Torus Quotients and Automorphism group of a Bott-Samelson-Demazure-Hansen variety

By

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DECLARATION

I declare that the thesis entitled **"Torus Quotients and Automorphism group of a Bott-Samelson-Demazure-Hansen variety"** submitted by me for the degree of **Doctor of Philosophy in Mathematics** is the record of academic work carried out by me under the guidance of Professor S. Senthamarai Kannan and this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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CERTIFICATE

I certify that the thesis entitled **"Torus Quotients and Automorphism group of a Bott-Samelson-Demazure-Hansen variety"** submitted for the degree of **Doctor of Philosophy in Mathematics** by B. Narasimha Chary is the record of research work carried out by him under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent his independent work in a very substantial measure.

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Date: August, 2015.

Professor S. Senthamarai Kannan
Thesis Supervisor.

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Dedicated to my Mother

Abstract

This thesis consists of two problems: Problem 1 deals with the study of the homogeneous coordinate ring of Torus quotient of the homogeneous space. More precisely, let G be a simple adjoint group over the field of complex numbers $\mathbb{C}$. We fix a maximal torus T of G. Let B be a Borel subgroup of G containing T. For any dominant character χ of T, let $\mathcal{L}_\chi$ be the corresponding T-linearized line bundle on the flag variety G/B. Let $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$ be the GIT quotient of G/B by T with respect to the line bundle $\mathcal{L}_\chi$. We are interested in the following question: When the homogeneous coordinate ring of $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$ is isomorphic to a polynomial ring; equivalently, when $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$ is isomorphic to a weighted projective space. We prove that it is a polynomial ring if χ satisfies a combinatorial property in terms of a "Coxeter element" of the Weyl group W of G.

Problem 2: We use the same notations as above. Let $Z(w, \underline{i})$ be the Bott-Samelson-Demazure-Hansen variety (the desingularization of the Schubert variety $X(w)$) corresponding to a reduced expression $\underline{i}$ of $w \in W$. We compute the connected component $Aut^0(Z(w, \underline{i}))$ of the automorphism group of $Z(w, \underline{i})$ containing the identity automorphism. In particular, the Bott-Samelson-Demazure-Hansen varieties corresponding to the different reduced expressions of w need not be isomorphic. We also prove that the Bott-Samelson-Demazure-Hansen varieties are rigid for simply laced groups and their deformations are unobstructed in general.

Contents

Chapter 1

Introduction

This thesis consists of two problems: The first problem is related to the torus quotients of homogeneous spaces and the second problem is about the automorphism group of the Bott-Samelson-Demazure-Hansen variety.

1.1 Torus Quotients

There are two central problems in the theory of invariants:
1. The study of the ring of invariants.
2. The study of the quotient variety under the action of an algebraic group.

1.1.1 Ring of Invariants

The standard setting of invariant theory is as follows: If G is a finite group acting linearly on a vector space V over an algebraic closed field k, then it induces an action on $k[V]$, the algebra of polynomial functions on V, the action is given by $(gf)(v) := f(g^{-1}.v)$ for all $g \in G$, for all $v \in V$, $f \in K[V]$. The ring of G-invariant polynomials is defined by $k[V]^G := \{f \in k[V] : gf = f \ \forall \ g \in G\}$. If G is a linear algebraic group acting on an affine variety X, then it defines an action on the coordinate ring $k[X]$ of X. Let $k[X]^G := \{f \in k[X] : gf = f \ \forall \ g \in G\}$. When $X = V$ is a representation of G, the action of G on $k[V]$ preserves degree and $k[V]^G \subseteq k[V]$ inherits the grading.

The basic question in invariant theory is the following: What is the structure of the ring $k[V]^G$? For example, when $k[V]^G$ is finitely generated? If it is finitely generated then find the generators and relations for $k[V]^G$, and the degree bounds for the generators. When is the ring $k[V]^G$ of invariants a polynomial ring ?

In general, it is a difficult problem to compute the ring of invariants. This was a major topic of research in nineteenth century. In 1868, Paul Gordan proved that (constructively) for the action of $G = SL_2(\mathbb{C})$ on the finite dimentional complex vector space V of homogeneous polynomial of degree n in two variables, the ring $\mathbb{C}[V]^G$ of invariants is finitely generated over $\mathbb{C}$. In 1890, David Hilbert proved that (in a non-contructive way) whenever G is linearly reductive acting on a finite dimensional complex vector space V, the ring $\mathbb{C}[V]^G$ of invariants is finitely generated $\mathbb{C}$-algebra and he proposed a general question of finite generation of invariant rings for arbitrary groups. This problem is now known as Hilbert's fourteenth problem.

The following theorem describes when the ring of invariants for the linear action of a finite group is a polynomial ring.

Theorem 1.1. *(Chevalley, Serre, Shephard-Todd, [Che55, Ser68, ST54]). Let V be a finite dimensional representation of a finite group G over a field k. Assume that the characteristic of k does not divide the order of G. Then, $k[V]^G$ is a polynomial algebra if and only if G is generated by pseudo-reflections. In such a case, $|G| = \prod_{i=1}^{n} deg(f_i)$, where $n = dim(V)$ and $\{f_1, f_2, \ldots, f_n\}$ is a set of algebraically independent generators of $k[V]^G$.*

If G is a subgroup of $GL(V)$ generated by pseudo reflections, but the characteristic of k divides the order of G, then the ring $k[V]^G$ of invariants need not be a polynomial algebra. For example, let k be an algebraic closed field of characteristic 3 and let W be the Weyl group of an algebraic group of type F_4 over k. Note that order $|W|$ of the Weyl group W is 1152 and 3 divides 1152. The ring of invariants is not a polynomial algebra (see [NS02, Chapter 7, page 192]).

The above results gives a characterization for $k[V]^G$ to be a polynomial algebra if G is finite, but there is no simple characterization for a semisimple algebraic group G.

Theorem 1.2 (Chevalley, [Hum72])**.** *For any semisimple algebraic group G over $\mathbb{C}$, the ring $\mathbb{C}[\mathfrak{g}]^G$ of G-invariants of the coordinate ring of the adjoint representation $\mathfrak{g}$ of G is a polynomial algebra (see [Hum72, page 127]).*

Theorem 1.3 (Steinberg, [Ste65])**.** *For any semisimple simply connected algebraic group G (over any algebraically closed field k) acting on itself by inner conjugation, the ring $k[G]^G$ of G-invariants is a polynomial algebra (see [Ste65, page 41]).*

When $G = T$ is a torus, D. Wehlau in [Weh94] gave two constructive criteria each of which determines those representations of T for which the ring of invariants is a polynomial ring (see [Weh94, Theorem 5.8]).

1.1.2 Torus action

Since torus is a reductive group, studying the invariants of torus is an interesting problem. The study of the action of a maximal torus $T \subset GL(n)$ on the Grassmannian $Gr(n,k)$ is connected to various interesting problems in geometry.

Classical problem in invariant theory is the study of binary quantics. The main object is to give an explicit description of the ring $k[V]^{SL_2}$, where V is the space of all homogeneous forms of degree n in two variables and to study the geometric properties of SL_2 quotients of projective space for a suitable choice of linearization. The natural generalization of this classical problem is the following;

Let $G = SL_n(k)$, the special linear group and P_2 is the maximal parabolic subgroup of $SL_n(k)$ associated to the simple root α_2, one knows that G/P_2 is the Grassmannian $G_{2,n}$ of all two- dimensional subspaces of an n-dimensional vector space over k. Let N be the normalizer of a maximal torus T in G. Then, one also has an isomorphism:

$$N\backslash\backslash(G/P_2)^{ss}(\mathcal{L}_2) = N\backslash\backslash(G_{2,n})^{ss}(\mathcal{L}_2) \simeq SL_2\backslash\backslash(\mathbb{P}(V))^{ss},$$

where V is the vector space of all homogeneous polynomials of degree n in two variables and $\mathcal{L}_2$ is the line bundle associated to the fundamental weight ω_2, and the variety $SL_2\backslash\backslash\mathbb{P}(V)^{ss}$ is precisely the space of binary quantics (see the proof of Theorem-1 and the proof of Theorem-4 of [Ses68]). Generally, one has the following isomorphism:

$$T\backslash\backslash(G/P_r)^{ss}(\mathcal{L}_r) = T\backslash\backslash(G_{r,n})^{ss}(\mathcal{L}_r) \simeq SL_r\backslash\backslash(\mathbb{P}^{r-1})^n,$$

where P_r is the maximal parabolic subgroup of $\subset G = SL_n(k)$ associated to the simple root α_r, $G_{r,n}$ is the Grassmannian of r-dimensional subspaces of an n dimensional vector space and $\mathcal{L}_r$ is the line bundle on $G/P_r = G_{r,n}$ associated to ω_r.

A more general question in this setting is the study of GIT related problems on the flag variety G/P associated to a semisimple algebraic group G for the action of a maximal torus T and its normalizer N in G.

When $G = SL_n(k)$ and P_r is the maximal parabolic subgroup of G associated to the simple root α_r and $\mathcal{L}_r$ the line bundle associated to the fundamental weight ω_r. S. Senthamarai Kannan proved in [Kan98] the following result:

$$(G/P_r)^{ss}_T(\mathcal{L}_r) = (G/P_r)^{s}_T(\mathcal{L}_r)$$

if and only if r and n are co-prime.

Further, in [Kan99], when G is semisimple algebraic group G, the parabolic subgroups P of G for which there is a T-linearized ample line bundle $\mathcal{L}$ on G/P such that

$$(G/P)^{ss}_T(\mathcal{L}) = (G/P)^{s}_T(\mathcal{L})$$

has been classified.

In [Str00], Strickland gave a shorter proof of these results. In [Zhg07] and [Zhg08], Zhgun studied that how the quotient vary as the line bundle varies. That is, there is a decomposition of the Weyl chamber C into GIT-equivalence classes of characters χ of B determining the same sets of semi-stable points $(G/B)^{ss}_T(\mathcal{L}_\chi)$. In [Sko09], Skorobogatov described the automorphism group of the quotient $T\backslash\backslash(G/P)^{ss}_T(\mathcal{L})$, where P is a maximal parabolic subgroup of G.

It is an interesting problem to study the minimal dimensional Schubert varieties in G/P admitting semi-stable points with respect to the T-linearized ample line bundle $\mathcal{L}$ on G/P. In [KP09b], when G is simply connected semisimple algebraic group, P is a maximal parabolic subgroup of G and $\mathcal{L} = \mathcal{L}_\omega$, where ω is a minuscule fundamental weight, it is shown that there exists a unique minimal Schubert variety $X(w)$ admitting semi-stable points with respect to the line bundle $\mathcal{L}$. Note that this includes the all maximal parabolic subgroups of a simple algebraic group of type A.

Let G be a simple algebraic group of type B, C or D and P is a maximal parabolic subgroup of G. Let $\mathcal{L}$ be an ample line bundle on G/P. In [KP09a], authors described all the minimal Schubert varieties in G/P admitting semi-stable points with respect to the line bundle $\mathcal{L}$. In the same paper [KP09a], they described all the Coxeter elements $w \in W$ for which the corresponding Schubert variety $X(w)$ admits a semi-stable point for the action of a maximal torus T with respect to a non trivial line bundle on G/B.

In [Pat14], for any simple, simply connected algebraic group G of exceptional types E_6, E_7, E_8, F_4, or G_2 and for any maximal parabolic subgroup P of G, the author describe all minimal Schubert varieties in G/P admitting semi-stable points for the action of a maximal torus T with respect to an ample line bundle on G/P.

In this thesis, we study the homogeneous coordinate ring of $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$ for some dominant character χ of B.

More precisely, let G be a simple algebraic group of adjoint type over $\mathbb{C}$. Let T be a maximal torus of G and let B be a Borel subgroup of G containing T. For any dominant character χ of T, let $\mathcal{L}_\chi$ be the corresponding T-linearized line bundle on G/B. Let $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$ be the GIT quotient of G/B by T with respect to $\mathcal{L}_\chi$. We are interested in studying the following question: For which dominant character χ of B, the homogeneous coordinate ring of $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$ is isomorphic to a polynomial ring; equivalently, for which character χ of B, the quotient $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$ is isomorphic to a weighted projective space.

Definition 1.4. A non trivial dominant character χ of T is said to be decomposable if there is a pair of non trivial dominant characters χ_1, χ_2 of T such that $\chi = \chi_1 + \chi_2$. Otherwise, we call χ is indecomposable.

Recall that w in W is said to be a Coxeter element if it has a reduced expression of the form $s_{i_1}s_{i_2}\cdots s_{i_n}$ such that $i_j \neq i_k$ whenever $j \neq k$ (refer to [Hum11]).

A further study about the dominant characters χ of T for which there is a Coxeter element w such that $X(w)^{ss}_T(\mathcal{L}_\chi)$ is non empty, we observed that in the case of A_2, given an indecomposable dominant character χ of T which is in the root lattice; $X(w)^{ss}_T(\mathcal{L}_\chi)$ is non empty for some coxeter element w in W if and only if χ must be one of the following: $\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$ and $\alpha_1 + 2\alpha_2$. We also observed that for all these three dominant characters χ, the ring of T-invariants of the homogeneous coordinate ring $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(\mathbb{P}GL_3(\mathbb{C})/B, \mathcal{L}_\chi^{\otimes d})$ is a polynomial ring.

In case of B_2 as well, given an indecomposable dominant character χ of T which is in the root lattice, $X(w)^{ss}_T(\mathcal{L}_\chi)$ is non empty for some coxeter element w in W if and only if χ must be one of the following: $\alpha_1 + \alpha_2$, $\alpha_1 + 2\alpha_2$. We also observed that for these two dominant characters χ, the ring of T-invariants of the homogeneous coordinate ring $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(SO(5,\mathbb{C})/B, \mathcal{L}_\chi^{\otimes d})$ is a polynomial ring.

The computations in the above mentioned special cases tempt us to ask the following question:

Let G be a simple adjoint group over $\mathbb{C}$, the field of complex numbers. Let T be a maximal torus of G, B be a Borel subgroup of G containing T. Then, for any indecomposable dominant character χ of T such that there is a Coxeter element w in W such that $X(w)^{ss}_T(\mathcal{L}_\chi)$ is non empty, is the ring of T- invariants of the homogeneous coordinate ring $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})$ a polynomial algebra ?

We prove the following result:

Theorem 1.5. *([KCP14, Theorem 4.8]). For any indecomposable dominant character χ of a maximal torus T of a simple adjoint group G such that there is a Coxeter element $w \in W$ for which $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$, the graded algebra $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring if and only if $dim(H^0(G/B, \mathcal{L}_\chi)^T) \leq$ rank of G.*

Equivalently, suppose that there exists a Coxeter element $w \in W$ such that $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$. Then $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$ is isomorphic to a weighted projective space if and only if $dim(H^0(G/B, \mathcal{L}_\chi)^T) \leq$ rank of G. In fact, when this holds, $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$ is isomorphic to a projective space in its antitautological embedding (that is, the polynomial generators of its homogeneous coordinate ring all lie in degree 1). In particular, the polarized variety $(T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi), \mathcal{M})$ is projectively normal, where $\mathcal{M}$ is the descent of $\mathcal{L}_\chi$ to the quotient $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$.

Now, we state the second result on torus quotients.

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of T. Let α_0 denote the highest root. Since $H^0(G/B, \mathcal{L}_{\alpha_0})$ is an irreducible self dual G module with highest weight α_0, the G modules $H^0(G/B, \mathcal{L}_{\alpha_0})$, $Hom(\mathfrak{g}, \mathbb{C})$ are isomorphic.

On the other hand, the natural T-invariant projection from $\mathfrak{g}$ to $\mathfrak{h}$ induces a isomorphism $Hom(\mathfrak{h}, \mathbb{C}) \to Hom(\mathfrak{g}, \mathbb{C})^T$. So, we have an isomorphism $Hom(\mathfrak{h}, \mathbb{C}) \to H^0(G/B, \mathcal{L}_{\alpha_0})^T$.

Thus, we have a homomorphism $f : \mathbb{C}[\mathfrak{h}] \longrightarrow \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ of $\mathbb{C}$-algebras.

We prove the following theorem.

Theorem 1.6. *([KCP14, Theorem 3.3]). The homomorphism* $f : \mathbb{C}[\mathfrak{h}] \longrightarrow \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ *is an isomorphism if and only if* $X(w)^{ss}_T(\mathcal{L}_{\alpha_0})$ *is non empty for some Coxeter element* w *in* W.

Equivalently, the GIT quotient $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_{\alpha_0})$ is isomorphic to the projective space $\mathbb{P}(\mathfrak{h})$ if and only if there exists a Coxeter element $w \in W$ such that $X(w)^{ss}_T(\mathcal{L}_{\alpha_0})$ is nonempty.

As a consequence, the polarized variety $(T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_{\alpha_0}), \mathcal{M})$ is projectively normal, where $\mathcal{M}$ is the descent of $\mathcal{L}_{\alpha_0}$ to the quotient $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_{\alpha_0})$.

Let $\mathbb{P}(\mathfrak{g})$ be the projective space of all one-dimensional subspaces of $\mathfrak{g}$.

By Chevalley restriction Theorem, we deduce the following:

Corollary 1.7. *([KCP14, Corollary 3.4]).* $N_G(T)\backslash\backslash(G/B(\mathcal{L}_{\alpha_0}))^{ss}_T \simeq G\backslash\backslash\mathbb{P}(\mathfrak{g})$ *if and only if* $X(w)^{ss}_T(\mathcal{L}_{\alpha_0})$ *is non empty for some Coxeter element* w *in* W.

1.2 Cohomology of the Tangent bundle of Bott-Samelson-Demazure-Hansen variety

Let G be a simple algebraic group of adjoint type over the field $\mathbb{C}$ of complex numbers. Let B be a Borel subgroup of G containing a maximal torus T of G. let W be the Weyl group of G. For $w \in W$, let $X(w) := \overline{BwB/B}$ denote the Schubert variety in G/B corresponding to w. Given a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ of w, with the corresponding tuple $\underline{i} := (i_1, \ldots, i_r)$, we denote by $Z(w, \underline{i})$, the desingularization of the Schubert variety $X(w)$, which is now known as the Bott-Samelson-Demazure-Hansen variety. It was first introduced by Bott and Samelson in a differential geometric and topological context (see [BS58]). Demazure in [Dem74] and Hansen in [Han73] independently adapted the construction in algebro-geometric situation, which explains the reason for the name. For the sake of simplicity, we will denote a Bott-Samelson-Demazure-Hansen variety by BSDH-variety.

In [Bot57], R. Bott proved that all the higher cohomology groups $H^i(G/B, T_{G/B})$ for the tangent bundle $T_{G/B}$ on the flag variety G/B vanish. Recall that the vanishing results of the cohomology groups of the restriction of the homogeneous vector bundle to the Schubert varieties have been an important area of the research in the theory of algebraic groups (see [And85], [BKS04], [BK07], [Dem76], [Jan07], [Lit98], [MR85] and [Ses07])

Recently, in [Kan13], author proved that the cohomology groups $H^i(X(w), T_{G/B}) = 0$ for all $i \geq 1$, where $T_{G/B}$ is the restriction of the tangent bundle of G/B to $X(w)$.

In this thesis, we prove the following vanishing results of the tangent bundle $T_{Z(w,\underline{i})}$ on $Z(w, \underline{i})$ (see [CKP15, Section 3]):

Theorem 1.8.

1. *$H^j(Z(w, \underline{i}), T_{Z(w,\underline{i})}) = 0$ for all $j \geq 2$.*
2. *If G is simply laced, then $H^j(Z(w, \underline{i}), T_{Z(w,\underline{i})}) = 0$ for all $j \geq 1$.*

As a consequence, it follows that the BSDH-varieties are rigid for simply laced groups and their deformations are unobstructed in general (see [CKP15, Section 3]). The above vanishing result is independent of the reduced expression $\underline{i}$ of w. By computing $H^1(Z(w, \underline{i}), T_{Z(w,\underline{i})})$ for non simply laced group, we observed that this cohomology group very much depend on the choice of a reduced expression $\underline{i}$ of w.

1.3 Automorphism group of a Bott-Samelson-Demazure-Hansen variety

The construction of the Bott-Samelson-Demazure-Hansen variety $Z(w, \underline{i})$ depends on the choice of the reduced expression $\underline{i}$ of w. So, it is natural to ask that for a given $w \in W$ whether the Bott-Samelson-Demazure-Hansen varieties corresponding to two different reduced expressions of w are isomorphic? We study the automorphism group of the Bott-Samelson-Demazure-Hansen varieties in order to answer this question.

We compute the connected component $Aut^0(Z(w,\underline{i}))$ of the automorphism group of $Z(w,\underline{i})$ containing the identity automorphism. We show that $Aut^0(Z(w,\underline{i}))$ contains a closed subgroup isomorphic to B if and only if $w^{-1}(\alpha_0) < 0$, where α_0 is the highest root of G with respect to T and B. We prove $Aut^0(Z(w_0,\underline{i}))$ is a parabolic subgroup of G, where w_0 denote the longest element of W . It is also shown that this parabolic subgroup depend very much on the chosen reduced expression $\underline{i}$ of w_0 and we describe all the parabolic subgroups of G that occur as $Aut^0(Z(w_0,\underline{i}))$. If G is simply laced, then we show that for every $w \in W$, and for every reduced expression $\underline{i}$ of w, $Aut^0(Z(w,\underline{i}))$ is a quotient of the parabolic subgroup $Aut^0(Z(w_0,\underline{j}))$ of G for a suitable choice of a reduced expression $\underline{j}$ of w_0 (see [CKP15, Theorem 7.3]). We also describe the kernel of the homomorphism $Aut^0(Z(w_0,\underline{j})) \longrightarrow Aut^0(Z(w,\underline{i}))$ of algebraic groups (see [CKP15, Corollary 7.4]). Thus, we have a complete description of $Aut^0(Z(w,\underline{i}))$ for any reduced expression $\underline{i}$ of w in the simply laced case.

We recall the following notation before describing the results: We denote the set of roots of G with respect to T by R. Let B^+ be a Borel subgroup of G containing T. Let B be the Borel subgroup of G opposite to B^+ determined by T. That is, $B = n_0B^+n_0^{-1}$, where n_0 is a representative in $N_G(T)$ of w_0. Let $R^+ \subset R$ be the set of positive roots of G with respect to the Borel subgroup B^+. Note that the set of roots of B is equal to the set $R^- := -R^+$ of negative roots. We use the notation $\beta > 0$ for $\beta \in R^+$ and $\beta < 0$ for $\beta \in R^-$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of all simple roots in R^+, where n is the rank of G. The simple reflection in the Weyl group corresponding to a simple root α is denoted by s_α. For simplicity of notation, the simple reflection corresponding to a simple root α_i is denoted by s_i.

Let $\mathfrak{g}$ denote the Lie algebra of G, let $\mathfrak{b} \subset \mathfrak{g}$ be the Lie algebra of B and $\mathfrak{h} \subset \mathfrak{b}$ be the Lie algebra of T. Let $X(T)$ denote the group of all characters of T. We have $X(T) \otimes \mathbb{R} = Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$, the dual of the real form $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}$. The positive definite W-invariant bilinear form on $Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form of $\mathfrak{g}$ is denoted by $(\ , \ )$. We use the notation $\langle \ , \ \rangle$ to denote $\langle \nu, \alpha \rangle = \frac{2(\nu,\alpha)}{(\alpha,\alpha)}$ for $\nu \in X(T) \otimes \mathbb{R}$ and $\alpha \in R$.

Given a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$, let $\underline{i} := (i_1, \ldots, i_r)$. Set

$$J'(w,\underline{i}) := \{l \in \{1,2,\ldots,r\} : \langle \alpha_{i_l}, \alpha_{i_k} \rangle = 0 \text{ for all } k < l\},$$

$$J(w,\underline{i}) := \{\alpha_{i_l} : l \in J'(w,\underline{i})\} \subset S.$$

Note that the simple reflections $\{s_{i_j} : j \in J'(w,\underline{i})\}$ commute with each other.

Let $W_{J(w,\underline{i})}$ be the subgroup of W generated by $\{s_j \in W \mid \alpha_j \in J(w,\underline{i})\}$. Let

$$P_{J(w,\underline{i})} := BW_{J(w,\underline{i})}B$$

be the corresponding standard parabolic subgroup of G. By abuse of notation, here $W_{J(w,\underline{i})}$ in the definition of the parabolic subgroup $P_{J(w,\underline{i})}$ means any lift of elements of $W_{J(w,\underline{i})}$ to $N_G(T)$. Let $N = |R^+|$.

Then, we have the following theorem:

Theorem ([CKP15]).

1. For any reduced expression $\underline{i}$ of w_0, $Aut^0(Z(w_0, \underline{i})) \simeq P_{J(w_0,\underline{i})}$.
2. For any reduced expression $\underline{i}$ of w, $Aut^0(Z(w, \underline{i}))$ contains a closed subgroup isomorphic to $P_{J(w,\underline{i})}$ if and only if $w^{-1}(\alpha_0) < 0$. In such a case, $P_{J(w,\underline{i})} = P_{J(w_0,\underline{j})}$ for any reduced expression $w_0 = s_{j_1}s_{j_2}\cdots s_{j_N}$ of w_0 such that $\underline{j} = (j_1, j_2, \ldots, j_N)$ and $(j_1, j_2, \ldots, j_r) = \underline{i}$.
3. If G is simply laced, $Aut^0(Z(w, \underline{i}))$ is a quotient of $Aut^0(Z(w_0, \underline{j}))$, where $\underline{j}$ is as in (2).
4. If G is simply laced, $Aut^0(Z(w, \underline{i})) \simeq P_{J(w,\underline{i})}$ if and only if $w^{-1}(\alpha_0) < 0$. In such a case, we have $P_{J(w,\underline{i})} = P_{J(w_0,\underline{j})}$ where $\underline{j}$ is as in (2).
5. The rank of $Aut^0(Z(w, \underline{i}))$ is at most the rank of G.

Let $g_w : Aut^0(Z(w_0, \underline{j})) \longrightarrow Aut^0(Z(w, \underline{i}))$ be the quotient map in the above Theorem part (3). Now, we will describe the kernel of the map g_w.

Recall that $\leq$ is the Bruhat-Chevalley ordering on W and $supp(w) := \{j \in \{1, 2, \ldots, n\} : s_j \leq w\}$, the support of w. For simplicity of notation we denote $supp(w)$ by A_w. Let $T(w) := \bigcap_{k \in A_w} Ker(\alpha_k)$. Set $J_1 := (\{1, 2, \ldots, n\} \setminus A_w) \cap J'(w_0, \underline{j})$. Let U^+ be the unipotent radical of B^+. For $j \in J_1$, let $U^+_{\alpha_j}$ denote the one-dimensional T-stable closed subgroup of U^+ (for the conjugation action of T on G) corresponding to α_j. Let $R^+(w) := \{\beta \in R^+ : w(\beta) \in R^-\}$ and $R_w := R^+ \setminus (\bigcup_{v \leq w} R^+(v^{-1}))$.

Corollary ([CKP15]). The connected component of the kernel of the map g_w is the closed subgroup of $Aut^0(Z(w_0, \underline{j}))$ generated by the torus $T(w)$, $\{U_{-\beta} : \beta \in R_w\}$ and $\{U^+_{\alpha_j} : j \in J_1\}$.

Consider the left action of T on G/B and let $w \in W$. Note that the Schubert variety $X(w^{-1})$ is T-stable. We use the notion of semi-stable points introduced by Mumford [MFK94]. Let α_0 be the highest root of G with respect to T and B^+. We denote by $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$, the set of all semi-stable points of $X(w^{-1})$ with respect to the T-linearized line bundle $\mathcal{L}_{\alpha_0}$ corresponding to the character α_0 of B.

The following result is a formulation of the above theorem using semi-stable points.

Corollary ([CKP15]).

1. $Aut^0(Z(w,\underline{i}))$ contains a closed subgroup isomorphic to $P_{J(w,\underline{i})}$ if and only if $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0}) \neq \emptyset$.

2. If G is simply laced, $Aut^0(Z(w,\underline{i})) \simeq P_{J(w,\underline{i})}$ if and only if $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0}) \neq \emptyset$.

Remark: By the above theorem, the automorphism group of the BSDH-variety $Z(w,\underline{i})$ depends on the choice of the reduced expression $\underline{i}$ of w.

Example: Let $G = PSL(4,\mathbb{C})$. Consider the following different reduced expressions for w_0:

1. $(w_0,\underline{i_1}) = s_1s_2s_1s_3s_2s_1$, $J(w_0,\underline{i_1}) = \{\alpha_1\}$.

2. $(w_0,\underline{i_2}) = s_2s_1s_2s_3s_2s_1$, $J(w_0,\underline{i_2}) = \{\alpha_2\}$.

3. $(w_0,\underline{i_3}) = s_3s_2s_3s_1s_2s_3$, $J(w_0,\underline{i_3}) = \{\alpha_3\}$.

4. $(w_0,\underline{i_4}) = s_1s_3s_2s_3s_1s_2$, $J(w_0,\underline{i_4}) = \{\alpha_1,\alpha_3\}$.

By the above Theorem, we see that $Aut^0(Z(w_0,\underline{i_1}))$, $Aut^0(Z(w_0,\underline{i_2}))$, $Aut^0(Z(w_0,\underline{i_3}))$ and $Aut^0(Z(w_0,\underline{i_4}))$ are isomorphic to $P_{\{\alpha_1\}}, P_{\{\alpha_2\}}, P_{\{\alpha_3\}}, P_{\{\alpha_1,\alpha_3\}}$ respectively.

Therefore, observe that $Aut^0(Z(w_0,\underline{i_1}))$ and $Aut^0(Z(w_0,\underline{i_4}))$ are not isomorphic and hence we conclude that the BSDH-varieties $Z(w_0,\underline{i_1})$ and $Z(w_0,\underline{i_4})$ are not isomorphic. And also observe that $Z(w_0,\underline{i_1})$ and $Z(w_0,\underline{i_2})$ are not isomorphic as $P_{\{\alpha_1\}}$ and $P_{\{\alpha_2\}}$ are not isomorphic.

Remark: Even if the automorphism groups of the BSDH-varieties are isomorphic, it is not clear that the BSDH-varieties are isomorphic.

1.4 Organization of the Thesis

The organization of the thesis is as follows:

In Chapters 2, we recall some basic concepts and preliminaries that will help to present our results in this thesis.

In Chapter 3, we present the results on the torus quotients of homogeneous spaces.

In Chapter 4, we prove some results on cohomology of line bundles and the vanishing results of cohomology of the tangent bundle of the Bott-Samelson-Demazure-Hansen variety.

In Chapter 5, we describe our results on the automorphism group of the Bott-Samelson-Demazure-Hansen variety.

Chapter 2

Preliminaries

To make this thesis self contained, in this chapter we review basic definitions and results on algebraic groups, Lie algebras, Schubert varieties and GIT, from the literature which will be used in the rest of the thesis. Good references for this are [Hum72], [Hum75], [Jan07], [BL00], [MFK94] and [New78].

2.1 Algebraic groups and Lie algebras

2.1.1 Algebraic groups

Throughout this thesis, we assume that all algebraic groups are affine.

Let G be an algebraic group over algebraically closed field k of arbitrary characteristic. Let H be a closed subgroup of G. Denote by G/H, the set of all left cosets of H in G. One would like to know whether the set G/H is endowed with a structure of an algebraic variety such that the natural map

$$\pi : G \to G/H$$

is a morphism of varieties.

The following theorem of Chevalley gives an affirmative answer to this question:

Theorem 2.1 (Chevalley)**.** *Let G be an algebraic group, H be a closed subgroup of G. Then, there is a rational representation $\rho : G \to GL(V)$ and a non-zero vector $v \in V$ such that $H = \{g \in G : \rho(g)v \in kv\}$.*

Let $\mathbb{P}(V)$ be the projective space corresponding to the vector space V. Then, the action of G on V via ρ (as above) induces an action of G on $\mathbb{P}(V)$. Further, the action map

$G \times \mathbb{P}(V) \longrightarrow \mathbb{P}(V)$ is a morphism. Also, there exists a point $[v] \in \mathbb{P}(V)$ such that the stabilizer of $[v]$ in G coincides with H. The orbit $G \cdot [v]$ is open in its closure and thus has a structure of a quasi-projective variety with an algebraic transitive G-action. The orbit map $G \to \mathbb{P}(V)$, $g \mapsto g \cdot [v]$ defines a bijection $G/H \to G \cdot [v]$, and induces a structure of a quasi-projective variety on G/H such that the natural map $G \to G/H$ is a morphism of varieties. In fact, we have :

Corollary 2.2.

1. *The set G/H admits a unique structure of a quasi-projective algebraic variety such that the natural map $G \to G/H$ is a morphism of varieties.*
2. *In addition, if H is a closed normal subgroup of G, then the quotient group G/H has a unique structure of an affine algebraic group such that the natural map $G \to G/H$ is a homomorphism of algebraic groups.*

Remark: The variety G/H is called a homogeneous space for G.

Definition 2.3. A Borel subgroup of G is a maximal closed connected solvable subgroup.

Theorem 2.4. *Let B be any Borel subgroup of G. Then,*
1. *The homogeneous space G/B is a projective variety.*
2. *Any two Borel subgroups of G are conjugate in G.*

Corollary 2.5.

1. *The maximal tori of G are those of Borel subgroups of G.*
2. *The maximal connected unipotent subgroups of G are those of Borel subgroups of G.*
3. *Any two maximal tori (respectively, maximal connected unipotent subgroups) of G are conjugate.*

Definition 2.6. A closed subgroup H of G is called a parabolic subgroup if G/H is projective.

Corollary 2.7.

1. *A closed subgroup of G is parabolic if and only if it contains a Borel subgroup.*
2. *In particular, a connected closed subgroup H of G is a Borel subgroup if and only if H is solvable and G/H is projective.*

Remark: The homogeneous space G/B is the largest homogeneous space for G having the structure of projective variety.

Definition 2.8.

1. A maximal closed connected normal solvable subgroup of G is called the radical of G and we denote it by $R(G)$.

2. A maximal closed connected normal unipotent subgroup of G is called the unipotent radical of G and we denote it by $R_u(G)$.

Remark: The radical (respectively, unipotent radical) of G is the identity component in the intersection of all Borel subgroups (respectively, of their unipotent parts).

An algebraic group G is called semisimple if its radical $R(G) = e$. For example, $G = SL(n, k)$. An algebraic group G is called reductive if its unipotent radical $R_u(G) = e$. For example, $G = GL(n, k)$ or G is any torus. Note that any semisimple algebraic group is reductive.

2.1.2 Lie Algebra of an Algebraic Group

Let G be an algebraic group. G acts on $k[G]$ by the left (respectively, right) translation, $(\lambda_x f)(y) = f(x^{-1}y)$ (respectively, $\rho_x f(y) = f(yx)$) for $x, y \in G$.

Let $Der(k[G])$ be the set of all derivations of $k[G]$. Note that $Der(k[G])$ admits a Lie algebra structure. Let $\mathcal{L}(G)$ be the space of all left invariant derivations of $k[G]$ (i.e $\mathcal{L}(G) = \{\delta \in Der(G) : \delta\lambda_x = \lambda_x\delta,\ for\ all\ x \in G\}$. Note that $\mathcal{L}(G)$ is a Lie subalgebra of $Der(k[G])$. We call $\mathcal{L}(G)$, the Lie algebra of G.

Theorem 2.9. *Let G be an algebraic group. Then,*
1. The Lie algebra $\mathcal{L}(G)$ of G is isomorphic to $T_e(G)$, the tangent space of G at the identity element e of G.
2. Let $\mathfrak{g} = \mathcal{L}(G)$. If $\phi : G \longrightarrow G'$ is a morphism of algebraic groups, then the induced map $d\phi_e : \mathfrak{g} \longrightarrow \mathfrak{g}'$ is a homomorphism of Lie algebras.

2.1.3 Abstract root system

Since we use frequently root systems in this thesis, we recall some definitions and results in this subsection.

Let V be a vector space over $\mathbb{R}$ with a positive definite bilinear form $(-,-)$. Define the reflection corresponding to a non zero vector $\alpha \in V$ to be the linear transformation on V given by

$$s_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha.$$

Note that $s_\alpha(\alpha) = -\alpha$ and it fixes the hyperplane (i.e a subspace of co-dimension one) perpendicular to α. Since the number $\frac{2(v,\alpha)}{(\alpha,\alpha)}$ occurs frequently, we denote it by $\langle v, \alpha \rangle$. Notice that $\langle -, - \rangle$ is linear only in the first variable.

