n^2$ for squarefree n (as α is a convolution of multiplicative functions and hence multiplicative), we have

$$\sum_{n\leq N}\frac{|\alpha(n)|}{n} \leq \sqrt{N}\sum_{r\leq N}\frac{d(r)}{r},$$

where $d(r)$ denotes the number of divisors of r. The claim is proved by the partial summation formula and the following asymptotic formula (see Appendix A.3.2)

$$\sum_{n\leq N} d(n) = N\log N + (2\gamma-1)N + O(\sqrt{N}).$$

We still have to prove that

$$\sum_{n>N}\frac{|\alpha(n)|}{n^2} = O(\frac{1}{\sqrt{N}}).$$

Similar to (3.23), we have

$$\lim_{M\to\infty}\sum_{N\leq n\leq M}\frac{|\alpha(n)|}{n^2} = \lim_{M\to\infty} O(\sum_{\substack{\sqrt{N}\leq n\leq\sqrt{M}\\ n \text{ squarefree}}}\frac{|\alpha(n^2)|}{n^4}) = O(\frac{1}{\sqrt{N}}).$$

The above expression is true as the series $L(2,d)$ is convergent. Consequently, $H(N) = cN^2 + o(N^{3/2+\epsilon})$ for any ϵ greater than zero for the explicit constant c given by $c = L(2,\alpha)/2$. □

We now complete the proof of Theorem 5.

Proof. For a fixed natural number N, note that

$$V_{\overline{\mathbb{Q}},N} \supseteq \sideset{}{'}\sum_{\substack{n \leq N \\ (n,P_\Omega)=1}} W^e_{\emptyset,\overline{\mathbb{Q}}}(n), \tag{3.24}$$

where the summation runs over squarefree integers n. We note that the vector space generated by the sum in the right side is given by

$$\overline{\mathbb{Q}} \left\langle L(1,\chi) \mid \chi \text{ non-trivial even character } \bmod n \text{ where } (n,P_\Omega)=1, n \leq N, n \text{ squarefree } \right\rangle.$$

If χ and χ' are two characters whose associated primitive character is same, then $L(1,\chi)$ and $L(1,\chi')$ differ by an algebraic multiple. Therefore, the generating set is reduced to the the special values $L(1,\chi)$ as χ varies over the squarefree primitive non-principal even characters $\bmod n$ for $n \leq N$ and co-prime to P_Ω. By Corollary 3, we note that these values are linearly independent over $\overline{\mathbb{Q}}$. Therefore,

$$\dim_{\overline{\mathbb{Q}}} \sideset{}{'}\sum_{\substack{n \leq N \\ (n,P_\Omega)=1}} W_{\emptyset,\overline{\mathbb{Q}}}(n)^e = \frac{1}{2} \sideset{}{'}\sum_{\substack{n \leq N \\ (n,P_\Omega)=1}} \Psi'(n) + O(N). \tag{3.25}$$

We get the term $O(N)$ by Proposition 3. To get the desired bound, from (3.25) we need to estimate $\sideset{}{'}\sum_{\substack{n \leq N \\ (n,P_\Omega)=1}} \Psi'(n)$.

$$\sideset{}{'}\sum_{\substack{n \leq N \\ (n,P_\Omega)=1}} \Psi'(n) = \sum_{d \mid P_\Omega} \mu(d) \sum_{\substack{n \leq N/d \\ (n,d)=1}} \Psi'(dn) = \sum_{d \mid P_\Omega} \mu(d)\Psi'(d) \sum_{\substack{n \leq N/d \\ (n,d)=1}} \Psi'(n), \tag{3.26}$$

as Ψ' is multiplicative. For any divisor $d \mid P_\Omega$, we denote

$$H_d(N) := \sum_{\substack{n \leq N \\ (n,d)=1}} \Psi'(n).$$

For a prime p not dividing d, we have :

$$H_{pd}(N) = H_d(N) - \Psi'(p) H_d\left(\left\lfloor \frac{N}{p} \right\rfloor\right).$$

Therefore,

$$H_{pd}(N) = \sum_{i=0}^{\lfloor \frac{\log N}{\log p} \rfloor} (-\Psi'(p))^i H_d\left(\lfloor \frac{N}{p^i} \rfloor\right) \tag{3.27}$$

If $H_d(N) = c_d N^2 + o(N^{3/2+\epsilon})$, then

$$H_{pd}(N) = \frac{c_d}{1 - \frac{\Psi'(p)}{p^2}} N^2 + o(N^{3/2+\epsilon}), \tag{3.28}$$

as $\Psi'(p) < p^2$. In particular, this holds true for $d = 1$. Thus for any divisor $d \mid P_\Omega$, by Lemma 7 we have

$$H_d(N) = \frac{c}{\prod_{p|d} (1 - \frac{\Psi'(p)}{p^2})} N^2 + o(N^{3/2+\epsilon}).$$

Substituting in (3.26), we get that .

$$\begin{aligned} \sum_{\substack{n \le N \\ (n,P_\Omega)=1 \\ n \text{ squarefree}}} \Psi'(n) &= c \sum_{d|P_\Omega} \frac{\mu(d)\Psi'(d)}{d^2} \frac{1}{\prod_{p|d} (1 - \frac{\Psi'(p)}{p^2})} N^2 + o(N^{3/2+\epsilon}). \\ &= c_\Omega N^2 + o(N^{3/2+\epsilon}). \end{aligned} \tag{3.29}$$

Here,

$$c_\Omega = \frac{L(2,\alpha)}{2} \sum_{d|P_\Omega} \frac{\mu(d)\Psi'(d)}{d^2} \frac{1}{\prod_{p|d} (1 - \frac{\Psi'(p)}{p^2})}.$$

From (3.24), along with the fact that $\dim_{\overline{\mathbb{Q}}} V_{\overline{\mathbb{Q}},N} = O(N^2)$ (as mentioned in [28]) we prove the theorem. □

Chapter 4

On p adic Euler Constants

The p-adic Dirichlet series $L_p(s,\chi)$ for a Dirichlet character χ was first constructed by Kubota and Leopoldt [36]. They interpolated values of $L(s,\chi)$ at certain negative integers and p-adically extended it to the whole of $\mathbb{Z}_p$. It was unknown if the p-adic regulator of the zeta function corresponding to the maximal real subfield of cyclotomic field is non-zero, and this was answered by Brumer [14] by proving the analogue of Baker's theorem of linear forms logarithms of non-zero algebraic numbers. Soon, Morita [44] had defined the p-adic analogue of the Gamma function, and Diamond [22] had introduced the p-adic analogue of the log Gamma function.

With the definitions of the p-adic zeta and gamma functions, we can define the p-adic Euler constants. The analogue by Briggs was defined by Diamond and he proved several properties akin to the ones proved by Lehmer. The transcendence of these constants were studied in [15, 48]. In [53], an analogue of p adic periodic Dirichlet series was defined. If f is an even periodic function of period F, the p-adic periodic Dirichlet series $L_p(s,f)$ is defined by

$$L_p(s,f) = \sum_{\substack{a=0 \\ (a,p)=1}}^{\mathcal{K}} f(a) H_p(s,a,\mathcal{K}),$$

where $\mathcal{K}$ denotes the least common multiple of F and p. Recall that $H_p(s,a,\mathcal{K})$ is the series given in (2.13). In [48], the transcendence of $L_p(1,f)$ for certain periodic functions f was also considered.

In this chapter, we first define the p-adic analogue of the infinite series $\sum_{n\geq 1} \frac{f(n)}{n}$ for a periodic function f of period F. The major issue with the function $L_p(s,f)$ is that it is not compatible with the definition of the Euler constants as defined by

Diamond [22] when $p \nmid F$. Hence, we need to formulate a p-adic analogue of the Hurwitz zeta function $H_p(s,a,F)$ when $p \nmid F$. We follow the underlying idea of Diamond as mentioned in 4.0.1.

The construction helps us to define $\overline{L_p}(s,f)$ when f is periodic of period F and we discuss about its value at $s = 1$. In fact, by using Gauss formula relating the constants $\gamma_p(a,F)$ with linear forms in logarithms of algebraic numbers (see Appendix A.2.6), we get an expression for $\overline{L_p}(1,f)$ exactly analogous to Theorem 14. We also introduce the p-adic analogue of generalised Euler Briggs constant under certain conditions on Ω. We discuss about the transcendence of families associated to the Euler-Briggs constant by appealing to a theorem of Baker, Birch and Wirsing in this context and finally compute the dimension estimates on $\gamma_p(\Omega,a,F)$ akin to the ones in the previous chapter. This chapter is mainly from [11].

This chapter is organised as follows. In Section 4.1, we construct a function $\overline{H_p}(s,a,F)$ which can be thought as an analogue of the Hurwitz zeta function in the p-adic setup and define the p-adic series $\overline{L_p}(s,f)$, for a periodic function f of period F. Section 4.2 consists of the p-adic analogue of generalised Euler Briggs constant and some of their properties. Finally in Section 4.3, we discuss about the theorem of Baker, Birch and Wirsing from which we conclude about the transcendence of the families associated to Euler constants. We begin with the motivation for defining the twisted Hurwitz zeta series.

4.0.1 Motivation

Throughout this chapter, p denotes an odd prime. Recall that for a positive integer F divisible by a prime p and an integer a such that $0 \leq a \leq F-1$ with $(a,p) = 1$, we have the following integral expression for $H_p(s,a,F)$ (see 2.16).

$$H_p(s,a,F) = \frac{< a >^{1-s}}{F(s-1)} \int_{\mathbb{Z}_p} < 1 + \frac{F}{a} t >^{1-s} dt. \tag{4.1}$$

We note that $H_p(s,a,F) = H_p(s,F-a,F)$ (see [56, Proposition 4, Pg. 268]). We also have a well known distribution formula for $H_p(s,a,F)$(see [20, Pg 286]):

$$H_p(s,a,F) = \sum_{n=0}^{r-1} H_p(s,a+nF,Fr), \tag{4.2}$$

where r is a natural number greater than one. The distribution relation states the following : If $\pi_{r,F}$ denotes the natural surjection from the groups $\mathbb{Z}/rF\mathbb{Z} \to \mathbb{Z}/F\mathbb{Z}$, then

$$H_p(s, \bar{a}_F, F) = \sum_{b \in \pi_{r,F}^{-1}(a)} H_p(s, \bar{b}_{rF}, rF). \tag{4.3}$$

where $\bar{a}_F$ denotes the representative of a mod F in the set $\{0, \ldots, F-1\}$.

When $p \nmid F$, we observe that the Volkenborn integral (4.1) does not exist. In order to define a p-adic analogue for the Hurwitz zeta function when $p \nmid F$, we first construct a suitable series in the Archimedean setup. Let $H(s, a, F)$ denote the following Dirichlet series :

$$H(s, a, F) = \sum_{n \geq 0} \frac{1}{(a+nF)^s} \qquad \text{where } \Re s > 1. \tag{4.4}$$

We note that the distribution relation mentioned in (4.3) is also valid for this Dirichlet series and therefore, we obtain the following equality when $r = p$.

$$H(s, a, F) - \frac{1}{p^s} H(s, \overline{ap^{-1}}_F, F) = \sum_{\substack{n=0 \\ n \neq \overline{-aF^{-1}}_p}}^{p-1} H(s, a+nF, pF). \tag{4.5}$$

As we vary a over the set $\{\overline{ap^{-i}}_F\}_{i=0}^{\phi-1}$ (ϕ denotes the order of p in $(\mathbb{Z}/FZ)^*$), we obtain the following identity.

$$(1 - \frac{1}{p^{\phi s}}) H(s, a, F) = \sum_{i=0}^{\phi-1} \frac{1}{p^{is}} \sum_{n=0}^{p-1}{}' H(s, \overline{ap^{-i}}_F + nF, pF), \tag{4.6}$$

where the last sum runs over integers n such that $\overline{ap^{-i}}_F + nF \neq 0$ mod p. Since we have the p-adic analogues for the Dirichlet series in the right side, it is enough to give the corresponding analogue for the Dirichlet polynomial $1/p^{is}$. To do so, we revert back to the p-adic setup. When $p \mid F_1$ and $p \nmid ad$, from (2.13) we have

$$H_p(s, da, dF_1) = \frac{< d >^{1-s}}{d} H_p(s, a, F_1).$$

This shows that $< d >^{1-s} /d$ is the required analogue of the Dirichlet polynomial d^{-s} and when $d = p$, we will have a factor $1/p$ as $< p >^{1-s} = 1$. This allows us to 'extend' the definition of $H_p(s, a, F)$ when $p \nmid F$.

We again consider (4.6). We have

$$(1-\frac{1}{p^{\phi s}})H(s,a,F) = \sum_{i=0}^{\phi-1}\sum_{n=0}^{p-1}{}' H(s,p^i(\overline{ap^{-i}}_F + nF), p^{i+1}F).$$

where the last sum runs over integers n such that $\overline{ap^{-i}}_F + nF \neq 0 \bmod p$. For each i in the given range, we apply distribution formula (4.3) with $r = p^{\phi-i-1}$ and $F = p^{i+1}F$. Hence, we can write

$$(1-\frac{1}{p^{\phi s}})H(s,a,F) = \sum_{n\in N(a,F)} H(s,a+nF,p^{\phi}F), \tag{4.7}$$

for a finite set $N(a,F)$ (depending on ϕ). It remains to determine the set $N(a,F)$. We note that for $p \nmid b$, and $0 \le i \le \phi-1$, we have

$$\pi^{-1}_{p^{\phi-i-1},p^{i+1}F}(p^ib) = \{p^ib + mp^{i+1}F \,:\, 0 \le m < p^{\phi-i-1}\}.$$

Therefore, the number of elements in $N(a,F)$ is $(p-1)(\sum_{i=0}^{\phi-1} p^{\phi-i-1}) = p^{\phi}-1$. All these elements are congruent to a mod F and not a multiple of p^{ϕ}. Therefore,

$$N(a,F) = \{n \,:\, 0 \le n < p^{\phi}, p^{\phi} \nmid (a+nF)\}.$$

The same principle was followed by Diamond [22], while defining the p-adic analogue of Euler constants in Definition 12.

4.1 On $\overline{H_p}(s,a,F)$ and $\overline{L_p}(s,f)$

Following Section 4.0.1, we define $\overline{H}_p(s,a,F)$ for non-negative integers a and F with $a < F$.

Definition 15. *We define $\overline{H_p}(s,0,1)$ as*

$$\overline{H_p}(s,0,1) = \frac{p}{p-1}\sum_{a=1}^{p-1} H_p(s,a,p).$$

For $F > 1$ and $0 \le a < F$ with $v_p(a) < v_p(F)$, we set

$$\overline{H_p}(s,a,F) = \frac{1}{p^k}H_p(s,\frac{a}{p^k},\frac{F}{p^k}) \qquad \text{if } p^k \;||\; (a,F).$$

If we further assume that $p \nmid F$ and $(a, F) = 1$, we set

$$\overline{H_p}(s, a, F) = \frac{p^{\phi}}{p^{\phi} - 1} \sum_{n \in N(a,F)} \overline{H_p}(s, a + nF, p^{\phi} F)$$

where ϕ is the order of $p \bmod F$ and

$$N(a, F) = \{0 \leq n < p^{\phi} \ : \ v_p(a + nF) < \phi\}.$$

Finally if $d = (a, F)$ and $d > 1$ we define

$$\overline{H_p}(s, a, F) = \frac{< d >^{1-s}}{d} \overline{H_p}(s, \frac{a}{d}, \frac{F}{d}).$$

Remark 6. *We have the following equivalent expressions for $\overline{H_p}(s, a, F)$ which we use frequently for computational reasons.*

6.1 *When $p \nmid F$, we have*

$$\overline{H_p}(s, a, F) = \frac{p^{\phi}}{p^{\phi} - 1} \sum_{i=0}^{\phi-1} \frac{1}{p^i} \sum_{n=0}^{p-1}{}' H_p(s, \overline{ap^{-i}}_F + nF, pF), \tag{4.8}$$

where the integers n runs over all elements such that $\overline{ap^{-i}} + nF \not\equiv 0 \bmod p$. ***We assume this condition implicitly whenever we write*** *$\sum'$. We obtain this expression due to the equality of (4.6) and (4.7).*

6.2 *For a fixed F co-prime to p, let ϕ_1 be such that $p^{\phi_1} \equiv 1 \bmod F$. Then we have*

$$\overline{H_p}(s, a, F) = \frac{p^{\phi_1}}{p^{\phi_1} - 1} \sum_{n \in N_1(a,F)} \overline{H_p}(s, a + nF, p^{\phi_1} F).$$

where $N_1(a, F) = \{0 \leq n < p^{\phi_1} \ : \ v_p(a + nF) < \phi_1\}$.
The above is true as (4.6) holds when we replace ϕ by ϕ_1.

We prove the following distribution formula for $\overline{H_p}(s, a, F)$ providing all the details. We remark here that any proof of Appendix A.2.5 will follow through here. The proof purely relies on the distribution formula (4.2) and appropriate rearrangements.

