HOLOGRAPHIC ASPECTS OF HYPERSCALING VIOLATING LIFSHITZ THEORIES

Ph.D. Thesis, Chennai Mathematical Institute, November 2018
Printed by: Later

Holographic Aspects of Hyperscaling violating Lifshitz Theories

Doctoral Thesis

Submitted to Chennai Mathematical Institute in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Ph.D)

Author

Debangshu Mukherjee

born on 14^{th} April, 1990.

Advisor:

Dr. K. Narayan, *Chennai Mathematical Institute (CMI).*

Doctoral Committee Members:

1. Dr. Alok Laddha, *Chennai Mathematical Institute (CMI).*
2. Dr. Balachandran Sathiapalan, *Institute of Mathematical Sciences (IMSc).*

Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

The work was done under the guidance of Prof. Dr. K. Narayan, at Chennai Mathematical Institute, India.

Debangshu Mukherjee

In our capacity as the committee members of the candidate's thesis, we certify that the above statements are true to the best of our knowledge.

Dr. K. Narayan
(Thesis supervisor)

Dr. Alok Laddha
(Committee Member)

Dr. Balachandran Sathiapalan
(Committee Member)

List of Publications

The present work is based on the following publications:

[P1] D. Mukherjee and K. Narayan, *AdS plane waves, entanglement and mutual information*, Phys. Rev. D **90**, no. 2, 026003 (2014), [arXiv:1405.3553].

[P2] K. S. Kolekar, D. Mukherjee and K. Narayan, *Hyperscaling violation and the shear diffusion constant*, Phys. Lett. B **760**, 86 (2016), [arXiv:1604.05092].

[P3] D. Mukherjee and K. Narayan, *Hyperscaling violation, quasinormal modes and shear diffusion*, JHEP **1712**, 023 (2017), [arXiv:1707.07490].

Other relevant publications:

[P'1] K. S. Kolekar, D. Mukherjee and K. Narayan, *Notes on hyperscaling violating Lifshitz and shear diffusion*, Phys. Rev. D **96**, no. 2, 026003 (2017), [arXiv:1612.05950].

Dedicated to
Ma, Baba, Titli and Ria

The story so far:
In the beginning the Universe was created.
This has made a lot of people very angry and been widely regarded as a bad move.

– Douglas Adams, *The Restaurant at the End of the Universe*

SYNOPSIS

The AdS/CFT correspondence provides a map between a quantum field theory and a gravitational theory. The correspondence is a powerful tool which can be used to study strongly coupled field theories using the dual gravity description. Over the last few years, taking motivation from various condensed matter systems the correspondence has been generalized and and applied to understand non-relativistic holography. An interesting class of non-relativistic spacetimes exhibits so-called hyperscaling violation. The gravity duals are conformally Lifshitz and arise as solutions to effective Einstein-Maxwell-dilaton theories. A certain subclass of hyperscaling violating theories can be constructed in string theory by null reduction of AdS plane waves which are effectively large boost, low temperature limit of boosted black branes. Some of the hyperscaling violating theories exhibit novel scaling behaviour for entanglement entropy which is also reflected in certain string realizations. This thesis can be broadly divided into two parts. After introducing the important aspects of AdS/CFT in Chapter 1, we will describe the work done in [P1] which involves the study of mutual information in the string constructions. We briefly summarize it as follows:

Entanglement Entropy (EE) and Mutual Information (MI):

Inspired by the area scaling of black hole entropy, Ryu and Takayanagi proposed a simple correspondence for calculating entanglement entropy in field theories with gravity duals: the EE for a subsystem in the d-dimensional field theory is the area in Planck units of a minimal surface bounding the subsystem, the bulk theory living in $(d+1)$-dimensions. Field theoretic techniques like "replica trick" can only be applied to study EE of ground state of 2-D systems. This geometric prescription thus gives us a calculable handle on what in field theory formalism is a rather complicated question.

In [P1] along with my advisor Prof. K. Narayan, we analysed mutual information (MI) for a certain class of CFT excited states: states with constant energy-momentum flux turned on ($T_{++} \sim Q$). The corresponding gravity duals are given by simple deformations of AdS, called AdS plane waves. Earlier works observed that the EE for a strip-subsystem is dependent on the orientation of the strip: for AdS_5 plane waves the EE grows logarithmically while for a general AdS_{d+1} with $d \neq 4$ the EE $\sim \sqrt{Q} l^{2-d/2}$ where l is the subsystem width. When the strip is orthogonal to the flux, we have a phase transition with the EE saturating for $l \gg Q^{-1/d}$.

Mutual information is another interesting information theoretic quantity defined as: $I[A:B] = S(A) + S(B) - S(A \cup B)$ where A and B are two disjoint subsystems. The Ryu-Takayanagi prescription implies a disentangling transition for mutual information in the large N classical gravity approximation with a critical separation. In [P1] we studied MI in AdS plane waves for the two different orientations of the strip subsysmtem. For wide strips $Ql^d \gg 1$ parallel to the flux we observed a disentangling transition at a critical separation (l_c) lesser than that of the ground state with l_c being independent of the characteristic energy scale $Q^{-1/d}$. This is quite distinct from the finite temperature case where for large enough subsystem size i.e. $lT \gg 1$ the linear extensive growth of entanglement implies that the subsystems disentangle for any finite separation independent of l. A phase transition for $l \gg Q^{-1/d}$ is observed when the strip is orthogonal to the flux and the MI vanishes. We also study the small width regime $Ql^d \ll 1$ treating Ql^d as a perturbative parameter; we observe the disentangling transition occurs at a lesser critical separation compared to the ground state. The intermediate regime i.e $Ql^d \sim 1$ is analysed numerically where we can see a non-trivial dependence of the disentangling point on Q and l. Overall, this is suggestive of the fact that these excited states are "partially ordered"–they disentangle faster than the ground state but slower than thermal states.

Non-relativistic holography and hydrodynamics:
In a series of work [P2, P'1, P3] done in collaboration with my advisor Prof. K. Narayan and fellow graduate student Kedar S. Kolekar, I studied the shear diffusion constant and the shear viscosity-to-entropy density ratio for hyperscaling violating Lifshitz (hvLif) theories. In the first couple of works [P2, P'1], the analysis was performed adapting a membrane paradigm-like approach used in *[JHEP* **0310**, *064 (2003)]* by *P. Kovtun, D. T. Son and A. O. Starinets.*

In the current thesis however, we will be interested in looking at the simpler where we do not perturb the background gauge field. The spatial directions $\{x_i\}$ enjoy translation invariance. Thus the diffusion of shear gravitational modes h_{xy}, h_{ty}, can be mapped to charge diffusion in an auxilliary theory obtained by compactifying the $(d_i + 2)$-dimn theory along one of the spatial directions, say y. We turn on plane wave modes for the perturbations $\propto e^{-\Gamma t + iqx}$ where Γ is the typical time scale over which the perturbation decays while q is the momentum along x. The y-compactification maps h_{xy}, h_{ty} to gauge fields $\mathcal{A}_t$ and $\mathcal{A}_x$ in the (d_i+1)-dimensional theory which in turn helps us in defining appropriate currents j^μ on the stretched horizon. A set of self-consistent approximations along with the equations of motion eventually leads to Fick's Law $j^x = -\mathcal{D}\partial_x j^t$ for charge diffusion of the gauge field $\mathcal{A}_\mu$ in the compactified background: we then identify $\mathcal{D}$ as the diffusion constant.

In hvLif theories with a gauge field, which is discussed in details in [P'1], the metric perturbations couple to the gauge field perturbations which complicates formulating Fick's Law. A crucial field redefinition helps us in formulating Fick's Law in terms of a modified current $\tilde{j}^\mu$. Similar to the earlier case of dilaton gravity, a self consistent set of approximations leads to the solutions of the redefined gauge fields $\tilde{\mathcal{A}}_t$ and $\tilde{\mathcal{A}}_x$ which in turn can be used to evaluate the shear diffusion constant $\mathcal{D}$ in terms of the metric components. However, we will not be concerned with this case in this thesis and will be presumably appear in a later work of my colleague Kedar S. Kolekar.

For a 4-dimensional hvLif theories with generic Lifshitz and hyperscaling-violating exponents (z, θ), following $z < 4-\theta$ the diffusion constant $\mathcal{D}$ exhibits a power-law behaviour with respect to the temperature T i.e $\mathcal{D} \sim T^{\frac{z-2}{2}}$ (T is the temperature of the field theory). Further, looking at various special cases motivated us to conjecture a relation between the shear viscosity (η) and shear diffusion ($\mathcal{D}$) eventually leading to $\frac{\eta}{s} = \frac{1}{4\pi}$ (where η is the viscosity and s is the entropy density) thus following the universal bound proposed by Kovtun-Son-Starinets [90] for relativistic field theories. The case where $z = 4-\theta$ seems interesting and the shear diffusion constant has a logarithmic scaling which appears to be a novel behaviour. The analysis seems to break down in the regime $z > 4-\theta$ failing to give us any insight into the hydrodynamic behaviour of these theories.
In our subsequent work [P3] we used holographic techniques to study hvLif theories. Generalizing the approach of *A.O. Starinets* as given in *[Phys. Lett. B* **670***, 442 (2009)]*, we studied the quasinormal modes of shear gravitational perturbations for hvLif theories. Quasinormal modes are diffusive solutions to linearized Einstein's equations that are regular at the horizon and vanishing at the boundary. Here again, we turn on the modes h_{ty}, h_{xy} and a_y of the form $e^{-i\omega t+iqx}$. Defining appropriate new field variables that are gauge invariant under residual gauge symmetry enables us to identify the relevant differential equations governing these modes. Restricting to the hydrodynamic regime allows the approximation of low frequency and momenta relative to the temperature scale which helps in writing a series solution for quasinormal modes. The lowest quasinormal modes for shear perturbations follows a dispersion relation of the form $\omega = -i\mathcal{D}q^2$ which can be used to read off the shear diffusion constant $\mathcal{D}$. This analysis vindicates our earlier results [P2, P'1] obtained using the membrane paradigm. Using the asymptotic behaviour of quasinormal modes, we further computed certain 2-point correlation

functions of the energy-momentum tensor. The poles of these retarded correlators are identical to the lowest quasinormal frequencies. We further use Kubo's formula to calculate the shear viscosity (η) vindicating our earlier result that $\frac{\eta}{s} = \frac{1}{4\pi}$ for $d_i + 2 - z - \theta < 0$. Interestingly, when $z = d_i + 2 - \theta$, we recover universal behaviour for viscosity i.e. $\eta/s = \frac{1}{4\pi}$ in the strict Kubo limit i.e. $(q = 0, \omega \to 0)$ while $\mathcal{D}$ scales logarithmically and can be presumably resummed.

Our work [P1], is suggestive of the fact that in the large N limit, the constant flux ($T_{++} \sim Q$) disorders the system so the subsystems disentangle at a lesser critical separation than the ground state. It will be interesting to have a better understanding of the nature of disentangling transition for an arbitrary excited state. The emergence of classically connected spacetime is inherently related to entanglement of the degrees of freedom. It will be interesting to explore these connections further.

Our work on the diffusion constant in hvLif theories hints at the fact that hydrodynamics for hvLif theories where $z = d_i + 2 - \theta$ is novel. As mentioned earlier, these class of hvLif theories exhibiting $z = d_i + 2 - \theta$ can be thought of as null reductions of low-temperature, high boost limit of boosted black branes. It would be interesting to apply the fluid-gravity paradigm to understand the logarithmic scaling for the shear diffusion constant and explore the hydrodynamics for a highly boosted observer.

Contents

Chapter 1

Introduction

The principal goal of research in theoretical physics in a broad sense is to capture a vast variety of natural phenomena in a mathematically consistent framework. Such endeavours often lead to interesting theoretical predictions which are subsequently verified through experiments. On ocassions, inputs from theoretical physics have led to novel results in mathematics. The past couple of centuries have witnessed enormous strides in the field of theoretical physics with two of its crowning achievements being:

General Relativity (GR) AND **Quantum Mechanics (QM)**

Both these theories work to an excellent degree of accuracy within their respective regimes of validity but curiously enough, these regimes are mutually exclusive. While QM provides a good description of the physics of atoms and subatomic particles, GR is a classical theory which describes physics over cosmological/astrophysical length scales which also reproduces Newtonian Mechanics under the non-relativistic limit ($c \to \infty$). Thus from a practical point of view QM and GR explains most of the natural phenomenon we encounter in our "everyday" life– from scattering experiments in high-energy colliders to the large scale structure of the cosmos. This however, seems to be a strange coincidence of nature wherein the fundamental constants $\hbar, c$ and G_N conspire to give a very large Planck scale ($\sim 10^{18}\ GeV$) compared to the laboratory scale. However, at super-Planckian scales quantum mechanical effects of gravity cannot be ignored. It is precisely the question of reconciliation between GR and QM or in other words the search for a correct theory of quantum gravity is what has driven a significant fraction of theoretical physics research over the last century. It is believed that such a "Theory of Everything" will unify the four fundamental forces of electromagnetism, strong force, weak force and gravity.

Although a complete quantum mechanical theory of gravity still remains elusive, the search for unification has led to the *Standard Model (SM) of particle physics* as we know it today. The SM is a mathematical formulation written in the language of quantum fields that contains internal symmetries of the unitary group $SU(3) \times SU(2) \times U(1)$. It has been hugely successful in accurately predicting the existence of various fundamental and composite particles which has been successfully detected in collider experiments. The SM received further validation as recently as 2012 when the Large Hadron Collider (LHC) at CERN announced the successful experimental detection of the Higgs Boson which was theoretically predicted back in 1964 [1, 2, 3].

In spite of enjoying such a success, the SM falls short of being a complete "Theory of Everything" since it does not incorporate gravitational forces. A consistent quantum theory of gravity must be able to describe physics beyond the Planck scale ($\sim 10^{18} GeV$) (e.g.: physics in the early universe near Big Bang singularity or final state of a black hole). At the same time the SM must appear as a low-energy effective theory below the GUT scale of $10^{16}\ GeV$.

String theory is currently considered to be the leading candidate for a consistent theory of quantum gravity. It assumes that the fundamental building blocks of matter are constructed out of extended objects like strings[1] instead of point particles. The mode excitations of these strings correspond to different fundamental particles that we observe in nature. A consistent quantum theory of strings predicts the existence of massless spin-2 modes having the same kinematical and dynamical property of gravitons. Thus, "quanta"s carrying information about gravitational interactions are automatically encoded by construction. Over the later part of the 80s and early 90s it became further clear that a consistent version of superstring theory also admits higher dimensional objects called D-branes. D-branes are non-perturbative objects that may be viewed from two different perspectives– D-branes are hyperplanes where open strings end. The open strings can possibly deform the D-branes to produce non-trivial gauge fields on it. They can also be interpreted as dynamical, massive objects that alter the ambient geometry. They are solutions that arise in the low-energy limit of string theory, supergravity. Using the dual pictures describing the same object, namely D-brane, in 1997 Maldacena

[1] We will later encounter other higher dimensional extended objects called D-branes

conjectured the "AdS/CFT correspondence"[4]–the first concrete realization of the gauge/gravity duality.

Dualities have played an important role in the development of string theory and quantum field theories in general. Dualities describe mapping between seemingly two different theories (with different Lagrangian and degrees of freedom) but are essentially equivalent at a deep and fundamental level. To be more precise, the two apparently differing theories must have the same Hilbert space structure and identical global symmetries. The AdS/CFT correspondence, proposed by Maldacena is a conjectured duality between a string theory living a certain background and a quantum field theory living in one dimension lesser. To be more specific, the so called Type IIB string theory in anti-de Sitter (AdS) space is dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with a certain gauge group which lives on the boundary of the AdS space. The couplings and other parameters are related accordingly. Naively speaking, QFT on flat background does not appear to describe quantum gravity. However, the AdS/CFT duality relates this apparently "simple" theory to a theory of quantum gravity which definitely makes it one of the most interesting discoveries in modern theoretical physics in the last couple of decades.

Since the AdS/CFT correspondence was conjectured, it has been used as a powerful computational tool to give us a better understanding of strongly coupled field theories. This is because the AdS/CFT correspondence is a strong-weak coupling duality i.e. the strongly coupled regime of the field theory is described by the low-energy limit of the dual gravitational theory which is classical and weakly curved. Thus certain questions pertaining to the strongly coupled regime of field theories become computationally tractable on the gravity side.

Since Maldacena's proposal, a flurry of activities in this field has led to the development of a holographic dictionary. The AdS/CFT correspondence has also been generalized further to describe strongly coupled regime of non-relativistic field theories too. A particular class of non-relativistic field theory of interest exhibit hyperscaling violation [5, 6, 7, 8, 9, 10, 11, 12]. More precisely, the dual metric is characterised by two parameters–the Lifshitz exponent, z and the hyperscaling violating exponent θ and can be explicitly stated in $(d_i + 2)$-dimensions as:

$$ds^2_{d_i+2} = r^{\frac{2\theta}{d_i}} \left(-\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \sum_{i=1}^{d_i} \frac{dx_i^2}{r^2} \right) \tag{1.1}$$

The above metric excluding the conformal factor of $r^{\frac{2\theta}{d_i}}$ is known as the Lifshitz metric and is invariant under the scaling

$$t \to \lambda^z t, \quad r \to \lambda r, \quad x_i \to \lambda x_i \tag{1.2}$$

However the hvLif metric (1.1) is not invariant under the above transformations, instead exhibiting a scaling behaviour of $ds \to \lambda^{\theta/d_i}\, ds$. Intuitively, this can be thought of as describing a strongly coupled field theory which exhibits a thermodynamic behaviour of the presence of a critical exponent z and living in $d_i - \theta$ dimensions. A certain subsector of these theories in the (z, θ) parameter space can be obtained from concrete string constructions. They can be obtained as null reductions of double scaling limit of highly boosted, low temperature black branes– AdS plane waves. The AdS/CFT correspondence has provided us with a powerful calculational handle towards understanding of hyperscaling violating Lifshitz theories which is indeed observed in certain condensed matter systems. This thesis is a study of certain aspects of mutual information in AdS plane waves and the hydrodynamics of hyperscaling violating Lifshitz theories (hvLif) within the general framework of gauge/gravity duality. The main computational techniques used in this thesis has parallels with the standard framework of the AdS/CFT correspondence. We will briefly review certain aspects of this duality before proceeding towards the findings in the context of AdS plane waves and hvLif spacetimes.

1.1 AdS_5/CFT_4 correspondence: Statement

This AdS_5/CFT_4 correspondence proposed by Maldacena in its strongest form conjectures:

- $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory with gauge group $SU(N)$ and a Yang-Mills coupling constant g_{YM}.

IS DUAL TO

- Type IIB string theory on $AdS_5 \times S^5$ with radius of curvature R and coupling constant g_s.

The free parameters in the two theories are related by

$$g_{YM}^2 = 2\pi g_s \quad \text{and} \quad R^4 = 2 g_{YM}^2 \alpha'^2 N \tag{1.3}$$

where $\sqrt{\alpha'} = l_s$ is the string length. The $\mathcal{N} = 4$ SYM theory is known as the 'CFT side' of the correspondence and exhibits conformal symmetry. Although first principle proof of this correspondence does not exist, Maldacena provided convincing evidences of such a duality. However, the fact that large N gauge theories and string theory share a deeper connection was first noticed by 't Hooft [13].

1.2 Large N Gauge Theories and String Theories

We briefly review the ideas of 't Hooft motivating the dual nature of string theory and large N gauge theories. Let us consider a generic *matrix* model constructed out of matrix fields Φ_a^b where the indices $\{a, b\}$ represent labels over the generators of the gauge group $SU(N)$. Schematically, we can propose a Lagrangian of the form

$$\mathcal{L} \sim \mathrm{Tr}(d\Phi_i d\Phi_i) + g_{YM} c^{ijk} \mathrm{Tr}(\Phi_i \Phi_j \Phi_k) + g_{YM}^2 d^{ijk} \mathrm{Tr}(\Phi_i \Phi_j \Phi_k \Phi_l)$$

By rescaling the fields as $\tilde{\Phi}_i \equiv g_{YM} \Phi_i$, we can rewrite the Lagrangian as

$$\mathcal{L} \sim \frac{1}{g_{YM}^2} \left[\mathrm{Tr}(d\tilde{\Phi}_i d\tilde{\Phi}_i) + c^{ijk} \mathrm{Tr}(\tilde{\Phi}_i \tilde{\Phi}_j \tilde{\Phi}_k) + d^{ijkl} \mathrm{Tr}(\tilde{\Phi}_i \tilde{\Phi}_j \tilde{\Phi}_k \tilde{\Phi}_l) \right] \tag{1.4}$$

where the coupling g_{YM} appears as an overall multiplicative constant in front of the action. For a $SU(N)$ theory, the propagator behaves as:

$$\langle \tilde{\Phi}_b^a \tilde{\Phi}_c^d \rangle \propto \delta_c^a \delta_b^d - \frac{1}{N} \delta_b^a \delta_c^d$$

Clearly, in the large N limit, the second term has subdominant contribution and hence can be neglected. The Feynman diagrams can now represented in the 't Hooft double line notation i.e. replace the propagator line with a double line having opposite orientations corresponding to the fundamental and anti-fundamental indices.

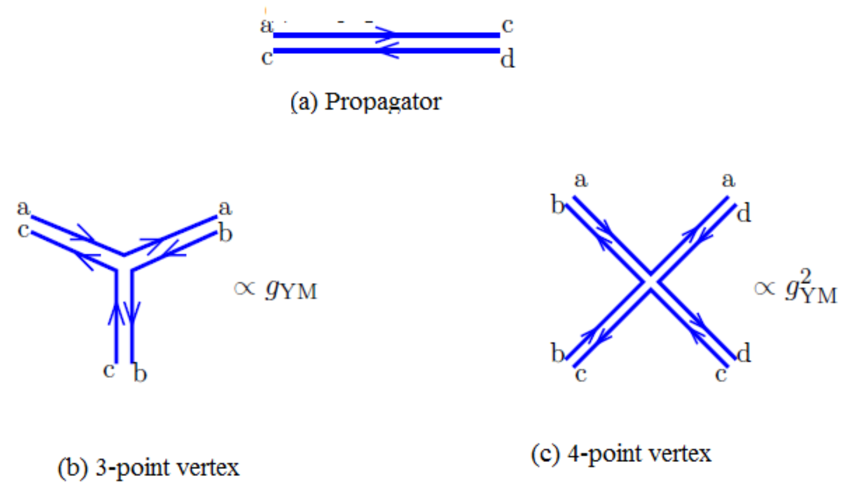

Figure 1.1: Propagators and vertices in the 't Hooft double-line notation

A few candidate diagrams that contribute to the vacuum to vacuum amplitude for a cubic interaction looks as follows in the double-line notation:

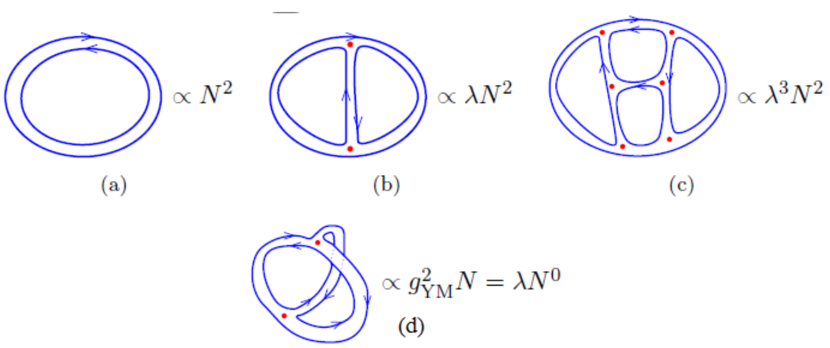

Figure 1.2: (a),(b) and (c) are planar diagrams while (d) is non planar. The arrows represent flow of color and the red dots represent interaction vertices

The first diagram (a) represents a simple gluon loop while the remaining diagrams consists of propagators, vertices and color loops. The schematic form of (1.4) can be used to estimate that the propagator is proportional to g^2_{YM} while the interaction vertex is proportional to $1/g^2_{YM}$; further each integration over a color loop gives a factor of N.

So, clearly diagram (a) $\propto N^2$. For diagram (b), a rough estimate of the amplitude can be made as $(g^2_{YM})^3\left(\frac{1}{g^2_{YM}}\right)^2 N^3 = g^2_{YM}N^3 = (g^2_{YM}N)N^2$ since, it has 3 propagators, 2 vertices and 3 color loops. Amplitude for diagram (c) $\sim (g^2_{YM})^9\left(\frac{1}{g^2_{YM}}\right)^6 N^5 = g^6_{YM}N^5 = (g^2_{YM}N)^3N^2$. However, the only non-planar diagram i.e. (d), has only a single color loop and the amplitude is proportional to $g_{YM}^2 N$. Far back in the '70 s, 't Hooft noticed that the factor of $g_{YM}^2 N$ apper in each of these diagrams contributing to the vacuum $\rightarrow$ vacuum amplitude. We consider that large N limit in a subtle way where $N \rightarrow \infty$ and $g_{YM} \rightarrow 0$ such that $\lambda = g_{YM}^2 N \rightarrow$ const.. The newly introduced parameter λ is known as the 't Hooft coupling which determines whether we are in the strongly or the weakly coupled regime of the field theory. Written in terms of this new parameter, the amplitude for diagram (b) $\sim \lambda N^2$ while for diagram (d) $\sim \lambda N^0$ even though both diagrams have the same number of propagators and vertices. In the 't Hooft limit ($g_{YM} \rightarrow 0, N \rightarrow \infty, \lambda \rightarrow$ const.) the contribution coming from diagram (d) will be scaled down by a factor of $1/N^2$ as compared to diagram (b). Although this is only an illustrative example, we will now argue that non-planar diagrams indeed will have subleading contribution as compared to their planar counterparts in the 't Hooft limit for generic diagrams.

Double line graphs and triangulation

Every Feynman diagram drawn in this 't Hooft double line notation can be thought of as a triangulation of a 2-dimensional surface Σ. There are two ways for the construction of such surfaces:

1. **Direct Surface:** This surface can be constructed by filling in the index loops with little plaquette like structures. By construction, this places that graphs on the manifold that is generated.

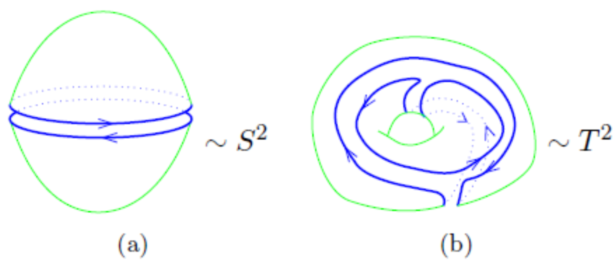

Figure 1.3: Direct surfaces for one-gluon loop and a non-planar graph. The former can be laid down upon S^2 while the latter on the torus T^2.

2. **Dual Surface:** This construction involves drawing a vertex inside every index loop and connecting all vertices through edges drawn across every

propagator. The loose end of the edges are identified with the point at infinity.

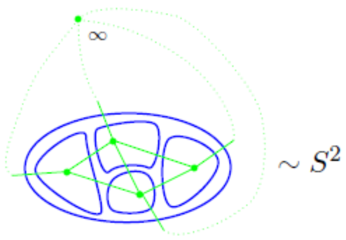

Figure 1.4: Dual surface for a planar diagram. Loose ends are identified with that point at ∞

The triangualtion effectively helps us in making the following identification:

Color loop $\Leftrightarrow$ Face
Propagators $\Leftrightarrow$ Edges
Interactions $\Leftrightarrow$ Vertices

Using λ and N to estimate vacuum to vacuum amplitude for a generic diagram with V vertices, E propagators (edges) and F loops (faces) gives:

$$\text{amplitude} \sim \left(\frac{\lambda}{N}\right)^E \left(\frac{N}{\lambda}\right)^V N^F = N^{V-E+F}\lambda^{E-V}$$

The quantity $V - E + F = \chi$, known as the *Euler charateristic* is a topological invariant of a manifold. For closed oriented surfaces, $\chi = 2 - 2g$ where g is the genus (the number of handles) of the surface. So, the perturbative expansion for the amplitude in the field theory may be written as a double expansion of the form

$$\sum_{g=0}^{\infty} N^{2-2g} \sum_{i=0}^{\infty} c_{gi}\lambda^i = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \tag{1.5}$$

where f_g is some polynomial in λ. In the large N limit, any computation will be dominated by surfaces with minimal genus, typically surfaces with topology of a sphere S^2, or equivalently of a plane (removing the point at infinity). Non-planar diagrams correspond to surfaces with handles and is supressed by a factor of $1/N^2$. The above arguments hold for any gauge theory coupled to adjoint matter fields, like the $\mathcal{N} = 4$ SYM.

The form of the expansion (1.5) is reminiscent of the topological expansion one finds with closed oriented strings, identifying $1/N$ as the string coupling constant. The analogy of (1.5) with perturbative string theory is one of the strongest motivations for believing that field theories and string theories are related, and it suggests that this relation would be more visible in the large N limit where the dual string theory may be weakly coupled. However, perturbative expansions typically do not converge and hence the identification is formal and far from a rigorous derivation. There are certainly instanton effects which are non-perturbative in the $1/N$ expansion, and an exact matching with string theory would require a matching of such effects with non-perturbative effects in string theory.

It turns out that the 't Hooft coupling λ acts like a kind of chemical potential for edges in our triangulation. So, if λ becomes large, then diagrams with lots of edges become significant. Hence, the triangualtions becomes more and more smoother. This is indicative of fewer quantum fluctuations on the world-sheet. We expect a relation of the form $\lambda^{-1} \sim \alpha'$ which is consistent with the intuition that large λ limit suppresses quantum fluctuations and takes us to the supergravity limit.

To check if the theory is still non-trivial in the large N limit, we should explore the beta function of the theory. The one-loop beta function equation for a pure $SU(N)$ YM theory is

$$\mu \frac{dg_{YM}}{d\mu} = -\frac{11}{3} N \frac{g_{YM}^3}{16\pi^2} + \mathcal{O}(g_{YM}^5)$$

In the large N limit, keeping $\lambda = g_{YM}^2 N$ fixed and defining $\beta_g = g_{YM}\lambda$, we can say,

$$\mu \partial_\mu \lambda = \beta_\lambda \sim \lambda^2$$

Thus, λ plays the role of the running coupling in the large N exhibiting a non-trivial RG flow.

1.3 Motivation for the AdS/CFT correspondence

As discussed earlier, besides fundamental strings, superstring theory also constains various non-perturbative solitonic higher dimensional objects called D-branes. Although these were first discovered as solutions to certain supergravity theories, Polchinski provided their interpretation within the framework of superstring theory. D-branes can be viewed from two different perspectives: the *open string* and the *closed string* perspective. Which perspective gives the correct description of the physics depends on the string coupling g_s, which controls interaction between open and closed strings.

1.3.1 D-branes

D-branes can be thought of as extended objects on which open strings end. We can choose to impose Neumann or Dirichlet boundary conditions on the string mode expansion. For an open string moving in D spacetime dimensions, we may choose to have Neumann boundary condition along $p+1$ spacetime dimensions while the other $D-p-1$ coordinates will respect the Dirichlet boundary condition. The end points of the string are free to move along $p+1$ spatial directions satisfying Neumann boundary condition and are fixed in the remaining directions. These can also be realized as solutions to the low-energy effective action of type IIB string theory. They have singularities extended in p space dimensions and are sometimes also denoted as Dp-branes. A Dp-brane may have a charge which is related to a $p+1$-form gauge potential of the supergravity approximation. These are generalized forms of the electric potential and in terms of the $p+1$ potential, we can write $F_{p+2} = dA_{p+1}$ where F_{p+2} is the *field strength*.

1.3.2 Open string perspective

Since, we treat the strings as small perturbations, this description is valid only when the coupling constant is small i.e. $g_s \ll 1$. We will also be in the limit where we can safely neglect massive excitations i.e $E \ll \alpha'^{-1/2}$. In this regime, the dynamics of open strings is described by a supersymmetric gauge theory living on the world volume of D-branes. For N coincident D-branes, the effective coupling is given by $g_s N$ and the open string perspective is valid when $g_s N \ll 1$.

We focus of type IIB superstring theory in flat (9+1) dimensional Minkowski background where N coincident $D3$-branes are also embedded. Since we are considering only massless excitations, we can neglect any strongy correction which appears at energies of order $\alpha'^{-1/2}$. The complete effective action for all massless string modes can be written as:

$$S = S_{closed} + S_{open} + S_{int} \tag{1.6}$$

where S_{closed} encodes the contribution of the closed string modes which can be schematically written as:

$$S_{closed} = \frac{s}{2\kappa^2} \int d^{10}x\sqrt{-g} \left(e^{-2\phi}(R + 4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) - \frac{1}{2}\sum_n \frac{1}{n!}F_n^2 \right) \tag{1.7}$$

where ϕ is the dilaton field. F_n are the Ramond-Ramond sector field strength, s is the signature e.g. -1 for Minkowski. The fermionic fields are not written out explicitly. Closed string modes can be thought of as fluctuations of the (9+1) dimensional Minkowski background. Expanding the metric around Minkowski background, $g = \eta + \kappa h$, where h is the perturbation, we see, (1.7) can be expanded as

$$S_{closed} \sim -\frac{1}{2}\int d^{10}x\partial_M h\partial^M h + \mathcal{O}(h) \tag{1.8}$$

Note here in the above expansion, the perturbed metric h is multiplied with κ to ensure canonical normalization in the kinetic term.

The action for the open strings and the interactions is captured by the DBI action which for a single $D3$-brane looks as

$$S_{DBI} = -\frac{1}{(2\pi)^3\alpha'^2 g_s}\int d^4x\ e^{-\phi}\sqrt{-\det(\mathcal{P}[g] + 2\pi\alpha'\mathcal{F}_{ab})} + \text{fermions} \tag{1.9}$$

Expanding $e^{-\phi}$ and $g = \eta + \kappa h$, we find that upto leading order in α',

$$S_{brane} = -\frac{1}{2\pi g_s}\int d^4x \left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^i + \mathcal{O}(\alpha') \right) \tag{1.10}$$

$$S_{int} = -\frac{1}{8\pi g_s}\int d^4x\ \phi F_{\mu\nu}F^{\mu\nu} \tag{1.11}$$

For N coincident $D3$-branes, the scalars and gauge fields are $U(N)$ valued i.e. $\phi^i = \phi^{ia}T_a$ and $A_\mu = A_\mu^a T_a$ (T_a are the generators of the gauge group) and hence to ensure gauge invariance we are required to trace over the gauge group. Thus, the gauge kinetic term in (1.10) will take the form $F_{\mu\nu}^a F^{a\mu\nu}$. The partial derivatives also requires to be replaced by covariant derivative and we need to add a scalar potential of the form

$$V = \frac{1}{2\pi g_s}\sum_{i,j}\text{Tr}[\phi^i, \phi^j]^2$$

to the action S_{brane} to lowest order in α'.

Taking the limit $\alpha' \to 0$ we find that S_{brane} is just the bosonic part of the action of $\mathcal{N} = 4$ SYM theory provided we make the identification $2\pi g_s = g_{YM}^2$. All other terms in S_{brane} are of $O(\alpha')$ or higher and hence vanishes in the limit when $\alpha' \to 0$ or in other words, the open and closed strings decouple. Similar to (1.8), in writing (1.11), we need to rescale the dilaton field ϕ by a factor of κ for canonical normalization. Thus, S_{int} is of order κ and vanishes for $\alpha' \to 0$.

To summarize, looking at a stack on N coincident D-branes when $g_s N \ll 1$ and considering their behaviour in the $\alpha' \to 0$ limit, one can observe that open and closed string modes decouple from each other. The dynamics of the open string modes in described by $\mathcal{N} = 4$ SYM theory while the closed strings are described by supergravity in flat $(9+1)$ dimensional spacetime. Thus we can schematically decompose (1.6) as

$$\begin{aligned} &S \xrightarrow{\alpha' \to 0} S_1 + S_2 \\ &S_1 : \mathcal{N} = 4 \text{ SYM theory with a gauge group } SU(N) \\ &S_2 : 10 \text{ -dimensional supergravity in flat spacetime} \end{aligned} \tag{1.12}$$

If we had started with $N+1$ $D3$ branes in flat $(9+1)$-dimensional spacetime and separated one of the branes from the other N coincident branes (say, along the direction x^9), the massless modes would not be described by a $U(N+1)$ gauge theory. Instead it will have a gauge group given by $U(N) \times U(1)$. If the separated brane is located at $x^9 = r$, we may think of the system as being in a Higgs phase with the vacuum expectation value of the Higgs scalar being $\langle \phi^9 \rangle = r/2\pi\alpha'$. We need to be cautious while taking the decoupling limit i.e. $\alpha' \to 0$ since we should keep all the field theory quantities fixed. Therefore, the correct decoupling limit, also known as the *Maldacena limit* is given by

$$\alpha' \to 0 \quad \text{such that} \quad u \equiv \frac{r}{\alpha'} \quad \text{is fixed} \tag{1.13}$$

where r is any distance.

1.3.3 Closed string perspective

In this picture, we will interchange the two limits i.e. the strong coupling and low-energy limits. We consider a stanck of N coincident D-branes in the strongly coupled limit when $g_s N \to \infty$ where the closed string perspective is relevant. The N $D3$-branes can be viewed as massive charged objects which sources various fields of Type IIB supergravity. We briefly review the solution in the following section.

Type IIB supergravity theory

Our starting point is the action of type IIB supergravity theory. A consistent truncation of the full action is given by the bosonic part as written in (1.7) while omitting terms coming from the NS-NS sector of the field strength and other fermionic fields. A conformal transformation of the form

$$g_{\mu\nu} \to e^{-\frac{1}{2}\phi} g_{\mu\nu}$$

takes the action to the *Einstein frame.*

$$S = -\frac{s}{2\kappa_D^2} \int d^D x \sqrt{-g} \left[R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \sum_n \frac{1}{n!} e^{a_n \phi} F_n^2 \right] \; ; \quad s = \pm 1 \quad (1.14)$$

The equations of motion following from the above action are:

$$\begin{aligned} R^\mu_\nu &= \frac{1}{2} \partial^\mu \phi \partial_\nu \phi + \frac{1}{2n!} e^{a_n \phi} \left(n F^{\mu \mu_2 \mu_3 .. \mu_n} F_{\nu \mu_2 .. \mu_n} - \delta^\mu_\nu \frac{n-1}{D-2} F_n^2 \right) \\ \partial_\mu (\sqrt{-g} & e^{a_n \phi} F^{\mu \nu_1 \nu_2 .. \nu_n}) = 0 \end{aligned} \quad (1.15)$$

The solution must have Poincaré symmetry in $p + 1$-dimension while enjoying spherical symmetry in the other $D - p - 1$ dimension. Imposing the above symmetry considerations, for 10 dimensional SUGRA, the resulting metric is

$$ds^2 = -\frac{f_+(\rho)}{\sqrt{f_-(\rho)}} dt^2 + \sqrt{f_-(\rho)} \sum_{i=1}^{p} dx^i dx^i + \frac{f_-(\rho)^{-\frac{1}{2} - \frac{5-p}{7-p}}}{f_+(\rho)} d\rho^2 + \rho^2 f_-(\rho)^{\frac{1}{2} - \frac{5-p}{7-p}} d\Omega^2_{8-p} \quad (1.16)$$

while the dilaton field is

$$e^{-2\phi} = g_s^{-2} f_-(\rho)^{-\frac{p-3}{2}} \; ; \qquad f_\pm(\rho) = 1 - \left(\frac{r_\pm}{\rho} \right)^{7-p} \quad (1.17)$$

r_+ is the horizon in the Einstein frame metric and r_- is a curvature singularity, which exists for $p \leq 6$. When $r_+ > r_-$, the singularity is covered by the horizon and the solution can be regarded as a black hole. In the case when $r_+ < r_-$, there is a timelike naked singularity which seems unphysical. Imposing the existence of an event horizon translates to a bound on the mass of the black brane i.e.

$$M \geq \frac{N}{(2\pi)^p g_s l_s^{p+1}} \quad (1.18)$$

Solutions saturating this inequality are dubbed *extremal Dp-branes* which also satisfies $r_+ = r_-$. The metric of an extremal Dp-brane is given by

$$ds^2 = \sqrt{f_+(\rho)}\left(-dt^2 + \sum_{i=1}^{p} dx^i dx^i\right) + f_+(\rho)^{-\frac{3}{2}-\frac{5-p}{7-p}} d\rho^2 + \rho^2 f_+(\rho)^{\frac{1}{2}-\frac{5-p}{7-p}} d\Omega_{8-p}^2$$

Outside the horizon, we define a new set of radial and isotropic coordinates $r^{7-p} \equiv \rho^{7-p} - r_+^{7-p}$ and $r^a = r\theta^a$. Finally, the extremal p-brane takes the form:

$$ds^2 = \frac{1}{\sqrt{H(r)}}\left(-dt^2 + \sum_{i=1}^{p} dx^i dx^i\right) + \sqrt{H(r)} \sum_{a=1}^{9-p} dr^a dr^a \tag{1.19}$$

where

$$\begin{aligned} e^{\phi} = g_s H(r)^{\frac{3-p}{4}} \ ; \quad H(r) = \frac{1}{f_+(\rho)} = 1 + \frac{r_+^{7-p}}{r^{7-p}} \ , \\ r_+^{7-p} = d_p g_s N l_s^{7-p} \ ; \quad d_p = 2^{5-p}\pi^{\frac{5-p}{2}}\Gamma\left(\frac{7-p}{2}\right) \end{aligned} \tag{1.20}$$

As can be seen from the above solutions, these are generalization of rotating black holes. Flux of the Ramond-Ramond charge on a $(8-p)$-sphere is quantized

$$\int_{S^{8-p}} *F_{p+2} = N \ ; \quad N \in \mathbb{Z}$$

Also, the tension of the brane is identified with the energy per unit volume in the p-spacelike directions.