Definition 2.10. An abstract root system R in V is defined by the following axioms:
1. R is a finite subset of V, spans V and $0 \notin R$.
2. If $\alpha \in R$, the only multiples of α in R are $\pm\alpha$.
3. $s_\alpha(R) = R$ for every $\alpha \in R$.
4. $\langle \alpha, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in R$.

The elements of R are called roots. The rank of the root system R is defined to be the dimension of the vector space V.

A subset $S = \{\alpha_1, \ldots, \alpha_n\}$ of R is called a basis of the root system R if,
1. S is a basis of the vector space V and
2. For any $\alpha = \sum_{i=1}^{n} c_i \alpha_i \in R$, where c_i is a non negative integer for all i or c_i is a non positive integer for all i.

The elements of S are called simple roots. The reflections corresponding to the simple roots are called the simple reflections. Let R^+ be the set of all $\alpha' s$ in R such that the coefficients of the simple roots S in the expression for α are non negative. R^+ is called the set of positive roots with respect to the simple roots S. Similarly, R^- is the set of all $\alpha' s$ in R such that the coefficients of the simple roots S in the expression for α are non positive. R^- is called the set of negative roots with respect to the simple roots S. Clearly, R is a disjoint union of R^+ and R^-.

The connected components of the complement of the union of the hyperplanes coresponding to the roots are called Weyl chambers. These chambers are in one-one correspondence with the set of all bases of the root system.

We recall that the Dynkin diagram is a graph with vertices indexed by the simple roots and the number of edges between α_i and α_j for $i \neq j$ is $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ with arrow pointing to the smaller of the two roots if they have different lengths. The integers $\langle \alpha_i, \alpha_j \rangle, 1 \leq i, j \leq n$, are called the Cartan integers, and these completely determine the root system up to an isomorphism. Note that the Dynkin diagrams contains the information of the Cartan integers. We extensively use in this thesis properties of the Cartan integers, for more details on Cartan integers we refer to [Hum72].

A root system R is said to be irreducible if it cannot be partitioned into a union of two proper subsets R_1 and R_2 such that each root α in R_1 is orthogonal to each root β in R_2. All irreducible root systems have been completely classified by the Dynkin diagrams.

We denote the subgroup of $GL(V)$ generated by the reflections s_α, $\alpha \in R$ by W. Note that W permutes the finite set R. Hence W is a subgroup of the symmetric group on R. In particular, W is finite. W is called the Weyl group of the root system R. Every $w \in W$ can be written as a product of simple reflections. The length $l(w)$ of $w \in W$ is defined to be the minimum among the lengths of all expressions for $w \in W$ as a product of simple reflections. The simple reflection in the Weyl group corresponding to a simple root α is denoted by s_α. For simplicity of notation, the simple reflection corresponding to a simple root α_i is denoted by s_i.

If $l(w) = r$, then the expression $w = s_{i_1}s_{i_2}\ldots s_{i_r}$ is called a reduced expression for w. Note that for a given $w \in W$, the reduced expression may not be unique. There is a unique element in W of largest length, denoted by w_0 and called the longest element of the Weyl group W. The longest element w_0 is characterized by the property $w_0(R^+) = R^-$.

For any vector $\lambda \in V$ with $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in R$ is called an abstract weight. The set of all abstract weights forms a lattice Λ called the weight lattice. Note that R is contained in Λ. The lattice Λ_r generated by R is called the root lattice. There is a partial order $\leq$ on Λ defined by $\mu \leq \lambda$ if $\lambda - \mu$ is non negative integer linear combination of simple roots. This is called a dominance ordering on Λ.

Fix a basis S of R. An element $\lambda \in \Lambda$ is called dominant if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in R^+$; and regular (or strongly) dominant if $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in R^+$. We denote the set of all dominant weights by Λ^+. Each weight in V is conjugate under W to one and only one dominant weight. If λ is dominant, then $w(\lambda) \leq \lambda$, for all $w \in W$. Moreover, for $\lambda \in \Lambda^+$ the number of dominant weights μ such that $\mu \leq \lambda$ is finite.

Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Note that $\{2\alpha_i/(\alpha_i, \alpha_i) : i = 1, 2, \ldots, n\}$ is forms a basis of V. Let $\{\omega_1, \omega_2, \ldots, \omega_l\}$ be the dual basis, i.e., $2(\omega_i, \alpha_j)/(\alpha_i, \alpha_i) = \delta_{ij}$. ω_i's are called the fundamental dominant weights. Every element $\lambda \in V$ can be written as $\lambda = \sum m_i \omega_i$, where $m_i = \langle \lambda, \alpha_i \rangle$. Therefore, $\Lambda = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_l$ and $\Lambda^+ = \mathbb{Z}_{\geq 0}\omega_1 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\omega_l$. Since Λ and Λ_r are of same rank, the group Λ/Λ_r is finite; called the *fundamental group* of the root system R.

2.1.4 Root system of algebraic groups

Throughout this section let G be a connected and semisimple algebraic group. Let T be a maximal torus in G.

Definition 2.11.

1. A homomorphism $\chi : T \longrightarrow \mathbb{G}_m$ (respectively, $\chi : B \longrightarrow \mathbb{G}_m$) of algebraic groups is said to be a character of T (respectively, of B), where $\mathbb{G}_m$ is the multiplicative group k^*.
2. A homomorphism $\lambda : \mathbb{G}_m \longrightarrow T$ (respectively, $\lambda : \mathbb{G}_m \longrightarrow B$) of algebraic groups is said to be a one-parameter of T (respectively, of B).

Let $X(T) := Hom(T, \mathbb{G}_m)$ be the group of all characters of T. Let $Y(T) := Hom(\mathbb{G}_m, T)$ be the group of all one-parameter subgroups of T. Note that these groups are free Abelian of rank $n = dim(T) = rank(G)$. There is a non-degenerate pairing $(-\ ,-\ ) : X(T) \times Y(T) \longrightarrow \mathbb{Z}$, sending the pair (χ, λ) to the integer $r = \langle \chi, \lambda \rangle$ which satisfy: $\chi(\lambda(t)) = t^r$ for all $t \in \mathbb{G}_m$.

Let V be a finite dimensional T-module. Then, we have a decomposition

$$V = \bigoplus_{\chi \in X(T)} V_\chi \ ,$$

where $V_\chi = \{v \in V : t \cdot v = \chi(t)v \ \text{ for all } t \in T\}$. The spaces V_χ are called the weight spaces with respect to T; and $\chi \in X(T)$ is called a weight in V if $V_\chi \neq 0$.

Adjoint representation of G**:** Let $\mathfrak{g}$ be the Lie algebra of G. For any $g \in G$, $Int(g) : G \longrightarrow G$ is the automorphism of the algebraic group G defined by $Int(g)(h) = ghg^{-1},\ for\ all\ h \in G$. This induces an isomorphism $Ad(g) : \mathfrak{g} \longrightarrow \mathfrak{g}$ of Lie algebras. Hence, we get the morphism of algebraic groups $Ad : G \longrightarrow GL(\mathfrak{g})$, called the adjoint representation of G.

By the above discussion, if we restrict the adjoint representation of G to the maximal torus T, we have the following decomposition of $\mathfrak{g}$, called Cartan decomposition.

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in X(T)} \mathfrak{g}_\alpha \ ,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : Ad(t)(x) = \alpha(t)x\ for\ all\ t \in T\}$. We denote the set of non zero weights of $\mathfrak{g}$ by R. The weight spaces $\mathfrak{g}_\alpha$ for $\alpha \in R$ are called root spaces and α are called roots relative to the maximal torus T.

Recall that for any $\lambda \in X(T)$ and $\phi \in Y(T)$, we have $\lambda \circ \phi \in End(\mathbb{G}_m) \simeq \mathbb{Z}$. Hence, there is a unique integer $\langle \lambda, \phi \rangle$ such that $\lambda \circ \phi : a \mapsto a^{\langle \lambda, \phi \rangle}$ for all $a \in \mathbb{G}_m$. Note that the paring $\langle -, - \rangle : X(T) \times Y(T) \to \mathbb{Z}$ is bilinear and induces an isomorphism $Y(T) \simeq Hom_{\mathbb{Z}}(X(T), \mathbb{Z})$. Therefore, we can identify $X(T)$ with $Y(T)^*$. Using this identification we get a W-invariant non-degenerate bilinear form on $X(T) \otimes \mathbb{R}$.

Then we have,

Theorem 2.12. *The set The set R forms an abstract root system by viewing in side the vector space $X(T) \otimes \mathbb{R}$.*

For each $\alpha \in R$, fix a basis x_α of the one-dimensional root space $\mathfrak{g}_\alpha$. The set $\{x_\alpha : \alpha \in R\}$ along with $\{h_i : 1 \leq i \leq n\}$ forms the Chevalley basis of $\mathfrak{g}$ (for definition of Chevalley basis see [Hum72, Chapter VII]). By theorem of Serre, $\mathfrak{g}$ is generated as a Lie algebra by Chevalley basis with some relations.

Let B be the Borel subgroup of G corresponding to R^+. Let U be the unipotent radical of B. Then, we have

$$\mathrm{U} \simeq \prod_{\alpha \in R^+} \mathrm{U}_\alpha \quad ,$$

product is taken in some order.

Remark: Note that $BwB = Bw'B$ if and only if $w = w'$ in W.

Theorem 2.13 (Bruhat Decomposition)**.** *Let B be a Borel subgroup of G, and let T be a maximal torus of G contained in B. Then, G is the disjoint union of the double cosets BwB, as w ranges over a set of representatives in $N_G(T)$ of the Weyl group W. i.e.*

$$G = \bigsqcup_{w \in W} BwB.$$

Corollary 2.14. The product map $\pi : \mathrm{U}^- \times B \longrightarrow G$ defines an isomorphism of $\mathrm{U}^- \times B$ onto an open subset Ω of G. The open subset Ω is called the big cell of G.

2.1.5 Root system of Parabolic Subgroups

Let P be a parabolic subgroup of G containing a Borel subgroup B of G. Let $R(P)$ be the radical of P, let $R_u(P)$ be the unipotent radical of P. Let R_P^+ be the subset of R^+ defined by $R^+ \setminus R_P^+ = \{\alpha \in R^+ : \mathrm{U}_\alpha \subset R_u(P)\}$. Let $R_P^- = -R_P^+$, $R_P = R_P^+ \cup R_P^-$ and $S_P = S \cap R_P$. Then R_P is a subroot system of R called the root system associated to the parabolic subgroup P, with S_P as a set of simple roots and R_P^+ (respectively, R_P^-) as the set of positive (respectively, negative) roots of R_P relative to S_P.

On the other hand, given a subset J of S, let $R_J^+ := (\{\sum_{\beta \in J} a_\beta \beta : a_\beta \in \mathbb{Z}_{\geq 0}\}) \cap R^+$. Now define the subgroup P of G generated by B and $\mathrm{U}_{-\alpha}$, $\alpha \in R_J^+$. Note that P is a parabolic subgroup of G containing B such that $S_P = J$. Thus, we have the following theorem:

Theorem 2.15.

The set of parabolic subgroups of G containing B is in bijection with the power set of S.

Remarks:
1. If $P = B$, then $S_P = \emptyset$.
2. If $P = G$, then $S_P = S$.

Levi decomposition:
The subgroup of P generated by T and $\{\mathrm{U}_\alpha : \pm\alpha \in S_P\}$ is called the *Levi subgroup* corresponding to S_P, and is denoted by L_P. Note that that P is the semidirect product of $R_u(P)$ and L_P; called the *Levi decomposition* of P.

The set of all maximal parabolic subgroups containing B is in one-to-one correspondence with S. Namely, given $\alpha \in S$, the parabolic subgroup P such that $S_P = S \setminus \{\alpha\}$ is a maximal parabolic subgroup, and conversely. We denote the maximal parabolic subgroup P such that $S_P = S \setminus \{\alpha_i\}$ by $P_{\hat{i}}$.

2.1.6 The Weyl Group of a Parabolic Subgroup

Given a parabolic subgroup P of G, let W_P be the subgroup of W generated by $\{s_\alpha : \alpha \in S_P\}$. W_P is called the Weyl group of P. Note that $W_P \simeq N_P(T)/T$, where $N_P(T)$ is the normalizer of T in P. In each coset $wW_P \in W/W_P$, there exists a unique element of minimal length. Let W_P^{min} be the set of all minimal length representatives of W/W_P . We have

$$W_P^{min} = \{w \in W : l(ww') = l(w) + l(w'), \text{for all } w' \in W_P\}.$$

In other words, each element $w \in W$ can be written uniquely as $w = uv$, where $u \in W_P^{min}, v \in W_P$ such that $l(w) = l(u) + l(v)$. The set W_P^{min} can also be characterized as

$$W_P^{min} = \{w \in W : w(\alpha) > 0, \text{for all } \alpha \in S_P\}.$$

W_P^{min} is also denoted by W^P. Similarly, in each coset $wW_P \in W/W_P$, there exists a unique element of maximal length and the set W_P^{max} of all maximal length representatives of W/W_P is equal to $\{w \in W : w(\alpha) < 0, \text{for all } \alpha \in S_P\}$. Further, if w_P is the unique element of maximal length in W_P, then we have

$$W_P^{max} = \{ww_P : w \in W_P^{min}\}.$$

If P is the parabolic subgroup corresponding to a subset I of S, then W_P (respectively, W^P) is also denoted by W_I (respectively, W^I).

2.1.7 Representations of algebraic groups.

Let G be a semisimple algebraic group. Let B be a Borel subgroup of G.

Definition: Let V be a G-module, let λ be a character of B. A non zero vector $v \in V$ is said to be a maximal weight vector of weight λ if;
(i) $b \cdot v = \lambda(b)v$ for all $b \in B$ and
(ii) v is fixed by U_β for all $\beta \in R^+$.

Remark: If V is non zero finite dimensional G-module, then maximal weight vector exists.

Theorem 2.16. *Let V be an irreducible G-module. Then, we have*
1. There is a unique B-stable one-dimensional subspace of V, spanned by a maximal weight vector for some $\lambda \in X(T)^+$.
2. The dimension of the weight space V_λ is 1.
3.The weights μ of V satisfy $\mu \leq \lambda$.
4. The Weyl group W permutes the set of all weights of V, and $dim(V_\mu) = dim(V_{w(\mu)})$ for $w \in W$.
5. If V' is another irreducible G-module, of highest weight λ then V is isomorphic to V'.
6. Let $\lambda \in X(T)$ be dominant. Then there exists an irreducible G-module of highest weight λ, and we denote it by $V(\lambda)$.

Theorem 2.17. *There is a one-one correspondence between $X(T)^+$ and the isomorphism classes of finite dimensional irreducible G-modules given by $\lambda \mapsto V(\lambda)$.*

Weyl Dimension Formula: Let $V(\lambda)$ be an irreducible representation of a semisimple algebraic group G with highest weight λ. Let $\rho = \frac{1}{2} \sum\limits_{\alpha \in R^+} \alpha$. Then, the dimension of $V(\lambda)$ is given by

$$dim(V(\lambda)) = \frac{\prod\limits_{\alpha \in R^+} \langle \lambda + \rho, \alpha \rangle}{\prod\limits_{\alpha \in R^+} \langle \rho, \alpha \rangle}.$$

Weyl Character Formula: We introduce symbols e^λ for $\lambda \in X(T)$ with the property $e^\lambda.e^\mu = e^{\lambda+\mu}$.

The character of an irreducible representation $V(\lambda)$ is given by

$$ch(V(\lambda)) = \frac{\sum\limits_{w \in W} (-1)^{l(w)} (e^{w(\lambda+\rho)})}{e^\rho \prod\limits_{\alpha \in R^+} (1 - e^{-\alpha})}.$$

2.2 Flag varieties and Schubert Varieties

In this section, we recall some basics on Schubert varieties and its geometric properties.

2.2.1 Schubert Varieties

Fix an algebraically closed field k of arbitrary characteristic. Let V be a vector space of dimension n over k.

Definition: The Grassmannian is the set of all r-dimensional subspaces of V, we denote it by $Gr(r, V)$. It is a smooth projective variety of dimension $r(n-r)$. A flag in a vector space

V is a chain $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_m = V$ of subspaces of V for some $1 \leq m \leq n$ with $V_i \neq V_{i+1}$ for every $i = 0, 1, \ldots, m-1$. A flag is called a full flag if $m = n$.

Let $\mathfrak{F}(V)$ be the set of all full flags in V. It is easy to see that the set $\mathfrak{F}(V)$ admits a natural structure of a projective variety. The variety $\mathfrak{F}(V)$ is called flag variety. Note that the flag variety $\mathfrak{F}(V)$ is isomorphic to the homogeneous space $SL(n,k)/B$, where B is the set of all upper triangular matrices in $SL(n,k)$.

More generally, let G be a connected semisimple algebraic group over k and B be a Borel subgroup of G. Let $\mathfrak{B}$ be the set of all Borel sub groups of G. The variety G/B can be identified with $\mathfrak{B}$. In fact, we have the following:

Proposition 2.18. *The set $\mathfrak{B}$ endowed with a structure of variety such that it is isomorphic to the homogeneous space G/B.*

Let P be a parabolic subgroup of G. The projective variety G/P is called a generalized flag variety.

Let G be a connected semisimple algebraic group over k of rank n. Let T be a maximal torus of G and B be a Borel subgroup of G containing T, let P be a parabolic subgroup of G. Let $N(T)$ be the normalizer of T in G and let $W = N(T)/T$ be the Weyl group of G.

Bruhat decomposition of G relative to P: For $w \in W$, let n_w be a lift of w in $N(T)$. Observe that the double coset Bn_wP in G depends only on the set wW_P in W but not on w or n_w. We write BwP for Bn_wP and we call it the open Bruhat cell in G associated to wW_P. The Zariski closure of BwP is called closed Bruhat cell in G associated to wW_P. The Bruhat decomposition of G relative to P is

$$G = \bigsqcup_{w \in W^P} BwP.$$

Note that when $P = B$ we get the Bruhat decomposition of G that we discusses before.

For $w \in W/W_P$, the B-orbit $C_P(w) = BwP/P$ in G/P is a locally closed subset of G/P, called the *Schubert cell or Bruhat cell*. The Zariski closure of $C_P(w)$ with the canonical reduced structure is the called *Schubert variety* associated to w, and is denoted by $X_P(w)$. Thus, the Schubert varieties in G/P are indexed by W^P.

Note that if $P = B$, then $W_P = \{id\}$, and the Schubert varieties in G/B are indexed by the elements of W. We denote the Schubert variety corresponding to $w \in W$ by $X(w)$.

Dimension of $X_P(w)$: If $P = B$, then for $w \in W$, we have

$$C_B(w) \simeq \prod_{\{\alpha \in R^+ : w^{-1}(\alpha) < 0\}} U_\alpha$$

Since $|\{\alpha \in R^+ : w^{-1}(\alpha) < 0\}| = l(w)$, $C_B(w)$ is isomorphic to the affine space $k^{l(w)}$. Hence we have

$$dimX(w) = dimC_B(w) = l(w).$$

For a general parabolic P, consider $w \in W/W_P$ and denote the unique representative for w in W_P^{min} (respectively, W_P^{max}) by w_P^{min} (resp. w_P^{max}). Now under the canonical projection $\pi_P : G/B \to G/P$, $X(w_P^{min})$ maps birationally onto $X_P(w)$, and $X(w_P^{max}) = \pi_P^{-1}(X_P(w))$. Hence we obtain

$$dimX_P(w) = dimX(w_P^{min}) = l(w_P^{min}).$$

Note that $G/B = X(w_0)$, w_0 being the longest element in W. The cell $C_B(w_0)$ is the unique cell of maximal dimension (= $l(w_0) = |R^+|$); it is affine, open and dense in G/B, called the *big cell* of G/B. It is denoted as $\mathcal{O}$. Let $B^- = w_0 B w_0^{-1}$ be the opposite Borel subgroup of B in G determined by T. The B^- orbit $B^- idB/B$ is affine, open and dense subset of G/B, and is called the *opposite big cell* of G/B, and it is denoted as $\mathcal{O}^-$. For a $w \in W$, $Y(w) = X(w) \cap \mathcal{O}^-$ is called the opposite cell in $X(w)$.

There is a partial order on W_P, known as the Bruhat order, induced by the partial order on the set of Schubert varieties given by inclusion, namely, for $w_1, w_2 \in W_P, w_1 \geq w_2 \iff X_P(w_1) \supseteq X_P(w_2)$.

The Bruhat decomposition of G/P and $X_P(w)$ are induced by the Bruhat decomposition of G/B.

$$G/P = \bigsqcup_{w \in W^P} BwP/P$$

and

$$X_P(w) = \bigsqcup_{\{w' \in W^P,\ w' \leq w\}} Bw'P/P.$$

2.2.2 Picard group of G/B

Let $\tilde{G}$ be a simply connected covering of G and let $\tilde{B}$ and $\tilde{T}$ be the Borel subgroup and maximal subgroups of $\tilde{G}$ corresponding to B and T in G.

Recall that the root system in $\tilde{G}$ with respect to $\tilde{T}$ is same as the root system in G with respect to T, $X(\tilde{T})$ is subgroup of $X(\tilde{T}) \otimes \mathbb{Q} = X(T) \otimes \mathbb{Q}$ generated by fundamental weights ω_i. Further, $\tilde{G}/\tilde{B}$ is isomorphic to G/B. In fact, $\tilde{G}/\tilde{P}$ is isomorphic to G/P for any parabolic subgroup P in G, $\tilde{P}$ being corresponding parabolic subgroup in $\tilde{G}$.

Note that the character group $X(\tilde{T})$ coincides with the weight lattice Λ. Since $X(\tilde{B}_u)$ is trivial; every character λ of T extends to a character of B. Hence, we have $X(T) = X(B)$. The canonical map $G \longrightarrow G/B$ is a principal B bundle.

Let $\lambda \in X(B)$. Set $G \times^B k = G \times k/\sim$, where $\sim$ is the equivalence relation defined by $(gb, \lambda(b)x) \sim (g, x), g \in G, b \in B, x \in k$. $G \times^B k$ is a total space of a line bundle over G/B and we denote this line bundle by $\mathcal{L}(\lambda)$. Let $Pic(G/B)$ be the Picard group of G/B which is by definition, the group of isomorphism classes of line bundles on G/B. Thus we get a map

$$\mathcal{L} : X(T) \to Pic(G/B),\ \lambda \mapsto \mathcal{L}(\lambda)$$

We have the following theorem due to Chevalley [Car05].

Theorem 2.19 (Chevalley)**.** *The map $\mathcal{L}$ is an isomorphism if G is simply connected.*

On the other hand, consider the prime divisors $X(w_0 s_i)$, $1 \leq i \leq n$ on G/B. Let $\mathcal{L}_i = \mathcal{O}_{G/B}(X(w_0 s_i))$ be the line bundle defined by $X(w_0 s_i)$, $1 \leq i \leq n$. Recall that the Picard group $Pic(G/B)$ is a free abelian group generated by the $\mathcal{L}_i$'s, and under the isomorphism $\mathcal{L} : X(T) \simeq Pic(G/B)$, we have $\mathcal{L}(\omega_i) = \mathcal{L}_i$, $1 \leq i \leq l$ (see [Car05]). Thus for $\lambda = \sum_{i=1}^{n} \langle \lambda, \alpha_i \rangle \omega_i$, we have $\mathcal{L}(\lambda) = \otimes_{i=1}^{n} \mathcal{L}_i^{\otimes \langle \lambda, \alpha_i \rangle}$.

For a general parabolic P, any $\lambda \in X(T)$ can not be lifted to a character of P always. To be a character of P the weight λ must be orthogonal to the positive roots of P. Therefore, λ must be an integral linear combination of the fundamental weights, $\omega_1, \ldots, \omega_r$ dual to the simple roots in $S \setminus S_P$. We call $\omega_1, \cdots, \omega_r$ the fundamental weights of P and the sublattice $\Lambda_P \subset \Lambda$ they generate the weights of P.

A line bundle $\mathcal{L}$ on an algebraic variety X is very ample if there exists an immersion $i : X \hookrightarrow \mathbb{P}^n$ such that $i^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{L}$. A line bundle $\mathcal{L}$ on X is ample if $\mathcal{L}^m$ is very ample for some positive integer $m \geq 1$. A line bundle $\mathcal{L}$ on X is said to be *numerically effective*, if the degree of the restriction to any algebraic curve in X is non-negative.

In the following theorem we recall some well known facts about line bundles on homogeneous spaces G/P (for example see [Sno93, Proposition 1.1]).

Theorem 2.20. *Let $X = G/P$, where G is a semisimple algebraic group and P is a parabolic subgroup. Let $\omega_1, \ldots, \omega_r$ be the fundamental weights of P and let $\lambda \in \Lambda_P$. Then*

1. $X = X_1 \times \cdots \times X_m$, where $X_i = G_i/P_i$, G_i is a simple algebraic group and P_i is a parabolic subgroup of G_i, $i = 1, \ldots, m$.

2. $\mathcal{L} = pr_1^* \mathcal{L}_1 \otimes \cdots \otimes pr_m^* \mathcal{L}_s$, where $\mathcal{L}_i$ is a line bundle on X_i, $i = 1, \ldots, s$.

3. $Pic(X) \simeq \Lambda_P$. In particular, $Pic(X) \simeq \mathbb{Z}$ if P is a maximal parabolic subgroup of G.

4. *$\mathcal{L}$ is numerically effective (nef) if and only if λ is dominant.*
5. *$\mathcal{L}$ is ample if and only if it is very ample if and only if $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in S \setminus S_P$.*

As we have described above, let E denote the total space of the line bundle $\mathcal{L}(\lambda)$ over G/B. Let $\sigma : E \to G/B$ be the canonical map $\sigma([g,c]) = gB$. Let

$$M_\lambda = \{f \in k[G] : f(gb) = \lambda(b)f(g),\ g \in G,\ b \in G\}.$$

Then M_λ can be identified with the space global of sections $H^0(G/B, L(\lambda)) := \{s : G/B \to E : \sigma \circ s = id_{G/B}\}$. This identification preserves the respective G-module structures.

2.2.3 Cohomology of Line bundles on Schubert varieties

We recall some results on cohomology of line bundles. We start this section by stating the Borel-Weil theorem which gives a geometric realization of irreducible representations of a semisimple algebraic group G.

Theorem 2.21 (Borel-Weil)**.** *Assume that char $k = 0$, let $\lambda \in X(B)$.*

1. *$H^0(G/B, \mathcal{L}(\lambda)) \neq 0$ if and only if λ is dominant.*
2. *If λ is dominant, $H^j(G/B, \mathcal{L}(\lambda) = 0$ for all $j \geq 1$.*

Theorem 2.22 (Borel-Weil-Bott)**.** *In characteristic 0, we have*

1. *If $\lambda + \rho$ is singular (i.e there is a $\beta \in R^+$ such that $\langle \lambda + \rho, \beta \rangle = 0$). Then, we have*

$$H^j(G/B, \mathcal{L}(\lambda) = 0 \text{ for all } j.$$

2. *If $\lambda + \rho$ is non singular,*
(i) $H^{l(w)}(G/B, \mathcal{L}_(\lambda)) = H^0(G/B, \mathcal{L}(w \cdot \lambda)$ for $w \cdot \lambda$ is dominant.
(ii) $H^j(G/B, \mathcal{L}(\lambda)) = 0$ for $j \neq l(w)$.

Theorem 2.23. *Let Char$k = 0$. If $\lambda \in X(T)$ dominant, then $H^0(G/B, \mathcal{L}(\lambda)) = V(\lambda)^*$.*

Corollary 2.24. *In characteristic zero, the map $\lambda \longmapsto H^0(G/B, \mathcal{L}(\lambda))$ gives a bijection between $X(T)^+$ and the set of all finite dimensional irreducible representaions of G.*

Thus in characteristic zero, we have a geometric realization of all irreducible representaions of G.

Assume that *Char* k= p. In general, the G-module $H^0(G/B, \mathcal{L}(\lambda))$ need not be irreducible. For example, let $G = SL(2,k)$, we have $G/B = \mathbb{P}^1$. Let $L = \mathcal{O}_{\mathbb{P}^1}(1)$. Note that $\mathcal{L}(\lambda) = L^r$ for some r and $H^0(G/B, L^r) = Sym^r(k^2)$. Let $r = np$ and $V = \{f^p, f \in Sym^n(k^2)\}$. Then, V is a G-stable proper subspace of $H^0(G/B, L^r)$ and hence $H^0(G/B, L^r)$ is not irreducible.

In any characteristic, we have

Theorem 2.25. *The followings are equivalent:*
1. λ is dominant.
2. $H^0(G/B, \mathcal{L}(\lambda))$ is non zero.

For any G-module M, the sum of all its simple submodules is called the *socle* of M and denoted by $soc_G(M)$. Set $E(\lambda) = soc_G(H^0(G/B, \mathcal{L}(\lambda)))$.

Theorem 2.26. *$E(\lambda)$ is irreducible G-module and conversely every irreducible G-module is isomorphic to $E(\lambda)$ for some $\lambda \in X(T)^+$.*

The following theorem gives the vanishing of cohomology of line bundles on Schubert varieties.

Theorem 2.27. *Let $\lambda \in X(T)^+$. Then we have:*
1. $H^i(X(w), \mathcal{L}(\lambda)) = 0$ for all $i \geq 1$.
2. The restriction map $H^0(G/B, \mathcal{L}(\lambda)) \longrightarrow H^0(X(w), \mathcal{L}(\lambda))$ is surjective.

The above theorem first proved in Char $k = 0$ by Demazure in [Dem74]. In Char $k = p > 0$, it was proved by various methods: 1. H.H Andersen proved by using "Characteristic p methods"(for example, see [And85]). 2. Mehta, Ramanan and Ramanathan proved by using Frobenius splitting methods (see [MR85] and [BK07]). 3. Lakshmibai, Musili and Seshadri proved by using standard monomial theory (see [BLPM+12]).

For non-dominant weights, the vanishing results of line bundle on Schubert varieties in characteristic zero has been studied in [BKS04]. In the case of Kac-Moody setting, the cohomology of line bundles on Schubert has been studied in [Kan07].

We recall the following vanishing results from [Kan13] (see [Kan13, Corollary 3.6] and [Kan13, Corollary 4.10]).

Assume that the base field is the field $\mathbb{C}$ of complex numbers.

Lemma 2.28. *Let $w \in W$, and $\alpha \in R^+$. Then, we have*
1. $H^i(w, \mathcal{L}(\alpha)) = 0$ for all $i \geq 2$.
2. If G is simply laced, $H^i(w, \mathcal{L}(\alpha)) = 0$ for all $i \geq 1$.

We recall the following result of Bott on cohomology of the tangent bundle $T_{G/B}$ of G/B from [Bot57].

Theorem 2.29 (Bott)**.**

1. $H^j(G/B, T_{G/B}) = 0$ *for all* $j \leq 1$.
2. $H^0(G/B, T_{G/B})$ *is the adjoint representation* $\mathfrak{g}$ *of* G.

By abuse of notation, we denote the restriction $T_{G/B}$ to $X(w)$ by $T_{G/B}$. Now, we state the following theorem from [Kan13] (see [Kan13, Theorem 3.7, Theorem 3.8 and Theorem 4.11]).

Theorem 2.30 (Senthamarai Kannan)**.** *Let* $w \in W$*. Then, we have*
1. $H^i(X(w), T_{G/B}) = (0)$ *for every* $i \geq 1$.
2. The adjoint representation $\mathfrak{g}$ *of* G *is a* B*-submodule of* $H^0(X(w), T_{G/B})$ *if and only if* $w^{-1}(\alpha_0) < 0$.
3. If G *is simply laced,* $H^0(X(w), T_{G/B})$ *is the adjoint representation* $\mathfrak{g}$ *of* G *if and only if* $w^{-1}(\alpha_0) < 0$.
4. Assume that G *is simply laced and* $X(w)$ *is a smooth Schubert variety. Let* $Aut^0(X(w))$ *be the connected component of the automorphism group of* $X(w)$ *containing the identity automorphism. Let* P_w *denote the stabilizer of* $X(w)$ *in* G*. Let* $\phi_w : P_w \longrightarrow Aut^0(X(w))$ *be the homomorphism induced by the action of* P_w *on* $X(w)$*. Then, we have*
(i) $\phi_w : P_w \longrightarrow Aut^0(X(w))$ *is surjective.*
(ii) $\phi_w : P_w \longrightarrow Aut^0(X(w))$ *is an isomorphism if and only if* $w^{-1}(\alpha_0) < 0$.

2.2.4 Geometry of Schubert varieties

In this section, we briefly recall some geometric properties of Schubert varieties. A very good reference for this is the book "Singular loci of Schubert varieties" by Sara Billey and V Lakshmibai [BL00].

Schubert varieties are non singular in co-dimension one (that is, the singular locus has dimension at least 2), (arithmetically) normal, (arithmetically) Cohen-Macaulay and have rational singularities (see [And85], [BK07], [BLPM+12] and [Ses85]). Note that in general, Schubert varieties are need not be smooth.

The first result on singular locus of Schubert varieties is due to Lakshmibai and Seshadri for classical groups by using the Standard monomial theory and the Jacobian criterion for smoothness. Lakshmibai started determining explicit basis of the tangent cones to the Schubert varieties which will be useful to determine singularities and multiplicities of singular points.

Geometry of Schubert varieties closely related to combinatorics of the Weyl group and representation theory. For example, smoothness, rational smoothness, Gorensteinness and local complete intersection properties are characterized using pattern avoidance (see [BL00], [Car11], [WY06] and [ÚW11]).

Using Kazhdan-Luszting polynomials, Nil-Hecke ring and tangent cone it has been studied the rational smoothness and multiplicity of a point in Schubert variety (for more details see [BL00]). Gorensteinness (respectively, local complete intersection) property of Schubert varieties are studied in [WY06](respectively, [ÚW11]) by using the combinatorics of the Weyl group W.

Demazure and Hansen independently gave a nice desingularization of Schubert varieties by iterated $\mathbb{P}^1$-fibrations that we will discuss in the next section.

2.2.5 Bott-Samelson-Demazure-Hansen Varieties

For $w \in W$, recall $X(w) := \overline{BwB/B}$ denote the Schubert variety in G/B corresponding to w. Given a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ of w, with the corresponding tuple $\underline{i} := (i_1, \ldots, i_r)$, we denote by $Z(w, \underline{i})$ the desingularization of the Schubert variety $X(w)$, which is now known as Bott-Samelson-Demazure-Hansen variety. This was first introduced by Bott and Samelson in a differential geometric and topological context (see [BS58]). Demazure in [Dem74] and Hansen in [Han73] independently adapted the construction in algebro-geometric situation, which explains the reason for the name. For the sake of simplicity, we will denote a Bott-Samelson-Demazure-Hansen variety by BSDH-variety.

For a simple root $\alpha \in S$, we denote by P_α the minimal parabolic subgroup of G generated by B and n_α, a lift of s_α in $N_G(T)$.

We recall that the BSDH-variety corresponding to a reduced expression $\underline{i}$ of $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ is defined by

$$Z(w, \underline{i}) = \frac{P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}}{B \times \cdots \times B},$$

where the action of $B \times \cdots \times B$ on $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}$ is given by $(p_1, \ldots, p_r)(b_1, \ldots, b_r) = (p_1 \cdot b_1, b_1^{-1} \cdot p_2 \cdot b_2, \ldots, b_{r-1}^{-1} \cdot p_r \cdot b_r)$, $p_j \in P_{\alpha_{i_j}}$, $b_j \in B$ and $\underline{i} = (i_1, i_2, \ldots, i_r)$ (see [Dem74, p.73, Definition 1], [BK07, p.64, Definition 2.2.1]).

For each reduced expression $\underline{i}$ of w, $Z(w, \underline{i})$ is a smooth projective variety and the orbit map

$$P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}} \longrightarrow Z(w, \underline{i})$$

is a locally trivial principal B^n bundle.

Define a morphism

$$\phi_w : Z(w, \underline{i}) \longrightarrow G/B$$

by

$$[p_1, p_2, \ldots, p_r] \longmapsto p_1 p_2 \cdots p_r B$$

This morphism can be seen as follows.

Let $m : P_{\alpha_1} \times P_{\alpha_2} \times \cdots \times P_{\alpha_r} \longrightarrow G$ be the multiplication map given by $(p_1, p_2, \ldots, p_r) \longmapsto p_1 p_2 \cdots p_r$.