Proposition 8. *Let N be a natural number greater than 1. We have the following distribution relation.*

$$\overline{H_p}(s,a,F) = \sum_{n=0}^{N-1} \overline{H_p}(s,a+nF,NF). \tag{4.9}$$

Proof. Without loss of generality, we assume that $(a,F)=1$ and by the surjective group homomorphism $\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ whenever $N \mid M$, it suffices to do the same when N is a prime. Indeed, if it is true for primes q and r, then it is true for $F = qr$ as shown below.

$$\begin{aligned}\overline{H_p}(s,a,F) &= \sum_{n=0}^{q-1} \overline{H_p}(s,a+nF,pF) = \sum_{n=0}^{q-1}\sum_{m=0}^{r-1} \overline{H_p}(s,a+nF+mqF,qrF). \\ &= \sum_{n=0}^{qr-1} \overline{H_p}(s,a+nF,qrF).\end{aligned}$$

If $p \mid F$, we have the distribution relation as mentioned in Equation (4.2) for $H_p(s,a,F)$, so we need to consider the case when $p \nmid F$. We have two cases namely when $N=p$ and $N \neq p$. Let us denote

$$\mathcal{F}(s) := \sum_{n=0}^{p-1} \overline{H_p}(s,a+nF,pF).$$

On evaluation of $\mathcal{F}(s)$, we obtain,

$$\begin{aligned}\mathcal{F}(s) &= \sum_{\substack{n=0 \\ n \neq \overline{-aF^{-1}}_p}}^{p-1} \overline{H_p}(s,a+nF,pF) + \overline{H_p}(s,a+\overline{-aF^{-1}}_pF,pF). \\ &= \sum_{\substack{n=0 \\ n \neq \overline{-aF^{-1}}_p}}^{p-1} \overline{H_p}(s,a+nF,pF) + \frac{1}{p}\overline{H_p}(s,\overline{ap^{-1}}_F,F).\end{aligned}$$

From Equation (4.8), we have

$$\overline{H_p}(s,\overline{ap^{-1}}_F,F) = \frac{p^\phi}{p^\phi - 1}\sum_{i=0}^{\phi-1}\frac{1}{p^i}{\sum_{n=0}^{p-1}}' H_p(s,\overline{ap^{-i-1}}_F+nF,pF).$$

Note that when $i = \phi - 1$, we have $\overline{ap^{-i-1}}_F + nF = a + nF$. Therefore, substituting this in the above, by Remark 6.1 we get

$$\begin{aligned}
\mathcal{F}(s) &= \sum_{\substack{n=0 \\ n \neq \overline{-aF^{-1}}_p}}^{p-1} H_p(s, a+nF, pF) + \frac{1}{p^\phi - 1} \sum_{\substack{n=0 \\ n \neq \overline{-aF^{-1}}_p}}^{p-1} H_p(s, a+nF, pF) \\
&+ \frac{1}{p}\Big(\frac{p^\phi}{p^\phi - 1} \sum_{i=0}^{\phi-2} \frac{1}{p^i} \sum_{n=0}^{p-1}{}' H_p(s, \overline{ap^{-i-1}}_F + nF, pF)\Big) \\
&= \frac{p^\phi}{p^\phi - 1}\Big(\sum_{\substack{n=0 \\ n \neq \overline{-aF^{-1}}_p}}^{p-1} H_p(s, a+nF, pF) \\
&+ \sum_{i=0}^{\phi-2} \frac{1}{p^{i+1}} \sum_{n=0}^{p-1}{}' H_p(s, \overline{ap^{-i-1}}_F + nF, pF)\Big). \\
&= \frac{p^\phi}{p^\phi - 1}\Big(\sum_{i=0}^{\phi-1} \frac{1}{p^i} \sum_{n=0}^{p-1}{}' H_p(s, \overline{ap^{-i-1}}_F + nF, pF)\Big) \\
&= \overline{H_p}(s, a, F).
\end{aligned}$$

We still need to consider the case when $N \neq p$ and N is a prime. We use the following notations :

$$\begin{aligned}
D_\infty = \mathbb{Q}\left\langle \frac{1}{d^s} \;:\; d \in \mathbb{N} \right\rangle, D_p = \mathbb{Q}\left\langle \frac{< d >^{1-s}}{d} \;:\; d \in \mathbb{N} \right\rangle \\
\mathcal{H}_\infty = \mathbb{Q}\left\langle H(s,a,F) \;:\; 0 \leq a \leq F-1, v_p(a) < v_p(F), F \in \mathbb{N} \right\rangle \\
\mathcal{H}_p = \mathbb{Q}\left\langle \overline{H_p}(s,a,F) \;:\; 0 \leq a \leq F-1, v_p(a) < v_p(F), F \in \mathbb{N} \right\rangle.
\end{aligned}$$

We define a group homomorphism $\iota_H : \mathcal{H}_\infty \to \mathcal{H}_p$ by

$$\iota_H(H(s,a,F)) = \overline{H_p}(s,a,F)$$

and extend this map $\mathbb{Q}$ linearly to whole of $\mathcal{H}_\infty$. This map is well defined as the distribution relation holds for both $H(s,a,F)$ and $H_p(s,a,F)$. Also, we note that this map is well behaved under the action of D_∞ on $\mathcal{H}_\infty$ and D_p on $\mathcal{H}_p$. More precisely, let $\iota_D : D_\infty \to D_p$ is given by the homomorphism :

$$\iota_D(\frac{1}{d^s}) = \frac{< d >^{1-s}}{d},$$

and extended $\mathbb{Q}$ linearly to D_∞. For $x \in D_\infty$ and $\alpha \in \mathcal{H}_\infty$, we have the following :

$$\iota_H(x\alpha) = \iota_D(x)\iota_H(\alpha).$$

For $p \nmid F$, from (4.7) with ϕ as order of p mod F, we have $(1 - \frac{1}{p^{\phi s}})H(s,a,F) \in \mathcal{H}_\infty$. From the definition of $\overline{H_p}(s,a,F)$, we have

$$\iota_H((1 - \frac{1}{p^{\phi s}})H(s,a,F)) = (1 - \frac{1}{p^{\phi}})\overline{H_p}(s,a,F). \tag{4.10}$$

Therefore, by (4.10) and with ϕ_1 as order of p mod NF, we have

$$\iota_H((1 - \frac{1}{p^{\phi_1 s}})H(s,a,F)) = (1 - \frac{1}{p^{\phi_1}})\overline{H_p}(s,a,F). \tag{4.11}$$

Now, applying the distribution formula for $H(s,a,F)$ in (4.11), we get

$$\begin{aligned}
\iota_H((1 - \frac{1}{p^{\phi_1 s}})H(s,a,F)) &= \iota_H((1 - \frac{1}{p^{\phi_1 s}})\sum_{i=0}^{N-1} H(s,a+iF,NF)) \\
&= \sum_{i=0}^{N-1} \iota_H((1 - \frac{1}{p^{\phi_1 s}})H(s,a+iF,NF)) \\
&= \sum_{i=0}^{N-1} (1 - \frac{1}{p^{\phi_1}})\overline{H_p}(s,a+iF,NF)
\end{aligned} \tag{4.12}$$

Comparing (4.11) with (4.12), we arrive at the result. □

Now, with the definition of $\overline{H_p}(s,a,F)$ and the above proposition, we can unambiguously define an analogue of periodic Dirichlet series $\overline{L_p}(s,f)$ in this context.

Definition 16. *Let $f : \mathbb{N} \to \mathbb{C}_p$ be a periodic function of period F. We define*

$$\overline{L_p}(s,f) := \sum_{a=0}^{F-1} f(a)\overline{H_p}(s,a,F).$$

Remark 7. *We make the following remarks.*

7.1 *If f and g are two periodic functions, then $\overline{L_p}(s,f) + \overline{L_p}(s,g) = \overline{L_p}(s,f+g)$. This is a consequence of Proposition 8.*

7.2 *The map $f \mapsto \overline{L_p}(s,f)$ need not be injective when the function f is even. Consider the example $f : \mathbb{N} \to \mathbb{Q}$ of period p, such that $f(n) = 1$ whenever $(n,p) = 1$ and $f(n) = -(p-1)$ otherwise. Therefore,*

$$\overline{L_p}(s,f) = \frac{f(0)}{p}\overline{H_p}(s,0,1) + \sum_{a=1}^{p-1} \overline{H_p}(s,a,p).$$

From the definition, we have $\overline{L_p}(s,f) \equiv 0$. More generally, the same is true whenever f is a periodic function of period p^k such that $f(n) = 1$ whenever $(n,p) = 1$, $f(np^k) = -\varphi(p^k)$ and $f(n) = 0$ otherwise.

7.3 *$\overline{L_p}(s,\chi)$ is indeed different from the Kubota p-adic L function $L_p(s,\chi)$ of conductor k. We have*

$$L_p(s,\chi) = \sum_{\substack{a=0 \\ (a,p)=1}}^{kp} \chi(a)H_p(s,a,kp).$$

On the contrary, we have,

$$\overline{L}_p(s,\chi) = \sum_{a=0}^{k-1} \chi(a)\overline{H_p}(s,a,k) = \sum_{a=0}^{kp-1} \chi(a)\overline{H_p}(s,a,kp)$$

$$= \sum_{\substack{a=0 \\ (a,p)=1}}^{kp} \chi(a)H_p(s,a,kp) + \sum_{0 \le a < k} \chi(pa)\overline{H_p}(s,pa,kp).$$

$$\implies (1 - \frac{\chi(p)}{p})\overline{L}_p(s,\chi) = L_p(s,\chi).$$

So upto an algebraic "Euler" factor, both the p-adic L functions are equal.

7.4 *Let f be an even periodic function of period F. If we consider $L_p(s,f)$ as mentioned in the introduction, we have, $L_p(s,f) = \overline{L_p}(s,f\chi_0)$ where χ_0 denotes the principal character $\bmod p$.*

As observed in Remark 7.2, there may exist non-zero even periodic functions f such that $\overline{L_p}(s,f) \equiv 0$. We rule out this scenario in the case when $p \nmid F$. More precisely,

Theorem 21. Let F be a natural number greater than one, co-prime to p and let f be a non-zero arithmetic function of period F taking algebraic values. Then we have $\overline{L_p}(s,f) \not\equiv 0$.

The strategy of the proof is to first show that the functions $\{H_p(s,a,pF) : 1 \le a \le pF, (a,p) = 1\}$ are linearly independent over $\overline{\mathbb{Q}}$. This is achieved by transferring the same to the real setup using analytic continuation. We then proceed to prove that if $f : \mathbb{N} \to \overline{\mathbb{Q}}$ is an arithmetic function of period F co-prime to p, such that

$$\sum_{a=0}^{F-1} f(a)\overline{H_p}(s,a,F) \equiv 0,$$

then $f \equiv 0$. The proof involves a valuation argument.

Proof of Theorem 21. Let F be greater than one such that $p \mid F$. Suppose there exists $\alpha_a \in \overline{\mathbb{Q}}$ such that

$$\sum_{\substack{a=1\\(a,p)=1}}^{F/2} \alpha_a H_p(s,a,F) = 0$$

Then on substituting $s = 1-k$ where k runs over the positive integers satisfying $k \equiv 0 \bmod (p-1)$ and noting that $H_p(1-k,a,F) = H(1-k,a,F)$ for such integers k (See [61, Theorem 5.9, Pg 55]), we have

$$\sum_{\substack{a=1\\(a,p)=1}}^{F/2} \alpha_a H(1-k,a,F) = 0 \tag{4.13}$$

We now construct an even arithmetic function g of period F as follows :

$$g(n) = \begin{cases} \alpha_a & \text{if } (n,p) = 1 \text{ and } n \equiv \pm a \bmod F \\ 0 & \text{otherwise} \end{cases}.$$

Note that from (4.13) and using $H(1-k,a,F) = H(1-k,F-a,F)$ we obtain

$$\sum_{a=1}^{F} g(a)H(1-k,a,F) = 0 \implies L(1-k,g) = 0, \text{ whenever } k \equiv 0 \bmod (p-1).$$

Now, from [50, Lemma 2.1], we have $L(k,\widehat{g}) = 0$ for positive integers $k \equiv 0 \bmod (p-1)$ and consequently $\widehat{g} \equiv 0$ (See [2, Theorem 11.3]). This implies $g \equiv 0$. Therefore, we have the following :

$$\sum_{\substack{a=1\\(a,p)=1}}^{F/2} \alpha_a H_p(s,a,F) = 0 \implies \alpha_a = 0 \text{ for all } a \text{ such that } 1 \le a \le F/2,\ (a,p) = 1 \tag{4.14}$$

Now let f be an even algebraic arithmetic function of period F coprime to p such that
$$\sum_{a=0}^{F-1} f(a)\overline{H_p}(s,a,F) = 0.$$
We therefore have

$$f(0)\frac{< F >^{1-s}}{F}\overline{H_p}(s,0,1) + \frac{p^\phi}{p^\phi - 1}\sum_{a=0}^{F-1} f(a)\sum_{i=0}^{\phi-1}\frac{1}{p^i}\sum_{n=0}^{p-1}{}' \overline{H_p}(s,\overline{ap^{-i}}_F + nF, pF) = 0,$$

where ϕ denotes the order of p in $(\mathbb{Z}/F\mathbb{Z})^*$. Re-writing the equation, we get

$$\frac{p}{p-1}f(0)\sum_{n=1}^{p-1} H_p(s,nF,pF) + \frac{p^\phi}{p^\phi - 1}\sum_{\substack{a=1\\(a,p)=1\\F\nmid a}}^{pF}\sum_{i=0}^{\phi-1}\frac{f(ap^i)}{p^i}H_p(s,a,pF) = 0.$$

Since f is even, as mentioned in (4.14) we have $f(0) = 0$ and for all a such that $(a,p) = 1$,
$$\sum_{i=0}^{\phi-1}\frac{f(ap^i)}{p^i} = 0. \tag{4.15}$$
Note that the above equality is also valid when $p \mid a$ as we can replace a by $a + F$.

Suppose $f(b) \neq 0$ for some b. Let $M_{[b]} = \min\{v_p(f(bp^i)) : 0 \leq i \leq \phi - 1\}$. We note that there exists j such that $M_{[b]} = v_p(f(bp^j))$. In (4.15), by replacing a with bp^{j+1}, we obtain
$$\sum_{i=0}^{\phi-1}\frac{f(bp^{i+j+1})}{p^i} = 0.$$
But this is not possible, as

$$v_p(\sum_{i=0}^{\phi-1}\frac{f(bp^{i+j+1})}{p^i}) = M_{[b]} - \phi + 1,$$

a contradiction to (4.15). Hence $f(b) = 0$ for all b. □

4.2 Euler constants and L-series

We begin this section, by producing the first two terms of the Laurent series expansion of $\overline{H_p}(s,a,F)$ around $s=1$. From [34], we recall that we can write

$$\gamma_p = \frac{p}{p-1}\sum_{a=1}^{p-1}\gamma_p(a,p). \tag{4.16}$$

In particular the above shows that γ_p is the derivative of $(s-1)\overline{H_p}(s,0,1)$ evaluated at $s=1$. We set $\gamma_p(0,F) := \frac{1}{F}\gamma_p - \frac{\log_p F}{F}$ so that we realise it as the derivative of $(s-1)\overline{H_p}(s,0,F)$ evaluated at $s=1$.

Corollary 7. *Let F be a natural number greater than or equal to 1 and let $0 \le a < F$. Around $s=1$, we have the following expansion of $\overline{H_p}(s,a,F)$.*

$$\overline{H_p}(s,a,F) = \frac{1}{F(s-1)} + \gamma_p(a,F) + O(s-1)$$

Proof. We need to consider the case when $a \ge 1$ as the case $a=0$ is already dealt with. Without loss of generality, it suffices to prove the Corollary when $(a,F)=1$. Indeed, by definition, if $(a,F)=d$ for $d>1$, we have

$$\lim_{s\to1}\overline{H_p}(s,a,F) = \frac{1}{d}\lim_{s\to1}\overline{H_p}(s,\frac{a}{d},\frac{F}{d})$$
$$\frac{d}{ds}(s-1)\overline{H_p}(s,a,F)|_{s=1} = -\frac{\log_p d}{d}\lim_{s\to1}\overline{H_p}(s,\frac{a}{d},\frac{F}{d}) + \frac{1}{d}\frac{d}{ds}(s-1)\overline{H_p}(s,\frac{a}{d},\frac{F}{d})|_{s=1}.$$

Therefore, if we have the corollary for the case when $(a,F)=1$, then we can apply Appendix A.2.3 to complete the proof. When $p \mid F$ and $p \nmid a$, we have the residue of $H_p(s,a,F)$ at $s=1$ to be $\frac{1}{F}$ and by Proposition 7, the constant term is given by $\gamma_p(a,F)$. So, we need to consider the case when $p \nmid F$. From Remark 6.1, we have

$\lim_{s\to1}(s-1)\overline{H_p}(s,a,F) =$

$$\frac{p^\phi}{p^\phi-1}\sum_{i=0}^{\phi-1}\frac{1}{p^i}\sideset{}{'}\sum_{n=0}^{p-1}\lim_{s\to1}(s-1)H_p(s,\overline{ap^{-i}}_F+nF,pF).$$

$$=\frac{p^\phi}{p^\phi-1}\sum_{i=0}^{\phi-1}\frac{1}{p^i}\frac{p-1}{pF} = \frac{1}{F},$$

thereby concluding the first part of the corollary. The second part simply follows from the definition by setting $F^* = F$ and noting that $\gamma_p(a_1, F_1) = \frac{1}{p}\gamma_p(\frac{a_1}{p}, \frac{F_1}{p})$ whenever $p \mid (a_1, F_1)$. □

4.2.1 Generalised Euler-Briggs constants

With the above, we motivate the definition of generalised Euler Briggs constants in the p-adic setup. Throughout, let Ω be a finite set of primes and we use the notations P_Ω, δ_Ω as given in Section 1.1. Just like p-adic Euler constants, we have to consider 2 cases, namely when $p \mid F$ and $p \nmid F$. For $p \nmid F$ and $1 \le a \le F-1$, we observe the following :

$$\gamma_p(a, F) \in \mathbb{Q} < \gamma_p(b, pF) \,:\, 1 \le b \le pF-1, (b, p) = 1 > . \tag{4.17}$$

We would like to retain the same property while defining the generalised Euler constants in the p-adic setup. So along with the natural restriction $(P_\Omega, F) = 1$, we would also require that $p \notin \Omega$.

Definition 17. *Let F be an integer greater than 1 such that $(F, P_\Omega) = 1$. If $v_p(a) < v_p(F)$ we set*

$$\gamma_p(\Omega, a, F) = -\lim_{k\to\infty} \frac{1}{P_\Omega F p^k} \sum_{\substack{n=0 \\ n\equiv a \bmod F \\ (n, P_\Omega)=1}}^{FP_\Omega p^k - 1} \log_p n.$$

When $v_p(a) \ge v_p(F)$, we write $F = p^k F^*$, with $(p, F^*) = 1$ and set $\phi = \varphi(F^*)$. We define

$$\gamma_p(\Omega, a, F) = \frac{p^\phi}{p^\phi - 1} \sum_{\substack{n=0 \\ v_p(a+nF)<\phi+k}}^{p^\phi - 1} \gamma_p(\Omega, a+nF, p^\phi F).$$

We immediately have the following proposition :

Proposition 9.
$$\gamma_p(\Omega, a, F) = \frac{d}{ds}(s-1)\overline{L_p}(s, f)|_{s=1}.$$
where $f : \mathbb{N} \to \{0, 1\}$ is the arithmetic function $f(n) = 1_{a,F}(n)1_\Omega(n)$.

Proof. Assume Ω to be a finite set of primes not containing the prime divisors of pF. We first evaluate $\frac{d}{ds}(s-1)\overline{L_p}(s,f)|_{s=1}$. We have

$$\begin{aligned}1_{a,F}(n)1_{\Omega}(n) &= \sum_{d|P_\Omega}\mu(d)1_{0,d}(n)1_{a,F}(n).\\ &= \sum_{d|P_\Omega}\mu(d)1_{a,F}(n)1_{0,d}(n) = \sum_{d|P_\Omega}\mu(d)1_{a_d,Fd}(n).\end{aligned}$$

where a_d is the representative of the congruences a mod F, 0 mod d in $(\mathbb{Z}/Fd\mathbb{Z})$ in the set $\{0,\ldots,d-1\}$. Therefore, we have

$$\begin{aligned}\overline{L_p}(s,f) &= \sum_{d|P_\Omega}\mu(d)\overline{L_p}(s,1_{a_d,Fd}).\\ &= \sum_{d|P_\Omega}\mu(d)\frac{<d>^{1-s}}{d}\overline{L_p}(s,1_{\overline{ad^{-1}}_F,F}).\end{aligned} \tag{4.18}$$

We obtain the following expression (4.19) with the help of Proposition 6 and Corollary 7.

$$\frac{d}{ds}(s-1)\overline{L_p}(s,f)|_{s=1} = -\frac{1}{F}\sum_{d|P_\Omega}\frac{\mu(d)}{d}\log_p d + \sum_{d|P_\Omega}\frac{\mu(d)}{d}\gamma_p(\overline{ad^{-1}}_F,F). \tag{4.19}$$

It remains to show that the above value is same as $\gamma(\Omega,a,F)$. We have three cases.