The metric (1.19) can be generalized to write multi-centered solutions which physically represent parallel extremal Dp-branes at k different locations $\mathbf{r} = \mathbf{r}_i$ by writing $H(r)$ as

$$H(r) = 1 + \sum_{i=1}^{k} \frac{r_{(i)+}^{7-p}}{|r - r_i|^{7-p}} \qquad r_{(i)+}^{7-p} = d_p g_s N_i l_s^{7-p}$$

Each brane carries N_i units of charge.

So far we have discussed the black Dp-brane in the context of classical supergravity. This description is appropriate when we can safely neglect stringy corrections or in other words, the curvature of the p-brane geometry is small compared to the string scale i.e. $r_+ \gg l_s$. The effective string coupling e^{ϕ} is also small. In particular, for $p = 3$, the dilaton is constant and can be consistently made small everywhere in the limit $g_s < 1 \ \Rightarrow \ l_P < l_s$. To summarize, for $D3$-branes, the supergravity approximation is a good physical description when $l_P < l_s \ll r_+$. Since, r_+ is related to the Ramond-Ramond charge N as $r_+^{7-p} = d_p g_s N l_s^{7-p}$, we can rewrite the regime of validity of solution (1.19) to

$$1 \ll g_s N < N \tag{1.21}$$

$D3$-branes

Focussing on the particular case of $D3$-brane i.e. where $p = 3$, (1.19) which boild down to

$$\begin{aligned} ds^2 &= H(r)^{-1/2} dx_\mu dx^\mu + H(r)^{1/2}(dr^2 + r^2 d\Omega^2) && (1.22)\\ e^\phi &= g_s && (1.23)\\ H(r) &= 1 + \frac{R^4}{r^4} && (1.24) \end{aligned}$$

where $R^4 = 4\pi g_s N \alpha'^2$. The world volume of the brane has a 4-dimensional Poincare symmetry $\mathbb{R}^4 \times SO(1,3)$. The dilaton and axion fields are constant while the geometry is regular at $r = 0$.

Even when N is large, we can demand $g_s \ll 1$, so that perturbative methods can be applied. We can study the following geometry is two regimes:

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} dx_\mu dx^\mu + \left(1 + \frac{R^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2) \tag{1.25}$$

- $r \gg R$: In this case, when studying the geometry far from the brane, we see that the deformation vanishes and we recover flat 10-dimensional space.
- $r \ll R$: In this case, the metric near the "throat" region in the deep IR is given by
$$ds^2 \simeq \frac{r^2}{R^2} dx_\mu dx^\mu + R^2 \frac{dr^2}{r^2} + r^2 d\Omega_5^2 \tag{1.26}$$
The geometry of the above is essentially: $AdS_5 \times S^5$.

Like before, as given in (1.13), we must approach the Maldacena limit in a consistent way such that physical observables are kept finite. Loosely speaking, this limit "zooms" into the near-horizon region of the brane solution. In terms of the newly defined coordinate $u \equiv r/\alpha'$, (1.26) in the near horizon limit takes the form:

$$ds^2 \simeq \alpha' \left[\frac{u^2}{\sqrt{4\pi g_s N}} dx_\mu dx^\mu + \sqrt{4\pi g_s N} \frac{du^2}{u^2} + \sqrt{4\pi g_s N} d\Omega_5^2 \right]$$

The fact that the spacetime $AdS_5 \times S^5$ deformation caused by the branes decouples from the 10-dimensional flat bulk, as $\alpha' \to 0$, can be seen from the function $H(u)$, which we now write as:

$$H(u) = 1 + \frac{4\pi g_s N}{\alpha'^2 u^4} \tag{1.27}$$

It should be noted that $\alpha' \to 0$ is not the point particle limit. Rather in this limit, the open strings becomes smaller thus bringing the $D3$-branes closer. Hence, this limit should just be considered as the process to obtain two decoupled emergent theories

$$\begin{aligned} &S \xrightarrow{\alpha' \to 0} S_1 + S_2 \\ &S_1 : \text{Type IIB supergravity on } AdS_5 \times S^5 \\ &S_2 : 10 \text{ -dimensional supergravity in flat spacetime} \end{aligned} \tag{1.28}$$

1.3.4 Combining both perspectives

Over the last couple of sections i.e. 1.3.2 and 1.3.3, in both the pictures we have found two decoupled effective theories in the low-energy limits as given in (1.12) and (1.28). Both these perspectives must be equivalent descriptions of the same physics while in the low-energy limit both has 10-dimensional supergravity over a flat background. This is suggestive of the fact that the other two theories must be equivalent. This is essentially what motivated Maldacena to conjecture that $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions in the large N limit is equivalent to type IIB supergravity on $AdS_5 \times S^5$. Relaxing the low-energy limit, we can make this statement stronger eventually arriving at the statement given in 1.1 which we restate again:

$\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory with gauge group $SU(N)$ and a Yang-Mills coupling constant g_{YM} is equivalent to type IIB superstring theory on $AdS_5 \times S^5$ with parameters of the two sides of the duality matched appropriately although the fundamental degrees of freedom differ significantly.

Comparing symmetries

As a basic check in support of this conjecture it is worthwhile to compare the symmetries of the two theories. $\mathcal{N} = 4$ Super Yang-Mills theory has conformal symmetry with vanishing β-function. In four dimensions, the conformal group is infact $SO(2,4)$. This theory also has an R-symmetry of $SO(6)$ which rotates 6 scalars into each other. This theory also preserves $\mathcal{N} = 4$ SUSY with 16 Poincaré supercharges grouped as four spinors (Q^a_α) and 16 superconformal supercharges (S^a_α). All the symmetries form the supergroup $PSU(2,2|4)$.

String theory on $AdS_5 \times S^5$ will be invariant under the isometry groups of AdS_5 and S^5 which are given by $SO(2,4)$ and $SO(6)$ respectively. This matches with the bosonic sector of the algebra of $PSU(2,2|4)$. It can be shown that the symmetries in the fermionic sector can also be mapped to the string theory. Thus, overall we can state that the global symmetries of type IIB string theory on $AdS_5 \times S^5$ is identical to the global symmetries of $\mathcal{N} = 4$ SYM theory in flat spacetime.

1.4 Bulk-Boundary Correspondence

Although the theory of quantized strings on AdS_5 background is still not well-understood, we can still use the AdS/CFT correspondence as a calculational toolbox to understand strongly coupled regime of gauge theories. Observables evaluated in a non-perturbative regime on the field theory side can essentially be mapped to certain quantities in a dual theory of classical gravity where computations are often tractable. However, before delving into any concrete calculation is it essential to understand the scheme of mapping between the two theories. The fields $\phi(r, x_\mu)$, living in the AdS_5 bulk is associated with a single trace operator $\mathcal{O}$, that belongs to the spectrum of the 4-dimensional SYM theory. There is a relation between the mass m of the field and the scale dimension Δ of the operator. We can view the value of the bulk field on the boundary as a source for the corresponding operator, producing its correlation functions. More precisely, we add in the field theory Lagrangian an interaction term between the operator and the boundary field, identified with the generating functional for connected correlation functions of $\mathcal{O}$. The precise mathematical relation stating the map of the AdS/CFT correspondence goes by the name of *GKPW relation* [14, 15] (named after Gubser, Klebanov, Ployakov and Witten).

$$e^{-W_{CFT}[\phi^{(0)}]} = \langle e^{-\int d^4x \phi^{(0)}(x)\mathcal{O}(x)} \rangle_{CFT} = Z[\phi(x,0) = \phi^{(0)}(x)]_{String} \tag{1.29}$$

where $r = 0$ is the boundary of AdS_5 in Poincaré patch. Since the last quantity on the RHS is not computationally tractable, a more practical version of the GKPW formula emerges in the classical limit:

$$W_{CFT}[\phi^{(0)}] = -\log\langle e^{-\int \phi^{(0)}\mathcal{O}}\rangle_{CFT} \approx \text{Extremum}_{\phi|_{r=\epsilon}\sim\phi^{(0)}}(N^2 I_{grav}) + O\left(\frac{1}{N^2}\right) + \cdots \tag{1.30}$$

It is evident from the above expansion that the gravity description is valid for large N and λ. I_{grav} is some dimensionless action. Thinking of $\phi^{(0)}$ as a small perturbation, we may write the following expansion

$$\begin{aligned} W_{CFT}[\phi^{(0)}] = W_{CFT}[0] + &\int d^D x \; \phi_0(x) G_1(x) \\ &+ \frac{1}{2}\int\int d^D x_1 \; d^D x_2 \; \phi^{(0)}(x_1) G(x_1, x_2) \phi^{(0)}(x_2) + \cdots \end{aligned} \tag{1.31}$$

where

$$\begin{aligned} G_1(x) = \langle \mathcal{O}(x)\rangle &= \frac{\delta W_{CFT}[\phi^{(0)}]}{\delta\phi^{(0)}(x)}\bigg|_{\phi^{(0)}=0} \\ G_2(x_1, x_2) = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_c &= \frac{\delta^2 W_{CFT}[\phi^{(0)}]}{\delta\phi^{(0)}(x_1)\delta\phi^{(0)}(x_2)}\bigg|_{\phi^{(0)}=0} \end{aligned} \tag{1.32}$$

Anticipating divergences near the boundary as $r \to 0$, we have introduced a cut-off in (1.30) and set bondary conditions there. Although (1.30) is written as if there is just one bulk field, in principle there exists a ϕ for every operator $\mathcal{O}$ in the dual field theory. In literature this is referred to as 'ϕ being coupled to $\mathcal{O}$'. The general n-point connected correlation function of the CFT is given by

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle = \frac{\delta}{\delta\phi_1^{(0)}(x_1)}\cdots\frac{\delta}{\delta\phi_n^{(0)}(x_n)} W_{CFT}|_{\phi_i^{(0)}=0} \tag{1.33}$$

where the operators $\mathcal{O}_i$ correspondes to *different* bulk fields ϕ_i. Since the sources are set to zero after the functional differentiation, interaction terms in the action with more than n fields will not contribute to the computation of the n-point function. The field $\phi^{(0)}(x)$ act as sources for the primary operators O_i which specify the spectrum of the CFT. There are a few simple examples of this correspondence e.g. local changes in the metric corresponds to change on the stress tensor of the boundary theory. The boundary action has a term of the form $\gamma_{\mu\nu}T^{\mu\nu}$ where $\gamma_{\mu\nu}$ is the induced metric on the boundary while $T^{\mu\nu}$ is the stress tensor; gauge fields A_μ in the bulk corresponds to currents J^μ on the boundary. The prescription (1.33) can be used in principal to compute arbitrary correlation functions given the classical bulk action. The reader can refer to [16] for a classic comprehensive review of the AdS/CFT correspondence. In the next two subsections, we will illustrate this technique in the context of free and massive scalar fields to calculate their two point correlation function. For a comprehensive review of correlator calculation from the gravity side, the reader can refer to [14, 15].

1.4.1 Correlation functions: A position space analysis

We consider a simple case of a probe massless scalar field $\phi(x)$ in AdS_{d+1} background. The equation of motion for this field is

$$\nabla_\mu \nabla^\mu \phi(x) = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \phi(x)) = 0$$

where the covariant derivative is with respect to the AdS metric

$$ds^2 = \frac{1}{r^2}(dr^2 + d\mathbf{x}^2)$$

The propagator $K(r)$ owing to translation invariance must be independent of the boundary coordinates ($\mathbf{x} \equiv x_1, ..., x_d$) the follows the equation

$$\frac{d}{dr}\left(\frac{1}{r^{d-2}}\frac{d}{dr}K(r)\right) = 0$$

Assuming a power-law ansatz $K(r) = cr^\alpha$ results in the constraint $\alpha(\alpha - d + 1) = 0$ with two possible solutions for α i.e. $\alpha = 0, d-1$. The constant solution is the non-normalizable mode while the other solution

$$K(r) = cr^{d-1} \tag{1.34}$$

vanishes at $r = 0$ but diverges at $r = \infty$ indicating singular behaviour. The singularity is infact a delta function which can be shown by mapping the point at infinity to the origin. This can be achieved through a $SO(1, d+1)$ transformation:

$$x^\mu \to \frac{x^\mu}{r^2 + \mathbf{x}^2}, \quad \mu = r, \mathbf{x} \tag{1.35}$$

This modifies the expression (1.34) to

$$K(\mathbf{x}, r) = c\frac{r^{d-1}}{(r^2 + \mathbf{x}^2)^{d-1}}$$

The integral over $\mathbf{x}$ is independent of r, since a simple scaling of $\mathbf{x}$ gives

$$\int d^{d-1}\mathbf{x} \frac{r^{d-1}}{(r^2 + \mathbf{x}^2)^{d-1}} \xrightarrow{\mathbf{x} \to r\mathbf{x}'} \int d^{d-1}\mathbf{x}' r^{d-1} \frac{r^{d-1}}{(r^2 + r^2\mathbf{x}'^2)^{d-1}} = \int \frac{d^{d-1}\mathbf{x}'}{(1 + \mathbf{x}'^2)^{d-1}}$$

Moreover, as $r \to 0$, K vanishes except at $x_1 = x_2 = ... = x_d = 0$. If now, $\phi^{(0)}(\mathbf{x})$ is the restriction of the bulk field $\phi(x^\mu)$ on the boundary, we can write the classical solution in AdS_{d+1} as

$$\phi(x) = c \int d\mathbf{y} \frac{r^{d-1}}{(r^2 + (\mathbf{x}-\mathbf{y})^2)^{d-1}} \phi^{(0)}(\mathbf{y}) \tag{1.36}$$

The n-point connected correlation function for this theory can be written down from the on-shell action as:

$$I[\phi^{(0)}] = -\frac{1}{2} \int d^{d+1}x \sqrt{-g} \partial_\mu \phi \partial^\mu \phi = -\frac{1}{2} \int d^{d+1}x \{\partial_\mu(\sqrt{-g}\phi \partial^\mu \phi) - \phi \partial_\mu(\sqrt{-g}\partial^\mu \phi)\}$$

The last term vanishes by virtue of the equation of motion while the other term can now be written as a boundary integral. To supress divergences, we evaluate this integral at the $z = \epsilon$ close to the boundary which can be interpreted as the UV cutoff on the CFT side. The on-shell action simplifies to

$$I[\phi^{(0)}] = \lim_{r \to \epsilon} -\frac{1}{2} \int d\mathbf{x}\ r^{-d+2} \phi(x) \partial_r \phi(x) \tag{1.37}$$

Close to the boundary, from (1.36) we can write

$$\lim_{r \to \epsilon} \partial_r \phi(x) = c(d-1) r^{d-2} \int d^{d-1}\mathbf{y} \frac{1}{|\mathbf{x}-\mathbf{y}|^{2d-2}} \phi^{(0)}(\mathbf{y}) \tag{1.38}$$

Using (1.36) and (1.38), we can see that the on-shell boundary action takes the form

$$I[\phi^{(0)}] = -\frac{c(d-1)}{2} \int d^{d-1}\mathbf{x}\ d^{d-1}\mathbf{y}\ \frac{\phi^{(0)}(\mathbf{x})\phi^{(0)}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{2d-2}} \tag{1.39}$$

The above expression eventually leads to the two point correlation function.

$$\begin{aligned} \langle \mathcal{O}(\mathbf{x}') \mathcal{O}(\mathbf{y}') \rangle &= \left[\frac{\delta}{\delta \phi^{(0)}(\mathbf{x}')} \frac{\delta}{\delta \phi^{(0)}(\mathbf{y}')} I[\phi^{(0)}] \right] \Bigg|_{\phi^{(0)}=0} \\ &= \frac{c(d-1)}{|\mathbf{x}' - \mathbf{y}'|^{2(d-1)}} \end{aligned}$$

This prescription thus reproduces the correct expected scaling of the 2-point correlation function for a conformal operator with scaling dimension $d-1$.

1.4.2 Correlation functions: A momentum space analysis

The AdS_{d+1} metric written in the Poincaré patch is given by

$$ds^2 = g_{AB}dx^A dx^B = R^2 \frac{dr^2 + dx^\mu dx_\mu}{r^2} \qquad A = 0, .., d \quad x^A = (r, x^\mu) \tag{1.40}$$

where R is the radius of curvature of the AdS space. We consider a massive probe scalar field in the bulk governed by the action:

$$S = -\frac{k}{2} \int d^{d+1}x \sqrt{-g}[g^{AB}\partial_A\phi\partial_B\phi + m^2\phi^2] \tag{1.41}$$

k is just a normalization constant. Clearly, the constant r-slices are Minkowski like and $\sqrt{-g} = (R/r)^{d+1}$. The equation of motion for the probe scalar is

$$(\Box - m^2)\phi = 0 \tag{1.42}$$

where $\Box$ is the Laplacian with respect to the metric (1.40) and is given by $\Box\phi = \nabla^A\nabla_A\phi = \frac{1}{\sqrt{-g}}\partial_A(\sqrt{-g}g^{AB}\partial_B)\phi$. Integrating (1.41) by parts, and using (1.42), the on-shell boundary action takes the form

$$S_{bdy} = \lim_{r\to\epsilon} -\frac{k}{2} \int_{\partial AdS} d^d x \sqrt{-g} g^{rB}\phi\partial_B\phi \tag{1.43}$$

where the d-dimensional measure runs over the boundary coordinates.

Taking advantage of translational invariance along the boundary coordinates we can decompose the bulk fields in terms of Fourier space variables as

$$\phi(r, x) = \int d^d k \; e^{ik_\mu x^\mu} f_k(r) \quad , \quad k_\mu x^\mu \equiv -\omega t + \mathbf{k}\cdot\mathbf{x}$$

$f_k(r)$ is governed by a wave equation of the form

$$[r^2k^2 - r^{d+1}\partial_r(r^{-d+1}\partial_r) + m^2R^2]f_k(r) = 0 \; ; \quad k^2 = -\omega^2 + \mathbf{k}\cdot\mathbf{k} \tag{1.44}$$

Assuming a scaling of the form $f_k = r^\Delta$, leads to the relation $(k^2r^2 - \Delta(\Delta - d) + m^2R^2)r^\Delta = 0$ which in the near-boundary region boils down to

$$\Delta(\Delta - d) = m^2R^2 \tag{1.45}$$

The above equation has the following two roots:

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2R^2} \; ; \qquad \Delta_+ + \Delta_- = d \tag{1.46}$$

Notice that this result is much stronger than the bound obtained from unitarity i.e. $\Delta_\pm \geq \frac{d-2}{2}$. Demanding reality of $\Delta_\pm$ translates to the condition

$$m^2R^2 \geq -\frac{d^2}{4} \tag{1.47}$$

This is known as the *Breitenlohner-Freedman* bound. Unlike flat space, in *AdS*, massive scalar fields are stable provided it is restricted by the above bound. It is clear from the fall-offs that near the boundary, we can write the solutions as

$$\phi(r,x) = \phi^{(0)}(x)r^{\Delta_-} + \phi^{(+)}(x)r^{\Delta_+} + \cdots \tag{1.48}$$

where the dots represent subleading terms in r. By definition, $\Delta_+ \geq \Delta_-$ and hence r^{Δ_-} is bigger as $r \to 0$ representing the non-normalizable mode while the second term captures the normalizable mode. More formally, we define the source at the boundary CFT as

$$\phi^{(0)}(x) \equiv \lim_{r\to 0} r^{-\Delta_-}\phi(r,x) = \lim_{r\to 0} r^{\Delta_+ - d}\phi(r,x) \tag{1.49}$$

The normalizable *AdS* mode i.e. $\phi^{(+)}$ can be thought of as the vacuum expectation value for a dual scalar field theory operator $\mathcal{O}$ of dimension $\Delta \equiv \Delta_+$ while the non-normalisable mode $\phi^{(0)}$ acts as the source.

Going back to (1.44), we can solve for $f_k(r)$ exactly. When k^μ is spacelike i.e. $k^2 > 0$, or in other words, in the context of Euclidean *AdS*, the solution is

$$f_k(r) = A_K r^{d/2} K_\nu(kr) + A_I r^{d/2} I_\nu(kr) \; ; \; \nu = \Delta - \frac{d}{2} \; ; \; k = \sqrt{k^2} \tag{1.50}$$

where $K_\nu(.)$ and $I_\nu(.)$ are modified Bessel functions. For a Lorentzian theory with timelike k^2 for on-shell states with $\omega^2 > \mathbf{k}^2$, we get two linearly independent solutions that have the same leading behaviour at the UV boundary:

$$r^{d/2}K_{\pm\nu}(iqr) \sim e^{\pm iqr} \; ; \quad q \equiv \sqrt{\omega^2 - \mathbf{k}^2} \tag{1.51}$$

In the deep IR region ($r \to \infty$) these modes exhibit oscillatory behaviour. This is a reflection of the ambiguity that arises in defining real-time Green's function for the CFT. The solution consistent with causality is the one which falls into the black-hole horizon i.e. the *infalling* condition–these solutions travel towards $r \to infty$ with time.

For (1.50), in the AdS bulk as $r \to \infty$, they behave as follows:

$$K(kr) \to e^{-kr} \quad ; \quad I(kr) \to e^{kr} \tag{1.52}$$

To ensure regularity in the interior we must set $A_I = 0$. Further the normalization condition $f_k(r = \epsilon) = 1$ gives the bulk-to-boundary propagator:

$$f_k(r) = \frac{r^{d/2} K_\nu(kr)}{\epsilon^{d/2} K_\nu(k\epsilon)} \tag{1.53}$$

Plugging this back in the boundary term for the action, we obtain

$$S[\phi] = -\frac{kR^{d-1}}{2} \int d^d k \phi_0(k,\epsilon)\phi_0(-k,\epsilon)\mathcal{F}_\epsilon(k) \tag{1.54}$$

Now, we use our GKPW Equation and get

$$\langle \mathcal{O}(k_1)\mathcal{O}(k_2)\rangle = -\frac{\delta}{\delta\phi_0(k_1)}\frac{\delta}{\delta\phi_0(k_2)} S = (2\pi)^d \delta^d(k_1 + k_2)\mathcal{F}_\epsilon(k_1) \tag{1.55}$$

where the quantity $\mathcal{F}_\epsilon$, also sometimes called the flux factor is given by

$$\mathcal{F}_\epsilon(k) = 2\epsilon^{-d+1}\partial_r \left(\frac{r^{d/2} K_\nu(kr)}{\epsilon^{d/2} K_\nu(k\epsilon)} \right)\Bigg|_{r=\epsilon} \tag{1.56}$$

The small value of the argument i.e. in the near boundary region, the behaviour of the modified Bessel function $K_\nu(.)$ is given by

$$K_\nu(u) = u^{-\nu}(a_0 + a_1 u^2 + ..) + u^\nu \ln u (b_0 + b_1 u^2 + ..) \tag{1.57}$$

The second term exists only for integer ν and is subleading compared to the first term. The coefficients a_i and b_i depend on ν and can be determined by imposing the correct boundary conditions as $r \to \infty$. Plugging (1.57) back in (1.56) and simplifying we get,

$$\begin{aligned} \mathcal{F}_\epsilon(k) = 2\epsilon^{-d} \Bigg[& \left\{ \frac{d}{2} - \nu(1 + c_2(\epsilon^2 k^2) + c_4(\epsilon^4 k^4) + \cdots \right\} \\ & + \left\{ \nu \frac{2b_0}{a_0}(\epsilon k)^{2\nu} \ln(\epsilon k)(1 + d_2(\epsilon k)^2 + \cdots \right\} \Bigg] \\ = & \ (\text{Analytic piece in } k) + (\text{Non-analytic piece in } k) \end{aligned} \tag{1.58}$$

The first line in the above expression is a Laurent series in ϵ containing only positive powers of k and is thus analytic in k at $k = 0$. These are contact terms and can be safely subtracted off. We can see this by noting

$$\int d^d k \; e^{-ikx} (\epsilon k)^{2m} \epsilon^{-d} = \epsilon^{2m-d} \Box_x^m \delta^d(x) \tag{1.59}$$

for $m > 0$. The ϵ^{2m-d} factor reinforces the notion that ϵ, which is an IR cutoff in AdS, is a UV cutoff for the QFT. The interesting bit of $\mathcal{F}_\epsilon(k)$ which gives the $x_1 \neq x_2$ behaviour of the correlator (1.55) is non-analytic in k and is encoded by the second line of (1.58). The non-analytic piece can be simplified further to give

$$\begin{aligned} \text{Non-analytic piece in } k &= -2\nu \frac{b_0}{a_0} k^{2\nu} \log(k\epsilon) \epsilon^{2\nu-d} (1 + \mathcal{O}(\epsilon^2)) \;, \\ \text{where} \quad \frac{b_0}{a_0} &= \frac{(-1)^{\nu-1}}{2^{2\nu} \nu \Gamma(\nu)^2} \quad \text{for} \quad \nu \in \mathbb{Z} \end{aligned} \tag{1.60}$$

The above result can be written in position space by performing a Fourier transform

$$\int d^d k \; e^{ikx} (\text{Non-analytic piece in } k) = \frac{2\nu \Gamma(\Delta_+)}{\pi^{d/2} \Gamma(\Delta_+ - d/2)} \frac{1}{x^{2\Delta_+}} \epsilon^{2\nu-d}$$

which is indeed the expected scaling for the 2-point correlation function of a CFT.

Looking carefully at the above expression, we can see there exists a dependence on the cut-off ϵ which can be taken care by defining renormalized field. It is to be noted that a naive application of (1.55) will not yield the two point correlator with the correct normalization and will not be consistent with the Ward identities. There infact exists a systematic procedure [17] called *holographic renormalization* where one adds counterterms to the bare action in order to get correlators that are manifestly well defined in the UV regime.

1.5 Non-relativistic holography and hyperscaling violation

Following the proposal of Maldacena, there was a flurry of activity focussing on concrete examples of AdS gravity and its conformal field theory dual. However, the class of metrics of interest in gauge/gravity duality has been considerably enlarged in recent years. As a simple generalization, we can consider metrics which are dual to field theories which are scale invariant but not conformally invariant. Such scaling properties are exhibited by interesting condensed matter systems which are strongly coupled. Thus it appears natural to use gauge/gravity duality in order to study these kind of systems as the gravity side is tractable. Of course the microscopic degrees of freedom of a condensed matter system is very different from that of a non-Abelian gauge theory at large N and is inherently non-relativistic in nature. Nevertheless, the idea is to make use of universality and consider theories at renormalization group fixed points where the microscopic details may not be important. These non-relativistic systems typically have reduced symmetries compared to AdS space theories. An interesting class of theories exhibits Lifshitz scaling symmetry of the form $t \to \lambda^z t$, $x_i \to \lambda x_i$, $r \to \lambda r$, with z known as the Lifshitz exponent and r is the radial coordinate of the gravity dual given by the metric

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dr^2 + dx_i^2}{r^2} \tag{1.61}$$

They can be realised as solutions to simple gravity theories with a negative cosmological constant coupled to appropriate matter and various constructions in string theory [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. The interested reader can see the reviews [36, 37] for a comprehensive discussion on holographic aspects of Lifshitz field theories. A simple subclass [20, 21] of such constructions involves the dimensional reduction of null deformations of $AdS \times X$ spacetimes that arise in familiar brane constructions: for instance the $AdS_5 \times X^5$ null deformation is of the form

$$ds^2 = \frac{1}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + g_{++}(dx^+)^2 + d\Omega_5^2 \tag{1.62}$$

where g_{++} is sourced by one or more fields. The long wavelength geometry upon a x^+ dimensional reduction resembles a $z = 2, d = 3 + 1$ dimensional Lifshitz spacetime, which are dual to a $2 + 1$ dimensional field theory. There exists a vast literature reviewing various holographic aspects of quantum field theories exhibiting Lifshitz scaling symmetry: [38, 39, 40, 41, 42, 43, 44].

With an abelian gauge field and a scalar dilaton one can engineer a larger class of metrics [5, 6, 7, 8, 9, 10, 11] known as *hyperscaling violating Lifshitz spacetimes* (hvLif) which are given by

$$ds^2_{d_i+2} = r^{\frac{2\theta}{d_i}} \left(-\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \sum_{i=1}^{d_i} \frac{dx_i^2}{r^2} \right) \tag{1.63}$$

Comparing with (1.62), we clearly see that this is conformal to Lifshitz spacetimes. Here d_i is the boundary spatial dimension (i.e. the dimension of x_i) and θ is the hyperscaling violating exponent. These spacetime exhibit the scaling

$$t \to \lambda^z t, \;\; x_i \to \lambda x_i, \;\; r \to \lambda r, \;\; ds \to \lambda^{\theta/d_i} ds \tag{1.64}$$

Various aspects of holography and entanglement entropy for hvLif spacetimes have been discussed in [18, 45, 46]. Further, in recent times there has been studies towards exploring various hydrodynamic transport coefficient of hvLif theories [47, 48, 49, 50].

A basic requirement to obtain physically sensible dual field theories to the above metric is the null energy condition $T_{\mu\nu}n^\mu n^\nu \geq 0$ where $n^\mu n_\mu \geq 0$ which along with Einstein's equations leads to the inequalities

$$(d_i - \theta)(d_i(z-1) - \theta) \geq 0 \; , \quad (z-1)(d_i + z - \theta) \geq 0 \; . \tag{1.65}$$

Roughly speaking, in a theory with hyperscaling violation, the thermodynamic behaviour is as if the theory enjoyed dynamical exponent z but lived in $d_i - \theta$ spacetime dimensions. In particular the gravitational description of the case $\theta = d_i - 1$ is interesting and is a promising candidate for field theories with hidden Fermi surface. For this particular case, the entanglement entropy has a logarithmic scaling which was also realised in certain string constructions [51]. Further details of the gravitational description of hvLif theories has been relegated to Appendix A.

This thesis can broadly be divided into three parts. Motivated by investigations done in [52] we have studied mutual information (MI) in he context of AdS plane waves which as mentioned earlier gives rise to hvLif theories under null reduction. The third chapter of the thesis adapts the approach pioneered by Kovtun, Son and Starinets in the context of hvLif theories. For the sake of simplicity, we have restricted out analysis for hvLif theories which has the corresponding gravity dual living in 3+1 spacetime dimensions. In Chapter 3, we study the shear diffusion constant for this class of non-relativistic field theories. Looking at various checkpoints for non-conforal Dp branes and gravity dual of Lifshitz field theories, a thermodynamic relation connection the shear diffusion constant and the shear viscosity is proposed. The fourth chapter of the thesis involves a study of the same transport properties using quasinormal modes which encodes information of hydrodynamic transport coefficients. Using the AdS/CFT correspondence, it was conjectured that the QNM spectrum of a fluctuation $\delta\phi$ of a higher dimensional gravitational background coincides with the location of the poles of the retarded correlators of the gauge theory operator dual $\mathcal{O}$ dual to $\delta\phi$. However, this conjecture was validated through various examples for gravity duals of relativistic field theories. In Chapter 4, we study the QNM of hvLif black branes and also find the two point correlator of the dual operator in the boundary theory (2-point function of the stress tensor). It is noteworthy that the gravitational fluctuations of these hvLif gravity duals couple to other matter fields like perturbations of the gauge field. Although the dual field theory is non-relativistic, we continue to see that the poles of retarded correlator which contains information of hydrodynamic transport coefficients in indeed encoded in the QNM of the gravitational fluctuations of the dual system. Using the standard prescription for computing correlators in Lorentzian signature, we compute the shear diffusion and shear viscosity and validate the thermodynamic relation connecting the two transport coefficients conjectured in [P2].

Chapter 2

Entanglement Entropy and Mutual Information

Entanglement is one of the most fascinating concepts of quantum mechanics. Classically, we can reconstruct a complicated system by studying the behaviour of its constituent non-interacting parts. However, such an intuition fails to carry over in the quantum realm–even the constituent non-interacting pieces may be entangled. Starting with the EPR gedanken experiments [53] which investigated this "spooky action at a distance" a significant amount of progress has been made to understand the nature of entanglement. Particularly with advances in the field of quantum information [54], entanglement has become an important resource and an interesting phenomenon for study in itself.

Entanglement can be observed in simple 2-qubit systems. Consider the Hilbert space formed out of direct product of single qubit systems i.e. $\mathcal{H} = \mathcal{H}_{qubit1} \otimes \mathcal{H}_{qubit2}$. This is spanned by a basis given by $\{|00\rangle, |01\rangle, |10\rangle |11\rangle\}$. Each one of the basis states are *separable* in the sense that they can be written as a direct product of two vectors each belonging to the two subsystems. However, such a decomposition is not possible for the *Bell state* given by $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The state of the two qubits are correlated i.e. if we have knowledge about one of the spin states, the other spin is automatically determined. Such states are called *entangled states.*

Although simple systems like spin models remains a powerful compuational laboratory for intuitive understanding of entanglement. it is believed that the basic concepts and principles carry over to the continuum limit. A good example are certain topological field theories in $(2 + 1)$-dimension which inspite of the absence of dynamical degrees of freedom exhibit a rich phase structure. It was realized that entanglement entropy can be used as an order parameter to study the phase structure of these theories [55, 56]. It can also be used to classify ground states of various interesting interacting many-body systems.

Entanglement entropy is fascinating in its own right within the context of field theory. However, with the advent of AdS/CFT [4], a deeper connection between gravitational dynamics and entanglement structure has emerged in the context of holography. Inspired by the area scaling of black hole entropy, Ryu and Takayanagi [57, 58] identified a simple geometric prescription for entanglement entropy (EE) in field theories with gravity duals in the large N limit: the EE for a subsystem in the d-dim field theory is the area in Planck units of a minimal surface bounding the subsystem, the bulk theory living in d+1-dimensions. For non-static situations, the prescription generalizes to the proposal of Hubeny-Rangamani-Takayanagi which involves finding the area of an appropriate bulk extremal surface with minimal area [59]. Inspired by all these developments, Swingle [60] and Van Raamsdonk [61, 62] argued that the emergence of spacetime geometry is somehow related to the entanglement structure of the quantum state on the QFT side. This was encoded in the statement "ER=EPR" by Maldacena and Susskind [63] which relates entangled states called EPR states (after Einstein, Podolsky and Rosen) at geometrical structures that arise in general relativity called ER bridges (after Einstein and Rosen). In the subsequent sections we will give a more precise definition to the notion of entanglement entropy in field theory and the Ryu-Takayanagi prescription.

2.1 Holographic entanglement entropy: a brief review

Let us consider the density matrix ρ of a quantum mechanical system which is in a pure state $|\Psi\rangle$ which is normalised as: $\langle\Psi|\Psi\rangle = 1$. In this case, the density matrix is given by $\rho = |\Psi\rangle\langle\Psi|$. For a system in a mixed state (e.g. thermal states are mixed states) the density matrix takes the form

$$\rho = \sum_n p_n |\Psi_n\rangle\langle\Psi_n| \tag{2.1}$$

where the state kets $|\Psi_n\rangle$ forms an orthonormal basis which satisfies $\sum_n p_n = 1$ (this is equivalent to the statement that the sum of probabilities of the system being found in one of the states $|\Psi_n\rangle$ is 1)

Suppose we divide our system into two parts: A and B. We will assume that this breaks the structure of the full Hilbert space $\mathcal{H}$ into a direct product of two independent Hilbert spaces i.e. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. For the subsystem A, we can define a reduced density matrix

$$\rho_A = \mathrm{Tr}_B \rho \tag{2.2}$$

where ρ is the density matrix of the full system. Tr_B traces over the basis vectors of $\mathcal{H}_B$. Physically, the reduced density matrix represents the lack of information that we have about the system B by "integrating" it out. The von Neumann entropy associated with ρ_A is defined as the *entanglement entropy* of the subsystem A i.e.

$$S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A) \tag{2.3}$$

This is a measure of entanglement. Note that even if the entire system is pure, it is possible for the entanglement entropy of a subsystem to be non-zero. The definition (2.3) immediately leads to certain interesting properties. If the global ystem is pure and B is the complement of A, we have $S_A = S_B$. This demonstrates that entanglement entropy is not an extensive quantity. For three disjoint subsystems A, B and C, we get the inequalities

$$\begin{aligned} S_{A+B+C} + S_B &\leq S_{A+B} + S_{B+C} \\ S_A + S_C &\leq S_{A+B} + S_{B+C} \end{aligned} \tag{2.4}$$

Choosing B to be an empty set leads us to what is dubbed as the *subadditivity relation*

$$S_{A+B} \leq S_A + S_B \tag{2.5}$$

In fact for two arbitrary systems, there exists a stronger relation known as the *strong subadditivity condition*

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B} \tag{2.6}$$

which reduces to (2.5) when A and B are disjoint.

In a quantum field theory living in d spacetime dimensions, we consider two complimentary regions A and B in space at a fixed time slice $t = t_0$. Akin to the earlier description, in this case too the we will essentially integrate out the degrees of freedom in B to get the entanglement entropy of A. The result is divergent and can be schematically written as

$$S_A = a_0(l\Lambda)^{d-2} + a_1(l\Lambda)^{d-3} + \cdots + [a_d \log(l\Lambda)] \tag{2.7}$$

where l is the typical size of subsystem A and Λ is the UV cutoff. The last term written in square brackets $[\cdot]$ is present only when d is even. The coefficients a_i are in general dependent on the theory we are interested in. The leading piece has a scaling of the form l^{d-2} which is identical to that of the boundary ∂A. This shows that entanglement entropy scales as the area of the boundary similar to black hole entropy. This motivated Ryu and Takayanagi to come up with a gravity dual for entanglement entropy which essentially is a certain codimension-2 surface in the bulk. The reader can find detailed accounts of the holographic viewpoint of entanglement entropy in the classic reviews [64, 65].

In particular, for 2-dimensional CFTs, we have

$$S = \frac{c}{3} \log(l\Lambda) \tag{2.8}$$

where l is the size of the subsystem A while c is the central charge of the CFT. To be a little more precise, for a 1-dimensional quantum many-body system at criticality (2-D CFT), the entanglement entropy is given by

$$S_A = \frac{c}{3} \log\left(\frac{L}{\pi a} \sin\left(\frac{\pi l}{L}\right)\right) \tag{2.9}$$

where l and L are the length of the subsystem A and the total subsystem $A \cup B$. Either ends of the full system i.e. $A \cup B$ has been identified perodically. a is the lattice spacing (UV cutoff $\Lambda \sim 1/a$) and c is the central charge. For field theories at a finite temperature, the above expression of entanglement entropy takes the form

$$S_A = \frac{c}{3} \log\left(\frac{\beta}{\pi a} \sinh\left(\frac{\pi l}{\beta}\right)\right) \tag{2.10}$$

where β now parametrizes the temperature scale i.e. $\beta = T^{-1}$.

2.1.1 Ryu-Takayanagi prescription

In the light of the AdS/CFT correspondence, Ryu and Takayanagi [57, 58] proposed the following: the entanglement entropy S_A in a CFT on $\mathbb{R}^{1,d}$ for a subsystem A that has an arbitrary $d-1$-dimensional boundary $\partial A \in \mathbb{R}^d$ is holographially given by

$$S_A = \frac{\text{Area of } \gamma}{4G_N^{(d+2)}} \tag{2.11}$$

where γ is the d-dimensional static minimal surface in AdS_{d+2} whose boundary is given by ∂A and $G_N^{(d+2)}$ is the $(d+2)$-dimensional Newton's constant. Intuitively, the minimal surface with the boundary ∂A can be thought of as a holographic screen since the observer has access to subsystem A alone.

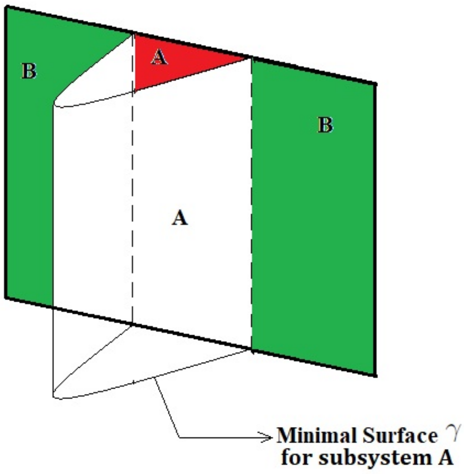

Figure 2.1: Schematic diagram of the minimal surface hinged on a strip shaped subsystem A. B denotes the environment which is traced out to write down a reduced density matrix ρ_A

It can be easily seen from (2.11) that $S_A = S_B$ where B is the complement of A and it also satisfies (2.5). The Ryu-Takayanagi prescription works for finite temperture systems too. However, for thermal systems ($T > 0$) we do not find $S_A = S_B$ since ρ i.e. the density matrix for the full system is in a mixed state where the equality does not hold. Holographically, thermal systems are decribed by the presence of a black hole horizon due to which the equality $S_A = S_B$ breaks down. Now, we will use the Ryu-Takayanagi prescription as stated in (2.11) to evaluate the entanglement entropy for thermal 2-dimensional CFTs and subsequently generalise for higher dimensions.