Now consider the following commutative diagram:

$$\begin{array}{ccc} P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \ldots \times P_{\alpha_{i_r}} & \xrightarrow{\quad m \quad} & G \\ \downarrow & & \downarrow \\ Z(w, \underline{i}) = P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \ldots \times P_{\alpha_{i_r}}/B \times \cdots \times B & \xrightarrow{\quad \phi_w \quad} & G/B \end{array}$$

Since $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is a reduced expression of w, we have

$$BwB = Bs_{i_1} Bs_{i_2} Bs_{i_3} B \cdots Bs_{i_r} B.$$

Since $Bs_{i_j}B$ is open in $P_{\alpha_{i_j}}$, $Z^0(w, \underline{i}) := (Bs_{i_1} Bs_{i_2} Bs_{i_3} B \cdots Bs_{i_r} B)/B^n$ is open in $Z(w, \underline{i})$.

Observe that the image $\phi_w(Z(w, \underline{i}))$ of $Z(w, \underline{i})$ in G/B is $X(w)$ and $Z^0(w, \underline{i})$ is isomorphic to the Schubert cell BwB/B.

Hence, ϕ_w is a birational surjective morphism from $Z(w, \underline{i})$ to $X(w)$.

Thus, we have

Theorem 2.31. *The morphism $\phi_w : Z(w, \underline{i}) \longrightarrow X(w)$ is a desingularization of $X(w)$.*

Let $f_r : Z(w, \underline{i}) \longrightarrow Z(ws_{i_r}, \underline{i}')$ denote the map induced by the projection

$$P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}} \longrightarrow P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_{r-1}}},$$

where $i' = (i_1, i_2, \ldots, i_{r-1})$. We first note that f_r is a $P_{\alpha_{i_r}}/B \simeq \mathbb{P}^1$-fibration. In fact we have, the following commutative diagram.

$$\begin{array}{ccc} Z(ws_{\alpha_{i_r}}) \times_{G/P_{\alpha_{i_r}}} G/B = Z(w) & \longrightarrow & G/B \\ {\scriptstyle f_r}\downarrow & & \downarrow \\ Z(ws_{\alpha_{i_r}}) & \longrightarrow & G/P_{\alpha_{i_r}} \end{array}$$

Then we observe that f_n is a $P_{\alpha_{i_r}}/B \simeq \mathbf{P}^1$-fibration.

Let $\sigma_r : P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_{r-1}}} \longrightarrow P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}$ be the inclusion map. It induces a closed immersion $Z(ws_{i_r}) \longrightarrow Z(w)$ and for convenience of notation we also denote it by σ_r.

Let L_α denote the Levi subgroup of P_α containing T for $\alpha \in S$. We denote by B_α the intersection of L_α and B. Then L_α is the product of T and a homomorphic image G_α of $SL(2,\mathbb{C})$ via a homomorphism $\psi : SL(2,\mathbb{C}) \longrightarrow L_\alpha$ (see [Jan07, II, 1.3]).

Homogeneous vector bundles and its cohomology groups:

Let $B'_\alpha := B_\alpha \cap G_\alpha \subset L_\alpha$. We note that the morphism $G_\alpha/B'_\alpha \longrightarrow L_\alpha/B_\alpha$ induced by the inclusion is an isomorphism. Since $L_\alpha/B_\alpha \hookrightarrow P_\alpha/B$ is an isomorphism, to compute the cohomology groups $H^i(P_\alpha/B, \mathcal{L}(V))$ for any B-module V; we treat V as a B_α-module and we compute $H^i(L_\alpha/B_\alpha, \mathcal{L}(V))$. Here, $\mathcal{L}(V)$ is the homogeneous vector bundle on P_α/B associated to the B-module V.

For a B-module V, let $\mathcal{L}(w, V)$ denote the restriction of the associated homogeneous vector bundle on G/B to $X(w)$. By abuse of notation we denote the pull back of $\mathcal{L}(w, V)$ via ϕ_w to $Z(w, \underline{i})$ also by $\mathcal{L}(w, V)$, when there is no cause for confusion. Then, we have

Lemma 2.32. *There is an isomorphism of B-linearized sheaves:*
1. $R^j f_{r*}\mathcal{L}(w, V) = \mathcal{L}(ws_{i_r}, (H^j(P_{\alpha_{i_r}}/B, \mathcal{L}(w, V) |_{(P_{\alpha_{i_r}}/B)}))$ *for all* $j \geq 0$.
2. $\sigma_r^*\mathcal{L}(w, V) = \mathcal{L}(ws_{i_r}, V)$

Let $\phi_w : Z(w, \underline{i}) \longrightarrow X(w)$ be the desingularization map as defined above.

We have the following theorem:

Theorem 2.33.

1. $R^i\phi_{w*}\mathcal{O}_{Z(w,\underline{i})} = 0$ *for* $i > 0$.
2. $\phi_{w*}\mathcal{O}_{Z(w,\underline{i})} = \mathcal{O}_{X(w)}$.
3. *For any locally free sheaf* $\mathcal{F}$ *on* $X(w)$*, we have* $H^i(X(w), \mathcal{F}) \simeq H^i(Z(w, \underline{i}), \phi_w^*\mathcal{F})$, $i \geq 0$.

We use the following *ascending 1-step construction* as a basic tool in computing cohomology modules.

For $w \in W$, let $l(w)$ denote the length of w. Let γ be a simple root such that $l(w) = l(s_\gamma w) + 1$. Let $Z(w, \underline{i})$ be a BSDH-variety corresponding to a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$, where $\alpha_{i_1} = \gamma$. Then we have an induced morphism

$$g : Z(w, \underline{i}) \longrightarrow P_\gamma/B \simeq \mathbb{P}^1,$$

with fibres $Z(s_\gamma w, \underline{i}')$, where $\underline{i}' = (i_2, i_3, \ldots, i_r)$.

By an application of the Leray spectral sequence together with the fact that the base is $\mathbb{P}^1$, we obtain for every B-module V, the following short exact sequence of P_γ-modules:

$$0 \to H^1(P_\gamma/B, R^{j-1}g_*\mathcal{L}(w,V)) \to H^j(Z(w,\underline{i}), \mathcal{L}(w,V)) \to H^0(P_\gamma/B, R^j g_*\mathcal{L}(w,V)) \to 0.$$

Since for any B-module V, the vector bundle $\mathcal{L}(w,V)$ on $Z(w,\underline{i})$ is the pull back of the homogeneous vector bundle from $X(w)$, we conclude that

$$H^j(Z(w,\underline{i}),\ \mathcal{L}(w,V)) \cong H^j(X(w),\ \mathcal{L}(w,V))$$

(see [BK07, Theorem 3.3.4 (b)]), and are independent of the choice of the reduced expression $\underline{i}$. Hence we denote $H^j(Z(w,\underline{i}),\ \mathcal{L}(w,V))$ by $H^j(w,V)$. For a character λ of B, we denote the one dimensional B-module corresponding to λ by $\mathbb{C}_\lambda$. Further, we denote the cohomology modules $H^j(Z(w,\underline{i}),\ \mathcal{L}(w,\mathbb{C}_\lambda))$ by $H^j(w,\lambda)$.

Rewriting the above short exact sequence using these simple notation, we have the following short exact sequence:

$$0 \to H^1(s_\gamma, H^{j-1}(s_\gamma w, V)) \to H^j(w,V) \to H^0(s_\gamma, H^j(s_\gamma w, V)) \to 0.$$

The B-modules V we deal with, satisfy $R^j g_*\mathcal{L}(w,V) = 0$ for all $j \geq 2$. Moreover, we use only the following two special cases of the above short exact sequence, which we denote by *SES*.

1. For $j = 0$, we have $H^0(w,V) \simeq H^0(s_\gamma, H^0(s_\gamma w, V))$.
2. For $j = 1$, we have $0 \to H^1(s_\gamma, H^0(s_\gamma w, V)) \to H^1(w,V) \to H^0(s_\gamma, H^1(s_\gamma w, V)) \to 0$.

The construction of BSDH-variety is depends on the choice of a reduced expression $\underline{i}$ of w. If we change the reduced expression, it is not clear that the BSDH-varieties are isomorphic or not. So, it is natural to ask that for a given $w \in W$ whether the BSDH-varieties corresponding to two different reduced expressions of w are isomorphic ? In order to address this question, in this thesis, we study the automorphism group of the BSDH-varieties.

2.3 Invariant Theory

In this section, we discuss the invariant theory of finite groups as well as reductive algebraic groups. We also recall some preliminaries and notation from Geometric invariant Theory.

2.3.1 Ring of Invariants

The standard setting of invariant theory is as follows: If G is a finite group acting linearly on a vector space V over an algebraically closed field k, then it induces an action on $k[V]$, the algebra of polynomial functions on V, the action is given by $(gf)(v) := f(g^{-1}.v)$ for all $g \in G, v \in V, f \in K[V]$. The ring of G-invariant polynomials is define by $k[V]^G := \{f \in k[V] : gf = f \; \forall\, g \in G\}$. If G is a linear algebraic group acting on an affine variety X, then it defines an action on the coordinate ring $k[X]$ of X and $k[X]^G := \{f \in k[X] : gf = f \; \forall\, g \in G\}$. When $X = V$ is a representation of G, the G action on $k[V]$ preserves degree and $k[V]^G \subseteq k[V]$ inherits the grading.

The basic question in invariant theory is the following:
What is the structure of the $k[V]^G$? For example, when $k[V]^G$ is finitely generated? If it is finitely generated then find the generators, relations for $k[V]^G$, and the degree bounds for the generators. When is the $k[V]^G$ a polynomial ring ?

2.3.2 Finite Generation of Ring of Invariants

In general, it is a difficult problem to compute the ring of invariants. This was a major topic of research in 19th century. In 1868, Paul Gordan proved (constructively) for the action of $G = SL_2(\mathbb{C})$ on the finite dimensional complex vector space V, the ring of invariants $\mathbb{C}[V]^G$ is finitely generated over $\mathbb{C}$. In 1890, David Hilbert proved that (in a non-constructive way) whenever G linearly reductive acting on a finite dimensional complex vector space V, the ring $\mathbb{C}[V]^G$ of invariants is finitely generated $\mathbb{C}$-algebra and he proposed a general question of finite generation of invariant rings for arbitrary groups. This problem is now known as Hilbert's 14th problem.

For finite groups, Hilbert's 14'th problem has been solved affirmatively: Emmy Noether firstly considered the problem specifically for the case when $k = \mathbb{C}$, where she was able to find constructive procedures to compute generaters and relations explicitly. She also proved that the invariant ring is finitely generated, if G is a finite group and k is an arbitrary field.

We recall the following.

Theorem 2.34. *(Hilbert [Hil90], Noether [Noe15, Noe26]). Let G be a finite group. Then the $k[V]^G$-module $k[V]$ is finitely generated and $k[V]^G$ is a finitely generated k-algebra.*

In general, the answer to Hilbert's 14'th problem is negative: In 1958, Masayoshi Nagata gave a counter example to this.

For non reductive groups, the ring of invariant may be finitely generated. For example:

Theorem 2.35 (Weitzenböck [Wei32]). *If V is a representation of $G = \mathbb{G}_a$, the one dimensional connected additive group over a field k of characteristic 0. Then $k[V]^G$ is finitely generated.*

Recall the following definitions:

Linear Reductivity : A linear algebraic group G is called linearly reductive if for any rational representation V of G and any nonzero invariant vector $v \in V$ there exists a G-invariant linear function f on V such that $f(v) \neq 0$. Equivalently, every rational representation V of G is completely reducible.

Geometric Reductivity: A linear algebraic group G is called geometrically reductive if for any rational representation V and any nonzero invariant vector $v \in V$ there exists a G-invariant homogeneous polynomial f on V such that $f(v) \neq 0$.

For any algebraic closed field, we have

Theorem 2.36. *The following are equivalent:*

1. G is reductive

2. G is geometric reductive.

If *char* $\mathrm{k} = 0$, then we have

Theorem 2.37. *Let G an algebraic group. The followings are equivalent:*

1. G is reductive.

2. G is linear reductive.

3. G is geometrically reductive.

Clearly, in any characteristic, if G is linear reductive then it is geometric reductive. But the converse is not true, for example, a non-trivial finite p-group in characteristic p is geometrically reductive but not linearly reductive.

We have the following characterization of linearly reductive groups in positive characteristic.

Theorem 2.38 (Nagata [N+61]). *Let G be an algebraic group, let char $k = p$. Then the following are equivalent:*

1. G is linearly reductive.

2. The connected component G^0 of G is a torus and $|G/G^0|$ is not divisible by $p =$ char k.

Thus, the linearly reductive groups are completely classified.

1. Finite groups whose order is not divisible by p,

2. Torus, and

3. Extensions of tori by finite groups whose order is not divisible by p.

In 1964 Nagata proved the following finiteness theorem for geometrically reductive groups.

Theorem 2.39. *(Nagata [N+63]). If X is an affine G-variety and G is a geometrically reductive group, then $k[X]^G$ is finitely generated. In particular if V is a representation of G, then $k[V]^G$ is a finitely generated k-algebra.*

The converse is also true. Popov proved the following.

Theorem 2.40. *(Popov [Pop79]). If $k[X]^G$ is a finitely generated k-algebra for every affine G-variety X, then G must be reductive.*

Note that any finite group G is linearly reductive. Let V be a finite dimensional representation of G. By theorem of Hilbert, we know that the ring $k[V]^G$ of invariants is finitely generated. Now it is natural to ask that when is the ring $k[V]^G$ is polynomial ring ?

First we recall

Pseudo reflection: Let V be a vector space of dimension n over a field k. A pseudo reflection g is a linear automorphism of V of finite order such that the set of fixed points V^g is a hyperplane.

We have the following theorem.

Theorem 2.41. *(Chevalley, Serre, Shephard-Todd, [Che55, Ser68, ST54]). Let V be a finite dimensional representation of a finite group G over a field k. Assume that the characteristic of k does not divides the order of G. Then G is generated by pseudo-reflections if and only if $k[V]^G$ is a polynomial algebra. In such a case, $|G| = \prod_{i=1}^{n} deg(f_i)$, where $n = dim(V)$ and $f_1, f_2, \ldots, f_n$ is a set of algebraically independent generators of $k[V]^G$.*

Broer gave an extension of the above theorem to positive characteristic.

Theorem 2.42. *(Broer [Bro10]). Suppose that V is an irreducible representation of a finite group G over a field k, then $k[V]^G$ is a polynomial algebra if and only if G is generated by pseudo-reflections and there is a surjective $k[V]^G$-linear map $\pi : k[V] \to k[V]^G$.*

The following criterion due to Kemper is valid over any field.

Theorem 2.43. *(Kemper [Kem96]). Let V be a n-dimensional representation of a finite group G over a field k. Then $k[V]^G$ is a polynomial ring if and only if there is a homogeneous system of parameter $h_1, h_2, \ldots, h_n$ of $k[V]^G$ with $|G| = \prod_{i=1}^{n} deg(h_i)$. In particular $k[V]^G = k[h_1, h_2, \ldots, h_n]$ implies $|G| = \prod_{i=1}^{n} deg(h_i)$.*

If G is a subgroup of $GL(V)$ generated by pseudo reflections, but the characteristic of k divides the order of G, then the ring $k[V]^G$ of invariants need not be a polynomial algebra. For example, let k be an algebraically closed field of characteristic 3 and let W be the Weyl group of an algebraic group of type F_4 over k. Note that the order $|W|$ of the Weyl group W is 1152 and 3 divides 1152. The ring of invariants is not a polynomial algebra (see [NS02, Chapter 7, page 192]).

The above results gives a characterization for $k[V]^G$ to be a polynomial algebra if G is finite, but there is no simple characterization for a semisimple algebraic group G.

Theorem 2.44 (Chevalley, [Hum72]). *For any semisimple algebraic group G over $\mathbb{C}$, the ring $\mathbb{C}[\mathfrak{g}]^G$ of G-invariants of the co-ordinate ring of the adjoint representation $\mathfrak{g}$ of G is a polynomial algebra (see [Hum72, page 127]).*

Theorem 2.45 (Steinberg, [Ste65]). *For any semisimple simply connected algebraic group G (over any algebraically closed field k) acting on itself by inner conjugation, the ring $k[G]^G$ of G-invariants is a polynomial algebra (see [Ste65, page 41]).*

When $G = T$ is a torus, D. Wehlau in[Weh94] proved a theorem giving a neccessary and sufficient condition for a rational representation V of a torus S for which the ring of G invariants of the co-ordinate ring $K[V]$ is a polynomial algebra (see Theorem 5.8 of [Weh94]).

2.4 Geometric Invariant Theory

In this subsection, we give some basic definitions and results on the Geometric Invariant theory. We always assume that G is an affine algebraic group over an algebraically closed field k. This material can found in [MFK94] and [New78].

2.4.1 Group Actions on Algebraic Varieties

Let G be an algebraic group acting on a variety X. The question whether the set X/G of orbits under this action can be given a geometric structure. That is, X/G has a structure of a variety such that the map $\pi : X \longrightarrow X/G$ is a morphism. Then in particular each orbit has to be closed, because π is continuous. But this need not always be the case as the following example shows.

Example: The action of $GL_n(k)$ on k^n has two orbits: $\{0\}$ and $k^n \setminus \{0\}$. The orbit $k^n \setminus \{0\}$ is not closed.

In most of the cases, we will see that there exists an open set $U \subset X$ such that U/G has the structure of algebraic variety and $U \to U/G$ is a morphism. In particular, U is a union of closed orbits.

2.4.2 Quotients

Definition: A categorical quotient of X by G is a pair (Y, ϕ), where Y is a variety and $\phi : X \to Y$ is a morphism such that ϕ is constant on orbits, and if $\psi : X \to Z$ is any morphism that is constant on orbits, then there exists a unique morphism $\eta : Y \to Z$ such that $\psi = \eta \circ \phi$.

Note that if a categorical quotient exists, it is unique up to isomorphism and has good functorial properties.

Definition: An orbit space is a categorical quotient (Y, ϕ) such that $\phi^{-1}(y)$ consists of a single orbit for all $y \in Y$.

Definition: A good quotient of X by G is a pair (Y, ϕ) such that Y is a variety and $\phi : X \to Y$ is a morphism such that:

1. ϕ is $G-$invariant,
2. ϕ is surjective,
3. ϕ is an affine morphism,
4. For any open subset $U \subset Y$, the homomorphism $\phi^* : k[U] \to k[\phi^{-1}(U)]$ induces an isomorphism from $k[U]$ onto $k[\Phi^{-1}(U)]^G$,
5. if W is a closed G-invariant subset of X, then $\phi(W)$ is closed in Y,
6. if W_1 and W_2 are closed G-invariant subsets of X with $W_1 \cap W_2 = \emptyset$, then $\phi(W_1) \cap \phi(W_2) = \emptyset$.

Lemma 2.46. *If (Y, ϕ) is a good quotient of X by G, then it is a categorical quotient.*

Definition: A geometric quotient of X by G is a good quotient (Y, ϕ) which is also an orbit space.

Notation: A good quotient is denoted by $X//G$ and geometric quotient is denoted by X/G.

In order to construct categorical quotients in general we first assume that X is an affine variety on which G acts morphically and, let $k[X]$ denote the algebra of morphisms $X \to k$. Then we have an rational action of G on $k[X]$, i.e $k[X]$ is a locally finite G-module.

Now, we can ask what is the right object for a categorical quotient Y. Suppose it exists and is affine, write $Y = Spec(B)$. The definition of the categorical quotient tells us that such a morphism factors through Y if and only if it is constant on orbits; which is equivalent to $B = k[X]^G$.

So, if Y should be affine, then $k[X]^G$ has to be finitely generated. We already discussed in previous section about the finite generation of the ring of invariants which has a positive answer if G is reductive.

We have the following theorem.

Theorem 2.47. *Let G be a reductive group acting on an affine variety X. Then $Y = Spec(k[X]^G)$, together with the map $\phi : X \to Y$ is a good quotient of X by G.*

Let G be a reductive group acting on projective variety X. Existence of the quotients of X by G is not as simple as in the affine case.

In general, to construct quotients we need the concept of linear actions, the actions induced from a representation of the group G on some ambient space (either affine or projective space) where the variety is embedded.

Definition: A linearization of an action of G on a variety X in $\mathbb{P}^n$ (resp. in $\mathbb{A}^n$) is a linear action (i.e a representation) of G on k^{n+1} (resp. k^n) which induces the given action on X. A linear action of G on X is an action of G together with a linearization of this action.

Note that a linear action on $X \subset \mathbb{P}^n$ of G determines an action on $k[x_0, x_1, \ldots x_n]$ and for any G-invariant homogeneous polynomial f of positive degree in $k[x_0, x_1, \ldots x_n]$, $X_f := \{x \in X : f(x) \neq 0\}$ is a G-invariant affine open subset of X.

Definition: Let X be a closed subvariety in $\mathbb{P}^n$. Let G be a reductive group acting linearly on X. A point $x \in X$ is a called

1. semi-stable if there exists a $G-$invariant homogeneous polynomial f in $k[x_1, x_2, \ldots, x_n]$ of positive degree such that $f(x) \neq 0$,
2. stable if there exists a G-invariant homogeneous polynomial f in $k[x_1, x_2, \ldots, x_n]$ of degree 1 such that $f(x) \neq 0$, $dim(G.x)) = dim(G)$ and the action of G on X_f is closed, where $X_f = \{x \in X : f(x) \neq 0\}$ i.e for all $y \in X_f, G.y \subset X_f$ closed,
3. unstable (or non semi-stable) if it not semi-stable.

Remark:

1. The set of semi-stale (respectively, stable) points of X depends on the embedding of X in $\mathbb{P}^n$ and the linearization of the action of G.
2. A unstable point is precisely one for which all G-invariants homogeneous polynomial vanish.
3. Let X^{ss} (respectively, X^s) be the set of all semi-stable (respectively, stable) points of X.

The subsets X^{ss} and X^s are $G-$invariant open subsets of X.
4. If Y is a closed subset of X, G acts on X and Y with the same linearization, then $Y^{ss} = X^{ss} \cap Y$ and $Y^s = X^s \cap Y$.
5. The morphism $\sigma_x : G \longrightarrow X$ given by $\sigma_x(g) = g.x$ is proper if and only if the orbit $G.x$ is closed and the stabilizer G_x is finite.

The following theorem says that the good quotient exists on the open subset X^{ss} of X.

Theorem 2.48. *Let G be a reductive group acting linearly on a projective variety X in $\mathbb{P}^n$. Then,*

1. There exists a good quotient (Y, ϕ) of X^{ss} by G, and Y is projective.
2. There exists an open subset Y^s of Y such that $\phi^{-1}(Y^s) = X^s$ and (Y^s, ϕ) is a geometric quotient of X^s.
3. For $x_1, x_2 \in X^{ss}$, $\phi(x_1) = \phi(x_2)$ if and only if $\overline{G.x_1} \cap \overline{G.x_2} \cap X^{ss} \neq \emptyset$.
4. For $x \in X^{ss}$, x is stable if and only if $dim(G.x) = dim(G)$ and $G.x$ is closed in X^{ss}.

2.4.3 Linearization

More generally, we can define stability and semi-stability on a quasi projective variety X with respect to a line bundle $\mathcal{L}$ on X.

Let X be a variety with an action of a group G, let $p : \mathcal{L} \longrightarrow X$ be a line bundle on X.

Definition: A G-linearization of a line bundle $\mathcal{L}$ on X is an action of G on $\mathcal{L}$ compatible with the action of G on X. That is, there is an action of G on $\mathcal{L}$ such that
1. for all $y \in \mathcal{L}, g \in G$, $p(gy) = gp(y)$ and
2. for all $x \in X, g \in G$, the map $\mathcal{L}_x \longrightarrow \mathcal{L}_{gx} : y \longmapsto gy$ (of fibers over x and gx) is linear.

A line bundle $\mathcal{L}$ with a linearization is called a G-linearized line bundle.

Remarks:

1. For a given G-linearized line bundle $\mathcal{L}$ on X and a G-invariant open set $U \subset X$, we have the induced action of G on $H^0(U, \mathcal{L})$.
2. If G is connected and group of characters $X(G)$ is trivial (for example, semisimple group), then every line bundle admits at most one linearization (see [MFK94, Proposition 1.4]).
3. Let X be a normal variety, G be a connected group acting on X and $\mathcal{L}$ is a line bundle on X. Then there exists $n \in \mathbb{Z}_{>0}$ such that $\mathcal{L}^n$ admits a G-linearization (see [MFK94, Corollary 1.6]).

A morphism of G-linearized line bundles is a G-equivariant morphism of line bundles. Thus we can speak of isomorphism classes of G-linearized line bundles on X and one can show that the set of all isomorphism classes of G-linearized line bundles on X has an abelian group structure (see [Dol03, Ch. 7]). We denote this group by $Pic^G(X)$, and we have a natural homomorphism

$$\theta : Pic^G(X) \to Pic(X)$$

which is forgetting the linearization. This homomorphism is not necessarily surjective.

Definition: Let X be a quasi-projective variety with an action of a reductive algebraic group G. Let $\mathcal{L}$ be a G-linearized line bundle on X. Let $x \in X$.

1. x is called semi-stable with respect to $\mathcal{L}$ if there exists $m > 0$ and $s \in H^0(X, L^{\otimes m})^G$ such that $X_s = \{y \in X | s(y) > 0\}$ is affine and contains x,
2. x is called stable if x is semi-stable, $dim(G.x) = dim(G)$ and the set $G.x$ is closed in X^{ss} for all semi-stable points.
3. x is called unstable with respect to $\mathcal{L}$ if x is not semi- stable.

Let $X^{ss}(\mathcal{L})$ be the set of all semi-stable points, and let $X^s(\mathcal{L})$ be the set of all stable points, let $X^{us}(\mathcal{L})$ be the locus of unstable points.

Remark:

1. $X^{ss}(\mathcal{L}) = X^{ss}(\mathcal{L}^n)$ and $X^s(\mathcal{L}) = X^s(\mathcal{L}^n)$ for any $n \in \mathbb{N}$.
2. If the line bundle $\mathcal{L}$ is ample, the set X_s is affine.
3. Let X be a projective variety in $\mathbb{P}^n$ and $\mathcal{L} = O_X(1)$ restriction of the hyperplane bundle. Note that any linear action G on X induces a G-linearization on $\mathcal{L}$. In that case, $X^{ss}(\mathcal{L}) = X^{ss}$ and $X^s(\mathcal{L}) = X^s$.
4. The unstable points are precisely the points on which all G-invariant homogeneous forms of positive degree vanish.

Now we have the following theorem.

Theorem 2.49. *(Mumford) Let G be a reductive group acting on a quasi-projective variety X. Let $\mathcal{L}$ be a G-linearized line bundle on X. Then there exists a good quotient*

$$\pi : X^{ss}(\mathcal{L}) \to X^{ss}(\mathcal{L})//G.$$

There exists an open set $U \subset X^{ss}(\mathcal{L})//G$ such that $X^s(\mathcal{L}) = \pi^{-1}(U)$ and the restriction of π to $X^s(\mathcal{L})$ is a geometric quotient of $X^s(\mathcal{L})$ by G. Moreover $X^{ss}(\mathcal{L})//G$ is a quasi-projective variety.

There exists a converse of Theorem 2.49 is true under some conditions: saying that some subsets $U \subset X$ for which a categorical quotient $U//G$ exist, are of the form $U = X^{ss}(\mathcal{L})$ for some linearization with respect to an ample line bundle $\mathcal{L}$ (see [MFK94, page 41]).

Now we have the following corollary.

Corollary 2.50. *Further assume that X is projective, and $\mathcal{L}$ is very ample. Let* $R = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} H^0(X, \mathcal{L}^{\otimes n})$. *Then we have*

$$X^{ss}(\mathcal{L})//G \simeq Proj(R^G).$$

In particular, $X^{ss}(\mathcal{L})//G$ *is a projective variety.*

Remark: Let

$$\mathcal{N} = X^{us}(\mathcal{L}) = \{x \in X : s(x) = 0 \text{ for all } s \in R^G\}.$$

Hilbert consided the set $\mathcal{N}$. It is called the *nullcone* and its elements are called *nullforms*. Nullforms can not be distinguished by invariant functions. In fact if we consider the special case that R^G is generated by generators $s_0, \ldots, s_k$ of the same degree, then the rational map $X \dashrightarrow \mathbb{P}^k$ given by

$$x \mapsto (s_0(x), \ldots, s_k(x))$$

is the quotient map (when restricted to the set of semi-stable points). The nullcone is the set of points where this map is not defined.

2.4.4 Hilbert-Mumford criterion

In general, finding invariants explicitly is difficult and hence so the semi-stablility. David Hilbert and David Mumford gave a criterion that says given a point is semi-stable or not, with out finding invariants explicitly. This is called numerical criterion for semi-stability. This numerical criterion reduces to the study of actions of G to the actions of one-dimensional torus via one-parameter subgroups of G.

Throughout this subsection, we assume that G is a reductive algebraic group acting linearly on a projective variety $X \subset \mathbb{P}^n$. We can consider the induced action of G on the affine cone $\hat{X} \subset k^{n+1}$. Let $\hat{x} \in \hat{X}$ be a point whose class is $x \in X$.

We have the lemma.

Lemma 2.51.

1. $x \in X$ *is semi stable if and only if* $0 \notin \overline{G \cdot \hat{x}}$.
2. $x \in X$ *is stable if and only if the morphism* $\sigma_{\hat{x}} : G \longrightarrow k^{n+1}$, $\sigma_{\hat{x}}(g) \longmapsto g\hat{x}$ *is proper.*

Recall that a one-parameter subgroup of G is a homomorphism $\lambda : \mathbb{G}_m \to G$ of algebraic groups. A one-parameter subgroup λ of G can be viewed as an action of G_m via λ on X, and hence we get the action of $\mathbb{G}_m$ on the cone $\hat{X} \subset k^{n+1}$ over X. Since any action of $\mathbb{G}_m$ on k^{n+1} can be diagonalized, we can choose coordinates such that the action on $\hat{X}$ is given by

$$\lambda(t) \cdot \hat{x} = (t^{r_0} x_0, \dots, t^{r_n} x_n)$$

for some integers r_i.

Now consider the map

$$\phi_x^* : \mathbb{G}_m \to k^{n+1}, \ t \mapsto \lambda(t) \cdot \hat{x}.$$

If this map can be extended to a map $\mathbb{A}^1 \to k^{n+1}$ by sending the origin to the origin then it is clear that 0 is in the closure of the orbit of $\hat{x}$ of the one-parameter subgroup λ of G, so that x is unstable.

Definition:

$$\mu(x, \lambda) := -min\{r_i : x_i \neq 0\}.$$

Note that the function μ doesn't depend on the diagonalization of the one-parameter action. This function μ is very helpful to check unstability. Given a point $x \in X$, if there is a one-parameter subgroup λ of G satisfying $\mu(x, \lambda) < 0$, then x is unstable. If we consider the action of G on X via a line bundle $\mathcal{L}$, we denote $\mu(x, \lambda)$ by $\mu^{\mathcal{L}}(x, \lambda)$.

The following numerical criterion is a very useful for computing the semi-stable and stable points.

Theorem 2.52. *(Hilbert-Mumford) Let G be a reductive algebraic group acting linearly on a projective variety $X \subset \mathbb{P}^n$. Let $x \in X$. Then,*

$$x \in X^{ss} \text{ if and only if } \mu^{\mathcal{L}}(x, \lambda) \geq 0 \text{ for all one-parameter subgroups } \lambda.$$

$$x \in X^{s} \text{ if and only if } \mu^{\mathcal{L}}(x, \lambda) > 0 \text{ for all non trivial one-parameter subgroups } \lambda.$$

2.4.5 G.I.T on Homogeneous space G/P and Schubert varieties

We assume throughout this section the base field is the field $\mathbb{C}$ of complex numbers.

Let G be a semisimple algebraic group over $\mathbb{C}$. Let T be a maximal torus of G. Let B be a Borel subgroup of G containing T. Let P be a parabolic subgroup of G containing B. In this section, we consider the action of T on G/P and Schubert varieties.

In [Kan98, Kan99], Senthamarai Kannan described all the parabolic subgroups of G for which there exist an ample line bundle $\mathcal{L}$ on G/P such that the set $(G/P)^{ss}_T(\mathcal{L})$ of semi-stable points is same as the set $(G/P)^{s}_T(\mathcal{L})$ stable points. Strickland gave a shorter proof of this result [Str00].

Recall that P is of the form $P = \cap_{r\in I} P_r$ for some subset I of S, where P_r is the maximal parabolic subgroup of G corresponding to the simple root α_r. Let (r,n) denote the greatest common divisor of r and n.

Theorem 2.53.

1. *Let $G = SL(n,\mathbb{C})$. In this case S is indexed by the set $\{1,2,\ldots,n-1\}$. Then, there is a line bundle $\mathcal{L}$ on G/P such that $(G/P)^{ss}_T(\mathcal{L}) = (G/P)^{s}_T(\mathcal{L})$ if and only if the least common multiple of $\{n/(r,n) : r \in I\}$ is n.*
2. *Let G be a simple algebraic group different from type A. Then, there is a line bundle on G/P such that $(G/P)^{ss}_T(\mathcal{L}) = (G/P)^{s}_T(\mathcal{L})$ if and only if $P = B$.*

The special case of a maximal parabolic subgroup of $SL(n.\mathbb{C})$ is studied in [Kan98].

We set the following notations:

Let $C(B)$ be the Weyl chamber in $Y(T)\otimes\mathbb{R}$ determined by B. Let $\mathcal{L}$ be a line bundle on G/B defined by a character $\chi \in X(T)$. If $x \in G/B$ then $x \in BwB/B$ for some $w \in W$. Let λ be a one-parameter subgroup of T which is in the closure $\overline{C(B)}$ of $C(B)$. Let $\langle -,-\rangle : X(T)\times Y(T) \longrightarrow \mathbb{R}$ be the canonical pairing. We define the natural action of W on $Y(T)$ by $w(\lambda) = n_w\lambda n_w^{-1}$, where n_w be a representative of w in $N_G(T)$.

Then, we have

Lemma 2.54 (Seshadri)**.** $\mu^{\mathcal{L}}(x,\lambda) = -\langle \chi, w(\lambda)\rangle$.

Let p_w be the unique section of the line bundle $\mathcal{L}(\chi)$ which does not vanish at the point wP/P. Let $s := \prod_{w\in W} p_w$. Then s is a non zero T-invariant section of $\mathcal{L}(\chi)^{\otimes|W|}$, i.e $0 \neq s \in H^0(G/P, \mathcal{L}(\chi)^{\otimes|W|})^T$. Thus, $(G/P)^{ss}_T(\mathcal{L}(\chi)) \neq \emptyset$. However, s vanishes on $X(w)_P$ for all $w \in W^P$ such that $X(w)_P \subsetneq G/P$.

In general, it is not clear whether there exist a semi-stable points in Schubert varieties with respect to a given line bundle for the action of maximal torus T. Therefore, it is a interesting problem to study for which Schubert variety $X(w)_P$; $(X(w)_P)^{ss}_T(\mathcal{L}(\chi)) \neq \emptyset$. By using above lemma we have the following proposition gives a criterion that Schubert varieties admitting semi-stable points due to Senthamarai Kannan and Santosh Pattanayak [KP09a].

Proposition 2.55. *Let $\chi = \sum\limits_{\alpha \in S} a_\alpha \omega_\alpha$ be a dominant character of T which is in the root lattice. Let $I = Supp(\chi) = \{\alpha \in S : a_\alpha \neq 0\}$ and let $w \in W^{I^c}$, where $I^c = S \setminus I$. Then $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ if and only if $w\chi \leq 0$.*

Let P be a maximal parabolic subgroup containing B. Let $\mathcal{L}$ be a ample line bundle on G/P. In [KP09b, KP09a] described all minimal dimensional Schubert varieties admitting semi-stable points with respect to the line bundle L.

Now, we describe all the Coxeter elements $w \in W$ for which $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ (due to Senthamarai Kannan and Santosh Pattanayak [KP09a]).

Theorem 2.56.

