Bounded Negativity and Harbourne Constants on Algebraic Surfaces

By

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DECLARATION

I declare that the thesis entitled **“Bounded Negativity and Harbourne Constants on Algebraic Surfaces"** submitted by me for the degree of **Doctor of Philosophy in Mathematics** is the record of academic work carried out by me under the guidance of Professor Krishna Hanumanthu and this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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CERTIFICATE

I certify that the thesis entitled **"Bounded Negativity and Harbourne Constants on Algebraic Surfaces"** submitted for the degree of **Doctor of Philosophy in Mathematics** by Aditya N K Subramaniam is the record of research work carried out by him under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent his independent work in a very substantial measure.

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Date: April, 2021.

Professor Krishna Hanumanthu
Thesis Supervisor.

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Dedicated to my Parents

Abstract

A smooth projective surface X over an algebraically closed field k is said to have *bounded negativity* if there exists an integer $b(X)$, depending only on X, such that $C^2 \geqslant -b(X)$ for all reduced curves C on X . The *Bounded Negativity Conjecture* (BNC) asserts that every smooth complex projective surface has bounded negativity. This conjecture is false in positive characteristic. While it is easy to prove BNC in some cases, it is open in general in characteristic 0. It is also not known if the existence of such a lower bound for self-intersections of reduced curves is preserved in the birational equivalence class of X.

The notion of *Harbourne constants* was defined in [5] as a way to investigate the occurrence of curves of negative self-intersection on blow ups of X. *Harbourne constants* measure the local negativity of curves on surfaces. In this thesis, we give lower bounds for the *Harbourne constants* of transversal arrangements of curves on a geometrically ruled surface X over a smooth curve. We define a *global* Harbourne constant as the infimum of Harbourne constants for arrangements of a specific type and give a lower bound for it. We also show that the surfaces associated to transversal arrangements on ruled surfaces that we consider in this thesis are not *ball quotients*, i.e., minimal smooth complex projective surfaces of general type satisfying equality in the Bogomolov-Miyaoka-Yau inequality.

Contents

Chapter 1

Introduction

1.1 Preliminaries

1.1.1 Basics

Definition 1.1. A *variety* is an integral, separated scheme of finite type over an algebraically closed field k. A *projective variety* is a variety which has a closed embedding into $\mathbb{P}^N$ for some positive integer N. A *curve* is a projective variety of dimension one and a *surface* is a projective variety of dimension two.

We say that a variety X is *smooth* if the local ring $\mathcal{O}_{X,x}$ is a regular local ring for all x in X. Smooth projective varieties over k are studied by means of divisors on them. We write $D = \sum n_i D_i$ for a divisor D on a smooth projective variety X, where D_i are projective codimension one subvarieties on X and n_i are integers such that only finitely many n_i are different from zero. If $n_i \geqslant 0$ for all i, then we say that D is an *effective divisor*.

Let X be a smooth projective variety, and denote by $\mathcal{K}_X = k(X)$ the constant sheaf of rational functions on X. We denote by $\mathcal{K}_X^*$ the sheaf (of multiplicative groups) of invertible elements in $\mathcal{K}_X$. The sheaf $\mathcal{K}_X$ contains the structure sheaf $\mathcal{O}_X$ as a subsheaf and so there is an inclusion $\mathcal{O}_X^* \subset \mathcal{K}_X^*$ of sheaves, where $\mathcal{O}_X^*$ is the sheaf of invertible elements in $\mathcal{O}_X$. We thus have the following exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^*/\mathcal{O}_X^* \longrightarrow 0, \tag{1.1}$$

where $\mathcal{K}_X^*/\mathcal{O}_X^*$ is the quotient sheaf on X. The global sections of $\mathcal{K}_X^*/\mathcal{O}_X^*$ are known as *Cartier divisors* on X. We denote by $\mathrm{Div}(X)$ the group of all global sections of $\mathcal{K}_X^*/\mathcal{O}_X^*$.

The short exact sequence in (1.1) gives rise to a connecting homomorphism in the long exact sequence of cohomology:

$$\mathrm{Div}(X) \longrightarrow H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X) \tag{1.2}$$

given by,

$$D \mapsto \mathcal{O}_X(D). \tag{1.3}$$

Here $\mathrm{Pic}(X)$ denotes the group of isomorphism classes of line bundles on X. The kernel of this map is precisely the set of all *principal divisors* of X. The group $\mathrm{Div}(X)$ modulo principal divisors is known as *divisor class group*, denoted as $\mathrm{Cl}(X)$. It is known that if X is integral, the induced map $\mathrm{Cl}(X) \to \mathrm{Pic}(X)$ is an isomorphism, see [30, Chapter II, Proposition 6.15]. Therefore from here onwards we will not differentiate between a divisor and the corresponding line bundle.

A distinguished divisor in a smooth projective variety X is the canonical divisor.

Definition 1.2 (Canonical Divisors)**.** Let X be a smooth projective variety of dimension n and let $\delta\colon X \to X \times X$ be the diagonal morphism. Let $\Delta := \delta(X)$ be the diagonal and $\mathcal{I}$ the ideal sheaf of Δ in $X \times X$. Then, the *sheaf of relative differentials* of X is defined to be:

$$\Omega_X := \delta^\star(\mathcal{I}/\mathcal{I}^2).$$

Its dual

$$\mathcal{T}_X := \Omega_X^\vee = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$$

is called the *tangent sheaf* of X and the *canonical sheaf* of X is defined to be its top exterior power

$$\omega_X = \bigwedge^n \Omega_X.$$

Note that ω_X is an invertible sheaf on X. A *canonical divisor* on X is any Cartier divisor K_X which corresponds to ω_X.

Given a divisor D on a smooth projective variety X, we associate a *linear system*, denoted by $|D|$, which consists of all effective divisors on X linearly equivalent to D. It has a structure of a projective variety, as it can be viewed as the projective space corresponding to the vector space of global sections of $\mathcal{O}_X(D)$ [30].

In this thesis, the main objects of study are smooth projective surfaces over an algebraically closed field k. From now on, we only look at smooth projective surfaces.

Intersection theory: An *intersection theory* should make it possible to calculate intersections of subvarieties, counted with "multiplicities". We give here an overview of the basic terminology for this area of study for surfaces. For a detailed introduction, see [21]. We will follow the axiomatic approach of [30, Appendix A], unless otherwise stated, and assume k to be an algebraically closed field.

Definition 1.3. Let X be a smooth projective surface over k. For $0 \leqslant r \leqslant 2$, let $Z^r(X)$ be the free abelian group generated by all closed subvarieties $Y \subseteq X$ of codimension r and define $Z(X)$ as $Z(X) := \oplus_{r=0}^{2} Z^r(X)$. An element of $Z(X)$ is called a *cycle*. A cycle is *positive* if each of its coefficients is a positive integer.

To be able to count intersections with multiplicities, we need to be able to "move" varieties around without changing the result of their intersection. The correct notion for this is *rational equivalence*.

Definition 1.4. If M is an A-module, we denote by $\text{length}_A(M)$ the *length* of M over A. It is the supremum of all length r chains $0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r = M$ of submodules $M_i \subseteq M$. We write $\text{length}(A)$ to denote the length of A as an A-module.

Definition 1.5. Let X be a smooth projective surface over k. If $Y \subseteq X$ is a closed subvariety and $f \in k(Y)$, we set

$$\mathrm{div}(f) := \sum_{\mathrm{codim}_Y(Z)=1} \mathrm{ord}_Z(f) \cdot Z.$$

Recall that the order of an element $f \in \mathcal{O}_{Y,Z}$ is defined to be

$$\mathrm{ord}_Z(f) := \text{length}_{\mathcal{O}_{Y,Z}}(\mathcal{O}_{Y,Z}/(f)).$$

We then extend this definition to the function field $k(Y) = \mathrm{Frac}(\mathcal{O}_{Y,Z})$ by defining $\mathrm{ord}(f/g) := \mathrm{ord}(f) - \mathrm{ord}(g)$.

A cycle of the form $\mathrm{div}(f)$ is called *rational*. The free abelian subgroup of $Z^r(X)$, generated by all rational cycles, is denoted $\mathrm{Rat}^r(X)$. For $W, V \in Z^r(X)$, we write $W \sim V$ if $W - V \in \mathrm{Rat}^r(X)$. We say that V and W are *rationally equivalent* in this case. The *Chow ring* of X is the graded ring $A(X) = \oplus_{r=0}^{2} A^r(X)$, where $A^r(X)$ is the factor group

$$A^r(X) := Z^r(X)/\mathrm{Rat}^r(X).$$

The elements of $A(X)$ are called *cycle classes*. A cycle class is *positive* if it can be represented by a positive cycle. We write $[Y]$ for the equivalence class of Y.

A cycle class can now be "moved" along rational cycles. Note that this is a generalization of the linear equivalence between the divisors $\mathrm{Div}(X) = Z^1(X)$. Hence, $A^1(X) = \mathrm{Pic}(X)$.

Theorem 1.1 in [30, Appendix A] assures the existence of a unique intersection theory that satisfies certain given properties. These properties imply that, given a smooth projective surface X, there exists a pairing

$$\begin{aligned} A^r(X) \times A^s(X) &\to A^{r+s}(X) \\ ([Y],[Z]) &\mapsto [Y]\cdot[Z] \end{aligned}$$

for all r and s, that turns $A(X)$ into a commutative graded ring with identity.

Definition 1.6. Let $\phi : X \to X'$ be a morphism of smooth projective surfaces and $Y \subseteq X$ a closed subvariety. If $\dim(\phi(Y)) < \dim(Y)$, we set $\phi_\star([Y]) := 0$. Otherwise, $k(Y)$ is a finite extension of $k(Y')$, where $Y' = \overline{\phi(Y)}$. We then set

$$\phi_\star([Y]) := [k(Y) : k(Y')] \cdot [Y'].$$

On the other hand, if $Y' \subseteq X'$ is any closed subvariety, denote by $\Gamma(\phi) \subseteq X \times X'$ the graph of ϕ and set

$$\phi^\star([Y']) := p_\star([\Gamma(\phi)] \cdot [q^{-1}(Y')]).$$

Here p and q are the projections from $X \times X'$ to X and X' respectively.

Proposition 1.7. *(See [30, A2]). For any morphism $\phi : X \to X'$ of smooth projective surfaces, $\phi^\star\colon A(X') \to A(X)$ is a ring homomorphism.*

Proposition 1.8. *(See [30, A3]). For any proper morphism $\phi : X \to X'$ of smooth projective surfaces, $\phi_\star\colon A(X) \to A(X')$ is a homomorphism of graded groups (which shifts degrees).*

The following theorem computes, in the more general language of divisors, the intersection product of two divisors on a smooth projective surface.

Theorem 1.9. *Let X be a smooth projective surface. There is a unique pairing*

$$\begin{aligned} \mathrm{Div}(X) \times \mathrm{Div}(X) &\longrightarrow \mathbb{Z} \\ (C,D) &\mapsto C\cdot D \end{aligned}$$

for any two divisors C and D on X such that

1. *if C and D are nonsingular curves meeting transversally, then $C \cdot D = \#(C \cap D)$, the number of intersection points of C with D,*
2. *it is symmetric: $C \cdot D = D \cdot C$,*
3. *it is additive: $(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$, and*
4. *it depends only on the linear equivalence classes: if $C_1 \sim C_2$ then $C_1 \cdot D = C_2 \cdot D$.*

Proof. We refer to [30, Section V.1, Theorem 1.1] for a proof. □

Two divisors D_1 and D_2 on a smooth projective surface X are said to be *numerically equivalent*, denoted $D_1 \equiv D_2$, if $D_1 \cdot C = D_2 \cdot C$ for every irreducible curve C in X. The *Néron Severi group* of X is the quotient $N^1(X)_{\mathbb{Z}} := \mathrm{Div}(X)/\equiv$. It is a basic fact that the Néron Severi group $N^1(X)_{\mathbb{Z}}$ is a free abelian group of finite rank [43, Proposition 1.1.16]. The rank of $N^1(X)_{\mathbb{Z}}$ is called the *Picard number* of X, and is denoted by $\rho(X)$.

In algebraic geometry, one of the most important problems is to classify objects considering a certain fixed set of invariants. Let X be a smooth complex projective surface. For a divisor D on X, we denote by $h^i(X, \mathcal{O}_X(D))$ the dimension of $H^i(X, \mathcal{O}_X(D))$ as a k-vector space. The following are some birational numerical invariants of X:

1. The geometric genus $p_g(X) := h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X(K_X))$.
2. The irregularity $q(X) := h^1(X, \mathcal{O}_X)$.
3. The m^{th} plurigenus $P_m(X) := h^0(X, \mathcal{O}_X(mK_X))$, $m > 0$.

Chern Classes: Chern classes can be defined for any smooth projective variety X over an algebraically closed field k. However, we will restrict to the case of surfaces and introduce Chern numbers of a smooth projective surface. We will follow the approach of [30, Appendix A], unless otherwise stated, and assume k to be an algebraically closed field.

Proposition 1.10. *Let $\mathcal{E}$ be a locally free sheaf of rank r on a smooth projective surface X. Let $\mathbb{P}(\mathcal{E})$ be the associated projective bundle (for the definition, see [30, Section II.7]). Let $h \in A^1(\mathbb{P}(\mathcal{E}))$ be the class of the divisor corresponding to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Let $\pi : \mathbb{P}(\mathcal{E}) \to X$*

be the projection. Then π^* makes $A(\mathbb{P}(\mathcal{E}))$ into a free $A(X)$-module generated by h^k for $0 \leqslant k \leqslant r-1$.

Definition 1.11. Let $\mathcal{E}$ be a locally free sheaf of rank r on a smooth projective surface X. Using the notation and statement of Proposition 1.10, for each $i = 0, \cdots, r$, we define the i-th *Chern class* $c_i(\mathcal{E}) \in A^i(X)$ as elements satisying the following : $c_0(\mathcal{E}) = 1$, and

$$\sum_{i=0}^{r} (-1)^i \pi^*(c_i(\mathcal{E})).h^{r-i} = 0$$

in $A^r(\mathbb{P}(\mathcal{E}))$. The *total Chern class* is the sum $c(\mathcal{E}) := \sum_{i=0}^{r} c_i(\mathcal{E})$. For a formal variable T, we define the *Chern polynomial*

$$c_T(\mathcal{E}) := \sum_{i=0}^{r} c_i(\mathcal{E}) \cdot T^i.$$

While this definition is formal, it can be shown that the Chern classes of a surface are subject to several useful properties:

C1. If $\mathcal{E}$ is a line bundle corresponding to a divisor class $[D] \in A^1(X)$, then $c_T(\mathcal{E}) = 1 + [D] \cdot T$. Indeed, in this case, $\mathbb{P}(\mathcal{E}) = X$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E}$, so $h = [D]$ in Proposition 1.10. Hence, by definition, $c_0(\mathcal{E}) \cdot [D] - c_1(\mathcal{E}) = 0$.

C2. If $\phi : X' \to X$ is a morphism and $\mathcal{E}$ is a locally free sheaf on X, then $c_i(\phi^*\mathcal{E}) = \phi^*(c_i(\mathcal{E}))$ for each i.

C3. If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is an exact sequence of locally free sheaves on X, then $c_T(\mathcal{E}) = c_T(\mathcal{E}') \cdot c_T(\mathcal{E}'')$.

One can show that these properties uniquely define a theory of Chern classes, which assigns to each locally free sheaf $\mathcal{E}$ on a smooth projective surface X, an element $c_i(\mathcal{E}) \in A^i(X)$ satisfying properties C1 to C3. For the proof of this, one requires the following:

Theorem 1.12 (Splitting Principle)**.** *Let $\mathcal{E}'$ be a locally free sheaf of rank r on a smooth projective surface X'. Then there exists a surface X and a morphism $\phi : X \to X'$ such that $\phi^* : A(X') \to A(X)$ is injective and $\mathcal{E} := \phi^*(\mathcal{E}')$ splits, i.e., has a filtration*

$$\mathcal{E} = \mathcal{E}_0 \supseteq \mathcal{E}_1 \supseteq \cdots \supseteq \mathcal{E}_r = 0$$

whose successive quotients $\mathcal{L}_i := \mathcal{E}_{i-1}/\mathcal{E}_i$ *are invertible sheaves.*

Then, one deduces the following property C4 from property C3. The uniqueness is then a result of property C1.

C4. If $\mathcal{E}$ splits and the filtration has the invertible sheaves $\mathcal{L}_1, \cdots, \mathcal{L}_r$ as quotients, then $c_T(\mathcal{E}) = \prod_{i=1}^{r} c_T(\mathcal{L}_i)$.

C5. Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves of rank r and s respectively on a smooth projective surface X. Using the splitting principle, we can write:

$$c_T(\mathcal{E}) = \prod_{i=1}^{r}(1 + a_i T), \qquad c_T(\mathcal{F}) = \prod_{j=1}^{s}(1 + b_j T),$$

where the a_k and b_k are just formal symbols. Then,

$$\begin{aligned} c_T(\mathcal{E}^{\vee}) &= \prod_{i=1}^{r}(1 - a_i T), \\ c_T(\bigwedge^{p} \mathcal{E}) &= \prod_{1 \leqslant i_1 < \cdots < i_p \leqslant r} (1 + (a_{i_1} + \cdots + a_{i_p})T), \\ c_T(\mathcal{E} \otimes \mathcal{F}) &= \prod_{i,j}(1 + (a_i + b_j)T). \end{aligned}$$

In the context of Hirzebruch-Riemann-Roch Theorem, the formal calculus of Chern classes is extended by the notions of exponential Chern character and Todd class:

Definition 1.13. Let $\mathcal{E}$ be a locally free sheaf of rank r on a smooth projective surface X over k and let $c_T(\mathcal{E}) = \prod_{i=1}^{r}(1 + a_i T)$ with formal variables a_i. We define the *exponential Chern character* as

$$\mathrm{ch}(\mathcal{E}) := \sum_{i=1}^{r} \exp(a_i),$$

where we formally set $\exp(a) := \sum_{k=0}^{\infty} \frac{a^k}{k!}$. Furthermore, the *Todd class* of $\mathcal{E}$ is the formal expression

$$\mathrm{td}(\mathcal{E}) := \prod_{i=1}^{r} \frac{a_i}{1 - \exp(-a_i)}.$$

Our interest is to be able to effectively compute the Chern classes of certain locally free sheaves on surfaces. The following lemma will be used to make explicit calculations:

Lemma 1.14. *(See [30, Section 4, Appendix A]). Let $\mathcal{E}$ be a locally free sheaf of rank r on a smooth projective surface X. Let $c_i = c_i(\mathcal{E})$ for $0 \leqslant i \leqslant r$ and $c_i = 0$ if $i > r$. Then:*

$$\mathrm{ch}(\mathcal{E}) = r + c_1 + \frac{(c_1^2 - 2c_2)}{2} + \frac{(c_1^3 - 3c_1c_2)}{6} + \frac{(c_1^4 - 4c_1^2c_2 + 2c_2^2)}{24} + \cdots$$
$$\mathrm{td}(\mathcal{E}) = 1 + \frac{c_1}{2} + \frac{(c_1^2 + c_2)}{12} + \frac{c_1c_2}{24} - \frac{(c_1^4 - 4c_1^2c_2 - 3c_2^2)}{720} + \cdots$$

We recall the following:

Definition 1.15. If $\mathcal{E}$ is a sheaf of $\mathcal{O}_X$-modules on a smooth projective surface X, then

$$\chi(\mathcal{E}) := \sum_{k \in \mathbb{Z}} (-1)^k.\mathrm{rank}(H^k(X, \mathcal{E}))$$

is defined as the *Euler characteristic* of $\mathcal{E}$.

We now mention a famous result proved by Hirzebruch over $\mathbb{C}$, and later generalized to any algebraically closed field k by Borel and Serre.

Theorem 1.16. *(The Hirzebruch-Riemann-Roch Theorem [10]). For a locally free sheaf $\mathcal{E}$ of rank r on a smooth projective surface X,*

$$\chi(\mathcal{E}) = \deg(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(\mathcal{T}_X))_2,$$

where $()_2$ denotes the component of degree 2 in $A(X) \otimes \mathbb{Q}$.

Definition 1.17. Let X be a smooth projective surface over an algebraically closed field k. Let c_1 and c_2 be the Chern classes of the tangent sheaf $\mathcal{T}_X$. As these numbers depend only on X, we can call them the *Chern classes* of X, and we will write $c_1(X) = c_1$, $c_2(X) = c_2$.

Definition 1.18 (Topological Euler characteristic)**.** Let X be a smooth complex projective surface. We define the i-th *Betti number* of X as

$$b_i(X) := \dim_{\mathbb{R}} H^i(X, \mathbb{R}).$$

The *topological Euler characteristic* of X is defined as

$$e(X) := \sum_i (-1)^i b_i(X).$$

We note that because of the Poincaré duality, the Betti numbers satisfy $b_i(X) = b_{4-i}(X)$. Hence, the topological Euler characteristic of a smooth complex projective surface X can be written as $e(X) = 2 - 2b_1(X) + b_2(X)$. For more details, see [8].

Let us now verify that, in the case of smooth complex projective surfaces, the topological Euler characteristic and self-intersection number of a canonical divisor correspond to c_2 and c_1^2 respectively.

Proposition 1.19. *Let X be a smooth projective surface. Then* $c_1(X) = c_1(\mathcal{T}_X) = -[K_X]$.

Proof. Let $c_T(\Omega_X) = \prod_{i=1}^2 (1 + a_i T)$ for formal variables a_i. By property C5, $c_T(\omega_X) = c_T(\bigwedge^2 \Omega_X) = 1 + (a_1 + a_2)T$. Together with property C1, this means $[K_X] = c_1(\omega_X) = c_1(\Omega_X)$. Again using property C5, we calculate $c_1(\mathcal{T}_X) = c_1(\Omega_X^\vee) = -c_1(\Omega_X) = -[K_X]$. □

Proposition 1.20 (Noether's formula)**.** *Let X be a smooth projective surface over an algebraically closed field k. Let K_X be a canonical divisor on X. Then*

$$\chi(\mathcal{O}_X) = \frac{K_X^2 + c_2(X)}{12}.$$

Proof. Let $\mathcal{E} = \mathcal{O}_X(D)$ be an invertible sheaf. By Lemma 1.14, we have $ch(\mathcal{E}) = 1 + D + \frac{D^2}{2}$. Also, by Proposition 1.19, we have that $c_1(\mathcal{T}_X) = -[K_X]$. Hence,

$$td(\mathcal{T}_X) = 1 - \frac{K_X}{2} + \frac{K_X^2 + c_2(X)}{12}.$$

We multiply this by $ch(\mathcal{E})$ and then take the component of degree 2. We let D^2 denote both the class in $A^2(X)$, and its degree. Thus, by Theorem 1.16, we get:

$$\chi(\mathcal{O}_X(D)) = \frac{D \cdot (D - K_X)}{2} + \frac{K_X^2 + c_2(X)}{12}.$$

In particular, for $D = 0$, we have the desired formula. □

Proposition 1.21. *Let X be a smooth projective surface over $\mathbb{C}$. Then* $c_2(X) = e(X)$.