1. When $p\mid F$, $v_p(a)<v_p(F)$. From Definition 17, we note that

$$\begin{aligned}\gamma_p(\Omega,a,F) &= -\lim_{k\to\infty}\frac{1}{P_\Omega Fp^k}\sum_{\substack{n=0\\ n\equiv a \bmod F}}^{FP_\Omega p^k-1}\sum_{d|(n,P_\Omega)}\mu(d)\log_p n.\\ &= -\sum_{d|P_\Omega}\frac{\mu(d)}{d}\lim_{k\to\infty}\frac{d}{P_\Omega Fp^k}\sum_{\substack{n=0\\ n\equiv a \bmod F\\ n\equiv 0 \bmod d}}^{FP_\Omega p^k-1}\log_p n.\\ &= -\frac{1}{F}\sum_{d|P_\Omega}\frac{\mu(d)}{d}\log_p d + \sum_{d|P_\Omega}\frac{\mu(d)}{d}\gamma_p(\overline{ad^{-1}}_F,F).\end{aligned} \tag{4.20}$$

2. When $p\nmid F$. Substituting (4.20) to the terms in Definition 17, we have

$$\gamma_p(\Omega, a, F) = \frac{p^\phi}{p^\phi - 1} \sum_{\substack{n=0 \\ v_p(a+nF)<\phi}}^{p^\phi - 1} \Big(-\frac{1}{p^\phi F} \sum_{d|P_\Omega} \frac{\mu(d)}{d} \log_p d + \sum_{d|P_\Omega} \frac{\mu(d)}{d} \gamma_p(\overline{(a+nF)d^{-1}}_{p^\phi F}, p^\phi F)\Big). \quad (4.21)$$

Note that there is exactly one n such that $v_p(a+nF) \geq \phi$. We can prove that there is at most one n by noting that if $0 \leq n_1 < n_2 \leq p^\phi - 1$, then

$$\phi > v_p((n_2 - n_1)F) = v_p(a + n_2F - a - n_1F) \geq \min\{v_p(a+n_2F),\ v_p(a+n_1F)\}.$$

The existence of n is also guaranteed by the same reason, as no two terms $a + n_1F$ and $a + n_2F$ can lie in the same equivalence class. For each $d \mid P_\Omega$, we note that the elements of the set $\{\overline{(a+nF)d^{-1}}_{p^\phi F} : 0 \leq n \leq p^\phi - 1, v_p(a+nF) < \phi\}$ satisfy the congruence $y \equiv ad^{-1} \bmod F, y \not\equiv 0 \bmod p^\phi$. With these observations, we can re-write (4.21) as follows.

$$\gamma_p(\Omega, a, F) = -\frac{1}{F} \sum_{d|P_\Omega} \frac{\mu(d)}{d} \log_p d + \sum_{d|P_\Omega} \frac{p^\phi}{p^\phi - 1} \sum_{\substack{n=0 \\ v_p(\overline{ad^{-1}}_F + nF)<\phi}}^{p^\phi - 1} \gamma_p(\overline{ad^{-1}}_F + nF, p^\phi F).$$

For each $d \mid P_\Omega$, note that the second summand is $\gamma_p(\overline{ad^{-1}}_F, F)$ from Definition 12 and the above expression coincides with (4.19).

3. When $p^k \ || \ (a, F)$. then we reduce the question to the case when $p \nmid (a, F)$ (one of the above cases) as

$$\gamma_p(\Omega, a, F) = \frac{1}{p^k} \gamma_p(\Omega, a', F'), \ \text{ and } \overline{L_p}(s, f) = \frac{1}{p^k} \overline{L_p}(s, f'),$$

where $f'(n) = 1_{a',F'}(n) 1_\Omega(n)$, with $a' = p^{-k}a, F' = p^{-k}F$.

□

4.3 Some non-vanishing and transcendental results

In this section, we first express $\overline{L_p}(1,f)$ as a linear form of p-adic logarithm of non-zero algebraic number whenever it exists, by appealing to Proposition 7 and the Gauss formula (Appendix A.2.6).

Corollary 8. *The residue of the function* $\overline{L_p}(s,f)$ *at* $s=1$ *is* $\frac{1}{F}\sum_{a=0}^{F-1} f(a)$. *In particular,* $\overline{L_p}(1,f)$ *exists if and only if* $\sum_{a=0}^{F-1} f(a) = 0$. *Under these assumptions, the explicit value is given by*

$$\overline{L_p}(1,f) = \sum_{a=0}^{F-1} f(a)\gamma_p(a,F) = -\sum_{r=1}^{F-1} \widehat{f}(r)\log_p(1-\zeta_F^r),$$

where $\widehat{f}(a) = \frac{1}{F}\sum_{r=0}^{F-1} f(a)\zeta_F^{-ar}$.

Proof. From Corollary 7, along with $\overline{H_p}(s,0,F) = \frac{< F >^{1-s}}{F} H_p(s,0,1)$, we have

$$\lim_{s\to 1}(s-1)\overline{L_p}(s,f) = \sum_{a=0}^{F-1} \lim_{s\to 1}(s-1)\overline{H_p}(s,a,F) = \sum_{a=0}^{F-1} \frac{f(a)}{F}.$$

This proves the first part and therefore, from now on, we assume that $\sum_{a=0}^{F-1} f(a) = 0$. For $0 \le a < F$, again from Corollary 7, we have

$$\overline{H_p}(s,a,F) = \frac{1}{F(s-1)} + \gamma_p(a,F) + O(s-1).$$

Substituting this expression in Definition 16, we get the first equality. The second equality is an immediate consequence of the first equality, Gauss Formula (Appendix A.2.6) and the identity $F = \prod_{r=1}^{F-1}(1-\zeta_F^r)$. □

We also have the following corollary owing to the distribution relation.

Corollary 9. *Let*

$$\Phi_p(F) = \sum_{\substack{r=1 \\ (r,F)=1}}^{F} \gamma_p(r,F).$$

We then have,

$$\Phi_p(F) = \frac{\varphi(F)}{F}\gamma_p + \frac{\varphi(F)}{F}\sum_{d|F}\frac{\log_p d}{d-1}.$$

where d runs over all the primes dividing F.

Proof. Let us denote $\Phi_p(s,F) := \sum_{\substack{a=1 \\ (a,F)=1}}^{F} \overline{H_p}(s,a,F)$.

$$\begin{aligned}\Phi_p(s,F) &= \sum_{a=0}^{F-1} \sum_{d|(a,F)} \mu(d)\overline{H_p}(s,a,F) = \sum_{d|F} \mu(d) \sum_{\substack{a=0 \\ d|a}}^{F-1} \overline{H_p}(s,a,F) \\ &= \sum_{d|F} \mu(d) \frac{< d >^{1-s}}{d} \sum_{a=0}^{F/d-1} \overline{H_p}(s,a,\frac{F}{d}) \\ &= \prod_{r|F} (1 - \frac{< r >^{1-s}}{r}) \overline{H_p}(s,0,1).\end{aligned}$$

where r runs over all the primes dividing F. The last equality is obtained by Proposition 8,

$$\overline{H_p}(s,0,1) = \sum_{a=0}^{F/d-1} \overline{H_p}(s,a,\frac{F}{d}).$$

Noting that $\overline{H_p}(s,0,1)$ has a pole at $s=1$ with residue 1, we arrive at the corollary by evaluating $\frac{d}{ds}(s-1)\Phi_p(s,F)|_{s=1}$, by Proposition 6 and Corollary 7. □

The above corollary was proved by Diamond [22] only for the case $v_p(F) > 0$. We remove this condition. This corollary together with Proposition 9 helps us in writing $\gamma_p(\Omega,a,F)$ as a $\overline{\mathbb{Q}}$ linear form of p-adic logarithm of non-zero algebraic numbers. This statement is the p-adic analogue of [29, Lemma 8].

Corollary 10. *With the same conditions on Ω and F, we have*

$$\begin{aligned}\gamma_p(\Omega,a,F) =& \frac{1}{\varphi(F)} \sum_{\substack{\chi \bmod F \\ \chi\neq\chi_0,\chi even}} (\chi(a))^{-1}\, \overline{L_p}(1,\chi) \prod_{r\in\Omega} (1 - \frac{\chi(r)}{r}) \\ &+ \frac{\delta_\Omega}{F}(\gamma_p + \sum_{r|FP_\Omega} \frac{\log r}{r-1}),\end{aligned}$$

where r runs over the primes dividing FP_Ω.

Proof. Let f be the arithmetic function defined in Proposition 9. From the orthogonality relations of the characters mod F, we note that

$$1_{a,F}(n) = \frac{1}{\varphi(F)} \sum_{\chi \bmod F} \chi(a)^{-1}\chi(n).$$

Now since the sum runs over the co-prime residue classes of P_Ω, we have

$$f(n) = 1_{a,F}(n)1_\Omega(n) = \frac{1}{\varphi(F)} \sum_{\chi \bmod F} \chi(a)^{-1}\chi(n)1_\Omega(n).$$

Evaluating $\overline{L_p}(s,f)$,

$$\overline{L_p}(s,f) = \frac{1}{\varphi(F)} \sum_{\chi \bmod F} \chi(a)^{-1}\overline{L_p}(s,\chi 1_\Omega),$$

as $\chi 1_\Omega$ is a periodic function of period $P_\Omega F$. Now, by the same argument mentioned in Corollary 9, we have

$$\overline{L_p}(s,\chi 1_\Omega) = \prod_{r\in\Omega}(1 - \frac{\chi(r) < r >^{1-s}}{r})\overline{L_p}(s,\chi),$$

$$\overline{L_p}(s,f) = \frac{1}{\varphi(F)}(\overline{L_p}(s,\chi_0)\prod_{r\in\Omega}(1 - \frac{< r >^{1-s}}{r})$$
$$+ \sum_{\substack{\chi \bmod F \\ \chi\neq\chi_0}} (\chi(a))^{-1}\overline{L_p}(s,\chi)\prod_{r\in\Omega}(1 - \frac{\chi(r) < r >^{1-s}}{r})).$$

where r runs over the primes in Ω. Again by the same reasoning we also have

$$\overline{L_p}(s,\chi_0) = \prod_{r|F}(1 - \frac{< r >^{1-s}}{r})H(s,0,1).$$

Noting that $\overline{L_p}(s,\chi)$ exists at $s = 1$ for $\chi \neq \chi_0$ and applying Corollary 9 for the first sum and that $\overline{L_p}(s,\chi) \equiv 0$ when χ is odd, we have the result. □

Finally using the above corollaries, we prove the theorem of Baker, Birch and Wirsing in this setup along with certain results about transcendence of families of $\{\gamma_p(a,F) : 1 \leq a \leq F, (a,F) = 1\}$ and their linear combinations.

Theorem 22. Let $f : \mathbb{N} \to \overline{\mathbb{Q}}$ be an even periodic function of period F such that $f(a) = 0$ whenever $1 < (a,F) < F$. If $\overline{L_p}(s,f) \not\equiv 0$, then $\overline{L_p}(1,f) \neq 0$.

Proof. By Corollary 9, we know that

$$\overline{L_p}(1,f) = f(0)\gamma_p(0,F) + \sum_{(a,F)=1} f(a)\gamma_p(a,F)$$

$$= \sum_{(a,F)=1} (f(a) + \frac{f(0)}{\varphi(F)})\gamma_p(a,F) - \frac{f(0)}{F} \sum_{r|F} \frac{\log_p r}{r-1} - \frac{f(0)}{F} \log_p F$$

where r runs over the primes dividing F. Let S be a maximal multiplicative independent set of $\{ \ : \ 1 - \zeta_F^i \ : \ \}_{i=1}^{F-1}$. We immediately observe that

$$\overline{L_p}(1,f) = \overline{L_p}(1,g) - \frac{f(0)}{F} \sum_{r|F} (\frac{1}{r-1} + v_r(F)) \log_p r. \tag{4.22}$$

where $g : \mathbb{N} \to \overline{\mathbb{Q}}$ is the arithmetic function

$$g(n) = \begin{cases} f(n) + \frac{f(0)}{\varphi(F)} & \text{if } (n,F) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Since g is a Dirichlet type even function, we note that $g \equiv \sum_{\substack{\chi \bmod F \\ \chi(-1)=1}} a_\chi \chi$, with $a_\chi \in \overline{\mathbb{Q}}$ and therefore,

$$\overline{L_p}(1,g) = \sum_{\substack{\chi \bmod F \\ \chi(-1)=1}} a_\chi \overline{L_p}(1,\chi),$$

In the above, note that $a_{\chi_0} = 0$, as $\sum_{a=0}^{F-1} g(a) = 0$. From Remark 7.3 and [48, Corollary 9], we note that $\overline{L_p}(1,g) \neq 0$ unless $g \equiv 0$. Thus if we assume $\overline{L_p}(1,f) = 0$, then we only have the following cases by Theorem 19.

1. If $F \neq p^k$. Then, $f(0) = 0$ and therefore $f(a) = 0$ (as $g \equiv 0$) whenever $(a,F) = 1$. Therefore $f \equiv 0$.
2. $F = p^k$. Then we have for all $(a,F) = 1$

   $$\frac{f(0)}{\varphi(F)} + f(a) = 0 \implies f(a) = -\frac{f(0)}{(p-1)p^{k-1}}.$$

   By the example mentioned in Remark 7.2, we have $\overline{L_p}(s,f) \equiv 0$.

□

Theorem 23. At most one of the elements of the set

$$\{\gamma_p, \gamma_p(a,F) \ : \ 1 \leq a \leq F, (a,F) = 1\}$$

is algebraic.

Proof. We begin by noting that at most one of the elements of the set

$$S_f := \{\gamma_p(a,F) \,:\, 1 \le a \le F, (a,F) = 1\}$$

is algebraic. Indeed, for any two equivalence classes $[a], [b]$ with $[b] \neq [-a]$, we can construct a periodic function f of period F as follows

$$f(n) = \begin{cases} 1 & \text{if } n \equiv \pm a \bmod F \\ -1 & \text{if } n \equiv \pm b \bmod F \\ 0 & \text{otherwise} \end{cases}.$$

Now we can apply Theorem 22 to conclude that $\overline{L_p}(1,f)$ is non-zero hence transcendental. However, we note that $\overline{L_p}(1,f) = 2(\gamma_p(a,F) - \gamma_p(b,F))$. Hence at most one elements of the set S_f is algebraic.

It remains to consider $S_f \cup \{\gamma_p\}$. Suppose γ_p and $\gamma_p(a,F)$ are algebraic. Then by Appendix A.2.6 and Corollary 5, we have $F\gamma_p(a,F) = \gamma_p$. Now we construct periodic functions f and g of period F as follows :

$$f(n) = \begin{cases} 1 & \text{if } n \equiv \pm a \bmod F \\ -2 & \text{if } n \equiv 0 \bmod F \\ 0 & \text{otherwise} \end{cases}, \qquad g(n) = \begin{cases} 1 & \text{if } (n,F) = 1 \\ -\varphi(F) & \text{if } n \equiv 0 \bmod F \\ 0 & \text{otherwise} \end{cases}.$$

By Corollaries 8 and 9, we have

$$\overline{L_p}(1,f) = +2\frac{\log_p F}{F}, \qquad \overline{L_p}(1,g) = \frac{\varphi(F)}{F}\sum_{r|F}\frac{\log_p r}{r-1} + \frac{\varphi(F)}{F}\log_p F.$$

Now consider the arithmetic function $h(n) := \varphi(F)f(n) - 2g(n)$. The function h is of Dirichlet type and therefore $\overline{L_p}(1,h)$ is a linear combination of logarithm of units. At the same time, we note that under the given assumption,

$$\overline{L_p}(1,h) = -2\sum_{r|F}\frac{\log_p r}{r-1}.$$

By Theorem 19, we obtain a contradiction if both $\gamma_p(a, F)$ and γ_p are algebraic provided $\overline{L_p}(1, h) \neq 0$. Thus we should have $\overline{L_p}(1, h) = 0$. However, this is not possible when $F \neq p^k$. When $F = p^k$, then, for $b \neq \pm a \bmod F$, we have $h(b) \neq h(a)$ and therefore again $\overline{L_p}(s, h) \neq 0$ and by Theorem 22, $\overline{L_p}(1, h) \neq 0$. Since $\overline{L_p}(1, h) \neq 0$, as mentioned earlier, we prove the theorem. □

We finally finish by proving Theorem 24.

Theorem 24. Let Ω be a finite set of primes not containing p and let

$$V_{\mathbb{Q},N} := \mathbb{Q} < \gamma_p(\Omega, a, m) \ : \ 1 \leq a < N, 1 \leq m \leq N \ (a, m) = (m, P_\Omega) = 1 > .$$

Then we have $\dim_{\mathbb{Q}} V_{\mathbb{Q},N} \asymp_\Omega N^2$ as $N \to \infty$.

Proof. We first note that for a fixed period $F > 1$ co-prime to P_Ω and a non-principal even Dirichlet character χ mod F, we have

$$\overline{L_p}(1, \chi) \in \overline{\mathbb{Q}} < \gamma_p(\Omega, a, F) \ : \ 1 \leq a \leq F, (a, F) = 1 > .$$

This is indeed true from Corollary 10 and the orthogonality relations of Dirichlet characters, as

$$\overline{L_p}(1, \chi) \prod_{r \in \Omega} (1 - \frac{\chi(r)}{r}) = \sum_{\substack{a=1 \\ (a,F)=1}}^{F} \chi(a) \gamma_p(\Omega, a, F).$$

The remaining part of the proof follow along the same lines as Theorem 5. Indeed, we note that as r varies over the squarefree integers $\mathcal{S}$ co-prime to P_Ω, the elements

$$\{\overline{L_p}(1, \chi) \ : \ r \in \mathcal{S}, \chi \bmod r, \chi \text{ is non-principal, even and primitive } \}$$

are linearly independent over $\overline{\mathbb{Q}}$. Therefore,

$$\dim_{\overline{\mathbb{Q}}} V_{\overline{\mathbb{Q}},N} \geq \frac{1}{2} \sum_{\substack{r \leq N \\ r \in \mathcal{S}}} \Psi'(r) = c_\Omega N^2 + O(N\sqrt{N}),$$

the last estimate as mentioned in (3.29). This proves the theorem. □

4.4 Concluding Remarks

In the study above, we have defined a function that imitates the construction of the p-adic Euler constants by Diamond and established some transcendence results. It should be noted that, from the viewpoint of irrationality questions associated to $\gamma_p(a, F)$, it is not necessary to define the functions $\overline{H_p}(s, a, F)$ and we can prove the Theorems 22 and 23 without these functions albeit with the same process involved. But defining these functions gives us a systematic view to consider these constants. For instance, the proof of Corollary 10 becomes more tedious when we try to prove the results without the definition of $\overline{H_p}(s, a, F)$. There were studies wherein the special values of the higher derivatives of $(s-1)H(s, a, F)$ at $s = 1$ were considered (known as Stieltjes constants, see [16]) and using these functions one can give a direct analogy in the p-adic setup.

The case $p = 2$ is more delicate. Not only is the constant γ_p different, certain functions like the logarithm and $H_2(s, a, 4F)$ have their domain as $1 + p^2\mathbb{Z}_p$ instead of $1 + p\mathbb{Z}_p$. This makes the construction of $\overline{H_p}(s, a, F)$ a bit more difficult to handle.

Chapter 5

On the arithmetic Chowla-Milnor Space

5.1 Introduction

For a fixed positive integer $k \geq 2$, the conjecture of Chowla-Chowla (Conjecture 2) predicts the rational valued arithmetic function f for which $L(k, f)$ is zero. As mentioned in [43], this is related to the linear independence of Hurwitz zeta values $\zeta(k, a/q)$. However, we can isolate a subspace of the vector space $\mathbb{Q}\langle\zeta(k, a/q) \mid 1 \leq a \leq q, (a, q) = 1\rangle$ which are algebraic multiples of π^k. The first instance was noted by Hecke [31] where he proved that $L(2k, \chi)$ is an algebraic multiple of π^{2k} whenever χ is an even quadratic character. In fact, the special values

$$\zeta(k, \frac{a}{q}) + (-1)^k \zeta(k, 1 - \frac{a}{q}). \qquad 1 \leq a < \frac{q}{2}, (a, q) = 1 \tag{5.1}$$

are algebraic multiples of π^k. The explicit formulae is mentioned in Appendix A.1.3.

In this chapter, we consider the linear relations between the special values of the Hurwitz zeta series (5.1) for a positive integer $q \geq 3$, over arbitrary number fields K. The $\mathbb{Q}$ linear independence associated to these values were observed in [30, 52]. The linear independence of these values over imaginary quadratic fields was observed in [59]. This chapter is from [8].

5.1.1 Notations

We recall the notations mentioned in Section 1.2. Let M be a Galois number field and $\{\gamma_i\}_{i=1}^m$ be $\mathbb{Q}$ linearly independent elements of M. For a number field $\mathbb{F}$, we define

$$\mathcal{L}(\mathbb{F}) := \left\{ \sum_{i=1}^{m} a_i \gamma_i \mid a_i \in \mathbb{F} \right\}.$$

We also define the $\mathbb{F}$-linear map $\Phi_{\mathbb{F}} : \mathbb{F}^m \to \mathcal{L}(\mathbb{F})$

$$\Phi(\bar{a}) = \sum_{i=1}^{m} a_i \gamma_i,$$

where $\bar{a} = (a_1, \dots a_m) \in \mathbb{F}^m$. We recall the definition of arithmetic Chowla-Milnor space.