Type A**:**
1. A_3**:** *For any Coxeter element w, $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ.*

2. $A_n, n \geq 4$**:** *If $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ and w is a Coxeter element, then w must be either $s_n s_{n-1} \ldots s_1$ or $s_i \ldots s_1 s_{i+1} \ldots s_n$ for some $1 \leq i \leq n-1$.*

Type B**:**
1. B_2**:** *For any Coxeter element w, $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ.*

2. $B_n, n \geq 3$**:** *If $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ and w is a Coxeter element, then $w = s_n s_{n-1} \ldots s_1$.*

(C) **Type** C_n**:** *If $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ and w is a Coxeter element, then $w = s_n s_{n-1} \ldots s_1$.*

(D) **Type** D**:**
1. D_4**:** *If w is a Coxeter element, then $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ if and only if $l(ws_2) = l(w) + 1$ and $l(ws_i) = l(w) - 1$ for exactly one $i \neq 2$.*

2. $D_n, n \geq 5$**:** *If $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ for some non-zero dominant weight χ and w is a Coxeter element, then $w = s_n s_{n-1} \ldots s_1$.*

(E) **Type** E_6, E_7 **or** E_8**:** *There is no Coxeter element w for which there exist a non-zero dominant weight χ such that $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$.*

(F) **Type** F_4**:** *There is no Coxeter element w for which there exist a non-zero dominant weight χ such that $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$.*

(G) **Type** G_2**:** *There is no Coxeter element w for which there exist a non-zero dominant weight χ such that $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$.*

In [Pat14], for any simple, simply connected algebraic group G of exceptional types E_6, E_7, E_8, F_4, or G_2 and for any maximal parabolic subgroup P of G, Santosh Pattanayak described all minimal Schubert varieties in G/P admitting semi-stable points for the action of a maximal torus T with respect to an ample line bundle on G/P.

Chapter 3

Torus quotients of flag varieties

In this chapter, we prove that for any indecomposable dominant character χ of a maximal torus T of a simple adjoint group G over $\mathbb{C}$ such that there is a Coxeter element w in the Weyl group W for which $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$, the graded algebra $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring if and only if $dim(H^0(G/B, \mathcal{L}_\chi)^T) \leq rank(G)$. Equivalently, if there exists a Coxeter element $c \in W$ such that $X(c)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$. Then, the GIT quotient $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$ is isomorphic to a weighted projective space if and only if $dim(H^0(G/B, \mathcal{L}_\chi)^T) \leq$ rank of G.

We also prove that the coordinate ring $\mathbb{C}[\mathfrak{h}]$ of the cartan subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G and $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ are isomorphic if and only if $X(w)^{ss}_T(\mathcal{L}_{\alpha_0})$ is non empty for some Coxeter element w in W, where α_0 denotes the highest long root. Equivalently, that the GIT quotient $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_{\alpha_0})$ is isomorphic to the projective space $\mathbb{P}(\mathfrak{h})$ if and only if there exists a Coxeter element $c \in W$ such that $X(c)^{ss}_T(\mathcal{L}_{\alpha_0})$ is nonempty.

3.1 Relationship between $\mathbb{C}[\mathfrak{h}]$ and the homogeneous coordinate ring of G/B associated to the highest long root

Recall that $\mathfrak{g} = Lie(G)$ is the adjoint representation of G, $\mathfrak{h} = Lie(T)$ and α_0 is the highest long root. Since G is simple, the adjoint representation $\mathfrak{g}$ of G is an irreducible representation with highest weight α_0. Let $\phi_1 : \mathfrak{g} \to \mathfrak{h}$ be the T-invariant projection. Then, ϕ_1 induces a natural isomorphism $Hom(\mathfrak{h}, \mathbb{C}) \to Hom(\mathfrak{g}, \mathbb{C})^T$. Since $H^0(G/B, \mathcal{L}_{\alpha_0})$ is an irreducible self dual G-module with highest weight α_0, the G-modules $H^0(G/B, \mathcal{L}_{\alpha_0})$ and $Hom(\mathfrak{g}, \mathbb{C})$ are isomorphic. So, we have an isomorphism $Hom(\mathfrak{h}, \mathbb{C}) \to H^0(G/B, \mathcal{L}_{\alpha_0})^T$.

Thus, we have a homomorphism $f : \mathbb{C}[\mathfrak{h}] \longrightarrow \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ *of* $\mathbb{C}$ *algebras. (1)*

In this section, *we show that the homomorphism* $f : \mathbb{C}[\mathfrak{h}] \longrightarrow \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ *as in (1) is injective*. Further, we also prove that $f : \mathbb{C}[\mathfrak{h}] \to \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is an isomorphism if and only if $X(w)^{ss}_T(\mathcal{L}_{\alpha_0})$ is non empty for some Coxeter element w in W.

We first set up some notation.

Recall that R denote the roots of G with respect to T, $R^+ \subset R$ be the set of positive roots with respect to B and $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset R^+$ denote the set of simple roots with respect to B.

For each positive root $\alpha \in R^+$, we denote by $U_{-\alpha}$, the T-stable root subgroup of $w_0 B w_0^{-1}$ corresponding to $-\alpha$. Let U^- be the unipotet radical of the opposite Borel subgroup $w_0 B w_0^{-1}$. Then, we have $U^- = \prod_{\alpha\in\Phi^+} U_{-\alpha}$.

We denote in this chapter $\{E_\beta : \beta \in R\} \bigcup \{H_\beta : \beta \in S\}$ be the Chevalley basis for $\mathfrak{g}$ (refer to [Hum72, Chapter VII])). Note that E_{α_0} is a highest weight vector of $\mathfrak{g}$ and we denote it by v^+. Recall that we denote $\langle \alpha, \beta \rangle := \frac{2(\alpha,\beta)}{(\beta,\beta)}$ for all $\alpha, \beta \in R$. Consider $U^- v^+ \subset \mathfrak{g}$ be the U^-orbit of v^+.

We have the following:

Lemma 3.1. The restriction map $\phi := \phi_1|_{U^- v^+} : U^- v^+ \to \mathfrak{h}$ is onto.

Proof. Since α_0 is dominant, we can choose a simple root γ_1 such that $\langle \alpha_0, \gamma_1 \rangle \geq 1$. Choose distinct simple roots $\gamma_2, \gamma_3, \ldots, \gamma_{n-1}$ such that for all $r = 1, 2, \ldots, n-1$, $\sum_{j=1}^r \gamma_j$ is a root. Let $\theta_r = \sum_{j=1}^r \gamma_j$, $r = 1, 2, \ldots, n-1$. Again since $\langle \alpha_0, \theta_r \rangle \geq 1$ for $1 \leq r \leq n-1$, each $\beta_r := \alpha_0 - \theta_r$ is a root. For every choices of $c_0, c_r, c'_r \in \mathbb{C}$, $1 \leq r \leq n-1$, we claim that
$\phi(exp(c_0 E_{-\alpha_0}))(exp(c_1 E_{-\beta_1}))(exp(c'_1 E_{-\theta_1}))(exp(c_2 E_{-\beta_2}))(exp(c'_2 E_{-\theta_2})) \cdots (exp(c_{n-1} E_{-\beta_{n-1}}))$
$(exp(c'_{n-1} E_{-\theta_{n-1}}))(v^+) = -c_0 H_{\alpha_0} - \sum_{r=1}^{n-1} c_r c'_r H_{\beta_r}$.

Take a typical monomial

$$M = \frac{c_0^{m_0}}{m_0!} E_{-\alpha_0}^{m_0} \frac{c_1^{a_1}}{a_1!} E_{-\beta_1}^{a_1} \cdots \frac{c_{n-1}^{a_{n-1}}}{a_{n-1}!} E_{-\beta_{n-1}}^{a_{n-1}} \frac{(c'_1)^{b_1}}{b_1!} E_{-\theta_1}^{b_1} \cdots \frac{(c'_{n-1})^{b_{n-1}}}{b_{n-1}!} E_{-\theta_{n-1}}^{b_{n-1}}$$

occuring in the expansion of
$(exp(c_0 E_{-\alpha_0}))(exp(c_1 E_{-\beta_1}))(exp(c'_1 E_{-\theta_1}))(exp(c_2 E_{-\beta_2}))(exp(c'_2 E_{-\theta_2})) \cdots (exp(c_{n-1} E_{-\beta_{n-1}})(exp(c'_{n-1} E_{-\theta_{n-1}})$.

Then, Mv^+ has weight zero if and only if $(1 - m_0)\alpha_0 = \sum_{j=1}^{n-1} a_j \beta_j + \sum_{k=1}^{n-1} b_k \theta_k$.

<u>Claim:</u> For all $j = 1, 2, \ldots, n-1$ and $k = 1, 2, \ldots, n-1$, there exist unique r in $\{1, 2, \ldots, n-1\}$ such that $a_r = b_r = 1$ and $a_j = b_j = 0$ for all $j \neq r$.

Now, Mv^+ has weight 0 implies $(1 - m_0)\alpha_0 - \sum_{j=1}^{n-1} a_j \beta_j - \sum_{k=1}^{n-1} b_k \theta_k = 0$.
$\Rightarrow (m_0 - 1)\alpha_0 + \sum_{j=1}^{n-1} a_j \beta_j + \sum_{k=1}^{n-1} b_k \theta_k = 0$.

$\Rightarrow (m_0 - 1)\alpha_0 + \sum_{j=1}^{n-1} a_j\beta_j + \sum_{k=1}^{n-1} b_k(\alpha_0 - \beta_k) = 0.$
$\Rightarrow (m_0 - 1)\alpha_0 + (\sum_{k=1}^{n-1} b_k)\alpha_0 + \sum_{j=1}^{n-1} a_j\beta_j - \sum_{k=1}^{n-1} b_k\beta_k = 0.$
$\Rightarrow ((m_0 - 1) + \sum_{k=1}^{n-1} b_k)\alpha_0 + \sum_{j=1}^{n-1}(a_j - b_j)\beta_j = 0.$

Since $\{\alpha_0, \beta_j : j = 1, 2, \ldots, n-1\}$ is a linearly independent subset of $X(T) \otimes \mathbb{R} = Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$, the dual of the real form $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}$, we have $(m_0 - 1) + \sum_{k=1}^{n-1} b_k = 0$ and $a_j = b_j$ for all $j = 1, 2, \ldots, n-1$. Since m_0 and $b_i's$ are non-negative integers, we have either $m_0 = 1$ and $a_j = b_j = 0$ for all j or $m_0 = 0$ and there exist unique k such that $a_k = b_k = 1$ and $a_j = b_j = 0$ for all $j \neq k$. Again $H_{\alpha_0}, H_{\beta_r}$ are linearly independent since α_0 and β_r are linearly independent.

So, we have a surjective map $\mathrm{U}^- v^+ \supseteq (\mathrm{U}_{-\alpha_0} \prod_{j=1}^{n-1} \mathrm{U}_{-\beta_j} \prod_{k=1}^{n-1} \mathrm{U}_{-\theta_k}) v^+ \to \mathfrak{h}$ given by $(u_{-\alpha_0}(c_0) \prod_{j=1}^{n-1} u_{-\beta_j}(c_j) \prod_{k=1}^{n-1} u_{-\theta_k}(c'_k)) v^+ \mapsto -c_0 H_{\alpha_0} - \sum_{r=1}^{n-1} c_r c'_r H_{\beta_r}$, where $u_\alpha(c) = exp(cE_\alpha)$, $\alpha \in R, c \in \mathbb{C}$. Hence $\phi : \mathrm{U}^- v^+ \to \mathfrak{h}$ is onto. This completes the proof of the lemma. $\square$

We have

Corollary 3.2. *The homomorphism $f : \mathbb{C}[\mathfrak{h}] \to \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ as in (1) is injective.*

Proof. By Lemma 3.1, we have $\phi : \mathrm{U}^- v^+ \to \mathfrak{h}$ is onto. So, $\phi^* : \mathbb{C}[\mathfrak{h}] \to \mathbb{C}[\mathrm{U}^- v^+]$ is injective. Let $[v^+]$ denote the point in G/B corresponding to the identity coset idB/B. Since the affine space $\mathrm{U}^-[v^+]$ is an open subset of G/B, we have the restriction map $H^0(\mathrm{G/B}, \mathcal{L}_{\alpha_0}^{\otimes d}) \longrightarrow \mathbb{C}[\mathrm{U}^- v^+]$ for all $d \in \mathbb{Z}_{\geq 0}$. So, we get a map $H^0(\mathrm{G/B}, \mathcal{L}_{\chi}^{\otimes d})^T \longrightarrow \mathbb{C}[\mathrm{U}^- v^+]$ for all $d \in \mathbb{Z}_{\geq 0}$. Hence, we have a homomorphism $g : \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T \longrightarrow \mathbb{C}[\mathrm{U}^- v^+]$ of $\mathbb{C}$-algebras.

Now, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{h}] & \xrightarrow{\quad f \quad} & \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T \\ & \searrow^{\phi^*} \quad \swarrow_{g} & \\ & \mathbb{C}[\mathrm{U}^- v^+] & \end{array}$$

So, we have $g \circ f = \phi^*$. Since ϕ is onto, ϕ^* is injective. Hence, the homomorphism $f : \mathbb{C}[\mathfrak{h}] \to \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is injective. $\square$

We now prove the following theorem.

Theorem 3.3. *The homomorphism $f : \mathbb{C}[\mathfrak{h}] \to \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is an isomorphism if and only if $X(w)_T^{ss}(\mathcal{L}_{\alpha_0})$ is non empty for some Coxeter element w in W.*

Proof. By Theorem 4.2 in [KP09a], $X(w)^{ss}_T(\mathcal{L}_{\alpha_0})$ is non empty for some Coxeter element w if and only if G is of type A_n, B_2 or C_n. Now, we prove that the homomorphism $f : \mathbb{C}[\mathfrak{h}] \to \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is an isomorphism if G is of type A_n, B_2 or C_n.

By Corollary 3.2, the graded homomorphism $f : \mathbb{C}[\mathfrak{h}] \to \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is injective. Hence we have $sym^d(\mathfrak{h}) \subset H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$. Let $\alpha_0 = \sum m_i\omega_i$. $J := \{i \in \{1, 2, \ldots, n\} : m_i \geq 1\}$.
Let $P = P_J$. Let U_P^- be unipotent radical of the opposite parabolic subgroup of P determined by T and B. Take the line bundle $\mathcal{L}_{\alpha_0}^{\otimes d}$ on G/P and restrict to U_P^-. Since U_P^- is an affine space, $\mathcal{L}_{\alpha_0}^{\otimes d}$ is trivial on U_P^-. So, we have $H^0(U_P^-, \mathcal{L}_{\alpha_0}^{\otimes d}) = \mathbb{C}[U_P^-]$, regular functions on U_P^-. Hence, $H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})$ is a T-submodule of $\mathbb{C}[U_P^-] \otimes \mathbb{C}_{-d_0\alpha_0}$.

Now, by considering the weights; the weight zero in $H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})$ corresponding to weight $d\alpha_0$ in $\mathbb{C}[U_P^-]$. Hence, $X_{-\alpha_0}^{a_0} X_{-\beta_1}^{a_1} X_{-\beta_2}^{a_2} \cdots X_{-\beta_{n-1}}^{a_{n-1}} X_{-\theta_1}^{b_1} \cdots X_{-\theta_{n-1}}^{b_{n-1}}$ has weight $d\alpha_0$ in $\mathbb{C}[U_P^-]$ if and only if $a_0\alpha_0 + \sum_{j=1}^{n-1} a_j\beta_j + \sum_{k=1}^{n-1} b_k\theta_k - d\alpha_0 = 0$.
$\Rightarrow (a_0 - d)\alpha_0 + \sum_{j=1}^{n-1} a_j\beta_j + \sum_{k=1}^{n-1} b_k\theta_k = 0$.
$\Rightarrow (a_0 - d)\alpha_0 + \sum_{j=1}^{n-1} a_j\beta_j + \sum_{k=1}^{n-1} b_k(\alpha_0 - \beta_k) = 0$.
$\Rightarrow (a_0 - d)\alpha_0 + (\sum_{k=1}^{n-1} b_k)\alpha_0 + \sum_{j=1}^{n-1} a_j\beta_j - \sum_{k=1}^{n-1} b_k\beta_k = 0$.
$\Rightarrow ((a_0 - d) + \sum_{k=1}^{n-1} b_k)\alpha_0 + \sum_{j=1}^{n-1}(a_j - b_j)\beta_j = 0$.

Since $\{\alpha_0, \beta_j : j = 1, 2, \ldots, n-1\}$ is a linearly independent subset of $X(T) \otimes \mathbb{R}$, we have $a_j = b_j$ for all $j = 1, 2, \ldots, n-1$. Since a_0 and $b_i's$ are non-negative integers, we have either $a_0 = d$, $b_k = 0$ for all $k = 1, 2, \ldots, n-1$ or $a_0 = 0$ and $\sum_{k=1}^{n-1} b_k = d$.

Let $V_d := \{X_{-\alpha_0}^{a_0}(X_{-\beta_1}X_{-\theta_1})^{a_1}(X_{-\beta_2}X_{\theta_2})^{a_2} \cdots (X_{-\beta_{n-1}}X_{-\theta_{n-1}})^{a_{n-1}} : \sum_{i=0}^{n-1} a_i = d\}$. In type A_n, B_2 and C_n , $dim(G/P) = 2n - 1$. So, we can identify $H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ with V_d. Also, we can idetify V_d as a subspace of $Sym^d(h)$. Therefore, we have $H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})^T \subset Sym^d(\mathfrak{h})$. So, $H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})^T = Sym^d(\mathfrak{h})$. Hence, the homomorphism $f : \mathbb{C}[\mathfrak{h}] \to \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is an isomorphism. Since $H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d}) = H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})$, the homomorphism $f : \mathbb{C}[\mathfrak{h}] \to \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is an isomorphism.

If G is not of type A_n, B_2 or C_n, we prove $dim(G/P) \geq 2n$.

Type $B_n, n \neq 2$: In this case, the highest long root α_0 is ω_2. So, we have $P = P_2$. The dimension of $U_{P_2}^- = \#\{\alpha \in R^+/\alpha \geq \alpha_2\} = 4n - 5$. Since $U_{P_2}^-$ is an affine open subset of G/P, $dim(G/P) = 4n - 5$. Hence, we have $dim(G/P) \geq 2n$ for $n \neq 2$.

Type D_n: In this case, the highest long root α_0 is ω_2. So we have $P = P_2$. The dimension of $U_{P_2}^- = \#\{\alpha \in R^+/\alpha \geq \alpha_2\} = 4n - 7$. Since $U_{P_2}^-$ is an affine open subset of G/P, $dim(G/P) = 4n - 7$. Hence we have $dim(G/P) \geq 2n$ for $n \geq 4$.

Type E_6: The highest long root $\alpha_0 = \omega_2$. Hence we have $P = P_2$. The dimension of $U_{P_2}^- = \#\{\alpha \in R^+/\alpha \geq \alpha_2\} = 21$. Then $dim(G/P) = 21$. Hence we have $dim(G/P) > 12$.

Type E_7: The highest long root $\alpha_0 = \omega_1$. Hence we have $P = P_1$. The dimension of $U^-_{P_1} = \#\{\alpha \in R^+/\alpha \geq \alpha_1\} = 33$. Then $dim(G/P) = 33$. Hence we have $dim(G/P) > 14$.

Type E_8: The highest long root $\alpha_0 = \omega_8$. Hence we have $P = P_8$. The dimension of $U^-_{P_8} = \#\{\alpha \in R^+/\alpha \geq \alpha_8\} = 57$. Then $dim(G/P) = 57$. Hence we have $dim(G/P) > 16$.

Type F_4: The highest long root $\alpha_0 = \omega_1$. Hence we have $P = P_1$. The dimension of $U^-_{P_1} = \#\{\alpha \in R^+/\alpha \geq \alpha_1\} \geq 8$. Hence we have $dim(G/P) > 8$.

Type G_2: The highest long root $\alpha_0 = \omega_2$. Hence we have $P = P_2$. The dimension of $U^-_{P_2} = \#\{\alpha \in R^+/\alpha \geq \alpha_2\} = 5$. Hence we have $dim(G/P) > 4$.

Since $dim(G/P) \geq 2n$, the Krull dimension of $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d}) > 2n$. Hence $dim(\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})^T) > n$. Therefore, the homomorphism $f : \mathbb{C}[\mathfrak{h}] \to \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is not an isomorphism if G is not of the type A_n, B_2 or C_n. This completes the proof of the theorem. □

Equivalently, we have the following.

Corollary 3.4. *The GIT quotient $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_{\alpha_0})$ is isomorphic to the projective space $\mathbb{P}(\mathfrak{h})$ if and only if there exists a Coxeter element $c \in W$ such that $X(c)^{ss}_T(\mathcal{L}_{\alpha_0})$ is nonempty.*

As a consequence, we have

Corollary 3.5. *The polarized variety $(T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_{\alpha_0}), \mathcal{M})$ is projectively normal, where $\mathcal{M}$ is the descent of $\mathcal{L}_{\alpha_0}$ to the quotient $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_{\alpha_0})$.*

Let $\mathbb{P}(\mathfrak{g})$ be the projective space corresponding to the affine space $\mathfrak{g}$.

Corollary 3.6. *If G is of type A_n, B_2 or C_n, $G\backslash\backslash\mathbb{P}(\mathfrak{g}) \simeq N_G(T)\backslash\backslash(G/B_T(\mathcal{L}_{\alpha_0}))^{ss}$.*

Proof. By Chevalley restriction theorem, we have the restriction map $\mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h}]^W$ is an isomorphism. So, we have $G\backslash\backslash\mathbb{P}(\mathfrak{g}) = W\backslash\backslash\mathbb{P}(\mathfrak{h})$. Since G is of type A_n, B_2 or C_n, by Theorem 3.3, we have $\mathbb{C}[\mathfrak{h}] \simeq \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$. Then, $\mathbb{C}[\mathfrak{h}]^W \simeq \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^{N_G(T)}$. Hence, we have $W\backslash\backslash\mathbb{P}(\mathfrak{h}) \simeq N_G(T)\backslash\backslash(G/B(\mathcal{L}_{\alpha_0}))^{ss}$. Therefore, $G\backslash\backslash\mathbb{P}(\mathfrak{g}) \simeq N_G(T)\backslash\backslash(G/B(\mathcal{L}_{\alpha_0}))^{ss}$. □

3.2 A description of line bundles $\mathcal{L}_\chi$ on G/B for which $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring

In this section, we prove that for any indecomposable dominant character χ of a maximal torus T of a simple adjoint group G such that there is a Coxeter element $w \in W$ for which

$X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$, the graded algebra $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring if and only if $dim(H^0(G/B, \mathcal{L}_\chi)^T) \leq rank(G)$.

Notation: Recall we use additive notation for the group $X(T)$ of characters of T. Let $X(T)^+$ denote the set of all dominant characters of T.

Definition 3.7. A non trivial dominant character χ of T is said to be decomposable if there is a pair of non trivial dominant characters χ_1, χ_2 of T such that $\chi = \chi_1 + \chi_2$. Otherwise we will call it indecomposable.

Let $X(T)^+_i$ denote the set of all indecomposable elements of $X(T)^+$.

Lemma 3.8. *Let G be a simple adjoint group of type A_{n-1}. Let $\chi = \sum_{i=1}^{n-1} a_i\alpha_i$, where $a_i \in \mathbb{N}$ for each $i = 1, 2, \ldots, n-1$ be an element of $X(T)^+_i$ such that $\langle \chi, \alpha_{n-1}\rangle = 0$. Suppose that $X(s_{n-1}\cdots s_1)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$, then*
(i) The coefficients a_i ,$i = 1, 2, \ldots, n-1$ satisfy the following inequality :

$$a_1 \geqslant a_2 \geqslant a_3 \geqslant \ldots \geqslant a_{n-2} = 2 \text{ and } a_{n-1} = 1.$$

(ii) χ must be of the form $i\omega_1 + \omega_{n-i}$ for some $2 \leq i \leq n-1$.

Proof. Since $X(s_{n-1}\cdots s_1)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$, we have $s_{n-1}\cdots s_1(\chi) \leq 0$. As χ dominant, we have $a_i \geqslant a_{i+1}$ for each $i = 1, 2, \ldots, n-2$.

Now we prove that $a_{n-1} = 1$.

If $a_{n-1} \geqslant 2$, let i be the largest positive integer such that $a_{n-i} = ia_{n-1}$. Since $2a_{n-1} - a_{n-2} = \langle \chi, \alpha_{n-1}\rangle = 0$, we must have $i \geqslant 2$. So, $a_{n-(i+1)} \neq (i+1)a_{n-1} \implies a_{n-(i+1)} = ia_{n-1} + c$, where $0 \leq c \leq a_{n-1} - 1$.

Case 1: If $c = 0$. Since χ dominant and $\langle \chi, \alpha_{n-1}\rangle = 0$, we have $a_{n-j} \leq ja_{n-1}, j = 1, 2, \ldots, n-1$ and also we have $a_j \geq a_{j+1}$ for each $j = 1, 2, \ldots, n-2$.

Claim: χ must be of the form $ia_{n-1}(\sum_{j=1}^{n-i}\alpha_j) + a_{n-1}(\sum_{j=n+1-i}^{n-1}(n-j)\alpha_j)$.

Since $c = 0$, $a_{n-(i+1)} = ia_{n-1}$. Now we prove $a_{n-(i+2)} = ia_{n-1}$. Since χ is dominant, $2a_{n-(i+1)} - a_{n-(i+2)} - a_{n-i} \geq 0. \implies 2ia_{n-1} - a_{n-(i+2)} - ia_{n-i} \geq 0. \implies ia_{n-i} \geq a_{n-(i+2)}$. Also, we have $a_{n-(i+2)} \geq a_{n-(i+1)} = ia_{n-i}$. So, $a_{n-(i+2)} = ia_{n-i}$.

Similarly, we can prove $a_{n-j} = ia_{n-1}$ for $j = i+3, \ldots, n-1$.

Now we prove $a_{n-(i-1)} = (i-1)a_{n-1}$. Since χ is dominant, $a_{n-(i-1)} + a_{n-3} \geq a_{n-2} + a_{n-i} = 2a_{n-1} + ia_{n-1} = (i+2)a_{n-1}$. Since $a_{n-j} \leq ja_{n-1}$, $a_{n-(i-1)} + a_{n-3} \leq (i+2)a_{n-1}$. So, we conclude that $a_{n-(i-1)} = (i-1)a_{n-1}$ and $a_{n-3} = 3a_{n-1}$. Similarly, we can prove that

$a_{n-j} = ja_{n-1}$ for $j = i-2, i-3, \ldots, 4$. χ is of the form $ia_{n-1}(\sum_{j=1}^{n-i}\alpha_j) + a_{n-1}(\sum_{j=n+1-i}^{n-1}(n-j)\alpha_j) = a_{n-1}(i\omega_1 + \omega_{n-i})$. This forces that χ is decomposable, since $a_{n-1} \geq 2$. This is a contradiction to the indecomposability of χ. This proves that if $c = 0$, then $a_{n-1} = 1$.

Case 2: If $c > 0$. $\langle \chi, \alpha_1 \rangle = 2a_1 - a_2 \geq a_1 \geq a_{n-(i+1)} = ia_{n-1} + c$. Similarly, $\langle \chi, \alpha_{n-i} \rangle \geq 2ia_{n-1} - ia_{n-1} - c - (i-1)a_{n-1} = a_{n-1} - c \geq 1$. Thus, $\chi - (i\omega_1 + \omega_{n-i})$ is still a non zero dominant weight which is in the root lattice. This contradicts the indecomposability of χ. So, $c = 0$ and $a_{n-1} = 1$ and this proves (i).

Proof of (ii). Using the above argument, we can see that χ is of the form $i\omega_1 + \omega_{n-i}$, where i is the largest positive integer such that $a_{n-i} = ia_{n-1}$. □

Lemma 3.9. *Let G be a simple adjoint group of type A_{n-1}. Let $\chi = \sum_{i=1}^{n-1} a_i\alpha_i$, where $a_i \in \mathbb{N}$ and $i = 1, \ldots, n-1$ be an element of $X(T)_i^+$ such that $\langle \chi, \alpha_1 \rangle = 0$. Suppose that $X(s_1 \cdots s_{n-1})_T^{ss}(\mathcal{L}_\chi) \neq \emptyset$, then*

(i) The coefficient $a_i, i = 1, \ldots, n-1$ satisfy the following inequality:

$$1 = a_1 \leq a_2 \leq \cdots \leq a_{n-1}.$$

(ii) χ must be of the form $\omega_i + i\omega_{n-1}$ for some $2 \leq i \leq n-1$.

Proof. Similar to the proof of Lemma 3.8. □

Lemma 3.10. *Let G be a simple adjoint group of type A_{n-1}. Let $\chi = \sum_{i=1}^{n-1} a_i\alpha_i$, where $a_i \in \mathbb{N}, i = 1, \ldots, n-1$ be an element of $X(T)_i^+$. If $X(s_{i+1} \cdots s_{n-1}s_i \cdots s_1)_T^{ss}(\mathcal{L}_\chi) \neq \emptyset$ for some $2 \leq i \leq n-3$, then $\chi = \alpha_1 + \cdots + \alpha_{n-1}$.*

Proof. Since $X(s_{i+1} \cdots s_{n-1}s_i \cdots s_1)_T^{ss}(\mathcal{L}_\chi) \neq \emptyset$ for some $2 \leq i \leq n-3$, $s_{i+1} \cdots s_{n-1}s_i \cdots s_1(\chi) \leq 0$.

So, we have $\sum_{j=1}^{i}(a_{j+1} - a_j)\alpha_j + (a_{i+1} - a_1 - a_{n-i})\alpha_{i+1} + \sum_{k=i+2}^{n-1}(a_{i+2} - a_{n-1})\alpha_k \leq 0$.

Since χ is dominant, we have $a_{i+1} \leq a_i \leq \cdots \leq a_2 \leq a_1$, $a_{i+1} \leq a_{i+2} \leq \cdots \leq a_{n-1}$ and $2a_{i+1} - a_i - a_{i+1} \geq 0$ then $a_{i+1} = a_i = a_{i+2}$. Similarly, we can prove that $a_1 = a_2 = \cdots = a_{n-1}$. Therefore $\chi = a_1(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1})$. Since χ is indecomposable and $a_i \in \mathbb{N}$, we have $a_1 = 1$. Hence $\chi = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$. □

Lemma 3.11. *Let G be a simple adjoint group of type A_{n-1}. Let $\chi = i\omega_1 + \omega_{n-i} \in X(T)_i^+$ for some $2 \leq i \leq n-3$, then we have $dim(H^0(G/B, \mathcal{L}_\chi)^T) > n-1$.*

Proof. Since $\chi = i\omega_i + \omega_{n-i}$, each integer in $\{1, 2, \ldots, n\}$ occurs in the standard young tableau corresponding to T-invariant standard monomial of shape χ (refer to [Ses85] for standard monomial) exactly once. Hence, $dim(H^0(G/B, \mathcal{L}_\chi)^T) = \binom{n-1}{i}$. Since $2 \leq i \leq n-3$,

$i < n-2$ and so $i-j < n-(j+2)$ for $j = 1, 2, \ldots, i-2$. So, we have $(n-2)(n-3)\cdots(n-i) > i!$. Hence, $\binom{n-1}{i} > n-1$. Therefore, $dim(H^0(G/B, \mathcal{L}_\chi)^T) > n-1$. □

Lemma 3.12. *Let G be a simple adjoint group of type A_{n-1}. Let $\chi = \omega_i + i\omega_{n-1} \in X(T)_i^+$ for some $2 \le i \le n-3$, then $dim(H^0(G/B, \mathcal{L}_\chi)^T) > n-1$.*

Proof. Let w_0 be longest Weyl group element. Since $\chi = \omega_i + i\omega_{n-1} \in X(T)_i^+, -w_0\chi = i\omega_1 + \omega_{n-i} \in X(T)_i^+$. Since $H^0(G/B, \mathcal{L}_\chi)^* = H^0(G/B, \mathcal{L}_{-w_0\chi})$, $dim(H^0(G/B, \mathcal{L}_\chi)^T) = dim(H^0(G/B, \mathcal{L}_{-w_0\chi})^T)$. By the previous Lemma, we have $dim(H^0(G/B, \mathcal{L}_{-w_0\chi})^T) > n-1$. Hence, we have $dim(H^0(G/B, \mathcal{L}_\chi)^T) > n-1$. □

Lemma 3.13. *Let G be a simple adjoint group of type A_{n-1}, $n \neq 4$. Let $\chi \in X(T)^+$. If $w \in W$ is Coxeter element such that $X(w)_T^{ss}(\mathcal{L}_\chi) \neq \emptyset$, then $\{i \in \{1, \cdots, n\} : l(ws_i) = l(w) - 1\} \subseteq \{1, n-1\}$.*

Proof. Let $\chi = \sum_{i=1}^{n-1} a_i\alpha_i$, where $a_i \in \mathbb{N}$. Suppose there is a $2 \le i \le n-2$ such that $l(ws_i) = l(w) - 1$. Since $w\chi \le 0$ we have $a_{i-1} + a_{i+1} \le a_i$. Since $\langle \chi, \alpha_{i-1} \rangle \ge 0$ and $\langle \chi, \alpha_{i+1} \rangle \ge 0$ we have $2a_{i-1} \ge a_{i-2} + a_i$ and $2a_{i+1} \ge a_i + a_{i+2}$. So, we have

$$2a_i \ge 2(a_{i-1} + a_{i+1}) \ge 2a_i + a_{i-2} + a_{i+2}.$$

Then, $a_{i-2} + a_{i+2} = 0$.
$\implies a_{i-2} = a_{i+2} = 0$.
$\implies i - 2 \le 0$ and $i + 2 \ge n$.
$\implies i = 2$ and $i = n-2$
$\implies i = 2$ and $n = 4$. This contradicts the assumption $n \neq 4$. This completes the proof of the lemma. □

We now prove the following theorem.

Theorem 3.14. *Let G be a simple adjoint group over $\mathbb{C}$. Let $\chi \in X(T)_i^+$ be such that there is a Coxeter element $w \in W$ for which $X(w)_T^{ss}(\mathcal{L}_\chi) \neq \emptyset$. Then, the graded algebra $\bigoplus_{d \in \mathbb{Z}_{\ge 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring if and only if $dim(H^0(G/B, \mathcal{L}_\chi)^T) \le rank(G)$.*

Proof. We prove the theorem by using case by case analysis.

For a given simple adjoint group G and for any indecomposable dominant character χ of T such that $X(w)_T^{ss}(\mathcal{L}_\chi) \neq \emptyset$ for some Coxeter element w in W, we prove that either the graded algebra $\bigoplus_{d \in \mathbb{Z}_{\ge 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring and $dim(H^0(G/B, \mathcal{L}_\chi)^T) \le rank(G)$ or the graded algebra
$\bigoplus_{d \in \mathbb{Z}_{\ge 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a not polynomial ring and $dim(H^0(G/B, \mathcal{L}_\chi)^T) > rank(G)$.

Type $A_{n-1}, n \neq 4$: By Lemma 3.9, if $w \in W$ is a Coxeter elements w such that $X(w)^{ss}_T(L_\chi)$ is non empty then $w = s_{i+1}. \cdots s_{n-1}s_i \cdots s_1$ for some $1 \leq i \leq n-2$ or $w = s_1 \cdots s_{n-1}$.

When $w = s_{n-1} \cdots s_1$ by using Lemma 3.8(ii), the indecomposable dominant character χ of T for which $X(w)^{ss}_T(L_\chi)$ is non empty are $\chi = i\omega_1 + \omega_{n-i}$ for $1 \leq i \leq n-1$.

If i=n-1: $\chi = n\omega_1$, in this case there is only one T-invariant monomial. Hence $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring in one variable.

If i=n-2: We have $\chi = (n-2)\omega_1 + \omega_2$. $dim(H^0(\mathrm{G/B}, \mathcal{L}_\chi)^T) = n-1$. Consider the map $\phi : \mathbb{C}[X_1, \cdots, X_{n-1}] \longrightarrow \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is given by $X_i \longmapsto p_{1i}p_2p_3 \cdots p_{i-1}p_{i+1} \cdots P_n$, where $p_{1i}, p_2, p_3, \ldots p_{i-1}, p_{i+1}, \ldots, P_n$ are Plüker coordinates. Using the standard monomial of shape $d\chi$ we can see that ϕ is surjective. So, we have

(1) the surjective map $\phi : \mathbb{C}[X_1, \cdots, X_{n-1}] \longrightarrow \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$.

Let $P = P_1 \cap P_2$. Since $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P, \mathcal{L}_\chi^{\otimes d})^T$, we have

(2) the Krull dimension of $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T = n-1$.

From (1) and (2), we conclude that the map $\mathbb{C}[X_1, \cdots, X_{n-1}] \longrightarrow \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is an isomorphism. Hence $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring.