Proof. We know that Ω_X is a locally free sheaf of rank 2. Let $c_i = c_i(\Omega_X)$. Then, by Lemma 1.14, we have that:

$$\mathrm{ch}(\Omega_X) = 2 + c_1 + \frac{c_1^2 - 2c_2}{2}.$$

By property C5, we have that $c_1(\Omega_X) = c_1(\bigwedge^2 \Omega_X) = K_X$. Using that $\mathcal{T}_X$ is the dual of Ω_X, property C5 gives $c_2(\Omega_X) = c_2(X)$. From Proposition 1.19, we know that $c_1(\mathcal{T}_X) = -[K_X]$. Using this and Lemma 1.14, we can write:

$$\mathrm{td}(\mathcal{T}_X) = 1 - \frac{K_X}{2} + \frac{K_X^2 + c_2(X)}{12}.$$

Using Noether's formula, we get:

$$\begin{aligned}(\mathrm{ch}(\Omega_X) \cdot \mathrm{td}(\mathcal{T}_X))_2 &= \frac{(c_1^2(X) - 2c_2(X))}{2} - \frac{K_X^2}{2} + \frac{(K_X^2 + c_2(X))}{6} \\ &= 2\chi(\mathcal{O}_X) - c_2(X).\end{aligned}$$

So, using Theorem 1.16, we get that $\chi(\Omega_X) = 2\chi(\mathcal{O}_X) - c_2(X)$. On the other hand, $\chi(\Omega_X) = 2q(X) - h^1(\Omega_X)$ and $\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X)$. Hodge theory (see [59]) says that the Betti numbers can be written as: $b_1(X) = 2q(X)$ and $b_2(X) = h^1(\Omega_X) + 2p_g(X)$. With all this, we get $c_2(X) = 2 - 2b_1(X) + b_2(X) = e(X)$. □

Definition 1.22 (Kodaira dimension)**.** Let X be a smooth complex projective surface and let ϕ_{mK_X} be the rational map from X to the projective space associated with the linear system $|mK_X|$. The *Kodaira dimension* of X, denoted $\kappa(X)$, is defined as the maximum dimension of the images of ϕ_{mK_X} for $m > 0$, or $-\infty$ if $|mK_X| = \emptyset$ for all $m > 0$.

The invariant $P_m(X)$ gives the following classification of surfaces (written in terms of the Kodaira dimension). This classification is known as the *Enriques classification*. Any smooth complex projective surface X falls in one of the classes below:

1. Kodaira dimension $-\infty$, i.e., $P_m(X) = 0$ for all m:
   - X is a ruled surface.
2. Kodaira dimension 0, i.e., $P_m(X)$ is either 0 or 1 for all m:
   - X is an abelian surface (projective quotient of $\mathbb{C}^2$ by a maximal rank lattice), or

- X is a K3 surface, or
- X is an Enriques surface, or
- X is a hyperelliptic surface.

3. Kodaira dimension 1, i.e., $P_m(X)$ grows linearly for $m \gg 0$:
   - X has an elliptic fibration.
4. Kodaira dimension 2, i.e., $P_m(X)$ grows quadratically for $m \gg 0$:
   - X is called a *surface of general type*.

In this thesis, we are mainly interested in ruled surfaces [Subsection 1.1.10]. These are surfaces of Kodaira dimension $-\infty$.

For a smooth complex projective surface X, one studies the pair $(K_X, e(X))$. The problem of establishing which pairs of integers (m, n) may appear as $m = K_X^2$ and $n = e(X)$ for a surface X of general type is known as the *geography problem*, and it is not completely solved yet.

As observed in the Enriques classification of surfaces, if

$$h^0(X, \mathcal{O}_X(mK_X)) \sim c \cdot m^2$$

for a positive constant c, then X is said to be a surface of general type.

A very important constraint on the geography of surfaces of general type is given by the Bogomolov-Miyaoka-Yau inequality [46]:

Theorem 1.23. *Let X be a smooth complex projective surface of general type. Then*

$$K_X^2 \leqslant 3e(X). \tag{1.4}$$

Remark 1.24. The inequality (1.4) is, in fact, true under the milder assumption that the canonical divisor has some sections asymptotically, i.e.,

$$h^0(X, \mathcal{O}_X(mK_X)) > 0 \tag{1.5}$$

for m sufficiently large.

It is natural to ask when equality holds in (1.4). It turns out that there is a topological answer to this question, see [Theorem 1.28]. We now recall a few definitions.

Definition 1.25. Let X be a topological space. A *covering space* of X is a topological space Y together with a continuous surjective map $p : Y \to X$ such that for every $x \in X$, there exists an open neighborhood U of x, such that $p^{-1}(U)$ is a union of disjoint open sets in Y, each of which is mapped homeomorphically onto U by p. The map p is called the *covering map*.

Definition 1.26. A covering $p : Y \to X$ of a topological space X is *universal* if Y is simply connected.

Definition 1.27 (Ball Quotients)**.** We say that a minimal smooth complex projective surface X of general type is a *ball quotient* if its universal cover is the 2-dimensional complex unit ball $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$.

The following is a fundamental result [45]:

Theorem 1.28. *Let X be a minimal smooth complex projective surface of general type. Then equality holds in (1.4) if and only if X is a ball quotient.*

In [35], Hirzebruch was interested in constructing ball quotients by starting with line arrangements on $\mathbb{P}^2$. We study ball quotients in Section 2.5.

There are different notions of positivity of a line bundle on a projective variety.

Definition 1.29. A line bundle L on a smooth complex projective variety X is said to be *very ample* if $L = \phi^*(\mathcal{O}_{\mathbb{P}^N}(1))$ for some closed embedding $\phi : X \hookrightarrow \mathbb{P}^N$ for some positive integer N. A line bundle L is called *ample* if an integral multiple $L^{\otimes m}$ of L is very ample, for some $m > 0$. A divisor D on X is said to be *ample* (resp. *very ample*) if the corresponding line bundle $\mathcal{O}_X(D)$ is *ample* (resp. *very ample*).

Definition 1.30. A line bundle L on a smooth complex projective variety X is said to be *numerically effective* or *nef*, if $L \cdot C \geqslant 0$ for every irreducible curve $C \subseteq X$. A Cartier divisor D on X is called *nef* if the corresponding line bundle $\mathcal{O}_X(D)$ is nef.

We now state some basic results, without proof, about smooth projective surfaces which we will use extensively in this thesis.

Proposition 1.31 (Adjunction formula [30])**.** *Let* C *be a reduced and irreducible curve on a smooth projective surface* X *and let* K_X *be the canonical divisor on* X*. Then there is a non-negative integer* $p_a(C)$*, called the arithmetic genus of* C*, such that*

$$C^2 + C \cdot K_X = 2p_a(C) - 2.$$

We next recall the Hodge Index theorem (see [30, Chapter V, Theorem 1.9]).

Theorem 1.32 (Hodge Index Theorem)**.** *Let* H *be an ample divisor on a smooth projective surface* X *and suppose that* D *is a divisor on* X *such that* $D \not\equiv 0$ *and* $D \cdot H = 0$*. Then* $D^2 < 0$*.*

In fact, the Hodge index theorem states that given a smooth projective surface X and a divisor $D \in N^1(X)_{\mathbb{Z}}$ with $D^2 > 0$, the intersection form on the space $D^{\perp} \subseteq N^1(X)_{\mathbb{Z}}$ of classes F with $F \cdot D = 0$ is negative definite.

In this thesis, we are mostly interested in curves with negative self-intersection on surfaces.

Definition 1.33 (Negative curve)**.** We say that a reduced and irreducible curve C on a smooth projective surface is *negative*, if its self-intersection number C^2 is less than zero.

Example 1.34 (Exceptional divisor, (-1)–curves)**.** Let X be a smooth projective surface and let $x \in X$ be a closed point. Let $\pi : \mathrm{Bl}_x X \to X$ be the blow up of X at the point x. Then the exceptional divisor E of π (i.e., the set of points in $\mathrm{Bl}_x X$ mapped by π to x) is a negative curve. More precisely by [30, Chapter V, Proposition 3.1], E is smooth rational and $E^2 = -1$.

In fact, Castelnuovo's result [30, Chapter V, Theorem 5.7] shows that the converse is also true.

Definition 1.35. A (-1)–*curve* on a smooth projective surface X is a smooth rational reduced and irreducible curve C such that $C^2 = -1$.

Theorem 1.36 (Castelnuovo's Contraction)**.** *Let* X *be a smooth projective surface defined over* $\mathbb{C}$*. If* C *is a* (-1)*–curve, then there exists a smooth projective surface* Y *and a projective morphism* $p : X \to Y$ *contracting* C *to a smooth point on* Y*. In other words,* X *is isomorphic to the blow up* $\mathrm{Bl}_y Y$ *for some point* $y \in Y$*.*

Thus, if $\pi : \mathrm{Bl}_x\mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ is a blow up of $\mathbb{P}^2_{\mathbb{C}}$ at a point $x \in \mathbb{P}^2_{\mathbb{C}}$ with exceptional divisor E_x, the self-intersection of the exceptional divisor E_x^2 is -1. In fact, E_x is the only reduced and irreducible curve on $\mathrm{Bl}_x\mathbb{P}^2_{\mathbb{C}}$ with negative self-intersection as $\mathrm{Pic}(\mathrm{Bl}_x\mathbb{P}^2_{\mathbb{C}})$ has a basis given by the pullback of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ and the exceptional divisor. In particular, this shows that $C^2 \geqslant -1$ for any reduced curve C on $\mathrm{Bl}_x\mathbb{P}^2_{\mathbb{C}}$. The Bounded Negativity conjecture [Conjecture 1.45] attempts at finding such lower bounds for self-intersection of reduced curves on an arbitrary nonsingular complex projective surface.

There are other situations in which negative curves on algebraic surfaces appear.

Example 1.37. ([30, Ex. V.1.6]) Let C be a smooth curve of genus $g(C) \geqslant 2$. Then the diagonal $\Delta \subset C \times C$ is a negative curve as its self-intersection is $\Delta^2 = 2 - 2g(C)$.

Remark 1.38. It is well-known that a blow up of $\mathbb{P}^2$ in 9 general points contains infinitely many (-1)–curves.

In fact, in [6, Theorem 4.1], it is shown that for every integer $m > 0$, there exist smooth projective complex surfaces containing infinitely many smooth irreducible curves of self-intersection $-m$. Thus one can produce surfaces with infinitely many negative curves of any given (fixed) negative self-intersection.

1.1.2 Bounded Negativity

Let X be a nonsingular projective surface over an algebraically closed field k. Given a reduced curve $C \subset X$, we wish to understand how negative the self-intersection C^2 can be. For example, consider the blow up X_r of $\mathbb{P}^2_{\mathbb{C}}$ at r distinct points on a line l. It is easy to see that $C^2 = 1 - r$, where C is the strict transform of l. Hence, C^2 can be made arbitrarily negative if we vary the surface X. The *Bounded Negativity Conjecture* is concerned with self-intersection C^2 of reduced curves C on a *fixed* surface X.

Definition 1.39 (Bounded Negativity)**.** Let X be a nonsingular projective surface over an algebraically closed field k. We say X has *bounded negativity* if there exists an integer $b(X)$, depending only on X, such that $C^2 \geqslant -b(X)$ for all reduced curves C on X.

To verify bounded negativity, it suffices to show that the self-intersection of reduced and irreducible curves is bounded below, by [6, Proposition 5.1]. We include the proof here [Proposition 1.41] for the convenience of the reader. The proof uses Zariski decomposition of effective divisors on any smooth projective surface.

The existence of Zariski decompositions was proved by Zariski [62] for effective divisors on any smooth projective surface X. He established the following result:

Theorem 1.40. *Let D be an effective $\mathbb{Q}$-divisor on a smooth projective surface X. Then there are uniquely determined effective (possibly zero) $\mathbb{Q}$-divisors P and N with*

$$D = P + N$$

such that

1. *P is nef,*
2. *N is zero or $N = \sum_{i=1}^{r} a_i C_i$, where C_i are irreducible, reduced curves, $a_i > 0$ and the intersection matrix $(C_i \cdot C_j)_{1 \leqslant i,j \leqslant r}$ is negative definite,*
3. *P is orthogonal to each of the components of N, i.e., $P \cdot C_i = 0$, for $i = 1, \cdots, r$.*

The decomposition $D = P + N$ is called the *Zariski decomposition* of D.

Proposition 1.41. *([6, Proposition 5.1]) Let X be a smooth projective surface (over an arbitrary algebraically closed ground field) for which there is a positive constant $b(X)$ such that $C^2 \geqslant -b(X)$ for every reduced, irreducible curve $C \subset X$. Then*

$$C^2 \geqslant -(\rho(X) - 1) \cdot b(X)$$

for every reduced curve $C \subset X$, where $\rho(X)$ is the Picard number of X.

Proof. Let C be a reduced curve in X. Let $C = P + N$ be the Zariski decomposition (see Theorem 1.40), so P is nef. If $N = 0$, then $C^2 \geqslant 0$. Suppose $N \neq 0$. Then $N = a_1 C_1 + \cdots + a_r C_r$, where the C_i are negative curves, a_i are positive rational numbers and $a_i \leqslant 1$ for all i (because C is reduced). Also, since the intersection matrix of N is negative definite, we have by Hodge Index Theorem [Theorem 1.32] that $r \leqslant \rho(X) - 1$.

Since P is nef and P and N are orthogonal, we have

$$C^2 = P^2 + 2P \cdot N + N^2 \geqslant N^2 \geqslant a_1^2 C_1^2 + \cdots + a_r^2 C_r^2 \geqslant -r \cdot b(X) \geqslant -(\rho(X) - 1) \cdot b(X),$$

as claimed. $\square$

We now give some examples of surfaces which are known to have bounded negativity.

Example 1.42. A smooth surface X has bounded negativity if $-mK_X$ is effective for some positive integer m, where $-K_X$ denotes the anti-canonical divisor on X. Indeed, since $-mK_X$ is effective, there are only finitely many reduced, irreducible curves C such that $-mK_X \cdot C < 0$. So, apart from these finitely many reduced, irreducible curves, we have $-mK_X \cdot C \geqslant 0$, in which case the adjunction formula [Proposition 1.31] gives $C^2 = 2p_a(C) - 2 - C \cdot K_X \geqslant -2$. Thus, in particular, bounded negativity holds for $K3$ surfaces, Enriques surfaces and abelian surfaces. We also remark that the same argument works for smooth surfaces X with $-K_X$ nef.

Example 1.43. Let $\pi : X \to \mathbb{P}^2_{\mathbb{C}}$ be a blow up of $\mathbb{P}^2_{\mathbb{C}}$ at atmost nine distinct points. Then X has bounded negativity. Indeed, since $-K_X$ is effective, bounded negativity follows by the arguments in Example 1.42.

There are surfaces without bounded negativity in positive characteristic.

Example 1.44. (Kollár) Let p be a prime number and C be a smooth projective curve of genus $g_C \geqslant 2$ defined over the finite field $\mathbb{F}_p$. We consider C as a curve over the algebraic closure k of $\mathbb{F}_p$. Let $f_n : C \to C$ be the k-linear Frobenius morphism obtained by taking p^n-th powers for $n \in \mathbb{N}$. Let $X = C \times C$ and $\Gamma_n \subset X$ be the graph of f_n. Let Δ be the diagonal of $C \times C$. Note that Δ is a negative curve as its self-intersection is $\Delta^2 = 2 - 2g_C$ (see Example 1.37).

Let $f_n \times \mathrm{id} : X = C \times C \to X = C \times C$ be given by $(x, y) \mapsto (f_n(x), y)$. It is then easy to see that $(f_n \times \mathrm{id})^{-1}(\Delta) = \Gamma_n$. Since $f_n \times \mathrm{id}$ is a finite morphism of degree $\deg(f_n).\deg(\mathrm{id}) = p^n$, we have $\Gamma_n^2 = p^n \cdot \Delta^2 = p^n(2 - 2g_C)$ (see [30, Exercise V.1.10]). Since $\Gamma_n^2 \to -\infty$ as $n \to \infty$, X does not have bounded negativity.

There is a long-standing open conjecture involving boundedness of negativity on surfaces. Its origins are unclear, but it goes back to F. Enriques.

Conjecture 1.45 (Bounded Negativity Conjecture (BNC)). Every smooth projective surface X in characteristic 0 has bounded negativity.

This conjecture is known to hold for some classes of surfaces, for instance, surfaces in Example 1.42 and Example 1.43 or for smooth projective complex surfaces equipped with a surjective endomorphism which is not an isomorphism (see [6, Proposition 2.1]).

It is known (see [19] and [47]) that a smooth complex projective surface admitting a surjective endomorphism that is not an isomorphism is one of the following types:

1. X is a toric surface;
2. X is a $\mathbb{P}^1$-bundle over a curve;
3. X is an abelian surface or a hyperelliptic surface;
4. X is an elliptic surface with Kodaira dimension $\kappa(X) = 1$ and topological Euler number $e(X) = 0$.

Thus, [6, Proposition 2.1] shows that all surfaces X in the above list have bounded negativity.

The BNC leads to a number of interesting questions. One such question is:

Question 1.46. *If X has bounded negativity and Y is birational to X, does Y have bounded negativity ?*

If the above question has an affirmative answer, then one can focus only on minimal surfaces to understand bounded negativity.

The following special case of Question 1.46 is also open:

Conjecture 1.47. Let $\pi : X_s \to \mathbb{P}^2_{\mathbb{C}}$ be a blow up of $\mathbb{P}^2_{\mathbb{C}}$ at $s \geqslant 10$ points. Then X_s has bounded negativity.

1.1.3 Bounded Negativity and the Nagata Conjecture

Let X_r be the blow up of $\mathbb{P}^2_{\mathbb{C}}$ at r distinct points. If the r points are collinear, then $C^2 = 1 - r$, where C is the strict transform of the line containing the r points. It is easy to see in this case that BNC holds for X_r. Also, if the blown up points lie on a cubic, BNC holds for X_r, by Example 1.42, since $-K_{X_r}$ is effective.

Bounded negativity of X_r is known for some other special sets of points in $\mathbb{P}^2$ as well (see examples in [15, Remark 3.13]). We include these examples here.

Example 1.48. Let C be an irreducible and reduced rational plane nodal sextic. Then, it is known that C has exactly 10 nodes. Let X be the blow up of the 10 nodes. Such surfaces are called *Coble surfaces* (these are smooth rational surfaces X such that $|-K_X| = \emptyset$ but $|-2K_X| \neq \emptyset$). Then it is known that BNC holds for X; see [12, Section 3.2].

Example 1.49. Let X be the blow up of 10 points of intersection of 5 general lines in $\mathbb{P}^2$. Then $-K_X$ is a big divisor (see Definition 1.56) and by [57, Theorem 1], X is a *Mori Dream Space*. For such surfaces, the submonoid of the Picard group generated by the effective classes is finitely generated. Hence BNC holds for X ([28, Proposition I.2.5]).

If the points blown up are very general, there is a connection between BNC and the *Nagata Conjecture*. We say that a set of points $x_1, ..., x_r$ on a smooth projective complex surface X is *very general* if the tuple $(x_1, ..., x_r)$ lies outside a countable union of proper Zariski closed sets in $X^r := X \times X \times ... \times X$.

Conjecture 1.50 (Nagata Conjecture). Let $p_1, ..., p_r \in \mathbb{P}^2$ be very general points with $r \geqslant 9$. If $C \subset \mathbb{P}^2$ is a reduced and irreducible curve of degree d passing through p_i with multiplicity m_i at p_i, then

$$\sqrt{r} \cdot d \geqslant \sum_{i=1}^{r} m_i.$$

If the points blown up are very general, the bounded negativity conjecture is still open, but there is a conjecture called the SHGH (*Segre-Harbourne-Gimigliano-Hirschowitz*) conjecture about linear systems of plane curves, which implies both the Nagata conjecture and BNC for very general blow ups of $\mathbb{P}^2$. We describe it below.

Let $\pi \colon X_r \to \mathbb{P}^2$ be a blow up of $\mathbb{P}^2$ at r very general points $p_1, p_2, ..., p_r \in \mathbb{P}^2$. Consider the linear system $\mathcal{L}_d(p_1^{m_1}, p_2^{m_2}, ..., p_r^{m_r})$ of plane curves in $\mathbb{P}^2$ of degree d passing through $p_1, p_2, ..., p_r$ with multiplicities at least $m_1, m_2, ..., m_r$, respectively. Let $\mathcal{L}$ denote the linear system on X_r which is the pullback of $\mathcal{L}_d(p_1^{m_1}, p_2^{m_2}, ..., p_r^{m_r})$ via π. Let K_{X_r} denote the canonical bundle on X_r. One then defines:

$$\mathcal{V}(\mathcal{L}) := \mathcal{X}(\mathcal{L}) - 1 = \frac{\mathcal{L} \cdot (\mathcal{L} - K_{X_r})}{2},$$

to be the *virtual dimension* of $\mathcal{L}$ and

$$e(\mathcal{L}) := \max\{\mathcal{V}(\mathcal{L}), -1\},$$

to be the *expected dimension* of $\mathcal{L}$. We call a system $\mathcal{L}$ to be *non-special* if $\dim(\mathcal{L}) = e(\mathcal{L})$, otherwise we call it *special*, i.e., when $\dim(\mathcal{L}) > e(\mathcal{L})$. It is not hard to see

$$\mathcal{L} \text{ is non-special} \quad \Leftrightarrow \quad h^0(X_r, \mathcal{L}) \cdot h^1(X_r, \mathcal{L}) = 0.$$

An example of special divisors is given by the following observation.
★ If the linear system $\mathcal{L}$ has a multiple of a (-1)–curve in its base locus, then $\mathcal{L}$ is special.