2.1.2 Entanglement Entropy for AdS_3/CFT_2

We are interested in the entangelement entropy of a 1-dimensional spatial subsystem at a crtical point. At a fixed time, the minimal surface is essentially the geodesic connecting two boundary points which defines the subsystem A which we are interested in. The Ryu-Takayanagi prescription (2.11) interprets a complicated quantity in a CFT as a geometric quantity in a dual gravity theory [57, 58].

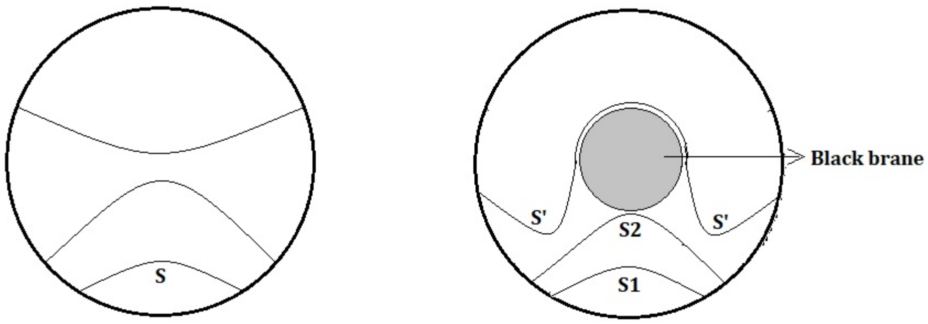

Figure 2.2: Schematic representation of minimal surfaces in AdS_3 at zero temperature and finite temperature. S: Minimal geodesic in AdS_3 at zero temperature; S1 and S2: Minimal geodesic for small subsystems in AdS_3 at finite temperature; S': Minimal geodesic for large subsystems in AdS_3 at finite temperature wraps the black brane.

The metric of AdS_3 written in global coordinates (t, ρ, θ) is

$$ds^2 = R^2(-\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\theta^2) \tag{2.12}$$

Anticipating divergences close to the boundary as $\rho \to \infty$, we introduce another constant-ρ surface at $\rho = \rho_0$ which acts as a regulator. In the dual field theory this is equivalent to introducing a UV cut-off. In fact for CFTs, if L is the full system size with the end-points identified and a is the lattice spacing then, we can write a relation of the form

$$e^{\rho_0} \sim \frac{L}{a} \quad \text{(upto numerical factors)} \tag{2.13}$$

The CFT_2 dual now lives on the (t, θ) cylinder on the $\rho = \rho_0$ surface. We define the subsystem A at a fixed time t to be the region $0 \leq \theta \leq \frac{2\pi l}{L}$. γ_A is the static geodesic as shown in Figure 2.2 travelling through the bulk of AdS_3 spacetime connecting the points $\theta = 0$ and $\theta = \frac{2\pi l}{L}$ having a length L_{γ_A}. The geodesic length (minimal length curve) is given by

$$\cosh\left(\frac{L_{\gamma_A}}{R}\right) = 1 + 2\sinh^2\rho_0 \sin^2\frac{\pi l}{L} \tag{2.14}$$

We can use the above expression to get a form of L_{γ_A} which can be plugged in (2.11) when $d = 1$ to obtain the entanglement entropy. Further, the AdS/CFT

correspondence relates the AdS radius R and the Newton's constant $G_N^{(3)}$ with the central charge c of the dual CFT through the relation

$$c = \frac{3R}{2G_N^{(3)}} \tag{2.15}$$

Using the above along with the assumption that $e^{\rho_0} \gg 1$ (physically this means that the system size is much larger compared to the lattice spacing) leads us to the expression

$$S_A \approx \frac{c}{3} \log\left(e^{\rho_0} \sin\frac{\pi l}{L}\right) \tag{2.16}$$

which matches with (2.9) upto some numerical factors. This prescription suggests that the curve denoting the Ryu-Takayanagi minimal surface γ_A acts like a black hole horizon and restricts an observer from probing subsystem B although the division between them is artificial. The grey region as given in Figure 2.2 is however depicts a black brane inaccessible to the minimal surface which now wraps around it)

One can carry out a similar computation in the context of thermal systems with certain simplifications. We assume that the system we are dealing with is infinitely long restricting us to the regime where $\beta/L \ll 1$. The dual gravity theory for a CFT at finite temperature is given by the Euclidean BTZ black hole with the metric

$$ds^2 = (r^2 - r_+^2)d\tau^2 + \frac{R^2}{r^2 - r_+^2}dr^2 + r^2 d\phi^2 \tag{2.17}$$

The Euclidean time and the angular coordinate has a periodicity given by the following identifications

$$\begin{aligned} \tau &\sim \tau + \frac{2\pi R}{r_+} \\ \phi &\sim \phi + 2\pi \end{aligned} \tag{2.18}$$

The parameters of the CFT and the BTZ black hole [66] can be related as $\frac{\beta}{L} = \frac{R}{r_+} \ll 1$. Similar to the zero temperature case, here again we define the subsystem A as $0 \leq \phi \leq \frac{2\pi l}{L}$ on the $\rho = \rho_0$ surface. In order to find the geodesic length in this case, we will use the result that an Euclidean BTZ black hole at a temperature T is equivalent to a thermal AdS_3 at temperature T^{-1}. Performing the transformations

$$r = r_+ \cosh\rho \ ; \quad r_+\tau = R\theta \ ; \quad r_+\phi = Rt \tag{2.19}$$

In terms of the new coodinates (t, ρ, θ) (2.17) indeed takes the form of thermal Euclidean AdS_3. The geodesic distance in this case i.e. the analog of (2.14) is given by

$$\cosh\left(\frac{L_{\gamma_A}}{R}\right) = 1 + 2\cosh^2\rho_0 \sinh^2\left(\frac{\pi l}{\beta}\right) \tag{2.20}$$

Using the above along with the relation (2.15) gives us (2.10), a well known result for thermal CFTs.

2.1.3 Higher dimensional entanglement entropy

The Ryu-Takayanagi prescription can be generalized to higher dimensions, giving a handle on the entanglement structure of CFTs living in odd dimensions where standard field theoretic techniques e.g. the replica trick fails. In higher dimensions, the geodesic distances (curves with minimal lengths connecting two boundary points) are replaced with the notion of minimal area surfaces. Unlike the earlier subsection, we will perform our calculations in the Poincaré patch of AdS_{d+2} where the metric is

$$ds^2 = \frac{R^2}{r^2}\left(dr^2 - dt^2 + \sum_{i=1}^{d} dx_i^2\right) \tag{2.21}$$

We will evaluate the entanglement entropy for two kinds of geometries for the subsystem A: when A is strip shaped and when A is disk shaped.

Let us first consider A to be a strip-shaped subsystem given by

$$A_{strip} = \{x_i | x_1 \in [-l/2, +l/2], x_{2,3,\ldots,d} \in [-\infty, +\infty]\} \tag{2.22}$$

at the boundary $r = 0$. Clearly, l is the width of the strip along x_1 while it is non-compact along all other boundary spatial directions. The minimal surface dips into the AdS bulk, where r_* represents the turning point of the minimal surface whose boundary is identical to that of A_{strip}. The minimal surface follows an equation of the form

$$\frac{dr}{dx_1} = \frac{\sqrt{r_*^{2d} - r^{2d}}}{r^d} \quad \text{where} \quad \frac{l}{2} = \int_0^{r_*} dr \frac{r^d}{\sqrt{r_*^{2d} - r^{2d}}} = r_*\sqrt{\pi}\frac{\Gamma(\frac{d+1}{2d})}{\Gamma(\frac{1}{2d})} \tag{2.23}$$

The area of the minimal surface for the strip-shaped subsystem defined as (2.22) finally takes the form

$$\text{Area of } A_{strip} = \frac{2R^d}{d-1}\left(\frac{L}{a}\right)^{d-1} - \frac{2^d \pi^{d/2} R^d}{d-1}\left(\frac{\Gamma(\frac{d+1}{2d})}{\Gamma(\frac{1}{2d})}\right)^d \left(\frac{L}{l}\right)^{d-1} \tag{2.24}$$

where L is the *regularized length* of the subsystem A along the non-compact directions $x_{2,3,...,d}$.

The same exercise can be repeated for disk-shaped geometry defined as

$$A_{disk} = \{x_i | r \leq l\} \tag{2.25}$$

The corresponding minimal surface generated in the bulk has an area given by

$$\begin{aligned}\text{Area of } A_{disk} &= C\int_{a/l}^{1} dy \frac{(1-y^2)^{\frac{d-2}{2}}}{y^d} \\ &= p_1\left(\frac{l}{a}\right)^{d-1} + p_3\left(\frac{l}{a}\right)^{d-3} + \cdots \\ &\quad \cdots + \begin{cases} p_{d-1}\left(\frac{l}{a}\right) + p_d + O\left(\frac{a}{l}\right) & d:\text{even} \\ p_{d-2}\left(\frac{l}{a}\right)^2 + q\log\left(\frac{l}{a}\right) + O(1) & d:\text{odd}\end{cases}\end{aligned} \tag{2.26}$$

where

$$\begin{aligned} C = \frac{2\pi^{d/2}R^d}{\Gamma(\frac{d}{2})} \;;\quad \frac{p_1}{C} = \frac{1}{d-1}\;; \\ \frac{p_d}{C} = (2\sqrt{\pi})^{-1}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1-d}{2}\right) \;;\quad \frac{q}{C} = (-1)^{\frac{d-1}{2}}\frac{(d-2)!!}{(d-1)!!}\end{aligned} \tag{2.27}$$

In the context of the AdS/CFT correspondence, the above expression can be used to write down the entanglement entropy of a subsystem of size l for $\mathcal{N}=4$ $SU(N)$ Yang-Mills theory in 4-dimensions at zero temperature which has a gravity dual given by (2.21) with $d=3$. This eventually leads us to the expressions

$$S_{A_{strip}} = \frac{N^2L^2}{2\pi a^2} - 2\sqrt{\pi}\left(\frac{\Gamma(2/3)}{\Gamma(1/6)}\right)^3\frac{N^2L^2}{l^2} \tag{2.28}$$

$$S_{A_{disk}} = N^2\left[\frac{l^2}{a^2} - \log\left(\frac{l}{a}\right) + O(1)\right] \tag{2.29}$$

which gives the entanglement entropy of a subsystem of size l.

From the expressions (2.24), (2.26), (2.28) and (2.29) it is evident that the UV divergent terms scale as $\sim a^{-(d-1)}$ which is reminiscent of the 'area law' for entanglement entropy in QFTs [67, 68]. Physically, this is expected since the correlation between the subsystem and the environment is maximal at the boundary separating the two. Although the presence of the area divergence term is a universal feature of entanglement entropy, the coefficient of this term depends on the UV cut-off. A more physically relevant quantity is one where there is no such cutoff dependence e.g. the second term in (2.24), (2.28) or the coefficient of the logarithmic term in (2.29). These terms are universal and depends only on the dimensionality of the theory that is under consideration.

Finally we consider the entanglement entropy of $\mathcal{N} = 4$ SYM theory on $\mathbb{R}^3$ at a finite temperature T. The dual geometric description has the metric

$$ds^2 = R^2 \left[\frac{dr^2}{h(r) r^2} + r^2 \left(-h(r) dt^2 + \sum_{i=1}^{3} dx_i^2 \right) + d\Omega_5^2 \right] \tag{2.30}$$

where

$$h(r) = 1 - \frac{r_0^4}{r^4} \; ; \quad r_0 = \pi T \; ; \quad \text{boundary: } r \to \infty \tag{2.31}$$

For a strip-shaped subsystem defined as $A_{strip} = \{x_i | x_1 \in [-l/2, l/2],\ x_{2,3} \in (-\infty, +\infty)\}$ we can compute the area of the minimal surface to be

$$\text{Area of } A_{strip} = 2R^3 L^2 \int_{r_*}^{a^{-1}} dr \, \frac{r^6}{\sqrt{(r^4 - r_0^4)(r^6 - r_*^6)}} \tag{2.32}$$

where the turning point r_* is given by

$$\frac{l}{2} = \int_{r_*}^{\infty} dr \, \frac{1}{\sqrt{(r^4 - r_0^4)\left(\frac{r^6}{r_*^6} - 1\right)}} \tag{2.33}$$

The area integral (2.32) scales as a^{-2} which is in conformity with the expected 'area law' divergence. As we increase the subsystem size, the minimal surface dips further into the bulk and gradually starts wrapping around the black hole horizon. From the bulk perspective, this is the limit when $r_* \sim r_0$ where the area integral simplifies to give $\sim \pi^3 R^3 L^2 l T^3$ where L denotes the regularized length along x_2 and x_3 directions. Finally, the Ryu-Takayanagi prescription dictates the finite part of entanglement entropy for the subsystem A to be

$$S_{A_{strip}}^{(finite)} \approx \frac{\pi^2 N^2 T^3}{2} L^2 l = \frac{\pi^2 N^2 T^3}{2} \times (\text{Area of } A_{strip}) \tag{2.34}$$

We can interpret the above quantity as a part of the Bekenstein-Hawking entropy for black 5-branes which is proportional to the area of the horizon at $r = r_0$. In the gravitational description, this part arises since the minimal surface wraps the black hole horizon. If we increase the size of subsystem A to coincide with the full system we indeed recover the Bekenstein-Hawking entropy exactly.

2.2 AdS plane waves and Mutual Information (MI)

We are interested in studying excited states of a certain kind, building on previous work. AdS plane waves [26, 30, 35, 51] are deformations of AdS which are dual to CFT excited states with constant energy-momentum flux $T_{++} \sim Q$ turned on. Upon x^+-dimensional reduction, these give rise to hyperscaling violating spacetimes [51], some of which exhibit violations [69, 70, 12] of the area law [67, 68]. In [52, 71], a systematic study of entanglement entropy for strip subsystems was carried out in AdS plane waves (with generalizations to nonconformal brane plane waves in [72]). The EE depends on the orientation of the subsystem i.e. whether the strip is parallel or orthogonal to the flux T_{++}. For the strip subsystem along the flux, the EE grows logarithmically with the subsystem width l for the AdS_5 plane wave (the corresponding hyperscaling violating spacetime lies in the family giving log-behaviour). The AdS_4 plane wave dual to plane wave excited states in the M2-brane Chern-Simons CFT exhibits an even stronger $\sqrt{l}$ growth. For the strip orthogonal to the flux, we have a phase transition with the EE saturating for $l \gg Q^{-1/d}$.

For two disjoint subsystems, an interesting information-theoretic object is mutual information (MI), defined as

$$I[A, B] = S[A] + S[B] - S[A \cup B] \ , \tag{2.35}$$

involving a linear combination of entanglement entropies. It measures how much two disjoint subsystems are correlated (both classical and quantum). The EE terms in $I[A, B]$ automatically cancel out the cutoff-dependent divergence thus making MI finite and positive semi-definite. A new divergence comes up when the subsystems collide. The term $S[A \cup B]$ in the above expression depends on the separation between the subsystems A and B: in the holographic context, there are two extremal surfaces of key interest. For large separation, the disconnected surface $S[A\cup B] = S[A]+S[B]$ having lower area is the relevant surface so that mutual information $I[A, B]$ vanishes. For nearby subsystems however, the connected surface has lower area. Thus the Ryu-Takayanagi prescription automatically implies a disentangling transition for mutual information in this large N classical gravity approximation [73], with a critical separation x_c.

In this chapter, we first discuss a phenomenological scaling picture for entanglement for CFT ground and some excited states, building on some renormalization-group like intuition described in [74] based on "entangling bits" or "partons" (sec. 3). In 2.4, we describe some generalities on holographic mutual information and then study mutual information in AdS plane waves for two parallel disjoint strip subsystems of width l each (sec. 5), first discussing the wide strip regime $Ql^d \gg 1$, exhibiting again a disentangling transition. Then we study the perturbative regime $Ql^d \ll 1$ and calculate the changes in the turning point and the entanglement area functional to $O(Q)$ treating the AdS plane wave as a perturbation to pure AdS, for the strip subsystem both parallel and orthogonal to the energy-momentum flux. This perturbative analysis has parallels with "entanglement thermodynamics" [75, 76, 77]. Finally, we perform some numerical analysis to gain some insights when Ql^d is $O(1)$. We discuss some similarities and key differences of our investigations with the study of mutual information for thermal excited states [78], which are somewhat different from these pure excited AdS plane wave states. In the next subsection, we briefly review entanglement entropy in the context of AdS plane waves.

2.2.1 Review: Entanglement Entropy for AdS plane waves

AdS plane waves [30, 35, 51] are rather simple deformations of AdS/CFT, dual to anisotropic excited states in the CFT with uniform constant energy-momentum density T_{++} turned on (with all other energy-momentum components vanishing),

$$ds^2 = \frac{R^2}{r^2}(-2dx^+dx^- + dx_i^2 + dr^2) + R^2 Q r^{d-2}(dx^+)^2 + R^2 d\Omega^2 \ , \tag{2.36}$$

with d the boundary spacetime dimension and $R^4 \sim g_{YM}^2 N \alpha'^2$ [AdS_5 plane wave], $R^6 \sim N l_P^6$ [AdS_4 plane wave]. These are normalizable deformations of $AdS_{d+1} \times S$ that arise in the near horizon limits of various conformal branes in string/M-theory. Structurally they are similar to the AdS null deformations [20, 21] that give rise to gauge/string realizations of $z = 2$ Lifshitz spacetimes [18, 19], except that these AdS plane waves are normalizable null deformations. Reducing on the sphere, these are solutions in a $d+1$-dim effective gravity theory with negative cosmological constant and no other matter, i.e. satisfying $R_{MN} = -\frac{d}{R^2} g_{MN}$. The parameter $Q > 0$ gives rise to a holographic energy-momentum density $T_{++} \propto Q$ in the boundary CFT. Dimensionally reducing (2.36) on the x^+-dimension (and relabeling $x^- \equiv t$) gives a hyperscaling violating metric $ds^2 = r^{\frac{2\theta}{d_i}}\big(-\frac{dt^2}{r^{2z}} + \frac{\sum_{i=1}^{2} dx_i^2 + dr^2}{r^2}\big)$, with exponents $z = \frac{d-2}{2} + 2$, $\theta = \frac{d-2}{2}$ and d_i is the boundary spatial dimension. These are conformal to Lifshitz space times and appear in various discussions of non-relativistic holography, arising in various effective Einstein-Maxwell-scalar theories e.g. [5, 6, 7, 8, 9, 10, 11]: see [12] for various aspects of holography with hyperscaling violation. It is known that these spacetimes for the special family "$\theta = d_i - 1$" exhibit a logarithmic violation of the area law [67, 68] of entanglement entropy, suggesting that these are signatures of hidden Fermi surfaces [69, 70]. For the special case of the AdS_5 plane wave, we have $\theta = 1, d_i = 2$, lying in this "$\theta = d_i - 1$" family.

This spacetime (2.36) can be obtained [26, 51, 30, 35] as a "zero temperature", highly boosted, double-scaling limit of boosted black branes, using [79]. For instance, AdS_5 Schwarzschild black brane spacetimes, with metric

$$ds^2 = \frac{R^2}{r^2}[-(1-r_0^4 r^4)dt^2 + dx_3^2 + \sum_{i=1}^{2} dx_i^2] + R^2 \frac{dr^2}{r^2(1-r_0^4 r^4)} \tag{2.37}$$

can be recast in boundary lightcone coordinates $x^\pm$ with $t = \frac{x^+ + x^-}{\sqrt{2}}$, $x_3 = \frac{x^+ - x^-}{\sqrt{2}}$. After boosting by λ as $x^\pm \to \lambda^{\pm 1} x^\pm$, we obtain

$$ds^2 = \frac{R^2}{r^2}\left[-2dx^+dx^- + \frac{r_0^4 r^4}{2}(\lambda dx^+ + \lambda^{-1}dx^-)^2 + \sum_{i=1}^{2} dx_i^2\right] + R^2 \frac{dr^2}{r^2(1-r_0^4 r^4)} \tag{2.38}$$

Now in the double scaling limit $r_0 \to 0$, $\lambda \to \infty$, with $Q = \frac{r_0^4 \lambda^2}{2}$ fixed, this becomes (2.36). For the near extremal AdS plane wave, from [79], we see that we have other energy-momentum components also turned on, $T_{++} \sim \lambda^2 r_0^4 \sim Q$, $T_{--} \sim \frac{r_0^4}{\lambda^2} \sim \frac{r_0^8}{Q}$, $T_{+-} \sim r_0^4$, $T_{ij} \sim r_0^4 \delta_{ij}$. Turning on a small r_0 about (2.36), this means T_{++} is dominant while the other components are small. In some sense, this is like a large left-moving chiral wave with $T_{++} \sim Q$, with a small amount of right-moving stuff turned on. Thus the near-extremal case (with small r_0) serves to regulate the AdS plane wave in the deep interior.

We now review certain aspects of holographic entanglement entropy in these AdS plane wave geometries [52]. First, it is worth recalling that the entanglement entropy for ground states ($Q = 0$) in the d-dim CFTs arising on the various conformal branes with strip-shaped subsystems has the form (upto numerical coefficients)

$$S_A \sim \frac{R^{d-1}}{G_{d+1}}\left(\frac{V_{d-2}}{\epsilon^{d-2}} - c_d \frac{V_{d-2}}{l^{d-2}}\right), \qquad \frac{R^3}{G_5} \sim N^2 \ (4d\ CFT), \quad \frac{R^2}{G_4} \sim N^{3/2} \ (3d\ CFT), \tag{2.39}$$

where $c_d > 0$ is some constant, l the strip width, V_{d-2} the longitudinal size and ϵ the ultraviolet cutoff. (We have used the relations $R_{D3}^4 \sim g_s N l_s^4$, $R_{M2}^6 \sim N l_P^6$, and those for the Newton constants $G_{10} \sim G_5 R_{D3}^5, G_{11} \sim G_4 R_{M2}^7$, where g_s is the string coupling, and l_s, l_P the string and Planck lengths.) The first term exhibiting the leading divergence represents the area law while the second term is a finite cutoff-independent part encoding a size-dependent measure of the entanglement [57, 58, 80]. With $Q \neq 0$, we have an energy flux in a certain direction: these are nonstatic spacetimes, and we therefore use the covariant formulation of holographic entanglement entropy [59] working in the higher dimensional theory (with x^+ noncompact), the strip geometry corresponding to a space-like subsystem on the boundary. Consider the strip to be along the flux direction, i.e. with width along some x_i direction [52]. Then the leading divergent term is the same as for ground states. The width scales as $l \sim r_*$, where r_* is the turning point of the bulk extremal surface, and the finite cutoff-independent piece in these excited states is

$$\pm\sqrt{Q} V_{d-2} l^{2-\frac{d}{2}} \frac{R^{d-1}}{G_{d+1}} \qquad [+:\ d<4, \quad -:\ d>4]\ ;$$
$$\sqrt{Q} V_2 N^2\ \log(lQ^{1/4})\ \ (D3)\ ; \qquad \sqrt{Q} L \sqrt{l}\ N^{3/2}\ \ (M2)\ . \tag{2.40}$$

Note that the logarithmic behavior for the 4-dim CFT is of the same form as for a Fermi surface, if the energy scale $Q^{1/4}$ is identified with the Fermi momentum k_F. Both 4- and 3-dim CFTs in these excited states thus exhibit a finite entanglement which grows with subsystem size l. In particular, for fixed cutoff, this finite part is larger than the leading divergence. Recalling that the finite entanglement for the thermal state (i.e. the AdS black hole) is extensive, of the form $V_{d-2} T^{d-1} l$, we see that these are states with subthermal entanglement. These are pure states in the large N gravity approximation since the entropy density vanishes.

It is worth noting that we regard the AdS plane wave spacetimes as a low temperature highly boosted limit of the AdS black brane: the scale $Q = \lambda^2 r_0^4 \gg r_0^4$ implies a large separation of scales between the flux in the AdS plane wave and the temperature of the black brane, with Q dominating the physics in the plane wave regime. The above estimates (2.40) for the finite part of entanglement arise if the bulk extremal surface dips deep enough in the radial direction to experience substantial deviation from the AdS geometry due to the plane wave, while still away from the regulating black brane horizon in the deep interior, i.e. the length scales satisfy $Q^{-1/d} \ll l \ll \frac{1}{r_0}$.

With the strip orthogonal to the flux direction, a phase transition was noted [52]: for large width l, there is no connected surface corresponding to a space-like subsystem, only disconnected ones.

This analysis can be extended [72] to the various nonconformal Dp-brane systems [81]. These have a ground state entanglement [58, 82] (after converting to field theory parameters) $S_A = N_{eff}(\epsilon)\frac{V_{d-2}}{\epsilon^{d-2}} - c_d N_{eff}(l)\frac{V_{d-2}}{l^{d-2}}$, with a scale-dependent number of degrees of freedom $N_{eff}(l) = N^2\left(\frac{g_{YM}^2 N}{l^{p-3}}\right)^{\frac{p-3}{5-p}}$ involving the dimensionless gauge coupling at scale l. For nonconformal Dp-brane plane waves, it turns out to be natural to redefine the energy density as $Q \to QN_{eff}(l)$ (i.e. Q in the conformal cases above is the energy density per nonabelian degree of freedom), and then the finite part of entanglement takes the form $S_A^{finite} \sim \frac{\sqrt{N_{eff}(l)}}{3-p}\frac{V_{p-1}\sqrt{Q}}{l^{(p-3)/2}}$ involving a dimensionless ratio of the energy density and the strip width/lengths and $N_{eff}(l)$ (the leading divergence is as for the ground state). This finite part is similar in structure to that for the conformal plane waves above, but is scale-dependent: analysing the UV-IR *Dp*-brane phase diagram [81] shows the finite part to be consistent with renormalization group flow [72].

2.3 A phenomenological scaling picture for entanglement

This is a generalization of an RG-like scaling picture in [74] for ground states. We assume a renormalization group type scaling behaviour with a notion of "entanglement per scale" as an organizing principle: i.e. in a CFT of spacetime dimension d, there are "entangling bits" or "partons" of all sizes s. Equivalently at scale s, we think of space as lattice-like with cell size s. In the ground state, each cell roughly contains one entangling parton. Entanglement arises from degrees of freedom straddling the boundary between the subsystem and the environment, in other words from partons partially within the subsystem and partly outside. Entanglement entropy arises from the fact that we trace over the environment and thus lose some information about the straddling partons. The scaling picture below is admittedly quite phenomenological and is only meant as an attempt at an intuitive picture that fits the holographic entanglement calculations.

We want to estimate the rough number of degrees of freedom contributing to entanglement at the interface between the subsystem and the environment which has area $V_{d-2} \equiv L^{d-2}$. At scale s, the rough number of cells of linear size s at the boundary is $(\frac{L}{s})^{d-2} = \frac{V_{d-2}}{s^{d-2}}$. For a CFT with nonabelian $N \times N$ matrix degrees of freedom, there are N^2 degrees of freedom per cell (we use N^2 with a SYM CFT in mind but this can be easily generalized to $N^{3/2}$ for the M2-brane CFT). We then integrate this over all scales greater than the UV cutoff ϵ with the logarithmic measure $\frac{ds}{s}$ and also we expect the IR cutoff is set by the subsystem size l. This gives (assuming $d > 2$)

$$S \sim \int_{\epsilon}^{l} \frac{ds}{s} \frac{V_{d-2}}{s^{d-2}} N^2 \ \sim \frac{N^2 V_{d-2}}{d-2} \left(\frac{1}{\epsilon^{d-2}} - \frac{1}{l^{d-2}} \right) . \tag{2.41}$$

This shows the leading area law divergence and the subleading cutoff-independent finite part. For $d = 2$, we obtain $S \sim \int_{\epsilon}^{l} \frac{ds}{s} N^2 \sim \ N^2 \log \frac{l}{\epsilon}$ which is the logarithmic behaviour characteristic of a 2-dim CFT: this can be used as a check that the logarithmic measure $\frac{ds}{s}$ is appropriate. This is a quantum entanglement, with contributions from various scales s.

Thus we see that there is a diverging number $\frac{V_{d-2}}{s^{s-2}}$ of ultra-small partons at short distances $s \to 0$ which essentially gives rise to the area law divergence [?]. For excited states, the energy-momentum density does not change the short distance behaviour but implies an enhanced number of partons at length scales much larger than the scale set by the energy-momentum, changing the IR behaviour of entanglement as we will see below.

Similar arguments can be made for the various nonconformal gauge theories arising on the various nonconformal Dp-branes. Now the gauge coupling is dimensionful and the number of nonabelian degrees of freedom at scale s is

$$N_{eff}(s) = N^2 \left(\frac{g_{YM}^2 N}{s^{p-3}} \right)^{\frac{p-3}{5-p}} \ . \tag{2.42}$$

For the ground state, the entanglement at the boundary of the subsystem is obtained as before by integrating over all scales the number $\mathcal{N}_{eff}(s)$ of entangling bits or partons at scale s

$$S \sim \int_{\epsilon}^{l} \frac{ds}{s} \frac{V_{d-2}}{s^{d-2}} N_{eff}(s) \ \ \sim \ \ (5-p) N_{eff}(\epsilon) \frac{V_{d-2}}{\epsilon^{d-2}} \ - \ (5-p) N_{eff}(l) \frac{V_{d-2}}{l^{d-2}} \ , \tag{2.43}$$

in agreement with the known holographic result for the ground state entanglement for the nonconformal brane theories, upto numerical factors. We see that the entanglement expression above breaks down for $p = 5$: these are nonlocal theories (e.g. little string theories for NS5-branes).

For the CFT_d at finite temperature T, the entanglement entropy has a finite cutoff-independent piece which is extensive and dominant in the IR limit of large strip width l: this is the thermal entropy, essentially a classical observable,

$$S \sim N^2 V T^{d-1} = N^2 \frac{V}{(1/T)^{d-1}} \ , \quad \text{and} \quad \rho \equiv \frac{E}{V} \sim N^2 T^d \ , \tag{2.44}$$

with ρ the energy density and we have used $\frac{1}{T} = \frac{\partial S}{\partial E}$. The energy density per nonabelian particle is $\frac{\rho}{N^2} = T^d = \frac{T}{(1/T)^{d-1}}$, which suggests that the characteristic size of the typical particle is $\frac{1}{T}$ with energy T. The CFT physics below this length scale $\frac{1}{T}$, in particular that of entanglement, will be indistinguishable from the ground state. Above this length scale, the presence of the energy density implies a larger number of entangling bits or partons and so a correspondingly larger entanglement. Thus the number of entangling partons $\mathcal{N}(s)$ for cell sizes $s \gg \frac{1}{T}$ is the number of partons of individual volume $(1/T)^{d-1}$ in the total cell volume s^{d-1}, i.e. $\mathcal{N}(s)|_{s\gg T^{-1}} \sim N^2 \frac{s^{d-1}}{(1/T)^{d-1}}$, thus $\mathcal{N}(s)$ is extensive for length scales larger than the inverse temperature. This implies a total entanglement

$$\begin{aligned} S \ &\sim \ \int_\epsilon^l \frac{ds}{s} \frac{V_{d-2}}{s^{d-2}} \mathcal{N}(s) \ \sim \ \frac{1}{d-1} \frac{N^2 V_{d-2}}{\epsilon^{d-2}} \ + \ N^2 \int \frac{ds}{s} \frac{V_{d-2}}{s^{d-2}} \frac{s^{d-1}}{(1/T)^{d-1}} \Big|_l \\ &\sim \ \frac{1}{d-2} \frac{N^2 V_{d-2}}{\epsilon^{d-2}} \ + \ N^2 T^{d-1} V_{d-2} l \ . \end{aligned} \tag{2.45}$$

The energy enhancement factor $\frac{s^{d-1}}{(1/T)^{d-1}}$ changes the IR behaviour as expected. The finite part of entanglement entropy is dominant for sufficiently large l and is essentially the thermal entropy in this regime. The linear growth with l of the entropy which is extensive is equivalent to the number of partons $\mathcal{N}(s)$ being extensive.

For the nonconformal theory in $d = p+1$ dim at finite temperature T, with $\rho = \frac{E}{V}$ being the energy density, the thermal entropy $S(\rho, V)$ and temperature $\frac{1}{T} = \frac{\partial S}{\partial E}$ are [81]

$$S \sim \ V g_{YM}^{(p-3)/(5-p)} \sqrt{N} \rho^{(9-p)/(2(7-p))} \ , \quad \rho \ \sim \ g_{YM}^{2(p-3)/(5-p)} N^{(7-p)/(5-p)} \ T^{2(7-p)/(5-p)} \ . \tag{2.46}$$

These can be recast as [82]

$$S \sim \ N_{eff}(1/T) V T^p, \quad \rho \sim \ N_{eff}(1/T) T^{p+1}, \quad N_{eff}(1/T) = N^2 (g_{YM}^2 N T^{p-3})^{\frac{p-3}{5-p}} \ . \tag{2.47}$$

Along the lines earlier, we could obtain the total entanglement by integrating the number of entangling partons over length scales longer than that set by the temperature: this gives ($d = p + 1$)

$$S^{finite} \sim \int \frac{ds}{s} \frac{V_{d-2}}{s^{d-2}} N_{eff}(1/T) \left(\frac{s}{(1/T)}\right)^{d-1} \sim V_{d-2} l T^{d-1} N_{eff}(1/T) \ . \quad (2.48)$$

It is important to note that the thermal entropy is essentially classical, with contributions from partons of size predominantly $\frac{1}{T}$ so that we do not integrate $N_{eff}(s)$ over all scales s: i.e. $N_{eff} = N_{eff}(1/T)$ above. In fact integrating the number of nonabelian degrees of freedom $N_{eff}(s)$ over scales $\epsilon < s < l$ in the above thermal context does not yield sensible results (e.g. giving logarithmic growth for the thermal entropy for $p = 1, 4$), in contrast with the ground state.

Now we want to interpret entanglement entropy for the pure CFT excited states dual to AdS plane waves within this scaling picture. The energy density $T_{++} = Q$ sets a characteristic length scale $Q^{-1/d}$: then the typical size of the partons is $Q^{-1/d}$. Thus for cells of size s much smaller than $Q^{-1/d}$, the parton distribution is similar to that in the ground state while for cells of size s much larger than $Q^{-1/d}$, there is an enhancement in the number of entangling partons per cell. The anisotropy induced by the flux which is along one of the spatial directions implies that the entangling partons have energy-momentum in that direction but can be regarded as essentially static in the other directions, as in the ground state. Consider first the case when the strip is along the flux direction: then as the strip width increases, the number of partons straddling the boundary increases since the partons move along the boundary. On the other hand, when the strip is orthogonal to the flux, the parton motion is orthogonal to the boundary: thus when the strip width is much larger than the characteristic size $Q^{-1/d}$ of the partons, the number of partons straddling the boundary is essentially constant since most of the partons enter the strip at one boundary and then shortly do not straddle the boundary but are completely encompassed within the strip. This reflects in the entanglement saturating for large width, with the strip orthogonal to the energy flux.

Now we consider the case of the strip along the flux in more detail. We again define the number of entangling bits or partons $\mathcal{N}(s)$ at scale s, with $\mathcal{N}(s)|_{s \ll Q^{-1/d}} \sim N^2$ for length scales much smaller than the characteristic length $Q^{-1/d}$: above this scale, we expect some nontrivial scaling of $\mathcal{N}(s)$ which will be a function of Qs^d on dimensional grounds. The precise functional form of $\mathcal{N}(s)$ for these AdS plane wave states is not straightforward to explain however: the known results for holographic entanglement entropy (2.40) suggest $\mathcal{N}(s) \sim N^2\sqrt{Qs^d}$. Although the AdS plane wave CFT states are simply the thermal CFT state in a low temperature large boost limit, this scaling of $\mathcal{N}(s)$ is not a simple boosted version of those for the thermal state (discussed below), but somewhat nontrivial. It would be interesting to explain this scaling of the AdS plane wave CFT states, perhaps keeping in mind the infinite momentum frame and Matrix theory. In this regard, we note that these AdS plane wave states preserve boost invariance, i.e. $x^{\pm} \to \lambda^{\pm 1} x^{\pm}$, $Q \to \lambda^{-2} Q$ is a symmetry of the bulk backgrounds. For the strip along the flux, the longitudinal size scales as $V_{d-2} \to \lambda V_{d-2}$ and the number of entangling partons is some function $f(Qs^d)$. Boost invariance then fixes $V_{d-2} f(Qs^d) = V_{d-2}\sqrt{Qs^d}$. Alternatively, imagine the collision of two identical plane wave states, moving in opposite directions. Assuming the resulting state has a number of partons $\mathcal{N}_L(s)\mathcal{N}_R(s) \propto Qs^d$ proportional to the energy-momentum density, we can estimate that either individual wave has $\mathcal{N}_L(s) \sim \mathcal{N}_R(s) \sim \sqrt{Qs^d}$. However this is a bit tricky since this makes $\mathcal{N}_L(s), \mathcal{N}_R(s)$ reminiscent of partition functions: a number of partons might instead be expected to be additive, as $\mathcal{N}_L(s) + \mathcal{N}_R(s)$.

Taking the number of entangling partons $\mathcal{N}(s)$ at the boundary at scale $s \gg Q^{-1/d}$ as $N^2 \frac{V_{d-2}}{s^{d-2}} \sqrt{Qs^d} = N^2 \frac{V_{d-2}}{s^{d-2}} \left(\frac{s}{Q^{-1/d}}\right)^{d/2}$, while for $s \ll Q^{-1/d}$ keeping $N^2 \frac{V_{d-2}}{s^{d-2}}$ as in the ground state, gives rise to an entanglement scaling as

$$\begin{aligned} S &\sim \int_{\epsilon}^{l} \frac{ds}{s} \frac{V_{d-2}}{s^{d-2}} \mathcal{N}(s) \sim \frac{1}{d-2} \frac{N^2 V_{d-2}}{\epsilon^{d-2}} + N^2 V_{d-2} \int \frac{ds}{s} \frac{\sqrt{Qs^d}}{s^{d-2}} \Big|_l \\ &\sim \frac{1}{d-2} \frac{N^2 V_{d-2}}{\epsilon^{d-2}} + \frac{N^2}{4-d} \sqrt{Q}\, V_{d-2} l^{2-\frac{d}{2}} \qquad [d \neq 4]\ , \\ &\sim \frac{1}{d-2} \frac{N^2 V_{d-2}}{\epsilon^{d-2}} + N^2 \sqrt{Q}\, V_2 \log(l Q^{1/4}) \qquad [d = 4]\ . \end{aligned} \tag{2.49}$$

For $d = 4$, the logarithmic growth in the finite part arises by integrating from scales longer than $Q^{-1/4}$ upto the IR scale l. Thus we see that the phenomenological scaling $\sqrt{Qs^d}$ is consistent with the holographic results. It would be interesting to understand this scaling better. Likewise for the nonconformal plane wave excited states (as in the conformal case) which we think of as chiral subsectors, the number of entangling partons at length scales s longer than that set by the energy density Q is proportional to $\sqrt{Qs^d}$ and the total finite part of entanglement for a strip subsystem of width l becomes

$$S_{finite} \sim \int \frac{ds}{s} \frac{V_{d-2}}{s^{d-2}} \sqrt{N_{eff}(s)}\ \sqrt{Qs^d} \Big|_l \sim \frac{5-p}{3-p} \frac{V_{d-2}}{l^{d-2}} \sqrt{N_{eff}(l)}\ \sqrt{Ql^d}\ , \tag{2.50}$$

recovering the holographic results [72].

It would be interesting to put the phenomenological discussions in this section on firmer footing with a view to gaining deeper insight into entanglement in field theory excited states.

2.4 Holographic mutual information: generalities

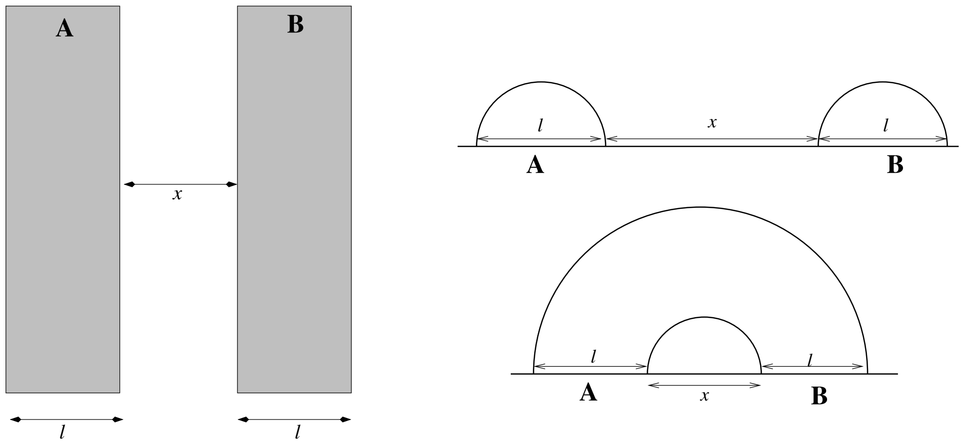

Figure 2.3: Two parallel disjoint strip subsystems of width l and separation x (and longitudinal size V_{d-2}) (left), with the disconnected extremal surface (top right) and the connected extremal surface (bottom right).