Definition 7. *Let $k > 1, q > 2$ be positive integers. The arithmetic space $V_{ar}(\mathbb{Q})$ is the $\mathbb{Q}$ linear space defined by*

$$V_{ar}(\mathbb{Q}) = \mathbb{Q} \left\langle \zeta(k, \frac{a}{q}) + (-1)^k \zeta(k, 1 - \frac{a}{q}) \ : \ (a, q) = 1, \quad 1 \leq a < \frac{q}{2} \right\rangle.$$

With the above definitions, for a fixed q, k, and setting

$$\gamma_a = \lambda_a = \frac{q^{k-1}}{(2k)!} \sum_{b=1}^{q} (\zeta_q^{ab} + (-1)^k \zeta_q^{-ab}) B_k(b/q),$$

by Appendix A.1.3, we have $V_{ar}(\mathbb{F}) = (2\pi i)^k \mathcal{L}(\mathbb{F})$ with $m = \phi(q)/2$.

5.2 Proof of the main theorem

We first prove the following theorem.

Theorem 25. Let K be a number field and let $L := K \cap M$. Then

$$dim_K \mathcal{L}(K) = dim_L \mathcal{L}(L).$$

Proof. Let $\{\alpha_1, \ldots, \alpha_l\}$ be a basis of L over $\mathbb{Q}$ and $\{\beta_1, \ldots \beta_n\}$ be a basis of K over L. Let $\overline{a} \in \ker\Phi_K$. Then we have $\sum_i a_i\gamma_i = 0,\ \ a_i \in K$, then writing a_i as basis elements,

$$\sum_i \sum_{j,k} b_{jk}^{(i)} \alpha_j \beta_k \gamma_i = 0 \implies \sum_k \big(\sum_{i,j} b_{jk}^{(i)} \alpha_j \gamma_i \big) \beta_k = 0.$$

We know that $\gamma_i \in M$ and hence $b_{jk}^{(i)} \alpha_j \gamma_i \in M$. So we need to check if $\{\beta_1, \ldots \beta_n\}$ forms a basis of KM over M. This is indeed the case as we have $[K : L] = [K(\zeta_q) : M]$ as M is Galois. Hence $\sum_{i,j} b_{jk}^{(i)} \alpha_j \gamma_i = 0$. This gives us a map

$$\iota : \ker\Phi_K \to K \otimes_L \ker\Phi_L, \qquad \iota(\overline{a}) = \sum_k \beta_k \otimes \overline{c^{(k)}},$$

where $c_i^{(k)} = \sum_j b_{jk}^{(i)} \alpha_j$. The map ι evidently is a K linear isomorphism as shown below.

1. (Isomorphism) For an element $\sum_{k=1}^n \beta_k \otimes \overline{c^{(k)}}$, with $\overline{c^{(k)}} = (c_i^{(k)})_i \in \ker\Phi_L$, we construct an element $a \in K^{\varphi(q)/2}$as follows : If $c_i^{(k)} = \sum_j b_{jk}^{(i)} \alpha_j$ with $b_{jk}^{(i)} \in \mathbb{Q}$, then $\overline{a} = (\sum_{j=1}^l \sum_{k=1}^n b_{j,k}^{(i)} \alpha_j \beta_k)_i$. We need to first check if $\overline{a} \in \ker\Phi_K$. To see the same, we need to check if $\sum_i a_i \lambda_i = 0$. Expressing $a_i = \sum_{j=1}^l \sum_{k=1}^n b_{j,k}^{(i)} \alpha_j \beta_k$, we evaluate the sum as $\overline{c^{(k)}} \in \ker\Phi_L$.

   $$\sum_i \sum_{j,k} b_{j,k}^{(i)} \alpha_j \beta_k \lambda_i = \sum_k \sum_{i,j} b_{j,k}^{(i)} \alpha_j \lambda_i \beta_k = 0$$

   The choice of $b_{jk}^{(i)}$ uniquely determines the choice of a_i and hence we proved the map is an isomorphism.

2. (K-linearity) For $y \in K^*$, the map $e_y : K \to K$ given by $x \mapsto yx$ is a K linear isomorphism and this induces isomorphisms $e_{y,\Phi_K} : \ker\Phi_K \to \ker\Phi_K$, and $e_{y,\Phi_L} : K \otimes_L \Phi_L \to K \otimes_L \Phi_L$ respectively given by $\overline{a} \mapsto \overline{ya}$ and $\sum_k \beta_k \otimes \overline{c^{(k)}} \mapsto \sum_k y\beta_k \otimes \overline{c^{(k)}}$. Since we have $\iota \circ e_{y,\Phi_K} = e_{y,\Phi_L} \circ \iota$, we obtain K linearity.

We have the following :

$$\ker\Phi_K \cong K \otimes_L \ker\Phi_L \text{ where } L = K \cap M. \tag{5.2}$$

Given a number field L in M, we have the exact sequence

$$0 \mapsto \ker \Phi_L \mapsto L^{\varphi(q)/2} \mapsto \mathcal{L}(L) \mapsto 0.$$

We also have the following exact sequence for the number field K :

$$0 \mapsto \ker \Phi_K \mapsto K^{\varphi(q)/2} \mapsto \mathcal{L}(K) \mapsto 0.$$

From (5.2), we have $\dim_K \ker \Phi_K = \dim_L \ker \Phi_L$ and therefore $\dim_K \mathcal{L}(K) = \dim_L \mathcal{L}(L)$. $\square$

Therefore, when we specialise the above to the case when $\gamma_a = \lambda_a$ for $1 \leq a \leq q/2$, $(a, q) = 1$, we have the following

$$\mathrm{dim_K V_{ar}(K) = dim_L V_{ar}(L)} \text{ where } L = K \cap \mathbb{Q}(\zeta_q).$$

We finally proceed to the proof of the main Theorem in this chapter.

Theorem 7. Let K be a number field contained in $\mathbb{Q}(\zeta_q)$. Then we have,

$$dim_K V_{ar}(K) = \begin{cases} \frac{\varphi(q)}{2[K:\mathbb{Q}]} & \text{if } K \text{ is totally real} \\ \frac{\varphi(q)}{[K:\mathbb{Q}]} & \text{if } K \text{ is totally imaginary} \end{cases} . \tag{1.10}$$

Proof. By Lemma 1 and Appendix A.1.3 we have $V_{ar}(K) \cong K < \lambda_a >$. It suffices to compute the dimension of $K < \lambda_a >$ as a $K-$vector space. To do so, we explicitly describe the set $K < \lambda_a >$. We define the $\mathbb{Q}-$vector space V_{odd} as the following. If we denote σ as the complex conjugation, then

$$V_{odd} := \{\alpha \in \mathbb{Q}(\zeta_q) \mid \sigma(\alpha) = -\alpha\}.$$

We note that V_{odd} is obtained as the kernel of the homomorphism $\mathbb{Q}(\zeta_q) \to \mathbb{Q}(\zeta_q)^+$ given by $x \to \frac{x+\sigma(x)}{2}$. Hence we have the following exact sequence of $\mathbb{Q}-$vector spaces:

$$0 \to V_{odd} \to \mathbb{Q}(\zeta_q) \to \mathbb{Q}(\zeta_q)^+ \to 0.$$

For any totally real subfield K of $\mathbb{Q}(\zeta_q)$, we note that V_{odd} is now a vector space over K because for all $\alpha \in V_{odd}$ and $x \in K$, we have $x\alpha \in V_{odd}$. From now on, we regard both V_{odd} and $\mathbb{Q}(\zeta_q)^+$ as $K-$vector spaces. If we fix a non zero element $\alpha \in V_{odd}$, the map

$$V_{odd} \to \mathbb{Q}(\zeta_q)^+ \quad x \mapsto \alpha x \tag{5.3}$$

is a $\mathbb{Q}$ linear isomorphism and hence also K linear. Hence, $V_{odd} \cong \mathbb{Q}(\zeta_q)^+$ as $K-$vector spaces.

Suppose k is odd. We note that $\lambda_a \in V_{odd}$ and from Lemma 1, we know that $\lambda_a's$ are linearly independent over $\mathbb{Q}$. Therefore,

$$\dim_{\mathbb{Q}} \mathbb{Q} < \lambda_a >= \frac{\varphi(q)}{2} = dim_{\mathbb{Q}} V_{odd}.$$

Hence $V_{odd} = \mathbb{Q} < \lambda_a >$. Since V_{odd} is a $K-$vector space for a totally real field K contained in $\mathbb{Q}(\zeta_q)$, we also obtain that $V_{odd} = K < \lambda_a >$. Hence in this case, we obtain $K < \lambda_a >\cong \mathbb{Q}(\zeta_q)^+$.

When K is a totally imaginary subfield of $\mathbb{Q}(\zeta_q)$, the existence of a non zero element α in K such that $\sigma(\alpha) = -\alpha$ ensures that $K < \lambda_a >= \mathbb{Q}(\zeta_q)$. This is indeed true as

$$V_{odd} = \mathbb{Q} < \lambda_a >\subset K < \lambda_a > \text{ and } \mathbb{Q}(\zeta_q)^+ = \alpha.V_{odd} \subset K < \lambda_a > .$$

The equality $\mathbb{Q}(\zeta_q)^+ = \alpha.V_{odd}$, mentioned above is obtained from (5.3). Hence in this case, we obtain $K < \lambda_a >= \mathbb{Q}(\zeta_q)$.

When k is even, the arguments of the preceding paragraphs go through by interchanging $\mathbb{Q}(\zeta_q)^+$ and V_{odd}. Summarizing these observations, we have :

$$V_{ar}(K) \cong \begin{cases} \mathbb{Q}(\zeta_q)^+ & \text{if } K \text{ is totally real} \\ \mathbb{Q}(\zeta_q) & \text{if } K \text{ is totally imaginary} \end{cases} . \tag{5.4}$$

Hence if K is totally real, we have $\dim_K V_{ar}(K) = \frac{\varphi(q)}{2[K:\mathbb{Q}]}$. If K is totally imaginary, then $\dim_K V_{ar}(K) = \frac{\varphi(q)}{[K:\mathbb{Q}]}$. □

Finally, with the above theorems, we prove the following corollary for a number field K.

Corollary 11. *One has*

$$dim_K V_{ar}(K) = \varphi(q)/2 \Leftrightarrow K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q} \text{ or } K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}(\sqrt{-d})$$

for some squarefree integer $d > 0$.

Proof. From Theorem 25, we have $\dim_K V_{ar}(K) = \dim_F V_{ar}(F) = \frac{\varphi(q)}{2}$ where $F = K \cap \mathbb{Q}(\zeta_q)$. From Theorem 7, If F is totally real, then $F = \mathbb{Q}$. If F is totally imaginary, then $[F : \mathbb{Q}] = 2$. □

Chapter 6

Linear independence of $L(1,\chi)$ over arbitrary number fields

6.1 Introduction

As alluded to in Chapter 1, when χ is an odd character, $L(1,\chi)$ is an algebraic multiple of π. Therefore, the special values $L(1,\chi)$ as χ varies over odd characters are linearly dependent over $\overline{\mathbb{Q}}$. However, the same result is no longer true when $\overline{\mathbb{Q}}$ is replaced by a number field K. In fact, for a fixed natural number n such that $(n,\varphi(n)) > 1$, it is still unknown if the values $L(1,\chi)$ as χ varies over all odd characters $\bmod n$ are linearly independent over $\mathbb{Q}$. When $(n,\varphi(n)) = 1$, this is a consequence of the Theorem 8 (see [5]). As highlighted earlier in the introduction, the special values $L(1,\chi)$ as χ ranges over the even primitive characters are linearly independent over $\overline{\mathbb{Q}}$. Hence the question of linear independence of $L(1,\chi)$ over $\mathbb{Q}$ remains unsolved when χ is an odd character.

In this chapter, we shall study the dimension estimates of the K-vector space spanned by $L(1,\chi)$ as χ varies over a fixed prime modulus p. The contents of this chapter is from [9].

6.1.1 Notations and Preliminary observations

Throughout, p will denote a prime number greater than 5. Any Dirichlet character χ modulo p is uniquely determined by a positive integer b where

$$1 \leq b < p \qquad \text{and} \qquad \chi(g) = \zeta_{p-1}^{b}.$$

In the above, we fix g to be generator (modulo p) for the cyclic group $(\mathbb{Z}/p\mathbb{Z})^{\times}$. Let us denote it by χ_b. Note that χ_b is an odd character if and only if b is odd. As mentioned in the appendix, the explicit value is

$$L(1,\chi) = (i\pi)\tau(\chi)\frac{B_{1,\overline{\chi}}}{p}. \tag{6.1}$$

From now on, for $1 \leq m \leq p-1$ and $(m,2) = 1$, let α_m denote the number $\alpha_m = \frac{L(1,\chi_m)}{i\pi}$. Note that $\alpha_m \in \mathbb{Q}(\zeta_p, \zeta_{p-1})$ as $\tau(\chi) \in \mathbb{Q}(\zeta_p, \zeta_{p-1})$ and $B_{1,\chi} \in \mathbb{Q}(\zeta_{p-1})$.

We require the following proposition (See [49, Lemma 10]):

Proposition 10. *Let $\alpha_1, \alpha_2, ..., \alpha_n$ be positive algebraic numbers.If $c_0, c_1, ..., c_n$ are algebraic numbers with $c_0 \neq 0$, then $c_0\pi + \sum_{j=1}^{n} c_j \log \alpha_j$ is a transcendental number and hence non-zero.*

For a number field K and a prime p, let $V(K,p)$ denote the K-vector space in $\mathbb{C}$ generated by the $p-2$ complex numbers $L(1,\chi)$ for non-trivial characters χ modulo p. Let $\dim\ V(K,p)$ denote its dimension. Proposition 10 in conjunction with Appendix A.1.2 asserts the following:

$$\overline{\mathbb{Q}}\left\langle L(1,\chi_{2m-1}) | 1 \leq m \leq \frac{p-1}{2} \right\rangle \cap \overline{\mathbb{Q}}\left\langle L(1,\chi_{2m}) | 1 \leq m \leq \frac{p-1}{2} - 1 \right\rangle = \{0\}.$$

This, along with Corollary 3 for a number field K, gives us the following inequality.

$$\frac{p-1}{2} \leq \dim\ V(K,p) \leq p-2.$$

Moreover, the lower bound is attained when $K = \mathbb{Q}(\zeta_p, \zeta_{p-1})$, and by Theorem 8, the upper bound is attained when $K \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$. The lower bound is attained when $K = \mathbb{Q}(\zeta_{p(p-1)})$. This is due to the fact that $\alpha_{2b-1} \in \mathbb{Q}(\zeta_p, \zeta_{p-1})$. Also, by Corollary

3 and Proposition 10 we conclude that

$$\dim \ V(K,p) = \frac{p-3}{2} + \dim \ V(K,p)^o,$$

where $V(K,p)^o$ is the subspace of $V(K,p)$ generated by the $L(1,\chi)$ values associated to the Dirichlet characters χ that are odd. We say that a field K satisfies property (NT) if

$$1 < \dim \ V(K,p)^o < \frac{p-1}{2} \tag{NT}$$

The breakup of this chapter is as follows : In Section 6.2, we construct explicit abelian number fields K contained in $\mathbb{Q}(\zeta_p, \zeta_{p-1})$ satisfying property (NT). By appealing to a theorem of Linnik (see Theorem 15), we establish a limsup estimate on $b(p)$ in Section 6.3. Finally, we study some questions for Fermat and Sophie Germain primes in Section 6.4.

6.2 Abelian Fields satisfying (NT)

We prove the following lemma.

Lemma 8. *Let p be an odd prime and $\mathbb{F}$ be a number field linearly disjoint with $\mathbb{Q}(\zeta_p)$. Then the $p-1$ Gauss sums associated to all the Dirichlet characters modulo p are linearly independent over F. In particular, when $\mathbb{F} = \mathbb{Q}(\zeta_{p-1})$, these elements form a basis for the field $\mathbb{Q}(\zeta_p, \zeta_{p-1})$ over the field $\mathbb{Q}(\zeta_{p-1})$.*

Proof. Let $\sum_{n=1}^{p-1} c_n \tau(\chi_n) = 0$ where $c_n \in F$. Thus we have,

$$\sum_{n=1}^{p-1} c_n \sum_{a=1}^{p-1} \chi_n(a) \zeta_p^a = 0,$$

and hence

$$\sum_{a=1}^{p-1} \left[\sum_{n=1}^{p-1} c_n \chi_n(a) \right] \zeta_p^a = 0.$$

Since the set $\{ \ \zeta_p^a : 1 \le a \le p-1\}$ is a basis of $F(\zeta_p)$ over F, we get

$$\sum_{n=1}^{p-1} c_n \chi_n(a) = 0, \qquad 1 \le a \le p-1$$

and hence the lemma follows from linear independence of characters (See Theorem 17). ☐

With this lemma, we prove the following theorem. We recall that a Sophie Germain prime is a prime $p = 2q + 1$, where q is also a prime.

Theorem 10. Let p be a prime greater than 7 and K be a number field such that $\mathbb{Q}(\zeta_{p-1}) \subseteq K \subseteq \mathbb{Q}(\zeta_{p(p-1)})$. Let $[K : \mathbb{Q}(\zeta_{p-1})] = d$. Then

$$\dim_K K \langle L(1,\chi) : \chi \text{ non-trivial Dirichlet character } \mod p\rangle = \frac{p-1}{2} + \frac{p-1}{2d}\delta_d - 1,$$

where $\delta_d = 1$ if d is odd and 2 if d is even.

The strategy for the proof of Theorem 10 is to use Lemma 8 with the ground field $F = \mathbb{Q}(\zeta_{p-1})$. For fields containing K containing F, we shall identify a subset of Gauss sums $\tau(\chi_m)$ which forms a basis of $\mathbb{Q}(\zeta_p, \zeta_{p-1})$ over K. Here χ_m varies over the Dirichlet characters $\bmod\, p$. The final step would be to restrict ourselves to odd characters in the above subset.