If $2 \leq i \leq n-3$: we have $\chi = i\omega_1 + \omega_{n-i}$, by Lemma 3.11 we have $dim(H^0(\mathrm{G/B}, \mathcal{L}_\chi)^T) > n-1$.

Claim: $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is not a polynomial ring.

Let $P = P_1 \cap P_{n-i}$. Since $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P, \mathcal{L}_\chi^{\otimes d})^T$, we have the krull dimension of $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T = 1 + i(n-1-i)$. On the other hand we have $dim(H^0(\mathrm{G/B}, \mathcal{L}_\chi)^T) = \binom{n}{i}$. With out loss of generality we assume $i \leq (n-1)/2$. So, we have $i(n-1-i) < (n-1)(n-i)/2$. Since $i \leq n-3, i-j < n-(j+2)$ for $j = 1, \ldots, i-3$. So, we have $(n-2)(n-3)\cdots(n-i+1) > 3\dot{4}\cdots i. \implies \binom{n}{i} > 1 + i(n-1-i)$. Then, we have $dim(H^0(\mathrm{G/B}, \mathcal{L}_\chi)^T) >$ Krull dimension of $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$. Hence $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is not a polynomial ring.

If $i = 1$: we have $\chi = \omega_1 + \omega_{n-1} = \alpha_1 + \cdots + \alpha_{n-1}$. By Theorem 3.3, we have the ring $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_1 + \cdots + \alpha_{n-1}}^{\otimes d})^T$ is a polynomial ring.

When $w = s_1 \cdots s_{n-1}$, By Lemma 3.9, the indecomposable dominant character χ of T for which $X(w)^{ss}_T(L_\chi)$ is non empty are $\chi = \omega_i + i\omega_{n-1}$ for $1 \leq i \leq n-1$.

If $i = n-1$: we have $\chi = n\omega_{n-1}$. Since $-w_0\chi = n\omega_1, \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ and $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{n\omega_1}^{\otimes d})^T$ are isomorphic. We proved that $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{n\omega_1}^{\otimes d})^T$ is a polynomial ring. Hence $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring.

If i=n-2: In this case $\chi = \omega_{n-2} + (n-2)\omega_{n-1}$. Since $-w_0\chi = \omega_2 + (n-2)\omega_1, \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ and $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\omega_2+(n-2)\omega_1}^{\otimes d})^T$ are isomorphic.
We proved that $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\omega_2+(n-2)\omega_1}^{\otimes d})^T$ is a polynomial ring. Hence $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring.
If $2 \leq i \leq n-3$: $\chi = \omega_i + i\omega_{n-1}$, by Lemma 3.12, we have $dim(H^0(\mathrm{G/B}, \mathcal{L}_\chi)^T) > n-1$.

claim: $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is not a polynomial ring.

Since $-w_0\chi = \omega_{n-i} + i\omega_1, \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ and $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\omega_{n-i}+i\omega_1}^{\otimes d})^T$ are isomorphic. We proved that $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\omega_{n-i}+i\omega_1}^{\otimes d})^T$ is not a polynomial ring. Hence $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is not a polynomial ring.

If $i = 1$: we have $\chi = \omega_1 + \omega_{n-1} = \alpha_1 + \cdots + \alpha_{n-1}$. By Theorem 3.3, we have the ring $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_1+\cdots+\alpha_{n-1}}^{\otimes d})^T$ is a polynomial ring.

When $w = s_{i+1}\cdots s_{n-1}s_i\cdots s_1$, where $2 \leq i \leq n-3$, by Lemma 3.10, the indecomposable dominant character χ such that $X(w)_T^{ss}(L_\chi)$ is non empty is $\alpha_1 + \cdots + \alpha_{n-1}$. By the theorem 2.3, the ring $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_1+\cdots+\alpha_{n-1}}^{\otimes d})^T$ is a polynomial ring.

Type A_3: The indecomposable dominant characters for which there is a Coxeter element w such that $X(w)_T^{ss}(L_\chi)$ is non empty are $\alpha_1+\alpha_2+\alpha_3$, $3\alpha_1+2\alpha_2+\alpha_3$, $\alpha_1+2\alpha_2+\alpha_3$ and $\alpha_1+2\alpha_2+3\alpha_3$.

When $\chi = \alpha_1+\alpha_2+\alpha_3 = \alpha_0$, by Theorem 3.3, we have $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring.

When $\chi = 3\alpha_1+2\alpha_2+\alpha_3 = 4\omega_1$, there is only one T-invariant monomial. Hence $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring in one variable.

When $\chi = \alpha_1+2\alpha_2+3\alpha_3 = 4\omega_3$. Since $-w_0\chi = 4\omega_1, \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ and $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{4\omega_1}^{\otimes d})^T$ are isomorphic. Hence $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring.

Now, we deal the special case of $\chi = 2\omega_2$ in A_3. In this case the Coxeter element is $w = s_1s_3s_2$. Let a typical monomial $\prod_{i<j} p_{ij}^{m_{ij}}$ in the Plüker coordinates which is T-invariant. Then it is easy to see that each of the indices $1,2,3,4$ occur same number of times. So, if p_{12} (resp. p_{13}) is a factor of T-invariant monomial M, then p_{34} (resp. p_{24}) is also a factor of M. Also, if p_{14} is a factor of T-invariant monomial M, then p_{23} also a factor of M. But by the Plüker relation we have

$$p_{14}p_{23} = p_{13}p_{24} - p_{12}p_{34}.$$

So, $p_{13}p_{24}$ and $p_{12}p_{34}$ generate the ring of T invariants of $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/P_2, \mathcal{L}_\chi^{\otimes d})$. where P_2 is the the maximal parabolic subgroup associated to α_2.

(1) Hence we have a surjective map $\mathbb{C}[p_{13}p_{24}, p_{12}p_{34}] \longrightarrow \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/P_2, \mathcal{L}_\chi^{\otimes d})^T$.

(2) The Krull dimension of $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/P_2, \mathcal{L}_\chi^{\otimes d})^T$ is two.

From (1) and (2), we conclude that the map $\mathbb{C}[p_{13}p_{24}, p_{12}p_{34}] \longrightarrow \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/P_2, \mathcal{L}_\chi^{\otimes d})^T$ is an isomorphism. So, $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/P_2, \mathcal{L}_{2\omega_2}^{\otimes d})^T$ is a polynomial algebra. Hence $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{2\omega_2}^{\otimes d})^T$ is a polynomial algebra. This completes the proof for the type A_{n-1}.

Type $B_n, n \neq 2$: By Theorem 4.2 in [KP09a], if G is of type B_n, the Coxeter elements w for which there is dominant character such that $X(w)^{ss}_T(L_\chi)$ is non empty is $s_n s_{n-1} \cdots s_2 s_1$. The indecomposable dominant character with this property is $\chi = \alpha_1 + \alpha_2 + \cdots + \alpha_n = \omega_1$. Now consider the standard representation $\mathbb{C}^{2n+1}$ of SO_{2n+1}. Then

$$(1) \qquad dim(Sym^2(\mathbb{C}^{2n+1})^*)) = (n+1)(2n+1).$$

By Weyl dimension formula, the dimension of the irreducible representation $V(2\omega_1)$ of SO_{2n+1} is

$$\prod_{\alpha\in\Phi^+} \frac{\langle 2\omega_1 + \rho, \alpha\rangle}{\langle \rho, \alpha\rangle}.$$

Again since $\langle 2\omega_1 + \rho, \alpha\rangle = \langle \rho, \alpha\rangle$ for $\alpha \ngeq \alpha_1$, we have

$$dim(V(2\omega_1)) = \prod_{\alpha\in\Phi^+, \alpha\geq\alpha_1} \frac{\langle 2\omega_1 + \rho, \alpha\rangle}{\langle \rho, \alpha\rangle}.$$

The set of $\alpha \in \Phi^+$ such that $\alpha \geq \alpha_1$ is $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_n, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + 2\alpha_n, \ldots, \alpha_1 + 2(\alpha_2 + \cdots + \alpha_n)\}$.
We now calculate $\frac{\langle 2\omega_1+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}$ for all $\alpha \geq \alpha_1$.

$$\frac{\langle 2\omega_1 + \rho, \alpha_1 + \cdots + \alpha_i\rangle}{\langle \rho, \alpha_1 + \cdots \alpha_i\rangle} = \frac{i+2}{i}, \quad 1 \leq i \leq n-1.$$

$$\frac{\langle 2\omega_1 + \rho, \alpha_1 + \cdots + \alpha_n\rangle}{\langle \rho, \alpha_1 + \cdots \alpha_n\rangle} = \frac{2n+3}{2n-1}.$$

$$\frac{\langle 2\omega_1 + \rho, \alpha_1 + \cdots + \alpha_{i-1} + 2(\alpha_i + \cdots \alpha_n)\rangle}{\langle \rho, \alpha_1 + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots \alpha_n)\rangle} = \frac{2n-j+2}{2n-j}, \quad 2 \leq j \leq n.$$

Hence, we have

$$(2) \qquad dim(V(2\omega_1)) = \prod_{\alpha\in\Phi^+,\alpha\geq\alpha_1} \frac{\langle 2\omega_1+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle} = n(2n+3).$$

From (1) and (2) we can conclude that $(Sym^2((\mathbb{C}^{2n+1})^*)^{SO_{2n+1}}$ is one dimensional, namely generated by the quadratic form q which defines the orthogonal group SO_{2n+1}. Hence we have

$$Sym^2(\mathbb{C}^{2n+1})^* = V(2\omega_1)^* + \mathbb{C}q,$$

where $q = \sum_{i=1}^n X_iX_{2n+2-i}$. Since q-vanishes on $SO_{2n+1}(\mathbb{C})/P_1$, where P_1 is the maximal parabolic associated to α_1, there is a unique quadratic relation among the variables X_iX_{2n+2-i}, $i = 1,2,\dots,n+1$ on $SO_{2n+1}(\mathbb{C})/P_1$, namely $aX_{n+1}^2 = \sum_i^n X_iX_{2n+2-i}$ for some non zero $a \in \mathbb{C}$ on $SO_{2n+1}(\mathbb{C})/P_1$ (refer to [LMS74]).

Now, we explain all the T-invariant polynomials restricted to $SO_{2n+1}(\mathbb{C})/P_1$. Take a T-invariant polynomial $X_1^{m_1}X_2^{m_2}\cdots X_{2n+1}^{m_{2n+1}}$ with $m_i = m_{2n+2-i}$. The above relatilon implies that every T-invariant polynomial restricted to $SO_{2n+1}(\mathbb{C})/P_1$ (restriction as a section) is a linear combination of the monomials of the form $(X_1X_{2n+1})^{r_1}(X_2X_{2n})^{r_2}\cdots(X_{n-1}X_{n+3})^{r_{n-1}}(X_nX_{n+2})^{r_n}$ for some $r_i's$ in $\mathbb{Z}_{\geq 0}$. Thus

(3) the map $\mathbb{C}[X_1X_{2n+1}, X_2X_{2n}, \dots X_{n-1}X_{n+3}, X_nX_{n+2}] \to \bigoplus_{d\in\mathbb{Z}_{\geq0}} H^0(G/P_1, \mathcal{L}_{2\omega_1}^{\otimes d})^T$ is onto.

On the other hand, we have $dim(U_{P_1}^-) = |\{\alpha \in R^+ : \alpha \geq \alpha_1\}| = 2n-1$, where $U_{P_1}^-$ be unipotent radical of the opposite parabolic subgroup of P_1 determined by T and B. Since $U_{P_1}^-$ is open subset of G/P_1, the dimension of the affine cone over G/P_1 is of dimension $2n$. So we have

(4) the Krull dimension of $\bigoplus_{d\in\mathbb{Z}_{\geq0}} H^0(G/P_1, \mathcal{L}_{2\omega_1}^{\otimes d})^T$ is $2n-n=n$.

From (3) and (4), we conclude that

$$\bigoplus_{d\in\mathbb{Z}_{\geq0}} H^0(G/P_1, \mathcal{L}_{\omega_1}^{\otimes d})^T = \mathbb{C}[X_1X_{2n+1}, X_2X_{2n}, \dots, X_{n-1}X_{n+3}, X_nX_{n+2}].$$

Hence $\bigoplus_{d\in\mathbb{Z}_{\geq0}} H^0(G/B, \mathcal{L}_{\omega_1}^{\otimes d})^T$ is a polynomial ring.

Now we prove that $dim(H^0(G/B, \mathcal{L}_{2\omega_1})^T) \leq rank(G)$. The T-invariant monomials in $Sym^2((\mathbb{C}^{2n+1})^*)$ are of the form X_iX_j for some $1 \leq i \leq n+1, j = 2n+2-i$. Hence we have $dim(Sym^2((\mathbb{C}^{2n+1})^*))^T)$ is $n+1$. Since $Sym^2((\mathbb{C}^{2n+1})^*) = V(2\omega_1)^* + \mathbb{C}q$, $dim(V(2\omega_1)^T) = n$. Hence $dim(H^0(G/B, \mathcal{L}_{2\omega_1})^T) = rank(G)$. This completes the proof for type $B_n, n \neq 2$.

Type B_2: By Theorem 4.2 in [KP09a], when G is of type B_2, the Coxeter elements w for which there is dominant character such that $X(w)^{ss}_T(L_\chi)$ is non empty are s_2s_1 and s_1s_2.

The indecomposable dominant character with this property for the Coxeter element s_2s_1 is $\chi = \alpha_1 + \alpha_2$. In this case using similar arguments as in type $B_n, n \neq 2$, we can prove that the ring $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring.

The indecomposable dominant character χ for the Coxeter element $w = s_1s_2$ for which $X(w)^{ss}_T(L_\chi)$ is non empty is $\chi = \alpha_1 + 2\alpha_2 = \alpha_0$. By Theorem 3.3, the ring $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is a polynomial ring.

Type C_n: When G is of the type C_n, by Theorem 4.2 of [KP09a], the only Coxeter element w for which there is dominant character such that $X(w)^{ss}_T(L_\chi)$ is non empty is $w = s_ns_{n-1}\cdots s_2s_1$. Further, the indecomposable dominant character χ with this property is $2\omega_1 = 2(\sum_{i\neq n}\alpha_i) + \alpha_n = \alpha_0$. By Theorem 3.3, the ring of T-invariants $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\omega_1}^{\otimes d})^T$ is a polynomial ring.

We now prove that $dim(H^0(G/B, \mathcal{L}_{2\omega_1})^T) = rank(G)$. Since $\chi = \alpha_0$, $H^0(G/B, \mathcal{L}_\chi) = \mathfrak{g}$. So, we have $H^0(G/B, \mathcal{L}_\chi)^T = \mathfrak{h}$. Hence, $dim(H^0(G/B, \mathcal{L}_{2\omega_1})^T) = rank(G)$.

Type D_4: By Theorem 4.2 of [KP09a], the only Coxeter elements $w \in W$ for which there exist a dominant weight χ such that $X(w)^{ss}_T(L_\chi)$ is non empty are $s_4s_3s_2s_1, s_4s_1s_2s_3$ and $s_3s_1s_2s_4$. The indecomposable dominant characters with this property are $2(\alpha_1+\alpha_2)+\alpha_3+\alpha_4$, $2(\alpha_3+\alpha_2)+\alpha_1+\alpha_4$ and $2(\alpha_4+\alpha_2)+\alpha_1+\alpha_3$ to the Coxeter elements $s_4s_3s_2s_1, s_4s_1s_2s_3$ and $s_3s_1s_2s_4$ respectively.

Since there is an automorphism of the Dynkin diagram sending α_1 to α_3 and fixing α_2 and α_4 and there is also an automorphism that sends α_1 to α_4 and fixing α_2 and α_3. If σ' is an automorphism of Dynkin diagram, we get a $\sigma : G \to G$ automorphism of algebraic groups such that $\sigma(B) = B$, $\sigma(T) = T$ and $\sigma(\alpha_i) = \sigma'(\alpha_i)$ for all $i = 1, \cdots, 4$. Further, we have $H^0(G/B, \mathcal{L}_\chi)$ and $H^0(G/B, \mathcal{L}_{\sigma(\chi)})$ are isomorphic as G-modules where the action of G on $H^0(G/B, \mathcal{L}_{\sigma(\chi)})$ via σ. Thus, $H^0(G/B, \mathcal{L}_\chi)^T = H^0(G/B, \mathcal{L}_{\sigma(\chi)})^T$. So it is sufficient to consider the case when $\chi = 2\omega_1 = 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$.

Now consider the standard representation $\mathbb{C}^8$ of SO_8. Then, we have

$$dim(Sym^2((\mathbb{C}^8)^*)) = 36. \tag{1}$$

By using Weyl dimension formula and by proceeding with similar calculation above we can see that the dimension of the irreducible representation $V(2\omega_1)$ is 35. (2)

From (1) and (2), we have

$$Sym^2((\mathbb{C}^8)^*) = V(2\omega_1)^* + \mathbb{C}q,$$

where $q = \sum_{i=1}^{8} X_iX_{9-i}$. Since q-vanishes on $SO_8(\mathbb{C})/P_1$, where P_1 is the maximal parabolic associated to α_1, there is a unique quadratic relation among the variables X_iX_{9-i}, $i = 1,2,3,4$ on $SO_8(\mathbb{C})/P_1$, namely $-X_1X_8 = \sum_i^4 X_iX_{9-i}$ on $SO_8(\mathbb{C})/P_1$(refer to [LMS74]).

Now we explain all the T-invariant polynomials restricted to $SO_8(\mathbb{C})/P_1$. Take a T-invariant polynomial $X_1^{m_1}X_2^{m_2}\cdots X_8^{m_8}$ with $m_i = m_{9-i}$. The above relatilon implies that every T-invariant polynomial restricted to $SO_8(\mathbb{C})/P_1$ is a linear combination of the monomials of the form $(X_2X_7)^{m_2}(X_3X_6)^{m_3}\cdots(X_4X_5)^{m_4}$. Thus

(3) the map $\mathbb{C}[X_2X_7, X_3X_6, X_4X_5] \to \bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/P_1, \mathcal{L}_{2\omega_1}^{\otimes d})^T$ is onto.

On the other hand we have $dim(U_{P_1}^-) = |\{\alpha \in R^+ : \alpha \geq \alpha_1\}| = 6$, where $U_{P_1}^-$ be unipotent radical of the opposite parabolic subgroup of P_1 determined by T and B.
Hence the dimension of the affine cone G/P_1 is of dimension 7.
So, we have

(4) the Krull dimension of $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/P_1, \mathcal{L}_{2\omega_1}^{\otimes d})^T$ is 3.

From (3) and (4), we conclude that $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/P_1, \mathcal{L}_{2\omega_1}^{\otimes d})^T$ is a polynomial ring. Hence $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{2\omega_1}^{\otimes d})^T$ is a polynomial ring.

Now, we prove that $dim(H^0(G/B, \mathcal{L}_{2\omega_1})^T) \leq rank(G)$.

The T-invariant monomials in $Sym^2((\mathbb{C}^8)^*)$ are of the form X_iX_j for $1 \leq i \leq 4, j = 9-i$. Hence, we have $dim(Sym^2((\mathbb{C}^8)^*)^T)$ is 4. Since $Sym^2((\mathbb{C}^8)^*) = V(2\omega_1)^* + \mathbb{C}q$, we have $dim(V(2\omega_1)^T) = 3$. Hence, $dim(H^0(G/B, \mathcal{L}_{2\omega_1})^T) \leq rank(G)$.

Type $D_n, n \neq 4$: By Theorem 4.2 of [KP09a], the Coxeter element w for which there is dominant character such that $X(w)_T^{ss}(L_\chi)$ is non empty is $s_ns_{n-1}\cdots s_2s_1$. The indecomposable dominant character with this property is $\chi = 2\omega_1 = 2(\alpha_1 + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$. Proof in this case is similar to that of $\chi = 2\omega_1$ in type B_n.

For other types : By Theorem 4.2 of [KP09a], there is no Coxeter element w and dominant character χ such that $X(w)_T^{ss}(L_\chi)$ is non empty. ☐

Suppose that there exists a Coxeter element $c \in W$ such that $X(c)_T^{ss}(\mathcal{L}_\chi) \neq \emptyset$. Then, we have

Corollary 3.15. *The GIT quotient $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$ is isomorphic to a weighted projective space if and only if dim$(H^0(G/B,\mathcal{L}_\chi)^T) \leq$ rank of G.*

In fact, when this holds, $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$ is isomorphic to a projective space in its anti-tautological embedding (that is, the polynomial generators of its homogeneous coordinate ring all lie in degree 1). In particular, we have

Corollary 3.16. *The polarized variety $(T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi),\ \mathcal{M})$ is projectively normal, where $\mathcal{M}$ is the descent of $\mathcal{L}_\chi$ to the quotient $T\backslash\backslash(G/B)^{ss}_T(\mathcal{L}_\chi)$.*

We prove the following.

Corollary 3.17. *Let G be a simple adjoint group over $\mathbb{C}$. Let $\chi \in X(T)^+_i$ be such that there is a Coxeter element $w \in W$ such that $X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset$ then*
(i) In type A_{n-1}: If further, the dim$(H^0(G/B,\mathcal{L}_\chi)^T) \geq$ rank(G) then the $N_G(T)$ invariants of the homogeneous coordinate ring $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B,\mathcal{L}_\chi^{\otimes d})$ is a polynomial ring.
(ii) For other types the $N_G(T)$ invariants of the homogeneous coordinate ring $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B,\mathcal{L}_\chi^{\otimes d})$ is a polynomial ring.

Proof. When G is of type A_n, B_2 or C_n, and $\chi = \alpha_0$, the highest long root, then, the ring of T-invariants of the homogeneous coordinate ring $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B,\mathcal{L}_\chi^{\otimes d})$ is isomorphic to $\mathbb{C}[\mathfrak{h}]$, and so the W-invariants of $\mathbb{C}[\mathfrak{h}]$ is a polynomial ring by [Chevalley, Shephard-Todd, Serre]. In the case of $\chi = 2\omega_1$ of type B_n, by Theorem 2.11, we have

$$\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B,\mathcal{L}_\chi^{\otimes d})^T = \mathbb{C}[X_1X_{2n+1}, X_2X_{2n}, \ldots X_nX_{n+2}].$$

The action of the Weyl group $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ on the space $\sum_{i=1}^n \mathbb{C}X_iX_{2n+2-i}$ factor through the natural map $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n \longrightarrow S_n$. Further the resulting action of S_n on $\sum_{i=1}^n \mathbb{C}X_iX_{2n+i}$ is given by the natural action of S_n on $\mathbb{C}^n$. So , the ring of W- invariants of $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B,\mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring generated by the elementary symmetric polynomials in the variables $\{X_iX_{2n+2-i} : i = 1,\ldots,n\}$.

When G is of type A_3, the Weyl group is S_4, and let V be the two dimensional vector space generated by $p_{12}p_{34}$ and $p_{13}p_{24}$. Then, the ring of T-invariants of $\bigoplus_{d\in\mathbb{Z}_{\geq 0}} H^0(G/B,\mathcal{L}^{\otimes d}_{\alpha_1+2\alpha_2+\alpha_3})$ is $Sym(V)$. So, we have the two dimensional representation of S_4. Actually, this representation is the standard two dimensional represenation of S_3, and it factors through the surjective homomorphism $\phi : S_4 \longrightarrow S_3$ given by the natural homomorphism from $S_4 \longrightarrow S_4/\{id,(12)(34),(14)(23),(13)(24)\}$.

For the case of D_n, the proof is similar to that of B_n. $\square$

Chapter 4

Cohomology of the Tangent bundle of BSDH-variety

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Let G be a simple algebraic group of adjoint type over the field of complex numbers, B be a Borel subgroup of G containing a maximal torus T of G, w be an element of the Weyl group W and $X(w)$ be the Schubert variety in G/B corresponding to w. Let $Z(w, \underline{i})$ be the Bott-Samelson-Demazure-Hansen variety (the desingularization of $X(w)$) corresponding to a reduced expression $\underline{i}$ of w.

In this chapter, we prove the vanishing results of cohomology of the tangent bunlde of the BSDH-variety $Z(w, \underline{i})$.

4.1 Cohomology of line bundles

Now, we recall the following result due to Demazure ([Dem76], Page 1) on a short exact sequence of B-modules:

Lemma 4.1. *Let α be a simple root and $\lambda \in X(T)$ be such that $\langle \lambda, \alpha \rangle \geq 0$. Let ev denote the evaluation map $H^0(s_\alpha, \lambda) \longrightarrow \mathbb{C}_\lambda$. Then we have*

1. *If $\langle \lambda, \alpha \rangle = 0$, then $H^0(s_\alpha, \lambda) \simeq \mathbb{C}_\lambda$.*
2. *If $\langle \lambda, \alpha \rangle \geq 1$, then $\mathbb{C}_{s_\alpha(\lambda)} \hookrightarrow H^0(s_\alpha, \lambda)$ and there is a short exact sequence of B-modules:*

$$0 \longrightarrow H^0(s_\alpha, \lambda - \alpha) \longrightarrow H^0(s_\alpha, \lambda)/\mathbb{C}_{s_\alpha(\lambda)} \xrightarrow{ev} \mathbb{C}_\lambda \longrightarrow 0.$$

Further more, $H^0(s_\alpha, \lambda - \alpha) = 0$ if $\langle \lambda, \alpha \rangle = 1$.

3. *Let $n = \langle \lambda, \alpha \rangle$. As a B-module, $H^0(s_\alpha, \lambda)$ has a composition series*

$$0 \subsetneq V_n \subsetneq V_{n-1} \subsetneq \ldots \subsetneq V_0 = H^0(s_\alpha, \lambda)$$

such that $V_i/V_{i+1} \simeq \mathbb{C}_{\lambda - i\alpha}$ for $i = 0, 1, \ldots, n-1$ and $V_n = \mathbb{C}_{s_\alpha(\lambda)}$.

Recall ρ as a half sum of positive roots. We define the dot action by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for any $w \in W$ and $\lambda \in X(T) \otimes \mathbb{R}$. Note that $s_\alpha \cdot 0 = -\alpha$ for $\alpha \in S$. As a consequence of the exact sequences of Lemma 4.1, we can prove the following.

Let $w \in W$, α be a simple root, and set $v = ws_\alpha$.

Lemma 4.2. *If $l(w) = l(v) + 1$, then we have*

1. *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^j(v, H^0(s_\alpha, \lambda))$ for all $j \geq 0$.*
2. *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^{j+1}(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*
3. *If $\langle \lambda, \alpha \rangle \leq -2$, then $H^{j+1}(w, \lambda) = H^j(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*
4. *If $\langle \lambda, \alpha \rangle = -1$, then $H^j(w, \lambda)$ vanishes for every $j \geq 0$.*

Proof. Choose a reduced expression of $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ with $\alpha_{i_r} = \alpha$. Hence $v = s_{i_1} s_{i_2} \cdots s_{i_{r-1}}$ is a reduced expression for v. Let $\underline{i} = (i_1, i_2, \ldots, i_r)$ and $\underline{i}' = (i_1, i_2, \ldots, i_{r-1})$. Now consider the morphism $f_r : Z(w, \underline{i}) \longrightarrow Z(v, \underline{i}')$ defined as above.

Proof of (1): Since $\langle \lambda, \alpha \rangle \geq 0$, we have $H^j(s_\alpha, \lambda) = 0$ for every $j > 0$. Hence using the isomorphism (*Iso*), we have $R^j f_{r*} \mathcal{L}(w, \lambda) = 0$ for every $j > 0$. Therefore, by [Har77, p.252, III, Ex 8.1] we have $H^i(w, \lambda) = H^i(v, H^0(s_\alpha, \lambda))$ for every $i \geq 0$.

Proof of (3): Since $\langle \lambda, \alpha \rangle \leq -2$, by using (Borel-Weil-Bott theorem) [Dem76, Theorem 2 (c)] for $L_\alpha/B_\alpha (\simeq P_\alpha/B)$; we have $H^i(s_\alpha, \lambda) = 0$ for $i \neq 1$ and $H^1(s_\alpha, \lambda) = H^0(s_\alpha, s_\alpha \cdot \lambda)$. By (*Iso*), we have $R^j f_{r*} \mathcal{L}(w, \lambda) = 0$ for every $j \neq 1$. Hence by using Leray spectral sequence, we see that $H^{j+1}(w, \lambda) = H^j(v, R^1 f_{r*} \mathcal{L}(w, \lambda)) = H^j(v, H^1(s_\alpha, \lambda))$ (see [Wei95, p.152, Section 5.8.6]). Hence $H^{j+1}(w, \lambda) = H^j(v, H^0(s_\alpha, s_\alpha \cdot \lambda))$ for every $j \geq 0$. Since $\langle s_\alpha \cdot \lambda, \alpha \rangle \geq 0$, by (1) we have $H^j(v, H^0(s_\alpha, s_\alpha \cdot \lambda)) = H^j(w, s_\alpha \cdot \lambda)$ for every $j \geq 0$. Hence we have $H^{j+1}(w, \lambda) = H^j(w, s_\alpha \cdot \lambda)$ for every $j \geq 0$.

Proof of (2): It follows from (3) by interchanging the role of λ and $s_\alpha \cdot \lambda$, because $\langle s_\alpha \cdot \lambda, \alpha \rangle = -\langle \lambda, \alpha \rangle - 2$.

Proof of (4): If $\langle \lambda, \alpha \rangle = -1$, then $H^i(s_\alpha, \lambda) = 0$ for every $i \geq 0$ (see [Jan07, p.218, Proposition 5.2(b)]). Now the proof of (4) follows by using similar arguments as in (1) and (3). □

The following consequence of Lemma 4.2 will be used to compute cohomology modules.

Let $\pi : \widetilde{G} \longrightarrow G$ be the simply connected covering of G. Let $\widetilde{L}_\alpha$ (respectively, $\widetilde{B}_\alpha$ be the inverse image of L_α (respectively, B_α) in $\widetilde{G}$ under π.

Lemma 4.3. *Let V be an irreducible $\widetilde{L}_\alpha$-module. Let λ be a character of $\widetilde{B}_\alpha$. Then, we have*

1. *As $\widetilde{L}_\alpha$-modules, $H^j(\widetilde{L}_\alpha/\widetilde{B}_\alpha, V \otimes \mathbb{C}_\lambda) \simeq V \otimes H^j(\widetilde{L}_\alpha/\widetilde{B}_\alpha, \mathbb{C}_\lambda)$ for every $j \geq 0$.*
2. *If $\langle \lambda, \alpha \rangle \geq 0$, $H^j(\widetilde{L}_\alpha/\widetilde{B}_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ for every $j \geq 1$.*
3. *If $\langle \lambda, \alpha \rangle \leq -2$, $H^0(\widetilde{L}_\alpha/\widetilde{B}_\alpha, V \otimes \mathbb{C}_\lambda) = 0$, and*
$$H^1(\widetilde{L}_\alpha/\widetilde{B}_\alpha, V \otimes \mathbb{C}_\lambda) \simeq V \otimes H^0(\widetilde{L}_\alpha/\widetilde{B}_\alpha, \mathbb{C}_{s_\alpha \cdot \lambda}).$$
4. *If $\langle \lambda, \alpha \rangle = -1$, then $H^j(\widetilde{L}_\alpha/\widetilde{B}_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ for every $j \geq 0$.*

Proof. Proof (1). By [Jan07, p.53, I, Proposition 4.8] and [Jan07, p.77, I, Proposition 5.12], for all $j \geq 0$, we have the following isomorphism of $\widetilde{L}_\alpha$-modules:

$$H^j(\widetilde{L}_\alpha/\widetilde{B}_\alpha, V \otimes \mathbb{C}_\lambda) \simeq V \otimes H^j(\widetilde{L}_\alpha/\widetilde{B}_\alpha, \mathbb{C}_\lambda).$$

Proof of (2), (3) and (4) follows from Lemma 4.2 by taking $w = s_\alpha$ and the fact that $\widetilde{L}_\alpha/\widetilde{B}_\alpha \simeq P_\alpha/B$. ☐

Recall the structure of indecomposable B_α-modules and $\tilde{B}_\alpha$-modules (see [BKS04, p.130, Corollary 9.1]).

Lemma 4.4.

1. *Any finite dimensional indecomposable $\widetilde{B}_\alpha$-module V is isomorphic to $V' \otimes \mathbb{C}_\lambda$ for some irreducible representation V' of $\widetilde{L}_\alpha$ and for some character λ of $\widetilde{B}_\alpha$.*
2. *Any finite dimensional indecomposable B_α-module V is isomorphic to $V' \otimes \mathbb{C}_\lambda$ for some irreducible representation V' of $\widetilde{L}_\alpha$ and for some character λ of $\widetilde{B}_\alpha$.*

Proof. Proof of (1) follows from [BKS04, p.130, Corollary 9.1].

Proof of (2) follows from the fact that every B_α-module can be viewed as a $\widetilde{B}_\alpha$-module via the natural homomorphism. ☐

Now, we prove the following:

Corollary 4.5. *Let $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression for w such that $\langle \alpha_{i_j}, \alpha_{i_r}\rangle = 0$ for every $j = 1, 2, \ldots, r-1$. Then, $H^0(w, \alpha_{i_r})$ is isomorphic to $H^0(s_{i_r}, \alpha_{i_r})(\simeq sl_{2,\alpha_{i_r}})$.*

Proof. Since $L_{\alpha_{i_r}}/B_{\alpha_{i_r}} \hookrightarrow P_{\alpha_{i_r}}/B$ is an isomorphism, we have

$$sl_{2,\alpha_{i_r}} \simeq H^0(L_{\alpha_{i_r}}/B_{\alpha_{i_r}}, \alpha_{i_r}) \simeq H^0(s_{i_r}, \alpha_{i_r}).$$

We note that $sl_{2,\alpha_{i_r}}$ gets a natural B-module structure via the above isomorphism $sl_{2,\alpha_{i_r}} \simeq H^0(s_{i_r}, \alpha_{i_r})$.

Let $v = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$. If $l(v) = 0$, then $w = s_{i_r}$ and we are done. Otherwise, let $v' = s_{i_2}\cdots s_{i_{r-1}}$. By induction on $l(v)$, we have

$$H^0(s_{i_2}\cdots s_{i_r}, \alpha_{i_r}) = H^0(s_{i_r}, \alpha_{i_r}).$$

By *SES*, we have

$$H^0(w, \alpha_{i_r}) = H^0(s_{i_1}, H^0(s_{i_2}\ldots s_{i_r}, \alpha_{i_r})) = H^0(s_{i_1}, H^0(s_{i_r}, \alpha_{i_r})).$$

Since $\langle \alpha_{i_r}, \alpha_{i_1}\rangle = 0$ and $\langle -\alpha_{i_r}, \alpha_{i_1}\rangle = 0$, by Lemma 4.4, $H^0(s_{i_r}, \alpha_{i_r})$ is the trivial $B_{\alpha_{i_1}}$-module of dimension 3. Hence, the vector bundle $\mathcal{L}(s_{i_1}, H^0(s_{i_r}, \alpha_{i_r}))$ on $X(s_{i_1}) \simeq \mathbb{P}^1$ is the trivial bundle of rank 3. Thus, we have

$$H^0(s_{i_1}, H^0(s_{i_r}, \alpha_{i_r})) = H^0(s_{i_r}, \alpha_{i_r}).$$

$\square$

We recall the following vanishing results from [Kan13] (see [Kan13, Corollary 3.6] and [Kan13, Corollary 4.10]).

Lemma 4.6. *Let $w \in W$, and $\alpha \in R^+$. Then, we have*

1. *$H^j(w, \alpha) = 0$ for all $j \geq 2$.*
2. *If G is simply laced, $H^j(w, \alpha) = 0$ for all $j \geq 1$.*

4.2 Vanishing of the Higher Cohomology of the Tangent Bundle of $Z(w, \underline{i})$

In this section, we prove that a BSDH variety has unobstructed deformations and it has no deformations whenever the group G is simply laced.

We recall that the BSDH-variety corresponding to a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ is denoted by $Z(w,\underline{i})$ and we denote the tangent bundle of $Z(w,\underline{i})$ by $T_{(w,\underline{i})}$, where $\underline{i} = (i_1,i_2,\ldots,i_r)$.