Mutual information is defined for two disjoint subsystems A and B as

$$I[A,B] = S[A] + S[B] - S[A \cup B] \ . \tag{2.51}$$

It is a measure of the correlation (both classical and quantum) between the degrees of freedom of two disjoint subsystems A and B. Mutual information is finite, positive semi-definite, and proportional to entanglement entropy when $B \equiv A^c$ (in that case, $S(A \cup A^c) = 0$). This linear combination of entanglement entropies ensures that the short distance area law divergence cancels between the various individual terms rendering the mutual information finite. There is a new cutoff-independent divergence however that arises when the two subsystems approach each other and collide, as we will see below.

The holographic prescription of Ryu-Takayanagi implies in a simple geometric way that mutual information vanishes when the two subsystems are widely separated: thus as discussed in [73], mutual information undergoes a disentangling phase transition as the separation between the two striplike subsystems A and B increases.

Recall that we choose that extremal surface which has minimal area, given the boundary conditions defined by the subsystem in question. In the case of the subsystem $A \cup B$ defined by two disjoint strips, there are two candidate extremal surfaces as in Figure 2.3. When the two subsystems are widely separated, the relevant extremal surface with lower area is simply the union of the two disconnected surfaces so that $S[A \cup B] = S[A] + S[B]$. However for nearby subsystems, the connected surface has lower area. For simplicity, we consider two disjoint parallel strip subsystems with longitudinal size V_{d-2}, and of the same width l each, with separation x. For fixed width l, we can vary the separation x. Then as we vary $\frac{x}{l}$ which is a dimensionless parameter, the behaviour of the extremal surface and its area $S[A \cup B]$ change: the extremal surface is

$$\begin{aligned} &(i) \quad \text{disconnected surface: area } S[A \cup B] = S(A) + S(B) = 2S(l)\ , \ \text{for large } \frac{x}{l}\ , \\ &(ii) \quad \text{connected surface: area } S[A \cup B] = S(2l + x) + S(x)\ , \ \text{for small } \frac{x}{l}\ . \end{aligned} \tag{2.52}$$

The Ryu-Takayanagi prescription of choosing the extremal surface of minimal area then leads to a change in the entangling surface for the combined subsystem $A \cup B$. Correspondingly the mutual information changes as

$$\begin{aligned} I[A, B] &> 0, \qquad & \frac{x}{l} < \frac{x_c}{l}\ , \\ &= 0, \qquad & \frac{x}{l} > \frac{x_c}{l}\ . \end{aligned} \tag{2.53}$$

The critical value $\frac{x_c}{l}$ is a dimensionless number, and depends on the field theory in question as well on the CFT state, as we discuss below. This critical value $\frac{x_c}{l}$ is thus the location of a sharp disentangling transition in the classical gravity approximation, since the mutual information vanishes for larger separations implying the subsystems are uncorrelated, especially in light of an interesting relation between the mutual information and correlation functions. It is known [83] that $I[A, B]$ sets an upper bound for 2-point correlation functions of operators, with one insertion at a point in region A and the other in B,

$$I[A, B] \geq \frac{(\langle \mathcal{O}_A \mathcal{O}_B \rangle - \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle)^2}{2|\mathcal{O}_A|^2 |\mathcal{O}_B|^2}\ . \tag{2.54}$$

This inequality implies that beyond the disentangling transition point all 2-point correlation functions also vanish (with one point in A and the other in B), since the mutual information vanishes. It is important to note that entanglement entropy and mutual information via the Ryu-Takayanagi prescription are $O(N^2)$ observables in the classical gravity approximation. However the 2-pt correlators are normalized as $O(1)$. One might imagine the mutual information decays as $I[A,B] \sim \sum \frac{c_\Delta}{r_{A,B}^{4\Delta}}$ and indeed the quantum $O(1)$ contributions effectively give a long distance expansion for mutual information [84] (see also [61, 70, 73, 85]). However the coefficients c_Δ at the classical level $O(N^2)$ vanish: this shows up in the large N approximation as the sharp disentangling transition in mutual information.

We now discuss this for large N conformal field theories in the ground and excited states. For the ground state, the mutual information for two strip shaped subsystems of width l parallel to each other and with separation x is

$$\begin{aligned} I[A,B] &= -cV_{d-2}\left(\frac{2}{l^{d-2}} - \frac{1}{(2l+x)^{d-2}} - \frac{1}{x^{d-2}}\right) \\ &= -c\frac{V_{d-2}}{l^{d-2}}\left(2 - \frac{1}{(2+\frac{x}{l})^{d-2}} - \frac{1}{(\frac{x}{l})^{d-2}}\right). \end{aligned} \tag{2.55}$$

This arises from the cutoff-independent parts of entanglement, the divergent terms cancelling. We see that for small separation x, the mutual information $I[A,B]$ grows as $I[A,B] \sim \frac{V_{d-2}}{x^{d-2}}$ and exhibits a divergence as $x \to 0$, i.e. when the subsystems collide with each other. As $\frac{x}{l}$ increases, $I[A,B]$ decreases and then vanishes at a critical value of $\frac{x}{l}$. Beyond this critical separation, the expression (2.55) for $I[A,B]$ as it stands is negative and is meaningless: this simply reflects the fact that the correct extremal surface for $A \cup B$ is in fact the disconnected surface, i.e. the subsystems disentangle, and mutual information actually vanishes beyond the disentangling point. This disentangling transition can be identified as the zero of $I[A,B]$ above, giving $\frac{x_c}{l} \simeq 0.732$ $[d=4]$, and 0.62 $[d=3]$, and so on.

Such a disentangling transition also happens at finite temperature, but the phase diagram is more complicated and has nontrivial dependence on the length scale $\frac{1}{T}$ set by the temperature T. For $l, x \ll \frac{1}{T}$, i.e. subsystem widths and separation small relative to the temperature scale, we only expect small corrections to the ground state behaviour above. Thus the disentangling transition point occurs at values which are "near" those for the ground state. However for large width l, the entanglement is well approximated by the extensive (linear) thermal entropy: thus

$$I[A,B] \sim T^{d-1}V_{d-2}\,\left(2l - (2l+x) - S^{fin}(x)\right) = T^{d-1}V_{d-2}\,\left(-S^{fin}(x) - x\right). \tag{2.56}$$

Thus we see that as the separation x increases, $-S^{fin}(x) > 0$ decreases and $I[A,B]$ decreases and eventually vanishes at a critical x_c, which turns out to be smaller in value than for the ground state. When the thermal entropy dominates the entanglement or equivalently the subsystem widths and separation are both large relative to the temperature scale, we see that

$$I[A,B] \sim \; T^{d-1}V_{d-2} \; (2l - (2l+x) - x) = -2T^{d-1}V_{d-2}x \; , \tag{2.57}$$

which is negative. This is a reflection of the fact that the two subsystems in fact are completely disentangled for any separation x larger than $\frac{1}{T}$. In some sense, the temperature "disorders" the system and the subsystems disentangle faster at finite temperature than in the ground state.

In what follows, we analyse holographic mutual information for AdS plane waves. We will see some similarities with the finite temperature case, but with nontrivial phase structure depending on the scale $Q^{-1/d}$. There are however some key differences as we will see below.

2.5 Mutual information in AdS plane waves

AdS plane waves exhibit anisotropy due to the energy flux in one direction. We are considering parallel disjoint strip subsystems that are either both along the flux or both orthogonal to the flux. We can analyse mutual information in two extreme regimes, where the strip widths l are large or small compared to the length scale set by the energy density flux Q. Eventually we will carry out some numerical analysis in intermediate regimes as well.

2.5.1 Wide strips: $lQ^{1/d} \gg 1$

Consider first the strip along the energy flux direction, with width direction along say x_1 (we assume $d \geq 3$). Then the spacelike strip subsystem A lying on a constant time slice has $0 \leq x_1 \leq l$, $(x^+, x^-) = (\alpha y, -\beta y) = (y, -y)$, $-\infty < y, x_2, x_3, \cdots, x_{d-2} < \infty$. The extremal surface γ_A is specified by the function $x_1 = x(r)$. V_{d-2} denotes the volume in the y and $(x_2, \cdots, x_{d-2})$ direction. ϵ is the UV cutoff. The subsystem width in terms of the turning point r_* is [52]

$$\Delta x_1 = l = 2\int_0^{r_*} dr \frac{Ar^{d-1}}{\sqrt{2 + Qr^d - A^2 r^{2(d-1)}}} \; , \tag{2.58}$$

while the entanglement entropy in terms of the area functional is

$$S_A = \frac{Area}{4G_{d+1}} = \frac{2V_{d-2}R^{d-1}}{4G_{d+1}} \int_{\epsilon}^{r_*} \frac{dr}{r^{d-1}} \frac{2+Qr^d}{\sqrt{2+Qr^d - A^2 r^{2(d-1)}}} \ . \tag{2.59}$$

The above integrals are however not exactly solvable for $d \geq 3$. There is a leading area law divergence from the contribution near the boundary $r = \epsilon$, with $EE \sim N^2 \frac{V_{d-2}}{\epsilon^{d-2}}$, where we have used $N^2 \sim \frac{R^{d-1}}{G_{d+1}}$. this integral can be expanded but, it will have contribution mostly For large energy density Q, and large width l, the turning point equation $2+Qr_*^d - A^2 r_*^{2d-2} = 0$ can be approximated as $Qr_*^d \simeq A^2 r_*^{2(d-1)} \gg 1$, so that $l \sim r_*$ from (2.58). The finite cutoff-independent piece of S_A is then estimated as

$$S_A^{finite} \sim \pm \frac{R^{d-1}}{G_{d+1}} V_{d-2}\sqrt{Q}\ l^{2-\frac{d}{2}} \qquad [d \neq 4] \tag{2.60}$$

$$\sim N^2 V_2 \sqrt{Q}\ \log(lQ^{1/4}) \qquad [d = 4] \ . \tag{2.61}$$

The sign in front of (2.60) is + for $d < 4$ and $-$ for $d > 4$.

Towards estimating mutual information for AdS plane waves, we must note that there are multiple regimes stemming from the various length scales $l,\ x,\ Q^{-1/d}$. When the strip widths and separations are large relative to the correlation length, i.e. $lQ^{1/d} \gg 1$ and $xQ^{1/d} \gg 1$, we can use the above estimates for the finite parts of entanglement entropy to estimate mutual information. For the AdS_5 plane wave, when the strips are not too far apart, we can assume mutual information is nonzero, obtaining from the finite parts above,

$$I[A,B] = 2S^{fin}(l) - S^{fin}(2l+x) - S^{fin}(x) \sim \ V_2\sqrt{Q}\log\left(\frac{l^2}{x(2l+x)}\right) . \tag{2.62}$$

The argument of the logarithm vanishes when

$$I[A,B] \to a0 \qquad \Rightarrow \qquad l^2 = 2lx + x^2, \qquad \text{i.e.}\ \ \frac{x_c}{l} = \sqrt{2} - 1 \simeq 0.414 \ . \tag{2.63}$$

Thus the subsystems disentangle at a separation less than that for the AdS_5 ground state, which has $\frac{x_c}{l} = 0.732$. It is also noteworthy that for any large Q, the subsystems disentangle only when they are sufficiently wide apart in comparison with the width, i.e. $x \geq 0.414l$, independent of the characteristic energy scale $Q^{-1/4}$: in particular the disentangling point x_c here could be substantially bigger than $Q^{-1/4}$. This transition location agrees with the analysis for hyperscaling violating spacetimes in [70] and [78], in accordance with the fact that the AdS_5

plane wave gives rise to the corresponding hyperscaling violating spacetime. The strips, being parallel to the flux, are unaffected by the reduction along the x^+-circle from that perspective. In the present case, we are studying this entirely from the higher dimensional AdS plane wave point of view. Note that this is quite distinct from the finite temperature case [78] in the corresponding regime $lT \gg 1,\ xT \gg 1$, i.e. sizes larger than the temperature scale $\frac{1}{T}$: in that case, the linear extensive growth of entanglement in this regime implied that the subsystems disentangled for any finite separation x independent of the width l (2.57).

Strictly speaking, we are thinking of the regulated AdS plane wave as a limit of the highly boosted low temperature AdS black brane, with a large separation of scales $Q \gg r_0^4$ between the energy density $Q = \lambda^2 r_0^4$ and the temperature r_0, with λ being the boost parameter. Over this wide range of length scales, the physics is dominated by the AdS plane wave description, with departures arising in the far infrared where the black brane horizon physics enters as a regulator. From this point of view, we are thinking of the strip subsystem widths as satisfying $Q^{-1/4} \ll l \ll \frac{1}{r_0}$, with the above behaviour of mutual information holding correpondingly: in the far IR when $l \gg \frac{1}{r_0}$ the behaviour of mutual information resembles that in the finite temperature case.

A similar analysis can be done for the AdS_4 plane wave, in the regime $lQ^{1/3} \gg 1$ and $xQ^{1/3} \gg 1$, taking again for simplicity both strips of equal width l with separation x. Then the mutual information arises from the finite parts of entanglement estimated (2.60) for large Q giving

$$I[A,B] \sim\ V_1\sqrt{Q}\left(2\sqrt{l} - \sqrt{2l+x} - \sqrt{x}\right). \tag{2.64}$$

This decreases as the separation x increases and finally vanishes when

$$I[A,B] \to 0 \qquad \Rightarrow \qquad \frac{x_c}{l} = \frac{1}{4}\,, \tag{2.65}$$

which is the location of the disentangling transition in this regime. Again we see that the subsystems disentangle when they are sufficiently wide apart in comparison to their widths l, without specific dependence on the energy scale $Q^{-1/3}$ as for the AdS_5 plane wave discussed above.

Nonconformal D-brane plane waves and entanglement entropy were studied in [72], with the emerging picture and scalings consistent with AdS plane waves in cases where comparison is possible. The analysis is more complicated in the nonconformal cases since there are multiple different length scales in the phase diagram. The structure of mutual information is still further complicated and we will not carry out a systematic study here. We can however make some coarse estimates in the large flux regime. For instance the D2-M2 ground state phase diagram [81] extends to a corresponding one for the D2-brane plane waves. The finite part of EE for a strip along the flux in the D2-brane supergravity regime is $S_{D2}^{fin} \sim V_1\sqrt{Q}\sqrt{l}\sqrt{N_{eff}(l)} \sim V_1\sqrt{Q}\sqrt{l}\sqrt{\frac{N^2}{(g_{YM}^2 Nl)^{1/3}}} \propto l^{1/3}$. Noting the D2-sugra regime of validity, it can be seen that this finite part is greater than $V_1\sqrt{Q}\sqrt{l}\sqrt{N^{3/2}}$ for the M2-brane (AdS_4) plane wave arising in the far IR [72]. In the D2-regime, we can approximate the mutual information as $MI_{D2} \sim V_1\sqrt{Q}(2l^{1/3}-(2l+x)^{1/3}-x^{1/3})$ which shows a disentangling transition at $\frac{x_c}{l} \sim 0.31$. Recalling that for the M2-brane regime, we have $\frac{x_c}{l} \sim 0.25$, we see that x_c decreases along the RG flow from the D2-brane sugra to the M2-brane regime. Similarly for the ground states also, it can be checked that in the D2-regime, we have $\frac{x_c}{l} \sim 0.66$ while in the M2-regime, we have $\frac{x_c}{l} \sim 0.62$. It is unclear if these are indications of some deeper structure for the "flow" of mutual information.

Now we make a few comments on mutual information in the case where the strips are orthogonal to the energy flux. In the large flux regime, we know [52] that entanglement entropy shows a phase transition for $l \gg Q^{-1/d}$ with no connected extremal surface but only disconnected ones. In this regime, we expect that mutual information simply vanishes since the connected surface of mutual information (2.52) is already disconnected: thus the entanglement is saturated for each of $S[l], S[2l+x], S[x] \sim S_{sat}$ so that $MI \sim 2S(l) - S[2l+x] - S[x] = 0$. In Sec. 2.5.2 and Sec. 2.5.3, we will study entanglement and mutual information in the perturbative regime $Ql^d \ll 1$: however in this regime, we do not expect any signature of the phase transition which is only visible for wide strips. It is then reasonable to expect some interesting interplay between the phase transition and the location of the disentangling transition for mutual information.

2.5.2 Narrow strips: $lQ^{1/d} \ll 1$, strips along flux

We would now like to understand the case of narrow strips, i.e. with the dimensionless quantity $lQ^{1/d} \ll 1$. In this limit, we expect that the entanglement entropy is only a small departure from the pure AdS case, since the energy density flux Q will only make a small correction to the ground state entanglement. We will first analyse the strip along the flux and obtain the entanglement correction to the ground state. This has parallels with "entanglement thermodynamics" [75, 76, 77] for these AdS plane waves, treating the g_{++} mode as a small deformation to AdS.

In the limit $Q^{1/d}l \ll 1$, we first calculate the change in the turning point r_* upto $O(Q)$, and then expand the width integral and area integral around AdS_{d+1}, using (2.58), (2.59). First we note that the pure AdS case, with s the turning point of the minimal surface, has the width integral

$$l = 2\int_0^s \frac{A}{\sqrt{\frac{2}{r^{2(d-1)}} - A^2}} = 2\int_0^s dr \frac{(r/s)^{d-1}}{\sqrt{1-\left(\frac{r}{s}\right)^{2(d-1)}}} = 2\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{d}{2d-2}\right)}{\Gamma\left(\frac{1}{2d-2}\right)}\right) s \;\equiv\; 2\eta s \;, \tag{2.66}$$

using $A^2 = \frac{2}{s^{d-1}}$ and $\eta = \int_0^1 \frac{x^{d-1}}{\sqrt{1-x^{2(d-1)}}}dx$. We want to calculate the change in the ground state entanglement entropy under the AdS plane wave perturbation to $O(Q)$, with the strip along the flux. With the entangling surface fixed at width l, the turning point s now changes to $r_* = s + \delta r_*$. We recast (2.58) and the turning point equation as

$$\frac{l}{2} = \int_0^{r_*} dr \frac{A}{\sqrt{\frac{g(r)}{r^{2(d-1)}} - A^2}} \quad \text{with} \quad g(r) = 2 + Qr^d\,, \quad \text{and} \quad A^2 = \frac{g(r_*)}{r_*^{2(d-1)}} \equiv \frac{g_*}{r_*^{2(d-1)}}\,. \tag{2.67}$$

Then we obtain

$$\frac{l}{2} = \int_0^{r_*} dr \frac{\frac{\sqrt{g_*}}{r_*^{d-1}}}{\frac{1}{r^{d-1}}\sqrt{g(r) - g_*\left(\frac{r}{r_*}\right)^{2(d-1)}}} = \int_0^{r_*} dr \frac{(r/r_*)^{d-1}}{f(r,r_*)} \frac{\left(1+\frac{Qr_*^d}{4}\right)}{\sqrt{1+\frac{Qr^d - Qr_*^d\left(\frac{r}{r_*}\right)^{2(d-1)}}{2f^2(r,r_*)}}}\,, \tag{2.68}$$

with the function

$$f(r,r_*) = \sqrt{1-\left(\frac{r}{r_*}\right)^{2(d-1)}}\,, \qquad 0 < f(r,r_*) < 1 \quad \text{for all } r < r_*\,. \tag{2.69}$$

The above expression has been obtained by taking $Qr_*^d \ll 1$ and expanding out the integrand. The above width integral can be further simplified to $O(Q)$ as

$$\begin{aligned} \frac{l}{2} &= \int_0^{r_*} dr \frac{(r/r_*)^{d-1}}{f(r,r_*)}\left(1+\frac{Qr_*^d}{4}\right)\left(1 - \frac{Qr^d - Qr_*^d\left(\frac{r}{r_*}\right)^{2(d-1)}}{4f^2(r,r_*)}\right) \\ &= \int_0^{r_*} dr \frac{(r/r_*)^{d-1}}{f(r,r_*)}\left(1+\frac{Q}{4f^2(r,r_*)}(r_*^d - r^d)\right) = s\eta = (r_* - \delta r_*)\eta \end{aligned} \tag{2.70}$$

the last expression arising since the width l is as in AdS. Using (2.66), we see that the leading AdS piece cancels giving

$$\delta r_* = -\frac{Q}{4\eta}\int_0^{r_*} dr \frac{(r/r_*)^{d-1}}{f^3(r,r_*)}\left(r_*^d - r^d\right) \sim -\frac{Qs^{d+1}}{4\eta}\int_0^1 dx \frac{x^{d-1}(1-x^d)}{(1-x^{2(d-1)})^{3/2}} . \quad (2.71)$$

As r_* happens to be the turning point of the minimal surface, $r < r_*$ which implies that $\delta r_* < 0$ always. Also since δr_* is $O(Q)$, we have approximated $r_* \sim s$ to obtain the second expression. Thus

$$\delta r_* \sim -\frac{Qs^{d+1}}{4\eta}\frac{\sqrt{\pi}}{(d-1)^2}\left(\frac{\Gamma(\frac{1}{d-1})}{\Gamma(\frac{1}{2}+\frac{1}{d-1})} - (d-1)\frac{\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})}\right) \equiv -\frac{Qr_*^{d+1}}{4\eta}\mathcal{N}_{r_*} . \quad (2.72)$$

We now calculate the change in the area integral and correspondingly the entanglement entropy upto $O(Q)$. For pure AdS, i.e. the CFT ground state, we have

$$4G_{d+1}S_0 = 2V_{d-2}R^{d-1}\int_0^s \frac{dr}{r^{d-1}}\frac{1}{f(r,s)} , \quad (2.73)$$

with $f(r,s) = \sqrt{1-\left(\frac{r}{s}\right)^{2(d-1)}}$ as in (2.69). We focus on the finite part of the above integral and use $l = 2s\eta$, obtaining

$$4G_{d+1}S_0 = \#R^{d-1}\frac{V_{d-2}}{\epsilon^{d-2}} - \frac{2^{d-1}\pi^{\frac{d-1}{2}}}{(d-2)}\left(\frac{\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})}\right)^{d-1}\frac{V_{d-2}}{l^{d-2}}R^{d-1} . \quad (2.74)$$

In our case of the AdS_{d+1} plane wave,

$$4G_{d+1}S = 2V_{d-2}R^{d-1}\int_0^{r_*}\frac{dr}{r^{d-1}}\frac{2+Qr^d}{\sqrt{2+Qr^d-A^2r^{2(d-1)}}} \quad (2.75)$$

Treating this as an infinitesimal g_{++}-deformation and expanding around pure AdS, we would like to obtain the $O(Q)$ change in EE, or equivalently the infinitesimal change for the plane wave excited state relative to the ground state. From the turning point equation, we have $A^2 = \frac{2+Qr_*^d}{r_*^{2(d-1)}}$ as before, giving

$$\begin{aligned}
4G_{d+1}S &= 2V_{d-2}R^{d-1}\int_0^{r_*}\frac{dr}{r^{d-1}}\frac{2+Qr^d}{\sqrt{2\Big(1-\big(\frac{r}{r_*}\big)^{2(d-1)}\Big)+Qr^d-Qr_*^d\big(\frac{r}{r_*}\big)^{2(d-1)}}} \\
&= 2\sqrt{2}V_{d-2}R^{d-1}\int_0^{r_*}\frac{dr}{r^{d-1}}\frac{1}{f(r,r_*)}\left(1+\frac{Qr^d}{2}\right) \\
&\qquad\qquad\times\left(1-\frac{Qr^d-Qr_*^d\left(\frac{r}{r_*}\right)^{2(d-1)}}{4f(r,r_*)^2}\right) \\
&= 4G_{d+1}S_0 \;+\; 2\sqrt{2}R^{d-1}\mathcal{N}_{EE}\;V_{d-2}Qr_*^2\;, \qquad (2.76)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{N}_{EE} &= \int_0^1 dx\left[\frac{x}{2\sqrt{1-x^{2(d-1)}}}+\frac{1}{4x^{d-1}\sqrt{1-x^{2(d-1)}}}\left(\frac{(1-x^d)}{(1-x^{2(d-1)})}-1\right)\right] \\
&= \frac{\sqrt{\pi}}{8(d-1)^2}\left(\frac{(d+1)\Gamma(\frac{1}{d-1})}{\Gamma(\frac{1}{2}+\frac{1}{d-1})}-\frac{2(d-1)\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})}\right)\;. \qquad (2.77)
\end{aligned}$$

It can be checked that the constant $\mathcal{N}_{EE}$ is positive, so that the correction to the entanglement entropy is positive. To $O(Q)$, we can replace r_* by s, the pure AdS turning point. Then using $l = 2s\eta$, we see that

$$\Delta S \sim\; +\frac{R^{d-1}}{G_{d+1}}\frac{\mathcal{N}_{EE}}{4\eta^2\sqrt{2}}\;V_{d-2}Ql^2 \;=+\frac{R^{d-1}}{G_{d+1}}\frac{\mathcal{N}_{EE}}{4\eta^2\sqrt{2}}\;\frac{V_{d-2}}{l^{d-2}}(Ql^d)\;, \qquad (2.78)$$

with $Ql^d \ll 1$. There are parallels of this analysis with "entanglement thermodynamics" [75, 76, 77] (see also [86, 87, 88]). In the present case, we have the energy change in the strip $\Delta E \sim \int \delta T_{tt} d^{d-1}x \sim\; QV_{d-2}l$, giving $T_E\Delta S_E \sim \Delta E$ with the "entanglement temperature" $T_E \sim \frac{1}{l}$. There is also an entanglement pressure. Although it is not crucial for our purposes here, it would be interesting to develop this further.

The above entanglement entropy change implies that the change in mutual information is negative (with $I_0[A,B]$ the mutual information in pure AdS):

$$\begin{aligned}
I[A,B] &= I_0[A,B] \;+\; \Delta I[A,B] \\
&= I_0[A,B] \;+\; \frac{R^{d-1}}{G_{d+1}}\frac{\mathcal{N}_{EE}}{\sqrt{2}}\;V_{d-2}Q\left(2l^2-(2l+x)^2-x^2\right) \\
&= I_0[A,B] \;-\; 2\frac{R^{d-1}}{G_{d+1}}\frac{\mathcal{N}_{EE}}{4\eta^2\sqrt{2}}\;V_{d-2}Ql^2\;\left(1+\frac{x}{l}\right)^2\;. \qquad (2.79)
\end{aligned}$$

Thus we see that mutual information strictly decreases, for a small T_{++} energy density flux perturbation along the strip subsystem. In this perturbative regime with the correction scaling as $O(Q)$ and as the area of the interface V_{d-2}, the entanglement and mutual information corrections involve the dimensionless quantity $V_{d-2}Ql^2$.

It is worth noting that unlike in the wide strip regime (2.62), the disentangling transition in this perturbative regime certainly depends on the energy density Q and the strip width through Ql^d. In particular, using (2.74), (2.55), we see that the mutual information (2.79) vanishes at

$$\mathcal{N}_{EE}^0\Big(\frac{1}{(\frac{x}{l})^{d-2}}+\frac{1}{(2+\frac{x}{l})^{d-2}}-2\Big)-\frac{\mathcal{N}_{EE}}{2\sqrt{2}\eta^2}Ql^d\Big(1+\frac{x}{l}\Big)^2=0\ , \tag{2.80}$$

where $\mathcal{N}_{EE}^0$ is the constant coefficient of the finite part in (2.74). A numerical study later (sec. 5.4) describes the location of the vanishing of mutual information and the disentangling transition for intermediate regimes as well, where $Ql^d \sim O(1)$.

2.5.3 Narrow strips: $lQ^{1/d} \ll 1$, strips orthogonal to flux

We describe the change in entanglement entropy and mutual information for the strips orthogonal to the flux in the perturbative regime $lQ^{1/d} \ll 1$ here. The analysis is similar to the previous case, but involves more calculation.

We first consider a single strip and study entanglement. In this case, the width direction of the strip A is parallel to x_{d-1}, with $x^\pm = \frac{t\pm x_{d-1}}{\sqrt{2}}$. The bulk extremal surface γ_A is specified by $x^+ = x^+(r),\ x^- = x^-(r)$, and the spacelike strip subsystem has width

$$\Delta x^+ = -\Delta x^- = \frac{l}{\sqrt{2}} > 0\ , \tag{2.81}$$

(spacelike implying $\Delta t = 0$) and longitudinal size $V_{d-2} \sim L^{d-2}$ with $L \gg l$ in the x_i directions. ϵ is the UV cut-off. Then the width integrals and the entanglement entropy area functional reduce to [52]

$$\frac{\Delta x^+}{2} = \int_0^{r_*}\frac{dr}{\sqrt{\frac{A^2B^2}{r^{2(d-1)}}+Qr^d-2B}}\ ,\qquad \frac{\Delta x^-}{2} = \int_0^{r_*}\frac{(Qr^d-B)\ dr}{\sqrt{\frac{A^2B^2}{r^{2(d-1)}}+Qr^d-2B}}\ ,$$
$$S_A = \frac{2R^{d-1}V_{d-2}}{4G_{d+1}}\int_\epsilon^{r_*}\frac{dr}{r^{d-1}}\frac{AB}{\sqrt{A^2B^2-2Br^{2(d-1)}+Qr^{3d-2}}}\ . \tag{2.82}$$

Unlike the previous case, here we have two parameters A, B and two integrals specifying the subsystem width l as a function of the turning point r_* of the extremal surface, given by (2.82). For pure AdS, with $Q = 0$, (2.82) alongwith (2.81) fixes $B = 1$, with $x^\pm$ treated "symmetrically" as expected in the absence of the energy flux. We will treat the AdS plane wave case in $O(Q)$ perturbation theory and expand both integrals around AdS. The turning point equation here is

$$\frac{A^2B^2}{r_*^{2(d-1)}} + Qr_*^d - 2B = 0 \qquad \Rightarrow \qquad \frac{A^2B^2}{r^{2(d-1)}} = \left(\frac{r_*}{r}\right)^{2(d-1)} (2B - Qr_*^d) \; . \tag{2.83}$$

This recasts the denominator of the width integrals in terms of

$$f(r, r_*) = \sqrt{1 - \left(\frac{r}{r_*}\right)^{2(d-1)}} \tag{2.84}$$

and B alone,

$$\left[\frac{A^2B^2}{r^{2(d-1)}} + Qr^d - 2B\right]^{1/2} = \left(\frac{r_*}{r}\right)^{d-1} f(r, r_*)\sqrt{2B}\left[1 - \frac{Qr_*^d\left(1 - \left(\frac{r}{r_*}\right)^{3d-2}\right)}{2Bf^2}\right]^{1/2} . \tag{2.85}$$

However unlike (2.67) earlier, we are still left with the parameter B here, so the turning point equation does not suffice. The other relation for recasting both A and B in terms of Q, r_* comes from the fact that we have a space-like subsystem, i.e. (2.81). Specifically with the pure AdS case corresponding to $B = 1$, in this perturbative regime with $Ql^d \ll 1$, we can safely assume that $B = 1 + \Delta B$ with $\Delta B \sim O(Q)$. Since the two width integrals for Δx^+ and Δx^- must obey the equality $\Delta x^+ = -\Delta x^- = \frac{l}{\sqrt{2}}$, we must have that the change in the turning point δr_* obtained from both is the same, which fixes $\Delta B \propto Qr_*^d$ as we will see.

To elaborate, from (2.81), (2.82), (2.85), we have

$$\frac{\Delta x^+}{\sqrt{2}} = \frac{l}{2} = \int_0^{r_*} dr \frac{(r/r_*)^{d-1}}{f(r, r_*)\sqrt{B}} \frac{1}{\left[1 - \frac{Qr_*^d\left(1 - (r/r_*)^{3d-2}\right)}{2Bf^2}\right]^{1/2}} \; . \tag{2.86}$$

Now, with $B = 1 + \Delta B = 1 + O(Q)$, we can expand this to $O(Q)$ obtaining

$$\frac{l}{2} = \int_0^{r_*} dr \frac{(r/r_*)^{d-1}}{f(r, r_*)} - \frac{\Delta B}{2}\int_0^{r_*} dr \frac{(r/r_*)^{d-1}}{f(r, r_*)} + Qr_*^d \int_0^{r_*} dr \frac{(r/r_*)^{d-1}(1 - (r/r_*)^{3d-2})}{4f^3(r, r_*)} \; . \tag{2.87}$$

As in the previous subsection, we keep our entangling surface fixed so $l = 2s\eta$, with s the pure AdS turning point. The new turning point is $r_* = s + \delta r_*$, so $l/2 = r_*\eta - \delta r_*\eta$. Thus

$$-\delta r_*\eta = -\frac{\Delta B r_*}{2}\int_0^1 dx \frac{x^{d-1}}{\sqrt{1-x^{2(d-1)}}} + Q r_*^{d+1}\int_0^1 dx \frac{x^{d-1}(1-x^{3d-2})}{4(1-x^{2(d-1)})^{3/2}} \; . \tag{2.88}$$

Starting with the Δx^- integral and using (2.81), (2.82), (2.85), we have analogous to (2.86),

$$\frac{l}{2} = \int_0^{r_*} dr \frac{(r/r_*)^{d-1}}{f(r,r_*)} \frac{B - Qr^d}{\sqrt{B}\Big(1 - \frac{Qr_*^d\left(1-(r/r_*)^{3d-2}\right)}{2Bf^2}\Big)^{1/2}} \; . \tag{2.89}$$

As above, expanding to $O(Q)$ gives

$$\begin{aligned}\frac{l}{2} = \int_0^{r_*} dr \frac{(r/r_*)^{d-1}}{f(r,r_*)} + \frac{\Delta B}{2}\int_0^{r_*} dr \frac{(r/r_*)^{d-1}}{f(r,r_*)} \\ + Qr_*^d \int_0^{r_*} dr \frac{(r/r_*)^{d-1}(1-(r/r_*)^{3d-2})}{4f^3(r,r_*)} - Q\int_0^{r_*} dr \frac{r^d (r/r_*)^{d-1}}{f(r,r_*)} \; .\end{aligned} \tag{2.90}$$

Then as above, the change in turning point is given by

$$\begin{aligned}-\delta r_*\eta = \frac{r_*\Delta B}{2}\int_0^1 dx \frac{x^{d-1}}{\sqrt{1-x^{2(d-1)}}} + Qr_*^{d+1}\int_0^1 dx \frac{x^{d-1}(1-x^{3d-2})}{4(1-x^{2(d-1)})^{3/2}} \\ - Qr_*^{d+1}\int_0^1 dx \frac{x^{2d-1}}{\sqrt{1-x^{2(d-1)}}} \; .\end{aligned} \tag{2.91}$$

For this spacelike subsystem, the above (2.91) should be identical to (2.88). Using (2.66), this gives

$$\Delta B = \alpha Q r_*^d \; , \quad \text{with} \quad \alpha = \frac{1}{\eta}\int_0^1 dx \frac{x^{2d-1}}{\sqrt{1-x^{2(d-1)}}} = \frac{\Gamma(\frac{1}{2d-2})\Gamma(\frac{1}{d-1})}{2(d-1)^2\Gamma(\frac{3}{2}+\frac{1}{d-1})\Gamma(\frac{d}{2d-2})} \; . \tag{2.92}$$

Using the above, we get

$$\delta r_* = \beta Q r_*^{d+1} \; , \qquad \text{with} \qquad \beta = \frac{1}{4(d-1)} - \frac{2^{\frac{1}{d-1}}}{8(d-1)^3\sqrt{\pi}} \frac{\Gamma(\frac{1}{2d-2})^2}{\Gamma(\frac{3}{2}+\frac{1}{d-1})} \; . \tag{2.93}$$

It can be checked that $\beta < 0$ ($\beta \to 0^-$ for large d): thus δr_* is negative.

We can do a similar perturbation for finding the $O(Q)$ change in the entanglement entropy S_0 for pure AdS given by (2.73). In the present AdS_{d+1} plane wave case with the strip orthogonal to the flux, the entanglement entropy is (2.82), i.e.

$$4G_{d+1}S = 2V_{d-2}R^{d-1}\int_{\epsilon}^{r_*}\frac{dr}{r^{d-1}}\frac{AB}{r^{d-1}\sqrt{\frac{A^2B^2}{r^{2(d-1)}}+Qr^d-2B}}\ . \tag{2.94}$$

From the turning point equation, we know that $AB = r_*^{d-1}\sqrt{2B-Qr_*^d}$. With $f(r,r_*)$ as defined before, the EE can be recast as

$$4G_{d+1}S = 2V_{d-2}R^{d-1}\int_{\epsilon}^{r_*}\frac{dr}{r^{d-1}}\frac{\left(1-\frac{Qr_*^d}{2B}\right)^{1/2}}{f(r,r_*)\left[1-\frac{Qr_*^d(1-(r/r_*)^{3d-2})}{2Bf^2}\right]^{1/2}}\ . \tag{2.95}$$

Now with $B = 1+\alpha Qr_*^d$, we see that the perturbation in EE is independent of ΔB to $O(Q)$, since B appears above only as $\frac{Q}{B}$. Expanding S to $O(Q)$, we obtain

$$\begin{aligned}
4G_{d+1}S &= 2V_{d-2}R^{d-1}\int_0^{r_*}\frac{dr}{r^{d-1}}\frac{1}{f(r,r_*)}\left[1-\frac{Qr_*^d}{4}+\frac{Qr_*^d}{4f^2}\left(1-(r/r_*)^{3d-2}\right)\right] \\
&= 4G_{d+1}S_0 + 2V_{d-2}R^{d-1}Qr_*^2\int_0^1 dx\left[\frac{1-x^{3d-2}}{4x^{d-1}(1-x^{2(d-1)})^{3/2}}\right. \\
&\qquad\qquad\left.-\frac{1}{4x^{d-1}(1-x^{2(d-1)})^{1/2}}\right] \\
&= 4G_{d+1}S_0\ +\ 2V_{d-2}R^{d-1}Qr_*^2\mathcal{M}_{EE}\ ,
\end{aligned} \tag{2.96}$$

with

$$\mathcal{M}_{EE} = \frac{\sqrt{\pi}}{4(d-1)^2}\left[\frac{\Gamma(\frac{1}{d-1})}{\Gamma(\frac{d+1}{2d-2})}-(d-1)\frac{\Gamma(\frac{d}{2d-2})}{\Gamma(\frac{1}{2d-2})}\right]\ . \tag{2.97}$$

It can be checked that $\mathcal{M}_{EE} > 0$ for $d > 1$. Thus the change in entanglement entropy is positive, as before. To $O(Q)$, we have $r_* \sim l$, so that as before,

$$\Delta S = \frac{R^{d-1}}{2G_{d+1}}\frac{\mathcal{M}_{EE}}{4\eta^2}\ V_{d-2}Ql^2\ =\ \frac{R^{d-1}}{2G_{d+1}}\frac{\mathcal{M}_{EE}}{4\eta^2}\ \frac{V_{d-2}}{l^{d-2}}(Ql^d)\ , \tag{2.98}$$

so that as in (2.79) previously, the mutual information decreases as

$$I[A,B] = I_0[A,B]\ -\ 2\frac{R^{d-1}}{G_{d+1}}\frac{\mathcal{M}_{EE}}{8\eta^2}\ V_{d-2}Ql^2\ \left(1+\frac{x}{l}\right)^2\ , \tag{2.99}$$

in this perturbative regime with $Ql^d \ll 1$. It should not be surprising that no hint of the phase transition is visible in this perturbative regime. For subsystem size well below the characteristic length scale set by the energy density, i.e. $l \ll Q^{-1/d}$, we only expect small corrections to the ground state entanglement and mutual information structure. The phase transition on the other hand corresponds to strips much wider than the characteristic length scale. In that regime, the two integrals for $\Delta x^{\pm}$ scale rather differently so that the spacelike subsystem requirement cannot be met: this leads to the absence of a connected surface and is the reflection of a phase transition. The corresponding entanglement saturation occurs since the degrees of freedom responsible for entanglement do not straddle the boundary for long if their size $\sim O(Q^{-1/d})$ is much smaller than the subsystem width, since they enter the strip and leave.