Proof of Theorem 10 : Let p be an odd prime and $L := \mathbb{Q}(\zeta_p, \zeta_{p-1})$, $F := \mathbb{Q}(\zeta_{p-1})$ be number fields in $\mathbb{C}$. Then $G = Gal(L/F)$ is cyclic isomorphic to $(\mathbb{Z}/p\mathbb{Z})^\times$. We fix the generator σ of G such that $\sigma(\zeta_p) = \zeta_p^g$, where we recall that g is the smallest positive generator of $(\mathbb{Z}/p\mathbb{Z})^\times$. Let $d \geq 1$ be an integer which divides $p-1$. Then there exists a unique subfield K such that $F \subseteq K \subseteq L$ with $[L : K] = d$. Further, the Galois group H of L/K is cyclic generated by $\sigma^{(p-1)/d}$. We now determine the dimension of $V(K,p)^o$. The action of the Galois element σ on a Gauss sum $\tau(\chi_m)$ is given by

$$\sigma(\tau(\chi_m)) = \sigma(\sum_{a=1}^{p-1} \chi_m(a)\zeta_p^a) = \sum_{a=1}^{p-1} \chi_m(a)\zeta_p^{ag} = \overline{\chi_m(g)}\tau(\chi_m).$$

Thus we see that,

$$\sigma^{\frac{p-1}{d}}(\tau(\chi_m)) = \left(\overline{\chi_m(g)}\right)^{\frac{p-1}{d}} \tau(\chi_m).$$

Recall that the Galois group H of L/K is generated by $\sigma^{(p-1)/d}$. This allows us to deduce that for $1 \leq m,n \leq p-1$,

$$\frac{\tau(\chi_m)}{\tau(\chi_n)} \in K \Leftrightarrow \sigma^{\frac{p-1}{d}}(\frac{\tau(\chi_m)}{\tau(\chi_n)}) = \frac{\tau(\chi_m)}{\tau(\chi_n)} \Leftrightarrow (\frac{\overline{\chi}_m(g)}{\overline{\chi}_n(g)})^{\frac{p-1}{d}} = 1 \Leftrightarrow m \equiv n \mod (d). \tag{6.2}$$

By Lemma 8, we know that $L = K < \tau(\chi_m) : 1 \leq m \leq p-1 >$. But by (6.2), we have that the Gauss sums $\{\tau(\chi_m), 1 \leq m \leq d\}$ form a basis of L over K and hence in particular:

$$L = K < \tau(\chi_m) : 1 \leq m \leq d > . \tag{6.3}$$

Now consider the following vector space

$$W = K < \frac{L(1,\chi)}{i\pi} : \chi \quad \text{odd} > .$$

As a K-vector space, it is isomorphic to the space $V(K,p)^o$ whose dimension we are interested in. But unlike $V(K,p)^o$, W is a subspace of L. Since $B_{1,\overline{\chi}}$ belongs to the field K, we conclude that

$$W = K < \tau(\chi_m) : 1 \leq m \leq p-1, m \quad \text{odd} > .$$

Suppose d is odd. Then for any even m with $1 \leq m \leq d$, $m+d$ is odd. Thus, by (6.2) and (6.3), we see that $W = L$ and hence dimension of the K-vector space $V(K,p)^o$ is d. On the other hand, if d is even, again by (6.2) and (6.3), we have

$$W = K < \tau(\chi_m) : 1 \leq m \leq d, m \quad \text{odd} >,$$

and hence dimension of the K-vector space $V(K,p)^o$ is $\frac{d}{2}$. Thus the dimension of the space $V(K,p)$ is equal to $\frac{p-1}{2} + d - 1$ if d is odd and is equal to $\frac{p-1}{2} + \frac{d}{2} - 1$ if d is even. $\square$

If p is not a Sophie Germain prime, $\frac{p-1}{2}$ has a proper divisor. Assuming this condition on p, we prove the following :
There exists a number field K containing $\mathbb{Q}(\zeta_{p-1})$ satisfying (NT).

In the forthcoming proposition, we choose elements from the set $\{\alpha_{2m-1} \mid 1 \leq m \leq p-1/2\}$ which spans $\mathbb{Q}(\zeta_p, \zeta_{p-1})$ as a $\mathbb{Q}(\zeta_p)$ vector space. Here, we recall that $\alpha_m = \frac{L(1,\chi_m)}{i\pi}$ with χ_m being an odd Dirichlet character mod p. This will allow us to construct another family of number fields K containing $\mathbb{Q}(\zeta_p)$, for which dim $V(K,p)$ is explicitly determined. With this, we state the second theorem in this section.

Theorem 26. For any odd prime $p > 5$, there exists a number field K such that $\mathbb{Q}(\zeta_p) \subset K$ satisfying property (NT). Moreover, we set

$$N(p) := \#\{n | \text{ there exists a number field } K \text{ such that } \dim\ V(K,p)^o = n\}.$$

Then we have,

$$N(p) \geq d(\varphi(p-1)).$$

We would require the following lemma.

Lemma 9. *The set* $\{\alpha_m : \quad 1 \leq m \leq p-1, \ (m, p-1) = 1\}$ *forms a basis for the field* $\mathbb{Q}(\zeta_{p-1}, \zeta_p)$ *over* $\mathbb{Q}(\zeta_p)$.

Proof. Let G denote the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{p-1}, \zeta_p)/\mathbb{Q}(\zeta_p))$. The elements σ_j of G are given by $\sigma_j(\zeta_{p-1}) = \zeta_{p-1}^j$ with $(j, p-1) = 1$. Now suppose that

$$\sum_i c_i \alpha_i = 0,$$

where i runs over co-prime classes mod $(p-1)$ and $c_i \in \mathbb{Q}(\zeta_p)$. Note that $\sigma_i(\alpha_1) = \alpha_i$. This implies that for any $\sigma_j \in G$ with $\sigma_j(\zeta_{p-1}) = \zeta_{p-1}^j$, we have the system of equations

$$\sum_i c_i(\sigma_j^{-1}\sigma_i)(\alpha_1) = 0, \quad 1 \leq j \leq p-1, \ (j, p-1) = 1.$$

But the determinant of the matrix $\left((\sigma_j^{-1}\sigma_i)(\alpha_1)\right)$ is a Dedekind determinant and is given by products of the form

$$\sum_{k \in (\mathbb{Z}/(p-1)\mathbb{Z})^\times} \rho(\sigma_k)\sigma_k(\alpha_1) = \sum_{k \in (\mathbb{Z}/(p-1)\mathbb{Z})^\times} \rho(\sigma_k)\alpha_k,$$

where ρ is a character of G. This is a linear combination of Gauss sums with coefficients in $\mathbb{Q}(\zeta_{p-1}, \zeta_{\varphi(p-1)})$. So by Lemma 8, we conclude $c_i = 0$ for all i. □

We proceed to prove Theorem 26.

Proof of Theorem 26. We work with a field K where $\mathbb{Q}(\zeta_p) \subseteq K \subseteq \mathbb{Q}(\zeta_{p(p-1)})$. By the previous lemma,

$$\mathbb{Q}(\zeta_p) < \alpha_m : 1 \leq m \leq p-1, (m, p-1) = 1 >= \mathbb{Q}(\zeta_{p(p-1)}).$$

Thus we have

$$K < \alpha_m : 1 \leq m \leq p-1, (m, p-1) = 1 >= \mathbb{Q}(\zeta_{p(p-1)}) \cong K^{[\mathbb{Q}(\zeta_{p(p-1)}):K]}.$$

We deduce that

$$\dim V(K,p)^o = [\mathbb{Q}(\zeta_{p(p-1)}) : K].$$

For any $m \mid \varphi(p-1)$, there exists a number field K such that $\mathbb{Q}(\zeta_p) \subseteq K \subseteq \mathbb{Q}(\zeta_{p(p-1)})$ and $[K : \mathbb{Q}(\zeta_p)] = m$ and hence

$$N(p) \geq d(\varphi(p-1)).$$

The above equation also shows that for any number field K with $\mathbb{Q}(\zeta_p) \subsetneq K \subsetneq \mathbb{Q}(\zeta_{p(p-1)})$, we have

$$1 + \frac{p-3}{2} < \dim \ V(K,p) = [\mathbb{Q}(\zeta_p, \zeta_{p-1}) : K] + \frac{p-3}{2} \leq \frac{\varphi(p-1)}{2} + \frac{p-3}{2} < p-2.$$

□

The above proof also gives us the following Corollary :

Corollary 12. *For any odd prime $p > 5$, there exists a number field $K \subset \mathbb{Q}(\zeta_{p(p-1)})$ such that* $\dim \ V(K,p) = \frac{p+1}{2}$.

Proof. Take $K = \mathbb{Q}(\zeta_p, \zeta_{p-1} + \zeta_{p-1}^{-1})$. □

6.3 Dimension Estimates

Definition 8. *For each prime p, let $B(p)$ be the set of integers*

$$B(p) := \left\{ n \;\middle|\; \frac{p-1}{2} < n < p-2 \text{ and } \dim \ V(K,p) = n \text{ for some number field } K \right\},$$

and $b(p)$ denotes the cardinality of $B(p)$.

We prove a dimension estimate on $\limsup b(p)$ by using congruence conditions.

Corollary 13. *We have* $\limsup_{p\to\infty} \ b(p) = \infty$.

Proof. For each $n \geq 1$, let us choose a prime p_n such that $p_n \equiv 1 \bmod (2^n)$ and $p_{n+1} > p_n$. As noted in the end of proof of Theorem 10, for every even d with $d \mid p_n - 1$, there exist a number field K such that dimension of the space $V(K, p_n)$ is equal to $\frac{p_n-1}{2} + \frac{d}{2} - 1$. Thus $b(p_n) \gg n$. This proves the Corollary. □

We now prove the following lemma which is a consequence of Linnik's theorem (See Theorem 15).

Lemma 10. *There exists infinitely many primes p such that*

$$d(p-1) > e^{(c\frac{\log p}{\log \log p})}.$$

Here c is an absolute constant.

Proof. For $n > 1$, let Q_n denote the product of first n primes. By Linnik's theorem, there exists a prime p such that

$$p \equiv 1 \mod (Q_n) \quad \text{and} \quad p < CQ_n^L. \tag{6.4}$$

Therefore, $p - 1$ has at least n distinct prime divisors and hence $d(p-1) \geq 2^n$. But by the Prime Number Theorem (See A.3.1), we have

$$Q_n = e^{(1+o(1))n \log n}.$$

Since $n >> \frac{\log Q_n}{\log \log Q_n}$, by (6.4), we get the result. □

We now state the theorem improving the limsup estimate of Corollary 13.

Theorem 27. There exists a constant $c > 0$ such that

$$b(p) > \exp\left(\frac{c \log p}{\log \log p}\right)$$

for infinitely many primes p. In particular, for any integer $N > 1$ we have

$$\limsup_{p \to \infty} \frac{b(p)}{(\log p)^N} = \infty.$$

Proof. Recall that in the course of the proof of Theorem 10, we showed that for every even divisor d of $p - 1$, there exist a number field K such that dim $V(K, p)$ is equal to $\frac{p-3}{2} + \frac{d}{2}$. Thus, we have

$$b(p) \geq d\left(\frac{p-1}{2}\right) - 2 \geq \frac{d(p-1)}{2} - 2.$$

But by Lemma 10, there are infinitely many p with $d(p-1) > e^{\left(\frac{c \log p}{\log \log p}\right)}$. This in particular proves the Theorem. □

6.4 Link to Fermat and Sophie Germain primes

In the final section, we link the dimension of the spaces $V(K, p)$ of certain cyclotomic fields to Fermat and Sophie Germain primes. For an odd prime $p > 5$, let $\mathbb{Q}(\zeta_p)$ be the p-th cyclotomic field and let d_p denote the dimension of the space $V(\mathbb{Q}(\zeta_p), p)$. We now have the following.

1. $d_p = p - 2$ if and only if p is a Fermat prime.

   Let p be a Fermat prime. Then $\varphi(p-1) = \frac{p-1}{2} = \dim\ V(\mathbb{Q}(\zeta_p), p)^o$. Hence $d_p = p - 2$. For the converse, let $d_p = p - 2$. Then $\dim\ V(\mathbb{Q}(\zeta_p), p)^o = \frac{p-1}{2}$. Hence $\varphi(p-1)$ is necessarily equal to $\frac{p-1}{2}$. This implies that 2 is the only prime factor of $p - 1$ and hence p is a Fermat prime.

2. $d_p = p - 3$ if and only if p is a Sophie Germain prime.

   Let p be a Sophie Germain prime, that is, $p = 2q + 1$ where q is a prime. Then as before,
   $$d_p = \frac{p-3}{2} + \varphi(2q) = 2q - 2 = p - 3.$$
   Conversely, if $d_p = p - 3$, then $\varphi(p-1) = \frac{p-1}{2} - 1$. But this implies that
   $$\frac{p-1}{2} \prod_{\substack{l \mid (p-1) \\ l \neq 2}} \left(1 - \frac{1}{l}\right) = \frac{p-1}{2} - 1$$
   which implies that $p - 1$ is square free and has only one odd prime factor q. Hence it is a Sophie Germain prime.

6.5 Concluding Remarks

The constructions in Section 6.2 rely heavily on linear independence of Gauss sums, and it is hard to imitate these constructions for non-cyclotomic fields. However we can say the following. By Theorem 25, for any number field K, we have

$$\dim V(K, p)^o = \dim V(K \cap \mathbb{Q}(\zeta_{p(p-1)}))^o$$

Thus to find the natural numbers $n \in B(p)$, it suffices to consider the fields K contained in $\mathbb{Q}(\zeta_p, \zeta_{p-1})$ and compute dim $V(K, p)$.

It is not necessary that all the integers in the interval $I_p = (\frac{p-1}{2}, p-2)$, should be attained as an element of $B(p)$ for all primes p. For instance, when $p = 47$, we note that the degree of the compositum $\mathbb{Q}(\zeta_p, \zeta_{p-1})$ over $\mathbb{Q}$ is $r = [\mathbb{Q}(\zeta_p, \zeta_{p-1}) : \mathbb{Q}] = 46 \times 22$. However, $d(r) = 12$, and therefore, at least one integer in the interval I_p is not attained. This is true for any prime p such that p is a Sophie Germain prime and the odd prime divisor of $(p-1)$ is also Sophie Germain. It is conjectured that there are infinitely many of such primes. Hence conjecturally, we have

$$\liminf b(p) = 12.$$

Chapter 7

On a conjecture of Erdős

7.1 Introduction

We recall the conjecture of Erdős concerning the non-vanishing of $L(1, f)$.

Conjecture 6. *If q is a positive integer and f is a number theoretic function with period q and $f(n) \in \{-1, 1\}$ when $n = 1, 2, \ldots q-1$ and $f(n) = 0$ whenever $n \equiv 0 \bmod q$ then $\sum \frac{f(n)}{n} \neq 0$.*

When $q \equiv 3 \bmod 4$, this conjecture was resolved by Murty and Saradha [49]. The case $q \equiv 1 \bmod 4$ remains open and no proof has been provided so far except for special cases (when q is a prime power, or product of two primes etc). Our goal is to provide conditions to ensure non-vanishing of the coefficient of π while expressing $L(1, f)$ as a linear form in logarithm of algebraic numbers, along with having consequences to the conjecture of Erdős and related questions.

In this chapter, we impose conditions on odd periodic rational valued functions f to arrive at the non-vanishing of $L(1, f)$. As seen in Appendix A.4, this value is an algebraic multiple of π. One of the main goals of this chapter is to prove Theorem 13.

Throughout this chapter, we restrict ourselves to odd functions f of odd square-free period N. The breakup of this chapter is as follows : In Section 7.2, we decompose the field $\mathbb{Q}(\zeta_N)$ as direct sum of 'primitive' subspaces W_d as d ranges over the divisors of N. Using this decomposition, we highlight an application to the non-vanishing of $L(1, f)$ when f is an odd rational periodic function satisfying certain conditions. In Section 7.3, we consider the aspects of valuation of elements of $\mathbb{Q}(\zeta_N)$. Finally, in Section 7.4, we represent the relative trace $\mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}$ of the element $\frac{1+\zeta_N}{1-\zeta_N}$ as $\sigma(\frac{1+\zeta_{N/p}}{1-\zeta_{N/p}})$ for an element σ of $\mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\zeta_{N/p})/\mathbb{Q})]$. This map σ will be computed explicitly. This helps us to prove Theorem 13 and find infinitely many natural numbers N for which Conjecture 6 is true (restricted to odd functions) .

7.2 On a primitive decomposition of $\mathbb{Q}(\zeta_N)$

The motivation for this section comes from the primitivity of characters. Here, we would like to decompose the vector space $\mathbb{Q}(\zeta_N)$ as direct sum of W_d where d divides N. The $\mathbb{Q}$-vector spaces W_d will play the role of 'primitivity' $\bmod d$. We start with the field $\mathbb{Q}(\zeta_p)$, where p is an odd prime. We have the following split exact sequence :

$$0 \longrightarrow W_p \xrightarrow{\quad i \quad} \mathbb{Q}(\zeta_p) \xrightarrow{\frac{1}{p-1} Tr_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}} \mathbb{Q} \longrightarrow 0$$

The kernel of the trace map is given by W_p. Hence, we have the following equality :

$$\mathbb{Q}(\zeta_p) = W_p + \mathbb{Q},$$

with any element $\alpha \in \mathbb{Q}(\zeta_p)$ decomposed uniquely as

$$\alpha = (\alpha - \frac{1}{p-1} Tr_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(\alpha)) + \frac{1}{p-1} Tr_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(\alpha).$$

We also have the isomorphism of $\mathbb{Q}$-algebras :

$$\otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_p) \to \mathbb{Q}(\zeta_N)$$

sending $\otimes_p \alpha_p \mapsto \prod_{p|N} \alpha_p$ (as N is squarefree here) where the prime p runs over the prime divisors of N. Substituting $\mathbb{Q}(\zeta_p) = W_p + \mathbb{Q}$ in the above isomorphism,

$$\mathbb{Q}(\zeta_N) = \sum_{d|N} W_d,$$

where $W_1 = \mathbb{Q}$, and for $d > 1$, $W_d = \prod_{p|d} W_p$. Note that the above sum is a direct sum and moreover $W_d \subset \mathbb{Q}(\zeta_d)$ as $W_p \subset \mathbb{Q}(\zeta_p)$.

Remark 8. *We enumerate the following remarks for further references.*

1. *We have the following characterisation of W_d.*

$$W_d = \{\alpha \in \mathbb{Q}(\zeta_d) \,:\, Tr_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_{d/p})}(\alpha) = 0 \text{ for all primes } p \mid d\}. \tag{7.1}$$

   To prove the above, we express the elements of $\mathbb{Q}(\zeta_d)$ as $\prod_{p|d} \alpha_p$ with $\alpha_p \in \mathbb{Q}(\zeta_p)$. We have the following implications.

$$Tr_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_{d/p})}\alpha = 0 \Leftrightarrow Tr_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}\alpha_p = 0 \Leftrightarrow \alpha_p \in W_p.$$

   We obtain the last implication from the definition of W_p. Hence, we conclude (7.1).

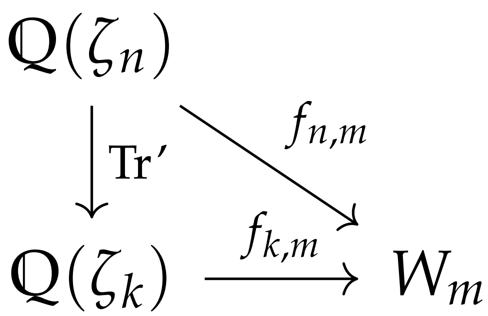
2. *By additive Hilbert's Satz 90, we can write $W_p = (1-\sigma_p)\mathbb{Q}(\zeta_p)$, where σ_p is a generator of $Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Since, $W_d = \prod_{p|d} W_p$, we can write any element y of W_d as $y = \prod_{p|d}(y_p - \sigma_p y_p)$, with $y_p \in \mathbb{Q}(\zeta_p)$. Extending σ_p to a generator of $Gal(\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_{d/p}))$, we can further write*

$$y = \Big(\prod_{p|d}(1-\sigma_p)\Big)\prod_{p|d} y_p.$$

   Hence, we can write $W_d = \big(\prod_{p|d}(1-\sigma_p)\big)\mathbb{Q}(\zeta_d)$, where σ_p is a generator of the subgroup $Gal(\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_{d/p}))$. Here p runs over the odd primes dividing d.