SHGH says ★ is the only way speciality can arise. An equivalent formulation of SHGH is:

SHGH Conjecture: Suppose $\mathcal{L}$ is a linear system on X_r which is *nonempty* and *reduced* (i.e., a general member of $\mathcal{L}$ is reduced), then $\mathcal{L}$ is non-special.

Let C be an irreducible and reduced curve of genus g on X_r. If the SHGH conjecture holds on X_r, then the virtual dimension of $\mathcal{O}_{X_r}(C)$ is non-negative, which is equivalent to

$$C^2 \geqslant g - 1.$$

In particular, we have that SHGH implies a following stronger version of the bounded negativity conjecture for X_r:

Conjecture 1.51. Let X_r be the blow up of $\mathbb{P}^2_{\mathbb{C}}$ at r very general points. If C is a reduced and irreducible curve in X_r, then $C^2 \geqslant -1$ and $C^2 = -1$ implies that C is a (-1)–curve.

Thus, if the points blown up are very general, there is a conjectural answer to the BNC for X_r, namely, Conjecture 1.51. Hence, we see that bounded negativity of X_r depends on the position of the points that are blown up.

In fact, Conjecture 1.51 can be seen as a strong form of the Nagata conjecture:

Proposition 1.52. *Let $\pi_r : X_r \to \mathbb{P}^2_{\mathbb{C}}$ be the blow up of $\mathbb{P}^2_{\mathbb{C}}$ at r very general points. If Conjecture 1.51 holds for X_r, then the Nagata conjecture (Conjecture 1.50) holds.*

Proof. Suppose that C is a reduced and irreducible curve violating the Nagata Conjecture, i.e., $r \geqslant 9$ and $\sqrt{r} \cdot d < \sum_{i=1}^{r} m_i$, where $m_i = \text{mult}_{p_i}(C)$ for very general points $p_1, \cdots, p_r$. Let $\widetilde{C}$ denote the strict transform of C on the blow up X_r of $\mathbb{P}^2_{\mathbb{C}}$ at $p_1, \cdots, p_r$. Then $\widetilde{C} = dH - \sum_{i=1}^{r} m_i E_i$, where $H = \pi_r^* \mathcal{O}_{\mathbb{P}^2}(1)$ and $E_1, ..., E_r$ are the exceptional divisors of the blow up. Then, for $m = \frac{m_1 + ... + m_r}{r}$ and using the Cauchy-Schwartz inequality $m^2 \leqslant \frac{m_1^2 + ... + m_r^2}{r}$, we have

$$d^2 < r\frac{(m_1 + ... + m_r)^2}{r^2} = rm^2 \leqslant m_1^2 + ... + m_r^2.$$

Thus, $\widetilde{C}^2 = d^2 - (m_1^2 + ... + m_r^2) < 0$ and hence, by Conjecture 1.51, we have $\widetilde{C}^2 = -1$ and $\widetilde{C}$ is a (-1)–curve.

We know that $-K_{X_r} = 3H - E_1 - \cdots - E_r$. So, $-K_{X_r} \cdot \widetilde{C} = 1$. But $-K_{X_r} \cdot \widetilde{C} = 3d - (m_1 + ... + m_r) \leqslant \sqrt{r}d - (m_1 + ... + m_r) < 0$, which is a contradiction. □

1.1.4 Variants of the Bounded Negativity Conjecture

We now mention a few variants of BNC. See [4, Section 3.3 and Section 3.7] for more details.

Conjecture 1.53 (Weak Bounded Negativity Conjecture (Weak BNC)). For any smooth projective surface X in characteristic zero and any integer g, there exists a positive constant $b(X, g)$ only depending on X and g, such that

$$C^2 \geqslant -b(X, g)$$

for any reduced curve $C = \sum C_i$ in X such that $g(C_i) \leqslant g$ for all i, where $g(C_i)$ denotes the geometric genus of C_i (i.e., the genus of the normalization of C_i).

In [26], the Weak BNC is proved for any smooth complex projective surface X. The Weak BNC is shown to hold for an arbitrary reduced curve in [26, Theorem 2.1] using Zariski decompositions.

We now quote some earlier results which are valid on specific surfaces, but which give very precise bounds for C^2.

Proposition 1.54 (Bogomolov). *Let X be a smooth projective surface with Kodaira dimension $\kappa(X) \geqslant 0$. Then for any smooth irreducible curve $C \subset X$ of genus $g(C)$, we* *have*

$$C^2 \geqslant K_X^2 - 4c_2(X) - 4g(C) + 4,$$

where c_2 *is the second Chern number of the surface* X.

The proof of Proposition 1.54 involves Bogomolov's criterion for unstable bundles on surfaces and the Bogomolov-Sommese vanishing theorem. See [4, Proposition 3.4.4] and [9, Section 5] for details.

Proposition 1.55. *([4, Proposition 3.5.3]). Let* X *be a smooth projective surface with* $\kappa(X) \geqslant 0$*. Then for every reduced, irreducible curve* $C \subset X$ *of geometric genus* $g(C)$ *we have*

$$C^2 \geqslant K_X^2 - 3c_2(X) + 2 - 2g(C),$$

where c_2 *is the second Chern number of the surface* X.

Note that Proposition 1.55 is actually a corollary of the generalized logarithmic Miyaoka-Yau inequality [46, Theorem 1.1].

There is yet another variant of the BNC. To get to that, we first look at the following definition:

Definition 1.56. An integral divisor D on a smooth projective surface X is called *big* if there is an ample divisor A on X, a positive integer $m > 0$ and an effective divisor N on X such that mD is numerically equivalent to $A + N$.

Conjecture 1.57 (Weighted Bounded Negativity Conjecture (Weighted BNC)). Let X be a smooth projective surface in characteristic zero. There exists a nonnegative integer $b_w(X) \in \mathbb{Z}$ such that

$$C^2 \geqslant -b_w(X)(H \cdot C)^2,$$

for all irreducible and reduced curves $C \subset X$ and all big and nef line bundles H satisfying $H \cdot C > 0$.

This conjecture is open in general.

In [42], the authors give bounds for self-intersection numbers of irreducible and reduced curves on blow ups of some algebraic surfaces at distinct points, where the bounds depend on the degree of the curve with respect to an explicitly constructed big and nef line bundle. We include the statements here.

Theorem 1.58. *([42, Theorem A]) Let X be a surface of non-negative Kodaira dimension and let $f : Y \to X$ be the blowing up of X along n distinct points. Then there exists a big and nef line bundle Γ that bounds negativity linearly, i.e.,*

$$C^2 \geqslant -\frac{1}{2}\big(\delta(X) + C.\Gamma\big) - n,$$

for every reduced and irreducible curve $C \subset Y$, where $\delta(X) = 3e(X) - K_X^2$.

Theorem 1.59. *([42, Theorem B]) Let $\sigma : Y \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at n distinct points in $\mathbb{P}^2$, and let C be an irreducible and reduced curve on Y. Then,*

$$C^2 \geqslant -2n(C.L),$$

where $L = \sigma^\star H$ and H is the class of a line in $\mathbb{P}^2$.

1.1.5 Bounded Negativity and Zariski Decomposition

There is another formulation of the BNC using Zariski decomposition, which we now discuss.

Let X be a smooth projective surface. Zariski [62] showed the existence of Zariski decomposition for effective divisors, see [Theorem 1.40]. Fujita [20] extended this notion of Zariski decomposition to the pseudo-effective case. Note that a divisor D is called *pseudo-effective* if its numerical class $[D] \in N^1(X)_{\mathbb{R}} := N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$ lies in the closure of the convex cone spanned by the classes of effective divisors in $N^1(X)_{\mathbb{R}}$, called the *pseudo-effective* cone and denoted by $\overline{\mathrm{Eff}}(X)$.

The geometric significance of Zariski decompositions lies in the fact that, given a pseudo-effective integral divisor D on X with Zariski decomposition $D = P + N$, one has for every sufficiently divisible integer $m \geqslant 1$, the equality

$$H^0(X, \mathcal{O}_X(mD)) = H^0(X, \mathcal{O}_X(mP)).$$

In other words, all sections of $\mathcal{O}_X(mD)$ come from the nef line bundle $\mathcal{O}_X(mP)$. The term "sufficiently divisible" here means that one needs to pass to a multiple mD that clears the denominators in P for the statement to hold.

Thus, we have the following natural question:

Question 1.60. Let X be a smooth projective surface. Does there exist an integer $d(X) \geqslant 1$ such that for every pseudo-effective integral divisor D, the denominators in the Zariski decomposition of D are bounded above by $d(X)$?

If such a bound $d(X)$ exists, then we say that X *has bounded Zariski denominators*. In this case, taking then the factorial $d(X)!$, one has in fact a uniform number that clears the denominators in all Zariski decompositions on X. It is an interesting question to ask whether a given smooth surface satisfies this boundedness condition.

In [7], the authors established the following theorem:

Theorem 1.61. *For a smooth projective surface X over an algebraically closed field, the following two statements are equivalent:*

(i) X *has bounded Zariski denominators.*

(ii) X *has bounded negativity.*

Remark 1.62. In [7, Theorem 2.2], it is shown that if the self-intersection of reduced and irreducible curves on a smooth projective surface X is bounded below by $-b(X)$, then $d(X) = b(X)^{\rho(X)-1}$ is a bound for the Zariski denominators of integral pseudo-effective divisors on X, where $\rho(X)$ is the Picard number of X. Further, in [7, Theorem 2.3], it is shown that if the Zariski denominators of a smooth projective surface X are bounded by $d(X)$, then the self-intersection of irreducible and reduced curves is bounded below by $-b(X)$, where

$$b(X) = d(X) \cdot d(X)! \cdot |\Delta|,$$

and Δ is the determinant of the intersection form on $N^1(X)_{\mathbb{Z}}$.

Example 1.63. ([7, Example 3.1]). Let C be a smooth projective curve of genus $g_C \geqslant 2$ defined over k which is the algebraic closure of a finite field of characteristic $p > 0$. Let $X = C \times C$. We saw in Example 1.44 that X does not have bounded negativity. Thus, by Theorem 1.61, X must have unbounded Zariski denominators. Let F' be a fiber of the second projection $X \to C$, and consider the divisor $D_n = F' + \Gamma_n$, where Γ_n is as defined in Example 1.44. The negative part of its Zariski decomposition has support Γ_n with coefficient

$$\frac{D_n \cdot \Gamma_n}{\Gamma_n^2} = \frac{1 + \Gamma_n^2}{\Gamma_n^2}.$$

Since numerator and denominator are coprime for all n, we see that the Zariski denominator is $-\Gamma_n^2 = p^n(2g_C - 2)$ and hence tends to infinity.

Example 1.64. Let X be a smooth projective surface with $-K_X$ nef. We can take the constant $b(X)$ in BNC as 2 in this case (see Example 1.42). Thus from Remark 1.62, we have that for every pseudo-effective integral divisor D on X, the Zariski decomposition of $(2^{\rho(X)-1})! \cdot D$ is integral.

1.1.6 Bounded Negativity Conjecture and Seshadri Constants

There is an interesting connection between bounded negativity and Seshadri constants.

The following are very useful numerical criteria to check the ampleness of a line bundle on a smooth projective surface.

Theorem 1.65. *(Nakai-Moishezon criterion [43]). Let L be a line bundle on a smooth projective surface X. Then L is ample if and only if $L^2 > 0$ and $L \cdot C > 0$ for all curves C on X.*

Theorem 1.66. *(Seshadri's criterion [29]). A line bundle L on a smooth projective surface X is ample if and only if there exists a positive number $\varepsilon > 0$ such that*

$$\frac{L \cdot C}{\text{mult}_x C} \geqslant \varepsilon,$$

for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x having multiplicity $\text{mult}_x(C)$ at x.

We now define the notion of a *Seshadri constant* at a point.

Definition 1.67 (Seshadri constant at a point)**.** Let X be a smooth projective surface and L be a nef line bundle on X. The *Seshadri constant of L at a point* $x \in X$, denoted $\varepsilon(X, L, x)$, is defined as

$$\varepsilon(X, L, x) := \inf_{x \in C} \frac{L \cdot C}{\text{mult}_x C},$$

where the infimum is taken over all reduced and irreducible curves $C \subset X$ passing through x having the multiplicity $\text{mult}_x(C)$ at x.

A lot of research is aimed at finding good lower bounds for Seshadri constants of ample line bundles on surfaces.

It is observed in [4, Section 3.6] that BNC gives a partial answer to the following question on Seshadri constants posed by Demailly [13, Question 6.9]:

Question 1.68. *Is the global Seshadri constant*

$$\varepsilon(X) := \inf_{x \in X} \varepsilon(X, x)$$

positive for every smooth projective surface X*, where*

$$\varepsilon(X, x) := \inf_{L \text{ ample}} \varepsilon(X, L, x)?$$

This question is still open. It is also unknown whether for every fixed $x \in X$, the quantity $\varepsilon(X, x)$ is always positive.

In [4, Proposition 3.6.2], it was shown that positivity of $\varepsilon(X, x)$ is a consequence of the Bounded Negativity Conjecture. We include the statement here with proof.

Proposition 1.69. *Let* X *be a smooth projective surface (in characteristic* 0*) and let* $f\colon Y \to X$ *be the blow up of* X *at a point* $x \in X$*. Suppose the Bounded Negativity Conjecture is true for* Y*. Let* $b(Y) > 0$ *be a positive integer such that* $C^2 \geqslant -b(Y)$ *for all reduced curves* C *on* Y*. Then,*

$$\varepsilon(X, x) \geqslant \frac{1}{\sqrt{b(Y) + 1}} \quad \forall x \in X.$$

Proof. Let $C \subset X$ be an irreducible curve of multiplicity m at x, and let $\widetilde{C} \subset Y$ be its proper transform on the blow up Y of X in x. Then

$$C^2 - m^2 = (f^*C - mE)^2 = \tilde{C}^2 \geqslant -b(Y)\ .$$

Note that if $m \leqslant \sqrt{b(Y)}$, then

$$\frac{L \cdot C}{m} \geqslant \frac{L \cdot C}{\sqrt{b(Y)}} \geqslant \frac{1}{\sqrt{b(Y)}}$$

On the other hand, if $m > \sqrt{b(Y)}$, we have

$$C^2 \geqslant m^2 - b(Y) > 0$$

and hence, using the Hodge Index Theorem [Theorem 1.32], we get

$$\frac{L \cdot C}{m} \geqslant \frac{\sqrt{L^2}\sqrt{C^2}}{m} \geqslant \sqrt{1 - \frac{b(Y)}{m^2}} \geqslant \sqrt{1 - \frac{b(Y)}{b(Y) + 1}} = \frac{1}{\sqrt{b(Y) + 1}}.$$

□

1.1.7 Harbourne constants

The notion of *Harbourne constants* was defined in [5] in an attempt to understand and clarify the bounded negativity conjecture.

To illustrate the concept, consider a blow up X of $\mathbb{P}^2_{\mathbb{C}}$ at r distinct points. It is clear that the occurrence of negative curves on X depends on the position of the points that are blown up. For example, if the points are very general, it is conjectured that $C^2 \geqslant -1$ for all reduced and irreducible curves $C \subset X$ (see Conjecture 1.51). On the other hand, $C^2 = 1 - r$ if the points are collinear and C is the strict transform of the line containing them. A natural approach to the BNC then is to divide by r and consider the ratio C^2/r for all reduced, not necessarily irreducible, curves C on X. The problem then is to bound these ratios C^2/r.

Definition 1.70 (Harbourne constants)**.** Let X be a nonsingular projective surface over a field k. Let $C \subset X$ be a reduced curve and let $P \subset X$ be a finite nonempty set. Then the *Harbourne constant of* C *at* P is defined as

$$H(X, C, P) := \frac{C^2 - \sum_{p_i \in P} (\mathrm{mult}_{p_i}(C))^2}{|P|}, \tag{1.6}$$

where $|P|$ denotes the cardinality of P.

We also define

$$H(X, C) := \inf_P H(X, C, P),$$

where the infimum is taken over all finite nonempty subsets $P \subset X$.

Remark 1.71. Note that if $f : Y \to X$ is the blow up of X at a finite nonempty set P and $\widetilde{C}$ is the proper transform of C, then

$$\widetilde{C}^2 = \left(f^{\star}C - \sum_{p_i \in P} (\mathrm{mult}_{p_i}(C))E_i \right)^2 = C^2 - \sum_{p_i \in P} (\mathrm{mult}_{p_i}(C))^2,$$

where $E_1, \ldots, E_{|P|}$ are the exceptional divisors of the blow up. Thus we have $\widetilde{C}^2 = |P|.H(X, C, P)$.

We now define the *global Harbourne constant* of a smooth projective surface X as follows:

Definition 1.72 (Global Harbourne constant)**.** Let X be a nonsingular projective surface over a field k. The *global Harbourne constant* of X is defined as

$$H(X) := \inf_{C} H(X, C), \tag{1.7}$$

where the infimum is taken over all *reduced* curves $C \subset X$.

Remark 1.73. In [5, Remark 2.4], it was observed that if C is a singular reduced curve and $H(X, C, P) < 0$ for a finite subset P of X which contains some singular points of C, then

$$H(X, C) = \inf_{P} H(X, C, P),$$

where the infimum is taken over all nonempty subsets $P \subseteq \mathrm{Sing}(C)$, where $\mathrm{Sing}(C)$ denotes the set of singular points of C. Thus, it follows that

$$H(X) = \inf_{C} H(X, C), \tag{1.8}$$

where the infimum is taken over all *reduced* singular curves $C \subset X$.

Remark 1.74. It is not known if $H(X) \neq -\infty$ for any surface X. (For example, this is not known for even $X = \mathbb{P}^2$.) We remark that if $H(X) \neq -\infty$, then for any $s \geqslant 1$ and any irreducible curve D on the blow up of X at s points, $D^2 \geqslant sH(X)$. Thus if the *global Harbourne constant* is finite on X, the BNC holds for blow ups of X at finite sets of points; see [5, Remark 2.3].

Example 1.75. In [5, Example 2.2], it was observed that $H(X) \leqslant -2$ for any surface X. To see this, we embed X in a projective space $\mathbb{P}^N$ as a surface of some degree d. Choose r general hyperplane sections $C_1, \ldots, C_r$ such that C_i and C_j meet transversally in d^2 distinct points, for $i \neq j$ and $i, j = 1, 2, \ldots, r$. Note that C_i is smooth $\forall i = 1, 2, \ldots, r$. Let $C = C_1 + C_2 + \ldots + C_r$. Thus C has $s = d^2\binom{r}{2}$ nodes.

Now, we have

$$C^2 = (C_1 + C_2 + \ldots + C_r)^2 = \textstyle\sum_{i=1}^{r} C_i^2 + 2\sum_{1 \leqslant i < j \leqslant r} C_i \cdot C_j = rd^2 + 2d^2\binom{r}{2}.$$

Thus,

$$\begin{aligned} H(X, C, \mathrm{Sing}(C)) &= \frac{C^2 - \sum_{i=1}^{s}(\mathrm{mult}_{P_i}(C))^2}{s} = \frac{rd^2 + 2d^2\binom{r}{2} - 4d^2\binom{r}{2}}{d^2\binom{r}{2}} \\ &= \frac{r(2-r)d^2}{d^2\binom{r}{2}} = -2 + \frac{2}{r-1}. \end{aligned}$$

Thus, $H(X, C) \leqslant -2 + \frac{2}{r-1}$ and hence $H(X) \leqslant -2$.

Remark 1.76. In fact, [56, Corollary 4] shows that $H(X) \leqslant -4$ for any smooth surface X.

From the definition of $H(X, C, P)$, it is clear that in order to obtain very negative values of $H(X, C, P)$, one should consider singular reduced curves C on the original surface, with the finite set P as the set of singular points of C. It is an open problem to determine how negative $H(X, C, \mathrm{Sing}(C))$ can be for a reduced singular curve $C \subset X$, where $\mathrm{Sing}(C)$ denotes the set of singular points of C. It is interesting to study Harbourne constants for reduced and irreducible singular curves on a surface.

When X is the projective plane $\mathbb{P}^2$, there is a variant of the Harbourne constant:

$$H_{irr}(\mathbb{P}^2) := \inf_{C} H(\mathbb{P}^2, C, \mathrm{Sing}(C)),$$

where the infimum is over all reduced and irreducible singular curves $C \subset \mathbb{P}^2$. One can show that $H_{irr}(\mathbb{P}^2) \leqslant -2$, see [5, Remark 2.4]. To see this, consider a general map of $\mathbb{P}^1$ into $\mathbb{P}^2$ of degree d. The image is a rational curve C_d of degree d with $\binom{d-1}{2}$ nodes. Then $H(\mathbb{P}^2, C_d, \mathrm{Sing}(C_d)) = -2 + \frac{6d-4}{d^2-3d+2}$. Thus we have that $H(\mathbb{P}^2, C_d, \mathrm{Sing}(C_d)) > -2$ and $\lim_{d\to\infty} H(\mathbb{P}^2, C_d, \mathrm{Sing}(C_d)) = -2$.

The following question appearing in [5, Remark 2.4] is still open:

Question 1.77. *Does there exist an irreducible reduced singular curve* $C \subset \mathbb{P}^2$ *such that* $H(\mathbb{P}^2, C, \mathit{Sing}(C)) \leqslant -2$?

In [14], the authors show that Question 1.77 has a negative answer in some cases.

On the other hand, it is also reasonable to consider singular reducible reduced curves on surfaces whose singularities arise as intersection points of the irreducible components. Further, one can also assume that the irreducible components are all smooth and intersect pairwise transversally. The technical advantage behind this assumption lies in the property that after blowing up all the intersection points just once, we obtain a simple normal crossing divisor. Also, transversal arrangements allow us to use some combinatorial identities, which fail when tangencies are allowed. For all these reasons, it is reasonable to keep this assumption.

Definition 1.78 (Transversal arrangement)**.** Let $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$ be an arrangement of curves on a smooth projective surface X. We say that $\mathcal{C}$ is a *transversal arrangement* if $d \geqslant 2$, all curves C_i are smooth and they intersect pairwise transversally.

Given a transversal arrangement $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$, we have a divisor $D = \sum_{i=1}^{d} C_i$ on X. We use the arrangement $\mathcal{C}$ and the divisor D interchangeably.

Let $\mathrm{Sing}(\mathcal{C})$ be the set of all intersection points of the curves in a transversal arrangement $\mathcal{C}$. Note that $\mathrm{Sing}(\mathcal{C})$ is precisely the set of intersection points of the irreducible components of the reduced curve D, since all the irreducible components of D are nonsingular by hypothesis. Let s denote the number of points in the set $\mathrm{Sing}(\mathcal{C})$.