Let $w = s_{i_1}s_{i_2}\cdots s_{i_r}$, $v = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$ and $\underline{i}' = (i_1,i_2,\ldots,i_{r-1})$. Note that $l(v) = l(w)-1$. Consider the fibration $f_r : Z(w,\underline{i}) \longrightarrow Z(v,\underline{i}')$ as in Section 2. One can easily see that this fibration is the fibre product of $\pi_r : G/B \to G/P_{\alpha_{i_r}}$ and $\pi_r \circ \phi_v : Z(v,\underline{i'}) \to G/P_{\alpha_{i_r}}$; namely, we have the following commutative diagram :

$$\begin{array}{ccc} Z(v,\underline{i}') \times_{G/P_{\alpha_{i_r}}} G/B = Z(w,\underline{i}) & \xrightarrow{\phi_w} & G/B \\ {\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle \pi_r} \\ Z(v,\underline{i}') & \xrightarrow{\pi_r\circ\phi_v} & G/P_{\alpha_{i_r}} \end{array}$$

The relative tangent bundle of π_r is the line bundle $\mathcal{L}(w_0,\alpha_{i_r})$. Hence the relative tangent bundle of f_r is $\phi_w^*\mathcal{L}(w_0,\alpha_{i_r})$. By taking the differentials of this smooth fibration f_r we obtain the following exact sequence:

$$0 \to \phi_w^*\mathcal{L}(w_0,\alpha_{i_r}) \to T_{(w,\underline{i})} \to f_r^*T_{(v,\underline{i}')} \to 0. \qquad (rel)$$

We use the above short exact sequence (rel) and Lemma 4.6 to prove the following:

Theorem 4.7. *Let $w \in W$, $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression for w and let $\underline{i} = (i_1,i_2,\ldots,i_r)$. Then, we have*

1. *$H^j(Z(w,\underline{i}),T_{(w,\underline{i})}) = 0$ for all $j \geq 2$.*
2. *If G is simply laced, $H^j(Z(w,\underline{i}),T_{(w,\underline{i})}) = 0$ for all $j \geq 1$.*

Proof. We start by proving (2). We first recall the following isomorphism (see [BK07, Theorem 3.3.4(b)]):

$$H^j(Z(w,\underline{i}),\phi_w^*\mathcal{L}(w_0,\alpha_{i_r})) \simeq H^j(X(w),\mathcal{L}(w,\alpha_{i_r})) = H^j(w,\alpha_{i_r}) \ \ for\ all \ \ j \geq 0.$$

Let $v = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$ and $\underline{i}' = (i_1,i_2,\ldots,i_{r-1})$. Since $f_r : Z(w,\underline{i}) \longrightarrow Z(v,\underline{i}')$ is a smooth fibration with fibre $\mathbb{P}^1$, by using [Har77, p.288, Corollary 12.9] and [Jan07, p.369, Section 14.6(3)] we have $H^j(Z(w,\underline{i}),f_r^*T_{(v,\underline{i}')}) = H^j(Z(v,\underline{i}'),T_{(v,\underline{i}')})$ for every $j \geq 0$.

By considering the long exact sequence associated to the short exact sequence (rel) and using above arguments, we have the following long exact sequence of B-modules:
$0 \longrightarrow H^0(w,\alpha_{i_r}) \longrightarrow H^0(Z(w,\underline{i}),T_{(w,\underline{i})}) \longrightarrow H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')}) \longrightarrow H^1(w,\alpha_{i_r}) \longrightarrow$

$H^1(Z(w,\underline{i}),T_{(w,\underline{i})}) \longrightarrow H^1(Z(v,\underline{i}),T_{(v,\underline{i}')}) \longrightarrow H^2(w,\alpha_{i_r}) \longrightarrow H^2(Z(w,\underline{i}),T_{(w,\underline{i})}) \longrightarrow$
$H^2(Z(v,\underline{i}'),T_{(v,\underline{i}')}) \longrightarrow H^3(w,\alpha_{i_r}) \longrightarrow \cdots$.
Since G is simply laced, by Lemma 4.6 (2), we have $H^j(w,\alpha_{i_r}) = 0$ for every $j \geq 1$. Thus we have $H^j(Z(w,\underline{i}),T_{(w,\underline{i})}) = H^j(Z(v,\underline{i}'),T_{(v,\underline{i}')})$ for every $j \geq 1$. Now the proof follows by induction on $l(w)$.

Proof of (1) is similar by using Lemma 4.6 (1). □

Note: The long exact sequence associated to the short exact sequence (rel) which is considered in the proof of the Proposition 4.7 will be used frequently in the future. We call this *LES*.

Theorem 4.7(1) yields $H^2(Z(w,\underline{i}),T_{(w,\underline{i})}) = 0$. Hence, we see that $Z(w,\underline{i})$ has unobstructed deformations. That is, $Z(w,\underline{i})$ admits a smooth versal deformation (see [Huy06, p.273, lines 19-21]).

If in addition G is simply laced, Theorem 4.7(2) yields $H^1(Z(w,\underline{i}),T_{(w,\underline{i})}) = 0$. Using [Huy06, p. 272, Proposition 6.2.10], we see that $Z(w,\underline{i})$ has no deformations. That is, a BSDH variety for a simply laced group G is rigid.

Chapter 5

Automorphism group of a BSDH-variety

In this chapter, we compute the connected component $Aut^0(Z(w, \underline{i}))$ of the automorphism group of $Z(w, \underline{i})$ containing the identity automorphism. We show that $Aut^0(Z(w, \underline{i}))$ contains a closed subgroup isomorphic to B if and only if $w^{-1}(\alpha_0) < 0$, where α_0 is the highest root. If w_0 denotes the longest element of W, then we prove that $Aut^0(Z(w_0, \underline{i}))$ is a parabolic subgroup of G. It is also shown that this parabolic subgroup depends very much on the chosen reduced expression $\underline{i}$ of w_0 and we describe all parabolic subgroups of G that occur as $Aut^0(Z(w_0, \underline{i}))$. If G is simply laced, then we show that for every $w \in W$, and for every reduced expression $\underline{i}$ of w, $Aut^0(Z(w, \underline{i}))$ is a quotient of the parabolic subgroup $Aut^0(Z(w_0, \underline{j}))$ of G for a suitable choice of a reduced expression $\underline{j}$ of w_0 (see Theorem 5.19).

5.1 Cohomology of the relative tangent bundle on $Z(w, \underline{i})$

In this section, we compute the cohomology groups of the relative tangent bundle on $Z(w, \underline{i})$.

We use the notation as in the previous chapter. For a B-module V and a character $\mu \in X(T)$, we denote by V_μ, the weight space for the action of T. By the definition, it is the space of all vectors v in V such that $t \cdot v = \mu(t)v$ for all $t \in T$. We denote by $dim(V_\mu)$ the dimension of the space V_μ.

Given a character $\lambda \in X(T)$ and a simple root $\gamma \in S$ such that $\langle \lambda, \gamma \rangle \geq 0$, we recall that the γ-string of λ is the set $\{\lambda, \lambda - \gamma, \ldots, \lambda - \langle \lambda, \gamma \rangle \gamma\}$ of weights, which by Lemma 4.1, is the set of weights occuring in $H^0(s_\gamma, \lambda)$.

Recall the partial order $\leq$ on $X(T)$ given by $\mu \leq \lambda$ if $\lambda - \mu$ is a non-negative integral linear combination of simple roots. We say that $\mu < \lambda$ if in addition $\lambda - \mu$ is non zero.

We begin by proving the following lemma:

Let R_s (respectively, R_s^-) be the set of short roots (respectively, negative short roots).

Lemma 5.1. *Let V be a B-module and let $w \in W$. Then we have*

1. *If there is a character $\lambda_0 \in X(T)$ such that $V_\mu = 0$ unless $\mu \leq \lambda_0$ (respectively, $\mu < \lambda_0$), then $H^0(w, V)_\mu = 0$ unless $\mu \leq \lambda_0$ (respectively, $\mu < \lambda_0$).*
2. *If $V_\mu = 0$ for every $\mu \in X(T) \setminus (R \cup \{0\})$, then $H^0(w, V)_\mu = 0$ for every $\mu \in X(T) \setminus (R \cup \{0\})$.*
3. *If $V_\mu = 0$ for every $\mu \in X(T) \setminus (R_s \cup \{0\})$, then $H^0(w, V)_\mu = 0$ for every $\mu \in X(T) \setminus (R_s \cup \{0\})$.*
4. *If $V_\mu = 0$ for every $\mu \in X(T) \setminus (R_s^- \cup \{0\})$, then $H^0(w, V)_\mu = 0$ for every $\mu \in X(T) \setminus (R_s^- \cup \{0\})$.*

Proof. Proof of (1): Let V be a B-module and $\lambda_0 \in X(T)$ such that $V_\mu = 0$ if $\mu \nleq \lambda_0$. Proof is by induction on $l(w)$. If $l(w) = 0$ there is nothing to prove. Otherwise, we can choose a $\gamma \in S$ such that $l(s_\gamma w) = l(w) - 1$. Let $u = s_\gamma w$. By *SES*, the B-modules $H^0(s_\gamma, H^0(u, V))$ and $H^0(w, V)$ are isomorphic.

Let $\mu \in X(T)$ be a weight of $H^0(w, V)$ (i.e, $H^0(w, V)_\mu \neq 0$). Then there is an indecomposable B_γ-summand V' of $H^0(u, V)$ such that $H^0(s_\gamma, V')_\mu \neq 0$. By Lemma 4.4, we have $V' = V'' \otimes \mathbb{C}_{\mu'}$ for some irreducible $\widetilde{L}_\gamma$-module V'' and for some character μ' of $\widetilde{B}_\gamma$. By Lemma 4.3, we have $H^0(s_\gamma, V') = V'' \otimes H^0(s_\gamma, \mu')$ and $\langle \mu', \gamma \rangle \geq 0$. Now, let μ'' be the highest weight of V''. Then, $H^0(s_\gamma, V') = H^0(s_\gamma, \mu'') \otimes H^0(s_\gamma, \mu')$. By the description of the weights of $H^0(s_\gamma, \mu'') \otimes H^0(s_\gamma, \mu')$, any weight λ of $H^0(s_\gamma, V')$ is of the form $\lambda = \mu_1 + \mu_2$ where $\mu_1 = \mu'' - a_1\gamma$ and $\mu_2 = \mu' - a_2\gamma$ for some integers $0 \leq a_1 \leq \langle \mu'', \gamma \rangle$, $0 \leq a_2 \leq \langle \mu', \gamma \rangle$. Thus, we have $\lambda = \mu'' + \mu' - (a_1 + a_2)\gamma$.

Hence, any weight λ of $H^0(s_\gamma, V')$ satisfies $\lambda \leq \mu' + \mu''$. In particular, $\mu \leq \mu' + \mu''$. Note that since $\mu' + \mu''$ is the highest weight of $H^0(s_\gamma, V')$, $H^0(u, V)_{\mu'+\mu''} \neq 0$. By induction on $l(w)$, $\mu' + \mu'' \leq \lambda_0$. Hence, we have $\mu \leq \lambda_0$.

Proof of $V_\mu = 0$ unless $\mu < \lambda_0 \implies H^0(w, V)_\mu = 0$ unless $\mu < \lambda_0$ is similar.

Proof of (2): Assume that $H^0(w, V)_\mu \neq 0$. We use the same notation as in the proof of (1). We have $H^0(s_\gamma, V') = H^0(s_\gamma, \mu') \otimes H^0(s_\gamma, \mu'')$. Since $V'_{\mu'+\mu''} \neq 0$, by induction on $l(w)$, $\mu' + \mu'' \in R \cup \{0\}$. By the proof of (1), the weights of $H^0(s_\gamma, V')$ are of the form

$\mu = \mu' + \mu'' - j\gamma$ for some integer $0 \leq j \leq \langle \mu' + \mu'', \gamma \rangle$. If $\mu' + \mu'' = 0$, then $j = 0$ and so $\mu = \mu' + \mu'' = 0$. Otherwise, $\mu' + \mu''$ is a root, it follows that μ is a root (see [Hum72, p.45, Section 9.4]).

Proof of (3) follows from the proof of (2) because any root in the γ-string of a short root is short.

Proof of (4) follows from (1) (by taking $\lambda_0 = 0$) and (3). □

Lemma 5.2. *Let $w \in W$. Then we have, $H^1(w, \mathfrak{b})_\mu = 0$ unless μ is a negative short root.*

Proof. If $l(w) = 0$, we are done. Otherwise, choose $\gamma \in S$ such that $l(s_\gamma w) = l(w) - 1$. Let $u = s_\gamma w$. Then by *SES*, we have

$$0 \longrightarrow H^1(s_\gamma, H^0(u, \mathfrak{b})) \longrightarrow H^1(w, \mathfrak{b}) \longrightarrow H^0(s_\gamma, H^1(u, \mathfrak{b})) \longrightarrow 0$$

By induction on $l(w)$, $H^1(u, \mathfrak{b})_\mu = 0$ unless μ is a negative short root. By Lemma 5.1 (4), $H^0(s_\gamma, H^1(u, \mathfrak{b}))_\mu = 0$ unless μ is a negative short root.

Now, we prove that $H^1(s_\gamma, H^0(u, \mathfrak{b}))_\mu = 0$ unless μ is a negative short root. Assume that $H^1(s_\gamma, H^0(u, \mathfrak{b}))_\mu \neq 0$. Then there exists an indecomposable B_γ-direct summand V_1 of $H^0(u, \mathfrak{b})$ such that $H^1(s_\gamma, V_1)_\mu \neq 0$. By Lemma 4.4, $V_1 = V' \otimes \mathbb{C}_{\mu'}$ for some irreducible $\widetilde{L}_\gamma$-module V' and for some character μ' of $\widetilde{B}_\gamma$. Since $H^1(s_\gamma, V_1) \neq 0$, by Lemma 4.3 we have $\langle \mu', \gamma \rangle \leq -2$ and $H^1(s_\gamma, V_1) = V' \otimes H^0(s_\gamma, s_\gamma \cdot \mu')$. Then any weight μ'' of $H^1(s_\gamma, V_1)$ is in the γ-string from $\mu_1 + \gamma = \mu_1 + \rho - s_\gamma(\rho) = s_\gamma(s_\gamma \cdot \mu_1)$ to $s_\gamma \cdot \mu_1$, where μ_1 is the lowest weight of V_1.

Note that by [Kan13, Lemma 2.6], the evaluation map $ev : H^0(u, \mathfrak{b}) \longrightarrow \mathfrak{b}$ is injective. Hence, if $H^0(u, \mathfrak{b})_{-\gamma} \neq 0$ then $\mathbb{C}.h_\gamma \oplus \mathbb{C}_{-\gamma}$ is an indecomposable B_γ-direct summand of $H^0(u, \mathfrak{b})$ (here h_γ is a basis vector of the zero weight space of $sl_{2,\gamma}$). By Lemma 4.4, we have

$$\mathbb{C}.h_\gamma \oplus \mathbb{C}_{-\gamma} = V \otimes \mathbb{C}_{-\omega_\gamma},$$

where V is the standard 2- dimensional representation of $\widetilde{L}_\gamma$. By Lemma 4.3, we have

$$H^0(s_\gamma, V \otimes \mathbb{C}_{-\omega_\gamma}) = V \otimes H^0(s_\gamma, -\omega_\gamma).$$

Since $\langle -\omega_\gamma, \gamma \rangle = -1$, by Lemma 4.2, $H^1(s_\gamma, \mathbb{C}.h_\gamma \oplus \mathbb{C}_{-\gamma}) = 0$.

Since V_1 is a B-submodule of $\mathfrak{b}$ and $H^1(s_\gamma, V') \neq 0$, by the above arguments, we see that V_1 is not isomorphic to $\mathbb{C}.h_\gamma \oplus \mathbb{C}_{-\gamma}$. In particular, we have $\mu_1 \in R^- \setminus \{-\gamma\}$. Let λ be the lowest weight of V'. Then, we have $\mu_1 = \lambda + \mu'$. Since $\langle \lambda, \gamma \rangle \leq 0$ and $\langle \mu', \gamma \rangle \leq -2$, we have $\langle \mu_1, \gamma \rangle \leq -2$. Further by [Hum72, p.45, Section 9.4], we have $-3 \leq \langle \mu_1, \gamma \rangle$. Then, the γ-string of μ is either $\mu+\gamma$ (if $\langle\mu_1,\gamma\rangle = -2$) or $\mu+\gamma, \mu+2\gamma$ (if $\langle\mu_1,\gamma\rangle = -3$). In particular, any weight μ'' of $H^1(s_\gamma, V_1)$ satisfies $|\langle\mu'',\gamma\rangle| \leq 1$ and μ'' is a negative short root. In particular, μ is a negative short root.

Hence by the above short exact sequence, we conclude that $H^1(w,\mathfrak{b})_\mu = 0$ unless μ is a negative short root. □

Recall from Section 2 that $h_{\alpha_{i_r}}$ is a basis vector of the zero weight space of $sl_{2,\alpha_{i_r}}$.

Lemma 5.3. *Let $w \in W$ and fix a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$. Then,*

1. *If there is a $1 \leq j < r-1$ such that $\alpha_{i_j} = \alpha_{i_r}$, then we have $H^0(w,\alpha_{i_r})_0 = 0$.*
2. *If $\alpha_{i_j} \neq \alpha_{i_r}$ for all $1 \leq j < r-1$, then $\mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}}$ is a $B_{\alpha_{i_r}}$-submodule of $H^0(w,\alpha_{i_r})$, and $H^0(w,\alpha_{i_r})_0 = \mathbb{C}.h_{\alpha_{i_r}}$. In particular, $dim(H^0(w,\alpha_{i_r})_0) = 1$.*

Proof. Proof of (1): If there is a $1 \leq j < r-1$ such that $\alpha_{i_j} = \alpha_{i_r}$, without loss of generality we may assume that there is no k such that $j < k < r-1$ and $\alpha_{i_k} = \alpha_{i_r}$. Since $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ is a reduced expression, there exists a $j < j' \leq r-1$ such that $\langle\alpha_{i_r},\alpha_{i_{j'}}\rangle \leq -1$ and $\langle\alpha_{i_r},\alpha_{i_k}\rangle = 0$ for every k such that $j' < k < r$. By Corollary 4.5, we have the following isomorphism of B-modules:

$$H^0(s_{i_{j'+1}}\cdots s_{i_r},\alpha_{i_r}) \simeq H^0(s_{i_r},\alpha_{i_r}) \simeq sl_{2,\alpha_{i_r}}.$$

By *SES*, we have $H^0(s_{i_{j'}}\cdots s_{i_r},\alpha_{i_r}) \simeq H^0(s_{i_{j'}}, H^0(s_{i_{j'+1}}\cdots s_{i_r},\alpha_{i_r}))$ as B-modules.

Then,

$$H^0(s_{i_{j'}}\cdots s_{i_r},\alpha_{i_r}) \simeq H^0(s_{i_{j'}}, H^0(s_{i_r},\alpha_{i_r})) \simeq \mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}} \oplus (\bigoplus_{m=1}^{-\langle\alpha_{i_r},\alpha_{i_{j'}}\rangle} \mathbb{C}_{-\alpha_{i_r}-m\alpha_{i_{j'}}}).$$

Since $\langle\alpha_{i_r},\alpha_{i_k}\rangle \leq 0$ for every $j+1 \leq k \leq j'-1$, we conclude that the indecomposable $B_{\alpha_{i_r}}$-summand $\mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}}$ is in the image of the evaluation map

$$ev : H^0(s_{i_{j+1}}\cdots s_{i_{j'-1}}, H^0(s_{i_{j'}}\cdots s_{i_r},\alpha_{i_r})) \longrightarrow H^0(s_{i_{j'}}\cdots s_{i_r},\alpha_{i_r}).$$

Since $H^0(s_{i_{j+1}}\cdots s_{i_r},\alpha_{i_r}) \simeq H^0(s_{i_{j+1}}\cdots s_{i_{j'-1}}, H^0(s_{i_{j'}}\cdots s_{i_r},\alpha_{i_r}))$, the indecomposable $B_{\alpha_{i_r}}$-module $\mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}}$ is a direct summand of $H^0(s_{i_{j+1}}\cdots s_{i_r},\alpha_{i_r})$. By similar arguments as in the proof of Lemma 5.2 and using Lemma 4.4, we have $H^0(s_{i_j}, \mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}}) = 0$.

Now, let $u_1 = s_{i_1}\cdots s_{i_{j-1}}$ and $u_2 = s_{i_j}\cdots s_{i_r}$. From the above arguments, we see that $H^0(u_2,\alpha_{i_r})_\mu = 0$ unless $\mu < -\alpha_{i_r}$ and $\mu \in R$. By Lemma 5.1, if $H^0(u_1, H^0(u_2,\alpha_{i_r}))_\mu \neq 0$ then $\mu < -\alpha_{i_r}$ and $\mu \in R$. Hence, the zero weight space of $H^0(w,\alpha_{i_r})$ is zero.

Proof of (2): Proof is similar to the proof of (1), for the completeness we will give the proof.
If $\langle \alpha_{i_j}, \alpha_{i_r} \rangle = 0$ for every $1 \leq j \leq r-1$, then by Corollary 4.5, we have $H^0(w, \alpha_{i_r}) = sl_{2,\alpha_{i_r}}$. Hence, (2) holds in this case.

Otherwise, there exists $1 \leq j \leq r-1$ such that $\langle \alpha_{i_j}, \alpha_{i_r} \rangle \neq 0$. Let $1 \leq k \leq r-1$ be the largest integer such that $\langle \alpha_{i_k}, \alpha_{i_r} \rangle \neq 0$. Then by *SES* and Corollary 4.5, we have

$$H^0(w, \alpha_{i_r}) \simeq H^0(s_{i_1}s_{i_2}\cdots s_{i_k}, H^0(s_{i_{k+1}}\cdots s_{i_r}, \alpha_{i_r})) \simeq H^0(s_{i_1}s_{i_2}\cdots s_{i_k}, sl_{2,\alpha_{i_r}}).$$

Since $\langle \alpha_{i_r}, \alpha_{i_k} \rangle \leq -1$, we have

$$H^0(w, \alpha_{i_r}) \simeq H^0(s_{i_1}s_{i_2}\cdots s_{i_{k-1}}, \mathbb{C}.h_{\alpha_{i_r}} \oplus \bigoplus_{m=0}^{-\langle \alpha_{i_k}, \alpha_{i_r} \rangle} \mathbb{C}_{-\alpha_{i_r} - m\alpha_{i_k}}).$$

Since $\alpha_{i_j} \neq \alpha_{i_r}$ for all $1 \leq j < r-1$, we see that $\mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}}$ is an indecomposable $B_{\alpha_{i_r}}$-submodule of $H^0(w, \alpha_{i_r})$. Further, $H^0(w, \alpha_{i_r})_0 = \mathbb{C}.h_{\alpha_{i_r}}$ and so $dim(H^0(w, \alpha_{i_r})_0) = 1$. This completes the proof of the lemma. □

Now onwards we denote by $M_{\geq 0}$ the semi subgroup of $Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$ generated by the set S of all simple roots .

Lemma 5.4. *Let $w \in W$. Let $\mu \in M_{\geq 0} \setminus \{0\}$ and let $\alpha \in S$. Then, we have*

1. *If $H^0(w, \alpha)_\alpha \neq 0$, then $dim(H^0(w, \alpha)_\alpha) = 1$.*
2. *$H^0(w, \alpha)_\mu \neq 0$ if and only if $\mu = \alpha$ and the evaluation map $ev : H^0(w, \alpha) \longrightarrow \mathbb{C}_\alpha$ is surjective.*

Proof. Proof of (1): Let $w_1 \in W$ be an element of minimal length such that $w_1(\alpha)$ is a dominant weight. Note that if $l(w_1) = 0$, then α is dominant. In particular, G is of rank 1 and $w \in \{id, s_\alpha\}$. Hence $dim(H^0(w, \alpha)_\alpha) = 1$. Otherwise, there exists a $\gamma \in S$ such that $l(w_1 s_\gamma) = l(w_1) - 1$ and $\langle \alpha, \gamma \rangle < 0$. Hence by Lemma 4.1, $\mathbb{C}_\alpha$ is a B-submodule of $H^0(s_\gamma, s_\gamma(\alpha))$. Then $H^0(w, \alpha)$ is a B-submodule of $H^0(w, H^0(s_\gamma, s_\gamma(\alpha)))$. Since $H^0(w, \alpha)_\alpha \neq 0$, by [BKS04, p.110, Theorem 3.3] (see also [Dab93] and [Pol89]) we have $l(ws_\gamma) = l(w) + 1$ (Note that since $\langle \alpha, \gamma \rangle < 0$, the regularity of λ as in [BKS04, p.110, Theorem 3.3] does not play a role). By Lemma 4.2, we have

$$H^0(ws_\gamma, s_\gamma(\alpha)) = H^0(w, H^0(s_\gamma, s_\gamma(\alpha))).$$

Hence $H^0(w, \alpha)$ is a B-submodule of $H^0(ws_\gamma, s_\gamma(\alpha))$. By induction on $l(w_1)$, $H^0(w, \alpha)$ is a B-submodule of $H^0(ww_1^{-1}, w_1(\alpha))$. Since $w_1(\alpha)$ is dominant, $H^0(ww_1^{-1}, w_1(\alpha))$ is a quotient of the B-module $H^0(w_0, w_1(\alpha))$. Further, since the multiplicity of the weight α in $H^0(w_0, w_1(\alpha))$ is 1, the multiplicity of the weight α in $H^0(ww_1^{-1}, w_1(\alpha))$ is at most 1. Hence, we conclude that $dim(H^0(w, \alpha)_\alpha) = 1$.

Proof of (2):

Assume that $H^0(w, \alpha)_\mu \neq 0$. If $l(w) = 0$, there is nothing to prove. Assume $l(w) > 0$. Therefore, we can choose a $\gamma \in S$ such that $l(s_\gamma w) = l(w) - 1$. Let $u = s_\gamma w$. By *SES*, we have $H^0(w, \alpha) = H^0(s_\gamma, H^0(u, \alpha))$.

Since $H^0(w, \alpha)_\mu \neq 0$, there exists an indecomposable B_γ-summand V of $H^0(u, \alpha)$ such that $H^0(s_\gamma, V)_\mu \neq 0$. Let μ' be the highest weight of V. By Lemma 4.4, we have $V = V' \otimes \mathbb{C}_\lambda$ for some character λ of $\widetilde{B}_\gamma$ and for some irreducible $\widetilde{L}_\gamma$-module V'. Let λ_1 be a highest weight of V'. By similar arguments as in the proof of Lemma 5.1, we have $\lambda_1 + \lambda = \mu'$, and $\mu = \mu' - a\gamma$ where $0 \leq a \leq \langle \mu', \gamma \rangle$. Therefore, $\mu' = \mu + a\gamma$ for some $a \in \mathbb{Z}_{\geq 0}$ and $H^0(u, \alpha)_{\mu'} \neq 0$. By induction on $l(w)$, $\mu' = \alpha$ and the evaluation map $ev : H^0(u, \alpha) \longrightarrow \mathbb{C}_\alpha$ is surjective. By (1), we see that $ev : H^0(u, \alpha)_\alpha \longrightarrow \mathbb{C}_\alpha$ is an isomorphism. Since $\mu \in M_{\geq 0} \setminus \{0\}$ and $\mu' = \alpha$, we have $a = 0$ and hence $\mu = \alpha$. By the above arguments, the restriction of the evaluation map $ev : H^0(w, \alpha)_\alpha \longrightarrow H^0(u, \alpha)_\alpha$ is surjective. Hence, the evaluation map $ev : H^0(w, \alpha) \longrightarrow \mathbb{C}_\alpha$ is surjective.

The other implication is straight forward. □

Corollary 5.5. *Let $w \in W$ and fix a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$. Let $\mu \in M_{\geq 0} \setminus \{0\}$. Then, we have*

1. *$H^0(w, \alpha_{i_r})_\mu \neq 0$ if and only if $\mu = \alpha_{i_r}$ and $\langle \alpha_{i_r}, \alpha_{i_k} \rangle = 0$ for $k = 1, 2, \ldots, r-1$.*
2. *In such a case, the evaluation map $ev : H^0(w, \alpha_{i_r}) \longrightarrow sl_{2,\alpha_{i_r}}$ is an isomorphism.*

Proof. Proof of (1): Assume that $H^0(w, \alpha_{i_r})_\mu \neq 0$. By Lemma 5.4, we have $\mu = \alpha_{i_r}$ and the evaluation map $ev : H^0(w, \alpha_{i_r}) \longrightarrow \mathbb{C}_{\alpha_{i_r}}$ is surjective. We now prove that $\langle \alpha_{i_r}, \alpha_{i_k} \rangle = 0$ for $k = 1, 2, \ldots, r-1$. Let $u = s_{i_2} s_{i_3} \cdots s_{i_r}$. Then, we have $l(u) = l(w) - 1$. Since the evaluation map $ev : H^0(w, \alpha_{i_r}) = H^0(s_{i_1}, H^0(u, \alpha_{i_r})) \longrightarrow \mathbb{C}_{\alpha_{i_r}}$ is non zero, the evaluation map $ev : H^0(u, \alpha_{i_r}) \longrightarrow \mathbb{C}_{\alpha_{i_r}}$ is non zero, because this evaluation map is the composition of the evaluation maps $H^0(s_{i_1}, H^0(u, \alpha)) \longrightarrow H^0(u, \alpha)$ and $H^0(u, \alpha) \longrightarrow \mathbb{C}_\alpha$. By induction on $l(w)$, $\langle \alpha_{i_r}, \alpha_{i_k} \rangle = 0$ for $k = 2, \ldots, r-1$. Hence, $w = s_{i_1} s_{i_r} s_{i_2} \cdots s_{i_{r-1}}$ is also a reduced expression for w. In particular, $\alpha_{i_1} \neq \alpha_{i_r}$ and hence $\langle \alpha_{i_1}, \alpha_{i_r} \rangle \leq 0$. By Corollary 4.5, we have $H^0(w, \alpha_{i_r}) = H^0(s_{i_1} s_{i_r}, \alpha_{i_r})$. Note that if $\langle \alpha_{i_1}, \alpha_{i_r} \rangle \leq -1$, by Lemma 4.3 we have $H^0(w, \alpha_{i_r})_{\alpha_{i_r}} = 0$, which is a contradiction. Thus, we have $\langle \alpha_{i_1}, \alpha_{i_r} \rangle = 0$. Hence $\langle \alpha_{i_r}, \alpha_{i_k} \rangle = 0$ for $k = 1, 2, \ldots, r-1$.

The other implication follows from Corollary 4.5.

Assertion (2) follows from the fact that $H^0(s_{i_r}, \alpha_{i_r})$ is the 3-dimensional cyclic B-submodule generated by a weight vector of weight α_{i_r}. ☐

Let $\mathfrak{p}$ be a B-submodule of $\mathfrak{g}$ containing $\mathfrak{b}$.

Lemma 5.6. *Let $w \in W$ and let $\mu \in M_{\geq 0} \setminus \{0\}$. If $H^0(w, \mathfrak{g}/\mathfrak{p})_\mu \neq 0$, then $\mu \in R^+$.*

Proof. If $l(w) = 0$, there is nothing to prove. Assume that $l(w) > 0$. Then, we can choose $\gamma \in S$ such that $l(s_\gamma w) = l(w) - 1$. Let $u = s_\gamma w$. By *SES*, we have $H^0(w, \mathfrak{g}/\mathfrak{p}) = H^0(s_\gamma, H^0(u, \mathfrak{g}/\mathfrak{p}))$.

Since $H^0(w, \mathfrak{g}/\mathfrak{p})_\mu \neq 0$, there exists an indecomposable B_γ-summand V of $H^0(u, \mathfrak{g}/\mathfrak{p})$ such that $H^0(s_\gamma, V)_\mu \neq 0$. Let μ' be the highest weight of V. By the same arguments as in the proof of Lemma 5.4, we have $\mu = \mu' - a\gamma$ where $0 \leq a \leq \langle \mu', \gamma \rangle$.

Since $l(u) = l(w) - 1$ and $V_{\mu'} \neq 0$, by induction on $l(w)$, $\mu' \in R^+$. Hence $\mu' - j\gamma \in R \cup \{0\}$ for every $0 \leq j \leq \langle \mu', \gamma \rangle$ (see [Hum72, p.45, Section 9.4]). Since $\mu \in M_{\geq 0} \setminus \{0\}$, we have $\mu \in R^+$. ☐

Proposition 5.7. *Let $w \in W$ and fix a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$. Fix $1 \leq j \leq r-1$. If $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for every $1 \leq k < j$, then the natural map $H^0(w, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}}$ is surjective.*

Proof. If $l(w) = 0$, there is nothing to prove. Assume $l(w) > 0$ and let $u = s_{i_1} w$. Then, we have $l(u) = l(w) - 1$. By *SES*, we have the evaluation map

$$ev : H^0(w, \mathfrak{g}/\mathfrak{p}) = H^0(s_{i_1}, H^0(u, \mathfrak{g}/\mathfrak{p})) \longrightarrow H^0(u, \mathfrak{g}/\mathfrak{p}).$$

We denote the restriction of the evaluation map ev to $H^0(w, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}}$ by ev_1.

First we will prove that ev_1 is an isomorphism.

Let v be a non zero vector in $H^0(w, \mathfrak{g}/\mathfrak{p})$ of weight α_{i_j}. Let $H^0(u, \mathfrak{g}/\mathfrak{p})) \simeq \bigoplus_{i=1}^m V_i$ be a decomposition as a sum of indecomposable $B_{\alpha_{i_1}}$-submodules. Since $v \in H^0(s_{i_1}, \bigoplus_{i=1}^m V_i) = \bigoplus_{i=1}^m H^0(s_{i_1}, V_i)$, $v = \sum_{i=1}^m v_i$ where $v_i \in H^0(s_{i_1}, V_i)$ $(1 \leq i \leq m)$, it follows that the weight of v_i is same as the weight of v. Hence, without loss of generality, we may assume that there exists an indecomposable $B_{\alpha_{i_1}}$-summand V of $H^0(u, \mathfrak{g}/\mathfrak{p})$ such that $v \in H^0(s_{i_1}, V)_{\alpha_{i_j}}$. Let μ be the highest weight of V. By the arguments as in the proof of Lemma 5.4, $\mu = \alpha_{i_j} + a\alpha_{i_1}$ for some $a \in \mathbb{Z}_{\geq 0}$. Since $H^0(u, \mathfrak{g}/\mathfrak{p})_\mu \neq 0$, by Lemma 5.6 we see that μ is a positive root. Since either $j = 1$, or $\langle \alpha_{i_j}, \alpha_{i_1} \rangle = 0$, we have $a = 0$. Hence $V = \mathbb{C}.v$. Thus, the map $ev_1 : H^0(w, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}} \longrightarrow H^0(u, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}}$ is injective. To prove ev_1 is surjective, let v' be a non zero vector in $H^0(u, \mathfrak{g}/\mathfrak{p})$ of weight α_{i_j}. By similar arguments, we may assume that there exists an indecomposable $B_{\alpha_{i_1}}$-summand V' of $H^0(u, \mathfrak{g}/\mathfrak{p})$ containing v'. Let μ' be the highest weight of V'. Then, by the arguments as in the proof of Lemma 5.4, $\mu' = \alpha_{i_j} + a\alpha_{i_1}$ for some $a \in \mathbb{Z}_{\geq 0}$. By the similar arguments as above, we see that $V' = \mathbb{C}.v'$. Hence, we conclude that v' is in the image of ev_1.

In particular, the restriction $ev_2 : H^0(w, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} \longrightarrow H^0(u, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}}$ of the evaluation map $H^0(w, \mathfrak{g}/\mathfrak{b}) \longrightarrow H^0(u, \mathfrak{g}/\mathfrak{b})$ is an isomorphism.