2.5.4 A more complete phase diagram and some numerical analysis

In the previous subsections, we have studied entanglement entropy and mutual information for large and small Ql^d, Qx^d. It is interesting to study the interpolation between these, including the regime where Ql^d, Qx^d are $O(1)$. Towards this, we perform a numerical study of the entanglement entropy integrals and thence mutual information (using Mathematica). The plots in Figure 2.4 and Figure 2.5 show the finite cutoff-independent part of entanglement entropy (black, green and blue curves) for the AdS_4 and AdS_5 plane waves, setting $Q = 1, 3, 10$ respectively, in the case of the strip along the energy flux: the red curves are those for pure AdS_4 and AdS_5. In the numerics, the area integrals have been regulated using a small UV cutoff regulator and subtracting off the area law divergence term, we obtain the finite part. by subtracting the area of the disconnected surface and

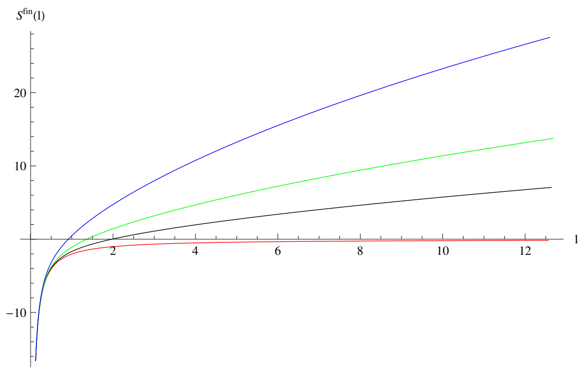

Figure 2.4: Plots of the finite parts of entanglement entropy for the AdS_4 ground state (red) and the AdS_4 plane wave (the black, green and blue curves correspond to the values $Q = 1, 3, 10$ respectively).

For small l, we see that the AdS plane wave (black, green, blue) curves lie "above" the pure AdS (red) curves, which means the finite entanglement is larger than for the ground state. This is of course consistent with the previous analytic studies in the perturbative and large Ql^d regimes but the plots show that this is also true for all Ql^d. Furthermore, the curves for larger Q values lie "above" those for smaller Q values, which is intuitively reasonable, implying that the finite entanglement increases with increasing energy density Q. The plot regions for large l are in reasonable agreement with fitted curves for $\sqrt{l}$ and $\log l$ (the fits improve with increasing accuracy, number of data points etc as expected with numerics).

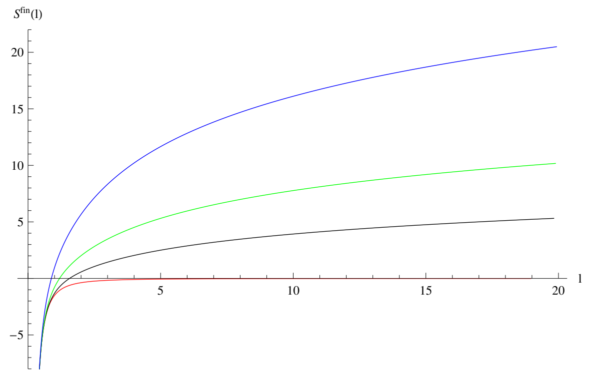

Figure 2.5: Plots of the finite parts of entanglement entropy for the AdS_5 ground state (red) and the AdS_5 plane wave (the black, green and blue curves correspond to the values $Q = 1, 3, 10$ respectively).

Likewise, Figure 2.6 shows the plot of mutual information vs the separation x for the AdS_5 plane wave with both strip subsystems along the flux (with fixed widths l taken as $l = 50$). The small x region shows a growth reflecting the divergence when the subsystems approach to collide (which is similar to the divergence for pure AdS_5). The mutual information vanishes at the critical value $\frac{x_c}{l} = 0.41$. We have also checked that the corresponding plot for pure AdS_5 behaves as expected, with the critical value $\frac{x_c}{l} \simeq 0.732$.

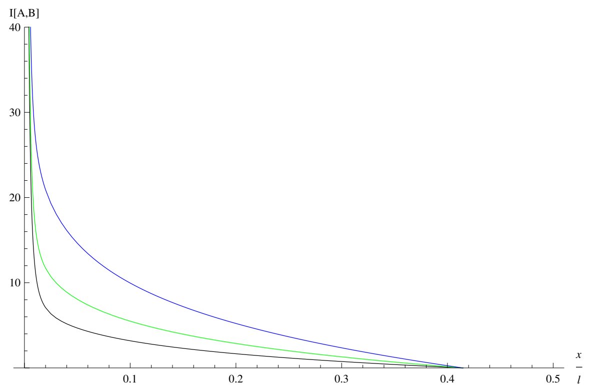

Figure 2.6: Plot of the MI vs $\frac{x}{l}$ with fixed width l for the AdS_5 plane wave.

Figure 2.7 shows the $\frac{x}{l}$ vs $lQ^{1/d}$ parameter space (shaded regions) with nonzero mutual information for the AdS_5 plane wave with both strip subsystems along the flux. We vary the width l and find the critical value x_c holding Q fixed: the three curves are for $Q = 1, 3, 10$ as before. We see that the critical value $\frac{x_c}{l}$ interpolates from about 0.732 ($lQ^{1/d} \ll 1$, approximately AdS_5 behaviour) to 0.41 for the AdS_5 plane wave. We see that the mutual information parameter space remains nonzero for large $lQ^{1/d}$, unlike the finite temperature case [78] where the curve has finite domain (with $x_c = 0$ for large lT). We have seen previously that in the wide strip regime $Ql^d \gg 1$, the mutual information disentangling transition location is independent of the energy density Q: this is reflected in Figure 2.7

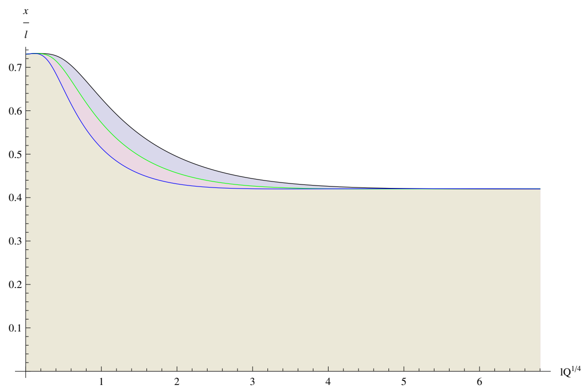

Figure 2.7: Plot of the $\frac{x}{l}$ vs $lQ^{1/d}$ parameter space with nonzero mutual information for the AdS_5 plane wave.

by the fact that the black, green, blue curves all flatten out for large l, signalling that the critical value $\frac{x_c}{l}$ is independent of the precise curve and corresponding Q value. However we note that in the intermediate $Ql^d \sim O(1)$ regime, the mutual information disentangling transition location $\frac{x_c}{l}$ certainly depends on the Q value, the different curves being distinct. Thus it is only in the $Ql^d \gg 1$ regime that the mutual information disentangling transition becomes effectively independent of the energy flux Q.

There are similar plots for the AdS_4 plane wave, which we have not shown.

Our discussion so far and the corresponding plots have been for strips parallel to the energy flux. For the strips orthogonal to the flux, entanglement shows a phase transition, corroborated in the corresponding plot (shown in [52]). Plotting mutual information appears more intricate with more technical challenges in general. For wide strips $l \gtrsim Q^{-1/d}$, the strip entanglements saturate: crude plots show the strips disentangling at critical $\frac{x_c}{l}$ values varying as Q varies, with all $\frac{x_c}{l}$ less than those for the strip along the flux (e.g. $\frac{x_c}{l} \sim 0.11$ with $Q = 1$, AdS_4 plane wave). It would be interesting to study this more completely.

2.6 Discussion

We have studied entanglement entropy and mutual information in AdS_{d+1} plane waves dual to CFT excited states with energy-momentum density $T_{++} = Q$, building on [51, 52], focussing on $d = 3, 4$ for two strips of width l and separation x, parallel and orthogonal to the flux.

For the strips parallel to the flux, mutual information exhibits a disentangling transition at a critical separation $\frac{x_c}{l}$ less than that for the ground state. For wide strips $Ql^d \gg 1$, we see that the subsystems disentangle only when they are sufficiently wide apart in comparison with the width: the critical separation $\frac{x_c}{l}$ is independent of the characteristic energy scale $Q^{-1/d}$ in this regime. This is quite distinct from the finite temperature case [78] where e.g. the linear extensive growth of entanglement in the corresponding regime $lT \gg 1$ implies the subsystems disentangle for any finite separation x independent of l. For the strips orthogonal to the flux, entanglement entropy shows a phase transition for $l \gg Q^{-1/d}$ [52]: in this case, entanglement is saturated and so mutual information also vanishes. In the perturbative regime $Ql^d \ll 1$ for the strips both parallel and orthogonal to the flux, we have seen that the change in entanglement entropy is $\Delta S \sim +V_{d-2}Ql^2$ with the analysis similar to "entanglement thermodynamics". Here the mutual information always decreases. Thus the disentangling transition in this regime again occurs for separations smaller than those for the ground state. In this perturbative regime, the critical separation $\frac{x_c}{l}$ certainly depends on Q and l. The numerical study shows the critical $\frac{x_c}{l}$ has nontrivial dependence on Q in intermediate regimes as well. As one approaches the wide strip regime $Ql^d \gg 1$, the mutual information curves approach each other and flatten out, signalling independence with Q.

Overall this suggests that the energy density disorders the system, so that the subsystems disentangle faster relative to the ground state. The thermal state is disordered, since in the regime with linear (extensive) entropy, the subsystems are disentangled or uncorrelated for any nonzero separation x. The AdS plane wave states are in some sense "partially ordered": the disentangling transition location occurs at critical values $\frac{x_c}{l}$ smaller than those for the ground state for the strip along the energy flux, but the critical value remains nonzero even for wide strips

$Ql^d \gg 1$. Perhaps this "semi-disordering" is also true for more general excited states that are "in-between" the ground and thermal states.

The AdS_5 plane wave gives rise to a hyperscaling violating spacetime exhibiting logarithmic violation of entanglement entropy, suggesting that perhaps these are indications of Fermi surfaces [69, 70]. In the regime where the strip widths and separation are large relative to the energy scale $Q^{-1/4}$, the logarithmic scaling of entanglement implies a corresponding scaling of mutual information, similar to the corresponding behaviour for Fermi surfaces. This regime is of course just one part of the full phase diagram thinking of these as simply excited states in AdS/CFT, as we have seen. It would be interesting to explore these further.

Chapter 3

Hydrodynamics in hyperscaling violating Lifshitz theories: A membrane paradigm approach

The principle of gauge/gravity duality has emerged as a highly successful toolbox for the understanding of hydrodynamics of strongly coupled field theories. Experimental realizations of such field theories have been observed in the Relativistic Heavy-Ion Collider (RHIC) at the Brookhaven National Laboratories, USA. Heavy-ion collision experiments produces an exotic phase of matter called the *quark-gluon plasma* which is better approximated as a strongly coupled fluid rather than a weakly interacting gas. Hence, gauge/gravity duality has proved to be successful in uncovering universal properties of various transport coefficients. One of the most famous example is the shear viscosity to entropy density ratio which has been conjectured to have a lower bound of $\frac{1}{4\pi}$ for relativistic theories admitting an Einstein gravity dual [89, 90]. The duality has also provided deep physical insight into relativistic and non-relativistic hydrodynamics through the fluid/gravity correspondence [91]

The AdS/CFT correspondence tell us that the field theory at thermodynamic equilibrium is described by a static black brane metric in the dual gravitational model. Basic principles of thermodynamics dictates that a perturbation away from equilibrium will result in thermalization–the system will gradually settle back into an equilibrium configuration. In the dual theory, this corresponds to fluctuations of the black brane background which eventually falls back into the brane. In the long wavelength, low frequency limit both these descriptions are consistent with the hydrodynamics of a relativistic CFT. In [P2], we used the technique pioneered by Kovtun, Son and Starinets [96] in order to evaluate the shear diffusion constant and shear viscosity of a non-relativistic fluid whose dual theory is described by (1.1). However, before discussing our work in further details, we will briefly review the work done in [96] in the context of relativistic theories for a better understanding of our generalization.

3.1 Review: Membrane paradigm approach for evaluating response function in relativistic theories

Kovtun, Son and Starinets formulated charge and shear diffusion for black brane backgrounds in terms of long-wavelength limits of perturbations on an appropriately defined *stretched horizon*, the broad perspective akin to the membrane paradigm [97]. Their analysis, which is quite general, begins with a background metric of the form

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = G_{tt}(r)dt^2 + G_{rr}(r)dr^2 + G_{xx}(r)\sum_{i=1}^{d-1} dx_i^2 \ , \tag{3.1}$$

which includes the hyperscaling violating backgrounds (4.10) as a subfamily. Charge difffusion of a gauge field perturbation A_μ in the background (3.1) is encoded by the charge diffusion constant D, defined through Fick's Law $j^i = -D\partial_i j^t$, where the 4-current j^μ is defined on the stretched horizon $r = r_h$ (with n the normal) as $j^\mu = n_\nu F^{\mu\nu}|_{r=r_h}$. Then current conservation $\partial_\mu j^\mu = 0$ leads to the diffusion equation $\partial_t j^t = -\partial_i j^i = D\partial_i^2 j^t$, with D the corresponding diffusion constant. Fick's law in turn can be shown to apply if the stretched horizon is localized appropriately with regard to the parameters Γ, q, T. Translation invariance along $x \in \{x_i\}$ allows considering plane wave modes for the perturbations $\propto e^{-\Gamma t + iqx}$, where Γ is the typical time scale of variation and q the x-momentum. In the IR regime, the modes vary slowly: this hydrodynamic regime is a low frequency, long wavelength regime. The diffusion of shear gravitational modes can be mapped to charge diffusion [96]: under Kaluza-Klein compactification of one of the directions along which there is translation invariance, tensor perturbations in the original background map to vector perturbations on the compactified background.

We turn on the metric fluctuations h_{xy} and h_{ty} ($x \equiv x_1$, $y \equiv x_2$) around (3.1), depending only on t, r, x, i.e. $h_{ty} = h_{ty}(t, x, r)$, $h_{xy} = h_{xy}(t, r, x)$. Other fluctuation modes can be consistently set to zero. There is translation invariance along the y-direction: thus after a y-compactification, the modes h_{xy} and h_{ty} become components of a $U(1)$ gauge field in the dimensionally reduced d-dim spacetime. The components are given by

$$g_{\mu\nu} = G_{\mu\nu}(G_{xx})^{\frac{1}{d-2}} \quad [\mu, \nu = 0, \ldots, d-1]; \quad A_0 = (G_{xx})^{-1}h_{ty} \ , \quad A_x = (G_{xx})^{-1}h_{xy} \ , \tag{3.2}$$

where $G_{\mu\nu}$ is the metric given by (3.1). A part of the gravitational action contains the Maxwell action with an r-dependent coupling constant,

$$\sqrt{-G}R \to \ -\frac{1}{4}\sqrt{-g}F_{\alpha\beta}F_{\gamma\delta}g^{\alpha\gamma}g^{\beta\delta}(G_{xx})^{\frac{d-1}{d-2}} \ . \tag{3.3}$$

The gauge field equations following from the action are

$$\partial_\mu\Big(\frac{1}{g_{\text{eff}}^2}\sqrt{-g}F^{\mu\nu}\Big) = 0 \ , \qquad \frac{1}{g_{\text{eff}}^2} = G_{xx}^{\frac{d-1}{d-2}} \ , \tag{3.4}$$

where we have read off the r-dependent g_{eff} from the compactified action. Analysing these Maxwell equations and the Bianchi identity assuming gauge field ansatze $A_\mu = a_\mu(r)e^{-\Gamma t + iqx}$ and radial gauge $A_r = 0$ as in [96] shows interesting simplifications in the near-horizon region. When $q = 0$, these lead to $\partial_r\big(\frac{\sqrt{-g}}{g_{\text{eff}}^2}g^{rr}g^{tt}\partial_r A_t\big) = 0$. We impose the boundary condition that the gauge fields vanish at $r = r_c \sim 0$. As in [96], for q nonzero but small, we assume an ansatz for A_t as a series expansion in $\frac{q^2}{T^{2/z}}$

$$\begin{aligned} A_t &= A_t^{(0)} + A_t^{(1)} + \ldots \ , \qquad\qquad A_t^{(1)} = O\Big(\frac{q^2}{T^{2/z}}\Big) \ , \\ A_t^{(0)} &= Ce^{-\Gamma t + iqx}\int_{r_c}^{r} dr' \frac{g_{tt}(r')g_{rr}(r')}{\sqrt{-g(r')}} \cdot g_{\text{eff}}^2(r') \\ &= Ce^{-\Gamma t + iqx}\int_{r_c}^{r} dr' \ \frac{G_{tt}(r')G_{rr}(r')}{G_{xx}(r')\sqrt{-G(r')}} \ , \end{aligned} \tag{3.5}$$

using (3.2), (3.4), with C some constant. Making a second assumption

$$|\partial_t A_x| \ll |\partial_x A_t| \tag{3.6}$$

as in [96], the gauge field component A_x, using the A_t solution, becomes

$$A_x = A_x^{(0)} + A_x^{(1)} + \dots \ ,$$
$$A_x^{(0)} = -\frac{i\Gamma}{q} C e^{-\Gamma t + iqx} \int_{r_c}^{r} dr' \frac{g_{xx}(r') g_{rr}(r')}{\sqrt{-g(r')}} \cdot g_{\text{eff}}^2(r') = -\frac{i\Gamma}{q} C e^{-\Gamma t + iqx} \int_{r_c}^{r} dr' \frac{G_{rr}(r')}{\sqrt{-G(r')}} \ , \tag{3.7}$$

again as a series expansion. As for A_t, we impose the boundary condition $A_x \to 0$ as $r \to r_c \sim 0$. In [96], the authors show that the above series expansions are self-consistent provided the following conditions hold on the location r_h of the stretched horizon and the parameters q, Γ and T (equivalently r_0):

$$e^{-\frac{T^2}{q^2}} \ll \frac{\frac{1}{r_0} - r_h}{\frac{1}{r_0}} \ll \frac{q^2}{T^2} \ll 1 \qquad \frac{1}{r_0} : \text{real event horizon.} \mathit{convo} \tag{3.8}$$

Physically the above condition means that although the stretched horizon is infinitesimally close to the real event horizon, it cannot be exponentially close to the same. This enables us to define Fick's law on the stretched horizon, and thereby the diffusion equation. The shear diffusion constant for the uncompactified higher dimensional theory then becomes

$$\begin{aligned} \mathcal{D} &= \frac{\sqrt{-g(r_h)}}{{g_{\text{eff}}}^2(r_h) g_{xx}(r_h) \sqrt{-g_{tt}(r_h) g_{rr}(r_h)}} \int_{r_c}^{r_h} dr \frac{-g_{tt}(r) g_{rr}(r) {g_{\text{eff}}}^2(r)}{\sqrt{-g(r)}} \\ &= \frac{\sqrt{-G(r_h)}}{\sqrt{-G_{tt}(r_h) G_{rr}(r_h)}} \int_{r_c}^{r_h} dr \frac{-G_{tt}(r) G_{rr}(r)}{G_{xx}(r) \sqrt{-G(r)}} \ . \end{aligned} \tag{3.9}$$

where r_c is the location of the boundary, and we are evaluating $\mathcal{D}$ at the stretched horizon which is at an infinitesimal distance away from the true event horizon located at r_h. This formalism essentially maps the shear diffusion constant for a $d+1$ dimensional theory to the charge diffusion of a d-dimensional theory.

Using the thermodynamic relation $\epsilon + P = Ts$, one can find a relation connecting the shear viscosity and shear diffusion given by

$$\eta = sT \frac{\sqrt{-G(r_h)}}{\sqrt{-G_{tt}(r_h) G_{rr}(r_h)}} \int_{r_c}^{r_h} dr \frac{-G_{tt}(r) G_{rr}(r)}{G_{xx}(r) \sqrt{-G(r)}} \ . \tag{3.10}$$

As an example, let us consider the metric of a stck on N non-extremal D3 branes in type IIB supergravity which in the near-horizon region takes the form

$$ds^2 = \frac{r^2}{R^2}(-f(r) dt^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{r^2 f(r)} dr^2 + R^2 d\Omega_5^2 \ . \tag{3.11}$$

where R is a constant (radius of the AdS space) and $f(r) = 1 - r_0^4/r^4$. The holographic dual theory for the above gravitational background is $\mathcal{N} = 4$ SYM theory with a gauge group $SU(N)$ in the limit of large N and large 't Hooft coupling living in 3+1 spacetime dimensions. The field theory is defined at the same temperature as the Hawking temperature of the gravitational background which is $T = r_0/\pi R^2$. The entropy density is $s = \frac{\pi^2}{2}N^2T^3$. Using (3.9) and (3.10), the shear diffusion constant and shear viscosity turns out to be

$$\mathcal{D} = \frac{1}{2\pi T} \quad \text{and} \quad \eta = \frac{\pi}{8}N^2T^3 \ . \tag{3.12}$$

This leads us to the relation

$$\frac{\eta}{s} = \frac{1}{4\pi} \ . \tag{3.13}$$

The remarkable regularity in the ratio of shear viscosity to entropy was also obtained for a stack of N M2 branes, M5 branes as well as Dp-branes. This universal behaviour of η/s led Kovtun, Son and Starinets to propose a lower bound on this ratio. Restoring the canonical dimensions, one can write the proposed bound as

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B} \approx 6.08 \times 10^{-13} K \cdot s \ . \tag{3.14}$$

Since this lower bound does not contain the speed of light c, [96] suggested that this is a lower bound for all systems, including non-relativistic ones. This motivated us to look into gravitational backgrounds exhibiting hyperscaling violation which has been proposed to be gravitational dual of non-relativistic systems.

3.2 Shear diffusion in the absence of gauge field: Dilaton gravity

In this section, we will adapt the technique pioneered by Kovtun, Son and Starinets in [96] in the context of hyperscaling violating Lifshitz theories. In order to simplify our understanding further, we will not introduce any perturbation in the gauge field sector and only consider shear gravitational perturbations. Although, this program was carried out in [P2] for a generic $d_i + 2$-dimensional gravitational background, for the sake of simplicity we will present the results focussing on 4 bulk dimensions. In the subsequent chapter, we will analyse the same system in arbitrary spacetime dimensions through studying the relevant quasinormal modes.

In this section, we will be concerned with the hyperscaling violating metric given by

$$ds^2 = r^\theta\left(-\frac{f(r)}{r^{2z}}dt^2 + \frac{dr^2}{f(r)r^2} + \frac{dx^2+dy^2}{r^2}\right) \ , \quad d_i = 2 \ , \quad d_{eff} = 2-\theta \ , \quad (3.15)$$

where $f(r) = 1-(r_0 r)^{2+z-\theta}$. The temperature for the dual field theory (i.e. the Hawking temperature for the black brane) is

$$T = \frac{2+z-\theta}{4\pi}r_0^z \ . \quad (3.16)$$

Similar to [96] we will make a gauge choice for the perturbations by setting $h_{\mu r}=0$ (radial gauge) and assume that the perturbations to be of the form $h_{\mu\nu}(t,x,r) = e^{-i\omega t+iq\cdot x}h_{\mu\nu}(r)$ where x is one of the spatial directions in the boundary theory. The shear mode h_{xy} couples to h_{ty} and decouples from the scalar mode φ giving us a system of three coupled equations,

$$\partial_r(r^{z+\theta-3}\partial_r(r^{2-\theta}h_{ty})) - \frac{r^{z+\theta-3}}{f}q(\omega r^{2-\theta}h_{xy} + qr^{2-\theta}h_{ty}) = 0 \ , \quad (3.17)$$

$$\partial_r(r^{-1-z+\theta}f\partial_r(r^{2-\theta}h_{xy})) + \frac{r^{z+\theta-3}}{f}\omega(\omega r^{2-\theta}h_{xy} + qr^{2-\theta}h_{ty}) = 0 \ , \quad (3.18)$$

$$q\partial_r(r^{2-\theta}h_{xy}) + \frac{\omega}{f}r^{2z-2}\partial_r(r^{2-\theta}h_{ty}) = 0 \ . \quad (3.19)$$

For the sake of completion, we have listed all the other equations of motions (including the gauge field perturbation) in Appendix A. Taking motivation from [96], we compactify the theory along y-direction to obtain an auxilliary theory in one lower dimension with gauge field $\mathcal{A}_t$ and $\mathcal{A}_x$. These gauge fields in the 3 dimensional theory capture the dynamics of the gravitational perturbations of the 3+1-dimensional hvLif theory. In terms of the y-compactified theory variables

$$\begin{aligned} &g_{\mu\nu} = r^{\theta-2}G_{\mu\nu} \qquad [\mu,\nu = t,x,r]; \qquad \mathcal{A}_t = r^{2-\theta}h_{ty} \ , \quad \mathcal{A}_x = r^{2-\theta}h_{xy} \ , \\ &\mathcal{F}_{rt} = \partial_r(r^{2-\theta}\mathcal{A}_t) \ , \qquad \mathcal{F}_{rx} = \partial_r(r^{2-\theta}\mathcal{A}_x) \ , \qquad \mathcal{F}_{tx} = -ir^{2-\theta}(\omega h_{xy} + qh_{ty}) \ , \end{aligned} \quad (3.20)$$

the above linearized Einstein equations become

$$\begin{aligned} \sqrt{-g}e^{4\psi}g^{tt}g^{xx}\partial_x\mathcal{F}_{tx} + \partial_r(\sqrt{-g}e^{4\psi}g^{rr}g^{tt}\mathcal{F}_{tr}) &= 0, && (3.21) \\ \sqrt{-g}e^{4\psi}g^{tt}g^{xx}\partial_t\mathcal{F}_{tx} + \partial_r(\sqrt{-g}e^{4\psi}g^{rr}g^{xx}\mathcal{F}_{rx}) &= 0 \ , && (3.22) \\ g^{tt}\partial_t\mathcal{F}_{tr} + g^{xx}\partial_x\mathcal{F}_{xr} &= 0 \ , && (3.23) \end{aligned}$$

where $e^{4\psi} = \frac{1}{g_{\rm eff}^2} = r^{2\theta-4}$. Other than these, we also have a Bianchi Identity

$$\partial_t \mathcal{F}_{rx} + \partial_x \mathcal{F}_{tr} - \partial_r \mathcal{F}_{tx} = 0 \ , \tag{3.24}$$

which is a trivial relation in the higher dimensional theory. Equation (3.19) is a constraint equation in the higher dimensional theory which can be mapped to (3.23) in the y-compactified theory. Defining currents as $j^\nu = n_\mu \mathcal{F}^{\mu\nu}$ (n_μ being the normal vector to the boundary $r = r_c$, with $g^{rr} n_r^2 = 1$) we can write them in terms of the perturbations of the higher dimensional theory,

$$j^x \ = \ n_r \mathcal{F}^{xr} = r^{6-3\theta} \sqrt{f} \ \partial_r (r^{2-\theta} h_{xy}) \ , \tag{3.25}$$

$$j^t \ = \ n_r \mathcal{F}^{tr} = -\frac{r^{4+2z-3\theta}}{\sqrt{f}} \ \partial_r (r^{2-\theta} h_{ty}) \ . \tag{3.26}$$

Identifying the ratio $\mathcal{D} \equiv -\frac{\omega}{iq^2}$ we can essentially write (3.19) in the form of Fick's Law as

$$j^x = -\mathcal{D} \partial_x j^t \ . \tag{3.27}$$

The formulation of Fick's Law in [P2, 96] is done entirely in terms of field variables of the y-compactified theory. Differentiating (3.23) w.r.t t we can eliminate $\mathcal{F}_{rx}$ using the Bianchi Identity (3.24) to get the following equation

$$\partial_t^2 \mathcal{F}_{tr} + r^{2-2z} f \partial_x (-\partial_x \mathcal{F}_{tr} + \partial_r \mathcal{F}_{tx}) = 0 \ . \tag{3.28}$$

In the the near horizon region approximating the thermal factor as $f(r) \approx (2 + z - \theta) \frac{(1/r_0) - r}{1/r_0}$ and parametrizing the frequency as $\omega = -i\Gamma$ for some positive Γ so that the perturbations decay in time, (3.28) can be written as

$$\left(1 + (2 + z - \theta) r_0^{2z-2} \frac{q^2}{\Gamma^2} \cdot \frac{\frac{1}{r_0} - r}{\frac{1}{r_0}} \right) \mathcal{F}_{tr} \approx -(2 + z - \theta) r_0^{2z-2} \frac{iq}{\Gamma^2} \cdot \frac{\frac{1}{r_0} - r}{\frac{1}{r_0}} \partial_r \mathcal{F}_{tx} \ . \tag{3.29}$$

Assuming

$$\frac{\frac{1}{r_0} - r}{\frac{1}{r_0}} \ll \frac{\Gamma^2}{q^2 r_0^{2z-2}} \ , \tag{3.30}$$

we differentiate both sides w.r.t x and approximate (3.29) further as

$$\partial_x \mathcal{F}_{tr} \approx (2 + z - \theta) \frac{q^2 r_0^{2z-2}}{\Gamma^2} \cdot \frac{\frac{1}{r_0} - r}{\frac{1}{r_0}} \partial_r \mathcal{F}_{tx} \ . \tag{3.31}$$

The assumption (3.30) implies

$$\partial_x \mathcal{F}_{tr} \ll \partial_r \mathcal{F}_{tx} \ , \tag{3.32}$$

which in turn simplifies the Bianchi Identity (3.24) to

$$\partial_t \mathcal{F}_{rx} = \partial_x \mathcal{F}_{rt} + \partial_r \mathcal{F}_{tx} \ \sim \ \partial_r \mathcal{F}_{tx} \ . \tag{3.33}$$

Differentiating (3.22) w.r.t t we get

$$\partial_r (r^{\theta-z-1} f \partial_t \mathcal{F}_{rx}) - \frac{r^{z+\theta-3}}{f} \partial_t^2 \mathcal{F}_{tx} = 0 \ . \tag{3.34}$$

Using the approximate Bianchi identity (3.33), to substitute for $\mathcal{F}_{rx}$ and then multiplying throughout with $-\frac{f}{r^{z+\theta-3}}$ we obtain a wave equation for the field strength $\mathcal{F}_{tx}$

$$\partial_t^2 \mathcal{F}_{tx} - \nu^2 \left(\frac{1}{r_0} - r\right) \partial_r \left(\left(\frac{1}{r_0} - r\right) \partial_r \mathcal{F}_{tx}\right) \approx 0 \ , \tag{3.35}$$

where ν is given by

$$\nu = (2 + z - \theta) r_0^z \ . \tag{3.36}$$

The horizon is a one-way membrane: we incorporate this by requiring that all perturbations obey ingoing boundary conditions at the horizon. This dissipative feature is of course at the heart of the diffusion equation that results from this near-horizon perturbations analysis. Thus, imposing ingoing boundary conditions on the wave equation amounts to choosing the ingoing solution, leading to

$$\mathcal{F}_{tx} = f_1 \left(t + \frac{1}{\nu} \log \left(\frac{1}{r_0} - r \right) \right) \ , \tag{3.37}$$

where f_1 is any arbitrary smooth function. If we now ensure that the perturbations decay as $t \to \infty$ we obtain

$$\mathcal{F}_{tx} + \nu \Big(\frac{1}{r_0} - r \Big) \mathcal{F}_{rx} = 0 \ . \tag{3.38}$$

The leading solutions for the fields $\mathcal{A}_t$ and $\mathcal{A}_x$ are

$$\begin{aligned}\mathcal{A}_t^{(0)} &= Ce^{-\Gamma t+iqx}\int_{r_c}^{r} dr' \frac{g_{tt}(r')g_{rr}(r')}{\sqrt{-g(r')}}\cdot g_{\text{eff}}^2(r')\\ &= Ce^{-\Gamma t+iqx}\int_{r_c}^{r} dr'\ \frac{G_{tt}(r')G_{rr}(r')}{G_{xx}(r')\sqrt{-G(r')}}\ ,\\ \mathcal{A}_x^{(0)} &= -\frac{i\Gamma}{q}Ce^{-\Gamma t+iqx}\int_{r_c}^{r} dr' \frac{g_{xx}(r')g_{rr}(r')}{\sqrt{-g(r')}}\cdot g_{\text{eff}}^2(r')\\ &= -\frac{i\Gamma}{q}Ce^{-\Gamma t+iqx}\int_{r_c}^{r} dr' \frac{G_{rr}(r')}{\sqrt{-G(r')}}\ ,\end{aligned} \tag{3.39}$$

where $r_c \sim 0$ is the boundary where we impose the boundary conditions that the perturbations die. In writing down the above expressions, we have used (3.2), (3.4), with C some constant. Using the metric components from (3.15), we can proceed to write down the leading order solutions $\mathcal{A}_t^{(0)}$ and $\mathcal{A}_x^{(0)}$.

When $4-z-\theta>0$ the leading solution $\mathcal{A}_t^{(0)}$ has a power-law behaviour

$$\mathcal{A}_t^{(0)}(t,x,r) = e^{-\Gamma t+iqx}\frac{C}{(4-z-\theta)}r^{4-z-\theta} \tag{3.40}$$

It is expected that close to the boundary i.e. near $r\approx r_c$ the hyperscaling violating phase breaks down and we require $r_0 r_c \ll 1$. The analogous statement for the boundary field theory will be to assume that the temperature is sufficiently below the UV cut-off. Thus, the condition $z<4-\theta$ arises from the boundary condition that $\mathcal{A}_t^{(0)}\to 0$ as $r\to 0$.

The expression for $\mathcal{A}_x^{(0)}$ yields

$$\mathcal{A}_x^{(0)} = \frac{i\Gamma}{q}\frac{Ce^{-\Gamma t+iqx}}{(2+z-\theta)r_0^{2+z-\theta}}\log(1-(r_0 r)^{2+z-\theta})\ . \tag{3.41}$$

However, the equivalent assumption to (3.6) in this case i.e.

$$|\partial_t\mathcal{A}_x| \ll |\partial_x\mathcal{A}_t| \tag{3.42}$$

restricts the validity of our solutions within the regime

$$\exp\left(-\frac{T^{2/z}}{q^2}\right) \ll \frac{\frac{1}{r_0}-r_h}{\frac{1}{r_0}} \ll \frac{q^2}{T^{2/z}} \ll 1\ . \tag{3.43}$$

We can now evaluate the shear diffusion constant on the stretched horizon for the hyperscaling violating theory with $4-z-\theta>0$ from Fick's Law and (3.38)

$$\mathcal{D} \equiv -\frac{j^x}{\partial_x j^t} = -\frac{g_{tt}}{g_{xx}}\frac{\mathcal{F}_{rx}}{\partial_x \mathcal{F}_{rt}} \approx -r_0^{z-1}\frac{\mathcal{F}_{tx}}{\partial_x \mathcal{F}_{rt}} = r_0^{z-1}\frac{\mathcal{A}_t}{\mathcal{F}_{rt}}\Big|_{r\sim r_h} \approx \frac{r_0^{z-2}}{4-z-\theta} + O(q^2) \ . \tag{3.44}$$

The solution for $\mathcal{A}_t^{(0)}$ is evaluated at the stretched horizon r_h: however $r_h \sim \frac{1}{r_0} + O(q^2)$ so to leading order $\mathcal{D}$ is evaluated at the horizon $\frac{1}{r_0}$. It is interesting that the effect of the hyperscaling violating exponent θ cancels in the final expression for $\mathcal{D}$ which is essentially the ratio of $\mathcal{A}_t$ to a field strength $\mathcal{F}_{rt}$ both of which has non-trivial θ-dependence.

Using the expression (3.16) we can express the diffusion constant in terms of the temperature as

$$\mathcal{D} = \frac{1}{4-z-\theta}\left(\frac{4\pi}{2+z-\theta}\right)^{\frac{z-2}{z}} T^{\frac{z-2}{z}} \tag{3.45}$$

which is identical to the one obtained in [P2] for the case without the gauge field, for $d_i = 2$ spatial dimensions.

For the family of hyperscaling violating solution where $z = 4-\theta$, from (3.39) it follows that the leading solution of $\mathcal{A}_t$ has logarithmic behaviour

$$\mathcal{A}_t^{(0)} = Ce^{-\Gamma t + iqx}\log\frac{r}{r_c} \ , \qquad z = 4-\theta \ . \tag{3.46}$$

Working further, we can evaluate the diffusion constant upto leading order as

$$\mathcal{D} = r_0^{z-2}\log\frac{1}{r_0 r_c} \ . \tag{3.47}$$

This implies that in the low temperature limit as $r_0 \to 0$, the diffusion constant vanishes if $z > 2$. The new condition on the exponents z and θ, namely $z < 4-\theta$ appears to be a new constraint which is separate from the null energy conditions

$$(2-\theta)(2(z-1)-\theta) \geq 0 \ , \qquad (z-1)(2+z-\theta) \geq 0 \ . \tag{3.48}$$

The regime of validity for this analysis (equivalently, the "thickness" of the stretched horizon) gets modified in this special case to

$$\exp\left(-\frac{T^{2/z}}{q^2}\frac{1}{\log\frac{1}{r_0 r_c}}\right) \ll \frac{\frac{1}{r_0} - r_h}{\frac{1}{r_0}} \ll \frac{q^2}{T^{2/z}}\log^2\frac{1}{r_0 r_c} \ . \tag{3.49}$$

However, since we are manifestly in the hydrodynamic regime, it means $r_c \ll \frac{1}{r_0}$ implying $\log \frac{1}{r_0 r_c} \gg 1$. This does not over-constrain the window of the stretched horizon: however the subleading terms contain the logarithmic piece affecting the validity of the series expansion. The subleading contributions to $\mathcal{A}_t$ and $\mathcal{A}_x$, namely $\mathcal{A}_t^{(1)}$ and $\mathcal{A}_x^{(1)}$ have been explicitly evaluated in Appendix.

The logarithmic scaling necessitates the presence of the UV scale r_c appearing in the diffusion constant in the hydrodynamic description which is manifestly a description at long wavelengths (physics at IR scale). The leading order value of the diffusion constant $\mathcal{D}$ is given by $\mathcal{A}_t^{(0)}$, which is obtained by solving the $q = 0$ and $\omega = 0$ sector of (3.17)

$$\partial_r(r^{z+\theta-3}\partial_r(r^{2-\theta}h_{ty})) = 0 \ . \tag{3.50}$$

The solution to the above equation is given by

$$h_{ty}(r) = c_1 r^{\theta-2} + c_2 r^{2-z} \ , \tag{3.51}$$

where c_1 and c_2 are arbitrary constants. For $z < 4 - \theta$, the $\theta - 2$ fall-off (non-normalizable mode) dominates over the $2 - z$ fall-off (normalizable mode) near the boundary $r \sim r_c$, while for $z > 4 - \theta$ we see the exact opposite behaviour. When $z = 4 - \theta$ there is a degeneracy in the two fall-offs and we have a new independent solution which scales logarithmically with r,

$$h_{ty}(r) = c_1 r^{\theta-2} + c_2 r^{\theta-2} \log \frac{r}{r_c} \ . \tag{3.52}$$

The swapping of roles between the normalizable and non-normalizable modes around the point $z = 4 - \theta$ gives some insight into the unusual logarithmic scaling for the diffusion constant when $z = 4-\theta$. It is in fact reminiscent of the alternative quantization of field modes [98] and thus holographically it is not surprising that the relevant correlation function exhibits logarithmic behaviour.

3.2.1 Infalling condition for perturbations

As should be clear, a key ingredient that goes in the formulation of Fick's Law is the relation (3.38) for the field strengths $\mathcal{F}_{tx}$ and $\mathcal{F}_{rx}$. In the context of the higher-dimensional hyperscaling violating theory where the perturbations satisfy (3.17), (3.18), (3.19), this relation can be derived exactly without any assumptions on the parameters q and ω. It turns out in this context it is a consequence of imposing a certain physical condition on the function $H(t,r,x)$ defined as

$$H(t,r,x) \equiv r^{\theta-z-1} f \cdot \partial_r (r^{2-\theta} h_{xy}) \ . \tag{3.53}$$

This condition that we impose is given by

$$\left(\partial_t + f \cdot r^{1-z} \partial_r\right) H = 0 \ . \tag{3.54}$$

Defining two new coordinates u and v as

$$\begin{aligned} v &= t + \frac{1}{\nu} \log\left(\frac{1}{r_0} - r\right) \ , \\ u &= t - \frac{1}{\nu} \log\left(\frac{1}{r_0} - r\right) \ . \end{aligned} \tag{3.55}$$

For $r \ll \frac{1}{r_0}$ expanding the log, we see that $v \sim t - \frac{r_0}{\nu} r$ so v is the ingoing coordinate (with r increasing towards the interior). We see that in the near horizon region the full wave operator is

$$4\partial_u \partial_v \equiv \partial_t^2 - \nu^2 \left(\frac{1}{r_0} - r\right) \partial_r \left(\left(\frac{1}{r_0} - r\right) \partial_r\right) \ , \tag{3.56}$$

while the linear differential operator acting on H in (3.54) is essentially $\partial_t + f \cdot r^{1-z} \partial_r \approx \partial_t + \nu \left(\frac{1}{r_0} - r\right) \partial_r = \partial_u$. With v the ingoing coordinate, this can be thus interpreted as the ingoing condition $\partial_u H = 0$ implying that the function has the form $H = H(v)$.