Definition 1.79 (Combinatorial invariants of transversal arrangements)**.** Let $\mathcal{C}$ be a transversal arrangement on a smooth projective surface X. For a point $p \in X$, let r_p denote the number of elements of $\mathcal{C}$ that pass through p. We call r_p the *multiplicity* of p in $\mathcal{C}$. We say p is an r*-fold* point of $\mathcal{C}$ if there are exactly r curves in $\mathcal{C}$ passing through p. For a positive integer $r \geqslant 2$, t_r denotes the number of r-fold points in $\mathcal{C}$.

These numbers satisfy the following well-known equality, which follows by counting incidences in a transversal arrangement in two ways:

$$\sum_{i<j} (C_i \cdot C_j) = \sum_{r \geqslant 2} \binom{r}{2} t_r. \tag{1.9}$$

Let

$$f_i = f_i(D) := \sum_{r \geqslant 2} r^i t_r.$$

In particular, $f_0 = s$ is the number of points in $\mathrm{Sing}(\mathcal{C})$.

Definition 1.80 (Harbourne constant of a transversal arrangement)**.** Let X be a smooth projective surface. Let $D = \sum_{i=1}^{d} C_i$ be a transversal arrangement of curves on X with $s > 0$. The rational number

$$H(\mathcal{C}) = H(D) := \frac{1}{s} \left(D^2 - \sum_{p \in \mathrm{Sing}(D)} r_p^2 \right)$$

is called the *Harbourne constant of the transversal arrangement* $\mathcal{C}$.

Remark 1.81. Comparing with Definition 1.70, for a transversal arrangement of curves $D = \sum_{i=1}^{d} C_i$ on a smooth projective surface X, we have $H(D) = H(X, D, \mathrm{Sing}(D))$.

Remark 1.82. We can rephrase Harbourne constants of a transversal curve arrangement $\mathcal{C}$ using the number of i-fold points t_i. More precisely, if $D = \sum_{i=1}^{d} C_i$ is a transversal arrangement of curves on X with $s > 0$, where s denotes the number of points in $\mathrm{Sing}(\mathcal{C})$, then

$$H(D) = \frac{1}{s}\left(D^2 - \sum_{r \geqslant 2} r^2 t_r\right).$$

Example 1.83. Let $X = \mathbb{P}^2$ and $\mathcal{C} = \{L_1, L_2, \ldots, L_d\}$ be d general lines in X. Then $r_p = 2$ for all $p \in \mathrm{Sing}(\mathcal{C})$. Let $D = \sum_{i=1}^{d} L_i$. Note that $s = \binom{d}{2}$. Also $D^2 = d + 2\binom{d}{2}$. Plugging these values in the definition of $H(\mathcal{C})$ and simplifying, we obtain $H(\mathcal{C}) = \frac{2}{d-1} - 2$.

Remark 1.84. If the Harbourne constants $H(\mathcal{C})$ for all transversal curve configurations $\mathcal{C}$ on a fixed surface X are uniformly bounded below by a number H, then BNC holds for all birational models $Y = \mathrm{Bl}_{\mathrm{Sing}(\mathcal{C})} X$ obtained from X by blowing up singular points of transversal arrangements $\mathcal{C}$ with $b(Y) = H \cdot s$. The reverse implication might fail i.e., it might happen that there is no uniform lower bound but nevertheless BNC holds for every such birational model of X.

1.1.8 Line arrangements and Harbourne constants

Line arrangements have played an important role in studying the bounded negativity problem and Harbourne constants [5].

Let $X = \mathbb{P}^2_k$, where k is a field. A *line arrangement* is a finite set of $d \geqslant 2$ distinct lines $\mathcal{C} = \{L_1, L_2, \ldots, L_d\}$ in the projective plane. We note that any line arrangement is a transversal arrangement.

In this case, there is a variant of Definition 1.80, called the *linear Harbourne constant*, see [5, Definition 3.1].

Definition 1.85 (Linear Harbourne constant)**.** Consider the projective plane $\mathbb{P}^2_k$, where k is a field. Then the *linear Harbourne constant* of $\mathbb{P}^2_k$ is defined as

$$H_{lin}(\mathbb{P}^2_k) = \inf_D \frac{1}{s}\left(D^2 - \sum_{p \in \mathrm{Sing}(D)} r_p^2\right),$$

where the infimum is over line arrangements $D = \sum_{i=1}^{d} L_i$ on $\mathbb{P}^2_k$ with $s > 0$. Note that r_p denotes the multiplicity of p in $\mathcal{C}$.

We now look at $H_{lin}(\mathbb{P}^2_k)$ for different fields k. We also include proofs here for completeness.

Proposition 1.86. *([27, Example 1.1.3]) Let* $\mathrm{char}(k) = p > 0$*, and let* $\overline{k}$ *be an algebraic closure of* k*. Then* $H_{lin}(\mathbb{P}^2_{\overline{k}}) = -\infty$.

Proof. For each finite field $\mathbb{F}_q \subseteq \overline{k}$ of order q, consider the arrangement of all lines $\mathcal{C}_q$ in $\mathbb{P}^2_{\overline{k}}$ defined over $\mathbb{F}_q$, where $q = p^r$ and r is a positive integer.

It is easy to see that there are $q^2 + q + 1$ lines in $\mathcal{C}_q$, $q^2 + q + 1$ points in $\mathrm{Sing}(\mathcal{C}_q)$ and $q + 1$ lines in $\mathcal{C}_q$ that pass through a given point $p \in \mathrm{Sing}(\mathcal{C}_q)$.

We now compute the Harbourne constant of the line arrangement $\mathcal{C}_q$.

$$\begin{aligned} H(\mathcal{C}_q) &= \frac{1}{s}\left(D^2 - \sum_{p \in \mathrm{Sing}(D)} r_p^2\right) \\ &= \frac{(q^2+q+1)^2 - (q^2+q+1)(q+1)^2}{q^2+q+1} \\ &= -q. \end{aligned}$$

Thus by taking larger and larger finite subfields, one can get arbitrarily negative Harbourne constants. Hence, $H_{lin}(\mathbb{P}^2_{\overline{k}}) = -\infty$. □

Remark 1.87. The line arrangement $\mathcal{C}_q$ in $\mathbb{P}^2_{\overline{k}}$ defined over $\mathbb{F}_q$ above is an example of a line arrangement with $t_2 = 0$, i.e., whenever two of the lines L_i in the arrangement cross, there is at least one other line in the arrangement that also goes through that crossing point.

Over $\mathbb{R}$, line arrangements with $d \geqslant 3$ concurrent lines (i.e., $d \geqslant 3$ lines through a point p) are the only line arrangements with $t_2 = 0$, due to the following result [44]:

Theorem 1.88 (Melchior)**.** *Given a real line arrangement of* d *lines with* $t_d = 0$ *(i.e., the lines are not concurrent), we have*

$$t_2 \geqslant 3 + \sum_{r>2} t_r(r-3).$$

Proposition 1.89. *([5, Theorem 3.15])* $H_{\mathrm{lin}}(\mathbb{P}^2_{\mathbb{R}}) = -3$.

Proof. We first show $H_{\mathrm{lin}}(\mathbb{P}^2_{\mathbb{R}}) \geqslant -3$.

Let $\mathcal{C} = \{L_1, L_2, \dots, L_d\}$ be a line arrangement in the real projective plane. If the lines are concurrent, we have $t_d = 1$ and $t_r = 0$ for $r \neq d$. Thus $H(\mathcal{C}) = 0$.

So suppose the lines are not concurrent. Then we have $d \geqslant 3$ and $t_d = 0$.

By Theorem 1.88, we have

$$t_2 = \alpha + 3 + \sum_{r\geqslant 3} t_r(r-3),$$

for some $\alpha \geqslant 0$.

By Equation (1.9), we also have

$$\binom{d}{2} = \sum_{r\geqslant 2} \binom{r}{2} t_r.$$

Thus,

$$\begin{aligned} H(\mathcal{C}) &= \frac{d - \sum_{r\geqslant 2} r t_r}{\sum_{r\geqslant 2} t_r} > \frac{-\sum_{r\geqslant 2} r t_r}{\sum_{r\geqslant 2} t_r} = \frac{-(2\alpha + 6 + 2\sum_{r\geqslant 3}(r-3)t_r) - \sum_{r\geqslant 3} r t_r}{\sum_{r\geqslant 2} t_r} \\ &= \frac{-(2\alpha + 6 + 3\sum_{r\geqslant 3}(r-2)t_r)}{\sum_{r\geqslant 2} t_r} = \frac{-(2\alpha + 6 + 3\sum_{r\geqslant 3}(r-2)t_r)}{\alpha + 3 + \sum_{r\geqslant 3}(r-2)t_r} \\ &= -3 + \frac{\alpha + 3}{\alpha + 3 + \sum_{r\geqslant 3}(r-2)t_r} > -3. \end{aligned}$$

Thus, we have $H_{\mathrm{lin}}(\mathbb{P}^2_{\mathbb{R}}) \geqslant -3$.

To show $H_{\mathrm{lin}}(\mathbb{P}^2_{\mathbb{R}}) \leqslant -3$, consider the line arrangement L_d of $2d$ lines, where d of the lines are the sides of a regular d-gon and the other d lines are the lines of bilateral symmetry of the d-gon (i.e., angle bisectors and perpendicular bisectors of the sides).

The arrangement L_d has one d-fold point at the center of the d-gon, given by the intersection of the lines of symmetry. The arrangement L_d further has $\binom{d}{2}$ triple points coming from intersections of pairs of sides with the line of symmetry between the sides of the pair. Also, L_d has d double points coming as midpoints of sides.

Thus, $t_d = 1$, $t_2 = d$, $t_3 = \binom{d}{2}$ and $t_r = 0$ for $r > 3$.

Hence, the Harbourne constant of L_d is $H(L_d) = \frac{2d-2t_2-3t_3-dt_d}{t_2+t_3+t_d} = -3 + \frac{4d+6}{d^2+d+2}$.

As $H(L_d)$ tends to -3 as $d \to \infty$, we conclude that $H_{lin}(\mathbb{P}^2_{\mathbb{R}}) \leqslant -3$. □

We now look at line arrangements in the complex projective plane.

Question 1.90 (Open Problem). *Classify all line arrangements in the complex projective plane with the* $t_2 = 0$ *property.*

This question has been studied in [1] and [24].

Remark 1.91. Over $k = \mathbb{C}$, only four kinds of line arrangements in $\mathbb{P}^2_{\mathbb{C}}$ seem to be known with $t_2 = 0$. (See [39] for Klein arrangements and [61] for Wiman arrangements.) We list these arrangements below. It is easy to compute Harbourne constants for each of these arrangements.

(1) Line arrangement $\mathcal{L}$ of $d \geqslant 3$ concurrent lines. In this case, it is easy to see that $H(\mathcal{L}) = 0$.

(2) The Fermat arrangement $\mathcal{C}_d$ of $3d$ lines for $d \geqslant 3$: the lines of this arrangement are defined by the factors of $(x^d - y^d)(x^d - z^d)(y^d - z^d)$. Each line contains $d+1$ crossing points, and we have $t_r = 0$ except for $t_3 = d^2$ and $t_d = 3$ when $d > 3$ or $t_3 = 12$ when $d = 3$.

Thus we have $H(\mathcal{C}_3) = -2.25$.

For $d > 3$, we have $H(\mathcal{C}_d) = \frac{3d-3t_3-dt_d}{t_3+t_d} = -3 + \frac{9}{d^2+3}$. Thus, $H(\mathcal{C}_d)$ tends to -3 as $d \to \infty$.

(3) The Klein arrangement $\mathcal{K}$ of 21 lines: here $t_r = 0$ except for $t_4 = 21$ and $t_3 = 28$. For this arrangement, each line contains 4 points where 3 lines cross and 4 points where 3 lines cross. Thus, we have $H(\mathcal{K}) = \frac{21-3t_3-4t_4}{t_3+t_4} = -3$.

(4) The Wiman arrangement $\mathcal{W}$ of 45 lines: here $t_r = 0$ except for $t_5 = 36$, $t_4 = 45$ and $t_3 = 120$. For this arrangement, each line contains 4 points where 5 lines

cross, 4 points where 4 lines cross and 8 points where 3 lines cross. Thus, we have $H(\mathcal{W}) = \frac{45-3t_3-4t_4-5t_5}{t_3+t_4+t_5} = \frac{-225}{67}$.

Over $\mathbb{C}$, the linear Harbourne constant of $\mathbb{P}^2$ has the following bounds, see [5, Theorem 3.3] for details:

Proposition 1.92.

$$-3.358 \approx -\frac{225}{67} \geqslant H_{lin}(\mathbb{P}^2_{\mathbb{C}}) \geqslant -4.$$

Remark 1.93. The bound $-\frac{225}{67} \geqslant H_{lin}(\mathbb{P}^2_{\mathbb{C}})$ comes from the Wiman arrangement. The bound $H_{lin}(\mathbb{P}^2_{\mathbb{C}}) \geqslant -4$ comes from applying an inequality due to Hirzebruch [35]: Given any complex arrangement of $d > 3$ lines such that $t_d = t_{d-1} = 0$, we have

$$t_2 + \frac{3}{4}t_3 \geqslant d + \sum_{r \geqslant 5}(r-4)t_r.$$

Harbourne constants for arrangements of d lines in $\mathbb{P}^2_k$ for arbitrary fields k are studied in [16]. In [16], the authors introduce another variant of the Harbourne constants called the *absolute linear Harbourne constant*.

Definition 1.94 (Absolute linear Harbourne constant)**.** The absolute linear Harbourne constant of d lines in $\mathbb{P}^2_k$ is defined as

$$H(d) = \min_k \min_D \frac{1}{s}\left(D^2 - \sum_{p \in \mathrm{Sing}(D)} r_p^2\right)$$

where the inner minimum is taken over all line arrangements $D = \sum_{i=1}^{d} L_i$ of d lines in $\mathbb{P}^2_k$ with $s > 0$ being the number of points in $\mathrm{Sing}(D)$ and the outer minimum is taken over all fields k.

In [16, Theorem 1.4], the value of $H(d)$ is computed for small values of d. Also in [16, Theorem 1.6], the authors give the following lower bound on Harbourne constants.

Theorem 1.95. *For $d \geqslant 6$, we have*

$$H(d) \geqslant -\frac{1}{2}\sqrt{4d-3} + \frac{1}{2}.$$

1.1.9 Curve arrangements and Harbourne constants

We now state some known results about Harbourne constants for curve arrangements on surfaces.

In [53], the authors describe a transversal arrangement $\mathcal{C}$ of curves in $\mathbb{P}^2$ for which $H(\mathcal{C}) \approx -3.571$.

The case of arrangements of conics on $\mathbb{P}^2$ was studied in [55]. The authors show the following:

Theorem 1.96. *([55, Theorem A]) Let $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$ be a transversal arrangement of conics in $\mathbb{P}^2$ with $t_d = 0$. Then $H(\mathcal{C}) \geqslant -4.5$.*

The author of [56] considers arrangements $\mathcal{C}$ of elliptic curves on an abelian surface or on $\mathbb{P}^2$. It is proved in [56, Theorem 1] that $H(\mathcal{C}) \geqslant -4$.

Further, the following is shown in [56, Theorem 5]:

Theorem 1.97. *There exists a sequence of reduced curves $D_n \subset \mathbb{P}^2$ (each of which is a union of elliptic curves) such that $\lim_n H(D_n) = -4$.*

In [54], the authors show the following:

Theorem 1.98. *([54, Theorem 4.2]) Let $D = C_1 + \ldots + C_d$ be a reduced divisor on $\mathbb{P}^2$, where C_i are smooth irreducible plane curves of degree $n \geqslant 3$ such that C_i and C_j meet transversally for all $i \neq j$. Assume also that $d \geqslant 4$ and $t_d = 0$. Let s be the number of singular points of D. Then*

$$H(D) \geqslant -4 + \frac{9nd - 5n^2 d}{2s}.$$

Harbourne constants for line arrangements on a smooth hypersurface X of degree $n \geqslant 3$ in $\mathbb{P}^3$ were first studied in [50]. The bounds obtained there were generalized in [41]. In [41, Theorem 3.2], it was shown that:

Theorem 1.99. *If X is a smooth hypersurface in $\mathbb{P}^3_{\mathbb{C}}$ of degree $n \geqslant 4$ and $\mathcal{L}$ is an arrangement of $d \geqslant 2$ lines on X such that the union of lines in $\mathcal{L}$ is connected as a subset of $\mathbb{P}^3_{\mathbb{C}}$, then*

$$H(\mathcal{L}) \geqslant -n(n-1).$$

Harbourne constants for transversal arrangements of smooth curves on a surface X with numerically trivial canonical class were studied in [40]. The bounds on Harbourne constants were given in terms of the number of curves and the second Chern class of X. This bound was generalized to surfaces with non-negative Kodaira dimension in [41, Theorem 2.2]. We include the statement here.

Theorem 1.100. *Let X be a smooth complex projective surface with non-negative Kodaira dimension. Let $D = \sum_{i=1}^{d} C_i$ be a transversal arrangement of curves on X with $d \geqslant 2$ and $s > 0$ as the number of singular points of D. Then*

$$H(D) \geqslant -4 + \frac{K_X^2 - 3c_2(X) + \sum_{i=1}^{d}(2 - 2g(C_i)) + t_2}{s}$$

As the above survey of the literature illustrates, most of the work on Harbourne constants for curve arrangements considered surfaces of non-negative Kodaira dimension or $\mathbb{P}^2$. In this thesis, we look at curve arrangements on ruled surfaces and prove lower bounds on their Harbourne constants.

1.1.10 Ruled surfaces

Definition 1.101. Let C be a smooth projective algebraic curve over an algebraically closed field k. A *geometrically ruled surface* or simply a *ruled surface*, is a smooth projective surface X, together with a surjective morphism $\phi : X \to C$ such that the fiber X_y is isomorphic to $\mathbb{P}^1_k$ for every closed point $y \in C$, and ϕ admits a section (i.e., a morphism $\alpha : C \to X$ such that $\phi \circ \alpha = id_C$).

The following proposition characterizes ruled surfaces:

Proposition 1.102. *([30, Proposition 2.2, Chapter V]) If $\phi : X \to C$ is a ruled surface, then $X \cong \mathbb{P}_C(E)$ over C for a vector bundle E of rank 2 over C. Conversely, every such $\mathbb{P}_C(E)$ is a ruled surface over C. Moreover, E_1 and E_2 are two vector bundles of rank 2 on C such that $X \cong \mathbb{P}_C(E_1) \cong \mathbb{P}_C(E_2)$ as ruled surfaces over C if and only if there is a line bundle N on C such that $E_1 \cong E_2 \otimes N$.*

We now mention the following theorem from [30, Chapter V, Section 2]:

Theorem 1.103. *If* $\phi : X \to C$ *is a ruled surface, then there is a vector bundle* F *of rank 2 on* C *such that* $X \cong \mathbb{P}_C(F)$ *as ruled surfaces over* C*, and* F *has the following property:* $H^0(C, F) \neq 0$*, but* $H^0(C, F \otimes N) = 0$ *for all line bundles* N *on* C *with deg*$(N) < 0$*.*

In this case, $e := -\deg(\wedge^2 F)$ is an invariant of X uniquely determined by X. Further in this case, there is a section $\alpha_0 : C \to X$, called the *normalised* section, such that Image$(\alpha_0) = C_0$ and $\mathcal{O}_X(C_0) = \mathcal{O}_X(1)$.

Thus from the above theorem, we have the following definition:

Definition 1.104. A vector bundle E of rank 2 on a smooth irreducible projective curve C is said to be *normalized* if $H^0(C, E) \neq 0$ and $H^0(C, E \otimes \mathcal{L}) = 0$ for all line bundles $\mathcal{L}$ on C with $\deg(\mathcal{L}) < 0$.

Let $\phi : X \to C$ be a ruled surface over a smooth complex curve C of genus g with invariant $e \geqslant 0$. We fix a section C_0 of X with $\mathcal{O}_X(C_0) = \mathcal{O}_X(1)$. Let f denote the numerical class of a fiber of ϕ.

Then

1. $\text{Pic}(X) \cong \mathbb{Z} \cdot \mathcal{O}_X(1) \oplus \phi^*(\text{Pic}(C))$, and
2. $N^1(X)_{\mathbb{Z}} \cong \mathbb{Z} \cdot C_0 \oplus \mathbb{Z} \cdot f$ satisfying $C_0^2 = -e$, $C_0 \cdot f = 1$ and $f^2 = 0$.

Thus any element of $N^1(X)_{\mathbb{Z}}$ has the form $aC_0 + bf$, for $a, b \in \mathbb{Z}$.

Any canonical divisor on X, denoted by K_X, is numerically equivalent to $-2C_0 + (2g - 2 - e)f$.

If an irreducible curve on X, different from C_0 and f, is numerically equivalent to $aC_0 + bf$, then $a > 0$ and $b \geqslant ae$. A divisor on X which is numerically equivalent to $aC_0 + bf$ is ample if and only if $a > 0$ and $b > ae$.

For more details, see [30, Chapter V, Section 2].

1.2 Aim of the thesis

In this thesis, we study the Bounded Negativity Conjecture (BNC) and Harbourne constants on ruled surfaces. We state our results in Section 1.2.1. These results are proved in Section 2.4 and Section 2.5 of Chapter 2. Section 1.2.1 is a joint work with Krishna Hanumanthu which has appeared in Manuscripta Math. [25].

1.2.1 Harbourne constants on ruled surfaces

We prove our results under the following assumption (see Assumption 2.41):

Assumption 1.105. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$ be a transversal arrangement of curves on a ruled surface X over a smooth curve with invariant $e \geqslant 4$ such that the following conditions hold:

1. All the curves C_i in $\mathcal{C}$ are linearly equivalent to a fixed divisor A on X, where A is numerically equivalent to $aC_0 + bf$, for $a, b \in \mathbb{Z}$ with $a > 0$ and $b \geqslant ae$.
2. $d \geqslant 4$ and $t_d = 0$, i.e., all curves do not go through a common point.
3. Either $a \geqslant 2$, or $a = 1$ and there exists a subset of four curves in $\mathcal{C}$ such that there is no point common to all the four curves.