Now, consider the following commutative diagram of T-modules:

$$\begin{array}{ccc} H^0(w, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} & \xrightarrow{f} & H^0(w, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}} \\ {\scriptstyle ev_2}\downarrow \wr & & {\scriptstyle ev_1}\downarrow \wr \\ H^0(u, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} & \xrightarrow{g} & H^0(u, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}} \end{array}$$

By the induction on $l(w)$, $g : H^0(u, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} \longrightarrow H^0(u, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}}$ is surjective. By the commutativity of the above diagram, it follows that the natural map

$$f : H^0(w, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p})_{\alpha_{i_j}}$$

is surjective. This completes the proof. □

Corollary 5.8. *Let $w \in W$ and fix a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$. Fix an integer $j \in \{1, \ldots, r-1\}$ such that for all $1 \leq k < j$, $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$. Then, $H^1(w, \alpha_{i_r})_{\alpha_{i_j}} = 0$.*

Proof. Let $\alpha = \alpha_{i_r}$. Now look at following short exact sequence of B-modules:

$$0 \longrightarrow \mathfrak{g}_\alpha \longrightarrow \mathfrak{g}/\mathfrak{b} \longrightarrow \mathfrak{g}/\mathfrak{p}_\alpha \longrightarrow 0$$

Note that by Theorem 5.16, $H^1(w, \mathfrak{g}/\mathfrak{b}) = 0$. Applying $H^0(w, -)$ to the above short exact sequence of B-modules and taking the α_{i_j} weight spaces, we have the exact sequence of T-modules:

$$0 \longrightarrow H^0(w, \alpha)_{\alpha_{i_j}} \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{b})_{\alpha_{i_j}} \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p}_\alpha)_{\alpha_{i_j}} \longrightarrow H^1(w, \alpha)_{\alpha_{i_j}} \longrightarrow 0.$$

By Proposition 5.7, we conclude that $H^1(w, \alpha)_{\alpha_{i_j}} = 0$. This completes the proof. □

5.2 Action of the minimal Parabolic subgroup $P_{\alpha_{i_1}}$ on $Z(w, \underline{i})$

Recall that ϕ_w denotes the birational morphism $Z(w, \underline{i}) \longrightarrow X(w)$. As in Section 2, the composition of inclusion $X(w)$ in G/B with ϕ_w will also be denoted by ϕ_w. Further, we denote the tangent bundle of $Z(w,\underline{i})$ by $T_{(w,\underline{i})}$, where $\underline{i}=(i_1,i_2,\ldots,i_r)$. By using the differential map, we see that $T_{(w,\underline{i})}$ is a subsheaf of $\phi_w^*(T_{G/B})$. Hence $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ is a B-submodule of $H^0(Z(w,\underline{i}),\phi_w^*(T_{G/B}))$.

Since the tangent bundle of G/B is the homogeneous vector bundle associated to the representation $\mathfrak{g}/\mathfrak{b}$ of B, we have

$$H^0(Z(w,\underline{i}),\phi_w^*(T_{G/B}))=H^0(w,\mathfrak{g}/\mathfrak{b}).$$

Therefore, $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ is a B-submodule of $H^0(w,\mathfrak{g}/\mathfrak{b})$.

Denote by $\mathfrak{p}_{\alpha_{i_1}}$, the Lie algebra of the minimal parabolic subgroup $P_{\alpha_{i_1}}$ of G containing B. Note that $\mathfrak{b}$ is contained in $\mathfrak{p}_{\alpha_{i_1}}$.

Lemma 5.9. *Let $w=s_{i_1}\cdots s_{i_r}$ be a reduced expression $\underline{i}$ for w. Then,*

1. *There is a non zero homomorphism $f_w:\mathfrak{p}_{\alpha_{i_1}}\longrightarrow H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ of B-modules (which is also a homomorphism of Lie algebras).*
2. *If $w=w_0$, the homomorphism $f_{w_0}:\mathfrak{p}_{\alpha_{i_1}}\longrightarrow H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})})$ in (1) is injective.*

Proof. Proof of (1): Consider the action of $P_{\alpha_{i_1}}$ on $Z(w,\underline{i})$ induced by the following left action of $P_{\alpha_{i_1}}$ on $P_{\alpha_{i_1}}\times P_{\alpha_{i_2}}\times\cdots\times P_{\alpha_{i_r}}$:
Let $p\in P_{\alpha_{i_1}}$ and $x=(p_1,p_2,\ldots,p_r)\in P_{\alpha_{i_1}}\times P_{\alpha_{i_2}}\times\cdots\times P_{\alpha_{i_r}}$ then $p.x:=(pp_1,p_2,\ldots,p_r)$.

Clearly, this action is non trivial. Hence, there is a non trivial homomorphism

$$\psi_w:P_{\alpha_{i_1}}\longrightarrow Aut^0(Z(w,\underline{i}))$$

of algebraic groups. Consider the action of B on $P_{\alpha_{i_1}}$ by conjugation and the action of B on $Aut^0(Z(w,\underline{i}))$ via ψ_w. Note that ψ_w is B-equivariant.

By [MO67, p. 17, Theorem 3.7], $Aut^0(Z(w,\underline{i}))$ is an algebraic group and by [MO67, p.13, Lemma 3.4], we have

$$Lie(Aut^0(Z(w,\underline{i})))=H^0(Z(w,\underline{i}),T_{(w,\underline{i})}).$$

Then, the induced homomorphism

$$f_w:\mathfrak{p}_{\alpha_{i_1}}\longrightarrow H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$$

of B-modules (homomorphism of Lie algebras) is non zero.

Proof of (2): Since $f_w:\mathfrak{p}_{\alpha_{i_1}}\longrightarrow H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ is a non zero homomorphism of B-modules (homomorphism of Lie algebras), $f_w(\mathfrak{p}_{\alpha_{i_1}})$ contains a B-stable line L. Let μ be the character of B such that $b.v = \mu(b).v$ for all $b \in B$ and for all $v \in L$. That is, L is the one-dimensional space generated by a lowest weight vector of weight μ.

Since $w = w_0$, $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a B-submodule of $H^0(G/B, T_{G/B})$. By Bott's theorem [Bot57, Theorem VII] we have $H^0(G/B, T_{G/B}) = \mathfrak{g}$. Hence $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a B-submodule of $\mathfrak{g}$. Since there is a unique B-stable one-dimensional subspace L of $\mathfrak{g}$ and the character of B is $-\alpha_0$, we conclude that $\mu = -\alpha_0$ and $L = \mathfrak{g}_{-\alpha_0} \subset f_{w_0}(\mathfrak{p}_{\alpha_{i_1}})$. By the similar arguments, the unique B-stable one-dimensional subspace in $\mathfrak{p}_{\alpha_{i_1}}$ is $\mathfrak{g}_{-\alpha_0}$. Hence f_{w_0} is injective (otherwise $Ker(f_{w_0}) \neq 0$ and hence the unique B-stable line $\mathfrak{g}_{-\alpha_0}$ is a subspace of $Ker(f_{w_0})$, which is a contradiction). □

Corollary 5.10.

1. *$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a Lie subalgebra of $\mathfrak{g}$.*
2. *Any Borel subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is isomorphic to $\mathfrak{b}$.*
3. *Any maximal toral subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is isomorphic to $\mathfrak{h}$.*

Proof. Proof of (1): By Lemma 5.9(2), $\mathfrak{b}$ is a Lie subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$. Since $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a B-submodule of $\mathfrak{g}$, for any $Y \in H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ and for any $X \in \mathfrak{b}$ the Lie bracket $[X, Y]$ in $\mathfrak{g}$ is same as the Lie bracket in $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$. It remains to prove that for every $\alpha, \beta \in R^+$ such that α, β are weights of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$, the Lie bracket $[x_\beta, x_\alpha]$ in $\mathfrak{g}$ is same as the Lie bracket in $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$.

Note that the Lie subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ generated by $\mathfrak{g}_\beta \cap H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ for $\beta \in R^+$ is same as the Lie subalgebra generated by $\mathfrak{g}_\alpha \cap H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ for $\alpha \in S$. Hence it is enough to prove that for every $\beta \in R^+$ and $\alpha \in S$ such that α, β are weights of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$, the Lie bracket $[x_\beta, x_\alpha]$ in $\mathfrak{g}$ is same as the Lie bracket in $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$.

Let $[-,-]'$ be the Lie bracket in $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$. For $\beta \in R^+, \alpha \in S$, by Jacobi identity we have

$$[x_{-\beta}, [x_\beta, x_\alpha]']' = [[x_{-\beta}, x_\beta]', x_\alpha]' + [x_\beta, [x_{-\beta}, x_\alpha]']'.$$

Since $x_{-\beta} \in \mathfrak{b}$ and $\mathfrak{b}$ is a Lie subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ and $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a B-submodule of $\mathfrak{g}$, we have

$$[x_{-\beta}, x_\beta]' = [x_{-\beta}, x_\beta] \text{ and } [x_{-\beta}, x_\alpha]' = [x_{-\beta}, x_\alpha].$$

Hence, we have

$$[x_{-\beta}, [x_\beta, x_\alpha]']' = [[x_{-\beta}, x_\beta], x_\alpha]' + [x_\beta, [x_{-\beta}, x_\alpha]]'. \tag{5.1}$$

Note that $[x_{-\beta}, x_\beta], [x_{-\beta}, x_\alpha] \in \mathfrak{b}$. Therefore, by (5.1) and Jacobi identity we have

$$[x_{-\beta}, [x_\beta, x_\alpha]']' = [[x_{-\beta}, x_\beta], x_\alpha] + [x_\beta, [x_{-\beta}, x_\alpha]] = [x_{-\beta}, [x_\beta, x_\alpha]].$$

Since $x_{-\beta} \in \mathfrak{b}$, we have $[x_{-\beta}, [x_\beta, x_\alpha]']' = [x_{-\beta}, [x_\beta, x_\alpha]']$. Hence, we have

$$[x_{-\beta}, [x_\beta, x_\alpha]'] = [x_{-\beta}, [x_\beta, x_\alpha]]. \tag{5.2}$$

If $[x_\beta, x_\alpha] = 0$, then $\alpha + \beta \notin R$. In particular, $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})_{\alpha+\beta} = 0$. Then, we have $[x_\beta, x_\alpha]' = 0$.

If $[x_\beta, x_\alpha] \neq 0$, then $\alpha + \beta \in R$. Further, we have

$$[x_{-\beta}, [x_\beta, x_\alpha]] = [x_\beta, [x_{-\beta}, x_\alpha]] - h_\beta \cdot x_\alpha.$$

If $[x_{-\beta}, x_\alpha] = 0$ and $h_\beta \cdot x_\alpha = 0$, then α, β are orthogonal and $\beta - \alpha \notin R$. Hence, we have $\alpha + \beta \notin R$. This contradicts the assumption. Hence, we have $[x_\beta, x_\alpha]' = c_1 x_{\alpha+\beta}$ and $[x_\beta, x_\alpha] = c_2 x_{\alpha+\beta}$, with $c_2 \neq 0$. Therefore, by (5.2) it follows that $c_1 = c_2$ and $[x_\beta, x_\alpha]' = [x_\beta, x_\alpha]$. Hence $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a Lie subalgebra of $\mathfrak{g}$.

Proof of (2): By (1), $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a Lie subalgebra of $\mathfrak{g}$. Now we claim that $\mathfrak{b}$ is a Borel subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$. Otherwise, there exists a Borel subalgebra $\mathfrak{b}'$ of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ properly containing $\mathfrak{b}$. Since $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a Lie subalgebra of $\mathfrak{g}$, we see that $\mathfrak{g}_\alpha \subset \mathfrak{b}'$ for some simple root α. Since $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{b}$ is a Lie subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$, the simple Lie algebra $sl_{2,\alpha}$ is a Lie subalgebra of $\mathfrak{b}'$, which is a contradiction to the solvability of $\mathfrak{b}'$. Hence $\mathfrak{b}$ is a Borel subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$. Since any two Borel subalgebras of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ are conjugate (see [Hum72, p.84, Theorem 16.4]), we conclude (2).

Proof of (3): Since any two maximal toral subalgebras of $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ are conjugate (see [Hum72, p.84, Corollary 16.4]), the proof follows from (2). □

Let $w \in W$, let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression $\underline{i}$ of w. Fix a reduced expression $w_0 = s_{j_1} s_{j_2} \cdots s_{j_r} s_{j_{r+1}} \cdots s_{j_N}$ of w_0 such that $\underline{j} = (j_1, j_2, \ldots j_N)$ and $\underline{i} = (j_1, j_2, \ldots j_r)$. Let $v = s_{j_{r+1}} s_{j_{r+2}} \cdots s_{j_N}$ and $\underline{j}' = (j_{r+1}, \ldots, j_N)$.

Since the $Z(v, \underline{j}')$-fibration $Z(w_0, \underline{j}) \longrightarrow Z(w, \underline{i})$ is $P_{\alpha_{i_1}}$ equivariant, it follows that

$$H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$$

is a homomorphism of $P_{\alpha_{i_1}}$-modules. Hence, it is a homomorphism of $\mathfrak{p}_{\alpha_{i_1}}$-modules. Thus, the restriction of this map to $\mathfrak{p}_{\alpha_{i_1}}$ is the same as the map induced by the action of $P_{\alpha_{i_1}}$ on $Z(w, \underline{i})$.

Note that since $f_{w_0} : \mathfrak{p}_{\alpha_{i_1}} \longrightarrow H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})})$ is injective (see Lemma 5.9(2)), we identify $\mathfrak{p}_{\alpha_{i_1}}$ as a Lie subalgebra of $H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})})$.

Hence, we have the following commutative diagram of $P_{\alpha_{i_1}}$-modules:

$$\begin{array}{ccc} \mathfrak{p}_{\alpha_{i_1}} & \hookrightarrow & H^0(Z(w_0, \underline{j}), T_{(w_0, \underline{j})}) \\ & \searrow^{f_w} & \downarrow \\ & & H^0(Z(w, \underline{i}), T_{(w, \underline{i})}) \end{array}$$

Further, the maps in the above diagram are homomorphisms of Lie algebras.

For simplicity of notation, we denote both the natural map

$$H^0(Z(w_0, \underline{j}), T_{(w_0, \underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w, \underline{i})})$$

and its restriction to $\mathfrak{p}_{\alpha_{i_1}}$ by f_w.

Let $d(w)$ be the number of distinct i_j's in $\underline{i} = (i_1, i_2, \ldots, i_r)$ (i.e, the number of distinct simple reflections s_{i_j}'s appearing in the reduced expression $\underline{i}$ of w). Let $\leq$ be the Bruhat-Chevalley ordering on W. Note that $d(w)$ is equal to the number of distinct Schubert curves in $X(w)$. That is, $d(w)$ is equal to the number of distinct $j \in \{1, 2, \ldots, n\}$ such that $s_j \leq w$. In particular, it is independent of the choice of the reduced expression $\underline{i}$ of w. Further, we also note that $d(w_0) = n$.

Now, we prove the following lemma:

Lemma 5.11.

1. *The dimension of the zero weight space* $H^0(Z(w, \underline{i}), T_{(w, \underline{i})})_0$ *is at most* $d(w)$.
2. *In particular,* $dim(H^0(Z(w, \underline{i}), T_{(w, \underline{i})})_0) \leq rank(G)$.

Proof. Consider the following short exact sequence of B-modules:

$$0 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{b} \longrightarrow 0$$

By applying $H^0(w, -)$ to the above short exact sequence, we have the following exact sequence of B-modules:

$$0 \longrightarrow H^0(w, \mathfrak{b}) \longrightarrow H^0(w, \mathfrak{g}) \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{b}) \longrightarrow H^1(w, \mathfrak{b}) \longrightarrow 0$$

(Note that $H^1(w, \mathfrak{g}) = 0$ (see [Kan13, Lemma 2.5(2)]))

By Lemma 5.2, we have $H^1(w, \mathfrak{b})_0 = 0$. Since $H^0(w, \mathfrak{g}) = \mathfrak{g}$, by taking the zero weight space to the above exact sequence we have the following short exact sequence of T-modules;

$$0 \longrightarrow H^0(w, \mathfrak{b})_0 \longrightarrow \mathfrak{h} \xrightarrow{\phi} H^0(w, \mathfrak{g}/\mathfrak{b})_0 \longrightarrow 0.$$

Claim: $dim(H^0(w, \mathfrak{b})_0) = rank(G) - d(w)$.

We use the similar arguments as in the proof of Lemma 5.2 and Lemma 5.3 to prove the claim.

Let $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression $\underline{i}$ of w. Since S is a basis for the complex vector space $\mathfrak{h}$, for every $1 \leq j \leq n$ there exists a $h(\alpha_j) \in \mathfrak{h}$ such that $\alpha_i(h(\alpha_j)) = \delta_{i,j}$ for $1 \leq i \leq n$. First note that for every $i \neq j$, the one-dimensional subspace $\mathbb{C}.h(\alpha_j)$ of $\mathfrak{h}$ is an indecomposable B_{α_i}-direct summand of $\mathfrak{b}$. Therefore, the image of the evaluation map $ev : H^0(w, \mathfrak{b}) \longrightarrow \mathfrak{b}$ contains $h(\alpha_j)$ for every $1 \leq j \leq n$ such that $s_j \not\leq w$. Let $1 \leq k \leq n$ such that $s_k \leq w$. Let $1 \leq j_0 \leq r$ be the largest integer such that $i_{j_0} = k$, let $u = s_{i_{j_0+1}}\cdots s_{i_r}$. Note that since $\alpha_{i_j}(h(\alpha_k)) = 0$ for $j_0 + 1 \leq j \leq r$, $\mathbb{C}.h(\alpha_k)$ is contained in the image of the evaluation map $ev : H^0(u, \mathfrak{b}) \longrightarrow \mathfrak{b}$. Therefore, $\mathbb{C}.h(\alpha_k) \oplus \mathbb{C}_{-\alpha_k}$ is an indecomposable B_{α_k}-direct summand of $H^0(u, \mathfrak{b})$ (see [Kan13, Lemma 3.3]).

Further, by Lemma 4.4

$$\mathbb{C}.h(\alpha_k) \oplus \mathbb{C}_{-\alpha_k} = V \otimes \mathbb{C}_{-\omega_k},$$

where V is the standard 2- dimensional representation of $\widetilde{L}_{\alpha_k}$. Therefore, by Lemma 4.3 and Lemma 4.2, $H^0(s_{i_{j_0}}, \mathbb{C}.h(\alpha_k) \oplus \mathbb{C}_{-\alpha_k}) = 0$.

Let $v = s_{i_{j_0}}u$. By *SES*, we conclude that $H^0(v, \mathfrak{b})_0 = \bigoplus_{\{i : s_i \not\leq v\}} \mathbb{C}.h(\alpha_i)$. In view of [Kan13, Lemma 2.6], $H^0(w, \mathfrak{b})_0 = \bigoplus_{\{i : s_i \not\leq w\}} \mathbb{C}.h(\alpha_i)$.

Then by the above claim and the short exact sequence, we have

$$dim(H^0(w, \mathfrak{g}/\mathfrak{b})_0) = d(w).$$

Since $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is a B-submodule of $H^0(w, \mathfrak{g}/\mathfrak{b})$, we have

$$dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) \leq d(w).$$

□

5.3 The B-module of the global sections of the tangent bundle on $Z(w,\underline{i})$

In this section, we study the B-module of the global sections of the tangent bundle on $Z(w,\underline{i})$. In particular, we prove that the dimension of the zero weight space of $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ is equal to $d(w)$, the number of Schubert curves in $X(w)$. We also prove that $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ contains a Lie subalgebra $\mathfrak{b}'$ isomorphic to $\mathfrak{b}$ if and only if $w^{-1}(\alpha_0)<0$.

We use the notation as in the previous Section.

Let $w\in W$ and fix a reduced expression $w=s_{i_1}s_{i_2}\cdots s_{i_r}$. Let $supp(w):=\{j\in\{1,2,\ldots,n\}:s_j\leq w\}$, the support of w. Note that $d(w)=|supp(w)|$.

We have the following proposition:

Proposition 5.12.

1. $\{f_w(h_{\alpha_{i_j}}):j\in supp(w)\}$ *forms a basis of* $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})_0$.
2. *In particular,* $dim(H^0(Z(w,\underline{i}),T_{(w,\underline{i})})_0)=d(w)$.
3. *The image* $f_w(\mathfrak{h})$ *is a maximal toral subalgebra of* $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$.

Proof. If $w=w_0$, then by Lemma 5.9(2), f_{w_0} is injective and by Corollary 5.10,

$$dim(H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})}))_0)=rank(G)=d(w_0).$$

Hence, $\{f_{w_0}(h_{\alpha_{i_j}}):j\in supp(w_0)\}$ forms a basis of $H^0(Z(w_0,\underline{i}),T_{(w_0,\underline{i})})_0$.

Otherwise, choose a reduced expression $w_0=s_{j_1}s_{j_2}\ldots s_{j_N}$ of w_0 such that $(j_1,j_2,\cdots,j_r)=\underline{i}$. Let $v=s_{j_1}s_{j_2}\cdots s_{j_{r+1}}$ and $\underline{i}'=(j_1,\ldots j_r,j_{r+1})$. Note that $l(v)=l(w)+1$. By descending induction on $l(w)$, $\{f_v(h_{\alpha_{i_j}}):j\in supp(v)\}$ forms a basis of $H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')})_0$ and

$$dim(H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')}))_0)=d(v).$$

Note that by Lemma 5.11, $dim(H^0(Z(w,\underline{i}),T_{(w,\underline{i})})_0)\leq d(w)$. By using *LES* and Lemma 4.6, we have the following exact sequence of B-modules:

$$0\longrightarrow H^0(v,\alpha_{i_{r+1}})\longrightarrow H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')})\longrightarrow H^0(Z(w,\underline{i}),T_{(w,\underline{i})})\longrightarrow H^1(v,\alpha_{i_{r+1}})\longrightarrow H^1(Z(v,\underline{i}'),T_{(v,\underline{i}')})\longrightarrow H^1(Z(w,\underline{i}),T_{(w,\underline{i})})\longrightarrow 0.$$

By taking the zero weight spaces, we have the following exact sequence of T-modules:

$$0 \longrightarrow H^0(v, \alpha_{i_{r+1}})_0 \longrightarrow H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})_0 \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0 \longrightarrow H^1(v, \alpha_{i_{r+1}})_0 \cdots .$$

First assume that there exists $1 \leq j \leq r$ such that $\alpha_{i_j} = \alpha_{i_{r+1}}$, so that $d(v) = d(w)$. By Lemma 5.3, we have $H^0(v, \alpha_{i_{r+1}})_0 = 0$. Hence

$$d(v) = dim(H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')}))_0) \leq dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) \leq d(w).$$

Since $d(w) = d(v)$, we have

$$dim(H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})_0) = d(v) = d(w) = dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0).$$

Hence, by the above exact sequence, we conclude that $\{f_w(h_{\alpha_{i_j}}) : j \in supp(w)\}$ forms a basis of $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0$.

Otherwise $d(w) = d(v) - 1$ and by Lemma 5.3(2), we see that $H^0(v, \alpha_{i_{r+1}})_0 = \mathbb{C}.h_{\alpha_{i_{r+1}}}$. By using the above exact sequence, we see that

$$dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) \geq d(v) - 1.$$

Since $dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) \leq d(w)$, we conclude that

$$dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0) = d(w)$$

and hence $\{f_w(h_{\alpha_{i_j}}) : j \in supp(w)\}$ forms a basis of $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0$. This completes the proof of (1) and (2).

Proof of (3):

By Lemma 5.9(2), $f_{w_0} : \mathfrak{p}_{\alpha_{i_1}} \longrightarrow H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})$ is an injective homomorphism of Lie algebras. By Corollary 5.10(1), $H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})$ is a Lie subalgebra of $\mathfrak{g}$. Hence, we have

$$H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})_0 = \mathfrak{h}.$$

Let $u = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$ and $\underline{i}' = (i_1, i_2, \ldots, i_{r-1})$. Note that $l(u) = l(w) - 1$.

Consider the homomorphism $f : H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(u, \underline{i}'), T_{(u,\underline{i}')})$ of Lie algebras induced by the $\mathbb{P}^1$-fibration $f_r : Z(w, \underline{i}) \longrightarrow Z(u, \underline{i}')$ as in Chapter 2, Section 2.2.5. By LES, $Ker(f) = H^0(w, \alpha_{i_r})$.

Note that by Lemma 5.1(1),

$$H^0(w, \alpha_{i_r})_\mu = 0 \ unless \ \mu \leq \alpha_{i_r}. \tag{5.3}$$

Case 1: If $s_{i_r} \leq u$, then by Lemma 5.3(1), $H^0(w, \alpha_{i_r})_0 = 0$. Hence by Corollary 5.5 and Lemma 5.1(2), we conclude that $H^0(w, \alpha_{i_r})_\mu = 0$ unless $\mu \in R^-$. Since for every $\beta \in R^+$, $ad(x_{-\beta})^r = 0$ in $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ for some $r \in \mathbb{N}$ (since for every positive root α, there is a $r \in \mathbb{N}$ such that $\alpha + k\beta \notin R$ for all $k \geq r$), we conclude that every element of $H^0(w, \alpha_{i_r}) \subseteq H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is nilpotent.

Case 2: Assume that $s_{i_r} \not\leq u$.

Sub case (a): If $\langle \alpha_{i_j}, \alpha_{i_r} \rangle \neq 0$ for some $1 \leq j \leq r-1$, then by Corollary 5.5(1), we have $H^0(w, \alpha_{i_r})_{\alpha_{i_r}} = 0$. Hence by (6.1), we have $H^0(w, \alpha_{i_r})_\mu = 0$ unless $\mu \leq 0$. Therefore, again by Lemma 5.1(2) $H^0(w, \alpha_{i_r})_\mu = 0$ unless $\mu \in R^- \cup \{0\}$. Further, by Lemma 5.3, $H^0(w, \alpha_{i_r})_0 = \mathbb{C}.h_{\alpha_{i_r}}$. Hence, a maximal toral subalgebra of $H^0(w, \alpha_{i_r}) \subseteq H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ lies in $\mathbb{C}.h_{\alpha_{i_r}} \oplus \mathbb{C}_{-\alpha_{i_r}}$ and so it is one-dimensional.

Sub case (b): If $\langle \alpha_{i_j}, \alpha_{i_r} \rangle = 0$ for all $1 \leq j \leq r-1$, then by Corollary 4.5, we have

$$H^0(w, \alpha_{i_r}) \simeq sl_{2,\alpha_{i_r}}.$$

Hence, any maximal toral subalgebra of the ideal $H^0(w, \alpha_{i_r}) \subseteq H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ lies in $sl_{2,\alpha_{i_r}}$ and so it is one-dimensional.

Hence, it follows that

$$f_w(\mathfrak{h}) \cap Ker(f) = Ker(f)_0 = H^0(w, \alpha_{i_r})_0$$

is a maximal toral subalgebra of $Ker(f)$ and its dimension is at most one.

By induction on $l(w)$ and by (1), $f_u(\mathfrak{h}) = H^0(Z(u, \underline{i}'), T_{(u,\underline{i}')})_0$ is a maximal toral subalgebra of $H^0(Z(u, \underline{i}'), T_{(u,\underline{i}')})$.

Now, consider the following commutative diagram of Lie algebras:

$$\begin{array}{ccc} H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) & & \\ \downarrow f_w & \searrow f_u & \\ H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) & \xrightarrow{f} & H^0(Z(u, \underline{i}'), T_{(u,\underline{i}')}) \end{array}$$

Note that by commutativity of the above diagram and by (1), it follows that $f_w(\mathfrak{h})$ is an extension of $f_u(\mathfrak{h})$ and $f_w(\mathfrak{h}) \cap Ker(f)$. Thus, we conclude that $f_w(\mathfrak{h}) = H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0$

is a maximal toral subalgebra of $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$. This completes the proof of the proposition.

□

Consider the restriction of the homomorphism $f_w : \mathfrak{p}_{\alpha_{i_1}} \longrightarrow H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$(as in Lemma 5.9) to $\mathfrak{b}$ and denote it also by f_w.

Lemma 5.13. *The homomorphism* $f_w : \mathfrak{b} \longrightarrow H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ *is injective if and only if* $w^{-1}(\alpha_0) < 0$.

Proof. Assume that f_w is injective. Since $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ is a B-submodule of $H^0(w,\mathfrak{g}/\mathfrak{b})$, we have $H^0(w,\mathfrak{g}/\mathfrak{b})_{-\alpha_0} \neq 0$.

Recall from the proof the Lemma 5.11, the following exact sequence of B-modules:

$$0 \longrightarrow H^0(w,\mathfrak{b}) \longrightarrow \mathfrak{g} \longrightarrow H^0(w,\mathfrak{g}/\mathfrak{b}) \longrightarrow H^1(w,\mathfrak{b}) \longrightarrow 0.$$

Note that if G is simply laced, by [Kan13, Lemma 3.4] $H^1(w,\mathfrak{b}) = 0$. If G is non simply laced, since $-\alpha_0$ is a long root by [Kan13, Lemma 4.8(2)], we have $H^1(w,\mathfrak{b})_{-\alpha_0} = 0$. Hence, we have the following short exact sequence of T-modules:

$$0 \longrightarrow H^0(w,\mathfrak{b})_{-\alpha_0} \longrightarrow \mathfrak{g}_{-\alpha_0} \longrightarrow H^0(w,\mathfrak{g}/\mathfrak{b})_{-\alpha_0} \longrightarrow 0.$$

Since $dim(\mathfrak{g}_{-\alpha_0}) = 1$, $H^0(w,\mathfrak{b})_{-\alpha_0} = 0$. Hence, we have $w^{-1}(\alpha_0) < 0$.

Now we prove the converse.

Let $\psi_w : B \longrightarrow Aut^0(Z(w,\underline{i}))$ be the homomorphism of algebraic groups induced by the action of B on $Z(w,\underline{i})$(as in the proof of Lemma 5.9). Let K be the kernel of ψ_w. Since

$$BwB/B = \prod_{\beta \in R^+(w)} U_{-\beta}wB/B$$

(see [Jan07, Section 13.1]) and $w^{-1}(\alpha_0) < 0$, we have

$$U_{-\alpha_0}wB/B \neq wB/B.$$

Since the desingularization map $\phi_w : Z(w,\underline{i}) \longrightarrow X(w)$ is B-equivariant and the restriction of ϕ_w to an open subset is an isomorphism onto BwB/B, we have $U_{-\alpha_0} \cap K = \{e\}$, where e is identity element in B.

Recall that $f_w : \mathfrak{b} \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ is the homomorphism of Lie algebras induced by ψ_w. Since $U_{-\alpha_0} \cap K = \{e\}$, we have

$$(Ker(f_w))_{-\alpha_0} = 0.$$

Since $Ker(f_w)$ is a B-submodule of $\mathfrak{b}$ and $\mathfrak{b}$ has a unique B-stable line $\mathfrak{g}_{-\alpha_0}$, we have $Ker(f_w) = 0$. Hence f_w is injective. $\square$

The following proposition describes the set of all positive roots occurring as a weight in $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$.

Proposition 5.14. *Let $w \in W$ and fix a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$. Let $\mu \in M_{\geq 0} \setminus \{0\}$. Then, we have*

1. *$H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_\mu \neq 0$ if and only if there exists an integer $1 \leq j \leq r$ such that $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$, and $\mu = \alpha_{i_j}$.*
2. *Fix $1 \leq j \leq r$ such that $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$. Then, we have $dim(H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_{\alpha_{i_j}}) = 1$ and $sl_{2,\alpha_{i_j}}$ is a $B_{\alpha_{i_j}}$-submodule of $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$.*

Proof. Proof of (1): Assume that $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_\mu \neq 0$. Let $v = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$ and let $i' = (i_1, i_2, \ldots, i_{r-1})$. By using *LES* and Lemma 4.6, we have the following exact sequence of B-modules:

$0 \longrightarrow H^0(w, \alpha_{i_r}) \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(v,\underline{i}'), T_{(v,\underline{i}')}) \longrightarrow H^1(w, \alpha_{i_r}) \longrightarrow H^1(Z(w,\underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(Z(v,\underline{i}'), T_{(v,\underline{i}')}) \longrightarrow 0.$

Since $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_\mu \neq 0$, either $H^0(w, \alpha_{i_r})_\mu \neq 0$ or $H^0(Z(v,\underline{i}'), T_{(v,\underline{i}')})_\mu \neq 0$.

Now, if $H^0(w, \alpha_{i_r})_\mu \neq 0$, then by Corollary 5.5, we are done.

Otherwise, we have $H^0(Z(v,\underline{i}'), T_{(v,\underline{i}')})_\mu \neq 0$. Then by the induction on $l(w)$, there exists $1 \leq j \leq r-1$ such that $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$ and $\mu = \alpha_{i_j}$.

We now prove the other implication:

Let $1 \leq j \leq r$ be such that $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$.

If $j = r$, then $\langle \alpha_{i_k}, \alpha_{i_r} \rangle = 0$ for all $1 \leq k \leq r-1$. By Corollary 5.5, we have

$$H^0(w, \alpha_{i_r})_{\alpha_{i_r}} \neq 0.$$

Hence, we conclude that

$$H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_{\alpha_{i_r}} \neq 0.$$

Otherwise, by Corollary 5.5, we have $H^0(w, \alpha_{i_r})_{\alpha_{i_j}} = 0$ and by Corollary 5.8, we have $H^1(w, \alpha_{i_r})_{\alpha_{i_j}} = 0$. By the above exact sequence, we get

$$H^0(Z(w, \underline{i}), T_{(w,i)})_{\alpha_{i_j}} \simeq H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})_{\alpha_{i_j}}.$$

Now the proof follows by induction on $l(w)$.

Proof of (2): Fix $1 \leq j \leq r$. Assume that $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$. Then, by (1), we have $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_{\alpha_{i_j}} \neq 0$.

Let $v = s_{i_1} s_{i_2} \cdots s_{i_{r-1}}$ and $i' = (i_1, i_2, \ldots, i_{r-1})$.

If $j = r$, then by Corollary 5.5 we have $H^0(w, \alpha_{i_r}) \simeq sl_{2,\alpha_{i_r}}$. Also, by using (1), we see that $H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})_{\alpha_{i_r}} = 0$. Hence, by the above exact sequence, we conclude that $dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_{\alpha_{i_r}}) = 1$ and $sl_{2,\alpha_{i_r}}$ is a $B_{\alpha_{i_r}}$-submodule of $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$.

On the other hand, if $j \neq r$ then by induction on $l(w)$,

$$dim(H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})_{\alpha_{i_j}}) = 1$$

and $sl_{2,\alpha_{i_j}}$ is a $B_{\alpha_{i_j}}$-submodule of $H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})$. Note that by Corollary 5.5, we have $H^0(w, \alpha_{i_r})_{\alpha_{i_j}} = 0$. Also, by Corollary 5.8, we have $H^1(w, \alpha_{i_r})_{\alpha_{i_j}} = 0$. Hence, by the above exact sequence, we see that

$$H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_{\alpha_{i_j}} \simeq H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})_{\alpha_{i_j}}$$

and $dim(H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_{\alpha_{i_j}}) = 1$.

Further, since $sl_{2,\alpha_{i_j}}$ is a cyclic $B_{\alpha_{i_j}}$-module generated by $x_{\alpha_{i_j}}$, it follows that $x_{\alpha_{i_j}}$ is in the image of the map $H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})$. Thus, we conclude that $sl_{2,\alpha_{i_j}}$ is a $B_{\alpha_{i_j}}$-submodule of $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$. ∎

Proposition 5.15. *Let $w \in W$ and $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression $\underline{i}$ of w. Then, $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ contains a Lie subalgebra $\mathfrak{b}'$ isomorphic to $\mathfrak{b}$ if and only if $w^{-1}(\alpha_0) < 0$.*

Proof. Recall from the proof of Lemma 5.13, $\psi_w : B \longrightarrow Aut^0(Z(w, \underline{i}))$ is the homomorphism of algebraic groups induced by the action of B on $Z(w, \underline{i})$ and $f_w : \mathfrak{b} \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is the induced homomorphism of Lie algebras.

Assume that $\mathfrak{b}'$ is a Lie subalgebra of $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ which is isomorphic to $\mathfrak{b}$, then there exists a closed subgroup B' of $Aut^0(Z(w, \underline{i}))$ such that B' is isomorphic to B and $Lie(B') = \mathfrak{b}'$.

Fix an isomorphism $g : B \longrightarrow B'$. Then, $g(T)(\simeq T)$ is a maximal torus in B'. Hence, we have

$$rank(Aut^0(Z(w,\underline{i}))) \geq dim(T).$$

By Proposition 5.12(3), $f_w(\mathfrak{h})$ is a maximal toral subalgebra of $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$. Hence, $\psi_w(T)$ is a maximal torus in $Aut^0(Z(w,\underline{i}))$. Thus, the restriction $\psi_w|_T : T \longrightarrow Aut^0(Z(w,\underline{i}))$ is injective.

Let T' be a maximal torus of B'. Since any two maximal tori in $Aut^0(Z(w,\underline{i}))$ are conjugate, there exists a $\sigma \in Aut^0(Z(w,\underline{i}))$ such that $T = \sigma T' \sigma^{-1}$. Now, let $B'' := \sigma B' \sigma^{-1}$. Then, we have $T \subset B''$. Since $Lie(B'')$ is a T-stable Lie subalgebra of $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$, by Proposition 5.14 we have

$$Lie(B'') = \mathfrak{h} \oplus \bigoplus_{\beta \in R'} \mathfrak{g}_\beta \oplus \bigoplus_{\alpha \in S'} \mathfrak{g}_\alpha$$

for some subset R' of R^- and for some subset S' of S.