Our next goal is to establish a homomorphism $f_{n,m} : \mathbb{Q}(\zeta_n) \mapsto W_m$ such that the following diagram commutes for any $m \mid k \mid n$. We set $G = Gal(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ and here, $\text{Tr}' : \mathbb{Q}(\zeta_n) \mapsto \mathbb{Q}(\zeta_k)$ is given by the following map

$$\text{Tr}' = \frac{1}{[\mathbb{Q}(\zeta_n) : \mathbb{Q}(\zeta_k)]} Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_k)}.$$



We first give a map from $\mathbb{Q}(\zeta_m) \mapsto W_m$. This is exactly as mentioned in Remark 2 with the map given by

$$x \mapsto \Big(\prod_{p|m}(1-\sigma_p)\Big)x. \tag{7.2}$$

Hence we have a $\mathbb{Q}[G]$ module homomorphism $f_{n,m} : \mathbb{Q}(\zeta_n) \mapsto W_m$ given by

$$f_{n,m}(x) = \frac{1}{[\mathbb{Q}(\zeta_n) : \mathbb{Q}(\zeta_m)]} \prod_{p|m} (1 - \sigma_p) \circ Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_m)}(x). \tag{7.3}$$

We now note that this map $f_{n,m}$ is unique for any choice of k such that $m \mid k \mid n$. Indeed, if we have $\mathbb{Q}(\zeta_k)$ such that $\mathbb{Q}(\zeta_m) \subset \mathbb{Q}(\zeta_k) \subset \mathbb{Q}(\zeta_n)$, then we have

$$\begin{aligned} f_{n,m} &\equiv \frac{1}{[\mathbb{Q}(\zeta_n) : \mathbb{Q}(\zeta_k)]} f_{k,m} \circ Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_k)} \\ &\equiv \frac{1}{[\mathbb{Q}(\zeta_n) : \mathbb{Q}(\zeta_k)]} \frac{1}{[\mathbb{Q}(\zeta_k) : \mathbb{Q}(\zeta_m)]} \prod_{p|m} (1 - \sigma_p) \circ Tr_{\mathbb{Q}(\zeta_k)/\mathbb{Q}(\zeta_m)} \circ Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_k)}, \end{aligned}$$

and by the transtivity of trace maps (clubbed with the fact that $\mathbb{Z}[G]$ is commutative), the expression depends only on n and m.

We would like to study the annihilators of the $\mathbb{Q}[G]$ modules W_d. Let $\widehat{\mathbb{Q}(\zeta_N)} = \mathrm{Hom}_{\mathbb{Q}[G]}(\mathbb{Q}(\zeta_N), \overline{\mathbb{Q}})$. Then $\widehat{\mathbb{Q}(\zeta_N)} \cong \mathbb{Q}[G]$ as $\mathbb{Q}(\zeta_N)$ has a normal basis. Henceforth, we identify $\widehat{\mathbb{Q}(\zeta_N)}$ with $\mathbb{Q}[G]$. For $x \in \mathbb{Q}(\zeta_N)$ and $f = \sum_{\sigma \in G} a_\sigma \sigma \in \mathbb{Q}[G]$, we have the natural action of f on x given by $f(x) = \sum_{\sigma \in G} a_\sigma \sigma(x)$. We recall the definition of annhilator of an element x in $\mathbb{Q}(\zeta_N)$.

Definition 18. *Let $x \in \mathbb{Q}(\zeta_N)$. The annihilator of x denoted by $Ann(x)$ is the set of elements*

$$Ann(x) = \{f \in \mathbb{Q}[G] \, : \, f(x) = 0\}$$

For a subset $S \subseteq \mathbb{Q}(\zeta_N)$, we define $Ann(S)$ as the intersection $\cap_{x \in S} Ann(x)$.

Note that $\mathrm{Ann}(S)$ is an ideal in $\mathbb{Q}[G]$. Moreover, $\mathrm{Ann}(\mathbb{Q}(\zeta_N)) = (0)$. We now compute the annhilator of W_N.

Proposition 11. *$Ann(W_N) = \sum_{p|N} (Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_{N/p})})$, where p runs over the prime divisors of N.*

Proof. For each prime divisor p of N, let us denote $g_p : V_N \to V_{N/p}$ by the relative trace map $Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_{N/p})}$. We have a split exact sequence given by

$$0 \mapsto \ker g_p \mapsto \mathbb{Q}(\zeta_N) \mapsto \mathbb{Q}(\zeta_{N/p}) \to 0.$$

$$\ker g_p = \{x \in \mathbb{Q}(\zeta_N) \, : \, Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_{N/p})} x = 0\} \tag{7.4}$$

We now compute $\mathrm{Ann}(\ker g_p)$:

$$\begin{aligned}\mathrm{Ann}(\ker g_p) &= \{f \in \mathbb{Q}[G] \,:\, f(\ker g_p) = \{0\}\} \\ &= \{f \in \mathbb{Q}[G] \,:\, f(\mathbb{Q}(\zeta_N)) \subseteq \mathbb{Q}(\zeta_{N/p})\} \\ &= \{f \in \mathbb{Q}[G] \,:\, \sigma_p f = f\}. \end{aligned} \tag{7.5}$$

We obtain the second equality by noting that $\mathbb{Q}(\zeta_N) = \ker g_p + \mathbb{Q}(\zeta_{N/p})$ and obtain the last equality as σ_p acts trivially on $\mathbb{Q}(\zeta_{N/p})$. Therefore, if we write $f = \sum_{i=0}^{p-2} \sigma_p^i f_i$, with $f_i \in \mathbb{Q}[Gal(\mathbb{Q}(\zeta_{N/p})/\mathbb{Q})]$, then by (7.5) clubbed with the fact that $\sigma_p^{p-1} = 1$, we note $f_i = f_{i+1}$ for all i. Hence $f = (\sum_{i=0}^{p-2} \sigma_p^i) f_0$. This proves that $\mathrm{Ann}(\ker g_p) = (Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_{N/p})})$.

By (7.4) and (7.1), we note that $W_N = \cap_{p|N} \ker g_p$, where p runs over the primes dividing N. Therefore, $\mathrm{Ann}(W_N) = \sum_{p|N}(Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_{N/p})})$. □

Now, for a natural number $N \not\equiv 2 \bmod 4$, we consider the following vector space over $\mathbb{Q}$:

$$V_N = \mathbb{Q}\,\langle x \,:\, x \in \mathbb{Q}(\zeta_N),\ \sigma(x) = -x\rangle\,,$$

where σ denotes the complex conjugation. Since the complex conjugation commutes with the trace, we construct a family of $\mathbb{Q}$ vector spaces $W_N^- \subset \mathbb{Q}(\zeta_N)$ such that for any $N \not\equiv 2 \bmod 4$ and squarefree; we can write

$$V_N = \oplus_{d|N} W_d^-, \tag{7.6}$$

Naturally $W_N^- = W_N \cap V_N$. Also, note that W_d^- is stable under the action of the Galois group $G = Gal(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. The proof of the preceding proposition and observations holds verbatim if we replace $\mathbb{Q}(\zeta_N)$ by V_N and $\mathbb{Q}[G]$ by $\mathbb{Q}[G]/(\sigma+1)$.

7.2.1 An application

We recall the value of $L(1, f)$ for an odd rational periodic function f of period N as mentioned in Appendix A.4.

$$L(1,f) = -\frac{\pi}{2N}\Big(\sum_{a=1}^{N} f(a)\cot\frac{2\pi a}{N}\Big) = \frac{\pi i}{N}\sum_{a=1}^{[N/2]} f(a)\frac{1+\zeta_N^a}{1-\zeta_N^a}.$$

Further assuming $f(a) = 0$ whenever $(a, N) > 1$, from Theorem 8, we have $L(1, f) \neq 0$ unless $f \equiv 0$. In other words, any $\mathbb{Q}$ linear combination

$$\sum_{\substack{j=1 \\ (j,N)=1}}^{[N/2]} a_i \frac{1+\zeta_N^j}{1-\zeta_N^j}$$

is not zero unless $a_j = 0$ for all j. From here, we conclude that

$$V_N = \mathbb{Q}[G]\langle \frac{1+\zeta_N}{1-\zeta_N} \rangle. \tag{7.7}$$

Since $V_N = (1-\sigma)\mathbb{Q}(\zeta_N)$, by Remark 2 we conclude :

$$W_N^- = \mathbb{Q}[G]\langle (\prod_{p|N}(1-\sigma_p))(\frac{1+\zeta_N}{1-\zeta_N}) \rangle. \tag{7.8}$$

Our goal is to understand the nature of the "individual components" of $L(1, f)$ for odd functions f of period pq where p and q are odd primes. We have the explicit expression of $L(1, f)$ namely,

$$L(1, f) = \frac{\pi i}{pq} \sum_{j=1}^{pq-1/2} f(j) \frac{1+\zeta_{pq}^j}{1-\zeta_{pq}^j} = \frac{\pi i}{pq} \sum_{d|pq} \beta_d,$$

where for each d dividing N, β_d is given by the following expression :

$$\beta_d := \sum_{\substack{i=1 \\ (i,d)=1}}^{[d/2]} f(\frac{pq}{d} i) \frac{1+\zeta_d^i}{1-\zeta_d^i}.$$

If $L(1, f) = 0$, then we have the following :

$$\sum_{d|pq} \beta_d = 0 \implies (1-\sigma_p)(1-\sigma_q) \sum_{d|pq} \beta_d = 0 \implies (1-\sigma_p)(1-\sigma_q)(\beta_{pq}) = 0.$$

Recall that σ_p is a generator of the group $\mathrm{Gal}(\mathbb{Q}(\zeta_{pq})/\mathbb{Q}(\zeta_q))$, and therefore $(1-\sigma_p)(\beta_q) = 0$. Also, note that the complex conjugation σ is given by $\sigma = \sigma_p^{p-1/2} \sigma_q^{q-1/2}$.

It was proved earlier by Okada [51] that Conjecture 6 is true when N is a product of two primes. We ask the following question :

Question 3. *Are there any rational valued periodic functions f of period pq satisfying $f(a) \in \{\pm 1\}$ whenever $(a, pq) = 1$ and $L(1, f) = 0$.*

With the conditions mentioned in the question, let $G = \mathrm{Gal}(\mathbb{Q}(\zeta_{pq})/\mathbb{Q})$ and

$$\beta_{pq} := \sum_{\substack{i=1 \\ (i,pq)=1}}^{(pq-1)/2} f(i) \frac{1+\zeta_{pq}^i}{1-\zeta_{pq}^i} = F_{pq}\left(\frac{1+\zeta_{pq}}{1-\zeta_{pq}}\right), \qquad f(a) \in \{\pm 1\} \tag{7.9}$$

where $F_{pq} = \sum_i f\chi_0(i)\sigma_i$ (Here σ_i denotes the element of G mapping ζ_{pq} to ζ_{pq}^i, χ_0 denotes the principal character).

Proposition 12. *Let $\beta_{pq} \in V_{pq}$ given as in (7.9). If $(1-\sigma_p)(1-\sigma_q)\beta_{pq} = 0$, then $\beta_{pq} \in V_p \cup V_q$.*

Proof. Since $(1-\sigma_p)(1-\sigma_q)\beta_{pq} = 0$, we have

$$\left(F_{pq}(1-\sigma_p)(1-\sigma_q)\right)\left(\frac{1+\zeta_{pq}}{1-\zeta_{pq}}\right) = 0.$$

We conclude that $F_{pq} \in \mathrm{Ann}(W_{pq}^-)$ and hence by Proposition 11, we can write

$$\begin{aligned} F_{pq} &= \left(\sum_{i=0}^{q-2} b_i\sigma_q^i\right)\left(\sum_{j=0}^{p-2}\sigma_p^j\right) + \left(\sum_{i=0}^{\frac{p-1}{2}-1} c_i\sigma_p^i\right)\left(\sum_{j=0}^{q-2}\sigma_q^j\right) \\ &= \left(\sum_{i=0}^{q-2}(b_i - b_{i+\frac{q-1}{2}})\sigma_q^i\right)\left(\sum_{j=0}^{\frac{p-1}{2}-1}\sigma_p^j\right) + \left(\sum_{i=0}^{\frac{p-1}{2}-1} c_i\sigma_p^i\right)\left(\sum_{j=0}^{q-2}\sigma_q^j.\right) \end{aligned} \tag{7.10}$$

In the above, the second equality is obtained by the substitution $\sigma_p^{p-1/2}\sigma_q^{q-1/2} = -1$ as the associated group ring is $\mathbb{Q}[G]/(\sigma_p^{p-1/2}\sigma_q^{q-1/2}+1)$. Assume $(1-\sigma_p)\beta_{pq} \neq 0$. Then $c_i \neq 0$ for some i as $(1-\sigma_p)$ annhilates the term $(\sum_{i=0}^{q-2} b_i\sigma_q^i)(\sum_{j=0}^{p-2}\sigma_p^j)$.

Now, comparing the coefficients of $\sigma_p^i\sigma_q^j$ and $\sigma_p^i\sigma_q^{j+q-1/2}$ for $0 \leq j \leq q-1/2$ in (7.10), we obtain the following:

$$c_i + (b_j - b_{j+\frac{q-1}{2}}) = \pm\left(c_i - (b_j - b_{j+\frac{q-1}{2}})\right) \neq 0.$$

The first equality holds true as the coefficients of F_{pq} in $\{\pm 1\}$. The solution of the above equation is given by $c_i = 0$ or $(b_j - b_{j+\frac{q-1}{2}}) = 0$. Since $c_i \neq 0$, we obtain $b_j - b_{j+\frac{q-1}{2}} = 0$ for all j in the given range. Substituting in (7.10), we get $(1-\sigma_q)\beta_{pq} = 0$, that is $\beta_{pq} \in V_p$. □

Remark 9. *The above proposition is motivated by the following : The number of equations ($\varphi(pq)/2$), is more than the number of unknowns ($\varphi(p)/2 + \varphi(q)/2$). It would be interesting to formulate the statement when the number of prime divisors of N exceeds two.*

To show that an element $\beta \in \mathbb{Q}(\zeta_N)$ is non-zero, we use the following principle: If there exists $F \in \mathbb{Q}[G]$ such that $F(\beta) \neq 0$, then $\beta \neq 0$. With the help of the above proposition, we prove the following corollary :

Corollary 14. *Let f be an odd periodic function of period pq with p, q being odd primes and further assume that β_{pq} is of the form (7.9). If $f(pn) \neq 0$ and $f(qm) \neq 0$ for some integer m and n, then $L(1, f) \neq 0$.*

Proof. To show the non-vanishing of $L(1, f)$, it suffices to prove that $S_{pq} := \sum_{d|pq} \beta_d \neq 0$. We first act S_{pq} by the operator $(1-\sigma_p)(1-\sigma_q)$.

$$\big((1-\sigma_p)(1-\sigma_q)\big)S_{pq} = \big((1-\sigma_p)(1-\sigma_q)\big)\beta_{pq}.$$

If $((1-\sigma_p)(1-\sigma_q))S_{pq} \neq 0$, then $S_{pq} \neq 0$. Hence, we consider the case $\big((1-\sigma_p)(1-\sigma_q)\big)\beta_{pq} = 0$. From Proposition 12, we have $\beta_{pq} \in V_p \cup V_q$. Assume that $\beta_{pq} \in V_p$ i.e. $(1-\sigma_q)\beta_{pq} = 0$. Now, note that

$$(1-\sigma_q)S_{pq} = (1-\sigma_q)\sum_{d|pq} \beta_d = (1-\sigma_q)\beta_q.$$

Since $\beta_q \neq 0$ (as $f(pn) \neq 0$), we have $(1-\sigma_q)\beta_q \neq 0$ ($(1-\sigma_q)$ is invertible on V_q as the elements of V_q are totally imaginary) and therefore $L(1, f) \neq 0$ as $\big(1-\sigma_q\big)S_{pq} \neq 0$. □

7.3 On ideals of group rings associated to $\mathbb{Q}(\zeta_p)$

We begin with the following elementary proposition giving a sufficient condition on the non-vanishing of $L(1, f)$ where f is an odd integer valued periodic function. We mention here that if p is an odd prime, then $p/(1-\zeta_p)$ is an algebraic integer.

Proposition 2. *Let N be square-free and $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}$ be an odd function. For each $d|N$, $d \neq 1$, we set*

$$\alpha_d := \sum_{\substack{i=1 \\ (i,\frac{N}{d})=1}}^{[N/2d]} f(di)\frac{1+\zeta_N^{di}}{1-\zeta_N^{di}}.$$

If $\alpha_{N/p}$ is not an algebraic integer for some prime p dividing N, then $L(1, f) \neq 0$.

Proof. Since $\alpha_{N/p}$ is not an algebraic integer for some prime p dividing N, we note that $N/p(\alpha_{N/p})$ is not an algebraic integer as $(N/p, p) = 1$ (Here we are using $p\alpha_{N/p}$ is an algebraic integer by Proposition 4). Therefore,

$$\frac{N}{p}L(1, f) = \frac{N}{p}\frac{\pi i}{N}\sum_{a=1}^{[N/2]} f(a)\frac{1+\zeta_N^a}{1-\zeta_N^a} = \frac{\pi i}{N}\big(\frac{N}{p}(\sum_{\substack{k|N \\ k\nmid p}} \alpha_{N/k}) + \frac{N}{p}\alpha_{N/p}\big).$$

Since every term in the above equation except $N/p(\alpha_{N/p})$ is an algebraic integer (upto the multiple of $\pi i/N$), we conclude that $L(1, f) \neq 0$. □

Hence, if there are primes p for which the none of the elements in the set

$$S_p := \{ \sum_{i=1}^{p-1/2} a_i \frac{1+\zeta_p^i}{1-\zeta_p^i} \ : \ a_i \in \{\pm 1\} \}, \tag{7.11}$$

is an algebraic integer, then by Proposition 2 we conclude Conjecture 6 (when restricted to odd periodic functions) is true mod N for all squarefree N divisible by the prime p. Using SAGE (See Appendix B.1.1 for the program), we check whenever p is either 3 or 5, there are no algebraic integers in S_p. Hence, the conjecture (for odd periodic functions f) is true whenever N is a squarefree integer divisible by either of these numbers.