We prove the following main theorems:

Theorem 1.106. *Let X be a ruled surface with $e \geqslant 4$ over a smooth curve of genus g. Let $\mathcal{C}$ be a transversal arrangement of curves satisfying Assumption 1.105. In particular, each curve in $\mathcal{C}$ is numerically equivalent to $aC_0 + bf$ with $a > 0$ and $b \geqslant ae$. Then we have the following bound on the Harbourne constant of $\mathcal{C}$:*

$$H(\mathcal{C}) \geqslant \frac{-9}{2} - \frac{8}{f_0} + \frac{d}{f_0}\left(\frac{(ae-2b)}{2}(3a-2) - 2a(g-1)\right) + \frac{16g + 4t_2 + t_4}{2f_0} + \frac{9t_3}{8f_0}.$$

If the curves in the arrangement $\mathcal{C}$ do not intersect the normalized section C_0, we obtain an improved bound for the Harbourne constants:

Theorem 1.107. *Let X be a ruled surface with $e \geqslant 4$ over a smooth curve of genus g. Let $\mathcal{C}$ be a transversal arrangement of curves satisfying Assumption 1.105. Assume further that no curve in $\mathcal{C}$ intersects the normalized section C_0. Then we have the following bound on the Harbourne constant of $\mathcal{C}$:*

$$H(\mathcal{C}) \geqslant \frac{-9}{2} + \frac{d}{f_0}\left(\frac{ae(2-3a) - 4a(g-1)}{2}\right) + \frac{16g + 4t_2 + t_4}{2f_0} + \frac{9t_3}{8f_0}.$$

For proving Theorem 1.106 and Theorem 1.107, we construct a surface Y of non-negative Kodaira dimension associated to a transversal arrangement of curves $\mathcal{C}$ on a ruled surface X with invariant $e \geqslant 4$ satisfying Assumption 1.105.

We now state a corollary which gives a lower bound on the self-intersection of the strict transform of the divisor associated to an arrangement of curves.

Corollary 1.108. *Let $\mathcal{C}$ be a transversal arrangement on the ruled surface X satisfying Assumption 1.105. Let $f : \widetilde{X} \to X$ be the blow up of X at Sing$(\mathcal{C})$. Let $\widetilde{D}$ denote the strict transform of D, which is the divisor defined as the sum of all the curves in $\mathcal{C}$. Then*

$$\widetilde{D}^2 \geqslant -8 - \frac{9}{2}s + d\left(\frac{(ae-2b)}{2}(3a-2) - 2a(g-1)\right) + 8g + 2t_2 + \frac{t_4}{2} + \frac{9t_3}{8}.$$

Further, if all curves in the arrangement do not intersect the normalized section C_0, then

$$\widetilde{D}^2 \geqslant \frac{-9}{2}s + d\left(\frac{ae(2-3a) - 4a(g-1)}{2}\right) + 8g + 2t_2 + \frac{t_4}{2} + \frac{9t_3}{8}.$$

In Section 2.4, we define the global Harbourne constant of a ruled surface for a fixed pair of integers a, b as follows:

Definition 1.109. Let X be a ruled surface with invariant $e \geqslant 4$. Let $a > 0$ and $b \geqslant ae$ be positive integers. We define the *global* Harbourne constant $H_{a,b}(X)$ of X as:

$$H_{a,b}(X) := \inf_{\mathcal{C}} H(\mathcal{C}),$$

where the infimum is over all transversal arrangements $\mathcal{C}$ satisfying Assumption 1.105.

We then give lower bounds for $H_{a,b}(X)$ for a ruled surface X.

Corollary 1.110. *Let X be a ruled surface over a smooth curve of genus g with invariant $e \geqslant 4$. Let $a > 0$ and $b > ae$ be positive integers. Then*

$$H_{a,b}(X) \geqslant \frac{-11}{2} + \frac{(ae-2b)}{2}(3a-2) - 2ag. \tag{1.10}$$

Further, if $ae = b$, then

$$H_{a,b}(X) \geqslant \frac{-9}{2} + \frac{ae(2-3a) - 4ag}{2}. \tag{1.11}$$

We also prove the following result:

Theorem 1.111. *Let X be a ruled surface with $e \geqslant 4$. There does not exist any transversal arrangement $\mathcal{C}$ on X satisfying Assumption 1.105 such that the associated surface Y is a ball quotient.*

1.3 Strategy of proofs

The basic tool in studying Harbourne constants for curve arrangements on surfaces is a method developed by Hirzebruch in [35]. The idea is to consider a branched abelian covering Z of X branched along the given configuration $\mathcal{C}$. In order to prove that such a branched covering does in fact exist for the ruled surface X, we use a result by Namba [48, Theorem 2.3.20]. We then consider the desingularization Y of Z. Assumption 1.105 ensures that Y has non-negative Kodaira dimension. Then one considers Hirzebruch-Miyaoka-Sakai type inequalities involving the Chern numbers of Y, see [37, Theorem 3, Page 144]. Hirzebruch described the Chern numbers of Y in terms of certain invariants of the surface X and certain combinatorial invariants of the arrangement $\mathcal{C}$. In the end, one obtains inequalities on combinatorial invariants of $\mathcal{C}$ which can then be used to obtain the required bounds on Harbourne constants $H(\mathcal{C})$ as in Theorem 1.106 and Theorem 1.107.

In order to prove Theorem 1.111, we use the theory of constantly branched covers developed in [3].

1.4 Organization of the thesis

In Chapter 2, we give detailed proofs of the results about Harbourne constants on ruled surfaces.

In Section 2.1, we recall basic definitions and properties about ramified and unramified morphisms. We also study the notion of constantly branched covers and regular constantly branched covers for surfaces and include some results on the behaviour of canonical divisors on such covers. We also state a few results on the topological Euler characteristic.

In Section 2.2, we introduce the curve arrangements that we study. We also mention some well-known combinatorial properties of these curve arrangements that we use.

In Section 2.3, using a result of Namba, we construct an abelian cover $Z \to X$ branched on the given curve arrangement and then consider the desingularization $Y \to Z$; see Figure 2.2. We also compute the Chern numbers of Y and relate these to the combinatorial data of the curve arrangement on X.

In Section 2.4, we first show that Y has non-negative Kodaira dimension which enables us to apply a Hirzebruch-Miyaoka-Sakai type inequality. Using this, we prove our results about Harbourne constants on ruled surfaces.

Finally, in Section 2.5, we show that the surface Y is not a ball quotient.

Chapter 2

Harbourne constants on ruled surfaces

2.1 Preliminaries

2.1.1 Ramified and Unramified Morphisms

Throughout this subsection, k denotes a field and all varieties are defined over k.

We recall several definitions and results about certain finite morphisms $\pi : Y \to X$ of varieties. These morphisms are the algebraic analogues of branched (ramified) covers in Euclidean topology. Our references are [23] and [34].

Definition 2.1. A *branched covering* $\pi : Y \to X$ is a finite surjective morphism between normal varieties. Denote by G the group of automorphisms $\alpha : Y \to Y$ such that $\pi(\alpha(y)) = \pi(y)$ for all $y \in Y$. The group G is called the *group of covering automorphisms* of π. If G acts transitively on all fibers of the cover π, then the covering is called *Galois*. We say that a branched covering $\pi : Y \to X$ is an *abelian covering* if $\pi : Y \to X$ is Galois and additionally the group of covering automorphisms is abelian.

For any finite surjective morphism $\pi : Y \to X$ of varieties, the *degree* of π, denoted $\mathrm{deg}(\pi)$, is the degree of the finite field extension $k(Y)$ over $\pi^* k(X)$. It is equal to the cardinality of the generic fibers of π. The closed set where the fibers are of smaller cardinality is the ramification locus of the covering. We now make this notion precise.

Definition 2.2. Let $\pi : Y \to X$ be a finite morphism of Noetherian k-schemes. Let $q \in Y$ be any point and set $p := \pi(q)$. We say that π is *unramified* at q if $\pi_q^\sharp : \mathcal{O}_{X,p} \to \mathcal{O}_{Y,q}$

satisfies $\mathfrak{m}_p \cdot \mathcal{O}_{Y,q} = \mathfrak{m}_q$ and the induced map between residue fields $k(p) \to k(q)$ is a finite separable extension. Otherwise, we say that π is *ramified* at q. We denote by $\mathcal{R}_\pi \subseteq Y$ the set of points where π is ramified and call it the *ramification locus* of π. The set $\mathcal{B}_\pi := \pi(\mathcal{R}_\pi)$ is called the *branch locus* of π. The morphism π is called *unramified* if it is nowhere ramified (i.e., $\mathcal{R}_\pi = \emptyset$).

A result of O. Zariski [63, Proposition 2] assures that $\mathcal{B}_\pi$ and $\mathcal{R}_\pi$ are divisors. We include the statement here without proof.

Theorem 2.3. *Suppose $\pi : Y \to X$ is a finite dominant morphism of smooth varieties such that $k(X) \to k(Y)$ is separable. If $\mathcal{R}_\pi \neq \emptyset$, then $\mathcal{R}_\pi$ and $\mathcal{B}_\pi$ are pure of codimension one (i.e., all irreducible components have codimension one).*

Definition 2.4. Let $\pi : Y \to X$ be a finite morphism of Noetherian k-schemes. Let $q \in Y$ be a closed point and set $p := \pi(q)$. Let $Y_p := Y \times_X \mathrm{Spec}(k(p))$ be the scheme-theoretic fiber of p under π. It is well-known that $\mathrm{sp}(Y_p)$ is homeomorphic to $\pi^{-1}(p)$, see [30, Exercise II.3.10] . We define

$$e_\pi(q) := \mathrm{length}_{\mathcal{O}_{Y_p,q}}(\mathcal{O}_{Y_p,q})$$

and call it the *ramification index* of π at q. If $Z = \overline{q}$ is the closure of q, we write $e_\pi(Z) := e_\pi(q)$. If $\mathrm{char}(k) = 0$, or if $\mathrm{char}(k) > 0$, but $\mathrm{char}(k)$ does not divide $e_\pi(q)$, we say that the ramification is *tame*, otherwise the ramification is *wild*.

Proposition 2.5. *With notation as in Definition 2.4, $\mathcal{O}_{Y,q}/(\mathfrak{m}_p \cdot \mathcal{O}_{Y,q}) \cong \mathcal{O}_{Y_p,q}$.*

Proof. We may assume that $X = \mathrm{Spec}(A)$ and hence, $Y = \pi^{-1}(X) = \mathrm{Spec}(B)$ is also affine. By definition, $\mathcal{O}_{Y_p,q} = (B \otimes_A k(p))_q$. Also,

$$\begin{aligned} (B \otimes_A k(p))_q = (B \otimes_A (A_p/\mathfrak{m}_p))_q &\xrightarrow{\sim} B_q/(\mathfrak{m}_p.B_q) \\ \frac{b \otimes (a + \mathfrak{m}_p)}{h} &\mapsto \frac{a.b}{h} + (\mathfrak{m}_p.B_q) \end{aligned}$$

is an isomorphism: For injectivity, $ab \in \mathfrak{m}_p.B_q$ implies $ab = a'b'$ with $a' \in \mathfrak{m}_p$ and $b' \in B_q$, but then $b \otimes (a + \mathfrak{m}_p) = b' \otimes (a' + \mathfrak{m}_p) = 0$. ☐

There is a connection between the ramification index and the notion of π being ramified:

Corollary 2.6. *Let* $\pi : Y \to X$ *be a finite morphism of smooth varieties. Let* $q \in Y$ *and* $p := \pi(q)$*. Then,* $e_\pi(q) = 1$ *if and only if* π *is unramified at* q*.*

Proof. Note that $e_\pi(q) = 1$ if and only if $\mathcal{O}_{Y_p,q}$ is a field, i.e., if and only if it is equal to $k(q)$. By Proposition 2.5, this is equivalent to

$$\mathcal{O}_{Y,q}/(\mathfrak{m}_p \cdot \mathcal{O}_{Y,q}) = \mathcal{O}_{Y_p,q} = k(q) = \mathcal{O}_{Y,q}/\mathfrak{m}_q.$$

□

The following corollary connects the above definition of the ramification index with the one given in [30, IV.2]:

Corollary 2.7. *Let* $\pi : Y \to X$ *be a finite, dominant morphism of regular integral Noetherian* k*-schemes. Let* $q \in Y$ *be a point of codimension one (i.e.,* $\overline{\{q\}}$ *is of codimension one) and* $p := \pi(q)$*. Let* f *be a uniformizing parameter at* p*, i.e.* $\mathfrak{m}_p = (f)$*. Let* $\nu_q : k(Y) \to \mathbb{Z}$ *denote the valuation corresponding to* $\mathcal{O}_{Y,q}$*. Then* $e_\pi(q) = \nu_q(\pi_q^\sharp(f))$*.*

Proof. By [17, Proposition 11.1], since Y is regular, ν_q can be evaluated on $\mathcal{O}_{Y,q}$ as follows: If h is a uniformizing parameter at q, i.e., $\mathfrak{m}_q = (h)$, then any element $y \in \mathcal{O}_{Y,q}$ can be written as $y = uh^\nu$ for some unit u and $\nu = \nu_q(y)$. Let $e := \nu_q(\pi_q^\sharp(f))$. Then Proposition 2.5 yields

$$\mathcal{O}_{Y_p,q} = \mathcal{O}_{Y,q}/(\pi_q^\sharp(f)) = \mathcal{O}_{Y,q}/(h^e),$$

which has length e over itself. □

Remark 2.8. By Theorem 2.3 and Corollary 2.6, there is a finite number of points q of codimension one where π is ramified, and these are the points with $e_\pi(q) > 1$.

Definition 2.9. Let $\pi : Y \to X$ be a finite morphism of Noetherian k-schemes. Let $q \in Y$ be a closed point. Let $p := \pi(q)$. Then we call

$$f_\pi(q) := [k(q) : k(p)]$$

the *inertia degree* of π at q. This is the degree of the restricted morphism $\overline{q} \to \overline{p}$.

A very important tool in the analysis of branched coverings is the following formula:

Theorem 2.10. *[Degree formula [23, Page 329]] Let* $\pi : Y \to X$ *be a finite, dominant morphism of integral regular Noetherian* k*-schemes. Then, for any closed point* $p \in X$*,*

$$\deg(\pi) = \sum_{\pi(q)=p} e_\pi(q) \cdot f_\pi(q).$$

As an application, one can show that an unramified morphism has constant fiber cardinality:

Corollary 2.11. *Let* $\pi : Y \to X$ *be an unramified, finite and surjective morphism of smooth varieties over an algebraically closed field* k*. Then,*

$$|\pi^{-1}(p)| = \deg(\pi)$$

for each closed point $p \in X$*.*

Proof. Since π is unramified, Corollary 2.6 and Theorem 2.10 give

$$\deg(\pi) = \sum_{\pi(q)=p} [k(q) : k(p)] = \sum_{\pi(q)=p} 1 = |\pi^{-1}(p)|,$$

since $k(q) \cong k(p) \cong k$ and k is algebraically closed. □

2.1.2 Constantly branched coverings

We will now restrict to a special class of finite morphisms. These *constantly branched coverings* for surfaces will be the objects of study for the rest of this section. Their branch locus is required to be a so-called *strict arrangement*. The notion of a constantly branched cover was introduced by Hirzebruch in [3] for complex surfaces.

Definition 2.12. Let $\mathcal{C} = \{C_0, C_1, C_2, ..., C_l\}$ be a transversal arrangement of curves on a smooth projective surface X, with $D = \sum_{i=0}^{l} C_i$ the associated divisor on X. We say that $\mathcal{C}$ is a *strict arrangement* if, in addition, the C_i intersect transversally (or equivalently, D has normal crossings). Recall that for a point $p \in X$, r_p denotes the number of elements of $\mathcal{C}$ that pass through p. We say that p is an r*-fold* point of $\mathcal{C}$ if $r_p = r$. Also for a positive integer $r \geqslant 2$, t_r denotes the number of r-fold points in $\mathcal{C}$.

Definition 2.13. Let $\mathcal{C} = \{C_0, C_1, C_2, ..., C_l\}$ be a transversal arrangement of curves on a smooth projective surface X. We say that a point $p \in X$ is a *redundant intersection* of

$\mathcal{C}$ if $r_p \geqslant 3$. We denote by $\mathrm{Rd}(D)$ the set of all redundant intersections of $\mathcal{C}$ and refer to as the *redundant part* of D.

Remark 2.14. Let $\mathcal{C} = \{C_0, C_1, C_2, ..., C_l\}$ be a transversal arrangement of curves on a smooth projective surface X. Then $\mathcal{C}$ is strict if and only if it has no redundant intersections.

In the next section, we will be interested in transversal curve arrangements on ruled surfaces satisfying an additional condition (see Assumption 2.34). These arrangements have a redundant part.

We can now define a constantly branched covering. Let $n > 0$ be an integer. Recall that for a field k containing all the n-th roots of unity, a *Kummer extension* is an algebraic extension of the form

$$k[\sqrt[n]{x_1}, \cdots \sqrt[n]{x_l}],$$

where $x_i \in k$. Our reference is [11, 4.9]. One assumes that $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > 0$ and $\mathrm{char}(k)$ does not divide n. In this case, the extension is automatically Galois.

Notation 2.15. Let A be a domain, $k := \mathrm{Frac}(A)$ and $\overline{k}$ its algebraic closure. For any nonzero $x \in A$, we understand $\sqrt[n]{x}$ as a set. More precisely, $\sqrt[n]{x} = \{y \in \overline{k} \mid y^n = x\}$.

Definition 2.16. Let k be a field. Let $n > 0$ be an integer. A finite, surjective morphism $\pi \colon Y \to X$ of smooth projective surfaces over k is called *n-fold locally Kummer* if $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > 0$ and $\mathrm{char}(k)$ does not divide n and for every closed point $q \in Y$, with $p = \pi(q)$, there exist germs $y_1, \cdots, y_l \in \mathcal{O}_{Y,q}$ with

$$\mathcal{O}_{Y,q} = \mathcal{O}_{X,p}[y_1, \cdots, y_l]$$

for $x_i = y_i^n \in \mathcal{O}_{X,p}$.

Definition 2.17. An n-fold locally Kummer morphism $\pi \colon Y \to X$ of surfaces is called an *n-fold constantly branched covering*, if $\mathcal{B}_\pi$ is a transversal arrangement with $\mathcal{O}_X(-\mathcal{B}_\pi)_p = (x_1 \cdots x_r)$. An *$n$-fold constantly branched covering* $\pi \colon Y \to X$ of surfaces is called *regular* if $\mathcal{B}_\pi$ is a strict arrangement.

Remark 2.18. For an *n-fold constantly branched covering* $\pi \colon Y \to X$ of surfaces, we may assume that $x_i \in \mathfrak{m}_p$ if and only if $i \leqslant r$.

Notation 2.19. Whenever we use the term "constantly branched covering", we will implicitly assume that the base field k is algebraically closed and X is smooth. We make this assumption as all constantly branched coverings in this thesis have a smooth base and we are only interested in working over algebraically closed fields.

Remark 2.20. If π is an n-fold constantly branched cover of surfaces, each component of the branch locus has ramification index n. To see this, just choose a closed 1-point of $\mathcal{B}_\pi$. By assumption, the ramification is always tame.

Proposition 2.21. *[33, Fact 2.20] Let $\pi : Y \to X$ be an n-fold regular constantly branched covering of surfaces. For any closed r-fold point p of $\mathcal{B}_\pi$ and any $q \in \pi^{-1}(p)$, there exist local coordinates $x_1, x_2 \in \mathcal{O}_{X,p}$ and $y_1, y_2 \in \mathcal{O}_{Y,q}$ such that*

1. *If $r = 1$, then $\mathcal{I}(\mathcal{B}_\pi)_p = (x_1)$ and $\mathcal{I}(\mathcal{R}_\pi)_q = (y_1)$ such that $x_1 = y_1^n$ and $x_2 = y_2$.*
2. *If $r = 2$, then $\mathcal{I}(\mathcal{B}_\pi)_p = (x_1, x_2)$ and $\mathcal{I}(\mathcal{R}_\pi)_q = (y_1, y_2)$ such that $x_1 = y_1^n$ and $x_2 = y_2^n$.*

Proof. We can find a coordinate system with the desired properties around p as $\mathcal{B}_\pi$ is a strict arrangement. Let $\xi_1, \cdots, \xi_l \in \mathcal{O}_{X,p}$ be such that

$$\mathcal{O}_{Y,q} = \mathcal{O}_{X,p}[\psi_1, \cdots, \psi_l]$$

with $\psi_i^n = \xi_i$. We may assume that $\xi_i = x_i$ for $1 \leqslant i \leqslant r$. For $i > r$, we know that ξ_i is a unit. Consequently, ψ_i is also invertible for $i > r$. If $r = 1$, replacing $\mathcal{O}_{X,p}$ by $\mathcal{O}_{X,p}[\psi_2, \cdots, \psi_l]$, we may therefore assume that

$$\mathcal{O}_{Y,q} = \mathcal{O}_{X,p}[y_1],$$

where $y_1^n = x_1$. Consequently,

$$\mathfrak{m}_q = \mathfrak{m}_p.\mathcal{O}_{Y,q} + (y_1) = (y_1, x_2).$$

If $r = 2$, replacing $\mathcal{O}_{X,p}$ by $\mathcal{O}_{X,p}[\psi_3, \cdots, \psi_l]$, we may therefore assume that

$$\mathcal{O}_{Y,q} = \mathcal{O}_{X,p}[y_1, y_2],$$

where $y_1^n = x_1$ and $y_2^n = x_2$. Consequently,

$$\mathfrak{m}_q = \mathfrak{m}_p.\mathcal{O}_{Y,q} + (y_1, y_2) = (y_1, y_2).$$

□

2.1.3 Euler Characteristic

In this subsection, we state some results about the Euler characteristic which will be used in the computations carried out in Section 2.3.

We recall that a morphism $\pi : Y \to X$ between complex varieties Y and X corresponds to a morphism $\pi_h : Y_h \to X_h$ between corresponding *associated complex analytic spaces* Y_h and X_h respectively. For more details, see [30, Appendix B].

Definition 2.22. (Euler characteristic) Let X be a smooth complex variety and X_h be the associated complex analytic space. The Euler characteristic of X is defined as

$$e(X) := e(X_h),$$

where $e(X_h)$ is the topological Euler characteristic of X_h.

Proposition 2.23. *If $\pi : Y \to X$ is an unramified finite surjective morphism of degree N between smooth complex varieties, then π_h is an N- fold covering map. In particular,* $e(Y) = N.e(X)$.