Likewise, choosing the solution (3.37) is equivalent to requiring that the field strength $\mathcal{F}_{tx}$ obeys the ingoing condition

$$\partial_t \mathcal{F}_{tx} + \nu \left(\frac{1}{r_0} - r\right) \partial_r \mathcal{F}_{tx} = 0 \ , \tag{3.57}$$

which can also be written as $\partial_u \mathcal{F}_{tx} = 0$, giving $\mathcal{F}_{tx} = \mathcal{F}_{tx}(v)$. Using (3.20) we can write

$$\partial_t \mathcal{F}_{tx} = -r^{2-\theta} \omega(\omega h_{xy} + q h_{ty}), \qquad \mathcal{F}_{rx} = \partial_r (r^{2-\theta} h_{xy}) = \frac{r^{z+1-\theta}}{f} H \ . \tag{3.58}$$

The above equalities in conjunction with (3.18) gives

$$\partial_r H = -\frac{r^{z-1}}{f} \omega(\omega h_{xy} + q h_{ty}) = \frac{r^{z+\theta-3}}{f} \partial_t \mathcal{F}_{tx} \ . \tag{3.59}$$

Also (3.54) naturally implies

$$\partial_r H = -\frac{r^{z-1}}{f}\partial_t H = -r^{\theta-2}\partial_t \mathcal{F}_{rx} \ . \tag{3.60}$$

Equating the above two expressions for $\partial_r H$, we recover the relation (3.38) as was obtained in [P2]. It should be noted that the relation between $\mathcal{F}_{rx}$ and $\mathcal{F}_{tx}$ was obtained in the y-compactified theory by making certain self-consistent approximations involving the parameters q and ω which is quite distinct from the derivation demonstrated here, using (3.17), (3.18), (3.19), directly.

3.3 Comments on $\frac{\eta}{s}$

We now make a few comments towards gaining insight into $\frac{\eta}{s}$:

(1) As a consistency check, we see that for pure AdS with $\theta = 0$, $z = 1$, we obtain $\mathcal{D} = \frac{1}{4\pi T}$. This corresponds to a relativistic CFT: the shear diffusion constant is $\mathcal{D} = \frac{\eta}{\varepsilon+P}$ and thermodynamics gives $\varepsilon + P = Ts$, where ε, P, s are energy, pressure and entropy densities. This gives the relation $\frac{\eta}{s} = \mathcal{D}T$ and thereby $\frac{\eta}{s} = \frac{1}{4\pi}$.

(2) Theories with metric (4.10) with $\theta = 0$ enjoy the Lifshitz scaling symmetry, $x_i \to \lambda x_i,\ t \to \lambda^z t$: Then the diffusion equation $\partial_t j^t = D\partial_i^2 j^t$ shows the diffusion constant to have scaling dimension $dim[\mathcal{D}] = 2 - z$. With temperature scaling as inverse time, we have $dim[T] = z$. Thus on scaling grounds, the temperature scaling in (3.45), which here is

$$\mathcal{D} = \frac{1}{4-z}\Big(\frac{4\pi}{2+z}\Big)^{\frac{z-2}{z}} T^{\frac{z-2}{z}} \ , \tag{3.61}$$

is expected, upto the z-dependent prefactors. For $z = 2$, the diffusion equation (structurally like a Schrodinger equation) already saturates the scaling dimensions, and $\mathcal{D}$ has apparently no temperature dependence. As T increases, $\mathcal{D}$ decreases for $z < 2$: however $\mathcal{D}$ increases with T for $z > 2$.

Aspects of Lifshitz hydrodynamics in $(d + 1)$ spacetime dimensions have been studied in e.g. [99], [48]. As discussed in [48], under the Lifshitz symmetry, we have the scalings $[T] = z,\ [\varepsilon] = z+d-1,\ [P] = z+d-1,\ [s] = d-1,\ [\eta] = d-1$. Indeed for Lifshitz black branes with horizon r_H and temperature (3.16), the entropy density is $s = \frac{r_H^{d-1}}{4G_{d+1}} = \frac{1}{4G_{d+1}}(\frac{4\pi}{d+z-1}T)^{\frac{d-1}{z}}$. The thermodynamic relations give $\varepsilon+P = Ts$. The shear viscosity [48] is $\eta = \frac{1}{16\pi G_{d+1}}T^{\frac{d-1}{z}}$ satisfying the universal bound $\frac{\eta}{s} = \frac{1}{4\pi}$. For this to arise from (3.61), we guess that the relation between shear viscosity and the shear diffusion constant is

$$\frac{\eta}{s} = \frac{(4-z)}{4\pi}\mathcal{D}r_0^{2-z} = \frac{(4-z)}{4\pi}\left(\frac{4\pi}{2+z}\right)^{\frac{2-z}{z}} \mathcal{D}T^{\frac{2-z}{z}} \ . \tag{3.62}$$

(3) For $\theta \neq 0$, the scaling analysis of the Lifshitz case is not applicable: however the temperature is θ-independent and the relation (3.45) continues to hold for generic θ. Towards guessing the hydrodynamics from the diffusion constant in this case, we first recall from [96] that nonconformal branes give $\mathcal{D} = \frac{1}{4\pi T}$, and thereby $\frac{\eta}{s} = \frac{1}{4\pi}$ continues to hold. On the other hand, [12] observed that nonconformal Dp-branes upon reducing on the sphere S^{8-p} give rise to hyperscaling violating theories with $z = 1$ and $\theta \neq 0$. It would therefore seem that the near-horizon diffusion analysis continues to exhibit this universal behaviour since the sphere should not affect these long-wavelength diffusive properties.
Happily, we see that (3.45) for $z = 1$ gives $\mathcal{D} = \frac{1}{4\pi T}$, with the θ-dependent prefactors cancelling precisely. Thus all hyperscaling violating theories with $z = 1$ appear to satisfy the universal viscosity bound

$$\frac{\eta}{s} = \mathcal{D}T = \frac{1}{4\pi} \ . \tag{3.63}$$

Putting this alongwith the Lifshitz case motivates us to guess the universal relation

$$\frac{\eta}{s} = \frac{(4-z-\theta)}{4\pi}\mathcal{D}r_0^{2-z} = \frac{(4-z-\theta)}{4\pi}\left(\frac{4\pi}{2+z-\theta}\right)^{\frac{2-z}{z}} \mathcal{D}T^{\frac{2-z}{z}} = \frac{1}{4\pi} \tag{3.64}$$

between $\eta, s, \mathcal{D}, T$, for general exponents z, θ. This reduces to (3.62) for the Lifshitz case $\theta = 0$. One might wonder if the prefactors for $\theta \neq 0$ somehow conspire to violate the universal bound: in this regard, it is worth noting that z, θ appear in linear combinations in the prefactors. Alongwith the previous subcases, this suggests consistency of (3.64).
Finally we know that the entropy density is $s = \frac{r_H^{d_{eff}}}{4G_4} \sim \frac{1}{4G_4}T^{\frac{2-\theta}{z}}$ in hyperscaling violating theories, with $d_{eff} = 2 - \theta$ the effective spatial dimension. Then (3.64) gives the shear viscosity as $\eta \sim \frac{1}{16\pi G_4}T^{\frac{2-\theta}{z}}$.

In the presence of a background gauge field, the analysis changes significantly. Had we perturbed the background gauge field as $A_\mu \to A_\mu + a_\mu$ with $a_\mu(t, x, r)$ denoting the perturbations, the analog of (3.19) is given by

$$q\partial_r(r^{2-\theta}h_{xy}) + \frac{\omega}{f}r^{2z-2}\partial_r(r^{2-\theta}h_{ty}) - k\frac{\omega}{f}r^{z-\theta+1}a_y = 0 \ . \tag{3.65}$$

Due to the presence of the gauge field perturbation a_y, we cannot map the above equation to Fick's Law by defining horizon currents as before. In [P'1], it was demonstrated that a field redefinition which involves a non-trivial combination of h_{ty} and $\int a_y\, dr$ gives us an equation which is similar in structure to Fick's Law in the dimensionally reduced theory.

3.4 Shear diffusion: highly boosted black brane

The analysis carried out in [P2] for a $d_i + 2$-dimensional hvLif spacetime showed that the shear diffusion constant has a logarithmic scaling with respect to the temperature when $z = d_i + 2 - \theta$. The parameters z and θ are related precisely in this way when the hyperscaling violating theory is constructed from the x^+-reduction of AdS plane waves (or highly boosted AdS_5 black branes), as well as nonconformal Dp-brane plane waves, as discussed in [51].

To be more precise, compactification of AdS_{d_i+3} plane wave along x^+ gives

$$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2 Q r^{d-1}(dx^+)^2 + R^2 d\Omega_S^2 \quad \longrightarrow (3.66)$$

$$ds^2 = r^{\frac{2\theta}{d_i}}\Big(-\frac{dt^2}{r^{2z}} + \frac{\sum_{i=1}^{d_i} dx_i^2 + dr^2}{r^2}\Big),\ z = \frac{d_i+4}{2},\ \theta = \frac{d_i}{2} \tag{3.67}$$

These can be obtained from a low-temperature, large boost limit [30, 51] of boosted black branes [79] arising from the near horizon limits of the conformal $D3$-, $M2$- and $M5$-branes. Similar features arise from reductions of nonconformal Dp-brane plane waves [30, 72], with exponents

$$z = \frac{2(p-6)}{p-5}\ , \qquad \theta = \frac{p^2-6p+7}{p-5}\ , \qquad d_i = p-1\ , \tag{3.68}$$

where the Dp-brane theory after dimensional reduction on the sphere S^{8-p} and the x^+-direction has bulk spacetime dimension $d+1 \equiv p+1$. The holographic entanglement entropy in these theories exhibits interesting scaling behaviour [52, 72, P1, 71].

To obtain the finite temperature theory, let us for simplicity consider the AdS_5 black brane

$$ds^2 = \frac{R^2}{r^2}\Big(-(1-r_0^4 r^4)dt^2 + dx_3^2 + \sum_{i=1}^{2} dx_i^2\Big) + R^2\frac{dr^2}{r^2(1-r_0^4r^4)}\ , \tag{3.69}$$

which is a solution to the action $S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g^{(5)}}(R^{(5)} - 2\Lambda)$. Rewriting (3.69) in lightcone coordinates and boosting as $x^\pm \to \lambda^\pm x^\pm$, we obtain

$$ds^2 = \frac{R^2}{r^2}\Big(-2dx^+dx^- + \frac{r_0^4 r^4}{2}(\lambda dx^+ + \lambda^{-1}dx^-)^2 + \sum_{i=1}^{2} dx_i^2\Big) + \frac{R^2 dr^2}{r^2(1-r_0^4 r^4)}\ . \quad (3.70)$$

Writing in Kaluza-Klein form

$$\begin{aligned} ds^2 = \frac{R^2}{r^2}\left[-\frac{(1-r_0^4r^4)}{Qr^4}(dx^-)^2 + dx^2 + dy^2 + \frac{dr^2}{(1-r_0^4r^4)}\right] \\ + QR^2r^2\left(dx^+ - \frac{(1-\frac{r_0^4r^4}{2})}{Qr^4}dx^-\right)^2 , \end{aligned} \quad (3.71)$$

where $Q = \dfrac{\lambda^2 r_0^4}{2}$ and compactifying along the x^+ direction gives

$$ds^2 = (Q^{1/2}R^3)r\Big[-\frac{(1-r_0^4r^4)}{Qr^6}(dx^-)^2 + \frac{dx^2+dy^2}{r^2} + \frac{dr^2}{r^2(1-r_0^4r^4)}\Big]. \quad (3.72)$$

This is simply the hyperscaling violating metric (4.10) with $z = 3,\ \theta = 1,\ d_i = 2$, in [51], but now at finite temperature. It is a solution to the equations stemming from the 4-dim Einstein-Maxwell-Dilaton action $S = \frac{1}{16\pi G_4}\int d^4x\sqrt{-g^{(4)}}\,(R^{(4)} - 2\Lambda e^{-\phi} - \frac{3}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}e^{3\phi}F^{\mu\nu}F_{\mu\nu})$ which arises upon dimensional reduction along the x^+-direction of the 5-dim Einstein action $S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g^{(5)}}(R^{(5)} - 2\Lambda)$. The scalar field has the profile $e^{2\phi} = R^2Qr^2$ while the gauge field is $\mathcal{A}_t = -\frac{1+f}{2Qr^4},\ \mathcal{A}_i = 0$ with $f = 1 - r_0^4r^4$. The finite temperature theory is of course obtained by taking the boost λ to be large but finite, and the temperature r_0 to be small but nonzero, while holding $Q = \frac{\lambda^2 r_0^4}{2}$ fixed. The boost simply serves to create a hierarchy of scales in the energy momentum components $T_{++} \sim \lambda^2 r_0^4 \sim Q,\ T_{--} \sim \frac{r_0^4}{\lambda^2} \sim \frac{r_0^8}{Q}\ ,\ T_{+-} \sim r_0^4,\ T_{ij} \sim r_0^4\delta_{ij}$, while keeping them nonzero.

The z, θ-exponents (3.67) arising in these reductions satisfy $z = d_i + 2 - \theta$, coinciding with the special case discussed earlier. This is also true for nonconformal Dp-brane plane waves (3.68). It is worth noting that this relation between the exponents is distinct from the window $d_i - 1 \le \theta \le d_i$ where the holographic entanglement entropy exhibits novel scaling behaviour: in particular the present relation involves the Lifshitz exponent. The diffusion constant for this class of hyperscaling violating theories then has the logarithmic behaviour (3.47) described earlier, provided we restrict to modes that describe the lower dimensional theory.

In the above x^+-compactification, we see that x^- above maps to the time coordinate t below. Thus mapping the perturbations between the higher dimensional description and the hyperscaling violating one, we see that the metric in KK-form (3.71) including the shear gravitational perturbations is of the form

$$ds^2 = \tilde{g}_{--}(dx^-)^2 + \tilde{g}_{ii}dx_i^2 + \tilde{g}_{rr}dr^2 + 2\tilde{h}_{-y}dx^-dy + 2\tilde{h}_{xy}dxdy + \tilde{g}_{++}(dx^+ + A_-dx^-)^2 \, , \tag{3.73}$$

where A_- is the background gauge field in the lower dimensional description. In other words, the perturbations map as $\tilde{h}_{-y} \to h_{ty}$, $\tilde{h}_{xy} \to h_{xy}$, upto the conformal factor arising from the x^+-reduction. In addition, the x^+-reduction requires that the perturbations h_{-y}, h_{xy} are x^+-independent. This in turn translates to the statement that the near horizon diffusive modes are of the form

$$h_{\mu y}(r)e^{-k_-x^- + ik_x x} \, , \qquad k_+ = 0 \, , \qquad [\mu = x^-, x] \, , \tag{3.74}$$

i.e. the nontrivial dynamics in the lower dimensional description arises entirely from the zero mode sector $k_+ = 0$ of the full theory.

Likewise, vector perturbations $\delta A_t, \delta A_y$ in the lower dimensional theory arise in (3.73) as

$$\ldots + g_{++}(dx^+ + A_-dx^- + \tilde{h}_{+-}dx^- + \tilde{h}_{+y}dy)^2.$$

We see that these arise from gravitational perturbations h_{+-}, h_{+y}.

To ensure that the massive KK-modes from the x^+-reduction decouple from these perturbations, it suffices to take the x^+-circle size L_+ to be small relative to the scale set by the temperature $T \sim r_0$, i.e. the temperature is small compared to the KK-scale $\frac{1}{L_+}$: in other words, we require $L_+ \ll \frac{1}{r_0}$. The ultraviolet cutoff near the boundary is $r_c \sim Q^{-1/4} \ll \frac{1}{r_0}$: the hyperscaling violating phase is valid for $r \gtrsim Q^{-1/4}$.

Finally to map (3.72) to (4.10) precisely, we absorb the factors of the energy scale Q by redefining $\tilde{x}^- = \frac{x^-}{\sqrt{Q}}$. Now the shear diffusion constant can be studied as in the hyperscaling violating theory previously discussed, by mapping it to charge diffusion in an auxiliary theory obtained from the finite temperature x^+-compactified theory by compactifying along say the y-direction. This requires mapping the shear gravitational perturbations to the lower dimensional auxiliary gauge fields as $A_t \propto \tilde{h}_{-y}$, $A_y \propto \tilde{h}_{xy}$, which can then be set up in a series expansion in the near horizon region. Thus finally the shear diffusion constant follows from (3.47) giving

$\mathcal{D} = r_0 \log(\frac{1}{r_0 Q^{-1/4}})$. For Q fixed, as appropriate for the lower dimensional theory, we see that the low temperature limit $r_0 \to 0$ gives a vanishing shear diffusion constant suggesting a violation of the viscosity bound. It is worth noting that the diffusion equation here is $\partial_{\tilde{x}^-} j^- = \tilde{\mathcal{D}} \partial_i^2 j^-$ where $\tilde{x}^- = \frac{x^-}{\sqrt{Q}}$ reflecting the Lifshitz exponent $z = 3$.

Noting that $Q \sim \lambda^2 r_0^4$, the diffusion equation and constant in the upstairs theory are

$$\partial_{x^-} j^- \sim \frac{\tilde{\mathcal{D}}}{\sqrt{Q}} \partial_i^2 j^- \ , \qquad \mathcal{D} r_0 \sim \frac{\tilde{\mathcal{D}}}{\sqrt{Q}} r_0 \sim \frac{1}{\lambda} \log \lambda \ . \tag{3.75}$$

The $r_0 \to 0$ limit of the lower dimensional theory (where $T \sim r_0^3$) implies a highly boosted limit $\lambda \to \infty$ of the black brane for fixed Q: here $\mathcal{D} r_0$ vanishes. However this appears to be a subtle limit of hydrodynamics. From the point of view of the upstairs theory of the unboosted black brane, shear gravitational modes are h_{ty}, h_{xy}. Upon boosting, it would appear that these mix with other perturbation modes as well, suggesting some mixing between shear and bulk viscosity. From the point of view of the boosted frame, this system has anisotropy generated by the boost direction. Previous studies of anisotropic systems and shear viscosity include e.g. [100, 101, 102, 103, 104, 105, 106]. (See also e.g. [107] for a review of the viscosity bound.) In the present case, the shear viscosity tensor can be analysed from a systematic study of the expansion of the energy-momentum tensor of the finite temperature Yang-Mills fluid in the highly boosted regime. However the scaling (3.75) is likely to be realized only after phrasing the boosted black brane theory in terms of the variables appropriate for the lower dimensional hyperscaling violating theory (which arises in the $k_+ = 0$ subsector as discussed above). It would be interesting to understand the hydrodynamics in the lower dimensional theory better, as a null reduction of the boosted black brane theory, perhaps similar in spirit to nonconformal brane hydrodynamics [108, 109] as a reduction of nonlinear hydrodynamics [91] of black branes in M-theory. We hope to explore this further.

Chapter 4

Hydrodynamics in hyperscaling violating Lifshitz Theories: A study of quasinormal modes

In the context of gauge/gravity duality for relativistic theories, various transport properties are encoded in the quasinormal modes of the dual gravitational black branes, see e.g. [110, 111, 112, 113]. Quasinormal modes are solutions to the linearized equations governing the gravitational perturbations that are ingoing at the horizon and vanishing at the boundary: these boundary conditions make the low lying hydrodynamic modes damped and diffusive, with a dispersion relation that encodes the hydrodynamic diffusive poles in certain 2-point correlation functions in the dual field theory.

Motivated by these earlier studies, in this chapter, we analyse the lowest quasinormal mode spectrum for shear gravitational perturbations in hyperscaling violating Lifshitz theories with Lifshitz exponent z and hyperscaling violating exponent θ. We turn on appropriate metric and gauge field perturbations h_{xy}, h_{ty} and a_y of the form $e^{-i\omega t+iqx}$. Defining appropriate new field variables $\mathcal{H}$ invariant under a residual gauge symmetry for such perturbations enables us to identify the relevant differential equations governing these modes. The hydrodynamic regime allows the approximation of low frequency and momentum relative to the temperature scale. Then using $\mathbf{\Omega} \sim \frac{\omega}{T} \ll 1$ and $\mathbf{Q} \sim \frac{q}{T^{1/z}} \ll 1$, as expansion parameters, we find series solutions for the quasinormal modes. The lowest quasinormal modes for these shear perturbations are of the form $\omega = -i\mathcal{D}q^2$ where $\mathcal{D}$ is the shear diffusion constant. Our analysis (sec. 2) of these quasinormal modes and the associated boundary conditions can be carried out provided the exponents satisfy $z \leq d_i + 2 - \theta$. In particular $\mathcal{D}$ exhibits power law scaling with temperature for $z < d_i + 2 - \theta$. The shear diffusion constant $\mathcal{D}$ for hvLif theories obtained thus is in agreement with that obtained previously in [P2, P'1], where $\mathcal{D}$ was obtained by adapting the "membrane paradigm" approach of [96]. To elaborate further, turning on perturbations h_{xy}, h_{ty} and a_y and further compactifying the theory along a spatial direction exhibiting translation invariance, we mapped near-horizon metric perturbations to gauge field perturbations in an auxilliary theory in one lower dimension. The gauge fields were used to define currents j^μ on a "stretched horizon", satisfying a diffusion equation $\partial_t j^t = \mathcal{D}\partial_x^2 j^t$. The shear diffusion constant $\mathcal{D}$ is obtained by solving for the perturbations using a set of self-consistent assumptions in a near horizon expansion. This membrane paradigm approach does not require holography as such.

In Sec. 4.4 using the asymptotic behaviour of the quasinormal mode perturbations in sec. 2 above, we adapt the prescription of [114, 115] to compute certain 2-point correlation functions of the dual energy-momentum tensor operators. The poles of these retarded correlators are identical to the lowest quasinormal frequencies above of the dual black brane for $z < d_i + 2 - \theta$, vindicating the correspondence between quasinormal mode frequencies, the shear diffusion constant and the poles of the retarded correlators for nonrelativistic theories with $z < d_i + 2 - \theta$, thereby giving $\frac{\eta}{s} = \frac{1}{4\pi}$.

For $z = d_i + 2 - \theta$, the shear diffusion constant above exhibits logarithmic scaling, the logarithm containing the ultraviolet cutoff. However the correlation functions obtained above, in the Kubo limit, continue to reveal universal behaviour for the viscosity bound with $\frac{\eta}{s} = \frac{1}{4\pi}$ as we discuss in sec. 4.4.1.

From a physical point of view, the transport coefficients under consideration, namely shear diffusion and shear viscosity measure the response of a system under small space and time dependent fluctuations about an equilibrium configuration at IR scales. This dovetails with the idea of *linear response* which precisely studies such fluctuations. The basic object in linear response theory is the retarded Green's function which relates linear fluctuation of sources to corressponding two-point function of operators which couple to the sources. More precisely, this leads to the celebrated Kubo's formula which can be used to calculate various transport coefficient from two-point correlation function. Before delving into a detailed analysis of shear diffusion and shear viscosity for hvLif systems, we will briefly review linear response theory.

4.0.1 Linear response theory

Linear response theory tries to describe the response of a system to small linear perturbations produced by controlled external forces e.g. an applied magnetic field, applied pressure etc. Applying a shear force to a fluid produces motion. How much the various layers of the fluid moves is determined by viscosity. Applying a temperature gradient results in flow of heat over the system till it equilibriates. The amount of heat flow is deterined by what is known as the thermal conductivity. In particular, we will look at time dependent perturbations and its effect over the system in frequency space.

Mathematically, we give a 'kick' to the system by adding a term of the form

$$H_{source}(t) = \phi_i(t)\mathcal{O}_i(t) \tag{4.1}$$

where $\phi_i(t)$ are called sources and the operators coupled to those fields i.e. $\mathcal{O}_i(t)$'s are the observables of the theory. Since the sources are small perturbations to the original system, we restrict our analysis such that the expectation value of the operators change by an amount that is linear in the perturbing source i.e.

$$\delta\langle\mathcal{O}_i(t)\rangle = \int dt'\ \chi_{ij}(t,t')\phi_j(t') \tag{4.2}$$

where $\chi_{ij}(t,t')$ is known as the *response function*. Clasically, these functions can be thought of as simply the Green's function for the operators $\mathcal{O}_i$. At this point, we assume that our system is invariant under time translations i.e.

$$\chi_{ij}(t,t') = \chi_{ij}(t-t') \tag{4.3}$$

Performing a Fourier transform[1] and thus going into frequency space, we get

$$\delta\langle\mathcal{O}_i(\omega)\rangle = \chi_{ij}(\omega)\phi_j(\omega) \tag{4.4}$$

Physically, this means a perturbation produced at frequency ω, also responds at the same frequency ω. Ignoring the indices $\{i,j\}$, we can decompose the response

[1] We have defined the Fourier transform and its inverse as $f(\omega) = \int dt\ e^{i\omega t} f(t)$ and $f(t) = \int \frac{d\omega}{2\pi}\ e^{-i\omega t} f(\omega)$

function into its real and imaginary parts

$$\begin{aligned}\chi(\omega) &= Re\ \chi(\omega) + i\ Im\ \chi(\omega) \\ &= \chi_R(\omega) + \chi_I(\omega)\end{aligned} \tag{4.5}$$

Imposing reality on the sources and expectation value of the perturbations gives us strong constraint on the functions χ_R and χ_I. The real part $\chi_R(\omega)$ is an even function in frequency space i.e. $\chi_R(-\omega) = \chi_R(\omega)$ called the reactive part of the response function while $\chi_I(\omega)$ is an odd function i.e. $\chi_I(-\omega) = -\chi_I(\omega)$ known as the *dissipative part*.

The sources that we turned on are perturbative in nature. Using time dependent perturbation theory by adding the term (4.1) leads us to the expression of change in expectation value of the operator $\mathcal{O}_i$:

$$\delta\langle\mathcal{O}_i(t)\rangle = i\int_{-\infty}^{+\infty} dt'\ \theta(t-t')\langle[\mathcal{O}_j(t'),\mathcal{O}_i(t)]\rangle\phi_j(t') \tag{4.6}$$

Comparing with (4.2) we can write the response function

$$\chi_{ij}(t-t') = -i\theta(t-t')\langle[\mathcal{O}_i(t),\mathcal{O}_j(t')]\rangle \tag{4.7}$$

This is known as the *Kubo formula*.

In the context of hydrodynamics, viscosity is associated with the transport of momentum. For field theories exhibiting translation invariance along temporal and spatial directions satisfy the conservation law $\partial_\mu T^{\mu\nu} = 0$ where $T^{\mu\nu}$ is the *energy-momentum tensor* or simply *stress tensor* in short. Viscosity essentially tells us how momentum along x (say) affects transport along the direction z. The relevant component of the stress tensor is T^{xz}. Since the viscosity is the result of a constant force perturbation, we should take the low frequency, low momenta limit in our final answer. Thus from (4.7), we can write

$$\chi_{xz,xz}(\omega,\mathbf{k}) = -i\int_{-\infty}^{+\infty} dt\ d^3x\ \theta(t)e^{i\omega t - i\mathbf{k}x}\langle[T_{xz}(\omega,\mathbf{k}),T_{xz}(0,0)]\rangle \tag{4.8}$$

The Kubo formula for viscosity (η) finally takes the form

$$\eta = \lim_{\omega\to 0}\frac{\chi_{xz,xz}(\omega,\mathbf{k}=0)}{i\omega} \tag{4.9}$$

We will make use of the above two expression extensively in this chapter. In particular, we will make use of the AdS/CFT correspondence in order to evaluate the R.H.S of (4.8).

4.1 Perturbations in hvLif background

The backgrounds of interest here are described by $(d+1)$-dimensional hyperscaling violating metrics at finite temperature given by

$$ds^2 = r^{2\theta/d_i}\left(-\frac{f(r)}{r^{2z}}dt^2 + \frac{dr^2}{f(r)r^2} + \sum_{d_i}\frac{dx_i^2}{r^2}\right), \quad d_i = d-1, \quad f(r) = 1-(r_0 r)^{d_i+z-\theta}. \tag{4.10}$$

$r = \frac{1}{r_0}$ is the location of the horizon, and d_i is the boundary spatial dimension. These are conformally Lifshitz solutions to Einstein-Maxwell-dilaton theories (see Appendix A for some details). Recall that the temperature of the field theory dual to the hvLif theory (4.10) is the Hawking temperature of the black brane

$$T = \frac{d_i + z - \theta}{4\pi} r_0^z \ . \tag{4.11}$$

We are interested in studying shear gravitational modes: these are the modes h_{xy} and h_{ty}, which in general couple to the gauge field perturbations a_y. We turn on perturbations of the form $e^{-i\omega t+iqx}h_{\mu\nu}(r)$, $e^{-i\omega t+iqx}a_\mu(r)$, and restrict ourselves to radial gauge ($h_{\mu r} = a_r = 0$). Then, as in [P2, P'1], shear diffusion can be studied by mapping it to charge diffusion in a theory in one lower dimension obtained by compactifying one of the spatial directions x_i enjoying translation invariance, say y. Motivated by this, we define the variables

$$H_{ty} = g^{xx}h_{ty} = r^{2-\frac{2\theta}{d_i}}h_{ty} \ , \qquad H_{xy} = g^{xx}h_{xy} = r^{2-\frac{2\theta}{d_i}}h_{xy} \ . \tag{4.12}$$

Then the equations of motion governing the perturbations are simply

$$\partial_r(r^{z+\theta-(d_i+1)}H'_{ty}) - ka'_y - \frac{r^{z+\theta-(d_i+1)}}{f}q(\omega H_{xy} + qH_{ty}) = 0 \ , \tag{4.13}$$

$$\partial_r(r^{\theta-z-d_i+1}fH'_{xy}) + \frac{r^{z+\theta-(d_i+1)}}{f}\omega(\omega H_{xy} + qH_{ty}) = 0 \ , \tag{4.14}$$

$$qr^{2-2z}H'_{xy} + \frac{\omega}{f}(H'_{ty} - kr^{(d_i+1)-z-\theta}a_y) = 0 \ , \tag{4.15}$$

$$\partial_r(r^{d_i+3-z-\theta}fa'_y) + \frac{r^{d_i+1+z-\theta}}{f}\omega^2 a_y - r^{d_i+3-z-\theta}q^2 a_y - kH'_{ty} = 0 \ , \tag{4.16}$$

where $k = (d_i + z - \theta)\alpha$ and $\alpha = -\sqrt{\frac{2(z-1)}{d_i+z-\theta}}$ (see (A.6)). The first three equations, namely (4.13),(4.14) and (4.15) are the three relevant components of the Einstein equations while (4.16) is the linearized Maxwell's equation. Now, following [110, 111], we note that there is a residual gauge invariance in these variables representing fluctuations of the form above: the metric fluctuations transform under infinitesimal diffeomorphisms as $h_{\mu\nu} \to h_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$, with the gauge functions $\xi_\mu(t,x,r) \equiv \xi_\mu(r) e^{-i\omega t + iqx}$. The residual gauge invariance then allows us to consider the following gauge invariant combination defined as

$$\mathcal{H} \ = \omega H_{xy} + q H_{ty} - kq \int_{r_c}^{r} s^{d_i+1-z-\theta} a_y(s) ds \ . \tag{4.17}$$

Note that the gauge field component a_y is invariant by itself. This combination is motivated by the investigation in [P'1] where a similar combination appears as the field variable (mixing h_{ty}, a_y perturbations) that allows a realization of the diffusion equation from the Einstein equations governing the near horizon shear perturbations: this is reviewed in Appendix C. Under a compactification of the y-direction, these metric components become gauge field components with a residual $U(1)$ gauge invariance. We will see the role of this variable $\mathcal{H}$ in what follows. The equations of motion i.e. (4.13)-(4.16) can be finally reduced to a system of two coupled second order equations using the field $\mathcal{H}$ defined above in (4.17), i.e. (primes denote r-derivatives)

$$\begin{aligned} &\mathcal{H}'' + \left[\partial_r \log r^{z+\theta-(d_i+1)} + \frac{\mathbf{\Omega}^2}{\mathbf{\Omega}^2 - (2\pi T)^{2/z-2}\mathbf{Q}^2 f r^{2-2z}} \partial_r \log(r^{2-2z} f)\right] \mathcal{H}' \\ &\quad + (2\pi T)^2 \left(\frac{r^{2z-2}}{f^2}\mathbf{\Omega}^2 - (2\pi T)^{2/z-2}\frac{\mathbf{Q}^2}{f}\right)\mathcal{H} \\ &\quad + (2\pi T)^{2+1/z} k\mathbf{Q} \left(\frac{r^{2z-2}}{f^2}\mathbf{\Omega}^2 - (2\pi T)^{2/z-2}\frac{\mathbf{Q}^2}{f}\right) \int_0^r ds\ s^{(d_i+1)-z-\theta} a_y = 0 \ , \end{aligned} \tag{4.18}$$

$$\begin{aligned} a_y'' + [\partial_r \log f r^{d_i+3-z-\theta}] a_y' + \left((2\pi T)^2 \frac{r^{2z-2}}{f^2}\mathbf{\Omega}^2 - (2\pi T)^{2/z}\frac{\mathbf{Q}^2}{f} - \frac{k^2}{r^2 f}\right) a_y \\ + \frac{(2\pi T)^{1/z-2} k\mathbf{Q}. r^{\theta-z-d_i-1}}{\mathbf{\Omega}^2 - (2\pi T)^{2/z-2}\mathbf{Q}^2 f r^{2-2z}} \mathcal{H}' = 0 \ . \end{aligned} \tag{4.19}$$

Here, $\mathbf{\Omega}$ and $\mathbf{Q}$ satisfying

$$\mathbf{\Omega} = \frac{\omega}{2\pi T} \ , \quad \mathbf{Q} = \frac{q}{(2\pi T)^{1/z}} \ , \qquad \mathbf{\Omega},\ \mathbf{Q} \ \ll 1 \ , \tag{4.20}$$

are combinations that are dimensionless for the $z = 1$ AdS case, and Lifshitz invariant for $\theta = 0$. The condition $\mathbf{\Omega}, \ \mathbf{Q} \ \ll 1$ in (4.20) is imposed to restrict to the hydrodynamic regime where we can identify appropriate quasinormal mode solutions and frequencies to the above equations. These are solutions to the above differential equations governing the perturbations which are ingoing at the horizon and vanishing at the boundary (far from the horizon): they are damped modes reflecting diffusion in these backgrounds and describe how the perturbed system "settles down". In relativistic theories, quasinormal modes are known to be closely related to hydrodynamic diffusive modes and the associated diffusive poles in the dual field theories. In what follows, we will generalize these studies to hvLif theories. Restricting to the hydrodynamic regime enables us to look for solutions to (4.18) and (4.19) in a perturbative series. It turns out that the leading and next-to-leading behaviour for the mode $\mathcal{H}$ can be determined independent of the a_y solution (which can then be solved for using the solution of $\mathcal{H}$).

4.2 hvLif in absence of gauge field: dilaton gravity, $z = 1, \ d_i = 2$

In this subsection, we will analyse a simple case of dilaton gravity in 4 bulk dimensions ($d_i = 2$) as a warmup example. It can be easily seen from (A.1), that in the absence of a background gauge field ($A_\mu = 0$), the hvLif theory reduces to a theory of a scalar field (dilaton) coupled to gravity with $z = 1$ and $k = 0$ (which follows from A.6). The gauge invariant combination $\mathcal{H}$ defined in (4.17) takes a simpler form and satisfies

$$\mathcal{H} = \omega H_{xy} + q H_{ty} : \qquad \mathcal{H}'' - \frac{P'}{P}\mathcal{H}' + (2\pi T)^2 \left(\frac{\mathbf{\Omega}^2}{f^2} - \frac{\mathbf{Q}^2}{f} \right) \mathcal{H} = 0 \ . \tag{4.21}$$

simplifying (4.18), and we have defined the function $P(r) \equiv \frac{\mathbf{\Omega}^2 - \mathbf{Q}^2 f(r)}{f(r) r^{\theta-2}}$. Close to the horizon (as $r \to \frac{1}{r_0}$), the above equation can be approximated as

$$\mathcal{H}'' + \frac{f'}{f}\mathcal{H}' + \frac{(2\pi T)^2 \mathbf{\Omega}^2}{f^2}\mathcal{H} = 0 \ . \tag{4.22}$$

Using an asymptotic scaling ansatz of the form $\mathcal{H} \sim f^A$ in this equation yields $A = \pm \frac{i\mathbf{\Omega}}{2}$. Choosing the exponent $A = -\frac{i\mathbf{\Omega}}{2}$, and restoring the explicit time-dependence, we see that $\mathcal{H} \sim e^{-i\omega\left(t + \frac{1}{4\pi T}\log f(r)\right)}$. With $z = 1, d_i = 2$, from (4.10) we have $f(r) = 1 - (r_0 r)^{3-\theta}$ with the boundary defined at $r \to 0$. The blackening factor turns off as $f(r) \to 1$ far from the horizon only if $\theta < 3$. Focussing therefore on $\theta < 3$ from now on, we see that as time evolves (increasing t), these modes carry energy towards the horizon, i.e. these are ingoing modes at the horizon. Taking the ansatz

$$\mathcal{H}(r, \mathbf{\Omega}, \mathbf{Q}) = f(r)^{-\frac{i\mathbf{\Omega}}{2}} F(r, \mathbf{\Omega}, \mathbf{Q}) \ , \tag{4.23}$$

and using in (4.21), we can obtain a second order equation governing $F(r, \mathbf{\Omega}, \mathbf{Q})$. Towards studying hydrodynamic modes, we analyse (4.21) in the regime $\mathbf{\Omega} \ll 1, \mathbf{Q} \ll 1$. To keep track of the order of the perturbative solution, we introduce a book-keeping parameter λ and rescale $\mathbf{\Omega} \to \lambda\mathbf{\Omega}$ and $\mathbf{Q} \to \lambda\mathbf{Q}$, following [110, 111]. Rescaling then gives

$$F'' - \left(i\lambda\mathbf{\Omega}\, \frac{f'}{f} + \frac{P'}{P}\right) F' + \left(-\frac{i\lambda\mathbf{\Omega}}{2}\left(\frac{f'}{f}\right)' - \frac{\lambda^2\mathbf{\Omega}^2}{4}\left(\frac{f'}{f}\right)^2 + \frac{i\lambda\mathbf{\Omega}}{2}\frac{f'}{f}\frac{P'}{P}\right) F \\ + (2\pi T)^2\lambda^2\left(\frac{\mathbf{\Omega}^2}{f^2} - \frac{\mathbf{Q}^2}{f}\right) F = 0 \ . \tag{4.24}$$

Assuming that the solution admits a series expansion in the perturbation parameter λ i.e.

$$F(r) = F_0(r) + \lambda F_1(r) + \lambda^2 F_2(r) + \cdots \ , \tag{4.25}$$

we can write a second order equation for $F_0(r)$ and its corresponding solution as

$$F_0'' - \frac{P'}{P}F_0' = 0 \, ; \qquad F_0(r) = C_0 + C_1 \int^r \frac{\mathbf{\Omega}^2 - \mathbf{Q}^2 f}{f r'^{\theta-2}} \cdot dr' \quad \xrightarrow{\text{regularity}} \quad F_0(r) = C_0 \ . \tag{4.26}$$

Near the horizon, $f(r)$ vanishes, giving a logarithmic divergence in F_0. Demanding regularity of the solution at the horizon forces us to set $C_1 = 0$, thus giving the solution as simply $F_0 = C_0$ in (4.26). Using this in (4.24) and collecting terms of $O(\lambda)$ gives an inhomogeneous differential equation for $F_1(r)$,

$$F_1'' - \frac{P'}{P}F_1' = C_0\frac{i\mathbf{\Omega}}{2}\left[\left(\frac{f'}{f}\right)' - \frac{P'}{P}\cdot\frac{f'}{f}\right] . \tag{4.27}$$

Integrating and multiplying throughout by P, we get

$$F_1' = \frac{i\mathbf{\Omega}}{2} C_0\ \partial_r \log f + \kappa_1 P \quad \Rightarrow \quad \kappa_1 = -\frac{iC_0}{2\mathbf{\Omega}} f'(1/r_0) r_0^{2-\theta} = \frac{iC_0}{2\mathbf{\Omega}}(3-\theta) r_0^{3-\theta} \ . \tag{4.28}$$

The above value for the constant κ_1 is required by demanding regularity of F_1 at the horizon which implies F_1' must be finite as $r \to \frac{1}{r_0}$. Using this value of κ_1, the solution to (4.27) is

$$F_1(r) - F_1(1/r_0) = \frac{iC_0\mathbf{Q}^2}{2\mathbf{\Omega}}(1-(r_0r)^{3-\theta}) \ . \tag{4.29}$$

We set the integration constant $F_1(1/r_0)$ to zero, as in [115]. This is consistent with the absence of any additional dependence on $\mathbf{\Omega}, \mathbf{Q}$ in the subleading terms $F_i(r)$ in (4.25), i.e. with fixing the normalization of the modes as simply $C_0e^{-i\omega t}f(r)^{-i\Omega/2}$ at the horizon.

Imposing the Dirichlet boundary condition $\mathcal{H}(0) = 0$, i.e. the fluctuations vanish on the boundary $r = r_c \to 0$, we obtain

$$1 + \frac{i\mathbf{Q}^2}{2\mathbf{\Omega}}\left(1-(r_0r)^{3-\theta}\right)\Big|_{r\sim 0} = 0 \ . \tag{4.30}$$

Using (4.20), at the boundary, we thus obtain the dispersion relation

$$\omega = -i\frac{1}{4\pi T}q^2 \equiv -i\mathcal{D}q^2 \ , \tag{4.31}$$

where $\mathcal{D} = \frac{1}{4\pi T}$ is the shear diffusion constant. This is consistent with $\mathcal{D}$ found in [P2, P'1], using a membrane-paradigm-like near horizon analysis (generalizing [96]), and the corresponding guess $\frac{\eta}{s} = \mathcal{D}T = \frac{1}{4\pi}$ for universal viscosity-to-entropy-density.