Fix $\alpha \in S'$, Then, we have $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_\alpha \neq 0$. Hence by Proposition 5.14, we have $dim(H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_\alpha) = 1$. Thus, the homomorphism $f_w : \mathfrak{b} \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ extends to $\widetilde{f_w} : \mathfrak{p}_\alpha \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ as T-modules such that $\widetilde{f_w}(\mathfrak{g}_\alpha) \neq 0$. Let $\mathfrak{l}_\alpha \subseteq \mathfrak{p}_\alpha$ be the Lie algebra of L_α. Consider the restriction $(f_w)_\alpha$ of $\widetilde{f_w}$ to $\mathfrak{l}_\alpha$. Clearly, $(f_w)_\alpha$ is injective homomorphism of Lie algebras. Let n_α be a representative of the simple reflection s_α in $N_G(T)$, let $(\psi_w)_\alpha : \widetilde{L}_\alpha \longrightarrow Aut^0(Z(w,\underline{i}))$ be the homomorphism of algebraic groups induced by $f_{w\alpha}$, where $\widetilde{L}_\alpha$ is a simply connected covering of L_α. Since $(f_w)_\alpha$ is injective, $\widetilde{n}_\alpha \notin Ker((\psi_w)_\alpha)$, where $\widetilde{n}_\alpha$ is a lift of n_α in $\widetilde{L}_\alpha$. Note that $(\psi_w)_\alpha(n_\alpha)$ normalizes T and hence $Ad((\psi_w)_\alpha(n_\alpha))(\mathfrak{h}) = \mathfrak{h}$.

Since $Lie(B'')$ is solvable Lie subalgebra and $\mathfrak{g}_\alpha \subseteq Lie(B'')$, $\mathfrak{g}_{-\alpha} \nsubseteq Lie(B'')$ (otherwise, $sl_{2,\alpha}$ would be Lie subalgebra of $Lie(B'')$). Hence, we have $R' \cap (-S') = \emptyset$.

Note that by Proposition 5.14, if $\alpha \in S'$, then $\alpha = \alpha_{i_j}$ for some integer $1 \leq j \leq r$ such that $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$. Hence, the elements in $\{s_\alpha : \alpha \in S'\}$ commute with each other. Thus, $(\prod_{\alpha \in S'} s_\alpha)(\beta) = -\beta$ for every $\beta \in S'$. Further, since $R' \cap (-S') = \emptyset$, we have $(\prod_{\alpha \in S'} s_\alpha)(R') \subseteq R^-$. Let $n = \prod_{\alpha \in S'} (\psi_w)_\alpha(\widetilde{n}_\alpha)$, where the product is taken in some ordering. Hence

$$Lie(nB''n^{-1}) = \mathfrak{h} \oplus \bigoplus_{\beta \in R''} \mathfrak{g}_\beta \oplus \bigoplus_{\gamma \in S'} \mathfrak{g}_\gamma,$$

where $R'' = (\prod_{\alpha \in S'} s_\alpha)(R')$. Note that for each $\alpha \in S'$, $s_\alpha(R') \cap (-S') = \emptyset$. Hence $R'' \cap (-S') = \emptyset$. Then, $Lie(nB''n^{-1}) \subseteq \mathfrak{b}$. Since $dim(\mathfrak{b}) = dim(Lie(nB''n^{-1}))$, we have $Lie(nB''n^{-1}) = \mathfrak{b}$.

In particular, we have $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})_{-\alpha_0} \neq 0$. Since $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ is a B-submodule of $H^0(w, \mathfrak{g}/\mathfrak{b})$, we have $H^0(w, \mathfrak{g}/\mathfrak{b})_{-\alpha_0} \neq 0$. Hence, we have $w^{-1}(\alpha_0) < 0$.

Proof of the converse follows from Lemma 5.13. □

5.4 Automorphism group of $Z(w, \underline{i})$:

In this section, we study the automorphism group of a BSDH variety.

We first recall the following theorem from [Kan13] (see [Kan13, Theorem 3.7, Theorem 3.8 and Theorem 4.11]). Let $T_{G/B}$ denote the tangent bundle of the flag variety G/B. By abuse of notation, we denote the restriction $T_{G/B}$ to $X(w)$ by $T_{G/B}$.

Theorem 5.16. *Let $w \in W$. Then*

1. *$H^i(X(w), T_{G/B}) = 0$ for every $i \geq 1$.*
2. *The adjoint representation $\mathfrak{g}$ of G is a B-submodule of $H^0(X(w), T_{G/B})$ if and only if $w^{-1}(\alpha_0) < 0$.*
3. *If G is simply laced, $H^0(X(w), T_{G/B})$ is the adjoint representation $\mathfrak{g}$ of G if and only if $w^{-1}(\alpha_0) < 0$.*
4. *Assume that G is simply laced and $X(w)$ is a smooth Schubert variety. Let $Aut^0(X(w))$ be the connected component of the automorphism group of $X(w)$ containing the identity automorphism. Let P_w denote the stabilizer of $X(w)$ in G. Let $\phi_w : P_w \longrightarrow Aut^0(X(w))$ be the homomorphism induced by the action of P_w on $X(w)$. Then, we have*
   (i) $\phi_w : P_w \longrightarrow Aut^0(X(w))$ is surjective.
   (ii) $\phi_w : P_w \longrightarrow Aut^0(X(w))$ is an isomorphism if and only if $w^{-1}(\alpha_0) < 0$.

Let $w \in W$ and fix a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$, let $\underline{i} = (i_1, i_2, \ldots, i_r)$.

Recall that for any reduced expression $w_0 = s_{j_1}s_{j_2}\cdots s_{j_N}$ of w_0 such that $\underline{j} = (j_1, j_2, \ldots, j_N)$ and $(j_1, j_2, \ldots, j_r) = \underline{i}$, there exits a natural homomorphism

$$f_w : H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$$

of Lie algebras from Section 5.

Recall the following notation:

$$J^{'}(w, \underline{i}) := \{l \in \{1, 2, \ldots, r\} : \langle \alpha_{i_l}, \alpha_{i_k} \rangle = 0 \text{ for all } k < l\},$$

$$J(w, \underline{i}) := \{\alpha_{i_l} : l \in J^{'}(w, \underline{i})\} \subset S.$$

Note that the simple reflections $\{s_{i_j} : j \in J'(w, \underline{i})\}$ commute with each other. For each α in $J(w, \underline{i})$, fix a representative n_α of s_α in $N_G(T)$ and let $P_{J(w,\underline{i})}$ be the subgroup of G generated by B and $\{n_\alpha : \alpha \in J(w, \underline{i})\}$. Let $\mathfrak{p}_{J(w,\underline{i})}$ be the Lie algebra of $P_{J(w,\underline{i})}$.

Then, we have

Theorem 5.17.

1. $\mathfrak{p}_{J(w_0,\underline{i})} \simeq H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$.
2. $\mathfrak{p}_{J(w,\underline{i})}$ *is isomorphic to a Lie subalgebra of* $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ *if and only if* $w^{-1}(\alpha_0) < 0$. *In such a case, we have* $\mathfrak{p}_{J(w,\underline{i})} = \mathfrak{p}_{J(w_0,\underline{j})}$ *for any reduced expression* $w_0 = s_{j_1} s_{j_2} \cdots s_{j_N}$ *of* w_0 *such that* $\underline{j} = (j_1, j_2, \ldots, j_N)$ *and* $(j_1, j_2, \ldots, j_r) = \underline{i}$.
3. *If G is simply laced,* $\mathfrak{p}_{J(w,\underline{i})} \simeq H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ *if and only if* $w^{-1}(\alpha_0) < 0$. *In such a case, we have* $\mathfrak{p}_{J(w_0,\underline{j})} \simeq H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$*, where* $\underline{j}$ *is as in (2).*
4. *If G is simply laced,* $f_w : H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ *is surjective, where* $\underline{j}$ *is as in (2).*

Proof. Proof of (1): By Lemma 5.9(2), $f_{w_0} : \mathfrak{b} \longrightarrow H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is injective. Also, by Corollary 5.10(1), $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is Lie subalgebra of $\mathfrak{g}$.

By Proposition 5.14, any $\mu \in M_{\geq 0} \setminus \{0\}$ such that $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})_\mu \neq 0$ is of the form $\mu = \alpha_{i_j}$ for some $1 \leq j \leq r$ such that $\langle \alpha_{i_j}, \alpha_{i_k} \rangle = 0$ for all $1 \leq k \leq j-1$. Hence, we conclude that $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is isomorphic to $\mathfrak{p}_{J(w_0,\underline{i})}$.

Proof of (2): If $\mathfrak{p}_{J(w,\underline{i})}$ is isomorphic to a Lie subalgebra of $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$, then by Proposition 5.15, we have $w^{-1}(\alpha_0) < 0$.

Conversely, assume that $w^{-1}(\alpha_0) < 0$. Let $w_0 = s_{j_1} s_{j_2} \cdots s_{j_N}$ be a reduced expression of w_0 such that $\underline{i} = (j_1, j_2, \ldots, j_r)$. Set $\underline{j} = (j_1, j_2, \ldots, j_N)$. Clearly, $J(w, \underline{i}) \subset J(w_0, \underline{j})$. Hence, we have $\mathfrak{p}_{J(w,\underline{i})} \subset \mathfrak{p}_{J(w_0,\underline{j})}$.

Therefore, by using (1), $\mathfrak{p}_{J(w,\underline{i})}$ is a Lie subalgebra of $H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})$.

Now, recall the following commutative diagram of Lie algebras:

$$\begin{array}{ccccc} \mathfrak{p}_{J(w,\underline{i})} & \hookrightarrow & H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) & \hookrightarrow & \mathfrak{g} \\ \uparrow & & \downarrow f_w & & \\ \mathfrak{b} & \xrightarrow{f_w|_{\mathfrak{b}}} & H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) & & \end{array}$$

(see Section 5.3).

Since the unique B-stable line $\mathfrak{g}_{-\alpha_0}$ in $H^0(Z(w_0,\underline{j}),T_{(w_0,\underline{j})})$ lies in $\mathfrak{b}$, by commutativity of the above diagram, we conclude that $f_w : H^0(Z(w_0,\underline{j}),T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ is injective if and only if its restriction $f_w|_{\mathfrak{b}}$ to $\mathfrak{b}$ is injective.

Since $w^{-1}(\alpha_0) < 0$, by Lemma 5.13, $f_w|_{\mathfrak{b}}$ to $\mathfrak{b}$ is injective. Hence, by the above arguments,

$$f_w : H^0(Z(w_0,\underline{j}),T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$$

is injective. Therefore, $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})_\alpha \neq 0$ for every $\alpha \in J(w_0,\underline{j})$. Thus, we conclude that $J(w_0,\underline{j}) = J(w,\underline{i})$.

Proof of (3): If G is simply laced, by Theorem 5.16 (3), we have $H^0(w,\mathfrak{g}/\mathfrak{b}) = \mathfrak{g}$ if and only if $w^{-1}(\alpha_0) < 0$. Recall from Section 5 that $H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ is a B-submodule of $H^0(w,\mathfrak{g}/\mathfrak{b})$. Hence, from the proof of (2), we conclude that $\mathfrak{p}_{J(w,\underline{i})} \simeq H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ if and only if $w^{-1}(\alpha_0) < 0$.

Proof of (4): Proof is by descending induction on $l(w)$. If $w = w_0$, we are done. Otherwise, let $w_0 = s_{j_1}s_{j_2}\cdots s_{j_N}$ be a reduced expression for w_0 such that $(j_1,j_2,\ldots,j_r) = \underline{i}$ and $r \leq N-1$. Let $v = s_{j_1}s_{j_2}\cdots s_{j_{r+1}}$ and let $\underline{i}' = (j_1,j_2,\ldots,j_{r+1})$. Note that $l(w) = l(v) - 1$.

Since G is simply laced, by using *LES* and Lemma 4.6 (2) we have the following short exact sequence of B-modules:

$$0 \longrightarrow H^0(v,\alpha_{i_{r+1}}) \longrightarrow H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')}) \longrightarrow H^0(Z(w,\underline{i}),T_{(w,\underline{i})}) \longrightarrow H^1(v,\alpha_{i_{r+1}}) = 0.$$

Consider the following commutative diagram of Lie algebras:

$$\begin{array}{ccc} H^0(Z(w_0,\underline{j}),T_{(w_0,\underline{j})}) & & \\ \big\downarrow{\scriptstyle f_v} & \searrow{\scriptstyle f_w} & \\ H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')}) & \longrightarrow & H^0(Z(w,\underline{i}),T_{(w,\underline{i})}) \end{array}$$

By descending induction on $l(w)$, $f_v : H^0(Z(w_0,\underline{j}),T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(v,\underline{i}'),T_{(v,\underline{i}')})$ is surjective. By commutativity of the above diagram and by the above short exact sequence, we conclude that $f_w : H^0(Z(w_0,\underline{j}),T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w,\underline{i}),T_{(w,\underline{i})})$ is surjective. This completes the proof of (4). $\square$

Recall that $\leq$ is the Bruhat-Chevalley ordering on W and $supp(w) := \{j \in \{1,2,\ldots,n\} : s_j \leq w\}$, the support of w. For simplicity of notation we denote $supp(w)$ by A_w. For $j \in A_w$, let n_j be a representative of s_j in $N_G(T)$. Let P_{A_w} be the standard parabolic subgroup of G containing B and $\{n_j : j \in A_w\}$. Let $\mathfrak{p}_{A_w}$ be the Lie algebra of P_{A_w}.

Let $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression of w and let $\underline{i} = (i_1, i_2, \ldots, i_r)$. Let $w_0 = s_{j_1}s_{j_2}\cdots s_{j_N}$ be a reduced expression for w_0 such that $(j_1, j_2, \ldots, j_r) = \underline{i}$.

Set $J_1 := (\{1, 2, \ldots, n\} \setminus A_w) \cap J'(w_0, \underline{j})$. Let $R_w = R^+ \setminus (\bigcup_{v \leq w} R^+(v^{-1}))$.

Let $f_w : H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ be the homomorphism as above.

Now, we will describe the kernel of the map f_w when G is simply laced. Let $Ker(f_w)$ be the kernel of f_w.

Corollary 5.18. *Let G be simply laced. Then, we have*

$$Ker(f_w) = (\bigcap_{k \in A_w} Ker(\alpha_k)) \oplus (\bigoplus_{\beta \in R_w} \mathfrak{g}_{-\beta}) \oplus (\bigoplus_{j \in J_1} \mathfrak{g}_{\alpha_j}).$$

Proof. Step 1: We will prove that for every $j \in A_w$, the restriction of f_w to the subspace $\mathbb{C}.h_{\alpha_j} \oplus \mathfrak{g}_{-\alpha_j}$ is injective.

Fix $j \in A_w$. Let k be the least positive integer in $\{1, 2, \ldots, r\}$ such that $j = i_k$. Let $v = s_{i_1}s_{i_2}\cdots s_{i_k}$ and set $\underline{i}' = (i_1, \ldots, i_k)$. Then, by Lemma 5.3(2), we see that $\mathbb{C}.h_{\alpha_j} \oplus \mathfrak{g}_{-\alpha_j}$ is a B_{α_j}-submodule of $H^0(v, \alpha_j)$. By *LES*, $H^0(v, \alpha_j)$ is a B-submodule of $H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})$. Let $g : H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')})$ be the homomorphism of B-modules induced by the fibration $Z(w, \underline{i}) \longrightarrow Z(v, \underline{i}')$.

Now, consider the following commutative diagram of B-modules:

$$\begin{array}{ccc} H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) & & \\ \downarrow f_w & \searrow^{f_v} & \\ H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) & \xrightarrow{g} & H^0(Z(v, \underline{i}'), T_{(v,\underline{i}')}) \end{array}$$

Note that $\mathbb{C}.h_{\alpha_j} \oplus \mathfrak{g}_{-\alpha_j}$ is a subspace of $H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})$. Therefore, by the above arguments, the restriction of f_v to the subspace $\mathbb{C}.h_{\alpha_j} \oplus \mathfrak{g}_{-\alpha_j}$ is injective. Hence, by commutativity of the above diagram, we conclude that the restriction of f_w to the subspace $\mathbb{C}.h_{\alpha_j} \oplus \mathfrak{g}_{-\alpha_j}$ is injective.

Step 2: Let $\mathfrak{l}_{A_w}$ be the Levi subalgebra of $\mathfrak{p}_{A_w}$, let $\mathfrak{z}(\mathfrak{l}_{A_w})$ be the center of $\mathfrak{l}_{A_w}$. We will prove that

$$\mathfrak{h} \cap Ker(f_w) = \mathfrak{z}(\mathfrak{l}_{A_w}) = \bigcap_{k \in A_w} Ker(\alpha_k).$$

First note that $\bigcap_{k\in A_w} Ker(\alpha_k) = \mathfrak{z}(\mathfrak{l}_{A_w})$ and the dimension of $\mathfrak{z}(\mathfrak{l}_{A_w})$ is $n - d(w)$ (since $|A_w| = d(w)$).

Now, we prove that $\mathfrak{h} \cap Ker(f_w)$ is contained in $\bigcap_{k\in A_w} Ker(\alpha_k)$.

Assume the contrary. Then, there exists a $k \in A_w$ and $h \in \mathfrak{h} \cap Ker(f_w)$ such that $\alpha_k(h) \neq 0$. Then,

$$x_{-\alpha_k} \cdot h = -[h, x_{-\alpha_k}] = \alpha_k(h)x_{-\alpha_k}$$

is a non zero multiple of $x_{-\alpha_k}$. Hence $\mathfrak{g}_{-\alpha_k}$ is contained in $Ker(f_w)$, which contradicts step 1. Therefore, $\mathfrak{h} \cap Ker(f_w)$ is contained in $\bigcap_{k\in A_w} Ker(\alpha_k)$.

By Proposition 5.12, we have $H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})_0 = \mathfrak{h}$ and $dim(\mathfrak{h} \cap Ker(f_w)) = n - d(w)$. Hence, we see that

$$f_w(H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})_0) = H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_0.$$

By the above arguments, $\mathfrak{h} \cap Ker(f_w)$ is a subspace of $\bigcap_{k\in A_w} Ker(\alpha_k)$ having the same dimension as that of $\bigcap_{k\in A_w} Ker(\alpha_k)$. Hence, we conclude that

$$\mathfrak{h} \cap Ker(f_w) = \bigcap_{k\in A_w} Ker(\alpha_k) = \mathfrak{z}(\mathfrak{l}_{A_w}).$$

Step 3: We will prove that for $j \in J_1$, sl_{2,α_j} is contained in $Ker(f_w)$.

Fix $j \in J_1$. By Theorem 5.17(2), it follows that sl_{2,α_j} is a B_{α_j}-submodule of $H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})})$. By Proposition 5.14(1), we see that $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_{\alpha_j} = 0$. Hence, $\mathfrak{g}_{\alpha_j} \subset Ker(f_w)$. Since sl_{2,α_j} is a cyclic B_{α_j}-module generated by $\mathfrak{g}_{\alpha_j}$, it follows that sl_{2,α_j} is contained in $Ker(f_w)$.

Step 4: The intersection of the nilradical of $\mathfrak{b}$ and $Ker(f_w)$ is equal to the direct sum $\bigoplus_{\beta\in R_w} \mathfrak{g}_{-\beta}$ of T-modules.

Consider the birational morphism $\phi_w : Z(w, \underline{i}) \longrightarrow X(w)$. Note that ϕ_w is a B-equivariant morphism for the natural left action of B on $Z(w, \underline{i})$ (respectively, on $X(w)$). Let $\phi : B \longrightarrow Aut^0(X(w))$ (respectively, $\phi' : B \longrightarrow Aut^0(Z(w, \underline{i}))$) be the homomorphism induced by the action of B on $X(w)$ (respectively, on $Z(w, \underline{i})$). Since ϕ_w is birational, we have $Ker(\phi) \cap B_u = Ker(\phi') \cap B_u$, where B_u is the unipotent radical of B.

Since G is simply laced, by [Kan13, Corollary 3.9], we conclude that $\mathfrak{b}_u \cap Ker(f_w) = \bigoplus_{\beta\in R_w} \mathfrak{g}_{-\beta}$, where $\mathfrak{b}_u$ is the nilradical of $\mathfrak{b}$.

From the steps 1 to 4, we conclude that

$$Ker(f_w) = (\bigcap_{k \in A_w} Ker(\alpha_k)) \oplus (\bigoplus_{\beta \in R_w} \mathfrak{g}_{-\beta}) \oplus (\bigoplus_{j \in J_1} \mathfrak{g}_{\alpha_j}). \qquad \square$$

Recall that if X is a smooth projective variety over $\mathbb{C}$, the connected component of the group of all automorphisms of X containing identity automorphism is an algebraic group (see [MO67, p.17, Theorem 3.7], [Gro61, p.268], which also deals the case when X may be singular or it may be defined over any field). Futher, the Lie algebrs of this automorphism group is isomorphic to the space of all vector fields on X, that is the space $H^0(X, T_X)$ of all global sections of the tangent bundle T_X of X (see [MO67, p.13, Lemma 3.4]).

We now prove the main results using Theorem 5.17.

Recall that $Aut^0(Z(w, \underline{i}))$ is the connected component of the identity element of the automorphism group of $Z(w, \underline{i})$.

Theorem 5.19.

1. $P_{J(w_0, \underline{i})} \simeq Aut^0(Z(w_0, \underline{i}))$.
2. *$Aut^0(Z(w, \underline{i}))$ contains a closed subgroup isomorphic to $P_{J(w,\underline{i})}$ if and only if $w^{-1}(\alpha_0) < 0$. In such a case, we have $P_{J(w,\underline{i})} = P_{J(w_0,\underline{j})}$ for any reduced expression $w_0 = s_{j_1} s_{j_2} \cdots s_{j_N}$ of w_0 such that $\underline{j} = (j_1, j_2, \cdots, j_N)$ and $(j_1, j_2, \cdots, j_r) = \underline{i}$.*
3. *If G is simply laced, $P_{J(w,\underline{i})} \simeq Aut^0(Z(w, \underline{i}))$ if and only if $w^{-1}(\alpha_0) < 0$. In such a case, we have $Aut^0(Z(w, \underline{i})) \simeq Aut^0(Z(w_0, \underline{j}))$, where $\underline{j}$ is as in (2).*
4. *The homomorphism $f_w : H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$ is induced by a homomorphism $g_w : Aut^0(Z(w_0, \underline{j})) \longrightarrow Aut^0(Z(w, \underline{i}))$ of algebraic groups, where $\underline{j}$ is as in (2).*
5. *If G is simply laced, the homomorphism $g_w : Aut^0(Z(w_0, \underline{j})) \longrightarrow Aut^0(Z(w, \underline{i}))$ of algebraic groups is surjective, where $\underline{j}$ is as in (2).*
6. *The rank of $Aut^0(Z(w, \underline{i}))$ is at most the rank of G.*

Proof. Recall that by [MO67, Theorem 3.7], $Aut^0(Z(w, \underline{i}))$ is an algebraic group and

$$Lie(Aut^0(Z(w, \underline{i}))) = H^0(Z(w, \underline{i}), T_{(w,\underline{i})}).$$

Let $\pi : \widetilde{G} \longrightarrow G$ be the simply connected covering of G. Let $\widetilde{P}_{J(w,\underline{i})}$ (respectively, $\widetilde{B}$) be the inverse image of $P_{J(w,\underline{i})}$ (respectively, of B) in $\widetilde{G}$.

Proof of (2): If $w^{-1}(\alpha_0) < 0$, then by Theorem 5.17(2), $\mathfrak{p}_{J(w,\underline{i})}$ is isomorphic to a Lie subalgebra of $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$. Hence, there is a homomorphism $\widetilde{\psi}_w : \widetilde{P}_{J(w,\underline{i})} \longrightarrow Aut^0(Z(w,\underline{i}))$ of algebraic groups. Since the center $Z(\widetilde{P}_{J(w,\underline{i})})$ of $\widetilde{P}_{J(w,\underline{i})}$ is same as $Z(\widetilde{B})$ and B acts on $Z(w,\underline{i})$, $Z(\widetilde{P}_{J(w,\underline{i})})$ acts trivially on $Z(w,\underline{i})$. Hence, the action of $\widetilde{P}_{J(w,\underline{i})}$ induces a homomorphism $\psi_w : P_{J(w,\underline{i})} \longrightarrow Aut^0(Z(w,\underline{i}))$ of algebraic groups. Since $\mathfrak{p}_{J(w,\underline{i})}$ is isomorphic to a Lie subalgebra of $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$, ψ_w is an isomorphism onto its image.

On the other hand, if $Aut^0(Z(w,\underline{i}))$ contains a closed subgroup isomorphic to $P_{J(w,\underline{i})}$, then there is an injective homomorphism $\psi_w : P_{J(w,\underline{i})} \longrightarrow Aut^0(Z(w,\underline{i}))$ of algebraic groups. Further, ψ_w induces an injective homomorphism $\widetilde{f}_w : \mathfrak{p}_{J(w,\underline{i})} \longrightarrow H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$ of Lie algebras. Hence, by Theorem 5.17(2), we have $w^{-1}(\alpha_0) < 0$. This completes the proof of (2).

Proofs of (1), (3) and (4) are similar to the proof of (2). For the sake of completeness we give proof here.

Proof of (1). By Theorem 5.17(1), $\mathfrak{p}_{J(w_0,\underline{i})}$ is isomorphic to the Lie algebra $H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$. Hence, there is a homomorphism $\widetilde{\psi}_{w_0} : \widetilde{P}_{J(w_0,\underline{i})} \longrightarrow Aut^0(Z(w_0,\underline{i}))$ of algebraic groups. Since the center $Z(\widetilde{P}_{J(w_0,\underline{i})})$ of $\widetilde{P}_{J(w_0,\underline{i})}$ is same as $Z(\widetilde{B})$ and B acts on $Z(w_0,\underline{i})$, $Z(\widetilde{P}_{J(w_0,\underline{i})})$ acts trivially on $Z(w_0,\underline{i})$. Hence, the action of $\widetilde{P}_{J(w_0,\underline{i})}$ induces a homomorphism $\psi_{w_0} : P_{J(w_0,\underline{i})} \longrightarrow Aut^0(Z(w_0,\underline{i}))$ of algebraic groups. Note that ψ_{w_0} induces an isomorphism $\widetilde{f}_{w_0} : \mathfrak{p}_{J(w_0,\underline{i})} \longrightarrow H^0(Z(w_0,\underline{i}), T_{(w_0,\underline{i})})$ of Lie algebras. Hence, we conclude that $\psi_{w_0} : P_{J(w_0,\underline{i})} \longrightarrow Aut^0(Z(w_0,\underline{i}))$ is an isomorphism of algebraic groups.

Proof of (3). By (2), we have the homomorphism $\psi_w : P_{J(w,\underline{i})} \longrightarrow Aut^0(Z(w,\underline{i}))$ of algebraic groups is injective if and only if $w^{-1}(\alpha_0) < 0$. Since G is simply laced, by Theorem 5.17(3), we conclude the proof of (3).

Proof of (4). By (1), we have $P_{J(w_0,\underline{j})} \simeq Aut^0(Z(w_0,\underline{j}))$.

Let

$$P_{J(w_0,\underline{j})} = LP_u = L_{ss}Z(L)P_u$$

be the Levi decomposition of $P_{J(w_0,\underline{j})}$ such that $T \subset L$, where L is the Levi factor of $P_{J(w_0,\underline{j})}$ containing T, L_{ss} is semi simple part of L and P_u is unipotent radical of $P_{J(w_0,\underline{j})}$.

Since $P_u \subset B$, we have the homomorphism $f_1 : P_u \longrightarrow Aut^0(Z(w,\underline{i}))$ of algebraic groups.

Since $Z(L) \subset T \subset B$, we have the homomorphism $f_2 : Z(L) \longrightarrow Aut^0(Z(w,\underline{i}))$.

For $j \in J(w,\underline{i})$, by Lemma 5.14, sl_{2,α_j} is contained in $H^0(Z(w,\underline{i}), T_{(w,\underline{i})})$. Hence for each $j \in J(w,\underline{i})$, we have $\phi_j : SL_{2,\alpha_j} \longrightarrow Aut^0(Z(w,\underline{i}))$.

For $j \in J(w_0, \underline{j}) \setminus J(w, \underline{i})$, by the proof of Corollary 5.18 (even though G is not necessarily simply laced), we have $\mathfrak{g}_{\alpha_j} \subset Ker(f_w)$. Hence, the homomorphism $\phi_j : SL_{2,\alpha_j} \longrightarrow Aut^0(Z(w, \underline{i}))$ is trivial. That is SL_{2,α_j} acts trivially on $Z(w, \underline{i}))$ for each $j \in J(w_0, \underline{j}) \setminus J(w, \underline{i})$.

Therefore, we have the homomorphism $\widetilde{L} \longrightarrow Aut^0(Z(w, \underline{i}))$ of algebraic groups, where $\widetilde{L}$ is inverse image of L in $\widetilde{G}$ by the universal cover $\pi : \widetilde{G} \longrightarrow G$.

Claim: For $j \in J(w_0, \underline{j})$, we have the following commutative diagram of algebraic groups:

$$\begin{array}{ccc} SL_{2,\alpha_j} & \xrightarrow{\phi_j} & Aut^0(Z(w, \underline{i})) \\ \downarrow & \nearrow & \\ PGL_{2,\alpha_j} & \longrightarrow & Aut^0(Z(w, \underline{i})) \end{array}$$

Let G_{α_j} be the image of SL_{2,α_j} in $Aut^0(Z(w, \underline{i}))$, let $B_{\alpha_j} = B \cap G_{\alpha_j}$. Let $\widetilde{B}_{\alpha_j} = \pi^{-1}(B_{\alpha_j})$, which is a Borel subgroup of SL_{2,α_j}.

Now consider the following commutative diagram:

$$\begin{array}{ccc} SL_{2,\alpha_j} & \xrightarrow{\phi_j} & Aut^0(Z(w, \underline{i})) \\ \uparrow & & \uparrow \\ \widetilde{B}_{\alpha_j} & \xrightarrow{\pi} & B_{\alpha_j} \end{array}$$

Since the kernel of π is contained in the kernel of ϕ_j, the action of $Z(\widetilde{B}_{\alpha_j})$ on $Z(w, \underline{i})$ is trivial. Since $Z(\widetilde{B}_{\alpha_j}) = Z(SL_{2,\alpha_j})$, we have the homomorphism $PSL_{2,\alpha_j} \longrightarrow Aut^0(Z(w, \underline{i}))$. This proves the claim.

From the above discussion, we conclude that the center $Z(\widetilde{P}_{J(w_0, \underline{j})})$ acts trivially on $Z(w, \underline{i})$. Hence, there is a homomorphism $g_w : P_{J(w_0, \underline{j})} \longrightarrow Aut^0(Z(w, \underline{i}))$ of algebraic groups which induces f_w. This completes the proof of (4).

Proof of (5) follows from Theorem 5.17(4).

Proof of (6) follows from Proposition 5.12. □

We use the same notation as before. Assume that G is simply laced.

Let $g_w : Aut^0(Z(w_0, \underline{j})) \longrightarrow Aut^0(Z(w, \underline{i}))$ be the natural map as in Theorem 5.19 (4). Let U^+ be the unipotent radical of B^+. For $j \in J_1$, let $U^+_{\alpha_j}$ denote the one-dimensional T-stable closed subgroup of U^+ (for the conjugation action of T on G) corresponding to α_j. Let $T(w) := \bigcap_{k \in A_w} Ker(\alpha_k)$. Since $\{\alpha_k : k \in A_w\}$ is a subset of the $\mathbb{Z}$-basis S of $X(T)$, $T(w)$ is connected.

Corollary 5.20. *The connected component of the kernel of the map g_w is the closed subgroup of $Aut^0(Z(w_0, \underline{j}))$ generated by the torus $T(w)$, $U_{-\beta} : \beta \in R_w\}$ and $\{U^+_{\alpha_j} : j \in J_1\}$.*

Proof. Let K be the kernel of the homomorphism g_w. Then, we have the following exact sequence of algebraic groups:

$$1 \longrightarrow K \longrightarrow Aut^0(Z(w_0, \underline{j})) \longrightarrow Aut^0(Z(w, \underline{i})) \longrightarrow 1.$$

By using the differentials, we have following exact sequence of Lie algebras:

$$0 \longrightarrow Lie(K) \longrightarrow H^0(Z(w_0, \underline{j}), T_{(w_0, \underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w, \underline{i})}) \longrightarrow 0.$$

By [Hum75, p.85, Theorem 12.5], the Lie algebra of K is $Ker(f_w)$. By Corollary 5.18, we have

$$Ker(f_w) = (\bigcap_{k \in A_w} Ker(\alpha_k)) \oplus (\bigoplus_{\beta \in R_w} \mathfrak{g}_{-\beta}) \oplus (\bigoplus_{j \in J_1} \mathfrak{g}_{\alpha_j}).$$

Let H be the closed subgroup of $Aut^0(Z(w_0, \underline{j}))$ generated by $T(w)$, $\{U_{-\beta} : \beta \in R_w\}$ and $\{U^+_{\alpha_j} : j \in J_1\}$. Note that H is connected (see [Hum75, p.56, Corollary 7.5]) and $Lie(H) \subset Ker(f_w)$. Since $dim(Lie(H)) = dim(Ker(f_w))$, we have

$$Lie(H) = Ker(f_w).$$

Hence, we conclude that $K^0 = H$. This completes the proof of the corollary. □

In the following corollary, for the simplicity of notation we denote the homogeneous vector bundle $\mathcal{L}(w, \mathbb{C}_{\alpha_0})$ on $X(w)$ corresponding to the character α_0 of B by $\mathcal{L}_{\alpha_0}$.

Consider the left action of T on G/B. Let $w \in W$. Note that the Schubert variety $X(w^{-1})$ is T-stable. We use the notion of semi-stable points introduced by Mumford [MFK94]. Let α_0 be the highest root of G with respect to T and B^+. We denote by $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ the set of all semi-stable points of $X(w^{-1})$ with respect to the T-linearized line bundle $\mathcal{L}_{\alpha_0}$ corresponding to the character α_0 of B (see [MFK94]).

The following result is a formulation of the Theorem 5.19 using semi-stable points.

Corollary 5.21.
1. *$Aut^0(Z(w, \underline{i}))$ contains a closed subgroup isomorphic to $P_{J(w,\underline{i})}$ if and only if $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0}) \neq \emptyset$.*
2. *If G is simply laced, $Aut^0(Z(w, \underline{i})) \simeq P_{J(w,\underline{i})}$ if and only if $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0}) \neq \emptyset$.*

Proof. By [KP09a, Lemma 2.1], we have $X(w^{-1})^{ss}_T(\mathcal{L}_{\alpha_0}) \neq \emptyset$ if and only if $w^{-1}(\alpha_0) < 0$. Proof of the corollary follows from Theorem 5.19 (2) and Theorem 5.19 (3). □

Remark: By Theorem 5.19, the automorphism group of the BSDH-variety $Z(w, \underline{i})$ depends on the choice of the reduced expression $\underline{i}$ of w.

Example: Let $G = PSL(4, \mathbb{C})$. Consider the following different reduced expressions for w_0:

1. $(w_0, \underline{i}_1) = s_1s_2s_1s_3s_2s_1$, $J(w_0, \underline{i}_1) = \{\alpha_1\}$.
2. $(w_0, \underline{i}_2) = s_2s_1s_2s_3s_2s_1$, $J(w_0, \underline{i}_2) = \{\alpha_2\}$.
3. $(w_0, \underline{i}_3) = s_3s_2s_3s_1s_2s_3$, $J(w_0, \underline{i}_3) = \{\alpha_3\}$.
4. $(w_0, \underline{i}_4) = s_1s_3s_2s_3s_1s_2$, $J(w_0, \underline{i}_4) = \{\alpha_1, \alpha_3\}$.

By Theorem 5.19, we see that $Aut^0(Z(w_0, \underline{i}_1))$, $Aut^0(Z(w_0, \underline{i}_2))$, $Aut^0(Z(w_0, \underline{i}_3))$ and $Aut^0(Z(w_0, \underline{i}_4))$ are isomorphic to $P_{\{\alpha_1\}}, P_{\{\alpha_2\}}, P_{\{\alpha_3\}}, P_{\{\alpha_1,\alpha_3\}}$ respectively.

"Problems cannot be solved at the same level of awareness that created them."

Albert Einstein

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