Proof. This follows from [60, Corollary 6.11] and Corollary 2.11. □

Proposition 2.24. *Let X be a complex smooth variety and $Y \subseteq X$ a closed subvariety. Let $U := X \backslash Y$, then $e(X) = e(Y) + e(U)$.*

Proof. See, [22, Exercise,Page 95]. □

Intuitively, the reason for this result is that Y is a neighborhood retract of X in the classical topology. An application of Mayer-Vietoris then yields the desired result.

Lemma 2.25. *If C is a nonsingular, complex curve of genus $g(C)$, then $e(C) = 2 - 2g(C)$.*

Proof. C has a cellular decomposition with $2g(C)$ cells in dimension one and one cell in each of the dimensions zero and two, as explained in [31, Chapter 0]. Thus, we are done by [31, Theorem 2.44]. □

Proposition 2.26. *If $D = C_0 + \cdots + C_l$ is any transversal arrangement inside a nonsingular complex surface X, then*

$$e(D) = \sum_{i=0}^{l} (2 - 2g(C_i)) - \sum_{r \geqslant 2} (r-1) \cdot t_r.$$

Proof. Let $Z_i \subset C_i$ be the (finite) set of points where C_i intersects with some other part of the arrangement. Let $Z := \cup_{i=0}^{l} Z_i$ and $Z' := \sqcup_{i=0}^{l} Z_i$, where $\sqcup$ denotes the disjoint union. Clearly,

$$|Z| = \sum_{r \geqslant 2} t_r \qquad \text{and} \qquad |Z'| = \sum_{r \geqslant 2} r t_r, \tag{2.1}$$

since in the disjoint union Z', each point $p \in Z$ is counted exactly r_p times. Hence, by Proposition 2.24, we have

$$e(D) = \sum_{i=0}^{l} e(C_i \backslash Z_i) + \sum_{p \in Z} e(P) = \sum_{i=0}^{l} e(C_i) - e(Z') + e(Z).$$

Since the Euler number of a finite set is the cardinality of the set, we get the desired result by (2.1) and Lemma 2.25. □

2.1.4 Canonical Divisors

In this subsection, we study the behaviour of canonical divisors under constantly branched coverings.

For a regular constantly branched cover $\pi : Y \to X$ of complex surfaces, we are going to express K_Y in terms of the pullbacks of K_X and the branch locus $\mathcal{B}_\pi$.

Theorem 2.27. *(Ramification Formula). Let $\pi : Y \to X$ be a finite dominant morphism between smooth complex projective varieties of same dimension. Denote by K_X and K_Y canonical divisors on X and Y, respectively. Then,*

$$K_Y \sim \pi^\star(K_X) + \sum_{\mathrm{codim}_Y(Z)=1} (e_\pi(Z) - 1) \cdot Z. \tag{2.2}$$

Proof. See [38, Theorem 5.5] and [2, Lemma 16.1]. □

Corollary 2.28. *If $\pi : Y \to X$ is an n-fold regular constantly branched cover of complex surfaces, then*

$$K_Y \sim \pi^\star(K_X) + \frac{n-1}{n} \pi^\star(\mathcal{B}_\pi). \tag{2.3}$$

Proof. Since $e_\pi = n$ on components of $\mathcal{R}_\pi$ and otherwise $e_\pi = 1$, Theorem 2.27 gives

$$K_Y \sim \pi^\star(K_X) + \sum_{\mathrm{codim}_Y(Z)=1} (e_\pi(Z) - 1) \cdot Z = \pi^\star(K_X) + (n-1).\mathcal{R}_\pi.$$

Also, $\pi^\star(\mathcal{B}_\pi) = n.\mathcal{R}_\pi$ by the local description in Proposition 2.21. □

From this representation (2.3) of K_Y, K_Y^2 can be expressed by the data in X. To do this, we use the following fact (see [2, Chapter II.10, Page 67]):

Fact 2.29. Let $\pi : Y \to X$ be a proper surjective morphism of smooth complex projective surfaces of degree $\deg(\pi)$, and let D, D' be two divisors on X. Then

$$\pi^\star D \cdot \pi^\star D = (\deg(\pi)) D \cdot D'.$$

Putting it all together now yields a formula for the self-intersection number of a canonical divisor of Y.

Corollary 2.30. *Let $\pi : Y \to X$ be an n-fold regular constantly branched cover of complex surfaces. Then,*

$$K_Y^2 = \deg(\pi) \cdot \left(K_X + \frac{n-1}{n} \mathcal{B}_\pi \right)^2.$$

Proof. This follows from Corollary 2.28 and Fact 2.29. □

In the case of complex curves, we have the following formula:

Proposition 2.31. *(Hurwitz formula) Let $f : Y \to X$ be a surjective morphism of smooth complex projective curves, and let $R = \sum_{p \in X} (e_p - 1) \cdot p$ be the ramification divisor of f. Let $n = \deg(f)$. Then*

$$2g(Y) - 2 = n.(2g(X) - 2) + \sum_{p \in X} (e_p - 1).$$

Proof. We take the degree of the divisors in (2.2). Note that degree of the canonical divisor of a smooth curve of genus g is $2g - 2$ by [30, IV, Example 1.3.3]. Also, by [30, II, Proposition 6.9],

$$\deg(f^\star K_X) = \deg(f) \cdot \deg(K_X).$$

□

Proposition 2.32. *Let $\pi : Y \to X$ be an n-fold regular constantly branched cover of complex surfaces with branch locus D. Denote by $\overline{D}$ the disjoint union of its components. Then,*

$$\frac{n^2}{\deg(\pi)} \cdot K_Y^2 = n^2 (K_X^2 + K_X \cdot D + T) - 2nT + (T - K_X \cdot D),$$

where $T = 2t_2 - e(\overline{D})$.

Proof. Let $D = C_0 \cup \cdots \cup C_l$ be the irreducible decomposition of D. By the adjunction formula and Lemma 2.25,

$$-e(C_i) = C_i^2 + C_i \cdot K_X,$$

for each i. Also, $2t_2 = \sum_{i=0}^{l} \sum_{i \neq j} C_i \cdot C_j$. Thus, we conclude

$$\begin{aligned} T = 2t_2 - \sum_{i=0}^{l} e(C_i) = \sum_{i=0}^{l} \left(\sum_{j \neq i} C_i \cdot C_j + C_i^2 + C_i \cdot K_X \right) \\ = D^2 + K_X \cdot D. \end{aligned}$$

By Corollary 2.30,

$$\frac{K_Y^2}{\deg(\pi)} = \left(K_X + \frac{n-1}{n} D \right)^2 = K_X^2 + \left(\frac{2n-2}{n} \right) K_X \cdot D + \left(\frac{n^2+1-2n}{n^2} \right) D^2.$$

Thus, we obtain

$$\frac{n^2}{\deg(\pi)} \cdot K_Y^2 = n^2 K_X^2 + 2(n^2 - n) K_X \cdot D + (n^2 + 1 - 2n)(T - K_X \cdot D).$$

□

2.1.5 Regularization

In Section 2.3, we will construct a constantly branched abelian covering $\pi : Z \to X$ of the ruled surface X, branched along an arrangement $\mathcal{C}$ satisfying Assumption 2.34. Such an arrangement $\mathcal{C}$ will have redundant intersections. Hence, the covering will not be regular. But we can transform such a covering into a regular one by resolving the singularities of Z. This is always possible by blowing up the redundant intersections and their preimages.

The following theorem shows that any n-fold constantly branched cover of a complex surface can be desingularized by a sequence of blow ups.

Theorem 2.33. *(Regularization) Let $\pi : Y \to X$ be an n-fold constantly branched cover of complex surfaces. Then there exists a commutative diagram (see Figure 2.1) such that the following properties hold:*

1. *Each π_i is an n-fold constantly branched cover with branch locus $\mathcal{B}_{\pi_i} = \alpha_i^{-1}(\mathcal{B}_{\pi_{i-1}})$.*

2. *Each α_{i+1} is the blow up of X_i along a redundant intersection p_i of $\mathcal{B}_{\pi_i}$ and β_{i+1} is the blow up along $\pi_i^{-1}(p_i)$.*

3. *The branch locus of $\tilde{\pi}$ is a strict arrangement.*

4. *Let $\beta := \beta_1 \circ \cdots \circ \beta_m$. The morphism β is a resolution of singularities, i.e., $\widetilde{Y}$ is a nonsingular complex surface and β is an isomorphism outside the singular locus of Y.*

Thus, $\tilde{\pi}$ is an n-fold regular constantly branched cover. We call $\tilde{\pi}$ a regularization of π.

$$
\begin{array}{ccccccccccccc}
\widetilde{Y} & = & Y_m & \xrightarrow{\beta_m} & Y_{m-1} & \xrightarrow{\beta_{m-1}} & \cdots & \xrightarrow{\beta_2} & Y_1 & \xrightarrow{\beta_1} & Y_0 & = & Y \\
\downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi_m} & & \downarrow{\scriptstyle \pi_{m-1}} & & \vdots & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi} \\
\widetilde{X} & = & X_m & \xrightarrow{\alpha_m} & X_{m-1} & \xrightarrow{\alpha_{m-1}} & \cdots & \xrightarrow{\alpha_2} & X_1 & \xrightarrow{\alpha_1} & X_0 & = & X
\end{array}
$$

FIGURE 2.1: Construction of the surface $\widetilde{Y}$

Proof. See [33, Theorem 2.35]. □

2.2 Combinatorics of transversal curve arrangements on ruled surfaces

In this section, we introduce the curve arrangements that we study on ruled surfaces and include some well-known combinatorial properties of these curve arrangements.

Assumption 2.34. Let X be a ruled surface over a smooth curve of genus $g \geqslant 0$ with invariant $e \geqslant 4$. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$ be a transversal arrangement of curves on X with $d \geqslant 4$ and $t_d = 0$. Suppose that all the curves C_i in $\mathcal{C}$ are linearly equivalent to a fixed divisor A on X, where A is numerically equivalent to $aC_0 + bf$, for $a, b \in \mathbb{Z}$ with $a > 0$ and $b \geqslant ae$. Note that under these assumptions, $C_i \cdot C_j = 2ab - a^2e$ for all curves $C_i, C_j \in \mathcal{C}$.

Lemma 2.35. *Let $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$ be a transversal arrangement of curves on a ruled surface X satisfying Assumption 2.34. Then we have the following.*

1. *For every curve $C_i \in \mathcal{C}$, we have $\sum_{p \in C_i}(r_p - 1) = (2ab - a^2e)(d-1)$.*

2. $f_2 - f_1 = \sum_{k \geqslant 2} k(k-1)t_k = (2ab - a^2e)d(d-1)$.

Proof. First we prove (1). Given a multiple point $p \in C_i$, $r_p - 1$ is the number of curves of the arrangement passing through p different from C_i. As every curve meets every other curve in $2ab - a^2e$ distinct points, the expression $\sum_{p \in C_i}(r_p - 1)$ counts all curves of the arrangement different from C_i, $2ab - a^2e$ times each. So (1) holds.

The first equality in (2) follows from the definition of f_2, f_1. As $\sum_{C_i \in \mathcal{C}} \sum_{p \in C_i}(r_p - 1) = \sum_{k \geqslant 2} k(k-1)t_k$, the second equality in (2) follows from (1). $\square$

2.3 Construction of the abelian cover

Our arguments follow the model developed by Hirzebruch in [35]. These ideas have been used by several recent authors. See [18, 50, 54–56], for example.

Let X be a ruled surface over a smooth curve C of genus g. Let $\mathcal{C} = \{C_1, \dots, C_d\}$ be a transversal arrangement of curves on X satisfying Assumption 2.34. Our goal is to give bounds for the Harbourne constant $H(\mathcal{C})$. The starting point is to consider a branched covering of X branched along the curves in $\mathcal{C}$. In order to prove that such a branched covering does in fact exist for the ruled surface X, we use a result of Namba, which we recall below.

As above, let $D = \sum_{i=1}^{d} C_i$. Let $\mathrm{Div}(X, D)$ be the subgroup of the $\mathbb{Q}$-divisors on X generated by all the integral divisors and the following $\mathbb{Q}$-divisors: $\frac{C_1}{2}, \frac{C_2}{2}, \dots, \frac{C_d}{2}$.

Let $\sim$ be linear equivalence in $\mathrm{Div}(X, D)$, where $G \sim G'$ if and only if $G - G'$ is an integral principal divisor. Let $\mathrm{Div}^0(X, D)/\sim$ denote the kernel of the first Chern class map:

$$\begin{array}{ccc} \mathrm{Div}(X, D)/\sim & \to & H^{1,1}(X, \mathbb{R}) \\ G & \mapsto & c_1(G) \end{array}$$

We use the following result of Namba [48, Theorem 2.3.20]. In our special case, it says the following.

Theorem 2.36 (Namba). *There exists a finite abelian cover $Z \to X$ with branch locus equal to D and ramification index 2 at each C_i if and only if for every $j = 1, \dots, d$, there exists an* *element of finite order* $\nu_j = \sum \frac{a_{ij}}{2} C_i + E_j$ *of* $\mathrm{Div}^0(X, D)/\sim$, *where* E_j *are integral divisors and* $a_{jj} \in \mathbb{Z}$ *is odd for every* $j = 1, \ldots, d$.

In this case, the subgroup of $\mathrm{Div}^0(X, D)/\sim$ *generated by the* ν_j *is isomorphic to the Galois group of the abelian cover* $Z \to X$.

Set $\nu_1 = \nu_2 = \frac{C_1 - C_2}{2}$ and $\nu_j = \frac{C_1 - C_j}{2}$ for $j = 3, \ldots, d$ and $E_j = 0$ for every j. Then, by Theorem 2.36, there exists an abelian cover $\pi : Z \to X$ ramified over $\mathcal{C}$ with ramification index 2. The Galois group G of π is generated by $\nu_1 = \nu_2, \nu_3, \ldots, \nu_d$ and no proper subset of $\{\nu_2, \ldots, \nu_d\}$ generates G. Note that every element of G has order 2. So the Galois group of π is $(\mathbb{Z}/2\mathbb{Z})^{d-1}$. Note that π is a 2-fold constantly branched cover with branch locus $\mathcal{C}$. We denote by $\rho : Y \to Z$ the minimal desingularization of Z.

For a singular point p of $\mathcal{C}$, recall that r_p denotes its multiplicity. Let $\tau : \widetilde{X} \to X$ be the blow up of X at the $f_0 - t_2 = \sum_{k \geqslant 3} t_k$ singular points of $\mathcal{C}$ with multiplicities $k \geqslant 3$. Let $\widetilde{D} = \sum_{i=1}^{d} \widetilde{C_i}$ be the strict transform of D in $\widetilde{X}$ and let $E_p := \tau^{-1}(p)$ be the exceptional divisor over the point p.

Note that the singular locus of Z is precisely the pre-image, under π, of the singular points of $\mathcal{C}$ of multiplicity at least 3 (see [49, Proposition 3.1], for example). Since τ is defined to be the blow up of the singular points of $\mathcal{C}$ of multiplicity at least 3, there exists a morphism $\sigma : Y \to \widetilde{X}$, by the universal property of blow ups. (See Theorem 2.33.) See the commutative diagram in Figure 2.2.

From the commutativity of the diagram, it is easy to see that σ is a regular constantly branched abelian cover with Galois group $(\mathbb{Z}/2\mathbb{Z})^{d-1}$, branch divisor $\widetilde{D}$ and ramification index 2 at every irreducible component of $\widetilde{D}$. Then $\sigma^{\star} E_p$ is a divisor in Y consisting of 2^{d-1-r_p} disjoint curves F_p, each with multiplicity 2. See [34, II.3.2] for more details. For a point $x \in E_p$ which is not in the branch locus of σ, $\sigma^{-1}(x)$ consists of 2^{d-1} distinct points and these are contained in the 2^{d-1-r_p} disjoint curves F_p. Since each F_p occurs with multiplicity 2 in $\sigma^{\star} E_p$, the number of elements in a single F_p that map to x is $\frac{2^{d-1}}{2(2^{d-1-r_p})} = 2^{r_p - 1}$. So each F_p is a finite cover of E_p of degree 2^{r_p-1}. The branch locus of the map $F_p \to E_p$ is precisely the r_p intersection points of E_p and $\widetilde{D}$. Since the ramification index is 2 and the degree of the map $F_p \to E_p$ is 2^{r_p-1}, there are $\frac{2^{r_p-1}}{2} = 2^{r_p-2}$ points in F_p that map to any point in the branch locus. Hence the degree of the ramification divisor is $2^{r_p-2} r_p$.

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & Z \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widetilde{X} & \xrightarrow{\tau} & X \end{array}$$

FIGURE 2.2: Construction of the surface Y

By the above discussion, we have $\sigma^\star E_p = \sum 2F_p$ with 2^{d-1-r_p} terms in the summand. So

$$-2^{d-1} = 2^{d-1}(E_p)^2 = (\sigma^\star E_p)^2 = 4(2^{d-1-r_p})F_p^2,$$

which implies that $F_p^2 = -2^{r_p-2}$ for every point $p \in \mathrm{Sing}(\mathcal{C})$ with $r_p \geqslant 3$. If a singularity p of D is a double point, then Y is smooth over p and the fiber of $\tau \circ \sigma$ above p has 2^{d-3} points.

Using the Hurwitz formula (see Proposition 2.31) to compute the Euler characteristic of F_p, we get

$$e(F_p) = 2 - 2g(F_p) = 2^{r_p-1}(2) - 2^{r_p-2}r_p = 2^{r_p-2}(4 - r_p). \tag{2.4}$$

We will calculate the Chern numbers c_2, c_1^2 of Y, where c_2 is same as the Euler characteristic $e(Y)$ of Y and c_1^2 is the self-intersection number of a canonical divisor of Y.

Note that

$$Y \setminus \bigcup_{p, r_p \geqslant 3} \sigma^{-1}E_p = (\tau \circ \sigma)^{-1}\left((X \setminus \mathcal{C}) \cup (\mathcal{C} \setminus \mathrm{Sing}(\mathcal{C})) \cup \{p \in \mathrm{Sing}(\mathcal{C}) | r_p = 2\}\right).$$

If $A \to B$ is an étale map of degree n, then $e(A) = ne(B)$ (see Proposition 2.23). Since σ is an étale map on $Y \setminus \bigcup_{p, r_p \geqslant 3} \sigma^{-1}E_p$, using Proposition 2.24 and Corollary 2.11, we get

$$e\left(Y \setminus \bigcup_{p, r_p \geqslant 3} \sigma^{-1}E_p\right) = 2^{d-1}e(X \setminus \mathcal{C}) + 2^{d-2}e(\mathcal{C} \setminus \mathrm{Sing}(\mathcal{C})) + 2^{d-3}t_2. \tag{2.5}$$

By Proposition 2.26, we have

$$e(\mathcal{C}) = 2\sum(1 - g(C_i)) - \sum_{k \geqslant 2}(k-1)t_k.$$

Using the additivity of the topological Euler characteristic (see Proposition 2.24), we have the following:

$$e(\mathcal{C} \setminus \mathrm{Sing}(\mathcal{C})) = 2\sum(1-g(C_i)) - \sum_{k\geqslant 2} kt_k,$$

$$e(X \setminus \mathcal{C}) = e(X) + 2\sum(g(C_i)-1) + \sum_{k\geqslant 2}(k-1)t_k.$$

Substituting these values in (2.5), we have

$$\begin{aligned} e\left(Y \setminus \bigcup_{p, r_p \geqslant 3} \sigma^{-1}E_p\right) &= 2^{d-1}\left(e(X) + 2\sum(g(C_i)-1) + \sum_{k\geqslant 2}(k-1)t_k\right) + \\ &\quad 2^{d-2}\left(-2\sum(g(C_i)-1) - \sum_{k\geqslant 2} kt_k\right) + 2^{d-3}t_2. \end{aligned}$$

It is easy to check that

$$e(X) = 4-4g \text{ and } 2g(C_i)-2 = -a^2e + 2ab + ae + a(2g-2) - 2b.$$

Note also that $\sum_{k\geqslant 2}(k-1)t_k = f_1 - f_0$.

So we get

$$\begin{aligned} e\left(Y \setminus \bigcup_{p, r_p \geqslant 3} \sigma^{-1}E_p\right) &= 2^{d-1}\left(4-4g+d(-a^2e+2ab+ae+a(2g-2)-2b)+f_1-f_0\right) + \\ &\quad 2^{d-2}\left(-d(-a^2e+2ab+ae+a(2g-2)-2b)-f_1\right) + 2^{d-3}t_2. \end{aligned}$$

There are 2^{d-1-r_p} curves with Euler characteristic $e(F_p)$ in Y over each exceptional divisor E_p in $\widetilde{X}$. So (2.4) gives

$$\begin{aligned} e(Y) &= e\left(Y \setminus \bigcup_{p, r_p \geqslant 3} \sigma^{-1} E_p\right) + \sum_{k \geqslant 3} 2^{d-1-k} t_k e(F_p) \\ &= e\left(Y \setminus \bigcup_{p, r_p \geqslant 3} \sigma^{-1} E_p\right) + \sum_{k \geqslant 3} 2^{d-1-k} t_k \left(2^{k-1}(2-k) + k 2^{k-2}\right) \\ &= e\left(Y \setminus \bigcup_{p, r_p \geqslant 3} \sigma^{-1} E_p\right) + 2^{d-3} \sum_{k \geqslant 3} t_k (4-k) \\ &= e\left(Y \setminus \bigcup_{p, r_p \geqslant 3} \sigma^{-1} E_p\right) + 2^{d-3}(4 f_0 - f_1 - 2 t_2) \end{aligned}$$

Now using the value of $e\left(Y \setminus \bigcup_{p, r_p \geqslant 3} \sigma^{-1} E_p\right)$ computed above and simplifying, we get

$$\frac{1}{2^{d-3}} e(Y) = 16 - 16g + d(-2a^2 e + 4ab + 2ae + 4ag - 4a - 4b) + f_1 - t_2. \tag{2.6}$$

Next we calculate $c_1^2(Y)$. By canonical divisor on a surface, we mean the class of the canonical divisor in the Neron-Severi group of that surface.