4.3 hvLif theory: generalized analysis

In this section, we will study hvLif theories in full generality. To study hydrodynamics, we will focus on the regime $\mathbf{\Omega} \ll 1$, $\mathbf{Q} \ll 1$, taking the ansatze

$$\mathcal{H}(r,\mathbf{\Omega},\mathbf{Q}) = f^{\frac{-i\mathbf{\Omega}}{2}}F(r,\mathbf{\Omega},\mathbf{Q}) \ ; \qquad a_y(r,\mathbf{\Omega},\mathbf{Q}) = f^{\frac{-i\mathbf{\Omega}}{2}}G(r,\mathbf{\Omega},\mathbf{Q}) \ . \tag{4.32}$$

The factor $f^{-\frac{i\mathbf{\Omega}}{2}}$ reflects the "ingoing" nature of these solutions, as in the previous $z = 1$ case. The null energy condition (A.2) implies $d_i + z - \theta > 0$ for theories with $z > 1$ so the factor $f^{-\frac{i\mathbf{\Omega}}{2}}$ always reflects "infalling" modes noting the form of $f(r)$ in (4.10). Rewriting (4.18) in terms of F and G and further rescaling $\mathbf{\Omega} \to \lambda\mathbf{\Omega}$,

$\mathbf{Q} \to \lambda \mathbf{Q}$, we end up with

$$F'' - \left(\frac{H'}{H} + i\lambda\mathbf{\Omega}\frac{f'}{f}\right)F' - \left[\frac{i\lambda\mathbf{\Omega}}{2}\left(\frac{f'}{f}\right)' + \frac{\lambda^2\mathbf{\Omega}^2}{4}\left(\frac{f'}{f}\right)^2 - \frac{i\lambda\mathbf{\Omega}}{2}\frac{f'H'}{fH} - \lambda^2(2\pi T)^2\frac{r^{2z-2}}{f^2}(\mathbf{\Omega}^2 - (2\pi T)^{\frac{2}{z}-2}\mathbf{Q}^2 f r^{2-2z})\right]F + \lambda^3(2\pi T)^{2+\frac{1}{z}}k\mathbf{Q}\frac{r^{2z-2}}{f^2}(\mathbf{\Omega}^2 - (2\pi T)^{\frac{2}{z}-2}\mathbf{Q}^2 f r^{2-2z})f^{\frac{i\lambda\mathbf{\Omega}}{2}}\int ds \cdot f^{-\frac{i\lambda\mathbf{\Omega}}{2}}\; s^{d_i+1-z-\theta}G = 0\,. \tag{4.33}$$

We also assume the solutions admit a series expansion in λ as following

$$\begin{aligned} F(r,\mathbf{\Omega},\mathbf{Q}) &= F_0(r,\mathbf{\Omega},\mathbf{Q}) + \lambda F_1(r,\mathbf{\Omega},\mathbf{Q}) + O(\lambda^2) + \cdots \\ G(r,\mathbf{\Omega},\mathbf{Q}) &= G_0(r,\mathbf{\Omega},\mathbf{Q}) + \lambda G_1(r,\mathbf{\Omega},\mathbf{Q}) + O(\lambda^2) + \cdots \end{aligned} \tag{4.34}$$

Gathering terms order-by-order, we see that F_0 follows a homogeneous second order differential equation while the nature of F_1 depends on F_0. We have argued in Appendix C that the last term in (4.33) becomes relevant only at $O(\lambda^3)$ and does not contribute to the F_0 and F_1 solutions. In Sec. 4.3.0.1, we demonstrate that the F_0 and F_1 solutions are consistent with (4.19) too and solve for the function $G_0(r,\mathbf{\Omega},\mathbf{Q})$ which is determined by F_0 and F_1. Further G_1 requires knowledge of F_2 also. Thus although the exact form of the perturbation solutions $\mathcal{H}, a_y$, is governed by the coupled equations (4.18), (4.19), restricting to $O(\lambda)$ essentially decouples the a_y terms from the equation governing $\mathcal{H}$ which we will solve for below.

Sticking (4.34) in (4.33) and gathering terms of $O(\lambda^0)$ gives

$$F_0'' - \frac{H'}{H}F_0' = 0 \quad \text{with} \quad H = \frac{\mathbf{\Omega}^2 - (2\pi T)^{2/z-2}\mathbf{Q}^2 f r'^{2-2z}}{f r'^{\theta-z-d_i+1}} \xrightarrow{\text{regularity}} F_0 = C_0\,. \tag{4.35}$$

To elaborate, the solution to the equation above (analogous to (4.26)) is $F_0 = C_0 + C_1\int H dr$ and regularity of F_0 requires $C_1 = 0$ giving $F_0 = C_0$ above. The next-to-leading solution F_1 satisfies an equation structurally similar to (4.27),

$$F_1'' - \frac{H'}{H}F_1' = C_0\frac{i\mathbf{\Omega}}{2}\left[\left(\frac{f'}{f}\right)' - \frac{H'}{H}\cdot\frac{f'}{f}\right]\,. \tag{4.36}$$

Integrating gives

$$F_1' = \frac{i\mathbf{\Omega} C_0}{2}\partial_r \log f + \kappa_2 \cdot \frac{\mathbf{\Omega}^2 - (2\pi T)^{2/z-2}\mathbf{Q}^2 f r^{2-2z}}{f r^{\theta-z-d_i+1}} \quad \Rightarrow \quad \kappa_2 = \frac{iC_0}{2\mathbf{\Omega}}(d_i+z-\theta) r_0^{d_i+z-\theta} . \tag{4.37}$$

The integration constant κ_2 is fixed as in (4.28) by demanding regularity of F_1 at the horizon $r \to 1/r_0$. This forces the singular part from the first term to be cancelled by the other singular piece coming from the $O(\mathbf{\Omega}^2)$ term, fixing κ_2 above. The value of κ_2 can be used to write down the solution to $F_1(r)$,

$$F_1(r) = -\frac{iC_0(d_i+z-\theta)\mathbf{Q}^2}{2\mathbf{\Omega}}(2\pi T)^{\frac{2}{z}-2} r_0^{d_i+z-\theta} \int_{\frac{1}{r_0}}^{r} r'^{d_i+1-z-\theta} dr' . \tag{4.38}$$

As in (4.29) and the comments following it, we have set the integration constant $F_1(1/r_0)$ to zero in the second line. Then the solution to (4.18) upto first order in the hydrodynamic expansion can be written down and varies depending on the value of (d_i, z, θ).

<u>$\boldsymbol{z < d_i + 2 - \theta}$</u>: This is the sector continuously connected to relativistic (AdS) theories which have $z = 1, \theta = 0$. This sector also includes hvLif theories arising from reductions of $p \le 4$ nonconformal Dp-branes where $z = 1,\ d_i = p,\ \theta = p - \frac{9-p}{5-p}$. The solution to (4.18) upto first order is given by

$$\mathcal{H} = C_0 f(r)^{-\frac{i\mathbf{\Omega}}{2}} \left[1 + \frac{iq^2}{(d_i+2-z-\theta)\omega} r_0^{z-2} \cdot \left(1 - (r_0 r)^{d_i+2-z-\theta}\right) \right] , \tag{4.39}$$

where r_0 is related to the temperature T as in (4.11). Imposing Dirichlet boundary conditions i.e. $\mathcal{H}(r \to 0) = 0$ at the UV cut-off boundary ($r = r_c \to 0$) using (4.20) gives

$$\omega = -iq^2 \cdot \frac{1}{d_i+2-z-\theta} \cdot \left(\frac{4\pi T}{d_i+z-\theta} \right)^{1-2/z} \equiv -i\mathcal{D}q^2 , \tag{4.40}$$

as the quasinormal mode frequency. This gives the leading shear diffusion constant

$$\mathcal{D} = \frac{r_0^{z-2}}{d_i+2-z-\theta} = \frac{1}{d_i+2-z-\theta} \cdot \left(\frac{4\pi}{d_i+z-\theta} \right)^{1-2/z} T^{\frac{z-2}{z}} , \tag{4.41}$$

which matches the result obtained using the membrane paradigm approach in [P2, P'1] (reviewed in Appendix C). This led to a guess for the relation between the shear diffusion constant $\mathcal{D}$ and shear viscosity η, consistent with various special cases,

$$\frac{\eta}{s} = \frac{d_i+2-z-\theta}{4\pi} \left(\frac{4\pi}{d_i+z-\theta} \right)^{\frac{2-z}{z}} \mathcal{D} T^{\frac{2-z}{z}} = \frac{1}{4\pi} . \tag{4.42}$$

We will later (sec. 3) evaluate the viscosity using holographic techniques, corroborating this. For $z = 1$, we see that $\frac{\eta}{s} = \mathcal{D}T = \frac{1}{4\pi}$ as in (4.31). It is worth noting that these quasinormal modes are diffusive damped modes. Our analysis and results hold in the regime (4.20) so in particular $\frac{q}{T^{1/z}} \ll 1$. The quasinormal mode frequency (4.40) can then be expressed as $\omega \sim -i(\frac{q}{T^{1/z}})^2 T$, and we are working at finite temperature in the hydrodynamic low frequency, low momentum regime. Then the time dependence of these modes is $\sim e^{-i\omega t} \sim e^{-\Gamma t}$, damped on long timescales.

$\underline{\boldsymbol{z = d_i + 2 - \theta}}$: Here, the integral in (4.38) gives the solution to (4.18) to first order as

$$\mathcal{H}(r) = C_0 f(r)^{-\frac{i\Omega}{2}} \left[1 + \frac{iq^2}{\omega} r_0^{z-2} \log \frac{1}{r_0 r}\right] , \tag{4.43}$$

where (4.11) now gives $T = \frac{z-1}{2\pi} r_0^z$. Then defining $\Lambda = \frac{z-1}{2\pi} \frac{1}{r_c^z}$ gives the low-lying quasinormal frequency

$$\omega = -iq^2 \cdot \frac{1}{z} \left(\frac{2\pi}{z-1}\right)^{1-2/z} \cdot T^{\frac{z-2}{z}} \log \frac{\Lambda}{T} \equiv -i\mathcal{D}q^2 . \tag{4.44}$$

This gives the shear diffusion constant $\mathcal{D} = \frac{1}{z}\left(\frac{2\pi}{z-1}\right)^{1-2/z} \cdot T^{\frac{z-2}{z}} \log \frac{\Lambda}{T}$ scaling logarithmically with temperature alongwith a power-law pre-factor. This also agrees with the results in [P2, P'1]. The logarithmic scaling necessitating the ultraviolet scale Λ perhaps suggests that this leading relation for the quasinormal mode frequency is subject to subleading corrections and possibly appropriate resummations. Nevertheless, recasting as $\omega \sim -i(\frac{q}{T^{1/z}})^2 T \log \frac{\Lambda}{T}$ shows that in the hydrodynamic regime $\frac{q}{T^{1/z}} \ll 1$, this leading mode is diffusive with damped time-dependence: in fact for $T \ll \Lambda$, the extra *log*-factor leads to additional damping. The hvLif theories arising from null reductions of AdS and nonconformal brane plane waves [51, 72] have exponents satisfying $z = d_i + 2 - \theta$: taking the quasinormal modes as a measure of stability of the backgrounds, we see that the diffusive frequencies suggest that low lying modes do not indicate any instability. The logarithmic behaviour of the leading shear diffusion constant then suggests a possibly novel limit of hydrodynamics in these theories, perhaps stemming from the large boost in the above string constructions.

$\underline{\boldsymbol{z > d_i + 2 - \theta}}$: The integral in (4.38) scales as $r_c^{d_i+2-\theta-z}$ thus acquiring dominant (divergent) contribution from high energy scales near $r_c \sim 0$. There is no universal low energy behaviour emerging from near horizon physics: it appears that these methods fail to yield insight on quasinormal modes, as does the membrane paradigm approach [P2, P'1]. This sector includes e.g. reductions of $D6$-branes $(d_i = 6,\ z = 1,\ \theta = 9)$ with ill-defined asymptotics.

4.3.0.1 Solving for gauge field perturbation a_y

Using the ansatze (4.32) and rescaling $\mathbf{\Omega} \to \lambda\mathbf{\Omega}$ and $\mathbf{Q} \to \lambda\mathbf{Q}$ we can recast (4.19) as

$$\begin{aligned} G'' &- \frac{i\lambda\mathbf{\Omega}}{2}\cdot\frac{f'}{f}G' + \partial_r \ln f r^{d_i+3-z-\theta}G' - \frac{i\lambda\mathbf{\Omega}}{2}\left(\frac{f'}{f}\right)'G - \frac{\lambda^2\mathbf{\Omega}^2}{4}\left(\frac{f'}{f}\right)^2 G \\ &- \frac{i\lambda\mathbf{\Omega}}{2}\frac{f'}{f}\partial_r \ln f r^{d_i+3-z-\theta}\ G + \lambda^2(2\pi T)^2\frac{r^{2z-2}}{f^2}(\mathbf{\Omega}^2 - (2\pi T)^{2/z-2}\mathbf{Q}^2 f r^{2-2z})\ G \\ &- \frac{k^2}{r^2 f}\ G + \frac{(2\pi T)^{1/z-2}k\mathbf{Q}.r^{\theta-z-d_i-1}}{\mathbf{\Omega}^2 - (2\pi T)^{2/z-2}\mathbf{Q}^2 f r^{2-2z}}\cdot\frac{1}{\lambda}\left(F' - \frac{i\lambda\mathbf{\Omega}}{2}\frac{f'}{f}F\right) = 0\ . \end{aligned} \tag{4.45}$$

Plugging in the series ansatz (4.34) we can construct the perturbative solution for a_y order-by-order. The leading order equation appears at $O(\frac{1}{\lambda})$ and is given by $F_0' = 0$ giving $F_0 = const$: this can be seen to be consistent with (4.35). We will subsequently see that G_0 is determined by F_1 and F_0 while G_1 is determined by F_2 and G_0, and so on. More generally, all subsequent equations involve more variables so there is no inconsistency in the solutions due to potential overconstraining in this system of equations.

Gathering all terms of $O(\lambda^0)$, we see G_0 follows the equation:

$$G_0'' + \partial_r \log f r^{d_i+3-z-\theta}G_0' - \frac{k^2}{r^2 f}G_0 + \frac{(2\pi T)^{1/z-2}k\mathbf{Q}.r^{\theta-z-d_i-1}}{\mathbf{\Omega}^2 - (2\pi T)^{2/z-2}\mathbf{Q}^2 f r^{2-2z}}\left(F_1' - \frac{i\mathbf{\Omega}}{2}\frac{f'}{f}F_0\right) = 0\ . \tag{4.46}$$

For $z < d_i + 2 - \theta$, using (4.35) and (4.38) we can write the most general solution to the above equation in terms of a new radial variable as

$$G_0(x) = \frac{A}{mn} + C_1 x^n\ {}_2F_1\left[2, \frac{n}{m}, 2 + \frac{n}{m}; x^m\right] - \frac{C_2}{n}x^{-m}(n + (m-n)x^m)\ , \tag{4.47}$$

where

$$x = r_0 r\ , \quad m = d_i + z - \theta\ , \quad n = 2z - 2\ , \quad A = iC_0 k r_0^{d_i-\theta}\frac{q}{\omega}\ , \tag{4.48}$$

while C_1 and C_2 are arbitrary constants which are to be fixed by demanding regularity of a_y at the horizon. Potential divergences in $a_y(r)$ and $a_y'(r)$ near the horizon can be removed by choosing $C_1 = 0$ and $C_2 = \frac{A}{m^2}$. In terms of the original radial coordinate r, the solution is

$$G_0(r) = -iC_0 k\, \frac{q}{\omega} \cdot \frac{r_0^{d_i-\theta}}{(d_i+z-\theta)^2} \cdot (r_0 r)^{-(d_i+z-\theta)} f(r)\ . \tag{4.49}$$

For relativistic theories, $(z=1, k=0)$ the above expression vanishes identically implying that the shear mode sector is governed exclusively by metric perturbations H_{xy} and H_{ty}.

The subleading term in a_y i.e. $G_1(r)$ can be determined by collecting terms of $O(\lambda)$ from (4.45). The inhomogeneous part of the equation governing G_1 involves F_2' and F_1. $F_2'(r)$ can be evaluated from $O(\lambda^2)$ terms of (4.33). Although we could find the general solution to G_1, finding the integration constants respecting regularity at the horizon seems difficult and cumbersome by analytic means. We discuss further details about F_2 and G_1 in Appendix C.

4.4 Dual field theory correlation functions

In this section, we will determine the energy-momentum tensor correlation functions $\langle TT \rangle$ following the prescription in [114, 115], and defining $T_{\mu\nu}$ as dual to the perturbation $h_{\mu\nu}$. The action governing the perturbations using the variables H_{ty}, H_{xy} and a_y in (4.12) is given by

$$\begin{aligned} S^{pert} = -\frac{1}{16\pi G_N^{(d_i+2)}} \int d^{d_i+2}x\, \Bigg[\frac{1}{2} r^{\theta-z-d_i+1} \Big(- r^{2z-2} (H_{ty}')^2 + \frac{r^{2z-2}}{f} (\omega H_{xy} + q H_{ty})^2 \\ + f (H_{xy}')^2 \Big) - k H_{ty} a_y' + \frac{1}{2} r^{d_i+3-z-\theta} f (a_y')^2 + \frac{1}{2} r^{d_i+3-z-\theta} \left(\frac{r^{2z-2}}{f} \omega^2 - q^2 \right) a_y^2 \Bigg] . \end{aligned} \tag{4.50}$$

The equations of motion from this action lead to (4.13)-(4.16) and we have suppressed contact terms. The above expression can be recast as a bulk piece (which vanishes by the equations of motion) and a boundary term. This boundary action takes the form

$$\begin{aligned} S^{bdy} = -\frac{1}{32\pi G_N^{(d_i+2)}} \int d^{d_i+1}x\, \Big[r^{\theta-z-d_i+1} \left(-r^{2z-2} H_{ty} H_{ty}' + f H_{xy} H_{xy}' \right) \\ + r^{d_i+3-z-\theta} a_y a_y' \Big] + \cdots \end{aligned} \tag{4.51}$$

again suppressing contact terms. Using the equation of motion (4.15) and the definition of $\mathcal{H}$ (4.17), we can recast the relevant terms of the above action as

$$S^{bdy} = \lim_{r\to r_c} -\frac{1}{32\pi G_N^{(d_i+2)}} \int d^{d_i+1}x \left[\frac{f r^{\theta-z-d_i+1}}{\omega^2 - q^2 f r^{2-2z}} \mathcal{H}'(r,x)\mathcal{H}(r,x)\right] + \cdots \quad (4.52)$$

In the above equation, we have exhibited only those terms that contribute to the 2-point function i.e. terms that are at least second order in $\mathcal{H}$. Using the Fourier decomposition of the bulk field as $\mathcal{H}(r,t,x) = \int d\omega\ dq\ e^{-i\omega t + iqx}\mathcal{H}(r,\omega,q)$ we can further recast as

$$S^{bdy} = \lim_{r\to r_c} -\frac{1}{32\pi G_N^{(d_i+2)}} \int d\omega\ dq\ \frac{f r^{\theta-z-d_i+1}}{\omega^2 - q^2 f r^{2-2z}} \mathcal{H}'(r,\omega,q)\mathcal{H}(r,-\omega,-q)\ . \quad (4.53)$$

For $z=1, \theta=0$, this agrees with the AdS case in [110]. Also, for theories with Lifshitz symmetry, it is clear that (4.53) is Lifshitz-invariant. The $\langle TT\rangle$ shear correlator in the boundary theory is defined as

$$G_{xy,xy} = \langle T_{xy}T_{xy}\rangle = \left.\frac{\delta^2 S^{bdy}}{\delta h_y^{x(0)}\delta h_y^{x(0)}}\right|_{r\approx r_c}\ . \quad (4.54)$$

We define the boundary fields through the $r\to r_c\sim 0$ limits of the bulk fields, i.e. $\mathcal{H}^{(0)}(\omega,q) = \mathcal{H}(r_c,\omega,q)$ and $h_y^{x(0)}(\omega,q) = h_y^{x(0)}(r_c,\omega,q)$. The asymptotics can be analysed by studying (4.18) in the limit $r\to 0$ at zero momenta and frequency ($\mathbf{Q}=\mathbf{\Omega}=0$) i.e.

$$\mathcal{H}'' + \frac{z+\theta-d_i-1}{r}\mathcal{H}' = 0\ . \quad (4.55)$$

The solutions are $\mathcal{H} = Ar^{\Delta}$ with $\Delta = 0,\ d_i+2-z-\theta$. Thus in the hydrodynamic regime i.e. $\mathbf{\Omega},\mathbf{Q} \ll 1$, we can schematically write the mode as $\mathcal{H} = \mathcal{A}(\omega,q) + \mathcal{B}(\omega,q)r^{d_i+2-z-\theta}$ where the functions $\mathcal{A}(\omega,q)$ and $\mathcal{B}(\omega,q)$ can be read off from (4.39),

$$\mathcal{A}(\omega,q) = C_0\left[1 + i\frac{q^2}{\omega}\frac{r_0^{z-2}}{d_i+2-z-\theta}\right]\ , \qquad \mathcal{B}(\omega,q) = -C_0\frac{iq^2}{\omega}\frac{r_0^{d_i-\theta}}{d_i+2-z-\theta}\ . \quad (4.56)$$

We can write the normalized bulk field $\mathcal{H}$ in terms of its source $\mathcal{H}^0$

$$\mathcal{H}(\omega,q) = \mathcal{H}^{(0)}(\omega,q)\frac{1}{\mathcal{N}}\left[1 + \frac{\mathcal{B}(\omega,q)}{\mathcal{A}(\omega,q)}r^{d_i+2-z-\theta}\right]\ , \qquad \mathcal{N} = 1+\frac{\mathcal{B}(\omega,q)}{\mathcal{A}(\omega,q)}r_c^{d_i+2-z-\theta}\ . \quad (4.57)$$

Note that the normalization factor $\mathcal{N}$ satisfies $\mathcal{N} \sim 1$ as $r_c \to 0$ for $z < 2 + d_i - \theta$. Using this solution in (4.53), we get

$$S^{bdy} = \lim_{r \to r_c} - \int \frac{d\omega\ dq}{32\pi G_N^{(d_i+2)}} \frac{f r^{2-2z}}{\omega^2 - q^2 f r^{2-2z}} \cdot \frac{d_i + 2 - z - \theta}{\mathcal{N}} \cdot \frac{\mathcal{B}(\omega, q)}{\mathcal{A}(\omega, q)} \cdot \mathcal{H}^{(0)}(\omega, q) \mathcal{H}^{(0)}(-\omega, -q) \tag{4.58}$$

As $r \to r_c \to 0$ with $z > 1$, we note that $\lim_{r\to 0} \frac{f r^{2-2z}}{\omega^2 - q^2 f r^{2-2z}} \frac{1}{\mathcal{N}^2} \frac{\mathcal{B}(\omega,q)}{\mathcal{A}(\omega,q)} = -\frac{1}{q^2} \frac{\mathcal{B}(\omega,q)}{\mathcal{A}(\omega,q)}$. From the definition of $\mathcal{H}$ in (4.17) and also noting $H_{xy} \equiv h^x_y = h^y_x$ and $H_{ty} \equiv h^y_t$ from (4.12), we see that $\frac{\delta}{\delta H^{(0)}_{xy}} = \frac{\delta}{\delta h^{x(0)}_y} = \omega \frac{\delta}{\delta \mathcal{H}^{(0)}}$. Thus the correlation function (4.54) becomes

$$\begin{aligned} G_{xy,xy} = \langle T^x_y(k) T^x_y(-k) \rangle = \frac{\delta^2 S^{bdy}}{\delta h^{x(0)}_y(k) \delta h^{x(0)}_y(-k)} &= \omega^2 \frac{\delta^2 S^{bdy}}{\delta \mathcal{H}^{(0)}(k) \delta \mathcal{H}^{(0)}(-k)} \\ &= \frac{1}{16\pi G_N^{(d_i+2)}} \frac{i\omega^2 r_0^{d_i-\theta}}{\omega + i\mathcal{D} q^2} \ , \end{aligned} \tag{4.59}$$

with $\mathcal{D}$ given in (4.41), and $k = (\omega, q)$. The Kubo formula then gives viscosity as

$$\eta = \lim_{\omega \to 0} \frac{G_{xy,xy}(\omega, q = 0)}{i\omega} = \frac{r_0^{d_i-\theta}}{16\pi G_N^{(d_i+2)}} \ . \tag{4.60}$$

With the entropy density given in terms of the horizon area $s = \frac{r_0^{d_i-\theta}}{4G_N^{(d_i+2)}}$, we obtain universal behaviour for the viscosity bound $\frac{\eta}{s} = \frac{1}{4\pi}$, as for relativistic theories [90]. This is consistent with [P2, P'1], where we conjectured the universal relation (4.42) saturating the proposed viscosity bound in [90]. Also, we can write down other correlators as follows:

$$\begin{aligned} G_{ty,ty} = \langle T^y_t(k) T^y_t(-k) \rangle &= \frac{1}{16\pi G_N^{(d_i+2)}} \frac{iq^2 r_0^{d_i-\theta}}{\omega + i\mathcal{D} q^2} \ , \\ G_{ty,xy} = \langle T^y_t(k) T^y_x(-k) \rangle &= \frac{1}{16\pi G_N^{(d_i+2)}} \frac{i\omega q r_0^{d_i-\theta}}{\omega + i\mathcal{D} q^2} \ . \end{aligned} \tag{4.61}$$

Each correlator above exhibits a pole at $\omega = -i\mathcal{D} q^2$ which is the lowest lying quasinormal mode as we have seen earlier. The viscosity (4.60) above agrees with the result in [116]: however what is noteworthy in our analysis is that this is obtained in the regime $z < d_i + 2 - \theta$.

4.4.1 Dual field theory correlation functions: $z = d_i + 2 - \theta$

hvLif theories with $z = d_i + 2 - \theta$ arise from the null reductions of highly boosted black branes in [51, 72] as mentioned previously. The asymptotic fall-offs in (4.55), (4.56), coincide in this case: this is the interface of the standard/alternative quantization in [98], and one of the solutions contains a logarithm. We see that in this case, (4.55) reduces to

$$\mathcal{H}'' + \frac{1}{r}\mathcal{H}' = 0 \ , \tag{4.62}$$

with the solution $\mathcal{H} = \mathcal{A}(\omega, q) + \mathcal{B}(\omega, q) \log \frac{1}{r_0 r}$ in (4.43). We define the normalized bulk field in terms of the source as

$$\mathcal{H}(r, \omega, q) = \mathcal{H}_0(\omega, q) \cdot \frac{1 + \frac{iq^2}{\omega} r_0^{z-2} \log \frac{1}{r_0 r}}{1 + \frac{iq^2}{\omega} r_0^{z-2} \log \frac{1}{r_0 r_c}} \ , \tag{4.63}$$

the source being the boundary value $\mathcal{H}^{(0)}(\omega, q) = \mathcal{H}(r_c, \omega, q)$. Note that unlike $\mathcal{N}$ in (4.57), the logarithm does not die near the boundary and the normalization above is less trivial. Using (4.63), the relevant part of the boundary action becomes

$$\begin{aligned} S^{bdy} &= \lim_{r \to r_c} -\frac{1}{32\pi G_N^{(d_i+2)}} \int d^{d_i+1}x \left[\frac{f r^{3-2z}}{\omega^2 - q^2 f r^{2-2z}} \ \mathcal{H}'(r, x)\mathcal{H}(r, x) \right] \\ &= \frac{1}{32\pi G_N^{(d_i+2)}} \int d\omega . dq \ \frac{i r_0^{z-2}}{\omega} \frac{1}{\left(1 + \frac{iq^2}{\omega} r_0^{z-2} \log \frac{1}{r_0 r_c}\right)} \mathcal{H}^{(0)}(\omega, q)\mathcal{H}^{(0)}(-\omega, -q) \ . \end{aligned} \tag{4.64}$$

As before, we obtain the energy-momentum tensor correlation functions as

$$\begin{aligned} G_{xy,xy} = \langle T_y^x(k) T_y^x(-k) \rangle = \frac{\delta^2 S^{bdy}}{\delta h_y^{x(0)}(k) \delta h_y^{x(0)}(-k)} &= \omega^2 \frac{\delta^2 S^{bdy}}{\delta \mathcal{H}^{(0)}(k) \delta \mathcal{H}^{(0)}(-k)} \\ &= \frac{1}{16\pi G_N^{d_i+2}} \frac{i\omega^2 r_0^{z-2}}{\omega + i\mathcal{D} q^2} \ , \end{aligned} \tag{4.65}$$

using the expression for the shear diffusion constant in (4.44). Using the Kubo formula, we again see that

$$\eta = \lim_{\omega \to 0} \frac{G_{xy,xy}(\omega, q = 0)}{i\omega} = \frac{r_0^{z-2}}{16\pi G_N^{(d_i+2)}} = \frac{r_0^{d_i-\theta}}{16\pi G_N^{(d_i+2)}} \ . \tag{4.66}$$

Note that the diffusive pole again coincides with the quasinormal mode frequency in (4.44): note that this here arises from the nontrivial normalization factor in (4.63). Now using the entropy density $s = \frac{r_0^{d_i-\theta}}{4G_N^{(d_i+2)}}$ given in terms of the horizon area, we again find universal behaviour $\frac{\eta}{s} = \frac{1}{4\pi}$ for the viscosity bound, again as for relativistic theories [90]. It is worth noting that in applying the Kubo formula, we first restrict to the zero momentum sector $q = 0$, which kills off the term containing the leading diffusion constant $\mathcal{D}$ which, strictly speaking, is logarithmically divergent as $r_c \to 0$.

4.5 Discussion

We have studied low lying hydrodynamic quasinormal modes for shear perturbations of hyperscaling violating Lifshitz black branes: these are of the form $\omega = -i\mathcal{D}q^2$ where $\mathcal{D}$ is the shear diffusion constant. This is consistent with $\mathcal{D}$ obtained in [P2, P'1] through a membrane-paradigm analysis of near horizon perturbations and the associated shear diffusion equation. This shear diffusion equation $\partial_t j^t = \mathcal{D}\partial_x^2 j^t$ following from the second order Einstein equations in a sense dictates the form of $\mathcal{D}$ above and is consistent with the low-lying quaisnormal mode spectrum. The analysis here and the associated boundary conditions are valid for theories with exponents satisfying $z \leq d_i+2-\theta$: this is the regime that is continuously connected to AdS theories ($z = 1, \theta = 0$). Using the asymptotics of these quasinormal modes, retarded correlators of dual operators can be obtained: we have seen that the poles of the retarded $\langle TT \rangle$ correlator at finite temperature coincide with the lowest quasinormal frequencies of the dual gravity theory. This analysis appears consistent with the Kubo formula for viscosity via the retarded Green's function at zero momentum only for theories with $z \leq d_i + 2 - \theta$. Perhaps this is not surprising given the asymptotic fall-offs of the quasinormal modes: for $z > d_i+2-\theta$, high energy modes appear to dominate, with no universal low energy behaviour for the diffusion expression. It would be interesting to understand this better.

hvLif spacetimes with $z = d_i + 2 - \theta$ exhibit more interesting hydrodynamic behaviour as we have seen: the asymptotic fall-offs of the bulk modes coincide here. While correlation functions in the Kubo limit continue to reveal universal behaviour $\frac{\eta}{s} = \frac{1}{4\pi}$ for the viscosity, the leading shear diffusion constant exhibits logarithmic scaling (involving the ultraviolet cutoff), perhaps suggesting that subleading contributions are important with some resummation required. The null reductions [51, 72] of highly boosted black branes give hvLif theories with $z = d_i + 2 - \theta$. It is worth noting that the boost induces anisotropy in the system (although the lower dimensional theory after compactification enjoys translation invariance in the spatial directions so that the zero momentum Kubo limit studied here is unambiguous). The large boost involved in these string constructions possibly leads to novel hydrodynamic behaviour. It would perhaps be interesting to study this directly from the null reduction of black brane hydrodynamics and the fluid/gravity correspondence [91].

Chapter 5

Conclusion and future directions

In this thesis, we have explored certain aspects of entanglement and hydrodynamics of the so called hyperscaling violating Lifshitz theories using the dual gravitational description. To reiterate again, we studied the mutual information (MI) between two strip shaped subsystems in the AdS plane wave background which are dual to excited states in CFT with constant energy momentum flux. In particular for AdS_5 plane waves, there exists a logarithmic scaling of entanglement entropy which is suggestive of hidden Fermi surfaces in the dual theory. These excited states are just a part of the full phase diagram and it will be interesting to explore this further.

The hydrodynamics of hvLif theories clearly led us to a constraint on the Lifshitz exponent i.e. $z \leq d_i+2-\theta$ for a field theory in d_i+1-dimensions. Interestingly this inequality saturates precisely for the class of hvLif theories that results from null reductions of AdS plane waves. Towards gaining more insight into the behaviour of hvLif theories, it will interesting to study the flow equations of various response functions adapting the methodology in [118]. In [118], the authors showed that at the level of linear response the low frequency limit of a strongly coupled field theory at finite temperature is determined by the horizon geometry of its gravity dual, i.e. by the "membrane paradigm" fluid of classical black hole mechanics. Thus generic boundary theory transport coefficients can be expressed in terms of geometric quantities evaluated at the horizon. Thinking of the radial direction as an energy scale, we can think of the flow equations as an RG equation. Given the results that has been discussed in the manuscript, it is expected that the flow equations exhibit possible pathologies in the regime $z > d_i + 2 - \theta$.

The correspondence between gravitational solutions and equations of fluid dynamics was put on a robust footing with the advent of the *fluid/gravity correspondence* [91, 92, 93, 94, 95]. Assuming that the dual field theory is in the hydrodynamic regime, we can map gravitational solutions to fluid dynamic solutions through the AdS/CFT correspondence. The map is constructed perturbatively by solving Einstein's equation order by order around a known stationary solution. The boundary stress tensor computed from the perturbed solution maps to equations of a viscous relativistic fluid. Using the standard constitutive relation for the stress tensor of a viscous fluid, one can read off the transport coefficients of the dual fluid. This program was adapted in the context of hvLif theories in [48] to write down the stress tensor of the dual non-relativistic fluid. It will be interesting to see how this follows from the higher dimensional uncompactified theory which is inherently relativistic in nature. Since AdS plane waves are high boost low temperature limit of boosted black branes, this will help us in mapping the transport coefficients of a boosted black brane to that of a non-relativistic system in one lower dimension.

Another interesting aspect for investigation would be the introduction of higher derivative corrections to hvLif theories. Recall that it was essentially the combination of the Lifshitz and hyperscaling violating exponents i.e. $z + \theta$ which resulted in the logarithmic behaviour of the diffusion constant. Introduction of higher derivative terms will correspondingly change this combination, presumably rendering a power-law scaling to the shear diffusion constant $\mathcal{D}$.

Appendix A

HvLif metric as solution to Einstein-Maxwell-dilaton action and linearized perturbations

The metric (4.10) is a solution to the Einstein-Maxwell-Dilaton action

$$S = -\frac{1}{16\pi G_N^{(d+1)}} \int d^{d+1}x \sqrt{-G} \left[R - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{Z(\phi)}{4} F_{\mu\nu} F^{\mu\nu} + V(\phi) \right] , \quad \text{(A.1)}$$

where $Z(\phi) = e^{\lambda\phi}$ and $V(\phi) = -2\Lambda e^{-\delta\phi}$ The null energy conditions following from (4.10) give constraints on the Lifshitz z and hyperscaling violating θ exponents

$$(z-1)(d_i + z - \theta) \geq 0 , \qquad (d_i - \theta)(d_i(z-1) - \theta) \geq 0 . \quad \text{(A.2)}$$

Varying with $G_{\mu\nu}$, A_μ and ϕ, we obtain the following equations of motion,

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu \phi \partial_\nu \phi - G_{\mu\nu} \frac{V(\phi)}{d-1} + \frac{Z(\phi)}{2} G^{\rho\sigma} F_{\rho\mu} F_{\sigma\nu} - \frac{Z(\phi)}{4(d-1)} G_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} , \quad \text{(A.3)}$$

$$\nabla_\mu (Z(\phi) F^{\mu\nu}) = 0 , \qquad \frac{1}{\sqrt{-G}} \partial_\mu (\sqrt{-G} G^{\mu\nu} \partial_\nu \phi) + \frac{\partial V(\phi)}{\partial \phi} - \frac{1}{4} \frac{\partial Z(\phi)}{\partial \phi} F_{\rho\sigma} F^{\rho\sigma} = 0 . \quad \text{(A.4)}$$

A particular stationary solution to the above equations of motion is given by

$$\begin{aligned} \phi &= \sqrt{2(d_i-\theta)(z-\theta/d_i-1)}\ \log r\ , \qquad &(A.5)\\ A_t &= \frac{\alpha f(r)}{r^{d_i+z-\theta}}\ , \qquad \alpha = -\sqrt{\frac{2(z-1)}{d_i+z-\theta}}\ , \qquad A_i = 0\ . &(A.6)\\ V(\phi) &= (d_i+z-\theta)(d_i+z-\theta-1)r^{-\frac{2\theta}{d_i}}\ ; \qquad Z(\phi) = r^{\frac{2\theta}{d_i}+2d_i-2\theta}\ . &(A.7) \end{aligned}$$

while the line element is given by

$$ds^2 = r^{\frac{2\theta}{d-1}}\left(-\frac{f(r)}{r^{2z}}dt^2 + \frac{dr^2}{f(r)r^2} + \sum_{d_i}\frac{dx_i^2}{r^2}\right), \qquad f(r) = 1-(r_0 r)^{d+z-\theta-1}. \quad (A.8)$$

Turning on gravitational, gauge field and scalar field perturbations $h_{\mu\nu}(x)$, $a_\mu(x)$ and $\varphi(x)$, the linearized Einstein's equations are given by

$$\begin{aligned} R^{(1)}_{\mu\nu} =& \frac{1}{2}\partial_\mu\phi\partial_\nu\varphi + \frac{1}{2}\partial_\mu\varphi\partial_\nu\phi - \frac{V}{2}(h_{\mu\nu} - G_{\mu\nu}\delta\varphi)\\ &+\frac{Z}{2}\left[G^{\rho\sigma}F_{\mu\rho}f_{\nu\sigma} + G^{\rho\sigma}f_{\mu\rho}F_{\nu\sigma} - h^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} + \lambda\varphi G^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}\right]\\ &-Z\left[\frac{1}{4}G_{\mu\nu}(F_{\rho\sigma}f^{\rho\sigma} - g^{\rho\alpha}h^{\sigma\beta}F_{\rho\sigma}F_{\alpha\beta}) + \frac{1}{8}h_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \frac{1}{8}\lambda\varphi G_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right]\ , \end{aligned} \quad (A.9)$$

where

$$\begin{aligned} R^{(1)}_{\mu\nu} &= \frac{1}{2}[\nabla_\alpha\nabla_\nu h^\alpha_\mu + \nabla_\alpha\nabla_\mu h^\alpha_\nu - \nabla_\alpha\nabla^\alpha h_{\mu\nu} - \nabla_\nu\nabla_\mu h]\ ;\\ f_{\mu\nu} &= \partial_\mu a_\nu - \partial_\nu a_\mu\ ; \qquad h = G^{\mu\nu}h_{\mu\nu}\ ; \qquad \delta = \frac{2\theta/d_i}{\sqrt{2(d_i-\theta)(z-\theta/d_i-1)}}\ . \end{aligned} \quad (A.10)$$

Similarly, the linearized Maxwell Equations (A.4) are

$$\nabla_\mu(Z\,f^{\mu\nu}) - \nabla_\mu(Z\,h^{\mu\rho}F_\rho{}^\nu) - Z(\nabla_\mu h^{\nu\sigma})F^\mu{}_\sigma + \frac{1}{2}(\nabla_\mu h)Z\,F^{\mu\nu} + \lambda\,Z\,F^{\mu\nu}\partial_\mu\varphi = 0\ . \quad (A.11)$$

The linearized scalar field equation is:

$$\begin{aligned} &\frac{1}{\sqrt{-G}}\partial_\mu(\sqrt{-G}G^{\mu\nu}\partial_\nu\varphi) - \frac{1}{\sqrt{-G}}\partial_\mu(\sqrt{-G}h^{\mu\nu}\partial_\nu\phi) + \frac{1}{2}G^{\mu\nu}\partial_\nu\phi\partial_\mu h + V\delta^2\varphi\\ &\qquad - \frac{\lambda Z}{4}(2F_{\mu\nu}f^{\mu\nu} - 2G^{\mu\rho}h^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + \lambda\varphi F_{\mu\nu}F^{\mu\nu}) = 0\ . \end{aligned} \quad (A.12)$$

In (A.9), (A.11), (A.12), indices are raised and lowered using (4.10).