We would also like to give an upper bound on the $\mathfrak{p}$-adic valuation of the elements of S_p purely depending on p where $\mathfrak{p} = (1 - \zeta_p)$ is the unique prime ideal above p in $\mathbb{Z}[\zeta_p]$. We prove the following lemma:

Lemma 11. *Let p be a prime. We can express $\frac{1+\zeta_p}{1-\zeta_p}$ as the following :*

$$\frac{1+\zeta_p}{1-\zeta_p} = -1 - \frac{2}{p}\sum_{i=1}^{p-1} i\zeta_p^i. \tag{7.12}$$

Proof. Since $\{\zeta_p^i\}_{i=1}^{p-1}$ forms a normal basis of $\mathbb{Q}(\zeta_p)$ over $\mathbb{Q}$, we write

$$\frac{1+\zeta_p}{1-\zeta_p} = \sum_{i=1}^{p-1} a_i \zeta_p^i. \tag{7.13}$$

Moreover we obtain

$$\frac{\zeta_p}{1-\zeta_p} = \frac{1+\zeta_p}{1-\zeta_p} - 1 = \sum_{i=1}^{p-1}(a_i+1)\zeta_p^i.$$

Therefore, we have another expression for (7.13) namely,

$$\frac{1+\zeta_p}{1-\zeta_p} = \sum_{i=1}^{p-2}(a_{i+1}-1)\zeta_p^i + (a_1+1) = \sum_{i=1}^{p-2}(a_{i+1}-a_1)\zeta_p^i - (a_1+1)\zeta_p^{p-1}. \tag{7.14}$$

Comparing the coefficients of (7.13) and (7.14), we have

$$a_i = a_{i+1} - a_1 \qquad \text{for all } 1 \le i \le p-2 \quad \text{and} \qquad a_{p-1} = -a_1 - 1. \tag{7.15}$$

From the above, we obtain $a_i = ia_1$ for all $1 \le i \le p-1$ and substituting in (7.15), we obtain $a_1 = \frac{-1}{p}$ and therefore $a_i = \frac{-i}{p}$. Hence, we have

$$\frac{1}{1-\zeta_p} = \frac{-1}{p}\sum_{i=1}^{p-1} i\zeta_p^i.$$

We get the lemma by noting that $\frac{1+\zeta_p}{1-\zeta_p} + 1 = 2\frac{1}{1-\zeta_p}$. □

Moreover, applying the automorphism $\zeta_p \to \zeta_p^j$, on (7.12) we obtain,

$$\frac{1+\zeta_p^j}{1-\zeta_p^j} = -1 - \frac{2}{p}\sum_{i=1}^{p-1} \overline{ij^{-1}}_p \zeta_p^i, \tag{7.16}$$

where $\overline{x}_p$ denotes the representative $i \in \mathbb{Z}$ $1 \le i \le p-1$ of x mod p. The above computation yields the following corollary :

Corollary 15. *For a fixed prime p greater than 5, we have*

$$\max_{y \in S_p} v_{\mathfrak{p}}(y) < p-1$$

Proof. Let $y \in \mathbb{Q}(\zeta_p)$ such that $v_{\mathfrak{p}}(y) \geq p-1$. Then,

$$(y) = \mathfrak{p}^{p-1}\mathcal{Q} = p\mathcal{Q} \qquad v_{\mathfrak{p}}(\mathcal{Q}) \geq 0.$$

Therefore, if we write $y = \sum_{i=1}^{p-1} a_{i,y}\alpha_i$ for some basis $\{\alpha_i\}_{i=1}^{p-1}$ of $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ with $v_{\mathfrak{p}}(\alpha_i) = 0$ for all i, then $p \mid a_{i,y}$. We proceed to show that the elements of S_p do not satisfy this condition. We fix a basis $\{\zeta_p^i\}_{i=1}^{p-1}$ of $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ to perform the computations. Using (7.16) with $j = 1$ and writing $-1 = \sum_{j=1}^{p-1} \zeta_p^j$, we have the following :

$$\frac{1+\zeta_p}{1-\zeta_p} = \sum_{i=1}^{p-1} (\frac{p-2i}{p})\zeta_p^i.$$

If we write the above coefficients as $\frac{\sum_{i=1}^{p-1} a_i\zeta_p^i}{p}$, then for all i we have $|a_i| < p-1$. For $i \neq \pm j \bmod p$, we have $a_i \neq a_j$ and hence, when we consider the conjugate $1+\zeta_p^j/(1-\zeta_p^j)$, we note that the coefficients of ζ_p^i are distinct from ζ_p^j. Therefore, the coefficient of any element in S_p with respect to the basis $\{\zeta_p^i\}_{i=1}^{p-1}$ is bounded by $\frac{1}{p}\sum_{i=1}^{p-1} i \leq (p-1)/2$. Thus, as mentioned in the first paragraph, if $y \in S_p$, then $v_{\mathfrak{p}}(y) < p-1$. □

7.3.1 Valuation Ideals

We denote $\mathbb{F}_p$ to be the field $\mathbb{Z}/p\mathbb{Z}$. Let p be an odd prime, $K = \mathbb{Q}(\zeta_p)$, O_K the ring of integers of K and $G = \mathrm{Gal}(K/\mathbb{Q})$. We set $\mathfrak{p} := (1-\zeta_p)$ the prime ideal in O_K above p and $v_{\mathfrak{p}}$, the associated normalised $\mathfrak{p}$-adic valuation in K^*. For $\alpha \in K^*$, let $I_{k,\alpha} = \{f \in \mathbb{Z}[G] : v_{\mathfrak{p}}(f(\alpha)) \geq k\}$. We would like to understand the generators of the $\mathbb{Z}[G]$ ideal I_k when $k \leq p-1$. Let τ_g be a fixed generator of G sending $\zeta_p \mapsto \zeta_p^g$. By abuse of notation, we will also denote $g \bmod p$ as g wherever required.

Proposition 13. *Fix an element α in K^* such that $-1 \leq v_{\mathfrak{p}}(\alpha) < p-2$. For $v_{\mathfrak{p}}(\alpha)+1 \leq k \leq (p-2)$, we have,*

$$I_{k,\alpha} = \left\langle p, \prod_{i=v_{\mathfrak{p}}(\alpha)}^{k-1} (\tau_g - g^i) \right\rangle.$$

Proof. Consider the $\mathbb{F}_p$ vector space $\mathfrak{p}^m/\mathfrak{p}^{m+1}$ for m in the given range. As an additive group, this is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and this vector space is also stable under the action of the Galois group G. We compute the action of $\tau_g : \zeta_p \mapsto \zeta_p^g$ on this vector space.

$$\tau_g(x(1-\zeta_p)^m + \mathfrak{p}^{m+1}) = x(1-\zeta_p^g)^m + \mathfrak{p}^{m+1} = g^m x(1-\zeta_p)^m + \mathfrak{p}^{m+1}, \tag{7.17}$$

where the first equality is obtained as the inertia group of $\mathfrak{p}$, $\mathcal{I}_{\mathfrak{p}} = G$ and the second equality is true since $(1-\zeta_p^g)/(1-\zeta_p) - g \in \mathfrak{p}$ (Note that $\sum_{i=0}^{g-1} \zeta_p^i = \sum_{i=0}^{g-1}(\zeta_p^i - 1) + g \in \mathfrak{p} + g$). Hence, we conclude the action of τ_g is the same as multiplication by g^m on the vector space $\mathfrak{p}^m/\mathfrak{p}^{m+1}$.

It remains to compute the ideal $I_{k,\alpha}$. Since, $p \in I_{k,\alpha}$ it suffices to find the generators modulo p i.e. to consider the ring $\mathbb{F}_p[G]$ and find the generator of $I_{k,\alpha}$ mod p. By Remark A.4.3, the ideal $\mathcal{I}_{k,\alpha}$ is of the form $\prod_{i\in S}(\tau_g - g^i)$. Since the element τ_g acts as g^m on $\mathfrak{p}^m/\mathfrak{p}^{m+1}$, we note that an element of $I_{k,\alpha}$ mod p due to successive iteration is $\left(\prod_{i=v_p(\alpha)}^{k-1}(\tau_g - g^i)\right)$. Moreover, we claim that this element is a generator of the ideal. If we consider a proper subset S of $\{v_{\mathfrak{p}}(\alpha), \ldots, k-1\}$, we write $S = S_1 \cup S_2$, with S_1 consisting of consecutive integers starting from $v_{\mathfrak{p}}(\alpha)$ to l and S_2 consisting of integers greater than $l+2$. The product $\prod_{i\in S}(\tau_g - g^i)$ maps α to $\mathfrak{p}^{l+1}$ as the element $\prod_{i\in S_2}(\tau_g - g^i)$ is invertible in $\mathfrak{p}^{l+1}/\mathfrak{p}^{l+2}$. This proves the claim and hence the proposition is proved by lifting the element $\left(\prod_{i=v_p(\alpha)}^{k-1}(\tau_g - g^i)\right)$ to $\mathbb{Z}[G]$. □

We consider the $\mathfrak{p}$-adic valuation of the elements in V_p (recall V_p is the $\mathbb{Q}$ vector space consisting of totally imaginary numbers in $\mathbb{Q}(\zeta_p)$). Since the prime ideal $\mathfrak{p}$ is totally ramified, for any non-zero element x of the totally real subfield $\mathbb{Q}(\zeta_p)^+$, we have $v_{\mathfrak{p}}(x) \in 2\mathbb{Z}$. It follows that the elements of V_p have odd $\mathfrak{p}$- adic valuation. Also, the element $\sigma - 1$ is invertible in V_p (recall σ is the complex conjugation) and due to this, the elements $(\tau_g - g^{2i})$ are invertible in V_p. Since $\mathrm{Ann} V_p = < \tau_g^{p-1/2} + 1 >$, we have the following corollary :

Corollary 16. *Let $\alpha \in V_p$ be as mentioned in the proposition. For $v_{\mathfrak{p}}(\alpha) + 1 \le k \le (p-2)$, we have,*

$$I_{k,\alpha} = \left\langle p, \prod_{\substack{i=v_{\mathfrak{p}}(\alpha)\\ i \text{ odd}}}^{k-1} (\tau_g - g^i), (\tau_g^{\frac{p-1}{2}} + 1) \right\rangle.$$

The above Corollary can also be used to show that Proposition 2 cannot be used to solve Erdős conjecture mod$N$ for almost all squarefree numbers N. In some sense, the fact that S_3 and S_5 do not have any algebraic integer are lucky instances.

Corollary 17. *Let p be an odd prime number which is not a Fermat prime. Then S_p contains an algebraic integer.*

Proof. Let L be a subfield of $\mathbb{Q}(\zeta_p)$ such that $[\mathbb{Q}(\zeta_p) : L]$ is odd. This is possible as p is not a Fermat prime and hence, $p - 1/2$ has an odd prime factor. We compute the relative trace $f = \mathrm{Tr}_{\mathbb{Q}(\zeta_p)/L} \in \mathbb{F}_p[G]$. If $\tau_g : \zeta_p \mapsto \zeta_p^g$ is a generator of $Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and $d = [\mathbb{Q}(\zeta_p) : L]$, then we note that $f = \sum_{i=0}^{d-1} \tau_g^{(p-1)i/d}$. We claim that $(\sigma_g - g^{-1})$ divides f in $\mathbb{F}_p[G]$. To see the same, consider a pullback of f (say $\widetilde{f}$) in the surjective ring homomorphism given by the following :

$$\mathbb{F}_p[X] \mapsto \mathbb{F}_p[G], \qquad X \to \tau_g, \alpha \mapsto \alpha \text{ for } \alpha \in \mathbb{F}_p.$$

We claim that the element $\widetilde{f}$ of $\mathbb{F}_p[G]$ given by $\widetilde{f} := \sum_{i=0}^{d-1} X^{(p-1)/d}$ belongs to $(X - g^{-1})$. Since $\widetilde{f} = \frac{X^{p-1}-1}{X^{p-1/d}-1}$ and $g^k = 1 \implies p-1 \mid k$, we will have $\widetilde{f}(g^{-1}) = 0$. Hence $f \in (\tau_g - g^{-1})$, and from previous corollary, $f(\frac{1+\zeta_p}{1-\zeta_p}) \in \mathbb{Z}[\zeta_p]$. From this computation, we conclude that $\alpha := \mathrm{Tr}_{\mathbb{Q}(\zeta_p)/L}(1+\zeta_p)/(1-\zeta_p) \in \mathbb{Z}[\zeta_p]$.

Let H be the subgroup of G fixing L and G^+ be the unique subgroup of G of index 2. Since the order of $[G : H] = (p-1)/d$ is even, we note that $H \subseteq G^+ \subset G$. Consider the sum $\beta := \sum_{[\tau] \in [G^+/H]} a_\tau \tau(\alpha)$ with $a_\tau \in \{\pm 1\}$. Since $\alpha \in L$, the element $\tau(\alpha)$ does not depend on the choice of the representative of τ in G/H. If we write

$$\tau(\alpha) = \sum_{j \in J_\tau} b_j \frac{1+\zeta_p^j}{1-\zeta_p^j}, \qquad b_j \in \{\pm 1\}$$

for $J_\tau \subset \{1, \dots (p-1)/2\}$, we note that for $[\tau_1] \neq [\tau_2]$, we have $J_{\tau_1} \cap J_{\tau_2} = \emptyset$. As we vary over the representatives $[\tau]$ in G^+/H, we cover over all the indices $\cup_{[\tau] \in G^+/H} J_\tau = \{1, \dots p-1/2\}$ (by order consideration) and since $J_{\tau_1} \cap J_{\tau_2} = \emptyset$, the coefficients of β are in $\{\pm 1\}$. This proves that the element β is in S_p. □

7.4 Trace of cotangents

In this section, we compute the relative trace $Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}$ of the element $(1+\zeta_n)/(1-\zeta_n)$ for squarefree n and an odd prime p. We proceed to provide a vanishing criteria of $L(1, f)$ for odd valued functions f of period N under additional assumptions. Through this computation, we find new family of infinitely many numbers N that satisfy ˝ecture (with few more restrictions on the functions f). Also, let $\mathrm{ord}_m p$ denote the order of p in $(\mathbb{Z}/m\mathbb{Z})^*$

Lemma 12. *Let n be an odd squarefree integer and p be a prime divisor of n.*

$$Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}(\frac{1+\zeta_n}{1-\zeta_n}) = \big(\sigma_b(p\Psi^{(p)}_{n/p} - 1)\big)(\frac{1+\zeta_{n/p}}{1-\zeta_{n/p}}).$$

where $\Psi^{(p)}_{n/p} \in Gal(\mathbb{Q}(\zeta_{n/p})/\mathbb{Q})$ is given by $\Psi^{(p)}_{n/p} : \zeta_{n/p} \mapsto \zeta^p_{n/p}$ and $\sigma_b \in Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is given by $\sigma_b(\zeta_n) = \zeta^b_n$.

Proof. Let a and b be integers such that $1/n = a/p + b/(n/p)$. We write

$$\zeta_n = \zeta^a_p \zeta^b_{n/p} \tag{7.18}$$

Consider the following sum

$$S = Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}(\frac{1}{1-\zeta_n} - \frac{1}{1-\zeta^p_n}).$$

Applying the formal identity

$$\frac{1}{1-x} - \frac{1}{1-x^c} = \frac{x-x^c}{(1-x)(1-x^c)} = \frac{x(1-x^{c-1})}{(1-x)(1-x^c)} = \frac{\sum_{i=1}^{c-1} x^i}{1-x^c},$$

by setting $c = p$ and $x = \zeta_n$, we obtain

$$S = Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})} \frac{\sum_{i=1}^{p-1} \zeta^i_n}{1-\zeta^p_n} = \frac{1}{1-\zeta^{pb}_{n/p}} Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}(\sum_{i=1}^{p-1} \zeta^i_n).$$

From (7.18), we have $Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}\zeta_n = -\zeta^b_{n/p}$, and re-writing the above equation, we get

$$S = -\frac{\sum_{i=1}^{c-1} \zeta^{ib}_{n/p}}{1-\zeta^{pb}_{n/p}} = \frac{-1}{1-\zeta^{pb}_{n/p}}(\frac{1-\zeta^{bp}_{n/p}}{1-\zeta^b_{n/p}} - 1) = -\frac{1}{1-\zeta^b_{n/p}} + \frac{1}{1-\zeta^{pb}_{n/p}}.$$

Therefore,

$$Tr_{\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_{n/p})}(\frac{1}{1-\zeta_n}) = -(1-p\Psi^{(p)}_{n/p})\frac{1}{1-\zeta^b_{n/p}} = \big(-\sigma_b(1-p\Psi^{(p)}_{n/p})\big)(\frac{1}{1-\zeta_{n/p}}).$$

Since the complex conjugation commutes with the Trace operator, we have the above lemma. □

We obtain the following corollary by the transitivity of the trace maps clubbed with commutativity of $\mathbb{Z}[Gal(\mathbb{Q}(\zeta_N)/\mathbb{Q})]$.

Corollary 18. *Let N be odd and squarefree and $m \mid N$. We have*

$$Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_m)}(\frac{1+\zeta_N}{1-\zeta_N}) = (\,\sigma \prod_{p|N/m} (1 - p\Psi_m^{(p)})\,)(\frac{1+\zeta_m}{1-\zeta_m}),$$

for some element $\sigma \in Gal(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ and $\Psi_m^{(p)} : \mathbb{Q}(\zeta_m) \to \mathbb{Q}(\zeta_m)$ denotes the automorphism $\Psi_m^{(p)}(\zeta_m) = \zeta_m^p$.

Remark 10. *There are some similarities and differences in the expression of cotangent [27]. In [27], naturally one has an explicit formula for the coefficients of $\cot \pi k/d$ in terms of $\cot \pi k/n$, whereas we have an expression in terms of elements of $\mathbb{Q}[G]$. By expanding the product*

$$\prod_{p|n/m} (p\Psi_p^m - 1)^{-1} = \prod_{p|n/m} \frac{1}{p^{ord_m p} - 1}(\sum_{i=0}^{ord_m p-1} p^i (\Psi_m^{(p)})^i\,), \tag{7.19}$$

and rearranging the terms we believe that a similar expression as that of [27, Theorem 1] would be obtained. It is curious to note that a similar expression was used in (15) ibidem. Our computation is purely based on elementary algebraic number theory.

7.4.1 Application to Erdős Conjecture

We reprove a statement by Chatterjee and Murty [17, Proposition 3.1] in our setup :

Lemma 13. *Let n be an odd squarefree integer and m be a divisor of n. Suppose we have*

$$S := \sum_{\substack{i=1 \\ (i,m)=1}}^{m/2} a_i \frac{1+\zeta_m^i}{1-\zeta_m^i} = \sum_{\substack{i=1 \\ (i,n)=1}}^{n/2} b_{i,m} \frac{1+\zeta_n^i}{1-\zeta_n^i},$$

where $a_i \in \{0, -1, 1\}$, then $|b_{i,m}| \leq 1/\phi(n/m)$.

Proof. If we write

$$\frac{1}{p\Psi_m^{(p)} - 1} \sum_{\substack{i=1 \\ (i,m)=1}}^{m/2} d_i\sigma_i = \frac{1}{p^{\mathrm{ord}_m p} - 1}(\sum_{i=0}^{\mathrm{ord}_m p-1} p^i(\Psi_m^{(p)})^i) \sum_{\substack{i=1 \\ (i,m)=1}}^{m/2} d_i\sigma_i = \sum_{\substack{i=1 \\ (i,m)=1}}^{m/2} c_{i,p}\sigma_i,$$

then we note that,

$$c_{i,p} = \frac{1}{p^{\text{ord}_m p} - 1} \sum_{\sigma_j(\Psi_m^{(p)})^k = \sigma_i} d_j p^k$$

and therefore,

$$|c_{i,p}| \leq \frac{1}{p-1} \max |d_j| \tag{7.20}$$

Now let $T := \sum_{\substack{i=1 \\ (i,m)=1}}^{m/2} a_i \sigma_i$, where $\sigma_i(\zeta_m) = \zeta_m^i$. We note that $T(\frac{1+\zeta_m}{1-\zeta_m}) = S$. Since $|a_i| \leq 1$, we have $|c_{i,p}| \leq \frac{1}{p-1}$. The lemma is proved by successive iteration of (7.20) as p varies over the prime divisors of n/m as mentioned in Corollary 18. □

We obtain Proposition 1 as a consequence of the above Lemma (for the case of odd functions).

Corollary 19. *Let q be a squarefree odd integer. If $1 > \sum_{\substack{d|q \\ d \geq 3, d \neq q}} \frac{1}{\varphi(d)}$, then Conjecture 6 is true modulo q.*

Proof. Applying the condition in the above lemma yields,

$$\sum_{\substack{d|q \\ d \geq 3, d \neq q}} |b_{1,d}| < 1$$

Let us denote the coefficient of $\frac{1+\zeta_q}{1-\zeta_q}$ as $a_{1,q}$. Since $a_{1,q} \in \{\pm 1\}$, we conclude $|a_{1,q} + \sum_{\substack{d|q \\ d \geq 3, d \neq q}} b_{i,m}| \neq 0$. Therefore Conjecture 6 is true. □

Remark 11. *This corollary is not very different from the proof of Okada's criterion modulo certain computations. Okada's criterion assumes that $L(1, f) = 0$, and proceeds to give a condition on coefficients of f but here, we obtain the same by basis computations. The proof of the above corollary requires the fact that $a_{i,q} \neq 0$ for some $1 \leq i \leq q/2$ with $(i, q) = 1$. It would be nice if we can remove this condition. The advantage of Corollary 18, is that we can attempt to formulate the "non-Archimedean" analogue.*

We now connect the trace computations to the vanishing of $L(1, f)$ for odd periodic functions f. In Proposition 2, we gave a sufficient criteria for the non-vanishing of $L(1, f)$ when f is odd periodic under some conditions on N. The criteria states that if $L(1, f) = 0$, then for all primes p dividing N, we have

$$\sum_{i=1}^{p-1/2} f(\frac{N}{p}i)\frac{1+\zeta_p^i}{1-\zeta_p^i} \text{ is an algebraic integer .}$$

By Corollary 16 (with $\alpha = (1+\zeta_p)/(1-\zeta_p)$ and $k = 0$) and Appendix A.4.2, the above condition can be re-written as

$$\sum_{i=1}^{p-1/2} f(\frac{N}{p}i)i^{-1} \equiv 0 \bmod p.$$

Question 4. *Can we impose additional conditions on N to get a stronger criteria of vanishing of $L(1, f)$ for odd integer valued periodic functions of period N?*

We provide a partial answer by proving the following theorem :

Theorem 13. Let N be an integer satisfying the following conditions

1. $(N, \varphi(N)) = 1$.
2. There exists a prime $p \equiv 1 \bmod 4$ dividing N satisfying the following condition:
   If $q \mid N$ is a prime and q is a square $\bmod p$.