For the divisor $D = \sum_{i=1}^d C_i$ on X, we know that $\tau^\star D - \sum_{\substack{p \in \mathrm{Sing}(\mathcal{C}), \\ r_p \geqslant 3}} r_p E_p$ is the strict transform of D in $\widetilde{X}$. The divisors $\sigma^\star(\tau^\star D - \sum r_p E_p)$ and $\sigma^\star E_p$ of $Y (p \in \mathrm{Sing}(\mathcal{C}), r_p \geqslant 3)$ are divisible by 2. For a canonical divisor K_X of X, $\tau^\star K_X + \sum E_p$ is a canonical divisor of $\widetilde{X}$. Applying Corollary 2.28 or [2, Page 42, Lemma 17.1] to the ramified covering $\sigma : Y \to \widetilde{X}$, we get the following:

Lemma 2.37. *Let Y be the surface constructed in Figure 2.2. The canonical divisor of Y is given by $K_Y = \sigma^\star T$ for the $\mathbb{Q}$-divisor T on $\widetilde{X}$ defined as*

$$T := \tau^\star K_X + \sum E_p + \frac{1}{2}\left(\sum E_p + \tau^* D - \sum r_p E_p\right),$$

where the summations are taken over all the points $p \in \mathrm{Sing}(\mathcal{C})$ such that $r_p \geqslant 3$.

Thus, $T^2 = K_X^2 + K_X \cdot D - \sum_{k \geqslant 3} t_k + \sum_{k \geqslant 3} (k-1) t_k + \frac{1}{4}(D^2 - \sum_{k \geqslant 3} (k-1)^2 t_k)$.

We have the following:

$K_X^2 = 8(1-g)$,

$K_X \cdot D = d\,(ae + a(2g-2) - 2b)$,

$\sum_{k\geqslant 3} t_k = f_0 - t_2$, $\sum_{k\geqslant 3}(k-1)t_k = f_1 - f_0 - t_2$, and

$D^2 - \sum_{k\geqslant 3}(k-1)^2 t_k = d(-a^2e + 2ab) + f_1 - f_0 + t_2$. For this equality, use Lemma 2.35(2).

Substituting these values in the expression for T^2 and noting that $c_1^2(Y) = 2^{d-1}T^2$, we get:

$$\frac{1}{2^{d-3}} c_1^2(Y) = 32 - 32g + d(-a^2e + 2ab + 4ae + 8ag - 8a - 8b) - 9f_0 + 5f_1 + t_2. \quad (2.7)$$

Now we have, by (2.6) and (2.7),

$$\frac{1}{2^{d-3}}(3e(Y) - c_1^2(Y)) = 16 - 16g + d[(2b - ae)(5a-2) + 4a(g-1)] + 9f_0 - 2f_1 - 4t_2. \quad (2.8)$$

Remark 2.38. By (2.4), F_p is rational if and only if $r_p = 3$ and F_p is elliptic if and only if $r_p = 4$. Thus we know that Y contains $2^{d-4}t_3$ disjoint (-2)-curves (above the 3-points) and contains $2^{d-5}t_4$ elliptic curves (above the 4-points), each of self-intersection -4.

2.4 Harbourne Constants

In this section, we will first show that the surface Y (constructed in the last section; see Figure 2.2) has non-negative Kodaira dimension. This will allow us to apply a Hirzebruch-Miyaoka-Sakai inequality involving the Chern numbers of Y and certain curves on Y coming from the arrangement $\mathcal{C}$ on X (see Theorem 2.44). Using this we obtain a Hirzebruch-type inequality (2.17). We prove our bound for the Harbourne constant of $\mathcal{C}$ in Theorem 2.45.

We will use the notation of Section 2.3. Recall that T is a $\mathbb{Q}$-divisor on $\widetilde{X}$ defined in Lemma 2.37. We start with the following.

Lemma 2.39. *Let X be a ruled surface with $e \geqslant 4$. Let $\mathcal{C}$ be a transversal arrangement of curves satisfying Assumption 2.34. Then $T \cdot E_p \geqslant 0$ for every $p \in Sing(\mathcal{C})$ such that $r_p \geqslant 3$.*

Proof. $T \cdot E_p = -1 + \frac{-1+r_p}{2} \geqslant -1 + \frac{-1+3}{2} = 0$. □

Lemma 2.40. *Let X be a ruled surface with $e \geqslant 4$. Let $\mathcal{C}$ be a transversal arrangement of curves satisfying Assumption 2.34. Let $C_j' = \tau^* C_j - \sum_{p \in C_j, r_p \geqslant 3} E_p$ be the strict transform of $C_j \in \mathcal{C}$, for $j = 1, 2, \ldots, d$. Then $T \cdot C_j' \geqslant 0$.*

Proof. Let f_0^j denote the number of multiple points on C_j and let t_k^j denote the number of k-fold points on C_j.

Now,

$$T \cdot C_j' = K_X \cdot C_j + \frac{D \cdot C_j}{2} - T \cdot \sum_{p \in C_j, r_p \geqslant 3} E_p. \tag{2.9}$$

We now compute each of the terms individually.

$$\begin{aligned} K_X \cdot C_j &= 2ae - 2b + (2g - 2 - e)a, \\ D \cdot C_j &= d(2ab - a^2 e), \\ T \cdot E_p &= \frac{r_p - 3}{2}; \quad p \in C_j, r_p \geqslant 3. \end{aligned}$$

By Lemma 2.35 (1), we have

$$\begin{aligned} T \cdot \sum_{p \in C_j, r_p \geqslant 3} E_p &= \sum_{p \in C_j, r_p \geqslant 3} \frac{r_p - 3}{2} \\ &= \sum_{p \in C_j, r_p \geqslant 2} \frac{r_p - 1}{2} - f_0^j + \frac{t_2^j}{2} \\ &= \frac{(2ab - a^2 e)(d-1)}{2} - f_0^j + \frac{t_2^j}{2}. \end{aligned}$$

Plugging the values computed above in (2.9), we get

$$T \cdot C_j' = 2ae - 2b + (2g - e - 2)a + \frac{2ab - a^2 e}{2} + f_0^j - \frac{t_2^j}{2}. \tag{2.10}$$

To prove the lemma, it suffices to show

$$f_0^j - \frac{t_2^j}{2} \geqslant -\left(\frac{2ab - a^2 e}{2}\right) - 2ae + a(e+2) + 2b. \tag{2.11}$$

Let k be the maximum of the multiplicities of the points on C_j. By Lemma 2.35 (1), we have

$$t_2^j + 2t_3^j + \ldots + (k-1)t_k^j = (2ab - a^2e)(d-1).$$

Now,

$$\begin{aligned} f_0^j - \frac{t_2^j}{2} &= \frac{t_2^j}{2} + t_3^j + \ldots + t_k^j \\ &\geqslant \frac{t_2^j + 2t_3^j + \ldots + (k-1)t_k^j}{k} = \frac{(2ab - a^2e)(d-1)}{k} \\ &\geqslant 2ab - a^2e, \end{aligned}$$

where last inequality holds since $k \leqslant d-1$.

Thus in order to show (2.11), it suffices to show the following inequality:

$$2ab - a^2e \geqslant -\left(\frac{2ab - a^2e}{2}\right) - 2ae + a(e+2) + 2b. \tag{2.12}$$

Now we have the following:

$$\begin{aligned} (2.12) \quad &\Leftrightarrow \quad 6ab - 4a - 4b \geqslant 3a^2e - 2ae \\ &\Leftarrow \quad b \geqslant \frac{4a}{3a-2} \\ &\Leftarrow \quad ae \geqslant \frac{4a}{3a-2} \\ &\Leftarrow \quad e \geqslant 4. \end{aligned}$$

The last inequality holds by Assumption 2.34. □

We now make a further assumption on our arrangement $\mathcal{C}$. This is required for our argument showing that K_Y is nef.

Assumption 2.41. Let X be a ruled surface over a smooth curve with $e \geqslant 4$. Let $\mathcal{C}$ be a transversal arrangement of curves on a ruled surface X satisfying Assumption 2.34. Assume further that $\mathcal{C}$ satisfies one of the following conditions:

1. $a \geqslant 2$, or

2. $a = 1$ and there exists a subset of four curves in $\mathcal{C}$ such that there is no point common to all the four curves.

Question 2.42. *We do not know any example of a transversal arrangement for which Assumption 2.41 does not hold. Does this assumption always hold for any arrangement satisfying Assumption 2.34?*

Theorem 2.43. *Let X be a ruled surface with $e \geqslant 4$ and let $\mathcal{C}$ be a transversal arrangement of curves satisfying Assumption 2.41. Let Y be the surface constructed in Figure 2.2. Then K_Y is nef.*

Proof. Recall (see Lemma 2.37) that T is a divisor on $\widetilde{X}$ given by

$$T := \tau^\star K_X + \frac{3}{2}\sum_{r_p \geqslant 3} E_p + \frac{1}{2}\sum C_i', \tag{2.13}$$

where C_i' is the strict transform of C_i by τ and $E_p = \tau^{-1}(p)$. Note that $K_Y = \sigma^\star T$. We have $\tau^\star C_i = C_i' + \sum_{p \in C_i, r_p \geqslant 3} E_p$.

We want to express T as a positive sum of effective divisors on $\widetilde{X}$. The negative terms in the expression occur because of the term involving $K_X = -2C_0 + (2g-2-e)f$. We consider two different cases.

Case (1): Assume $a \geqslant 2$. Let $C_1, C_2 \in \mathcal{C}$.

For $q := a-2 \geqslant 0, p := 2g-e-2+b \geqslant 0$, we have $K_X = pf + qC_0 - \frac{C_1+C_2}{2}$. Note that $p > 0$, since $b \geqslant ae$ and $e \geqslant 4$.

Thus, (2.13) becomes,

$$\begin{aligned} T &= \tau^\star(pf+qC_0) - \frac{1}{2}\left(C_1' + \sum_{p \in C_1, r_p \geqslant 3} E_p + C_2' + \sum_{p \in C_2, r_p \geqslant 3} E_p\right) + \frac{3}{2}\sum_{r_p \geqslant 3} E_p + \frac{1}{2}\sum_{i=1}^{d} C_i' \\ &= \tau^\star(pf+qC_0) + \frac{1}{2}\sum_{i=3}^{d} C_i' + \sum \lambda_p E_p, \text{ for some } \lambda_p. \end{aligned}$$

Note that λ_p is non-negative for every point $p \in \text{Sing}(\mathcal{C})$ with $r_p \geqslant 3$. Indeed, $\lambda_p = \frac{3}{2}$ if $p \notin C_1 \cup C_2$; $\lambda_p = 1$ if p belongs to exactly one of the curves C_1 or C_2; and $\lambda_p = \frac{1}{2}$ if $p \in C_1 \cap C_2$. Thus T is effective and we have

$$K_Y = \sigma^\star T = \sigma^\star\tau^\star(pf+qC_0) + \sigma^\star\left(\frac{1}{2}\sum_{i=3}^{d} C_i'\right) + \sigma^\star\left(\sum \lambda_p E_p\right).$$

If C is a curve in Y not contained in $\sigma^\star E_p$ for all p and not contained in $\sigma^\star C_i'$ for all i,

$$K_Y \cdot C = \sigma^\star\tau^\star(pf + qC_0) \cdot C + \sigma^\star \left(\frac{1}{2} \sum_{i=3}^{d} C_i' \right) \cdot C + \sigma^\star \left(\sum \lambda_p E_p \right) \cdot C \geqslant 0.$$

If C is a curve in Y such that C is either contained in $\sigma^\star C_i'$ for some i or contained in $\sigma^\star E_p$ for some p, Lemma 2.39 and Lemma 2.40 imply that $K_Y \cdot C \geqslant 0$. Thus $K_Y \cdot C \geqslant 0$ for every curve C in Y. Hence, K_Y is nef.

Case (2): Suppose that $a = 1$. By Assumption 2.41, there are four curves, say C_1, C_2, C_3, C_4, in $\mathcal{C}$ such that no point is contained in all the four curves.

Let $p := 2g - 2 - e + 2b > 0$. Then $K_X = pf - \frac{C_1+C_2+C_3+C_4}{2}$.

Thus,

$$\begin{aligned}
T &= \tau^\star(pf) - \frac{1}{2}\left(\sum_{i=1}^{4} C_i' + \sum_{p \in C_i, r_p \geqslant 3} E_p \right) + \frac{3}{2} \sum_{r_p \geqslant 3} E_p + \frac{1}{2} \sum_{i=1}^{d} C_i'. \\
&= \tau^\star(pf) - \frac{1}{2}\left(\sum_{p \in C_i, r_p \geqslant 3} E_p \right) + \frac{3}{2} \sum_{r_p \geqslant 3} E_p + \frac{1}{2} \sum_{i=5}^{d} C_i'. \\
&= \tau^\star(pf) + \frac{1}{2} \sum_{i=5}^{d} C_i' + \sum \lambda_p' E_p, \text{ for some } \lambda_p'.
\end{aligned}$$

We have $\lambda_p' = \frac{3}{2}$ if $p \notin C_1 \cup C_2 \cup C_3 \cup C_4$. By Assumption 2.41 and the choice of C_1, C_2, C_3, C_4, there are no points in the intersection $C_1 \cap C_2 \cap C_3 \cap C_4$. If p belongs to three of them, then $\lambda_p' = \frac{3}{2} - \frac{3}{2} = 0$. So we have $\lambda_p' \geqslant 0$ for all $p \in \mathrm{Sing}(\mathcal{C})$ with $r_p \geqslant 3$. Thus T is effective and we have

$$K_Y = \sigma^\star\tau^\star(pf) + \frac{1}{2}\sigma^\star \left(\sum_{i=5}^{d} C_i' \right) + \sigma^\star \left(\sum \lambda_p' E_p \right).$$

If C is a curve in Y not contained in $\sigma^\star E_p$ for all p and not contained in $\sigma^\star C_i'$ for all i,

$$K_Y \cdot C = \sigma^\star\tau^\star(pf) \cdot C + \sigma^\star \left(\frac{1}{2} \sum_{i=5}^{d} C_i' \right) \cdot C + \sigma^\star \left(\sum \lambda_p' E_p \right) \cdot C \geqslant 0.$$

If C is a curve in Y such that C is either contained in $\sigma^\star C_i'$ for some i or contained in $\sigma^\star E_p$ for some p, Lemma 2.39 and Lemma 2.40 imply that $K_Y \cdot C \geqslant 0$. Thus $K_Y \cdot C \geqslant 0$ for every curve C in Y. Hence, K_Y is nef. □

The following result of Hirzebruch [37, Theorem 3, Page 144] is crucial in our computations. It strengthens earlier results of Miyaoka and Sakai.

Theorem 2.44 (Hirzebruch). *Let X be a smooth surface of non-negative Kodaira dimension and $E_1, \ldots, E_k$ configurations (disjoint to each other) of rational curves on X (arising from quotient singularities) and $C_1, \ldots, C_p$ smooth elliptic curves (disjoint to each other and disjoint to the E_i). Let $c_1^2(X), c_2(X)$ be the Chern numbers of X. Then*

$$3c_2(X) - c_1^2(X) \geqslant \sum_{j=1}^{p} (-C_j^2) + \sum_{i=1}^{k} m(E_i).$$

Hirzebruch proved Theorem 2.44 under the assumption that X is of general type and remarks that the theorem also holds when X has non-negative Kodaira dimension. We use the theorem in this case.

The numbers $m(E_i)$ mentioned in the theorem are positive numbers defined using certain invariants (Euler characteristics, self-intersections) of the arrangements E_i. Hirzebruch gives a formula to compute them in [37, Page 144, (5)] which shows that if E_i is a single (-2)-curve, then $m(E_i) = \frac{9}{2}$. See also [32].

Now we are ready to prove the main result of this thesis.

Theorem 2.45. *Let X be a ruled surface with $e \geqslant 4$ over a smooth curve of genus g. Let $\mathcal{C}$ be a transversal arrangement of curves satisfying Assumption 2.41. In particular, each curve in $\mathcal{C}$ is numerically equivalent to $aC_0 + bf$ with $a > 0$ and $b \geqslant ae$. Then we have the following bound on the Harbourne constant of $\mathcal{C}$:*

$$H(\mathcal{C}) \geqslant \frac{-9}{2} - \frac{8}{f_0} + \frac{d}{f_0}\left(\frac{(ae-2b)}{2}(3a-2) - 2a(g-1)\right) + \frac{16g + 4t_2 + t_4}{2f_0} + \frac{9t_3}{8f_0}. \quad (2.14)$$

Proof. By Remark 2.38, the surface Y (constructed in Figure 2.2) contains $2^{d-4}t_3$ disjoint rational (-2)-curves E_i (above the 3-points) and contains $2^{d-5}t_4$ elliptic curves C_j (above the 4-points), each of self-intersection -4.

By Theorem 2.43, K_Y is nef. Thus, by Theorem 2.44:

$$\frac{3c_2(Y) - c_1^2(Y)}{2^{d-3}} \geqslant \frac{\sum(-C_j^2) + \sum m(E_i)}{2^{d-3}}. \tag{2.15}$$

As noted earlier, $m(E_i) = \frac{9}{2}$ for all rational curves E_i of self-intersection -2.

From (2.8), we have,

$$\frac{1}{2^{d-3}}(3e(Y) - c_1^2(Y)) = 16 - 16g + d[(2b - ae)(5a - 2) + 4a(g-1)] + 9f_0 - 2f_1 - 4t_2.$$

Also, from our discussion above,we have

$$\sum m(E_i) = \frac{9}{2}2^{d-4}t_3, \text{ and}$$
$$\sum(-C_j^2) = 4t_4 2^{d-5}.$$

Plugging these values in (2.15) and simplifying, we have :

$$16 - 16g + d(2ae - 5a^2e + 10ab + 4ag - 4a - 4b) + 9f_0 - 2f_1 - 4t_2 - t_4 - \frac{9}{4}t_3 \geqslant 0. \tag{2.16}$$

Simplifying and re-arranging (2.16), we obtain the following Hirzebruch-type inequality for $\mathcal{C}$:

$$t_2 + \frac{3}{4}t_3 \geqslant -16 + 16g + \sum_{k \geqslant 5}(2k-9)t_k + d(e(5a^2 - 2a) - 10ab - 4ag + 4a + 4b). \tag{2.17}$$

Now we bound $H(\mathcal{C})$. We have

$$H(\mathcal{C}) = \frac{(2ab - a^2e)d^2 - \sum_{k\geqslant 2} k^2 t_k}{f_0} = \frac{(2ab - a^2e)d^2 - f_2}{f_0} = \frac{(2ab - a^2e)d - f_1}{f_0},$$

where the last equality follows from Lemma 2.35(2).

From (2.16), we have

$$-f_1 \geqslant \frac{-16 + 16g + d\left(e(5a^2 - 2a) - 10ab - 4ag + 4a + 4b\right) - 9f_0 + 4t_2 + \frac{9}{4}t_3 + t_4}{2}.$$

Thus,

$$H(\mathcal{C}) \geqslant \frac{d(-a^2e+2ab)-8+8g+\frac{d(e(5a^2-2a)-10ab-4ag+4a+4b)-9f_0}{2}+2t_2+\frac{9}{8}t_3+\frac{t_4}{2}}{f_0}$$
$$= \frac{-9}{2} - \frac{8}{f_0} + \frac{d}{f_0}\left(\frac{ae}{2}(3a-2)-2ag-3ab+2a+2b\right) + \frac{16g+4t_2+t_4}{2f_0} + \frac{9t_3}{8f_0}.$$

This completes the proof of the theorem. □

If the curves in the arrangement $\mathcal{C}$ do not intersect the normalized section C_0, then we obtain an improved bound for the Harbourne constants as shown in the following proposition. We obtain an improved bound in this case because Y contains some additional rational curves.

Proposition 2.46. *Let X be a ruled surface with $e \geqslant 4$ over a smooth curve of genus g. Let $\mathcal{C}$ be a transversal arrangement of curves satisfying Assumption 2.41. Assume further that no curve in $\mathcal{C}$ intersects the normalized section C_0. Then we have the following bound on the Harbourne constant of $\mathcal{C}$:*

$$H(\mathcal{C}) \geqslant \frac{-9}{2} + \frac{d}{f_0}\left(\frac{ae(2-3a)-4a(g-1)}{2}\right) + \frac{16g+4t_2+t_4}{2f_0} + \frac{9t_3}{8f_0}. \tag{2.18}$$

Proof. As in the previous theorem, by Remark 2.38, the surface Y contains $2^{d-4}t_3$ disjoint rational (-2)-curves E_i (above the 3-points), $2^{d-5}t_4$ elliptic curves C_j (above the 4-points), each of self-intersection -4. Further, since the curves in the arrangement do not intersect C_0, the surface $\tilde{X}$ has an isomorphic copy of C_0. Hence Y contains 2^{d-1} copies of a rational curve H of self-intersection $-e$.

Hirzebruch gives a formula to compute the value $m(H)$ in [37, Page 144, (4)]. Applying this formula, we have that for rational curves H of self-intersection $-e$, $m(H) = 2 + e + \frac{1}{e}$.

By Theorem 2.43, K_Y is nef. Thus, by Theorem 2.44, the inequality in (2.15) is satisfied.

From (2.8), we have,

$$\frac{1}{2^{d-3}}(3e(Y) - c_1^2(Y)) = 16 - 16g + d[(2b-ae)(5a-2)+4a(g-1)] + 9f_0 - 2f_1 - 4t_2.$$

We have

$$\sum m(E_i) + \sum m(H) = \frac{9}{2}2^{d-4}t_3 + 2^{d-1}(2+e+\frac{1}{e}), \text{ and}$$
$$\sum(-C_j^2) = 4t_4 2^{d-5}.$$

Plugging these values in (2.15) and simplifying, we have:

$$16-16g+d(2ae-5a^2e+10ab+4ag-4a-4b)+9f_0-2f_1-4t_2-t_4-\frac{9}{4}t_3-4(2+e+\frac{1}{e}) \geqslant 0. \tag{2.19}$$

Simplifying (2.19), with $ae = b$, we arrive at the following modified Hirzebruch-type inequality for $\mathcal{C}$:

$$t_2 + \frac{3}{4}t_3 \geqslant 4(e+\frac{1}{e}) - 8 + 16g + \sum_{k\geqslant 5}(2k-9)t_k + d(-5a^2e + 2ae - 4ag + 4a). \tag{2.20}$$

Since $e \geqslant 4$, we have $4(e+\frac{1}{e}) \geqslant 17$. So (2.20) becomes:

$$t_2 + \frac{3}{4}t_3 \geqslant 9 + 16g + \sum_{k\geqslant 5}(2k-9)t_k + d\left(-5a^2e + 2ae - 4ag + 4a\right). \tag{2.21}$$

From the above inequality (2.21), we have

$$-f_1 \geqslant \frac{9+16g+d\left(e(2a-5a^2)-4ag+4a\right)-9f_0+4t_2+\frac{9}{4}t_3+t_4}{2}.$$

We now bound $H(\mathcal{C})$.

$$\begin{aligned} H(\mathcal{C}) &\geqslant \frac{d(-a^2e+2ab)+8g+\frac{d(e(2a-5a^2)-4ag+4a)-9f_0+9}{2}+2t_2+\frac{9}{8}t_3+\frac{t_4}{2}}{f_0} \\ &\geqslant \frac{-9}{2}+\frac{d}{f_0}\left(\frac{-7a^2e}{2}+2ab+ae-2ag+2a\right)+\frac{16g+4t_2+t_4}{2f_0}+\frac{9t_3}{8f_0}. \end{aligned}$$

Since $ae = b$, we get

$$H(\mathcal{C}) \geqslant \frac{-9}{2}+\frac{d}{f_0}\left(\frac{ae(2-3a)-4a(g-1)}{2}\right)+\frac{16g+4t_2+t_4}{2f_0}+\frac{9t_3}{8f_0},$$

as required. □

We now define the global Harbourne constant of a ruled surface for a fixed pair of integers a, b as follows.