The relevant equations of motion has already been stated explicitly in (3.17)-(3.19). For the sake of completion, we list the other equations here. The t, x and r components of the linearized Maxwell's equation (A.11), respectively, give

$$\partial_r(r^{3+z-\theta}\partial_r a_t) - \frac{r^{3+z-\theta}}{f}(q^2 a_t + q\omega a_x) - \frac{k}{2}\Big[\partial_r(r^{2-\theta}(h_{xx}+h_{yy})) + \partial_r\Big(\frac{r^{2z-\theta}}{f}h_{tt}\Big)\Big] - k\lambda\partial_r\varphi = 0\ , \tag{A.13}$$

$$\partial_r(r^{5-z-\theta}f\partial_r a_x) + \frac{r^{3+z-\theta}}{f}(q\omega a_t + \omega^2 a_x) - k\partial_r(r^{2-\theta}h_{tx}) = 0\ , \tag{A.14}$$

$$q[r^{5-z-\theta}f\partial_r a_x - kr^{2-\theta}h_{tx}] + \omega\Big[r^{3+z-\theta}\partial_r a_t - \frac{k}{2}\Big(r^{2-\theta}(h_{xx}+h_{yy}) + \frac{r^{2z-\theta}}{f}h_{tt}\Big) - k\lambda\varphi\Big] = 0\ . \tag{A.15}$$

The tt-component of the linearized Einstein's equation (A.9) gives

$$\begin{aligned}&\partial_r^2 h_{tt} - \Big(\frac{2-4z+\theta}{r} + \frac{\partial_r f}{f}\Big)\frac{\partial_r h_{tt}}{2} - \frac{q^2}{f}h_{tt} - \frac{2q\omega}{f}h_{tx} - kr^{1-z}\partial_r a_t\\ &+\Big[\frac{-2(1+z-\theta)(2+z-\theta)}{r^2 f} - \frac{2(2z-\theta)\partial_r f}{rf} + \frac{(\theta-2z)^2}{r^2} + \frac{(\partial_r f)^2}{f^2}\Big]\frac{h_{tt}}{2}\\ &-\frac{1}{2}\partial_r(r^{\theta-2z}f)\partial_r(r^{2-\theta}(h_{xx}+h_{yy})) - \frac{\omega^2}{f}(h_{xx}+h_{yy}) + (2+z-\theta)\beta r^{-2-2z+\theta}\varphi = 0\ .\end{aligned} \tag{A.16}$$

The tx-component of (A.9) gives

$$\partial_r(r^{z+\theta-3}\partial_r(r^{2-\theta}h_{tx})) + \frac{r^{z-1}}{f}q\omega h_{yy} - k\partial_r a_x = 0\ . \tag{A.17}$$

The tr-component of (A.9) gives

$$q\partial_r\Big(\frac{r^{2z-\theta}}{f}h_{tx}\Big) + \frac{\omega}{2}\Big[\partial_r\Big(\frac{r^{2z-\theta}}{f}(h_{xx}+h_{yy})\Big) + \frac{r^{2z-2}}{f}\partial_r(r^{2-\theta}(h_{xx}+h_{yy}))\Big] + \omega\frac{r^{2z-3}}{f}\beta\varphi = 0\ . \tag{A.18}$$

Adding xx-component to yy-component of (A.9) gives

$$\begin{aligned}&\partial_r(r^{2\theta-z-3}f\partial_r(r^{2-\theta}(h_{xx}+h_{yy}))) + \omega^2\frac{r^{z+\theta-3}(h_{xx}+h_{yy})}{f} - 2r^{\theta-z-1}q^2 h_{yy}\\ &+\frac{2r^{z+\theta-3}}{f}q\omega h_{tx} - 2kr^{\theta-2}\partial_r a_t - (\theta-2)r^{2\theta-z-4}f\partial_r\Big(\frac{r^{2z-\theta}h_{tt}}{f}\Big) + \frac{k^2 r^{z+\theta-5}}{f}h_{tt}\\ &+q^2\frac{r^{z+\theta-3}h_{tt}}{f} - \frac{2(2+z-\theta)(4-4z-2\theta+\theta^2)}{\beta}r^{2\theta-z-5}\varphi = 0\ .\end{aligned} \tag{A.19}$$

Subtracting yy-component from xx-component of (A.9) gives

$$\partial_r(r^{\theta-z-1}f\partial_r(r^{2-\theta}(h_{xx}-h_{yy})))+\frac{r^{z-1}}{f}\omega^2(h_{xx}-h_{yy})+\frac{r^{z-1}}{f}q^2h_{tt}+\frac{2r^{z-1}}{f}q\omega h_{tx}=0\ . \tag{A.20}$$

The xr-component of (A.9) gives

$$\begin{aligned} q[r^{2-\theta}\Big(\partial_r h_{tt}+\frac{1+z-\theta}{r}h_{tt}-\frac{\partial_r f}{2f}h_{tt}\Big)-r^{2-2z}f\partial_r(r^{2-\theta}h_{yy})-kr^{3-z-\theta}a_t \\ -\beta r^{1-2z}f\varphi]+\omega[\partial_r(r^{2-\theta}h_{tx})-kr^{3-z-\theta}a_x]=0\ . \end{aligned} \tag{A.21}$$

The rr-component of (A.9) gives

$$\begin{aligned} &\partial_r^2(h_{xx}+h_{yy})+\Big(\frac{3(2-\theta)}{2r}+\frac{\partial_r f}{2f}\Big)\partial_r(h_{xx}+h_{yy})+(\theta-2)\Big(\frac{\theta}{2r^2}-\frac{\partial_r f}{2rf}\Big)(h_{xx}+h_{yy}) \\ &-\frac{r^{2z-2}}{f}\partial_r^2h_{tt}+r^{2z-2}\partial_r h_{tt}\Big(\frac{-2-4z+3\theta}{2rf}+\frac{\partial_r f}{2f^2}\Big)+\alpha(2+z-\theta)\frac{r^{z-1}}{f}\partial_r a_t \\ &+\Big[\frac{\theta(2z-\theta)}{2r^2f}-\frac{(\partial_r f)^2}{2f^3}-\frac{1}{r^2f^2}\Big((z-1)(2+z-\theta)+(z-1)r\partial_r f-r^2\partial_r^2 f\Big)\Big]r^{2z-2}h_{tt} \\ &+2\beta r^{\theta-3}\partial_r\varphi-\frac{(2+z-\theta)\beta}{f}r^{\theta-4}\varphi=0\ . \end{aligned} \tag{A.22}$$

The linearized scalar field equation (A.12) gives

$$\begin{aligned} &\partial_r(r^{\theta-z-1}f\partial_r\varphi)+\Big(\frac{k^2\lambda^2}{2}-2\Lambda\delta^2\Big)r^{\theta-z-3}f\varphi+r^{\theta-z-1}\Big(\frac{r^{2z-2}\omega^2}{f^2}-\frac{q^2}{f}\Big)\varphi \\ &+\frac{\beta r^{\theta-z-2}f}{2}\Big[\partial_r(r^{2-\theta}(h_{xx}+h_{yy}))-\partial_r\Big(\frac{r^{2z-\theta}h_{tt}}{f}\Big)\Big]+\frac{k^2\lambda r^{z-3}h_{tt}}{2f}-k\lambda\partial_r a_t=0\ . \end{aligned} \tag{A.23}$$

Appendix B

Subleading terms in membrane paradigm approach: $\mathcal{A}_t^{(1)}$ and $\mathcal{A}_x^{(1)}$

B.1 Subleading terms when $z < 4 - \theta$

Using the expansion over q^2 for $\mathcal{A}_t$ and $\mathcal{A}_x$ in the gauge field equations, it can be checked that the leading $O(q^0)$ terms are consistent with the ansatz for $\mathcal{A}_t^{(0)}, \mathcal{A}_x^{(0)}$ in the regime (3.43). Likewise the subleading terms can be evaluated: collecting terms of $O(q^2)$ consistently, we further note that $\mathcal{D} \sim r_0^{z-2} \sim T^{\frac{z-2}{z}}$ and $\Gamma \sim \mathcal{D}q^2$ which leads to $\frac{\Gamma}{q} \sim \frac{q}{T^{2/z-1}}$. Using this heuristic estimate and simplifying gives

$$\partial_r \mathcal{A}_t^{(1)} \sim r_0 \Big[\frac{q^2}{T^{2/z}} \log\left(\frac{1/r_0}{(1/r_0) - r}\right) + \frac{q^4}{T^{4/z}} \log^2\left(\frac{1/r_0}{(1/r_0) - r}\right)\Big] \mathcal{A}_t^{(0)} \ . \tag{B.1}$$

Using (3.43), we see that $\partial_r \mathcal{A}_t^{(1)} \ll \mathcal{A}_t^{(0)}$, and after integrating, that $\mathcal{A}_t^{(1)} \ll \mathcal{A}_t^{(0)}$, verifying that these are indeed subleading. Likewise, we find

$$\mathcal{A}_x^{(1)} \sim \Big[\frac{q^2}{T^{2/z}} \log\left(\frac{1/r_0}{1/r_0 - r}\right) + \frac{q^4}{T^{4/z}} \log^2\left(\frac{1/r_0}{1/r_0 - r}\right)\Big] \mathcal{A}_x^{(0)} \ll \mathcal{A}_x^{(0)} \ . \tag{B.2}$$

B.2 Subleading terms when $z = 4 - \theta$

For the case $z = 4 - \theta$, we obtain $\frac{\mathcal{A}_x^{(0)}}{\mathcal{A}_t^{(0)}} \sim \frac{1}{r_0^{2(z-1)}} \frac{\Gamma}{q} \frac{\log(\frac{1/r_0}{1/r_0 - r})}{\log(\frac{1}{r_0 r_c})}$ so that imposing (3.6) gives

$$\frac{1}{r_0^{2(z-1)}} \cdot \frac{\Gamma^2}{q^2} \cdot \frac{\log(\frac{1/r_0}{1/r_0 - r})}{\log(\frac{1}{r_0 r_c})} \ll 1 \ . \tag{B.3}$$

From the estimates obtained for $\mathcal{D}$ from the diffusion equation and the diffusion integral, we obtain $\frac{\Gamma}{q} \sim \frac{q}{T^{\frac{2}{z}-1}} \log(\frac{1}{r_0 r_c})$. Using along with $\mathcal{D} \sim \Gamma q^2$, we obtain the modified bound $\frac{(1/r_0)-r_h}{1/r_0} \ll \frac{q^2}{T^{2/z}} \log^2(\frac{1}{r_0 r_c})$. The above estimates leads to the inequality $\frac{q^2}{T^{2/z}} \log(\frac{1/r_0}{(1/r_0)-r_h}) \log \frac{1}{r_0 r_c} \ll 1$. The subleading terms now give

$$\partial_r \mathcal{A}_t^{(1)} \sim r_0 \Big[\frac{q^2}{T^{2/z}} \log\left(\frac{1/r_0}{(1/r_0)-r}\right) + \frac{q^4}{T^{4/z}} \log^2\left(\frac{1/r_0}{(1/r_0)-r}\right) \log\left(\frac{1}{r_0 r_c}\right)\Big] \mathcal{A}_t^{(0)} \; . \tag{B.4}$$

Within the regime , it would appear that $\partial_r \mathcal{A}_t^{(1)} \ll \mathcal{A}_t^{(0)}$: however $r_0 r_c \ll 1$ implies that $\log(\frac{1}{r_0 r_c})$ is large so that the $O(q^4)$ term need not be small even if $\frac{q^2}{T^{2/z}} \ll 1$, suggesting a breakdown of the series expansion.

Appendix C

Solution for gauge field perturbation in QNM approach

Using (4.35) and (4.37) we can simplify the inhomogeneous terms in (4.46) significantly. In terms of the new radial coordinate x and the parameters m, n and A as defined in (4.48), the equation (4.46) can be recast as

$$\frac{d^2G_0}{dx^2} + \left(\frac{2m-n+1}{x} - \frac{m}{x(1-x^m)}\right)\frac{dG_0}{dx} - \frac{mn}{x^2(1-x^m)}G_0 + \frac{A}{x^2(1-x^m)} = 0 \ . \tag{C.1}$$

The solution to the above equation is given by (4.47).

$F_2(r,t,x)$ solution: The equation governing $F_2(r)$ follows from $O(\lambda^2)$ terms of (4.33). The relevant terms following from the first two lines of (4.33) are computationally straightforward to derive. Let us concentrate on the last line of (4.33). Concentrating on the powers of λ, the integral in the last term can be integrated by parts and rewritten as

$$\lambda^3 f^{\frac{i\lambda\Omega}{2}} \int ds\ f^{-\frac{i\lambda\Omega}{2}}\ s^{d_i+1-z-\theta} G \approx \lambda^3 f^{\frac{i\lambda\Omega}{2}} \left[f^{-\frac{i\lambda\Omega}{2}} \int ds\ s^{d_i+1-z-\theta} G_0 + \frac{i\lambda\Omega}{2} \int ds\ f^{-\frac{i\lambda\Omega}{2}}\ \frac{f'}{f} \int ds'\ s'^{d_i+1-z-\theta} G_0 \right] \sim O(\lambda^3) + O(\lambda^4) \ . \tag{C.2}$$

The above expression shows that the leading contribution from the last term of (4.33) becomes relevant at $O(\lambda^3)$ and has no role to play in determining $F_2(r)$.

We can write the equation governing F_2 as

$$F_2'' - \frac{H'}{H}F_2' + \Bigg[-i\mathbf{\Omega}\frac{f'}{f}\ F_1' - \frac{i\mathbf{\Omega}}{2}\left(\frac{f'}{f}\right)' F_1 - \frac{\mathbf{\Omega}^2}{4}\left(\frac{f'}{f}\right)^2 C_0 + \frac{i\mathbf{\Omega}}{2}\frac{f'}{f}\frac{H'}{H}F_1 \\ +(2\pi T)^2\frac{r^{2z-2}}{f^2}\cdot(\mathbf{\Omega}^2 - (2\pi T)^{2/z-2}\mathbf{Q}^2 f r^{2-2z})C_0\Bigg] = 0\ . \tag{C.3}$$

Using (4.38) and integrating the above once, we get

$$F_2'(r) = \frac{i\mathbf{\Omega}}{2}\frac{f'}{f}F_1 - H\tilde{C}_1 \\ - \frac{C_0 m^2 r_0^m}{4}\left(\frac{(r_0 r)^{n+2-m}}{n+2-m}\ {}_2F_1\left[1, \frac{n+2-m}{m}, \frac{n+2}{m}, (r_0 r)^m\right] + \frac{1}{m}\log f\right)\cdot H \tag{C.4}$$

where $\bar{C}_1$ is an arbitrary integration constant. Choosing $\tilde{C}_1 = \frac{C_0}{4} m r_0^m\left(\gamma + \psi\left(\frac{n+2}{m}\right)\right)$ where γ is the Euler-Mascheroni constant and ψ is the digamma function, ensures $F_2'(r)$ is finite at the horizon. The above expression can be integrated again subject to the boundary condition $F_2(r \sim \frac{1}{r_0}) = 0$ to obtain an explicit expression for the function $F_2(r)$.

$G_1(r, t, x)$ solution: Collecting $O(\lambda)$ terms from (4.45), we can write the equation governing $G_1(r)$ is given by

$$G_1'' + \partial_r \log f r^{d_i+3-z-\theta}\ G_1' - \frac{k^2}{r^2 f}\ G_1 = \frac{i\mathbf{\Omega}}{2}\frac{f'}{f}(G_0' + \partial_r \log f r^{d_i+3-z-\theta}\ G_0) \\ + \frac{i\mathbf{\Omega}}{2}\left(\frac{f'}{f}\right)' G_0 + (2\pi T)^{\frac{1}{z}-2}\frac{\mathbf{Q}k}{r^2 f}\times \\ \left(\frac{C_0}{4}m^2 r_0^m\left[\frac{(r_0 r)^{n+2-m}}{n+2-m}\ {}_2F_1\left[1, \frac{n+2-m}{m}, \frac{n+2}{m}; (r_0 r)^m\right] + \frac{1}{m}\log f\right] + \tilde{C}_1\right) = 0\ . \tag{C.5}$$

The homogeneous part of the above equation is identical to the homogeneous part of (4.46). This helps us in writing down the solution for G_1 as

$$G_1(x) = \bar{C}_1\ y_1(x) - y_1(x)\int \frac{h(x) y_2(x)}{W(x)}\ dx + y_2(x)\int \frac{h(x) y_1(x)}{W(x)}\ dx \tag{C.6}$$

where x is the radial variable defined in (4.48). Also,

$$
\begin{aligned}
&y_1(x) = x^{-m}\,\frac{n+(m-n)x^m}{n}\,, y_2(x) = x^n\ {}_2F_1\left[2, \frac{n}{m}, 2+\frac{n}{m}; x^m\right] \\
&W(x) = \frac{(m+n)x^{n-m-1}}{1-x^m} \\
&h(x) = \frac{\Lambda_1}{x^2} + \frac{\Lambda_2}{x^2(1-x^m)} + \frac{\Lambda}{x^2(1-x^m)}\left[\frac{x^{n+2-m}}{n+2-m}\ {}_2F_1\left[1, \frac{n+2-m}{m}, \frac{n+2}{m}; x^m\right]\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left. + \frac{\log f}{m}\right]
\end{aligned}
\tag{C.7}
$$

where the constants Λ_1, Λ_2 and Λ are given by $\Lambda_1 = -\frac{\Lambda}{m}(2-\frac{n}{m})$, $\Lambda_2 = \Lambda(\gamma + \psi(\frac{2+n-m}{m}))$, $\Lambda = C_0 kq\, r_0^{m-n-2}$. In principle we can fix the constant $\bar{C}_1$ by demanding regularity of the solution near horizon i.e. as $x \sim 1$.

The crucial thing to note about the a_y solution is that even the leading order piece $G_0 \sim \frac{q}{\omega}$ vanishes in the zero momentum sector i.e. when $q = 0$. However, Kubo's formula to evaluate the response functions i.e. in this case the $\langle a_y(k)a_y(-k)\rangle$ correlator is evaluated at zero momenta which leads to $\langle a_y(k)a_y(-k)\rangle = 0$. This is very similar to the behaviour of the correlators $\langle T^y_t(k)T^y_t(-k)\rangle$ and $\langle T^y_t(k)T^y_x(-k)\rangle$ given by (4.61) which vanishes in the $q \to 0$ limit. Indeed from (4.13)-(4.16) we can observe that when $q = 0$ the H_{ty} and a_y fields decouple and the shear mode sector is governed exclusively by the dynamics of H_{xy}. This results in a non-trivial $\langle T^y_x(k)T^y_x(-k)\rangle$ correlator i.e. (4.59) which is indeed non-zero in the $q = 0$ sector eventually leading to (4.60).

Bibliography

[1] F. Englert, F. and R. Brout, "Broken Symmetry and the Mass of Gauge Vector Mesons", Physical Review Letters, 1964, Aug, Volume 13, 321-323.

[2] Higgs, P. W., "Broken Symmetries and the Masses of Gauge Bosons", Physical Review Letters, 1964,Oct, Volume 13, 508-509,

[3] Guralnik, G. S. and Hagen, C. R. and Kibble, T. W., "Global Conservation Laws and Massless Particles", Physical Review Letters, 1964, Nov, Volume 13, 585-587,

[4] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] [hep-th/9711200].

[5] K. Goldstein, S. Kachru, S. Prakash, S. P. Trivedi, "Holography of Charged Dilaton Black Holes," JHEP **1008**, 078 (2010) [arXiv:0911.3586 [hep-th]];

[6] M. Cadoni, G. D'Appollonio and P. Pani, "Phase transitions between Reissner-Nordstrom and dilatonic black holes in 4D AdS spacetime," JHEP **1003**, 100 (2010) [arXiv:0912.3520 [hep-th]];

[7] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis and R. Meyer, "Effective Holographic Theories for low-temperature condensed matter systems," JHEP **1011**, 151 (2010) [arXiv:1005.4690 [hep-th]]

[8] E. Perlmutter, "Domain Wall Holography for Finite Temperature Scaling Solutions," JHEP **1102**, 013 (2011) [arXiv:1006.2124 [hep-th]].

[9] G. Bertoldi, B. A. Burrington and A. W. Peet, "Thermal behavior of charged dilatonic black branes in AdS and UV completions of Lifshitz-like geometries," Phys. Rev. D **82**, 106013 (2010) doi:10.1103/PhysRevD.82.106013 [arXiv:1007.1464 [hep-th]].

[10] N. Iizuka, N. Kundu, P. Narayan and S. P. Trivedi, "Holographic Fermi and Non-Fermi Liquids with Transitions in Dilaton Gravity," JHEP **1201**, 094 (2012) [arXiv:1105.1162 [hep-th]].

[11] B. Gouteraux and E. Kiritsis, "Generalized Holographic Quantum Criticality at Finite Density," JHEP **1112**, 036 (2011) [arXiv:1107.2116 [hep-th]];

[12] X. Dong, S. Harrison, S. Kachru, G. Torroba and H. Wang, "Aspects of holography for theories with hyperscaling violation," JHEP **1206**, 041 (2012) [arXiv:1201.1905 [hep-th]].

[13] Coleman S. (1982) "$1/N$": Book chapter from Aspects of Symmetry.

[14] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428**, 105 (1998) [hep-th/9802109].

[15] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150];

[16] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [hep-th/9905111].

[17] S. de Haro, S. N. Solodukhin and K. Skenderis, "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence," Commun. Math. Phys. **217**, 595 (2001) [hep-th/0002230].

[18] S. Kachru, X. Liu and M. Mulligan, "Gravity duals of Lifshitz-like fixed points," Phys. Rev. D **78**, 106005 (2008) [arXiv:0808.1725 [hep-th]].

[19] M. Taylor, "Non-relativistic holography," arXiv:0812.0530 [hep-th].

[20] K. Balasubramanian and K. Narayan, "Lifshitz spacetimes from AdS null and cosmological solutions," JHEP **1008**, 014 (2010) [arXiv:1005.3291 [hep-th]].

[21] A. Donos and J. P. Gauntlett, "Lifshitz Solutions of D=10 and D=11 supergravity," JHEP **1012**, 002 (2010) [arXiv:1008.2062 [hep-th]].

[22] R. Gregory, S. L. Parameswaran, G. Tasinato and I. Zavala, "Lifshitz solutions in supergravity and string theory," JHEP **1012**, 047 (2010) [arXiv:1009.3445 [hep-th]].

[23] A. Donos, J. P. Gauntlett, N. Kim and O. Varela, "Wrapped M5-branes, consistent truncations and AdS/CMT," JHEP **1012**, 003 (2010) [arXiv:1009.3805 [hep-th]].

[24] D. Cassani and A. F. Faedo, "Constructing Lifshitz solutions from AdS," arXiv:1102.5344 [hep-th].

[25] N. Halmagyi, M. Petrini and A. Zaffaroni, "Non-Relativistic Solutions of $\mathcal{N} = 2$ Gauged Supergravity," arXiv:1102.5740 [hep-th].

[26] H. Singh, "Special limits and non-relativistic solutions," JHEP **1012**, 061 (2010) [arXiv:1009.0651 [hep-th]]; H. Singh, "Holographic flows to IR Lifshitz spacetimes," arXiv:1011.6221 [hep-th].

[27] K. Narayan, "Lifshitz-like systems and AdS null deformations," Phys. Rev. D **84**, 086001 (2011) [arXiv:1103.1279 [hep-th]].

[28] W. Chemissany and J. Hartong, "From D3-Branes to Lifshitz Space-Times," Class. Quant. Grav. **28**, 195011 (2011) [arXiv:1105.0612 [hep-th]].

[29] B. S. Kim, "Schródinger Holography with and without Hyperscaling Violation," JHEP **1206**, 116 (2012) [arXiv:1202.6062 [hep-th]];

[30] H. Singh, "Lifshitz/Schrödinger Dp-branes and dynamical exponents," JHEP **1207** (2012) 082 [arXiv:1202.6533 [hep-th]];

[31] P. Dey and S. Roy, "Lifshitz-like space-time from intersecting branes in string/M theory," arXiv:1203.5381 [hep-th];

[32] E. Perlmutter, "Hyperscaling violation from supergravity," JHEP **1206**, 165 (2012) [arXiv:1205.0242 [hep-th]];

[33] M. Ammon, M. Kaminski and A. Karch, "Hyperscaling-Violation on Probe D-Branes," arXiv:1207.1726 [hep-th];

[34] M. Kulaxizi, A. Parnachev and K. Schalm, "On Holographic Entanglement Entropy of Charged Matter," JHEP **1210**, 098 (2012) [arXiv:1208.2937 [hep-th]].

[35] K. Narayan, "AdS null deformations with inhomogeneities," Phys. Rev. D **86**, 126004 (2012) [arXiv:1209.4348 [hep-th]].

[36] D. T. Son, “Applied holography,” Nucl. Phys. Proc. Suppl. **195**, 217 (2009).

[37] M. Taylor, “Lifshitz holography,” Class. Quant. Grav. **33**, no. 3, 033001 (2016) [arXiv:1512.03554 [hep-th]].

[38] J. Bhattacharya, S. Cremonini and A. Sinkovics, “On the IR completion of geometries with hyperscaling violation,” JHEP **1302**, 147 (2013) [arXiv:1208.1752 [hep-th]].

[39] S. F. Ross, “Holography for asymptotically locally Lifshitz spacetimes,” Class. Quant. Grav. **28**, 215019 (2011) [arXiv:1107.4451 [hep-th]].

[40] M. H. Christensen, J. Hartong, N. A. Obers and B. Rollier, “Boundary Stress-Energy Tensor and Newton-Cartan Geometry in Lifshitz Holography,” JHEP **1401**, 057 (2014) [arXiv:1311.6471 [hep-th]].

[41] W. Chemissany and I. Papadimitriou, “Lifshitz holography: The whole shebang,” JHEP **1501**, 052 (2015) [arXiv:1408.0795 [hep-th]].

[42] J. Hartong, E. Kiritsis and N. A. Obers, “Field Theory on Newton-Cartan Backgrounds and Symmetries of the Lifshitz Vacuum,” JHEP **1508**, 006 (2015) [arXiv:1502.00228 [hep-th]].

[43] D. W. Pang, “Conductivity and Diffusion Constant in Lifshitz Backgrounds,” JHEP **1001**, 120 (2010) [arXiv:0912.2403 [hep-th]].

[44] S. Cremonini and P. Szepietowski, “Generating Temperature Flow for eta/s with Higher Derivatives: From Lifshitz to AdS,” JHEP **1202**, 038 (2012) [arXiv:1111.5623 [hep-th]].

[45] M. Alishahiha and H. Yavartanoo, “On Holography with Hyperscaling Violation,” JHEP **1211**, 034 (2012) [arXiv:1208.6197 [hep-th]].

[46] S. A. Hartnoll, A. Lucas and S. Sachdev, “Holographic quantum matter,” arXiv:1612.07324 [hep-th].

[47] J. Sadeghi and A. Asadi, “Hydrodynamics in a black brane with hyperscaling violation metric background,” Can. J. Phys. **92**, no. 12, 1570 (2014) [arXiv:1404.5282 [hep-th]].

[48] E. Kiritsis and Y. Matsuo, “Charge-hyperscaling violating Lifshitz hydrodynamics from black-holes,” JHEP **1512**, 076 (2015) [arXiv:1508.02494 [hep-th]]; E. Kiritsis and Y. Matsuo,

"Hyperscaling-Violating Lifshitz hydrodynamics from black-holes: Part II," arXiv:1611.04773 [hep-th].

[49] M. Blake, "Universal Charge Diffusion and the Butterfly Effect," arXiv:1603.08510 [hep-th].

[50] A. A. Patel, A. Eberlein and S. Sachdev, "Shear viscosity at the Ising-nematic quantum critical point in two dimensional metals," arXiv:1607.03894 [cond-mat.str-el].

[51] K. Narayan, "On Lifshitz scaling and hyperscaling violation in string theory," Phys. Rev. D **85**, 106006 (2012) [arXiv:1202.5935 [hep-th]].

[52] K. Narayan, T. Takayanagi and S. P. Trivedi, "AdS plane waves and entanglement entropy," JHEP **1304**, 051 (2013) [arXiv:1212.4328 [hep-th]];

[53] A. Einstein, B. Podolsky, N. Rosen, "Can quantum mechanical description of physical reality be considered complete?" Phys. Rev. 47, 777–780 (1935)

[54] M.A. Nielsen, I.L. Chuang, "Quantum Computation and Quantum Information" (Cambridge University Press, New York, 2010)

[55] A. Kitaev and J. Preskill, "Topological entanglement entropy," Phys. Rev. Lett. **96**, 110404 (2006) doi:10.1103/PhysRevLett.96.110404 [hep-th/0510092].

[56] M. Levin and X. G. Wen, "Detecting Topological Order in a Ground State Wave Function," Phys. Rev. Lett. **96**, 110405 (2006) doi:10.1103/PhysRevLett.96.110405 [cond-mat/0510613 [cond-mat.str-el]].

[57] S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from AdS/CFT," Phys. Rev. Lett. **96**, 181602 (2006) [hep-th/0603001].

[58] S. Ryu and T. Takayanagi, "Aspects of Holographic Entanglement Entropy," JHEP **0608**, 045 (2006) [hep-th/0605073].

[59] V. E. Hubeny, M. Rangamani and T. Takayanagi, "A Covariant holographic entanglement entropy proposal," JHEP **0707** (2007) 062 [arXiv:0705.0016 [hep-th]].

[60] B. Swingle, "Entanglement Renormalization and Holography," Phys. Rev. D **86**, 065007 (2012) doi:10.1103/PhysRevD.86.065007 [arXiv:0905.1317 [cond-mat.str-el]].

[61] M. Van Raamsdonk, "Comments on quantum gravity and entanglement," arXiv:0907.2939 [hep-th].

[62] M. Van Raamsdonk, "Building up spacetime with quantum entanglement," Gen. Rel. Grav. **42**, 2323 (2010) [Int. J. Mod. Phys. D **19**, 2429 (2010)] [arXiv:1005.3035 [hep-th]].

[63] J. Maldacena and L. Susskind, "Cool horizons for entangled black holes," Fortsch. Phys. **61**, 781 (2013) [arXiv:1306.0533 [hep-th]].

[64] T. Nishioka, S. Ryu and T. Takayanagi, "Holographic Entanglement Entropy: An Overview," J. Phys. A **42** (2009) 504008;

[65] T. Takayanagi, "Entanglement Entropy from a Holographic Viewpoint," Class. Quant. Grav. **29**, 153001 (2012) [arXiv:1204.2450 [gr-qc]].

[66] M. Banados, C. Teitelboim and J. Zanelli, "The Black hole in three-dimensional space-time," Phys. Rev. Lett. **69**, 1849 (1992) [hep-th/9204099].

[67] L. Bombelli, R. K. Koul, J. Lee and R. D. Sorkin, "A Quantum Source of Entropy for Black Holes," Phys. Rev. D **34** (1986) 373;

[68] M. Srednicki, "Entropy and area," Phys. Rev. Lett. **71** (1993) 666 [hep-th/9303048].

[69] N. Ogawa, T. Takayanagi and T. Ugajin, "Holographic Fermi Surfaces and Entanglement Entropy," JHEP **1201**, 125 (2012) [arXiv:1111.1023 [hep-th]].

[70] L. Huijse, S. Sachdev and B. Swingle, "Hidden Fermi surfaces in compressible states of gauge-gravity duality," Phys. Rev. B **85**, 035121 (2012) [arXiv:1112.0573 [cond-mat.str-el]].

[71] K. Narayan, "On a lightlike limit of entanglement," Phys. Rev. D **91**, no. 8, 086010 (2015) [arXiv:1408.7021 [hep-th]].

[72] K. Narayan, "Non-conformal brane plane waves and entanglement entropy," Phys. Lett. B **726**, 370 (2013)

[73] M. Headrick, "Entanglement Renyi entropies in holographic theories," Phys. Rev. D **82**, 126010 (2010) [arXiv:1006.0047 [hep-th]].

[74] B. Swingle, “Mutual information and the structure of entanglement in quantum field theory,” arXiv:1010.4038 [quant-ph].

[75] J. Bhattacharya, M. Nozaki, T. Takayanagi and T. Ugajin, “Thermodynamical Property of Entanglement Entropy for Excited States,” Phys. Rev. Lett. **110**, no. 9, 091602 (2013) [arXiv:1212.1164].

[76] D. Allahbakhshi, M. Alishahiha and A. Naseh, “Entanglement Thermodynamics,” JHEP **1308**, 102 (2013) [arXiv:1305.2728 [hep-th]].

[77] G. Wong, I. Klich, L. A. Pando Zayas and D. Vaman, “Entanglement Temperature and Entanglement Entropy of Excited States,” JHEP **1312**, 020 (2013) [arXiv:1305.3291 [hep-th]].

[78] W. Fischler, A. Kundu and S. Kundu, “Holographic Mutual Information at Finite Temperature,” Phys. Rev. D **87**, no. 12, 126012 (2013) [arXiv:1212.4764 [hep-th]].

[79] J. Maldacena, D. Martelli and Y. Tachikawa, “Comments on string theory backgrounds with non-relativistic conformal symmetry,” JHEP **0810**, 072 (2008) [arXiv:0807.1100 [hep-th]].

[80] H. Casini and M. Huerta, “Entanglement and alpha entropies for a massive scalar field in two dimensions,” J. Stat. Mech. **0512**:P12012,2005, [cond-mat/0511014].

[81] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, “Supergravity and the large N limit of theories with sixteen supercharges,” Phys. Rev. D **58**, 046004 (1998) [hep-th/9802042].

[82] J. L. F. Barbon and C. A. Fuertes, “Holographic entanglement entropy probes (non)locality,” JHEP **0804**, 096 (2008) [arXiv:0803.1928 [hep-th]].

[83] M.M. Wolf, F. Verstraete, M.B. Hastings, J.I. Cirac, “Area laws in quantum systems: mutual information and correlations”, Phys.Rev.Lett., 100, 070502 (2008), [arXiv:0704.3906].

[84] T. Faulkner, A. Lewkowycz and J. Maldacena, “Quantum corrections to holographic entanglement entropy,” JHEP **1311**, 074 (2013) [arXiv:1307.2892].

[85] H. Casini and M. Huerta, “Remarks on the entanglement entropy for disconnected regions,” JHEP **0903**, 048 (2009) doi:10.1088/1126-6708/2009/03/048 [arXiv:0812.1773 [hep-th]].

[86] D. D. Blanco, H. Casini, L. Y. Hung and R. C. Myers, "Relative Entropy and Holography," JHEP **1308**, 060 (2013) doi:10.1007/JHEP08(2013)060 [arXiv:1305.3182 [hep-th]].

[87] N. Lashkari, M. B. McDermott and M. Van Raamsdonk, "Gravitational dynamics from entanglement 'thermodynamics'," JHEP **1404**, 195 (2014) doi:10.1007/JHEP04(2014)195 [arXiv:1308.3716 [hep-th]].

[88] T. Faulkner, M. Guica, T. Hartman, R. C. Myers and M. Van Raamsdonk, "Gravitation from Entanglement in Holographic CFTs," JHEP **1403**, 051 (2014) doi:10.1007/JHEP03(2014)051 [arXiv:1312.7856 [hep-th]].

[89] G. Policastro, D. T. Son and A. O. Starinets, "The Shear viscosity of strongly coupled N=4 supersymmetric Yang-Mills plasma," Phys. Rev. Lett. **87**, 081601 (2001) [hep-th/0104066].

[90] P. Kovtun, D. T. Son and A. O. Starinets, "Viscosity in strongly interacting quantum field theories from black hole physics," Phys. Rev. Lett. **94**, 111601 (2005) [hep-th/0405231].

[91] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, "Nonlinear Fluid Dynamics from Gravity," JHEP **0802**, 045 (2008) [arXiv:0712.2456 [hep-th]].

[92] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, S. P. Trivedi and S. R. Wadia, "Forced Fluid Dynamics from Gravity," JHEP **0902**, 018 (2009) [arXiv:0806.0006 [hep-th]].

[93] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma, "Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions," JHEP **0812**, 116 (2008) [arXiv:0809.4272 [hep-th]].

[94] J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, "Fluid dynamics of R-charged black holes," JHEP **0901**, 055 (2009) [arXiv:0809.2488 [hep-th]].

[95] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, "Hydrodynamics from charged black branes," JHEP **1101**, 094 (2011) [arXiv:0809.2596 [hep-th]].

[96] P. Kovtun, D. T. Son and A. O. Starinets, "Holography and hydrodynamics: Diffusion on stretched horizons," JHEP **0310**, 064 (2003) [hep-th/0309213].

[97] K. S. Thorne, R. H. Price and D. A. Macdonald, "Black Holes: The Membrane Paradigm," NEW HAVEN, USA: YALE UNIV. PR. (1986) 367p

[98] I. R. Klebanov and E. Witten, "AdS / CFT correspondence and symmetry breaking," Nucl. Phys. B **556**, 89 (1999) doi:10.1016/S0550-3213(99)00387-9 [hep-th/9905104].

[99] C. Hoyos, B. S. Kim and Y. Oz, "Lifshitz Field Theories at Non-Zero Temperature, Hydrodynamics and Gravity," JHEP **1403**, 029 (2014) [arXiv:1309.6794 [hep-th], arXiv:1309.6794].

[100] D. Mateos and D. Trancanelli, "Thermodynamics and Instabilities of a Strongly Coupled Anisotropic Plasma," JHEP **1107**, 054 (2011) [arXiv:1106.1637 [hep-th]].

[101] J. Erdmenger, P. Kerner and H. Zeller, "Transport in Anisotropic Superfluids: A Holographic Description," JHEP **1201**, 059 (2012) [arXiv:1110.0007 [hep-th]].

[102] A. Rebhan and D. Steineder, "Violation of the Holographic Viscosity Bound in a Strongly Coupled Anisotropic Plasma," Phys. Rev. Lett. **108**, 021601 (2012) [arXiv:1110.6825 [hep-th]].

[103] J. Polchinski and E. Silverstein, "Large-density field theory, viscosity, and '$2k_F$' singularities from string duals," Class. Quant. Grav. **29**, 194008 (2012) [arXiv:1203.1015 [hep-th]].

[104] K. A. Mamo, "Holographic RG flow of the shear viscosity to entropy density ratio in strongly coupled anisotropic plasma," JHEP **1210**, 070 (2012) [arXiv:1205.1797 [hep-th]].

[105] S. Jain, N. Kundu, K. Sen, A. Sinha and S. P. Trivedi, "A Strongly Coupled Anisotropic Fluid From Dilaton Driven Holography," JHEP **1501**, 005 (2015) [arXiv:1406.4874 [hep-th]].

[106] S. Jain, R. Samanta and S. P. Trivedi, "The Shear Viscosity in Anisotropic Phases," JHEP **1510**, 028 (2015) [arXiv:1506.01899 [hep-th]].

[107] S. Cremonini, "The Shear Viscosity to Entropy Ratio: A Status Report," Mod. Phys. Lett. B **25**, 1867 (2011) [arXiv:1108.0677 [hep-th]].

[108] I. Kanitscheider, K. Skenderis and M. Taylor, "Precision holography for non-conformal branes," JHEP **0809**, 094 (2008) [arXiv:0807.3324 [hep-th]].

[109] I. Kanitscheider and K. Skenderis, "Universal hydrodynamics of non-conformal branes," JHEP **0904**, 062 (2009) [arXiv:0901.1487 [hep-th]].

[110] P. K. Kovtun and A. O. Starinets, "Quasinormal modes and holography," Phys. Rev. D **72**, 086009 (2005) doi:10.1103/PhysRevD.72.086009 [hep-th/0506184].

[111] A. O. Starinets, "Quasinormal spectrum and the black hole membrane paradigm," Phys. Lett. B **670**, 442 (2009) [arXiv:0806.3797 [hep-th]].

[112] A. Nunez and A. O. Starinets, "AdS / CFT correspondence, quasinormal modes, and thermal correlators in N=4 SYM," Phys. Rev. D **67**, 124013 (2003) [hep-th/0302026].

[113] E. Berti, V. Cardoso and A. O. Starinets, Class. Quant. Grav. **26**, 163001 (2009) doi:10.1088/0264-9381/26/16/163001 [arXiv:0905.2975 [gr-qc]].

[114] D. T. Son and A. O. Starinets, "Minkowski space correlators in AdS / CFT correspondence: Recipe and applications," JHEP **0209**, 042 (2002) [hep-th/0205051];

[115] G. Policastro, D. T. Son and A. O. Starinets, "From AdS / CFT correspondence to hydrodynamics," JHEP **0209**, 043 (2002) [hep-th/0205052].

[116] X. M. Kuang and J. P. Wu, "Transport coefficients from hyperscaling violating black brane: shear viscosity and conductivity," arXiv:1511.03008 [hep-th].

[117] D. Roychowdhury, "Holographic charge diffusion in non relativistic branes," Phys. Lett. B **744**, 109 (2015) [arXiv:1412.0911 [hep-th]]; "Hydrodynamics from scalar black branes," JHEP **1504**, 162 (2015) [arXiv:1502.04345 [hep-th]].

[118] N. Iqbal and H. Liu, "Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm," Phys. Rev. D **79**, 025023 (2009) [arXiv:0809.3808 [hep-th]].