If $L(1, f) = 0$, for an integer valued odd periodic function f of period N, then the following conditions are satisfied :

$$\sum_{a=1}^{(p-1)/2} f(\frac{N}{p}a)a^{-1} \equiv 0 \bmod p \quad \text{and} \quad \sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right) f(\frac{N}{p}a)a^{-1} \equiv 0 \bmod p. \quad (1.13)$$

Here $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol $\bmod p$.

Let $G = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. The idea of the proof is to consider the trace map $Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_p)}$, apply reduction mod p in $\mathbb{Z}[G]$ to observe that the image of the trace map modulo p is a proper subspace of $\mathbb{F}_p[G]$. For an element $F \in \mathbb{Z}[G]/(\sigma+1)$, we denote the reduction by $\overline{F}$ in the ring $\mathbb{F}_p[G]/(\sigma+1)$. Recall that σ denotes the complex conjugation.

Proof. Let us write $L(1,f) = \frac{\pi i}{N} \sum_{\substack{d|N \\ d\neq 1}} \alpha_d$, where

$$\alpha_{N/d} := \sum_{\substack{i=1 \\ (i,N/d)=1}}^{\lfloor N/2d \rfloor} f(di) \frac{1+\zeta_N^{di}}{1-\zeta_N^{di}} = F_{N/d}\left(\frac{1+\zeta_N^d}{1-\zeta_N^d}\right), \qquad F_{N/d} = \sum_{\substack{i=1 \\ (i,N/d)=1}}^{\lfloor N/2d \rfloor} f(di)\sigma_i \quad (7.21)$$

with $\sigma_i \in Gal(\mathbb{Q}(\zeta_{N/d})/\mathbb{Q})$ given by $\sigma_i(\zeta_{N/d}) = \zeta_{N/d}^i$. Thus the element $F_{N/d} \in \mathbb{Z}[Gal(\mathbb{Q}(\zeta_{N/d})/\mathbb{Q})]$. Since, $L(1,f) = 0$, we have $\sum_{\substack{d|N \\ d\neq 1}} \alpha_d = 0$.

For a divisor d of N not divisible by p, we claim that $Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_p)}\alpha_d = 0$. Indeed this is true as $Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_p)}\alpha_d \in V_d \cap V_p$ and since $p \nmid d$, the intersection is trivial. Therefore, we have

$$Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_p)}\Big(\sum_{d|N} \alpha_d\Big) = \sum_{d|N} Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_p)}\alpha_d = \sum_{p|d|N} Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_p)}\alpha_d$$

However,

$$\begin{aligned} \sum_{p|d|N} Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_p)}\alpha_d &= \sum_{p|d|N} Tr_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_p)} \circ Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_d)}\alpha_d \\ &= \sum_{p|d|N} \frac{\varphi(N)}{\varphi(d)} Tr_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_p)}\alpha_d. \qquad (7.22) \end{aligned}$$

Recall that $\alpha_d = F_d(\frac{1+\zeta_d}{1-\zeta_d})$ with F_d as given in (7.21). Since F_d can be realised as an element of $\mathbb{Z}[Gal(\mathbb{Q}(\zeta_N)/\mathbb{Q})]$ (by linear disjointness), we have,

$$Tr_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_p)} F_d\left(\frac{1+\zeta_d}{1-\zeta_d}\right) = F_d Tr_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_p)}\left(\left(\frac{1+\zeta_d}{1-\zeta_d}\right)\right) = \sigma_d F_d \prod_{\substack{q|d \\ q\neq p}} \left(1 - q\Psi_p^{(q)}\right)\left(\frac{1+\zeta_p}{1-\zeta_p}\right),$$

where q runs over the prime divisors of d/p and σ_d is some element of $Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Moreover, $Tr_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_p)}$ is invariant under the action of σ_q for $q \neq p$, and hence this invariance also holds for $F_d \circ Tr_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_p)}$. Therefore, we have

$$F_d \circ Tr_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_p)} = G_d \circ Tr_{\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_p)},$$

where G_d is the evaluation of F_d at $(\sigma_p, 1, \ldots, 1)$. Applying this result in (7.22), we obtain

$$Tr_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_p)}\big(\sum_{d|N} \alpha_d\big) = \big(\sum_{p|d|N} c_d \sigma_d G_d \circ \prod_{\substack{q|d \\ q\neq p}} (1 - q\Psi_p^{(q)})\big)\big(\frac{1+\zeta_p}{1-\zeta_p}\big), \tag{7.23}$$

where $c_d = \varphi(N)/\varphi(d)$. We arrive at the following situation :

$$F\big(\frac{1+\zeta_p}{1-\zeta_p}\big) = 0 \text{ where } F = \sum_{p|d|N} c_d \sigma_d G_d \circ \prod_{\substack{q|d \\ q\neq p}} (1 - q\Psi_p^{(q)}).$$

Now we reduce the coefficients of F mod p i.e. consider the element $\overline{F}$ in $\mathbb{F}_p[G]/(\sigma_p^{p-1/2} + 1)$. Since $(N, \varphi(N)) = 1$, we note that $c_d \neq 0 \bmod p$. Moreover, since q is a square $\bmod p$, we have

$$\overline{(1 - q\Psi_p^{(q)})} \in \overline{(1 - \tau_g^2 g^2)}, \tag{7.24}$$

where we recall that τ_g is a fixed generator of $G := Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Since p is a prime congruent to 1 mod 4, if g is a generator of $(\mathbb{F}_p)^*$, then so is $-g$. Combining all these observations with the fact $\sum_{d|N} \alpha_d = 0$, we obtain

$$\begin{aligned}
&\overline{\big(\sum_{p|d|N} c_d \sigma_d G_d \circ \prod_{\substack{q|d \\ q\neq p}} (1 - q\Psi_p^{(q)})\big)} \equiv 0 \bmod \mathbb{F}_p[G]/(\tau_g^{p-1/2} + 1) \\
\implies &\big(\sum_{\substack{p|d|N \\ d\neq p}} \overline{c_d \sigma_d G_d \circ \prod_{\substack{q|d \\ q\neq p}} (1 - q\Psi_p^{(q)})}\big) + \overline{c_p F_p} \equiv 0 \bmod \mathbb{F}_p[G]/(\tau_g^{p-1/2} + 1) \\
\implies &\overline{c_p F_p} \equiv -\big(\sum_{\substack{p|d|N \\ d\neq p}} \overline{c_d \sigma_d G_d} \circ \prod_{\substack{q|d \\ q\neq p}} \overline{(1 - q\Psi_p^{(q)})}\big) \bmod \mathbb{F}_p[G]/(\tau_g^{p-1/2} + 1) \\
\implies &\overline{F_p} \in (1 - \tau_g^2 g^2),
\end{aligned} \tag{7.25}$$

with the last equality due to (7.24) and observing that

$$(\tau_g^{p-1/2} + 1) = \prod_{\substack{r=1 \\ r\text{odd}}}^{p-1} (\tau_g - g^r) \text{ in } \mathbb{F}_p[G].$$

Therefore $\overline{F_p} \in \overline{(\tau_g^2 - g^{-2})}$. Writing $F_p = \sum_{a=1}^{p-1/2} f(Na/p)\sigma_a$ (Here $\sigma_a : \zeta_p \mapsto \zeta_p^a$) by Remark A.4.3, we obtain the theorem. □

Corollary 4. *Let N be an integer divisible by 17 satisfying the following conditions :*

1. $(N, \varphi(N)) = 1.$
2. *If $q \mid N$ is a prime, then q is a square* $\bmod 17$.

Then, for all Erdősian functions f such that the arithmetic function $g(a) := f(Na/17)$ is not even , we have $L(1, f) \neq 0$. Hence, we obtain infinitely many natural numbers q satisfying (1.12) *and Conjecture 6 under some additional restrictions.*

Proof. We write $f = f_o + f_e$ where

$$f_o(n) := \frac{f(n) - f(-n)}{2} \qquad f_e(n) := \frac{f(n) + f(-n)}{2}.$$

Clearly f_o is odd and f_e is even function, and to prove the non-vanishing of $L(1, f)$, it suffices to prove $L(1, f_o)$ is not zero (See [18]). By the hypothesis, we are considering all non-even functions f such that $f_o(Na/17) \neq 0$ for some integer a. From SAGE(See Appendix B.2.1 for the program), we verify that all the non-even Erdősian functions of period $\bmod 17$ do not satisfy the criteria mentioned in Theorem 13 (1.13). Therefore, $L(1, f) \neq 0$. To prove the second part, we choose primes $q \equiv 4 \bmod 17$. From Appendix A.3.3, we note that

$$\sum_{\substack{q \text{ prime} \\ q \equiv 4 \bmod 17}} \frac{1}{q} \text{ diverges },$$

and therefore we can find infinitely many N divisible by these primes q and 17 such that

$$\sum_{\substack{d \mid N \\ d \geq 3}} \frac{1}{\varphi(d)} > 1.$$

□

Remark 12. *If the "algebraic part" of $L(1, g_o)$ is not an algebraic integer, we could have proved the above Corollary with Proposition 2. However, using SAGE, we see that there are* 16 *algebraic integers in S_{17} and* 384 *algebraic integers in the extended set*

$$S'_{17} = \{\sum_{i=1}^{8} a_i \frac{1 + \zeta_{17}^i}{1 - \zeta_{17}^i} \;:\; a_i \in \{\pm 1, 0\}\} \setminus \{0\}.$$

The above corollary helps us to show that $L(1, f) \neq 0$ *under these additional assumptions, and this does not seem to be covered by the earlier methods.*

Appendix A

Explicit Formulaes and Remarks

A.1 Special values of Dirichlet series

A.1.1 The ordinary Bernoulli numbers B_n are defined by the following generating series

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n}{n!} t^n, \tag{A.1}$$

and the Bernoulli polynomials $B_n(X)$ are defined as

$$\frac{te^{Xt}}{e^t - 1} = \sum_{n \geq 0} \frac{B_n(X)}{n!} t^n, \tag{A.2}$$

A.1.2 Let χ be a non-principal primitive Dirichlet character of conductor N and $\tau(\chi)$ denote the Gauss sum i.e. $\tau(\chi) = \frac{1}{N} \sum_{a=1}^{N} \chi(a) \zeta_N^{-a}$. We then have :

$$L(1, \chi) = \begin{cases} \pi i \frac{\tau(\chi)}{N} B_{1,\overline{\chi}} & \text{if } \chi(-1) = -1, \\ -\frac{\tau(\chi)}{N} \sum_{a=1}^{N-1} \overline{\chi}(a) \log |1 - \zeta_N^a| & \text{if } \chi(-1) = 1, \end{cases} \tag{A.3}$$

where the generalised Bernoulli number $B_{1,\chi}$ is given by the following expression.

$$B_{1,\chi} = \frac{1}{N} \sum_{a=1}^{N} \chi(a) a.$$

A.1.3 The element $\zeta(k,\frac{a}{q})+(-1)^k\zeta(k,1-\frac{a}{q})$ is an algebraic multiple of π^k. More precisely, For any $1 \le a < \frac{q}{2}$ with $(a,q)=1$, the number

$$\lambda_a = \frac{\zeta(k,\frac{a}{q})+(-1)^k\zeta(k,1-\frac{a}{q})}{(2\pi i)^k} = \frac{1}{(2\pi i)^k}\frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\pi cot(\pi z)|_{z=a/q}$$

is an element of $\mathbb{Q}(\zeta_q)$. The explicit value of λ_a is

$$\lambda_a = \frac{q^{k-1}}{(2k)!}\sum_{b=1}^{q}(\zeta_q^{ab}+(-1)^k\zeta_q^{-ab})B_k(b/q)$$

where $B_k(x)$ is the $k-th$ Bernoulli polynomial. See [30] for further details. This result also works when $k=1$. As a consequence, when f is an odd function of period N, we have

$$L(1,f) = \sum_{n=1}^{\infty}\frac{f(n)}{n} = \frac{\pi i}{N}\sum_{b=1}^{[N/2]}f(b)\frac{1+\zeta_q^b}{1-\zeta_q^b}. \tag{A.4}$$

A.2 Euler Constants - Classical Setup and p-adic setup

We enumerate a few results as mentioned in [29, 40].

A.2.1 Let $q \ge 1$ be an integer and let $1 \le a \le q$ with $d=(a,q)$. Then,

$$\gamma(a,q) = \frac{1}{d}\gamma(\frac{a}{d},\frac{q}{d}) - \frac{1}{q}\log(d).$$

A.2.2 Let f be an arithmetic function of period $q \ge 1$ and M be a natural number co-prime to q. Then

$$\sum_{\substack{n=1\\(n,M)=1}}^{\infty}\frac{f(n)}{n}$$

converges if and only if $\sum_{a=1}^{q} f(a) = 0$. Moreover when the above series converges, the value of the infinite sum is given by $\sum_{a=1}^{q} f(a)\gamma(\Omega,a,q)$, where Ω consists of the prime divisors of M.

Similarly, we have a few results proved by Diamond in the p-adic setup. We recall the definition of p-adic Euler constants as mentioned in Definition 12. Let F be a natural number greater than one and r be a positive integer less than F. We have the following properties.

A.2.3 If $d \mid (r, F)$, then $F\gamma_p(r, F) = \frac{F}{d}\gamma_p(\frac{r}{d}, \frac{F}{d}) - \log_p d$.
A.2.4 $\gamma_p(r, F) = \gamma_p(F - r, F)$.
A.2.5 If $b \in \mathbb{Z}^+$, then $\gamma_p(r, F) = \sum_{n=0}^{b-1} \gamma_p(r + nF, bF)$.
A.2.6 (Gauss Formula):

$$F\gamma_p(r, F) = \gamma_p - \sum_{a=1}^{F-1} \zeta_F^{-ar} \log_p(1 - \zeta_F^a).$$

A.3 Some Results in Analytic Number Theory

We recall some results from [2].

A.3.1 Prime Number Theorem : If $1_{\mathcal{P}} : \mathbb{N} \to \{0, 1\}$ denotes the indicator function of the primes, we have

$$\pi(x) = \sum_{n \leq x} 1_{\mathcal{P}}(n) = \frac{x}{\log x} + O(\frac{x}{\log^2 x})$$

If p_n denoes the n^{th} prime, the above statement is equivalent to proving that $p_n \sim n \log n$.

A.3.2 If $d(n)$ denotes the number of divisors of n, then we have $\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x})$.

A.3.3 With the help of Prime Number Theorem, we mention the following result for $(h, k) = 1$ (See [2, Pg 156, Ex 6])

$$\sum_{\substack{p \leq x \\ p \equiv h \bmod k}} \frac{1}{p} = \frac{1}{\varphi(k)} \log_2 x + A_{h,k} + O(\frac{1}{\log x}).$$

Here $A_{h,k}$ is an explicit constant depending on h and k.

A.4 Remarks about the group ring $\mathbb{F}_p[G]$ where G is a cyclic group

For an odd prime p, let G denote the Galois group $Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. We begin with a few remarks about $\mathbb{F}_p[G]$.

A.4.1 We have the following isomorphism :

$$\mathbb{F}_p[G] \cong \mathbb{F}_p[x]/(x^{p-1}-1) \cong \prod_{i=1}^{p-1} \mathbb{F}_p[x]/(x-g^i),$$

for a generator g of $\mathbb{F}_p$. We obtain the first isomorphism as G is a cyclic group of order $p-1$, and the second isomorphism is obtained as the elements t of $\mathbb{F}_p$ satisfy $t^p = t$.

A.4.2 Since $\mathbb{F}_p[x]$ is an Euclidean domain, we can describe the pullback of the ideal $(x-g^k)$ in $\mathbb{F}_p[G]$ in the following way : If we denote σ_j by the automorphism $\sigma_j(\zeta_p) = \zeta_p^j$, then the pullback of the ideal $(x-g^k)$ in $\mathbb{F}_p[G]$ is given by the collection of elements

$$\{\sum_j a_j\sigma_j \in \mathbb{F}_p[G] \mid \sum_j a_j j^k = 0\}.$$

A.4.3 From the isomorphism, we also conclude every ideal of $\mathbb{F}_p[G]$ is a principal ideal as every proper ideal $\mathcal{I}$ can be expressed as $\prod_{i\in S}(x-g^i)$ for $S \subset \{0,\dots,p-2\}$. Pulling back, we can write

$$\mathcal{I} = \{\sum_j a_j\sigma_j \mid \sum_j a_j j^k = 0 \text{ for all } k \in S\}.$$

Appendix B

Sage Code

B.1 Program for checking algebraic integer in S_p

We recall the definition of S_p as given in Chapter 7, (7.11). We would like to know whether this Set contains an algebraic integer for which we use Corollary 16 with $\alpha = \frac{1+\zeta_p}{1-\zeta_p}$ and $k = 1$. Note that α is not an algebraic integer as $N_{K/\mathbb{Q}}(1-\zeta_p) = p$, and therefore $\alpha \in \mathfrak{p}^{-1} \setminus O_K$. We appeal to the condition mentioned in Appendix I A.4.2 to do the required computation.

B.1.1 Program

```
p=5; # Fixing the prime p
F=Integers(p); #The Field Z/pZ
def dot(A, B): # Defines the dot product of two lists
return sum(i[0] * i[1] for i in zip(A, B))
A=[F(1/i) for i in [1..(p-1)/2]];
# Constructing the elements [1,1/2,...,1/8] (Modulo p)
LIST=[F(1),F(-1)];
# Required for the polynomial with coefficients in LIST
d=(p-1)/2;
# Required for the count in Cartisean product of LIST x LIST X ... X LIST
from itertools import product
P = product(LIST, repeat=d)
```

```
#To construct all possible combinations of length d with coefficients in LIST
count=0; # Set the count to 0
for coeffs in P:
if dot(A,coeffs)==0:
#Taking dot product i.e. computing the conditions mentioned in (8.16)
count=count+1; # If condition is satisfied the count is incremented
count = count-1 # As the Product P contains 0.
print count;
```

B.2 Program for verifying the condition (1.13) for Erdösian functions

B.2.1 Program

We shall verify the condition (1.13) for prime $p = 17$. We shall first construct the tuple $(1/1, 1/2, ..., 1/(p-1/2)) \bmod p$ and take its dot product with the tuple $(f(1), \ldots, f(\frac{p-1}{2}))$ and $(\left(\frac{1}{p}\right) f(1), \ldots, \left(\frac{p-1/2}{p}\right) f(\frac{p-1}{2}))$ with the coefficients of f in $\{\pm 1, 0\}$. Instead of checking $\bmod p$, we work with the coefficients in $\mathbb{F}_p$. We proceed with the program here.

```
p=17; # Fixing the prime p
F=Integers(p); #The Field Z/pZ
def dot(A, B): # Defines the dot product of two lists
   return sum(i[0] * i[1] for i in zip(A, B))
A=[F(1/i) for i in [1..(p-1)/2]];
# Constructing the elements [1,1/2,...,1/8] (Modulo p)
B=[kronecker(i,p)*F(1/i) for i in [1..(p-1)/2]];
# Constructing the elements [1,(2:p)1/2,...,(8:p)1/8] (Modulo 17)
# (a:p) denotes the Legendre Symbol of a mod p
LIST=[F(1),F(-1),F(0)];
# Required for the polynomial with coefficients in LIST
d=(p-1)/2;
# Required for the count in Cartisean product of LIST x LIST X ... X LIST
from itertools import product
```

```
P = product(LIST, repeat=d)
#To construct all possible combinations of length d with coefficients in LIST
count=0; # Set the count to 0
for coeffs in P:
    if dot(A,coeffs)==0 and dot(B,coeffs)==0:
    #Taking dot product i.e. computing the conditions mentioned in (8.16)
        count=count+1; # If condition is satisfied the count is incremented
count = count-1 # As the Product P contains 0.
print count;
```

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