Definition 2.47. Let X be a ruled surface with invariant $e \geqslant 4$. Let $a > 0$ and $b \geqslant ae$ be positive integers. We define the *global* Harbourne constant $H_{a,b}(X)$ of X as :

$$H_{a,b}(X) := \inf_{\mathcal{C}} H(\mathcal{C}),$$

where the infimum is over all transversal arrangements $\mathcal{C}$ satisfying Assumption 2.41.

In order to bound the constant $H_{a,b}(X)$, we make the following observation.

Lemma 2.48. *Let $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$ be a transversal arrangement on the ruled surface X satisfying Assumption 2.41. Then $f_0 \geqslant d$.*

Proof. This is proved in [18, Lemma 6.1]. We write the proof here for the convenience of the reader.

Let $s = f_0$ and $h = 2ab - a^2e$. Let $\mathrm{Sing}(\mathcal{C}) = \{p_1, \ldots, p_s\}$. Consider the $\mathbb{Q}-$vector space $\mathbb{Q}^s$ with the usual dot product: if $v = (a_1, \ldots, a_s)$ and $u = (b_1, \ldots, b_s)$, then $v \cdot u := a_1b_1 + \ldots + a_sb_s$.

For every curve $C_i \in \mathcal{C}$, we associate a vector $v_i \in \mathbb{Q}^s$ by setting the l-th entry of v_i equal to 1, if C_i passes through p_l, and 0 otherwise.

Note that if $i \neq j$, then $v_i \cdot v_j$ is precisely the number of points common to C_i and C_j. By our hypothesis, we have $v_i \cdot v_j = h$. Also $v_i \cdot v_i$ is the number of multiple points that are contained in C_i.

We claim that each curve C_i contains at least $h + 1$ intersection points with other curves in the arrangement. Since there are at least two curves in $\mathcal{C}$, we have $v_i \cdot v_i \geqslant h$. If $v_i \cdot v_i = h$, then all the curves in $\mathcal{C}$ intersect C_i in the same h points. This contradicts the assumption $t_d = 0$. Thus $v_i \cdot v_i > h$ for all i.

To prove the lemma, it suffices to show that the set $\{v_1, v_2, \ldots, v_d\}$ is linearly independent. If it is not linearly independent, without loss of generality, let $v_1 = \sum_{j=2}^{d} a_jv_j$ for $a_j \in \mathbb{Q}$.

Consider $v_1 \cdot (v_1 - v_q)$ where $q \geqslant 2$. Then

$$\begin{aligned}(v_1 \cdot v_1) - h &= v_1 \cdot (v_1 - v_q) \\ &= \left(\sum_{j=2}^{d} a_j v_j\right) \cdot (v_1 - v_q) \\ &= \sum_{j=2}^{d} a_j \Big(h - (v_j \cdot v_q)\Big) \\ &= a_q \left(h - (v_q \cdot v_q)\right)\end{aligned}$$

So $a_q = \frac{(v_1 \cdot v_1) - h}{h - (v_q \cdot v_q)} < 0$. Since this holds for all $q \geqslant 2$, v_1 is a linear combination of $v_2, \ldots, v_d$ with negative coefficients. But the entries of v_i for any $i = 1, \ldots, d$ are either 0 or 1 and we obtain the required contradiction. □

Corollary 2.49. *Let X be a ruled surface over a smooth curve of genus g with invariant $e \geqslant 4$. Let $a > 0$ and $b > ae$ be positive integers. Then*

$$H_{a,b}(X) \geqslant \frac{-11}{2} + \frac{(ae - 2b)}{2}(3a - 2) - 2ag. \tag{2.22}$$

Further, if $ae = b$, then

$$H_{a,b}(X) \geqslant \frac{-9}{2} + \frac{ae(2 - 3a) - 4ag}{2}. \tag{2.23}$$

Proof. We first claim that $f_0 > 2ab - a^2e + 1$. Indeed, if not, $f_0 \leqslant 2ab - a^2e + 1$. Then

$$\begin{aligned}(2ab - a^2e)d(d-1) &= \sum_{k \geqslant 2} k(k-1)t_k, \text{by Lemma 2.35(2)} \\ &\leqslant (d-1)(d-2)f_0, \text{ since } k \leqslant d-1 \\ &\leqslant (d-1)(d-2)(2ab - a^2e + 1).\end{aligned}$$

This gives

$$\begin{aligned}(2ab - a^2e)d &\leqslant (d-2)(2ab - a^2e + 1) \\ \Rightarrow 2(2ab - a^2e) &\leqslant (d-2) \\ \Rightarrow 2(d-1) &\leqslant 2(2ab - a^2e) \leqslant (d-2), \text{ by Lemma 2.48} \\ \Rightarrow d &\leqslant 0.\end{aligned}$$

This is a contradiction and the claim follows.

Now, since $b > ae$, $e \geqslant 4$ and $a > 0$ by our assumptions, the claim gives $f_0 \geqslant 2ab - a^2e + 2 \geqslant 2a(ae+1) - a^2e + 2 \geqslant 4a^2 + 2a + 2 \geqslant 8$. Thus $f_0 \geqslant 8$ and hence $\frac{-8}{f_0} \geqslant -1$.

By Theorem 2.45, we have $H(\mathcal{C}) \geqslant \frac{-9}{2} - \frac{8}{f_0} + \frac{d}{f_0}\left(\frac{(ae-2b)}{2}(3a-2) - 2a(g-1)\right)$. Note that $\frac{(ae-2b)}{2}(3a-2) - 2ag$ is a negative number as $b > ae$. Hence, as $\frac{-8}{f_0} \geqslant -1$, Lemma 2.48 gives (2.22).

Similarly, by Proposition 2.46, we have $H(\mathcal{C}) \geqslant \frac{-9}{2} + \frac{d}{f_0}\left(\frac{ae(2-3a)-4a(g-1)}{2}\right)$. Since $\frac{ae(2-3a)-4ag}{2}$ is a negative number, Lemma 2.48 gives (2.23). □

We now state a corollary which gives a lower bound on the self-intersection of the strict transform of the divisor associated to an arrangement of curves.

Corollary 2.50. *Let $\mathcal{C}$ be a transversal arrangement on the ruled surface X satisfying Assumption 2.41. Let $f : \widetilde{X} \to X$ be the blow up of X at Sing$(\mathcal{C})$. Let $\widetilde{D}$ denote the strict transform of D, which is the divisor defined as the sum of all the curves in $\mathcal{C}$. Then*

$$\widetilde{D}^2 \geqslant -8 - \frac{9}{2}s + d\left(\frac{(ae-2b)}{2}(3a-2) - 2a(g-1)\right) + 8g + 2t_2 + \frac{t_4}{2} + \frac{9t_3}{8}.$$

Further, if all curves in the arrangement do not intersect the normalized section C_0, then

$$\widetilde{D}^2 \geqslant \frac{-9}{2}s + d\left(\frac{ae(2-3a) - 4a(g-1)}{2}\right) + 8g + 2t_2 + \frac{t_4}{2} + \frac{9t_3}{8}.$$

Proof. Indeed, note that $f_0 = s$ and $\widetilde{D}^2 = sH(\mathcal{C})$. The corollary now follows from (2.14) and (2.18). □

2.4.1 Examples

It is not easy to construct arrangements which have small Harbourne constants. Most easy to construct examples of curve arrangements have much larger Harbourne constants than our bounds predict. For example, if $\mathcal{C} = \{C_1, \dots, C_d\}$ is a general arrangement of curves on a ruled surface X satisfying our assumptions, then it is easy to see that $H(\mathcal{C}) = \frac{-2(d-2)}{d-1}$. Indeed, all singular points of $\mathcal{C}$ have multiplicity 2 and consequently, $t_2 = \binom{d}{2}C_1^2$ and $t_k = 0$ for $k \geqslant 3$. Now an easy calculation gives

$H(\mathcal{C}) = \frac{-2(d-2)}{d-1}$. But this value is much larger than the bounds given by our main results Theorem 2.45 or Corollary 2.49.

This situation is analogous to the case of line arrangements in $\mathbb{P}^2$. The best bound we have in this case is given in [5, Theorem 3.3] which proves that $H(\mathcal{L}) > -4$ for all line arrangements $\mathcal{L}$ in $\mathbb{P}^2$. But for a general line arrangement or for many simple examples, the Harbourne constant is at least -2. However, there do exist line arrangements in the plane which have small Harbourne constants. We can use these to obtain fairly small Harbourne constants for curve arrangements on ruled surfaces. We illustrate this with two examples below.

Example 2.51. Let $X = X_e$ be a rational ruled surface with invariant $e \geqslant 1$. Given a line arrangement in $\mathbb{P}^2$, one can obtain an arrangement of curves on X_e, following a construction outlined in [18, Example 15]. Let $\pi : X \to \mathbb{P}^1$ be the natural projection map given by $[s_1 : s_2; x_1 : x_2] \mapsto [s_1 : s_2]$. Let ψ be an isomorphism between a fiber of π and $\mathbb{P}^1$.

Let $\rho : X_1 \to \mathbb{P}^2$ be the blow up of a point p that is not on the line arrangement. Define the morphism $\eta : X \to X_1$ as:

$$\eta([s_1 : s_2; x_1 : x_2]) = [s_1^e : s_2^e; \psi([s_1 : s_2; x_1 : x_2])].$$

The morphism η is of degree e and is branched upon two fibers of X_1. Choosing appropriate coordinates on $\mathbb{P}^2$, we can choose two lines passing through p that intersect the line arrangement transversally such that the pull-back of these lines by ρ gives the two fibers of the branch locus of η.

So we can pull-back lines in $\mathbb{P}^2$ by $\rho \circ \eta$ to X_e which are in the class $(1, e)$. If $\mathcal{L}$ is a line arrangement of d lines in the plane, its pull-back gives a curve arrangement $\mathcal{C}$ of d curves in X_e.

To be more precise, suppose that $\mathcal{L}$ has s singularities and t_k denotes the number of singular points of $\mathcal{L}$ of multiplicity k. Then the singular points of $\mathcal{C}$ are precisely the pre-images of singularities of $\mathcal{L}$. So $\mathcal{C}$ has es singular points and the number of singular points of multiplicity k is et_k. Note that each curve in $\mathcal{C}$ is in the class $(1, e)$ and has self-intersection e. So the self-intersection of the divisor associated to $\mathcal{C}$ is d^2e.

Hence we have

$$H(\mathcal{C}) = \frac{d^2 e - e \sum\limits_{p \in \mathrm{Sing}(\mathcal{L})} r_p^2}{se} = \frac{d^2 - \sum\limits_{p \in \mathrm{Sing}(\mathcal{L})} r_p^2}{s} = H(\mathcal{L}).$$

We now assume $e \geqslant 4$. First we consider the Klein arrangement [39], denoted by $\mathcal{L}_1$. This arrangement consists of 21 lines with $t_3 = 28, t_4 = 21$ and $t_k = 0$ for $k \neq 3, 4$. It is easy to see that $H(\mathcal{L}_1) = -3$. So if $\mathcal{C}_1$ is the curve arrangement in X obtained from $\mathcal{L}_1$, then $H(\mathcal{C}_1) = -3$.

Now we calculate the bound given by Proposition 2.46. (Note that since $ae = b$, this bound is better than the one given by Theorem 2.45.) We have $d = 21, f_0 = 49e, a = 1, b = e, g = 0, t_2 = 0, t_3 = 28e, t_4 = 21e$. So Proposition 2.46 gives

$$H(\mathcal{C}_1) \geqslant \frac{-9}{2} + \frac{21}{49e}\left(\frac{4-e}{2}\right) + \frac{21e}{98e} + \frac{9(28)}{8(49)} = \frac{42}{49e} - 3.857.$$

Next let $\mathcal{L}_2$ denote the Wiman configuration [61]. This arrangement consists of 45 lines with $t_3 = 120, t_4 = 45, t_5 = 36$ and $t_k = 0$ for $k \neq 3, 4, 5$. It is easy to check that $H(\mathcal{C}_2) = -3.359$, where $\mathcal{C}_2$ is the arrangement of curves in X given by $\mathcal{L}_2$.

As above, using Proposition 2.46, we obtain

$$H(\mathcal{C}_2) \geqslant \frac{-9}{2} + \frac{45}{201e}\left(\frac{4-e}{2}\right) + \frac{45e}{402e} + \frac{9(120)}{8(201)} = \frac{90}{201e} - 3.828.$$

2.5 Ball quotients

Ball quotients are algebraic surfaces for which the universal cover is the 2-dimensional unit ball. Equivalently, ball quotients are minimal smooth complex projective surfaces Y of general type satisfying equality in the Bogomolov-Miyaoka-Yau inequality. In other words, they are minimal smooth complex projective surfaces Y such that K_Y is nef and big and $K_Y^2 = 3e(Y)$, where K_Y denotes the canonical divisor and $e(Y)$ is the topological Euler characteristic. See [58] for more details on ball quotients.

Hirzebruch [37] gave examples of ball quotients using line arrangements in $\mathbb{P}^2$. To a line arrangement in $\mathbb{P}^2$, he associated a surface Y (by first an abelian cover of

$\mathbb{P}^2$ branched on that line arrangement and then taking a desingularization). He exhibited three specific line arrangements whose associated surfaces Y are ball quotients.

In this section, we show that the surfaces associated to transversal arrangements on ruled surfaces that we consider in this thesis are not ball quotients. In order to do this, we use the theory of constantly branched covers developed in [3]. The crucial idea is the following. Let Y be a ball quotient which arises from the abelian cover construction we used in Section 2.3. Then if E is a curve contained in the (reduced) ramification divisor of $\sigma : Y \to \widetilde{X}$, then the *relative proportionality* of E is zero. This is defined as $\operatorname{prop}(E) := 2E^2 - e(E)$. For more details, see [3, Section 1.3]. See also [36] for a nice introduction. In the notation of [36], one says that Y is a *good covering* of $\widetilde{X}$ via σ.

The same method was used in [51] and [52] to study ball quotients.

Let X be a ruled surface with $e \geqslant 4$. Let $\mathcal{C} = \{C_1, C_2, \dots, C_d\}$ be a transversal arrangement of curves on the ruled surface X satisfying Assumption 2.41. Let Y be the associated surface constructed in Section 2.3; see Figure 2.2. By Theorem 2.43, K_Y is nef and consequently, Y is a minimal surface of non-negative Kodaira dimension. In fact, Y is a surface of general type most of the time as the following remark shows.

Remark 2.52. Let $\mathcal{C}$ be a transversal arrangement on the ruled surface X satisfying Assumption 2.41. Assume in addition that $a \geqslant 8$. By (2.7), we have

$$K_Y^2 = 2^{d-3}\left(32 + (8ad - 32)g + d(a(2b - ae) + 4a(e-2) - 8b) + 5f_1 - 9f_0 + t_2\right).$$

Using $a \geqslant 8$ and Assumption 2.41, it is easy to see that $K_Y^2 > 0$. Thus Y is a minimal surface of general type.

We define the Hirzebruch polynomial as

$$H_{\mathcal{C}}(2) := \frac{1}{2^{d-3}}(3e(Y) - c_1^2(Y)).$$

Note that by equation (2.8), we have

$$H_{\mathcal{C}}(2) = 16 - 16g + d\left((2b - ae)(5a - 2) + 4a(g-1)\right) + 9f_0 - 2f_1 - 4t_2.$$

Since K_Y is nef, we have by Remark 1.24 and (1.4) that $H_{\mathcal{C}}(2) \geqslant 0$. If Y is a ball quotient then $H_{\mathcal{C}}(2) = 0$.

We now check whether there exists a transversal arrangement $\mathcal{C}$ on X satisfying Assumption 2.41 such that the associated surface Y is a ball quotient.

As noted above, the relative proportionality of curves contained in the (reduced) ramification divisor of σ is zero. There are two kinds of curves which are contained in the ramification divisor of σ. The first kind are the irreducible components F_p of $\sigma^\star E_p$ for $p \in \mathrm{Sing}(\mathcal{C})$ with $r_p \geqslant 3$. Since $F_p^2 = -2^{r_p-2}$, (2.4) gives $\mathrm{prop}(F_p) = 2^{r_p-2}(r_p - 6)$.

So, if the associated surface Y is a ball quotient, then for any point $p \in \mathrm{Sing}(\mathcal{C})$ with $r_p \geqslant 3$, we have $r_p = 6$. Hence the arrangement $\mathcal{C}$ satisfies $t_k = 0$ for $k \neq 2, 6$.

For any $C_i, C_j \in \mathcal{C}$, let $a' := C_i \cdot C_j = 2ab - a^2 e$ and $b' := K_X \cdot C_i = 2ae + a(2g - 2 - e) - 2b$.

For any $j \in \{1, \ldots, d\}$, let t_k^j denote the number of k-fold points of C_j. Since $t_k = 0$ for $k \neq 2, 6$, Lemma 2.35(1) gives

$$a'(d-1) = 5t_6^j + t_2^j. \tag{2.24}$$

The second kind of curves contained in the (reduced) ramification divisor of σ are irreducible components of $D_j := \sigma^\star(C_j')$, where C_j' is the strict transform of C_j under the blow up τ. We note that $\sigma^\star(C_j')$ consists of disjoint union of irreducible curves which are taken to one another by automorphisms of Y.

We now calculate the relative proportionality $\mathrm{prop}(D_j)$.

Note that $K_Y = \sigma^\star(T)$, where T was defined in Lemma 2.37. We also recall that, by (2.10), we have $T \cdot C_j' = b' + \frac{a'}{2} + f_0^j - \frac{t_2^j}{2}$. Finally, note that $C_j'^2 = C_j^2 - \sum_{k \geqslant 3} t_k^j = a' - \sum_{k \geqslant 3} t_k^j$.

Then $\mathrm{prop}(D_j) = 2D_j^2 - e(D_j) = 3D_j^2 + K_Y \cdot D_j = 3\left((\frac{2^{d-1}}{2^2})C_j'^2\right) + (\frac{2^{d-1}}{2})(T \cdot C_j') = 2^{d-3}\left(3a' - 3\sum_{k \geqslant 3} t_k^j\right) + 2^{d-3}\left(2b' + a' + 2f_0^j - t_2^j\right) = 2^{d-3}\left(4a' + 2b' - t_6^j + t_2^j\right)$.

For the final equality above, we use the fact that $t_k = 0$ for $k \neq 2, 6$. If Y is a ball quotient, then $\mathrm{prop}(D_j) = 0$. This gives

$$4a' + 2b' = t_6^j - t_2^j. \tag{2.25}$$

Solving the linear equations (2.24) and (2.25) for t_2^j and t_6^j, and using the easy combinatorial identity $\sum_{j=1}^d t_k^j = kt_k$, we get

$$t_2 = \frac{a'd^2 - 21a'd - 10b'd}{12}, \; t_6 = \frac{a'd^2 + 3a'd + 2b'd}{36}. \tag{2.26}$$

If there exists an arrangement $\mathcal{C}$ on X satisfying Assumption 2.41 and having only double and sixfold points such that the associated surface Y is a ball quotient, then $H_{\mathcal{C}}(2) = 0$. This gives

$$16 - 16g + d[(2b - ae)(5a - 2) + 4a(g - 1)] + t_2 = 3t_6. \tag{2.27}$$

Plugging the values of t_2 and t_6 obtained above in (2.27) and simplifying, we get

$$16 - 16g = -d\left((3a - 1)(2b - ae) + 2a(g - 1)\right). \tag{2.28}$$

We can rewrite (2.28) as

$$-16 = d[(3a - 1)(2b - ae) - 2a] + (2ad - 16)g. \tag{2.29}$$

Thus by our assumptions, we have

$$\begin{aligned} d[(3a-1)(2b-ae)-2a] \geqslant d[(3a-1)ae-2a] = ad[e(3a-1)-2] \\ > 0. \end{aligned}$$

Note that $d \geqslant 4$ by Assumption 2.34. So if $a \geqslant 2$ or if $a = 1, d \geqslant 8$, then $(2ad - 16)g \geqslant 0$ and thus the right-hand side of (2.29) is a positive number, a contradiction.

Let $a = 1$ and $4 \leqslant d \leqslant 7$. Then it is easy to directly check that (2.27) is not possible. First note that the largest value of t_6 is attained when $t_k = 0$ for $k \neq 6$ and in this case we have $t_6 = \frac{a'd(d-1)}{30}$, by Lemma 2.35(2).
If Y is a ball quotient, then (2.27) holds and we have

$$\begin{aligned} 0 &= 16 - 16g + d[(2b - ae)(5a - 2) + 4a(g - 1)] + t_2 - 3t_6 \\ &\geqslant 16 - 16g + d(6b - 3e + 4g - 4) - \frac{a'd(d-1)}{10} \\ &\geqslant 16 - 16g + 4gd - 4d + (2b - e)\left(3d - \frac{d(d-1)}{10}\right) \\ &\geqslant 16 - 4d + 4\left(3d - \frac{d(d-1)}{10}\right), \qquad \text{since } d \geqslant 4, b \geqslant e \geqslant 4. \end{aligned}$$

Now it is easy to check that the last term above is positive for $4 \leqslant d \leqslant 7$, giving a contradiction.

The above arguments prove the following theorem.

Theorem 2.53. *Let* X *be a ruled surface with* $e \geqslant 4$*. There does not exist any transversal arrangement* $\mathcal{C}$ *on* X *satisfying Assumption 2.41 such that the associated surface* Y *is a ball quotient.*

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