Doctoral Thesis

Asymptotic Symmetries, Horizon Hair, and Memory Effect

By

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to

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Amitabh Virmani
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Abstract

The quest for the correct theory of quantum gravity has occupied many minds for the last few decades. Recently, it was realised by Strominger and his collaborators that diverse aspects of infrared physics are related to one another. These relations, well-known as the infrared (IR) triangle have successfully predicted new symmetries of asymptotically flat spacetimes. These predictions may prove to be the pillars of establishing the holographic dual to flat spacetime. In that context, this thesis explores the theory of asymptotic symmetries of asymptotically flat spacetimes and the spin memory effect in $D = 4$ dimensions.

The thesis starts by exploring the supertranslations at timelike infinity where, we propose a definition of asymptotic flatness at timelike infinity in four spacetime dimensions. We present a detailed study of the asymptotic equations of motion and the action of supertranslations on asymptotic fields. We show that the Lee-Wald symplectic form $\Omega(g, \delta_1 g, \delta_2 g)$ does not get contributions from future timelike infinity with our boundary conditions. As a result, the "future charges" can be computed on any two-dimensional surface surrounding the sources at timelike infinity. We present expressions for supertranslation and Lorentz charges.

In a separate attempt to explore the asymptotic symmetries of asymptotically flat spacetimes, we study the dynamics of a probe Maxwell field on the extreme Reissner-Nordström solution. The extreme Reissner-Nordström solution has a discrete conformal isometry that maps the future event horizon to future null infinity and vice versa, the Couch-Torrence inversion isometry. We present a gauge fixing that is compatible with the inversion symmetry. The gauge fixing allows us to relate the gauge parameter at the future horizon to future null infinity, which further allows us to study global charges for large gauge symmetries in the exterior of the extreme Reissner-Nordström black hole. Along the way, we construct Newman-Penrose and Aretakis like conserved quantities along future null infinity and the future event

horizon, respectively, and relate them via the Couch-Torrence inversion symmetry.

Finally, we explore the other corner of Strominger's IR triangle, the memory effect. We derive the leading spin-dependent gravitational tail memory, which appears at the second post-Minkowskian (2 PM) order and behaves as u^{-2} for large retarded time u. This result follows from classical soft graviton theorem at order $\omega \ln \omega$ as a low-frequency expansion of gravitational waveform with frequency ω. First, we conjecture the gravitational waveform from the classical limit of quantum soft graviton theorem up to sub-subleading order in soft expansion and then we derive it for a classical scattering process without any reference to the soft graviton theorem. We show that the final result of the gravitational waveform in the direct derivation completely agrees with the conjectured waveform.

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Publications

The thesis is based on the central part of the Doctoral studies conducted by the author and includes the following publications in the order that they appear in this thesis.

1) S. Chakraborty, Debodirna Ghosh, Sk J. Hoque, A. Khairnar, A. Virmani *Supertranslations at timelike infinity*, JHEP 02 (2022) 022

2) K. Fernandes, Debodirna Ghosh and A. Virmani, *Horizon Hair from Inversion Symmetry*, Class. Quant. Grav. **38** (2020) no.5, 055006.

3) Debodirna Ghosh, B. Sahoo, *Spin dependent gravitational tail memory in $D=4$ dimensions*, Phys.Rev.D 105 (2022) 2, 025024

In addition to the topics covered here, the author has also worked on the Quasinormal modes of Supersymmetric microstate geometries and classical soft theorems from scattering amplitudes, which led to the following publications:

1) Bidisha Chakrabarty, Debodirna Ghosh, Amitabh Virmani, *Quasinormal modes of supersymmetric microstate geometries from the D1-D5 CFT, J. High Energ. Phys. 2019, 72 (2019).*

2) A. Manu, Debodirna Ghosh, A. Laddha and P. V. Athira, *Soft radiation from scattering amplitudes revisited, JHEP **05** (2021), 056.*

List of Symbols

$\cup$	union of two sets
$\mathcal{J}^+(\mathcal{J}^-)$	Future (past) null infinity
$\mathcal{J}^+_+(\mathcal{J}^+_-)$	Future (past) endpoint of future null infinity
$\mathcal{J}^-_+(\mathcal{J}^-_-)$	Future (past) endpoint of past null infinity
i^0	Spacelike infinity
$i^+(i^-)$	Future(Past) timelike infinity
H^+	Future event horizon
$H^+_+(H^+_-)$	Future(Past) endpoint of future event horizon
$Y_{lm}(\theta,\varphi)$	Spherical harmonics
G	Gravitational constant

Contents

Dedicated to Ma and Baba...

CHAPTER 1

Introduction

Diverse studies of infrared physics namely, asymptotic symmetries, memory effects, and soft theorems form a triangular relation, known as the infrared triangle [1], shown in Fig. 1.1. These relations have successfully illuminated symmetries of asymptotically flat spacetimes. The three main ideas of infrared physics reside at the three corners of the infrared triangle. This thesis explores few technical questions at various corners of the infrared triangle.

The infrared triangle is as follows. The lower left corner is represented by soft theorems. Soft theorems are well established in quantum electrodynamics (QED) [2, 3] which were later been generalised to gravity by Weinberg [4]. Soft theorems in quantum field theories are the exact statement of the factorisation property of $(n+1)$-particle to n-particle scattering amplitudes in gauge and gravity theories, when one of the particles are taken to be soft (energy taken to be zero).

The lower right corner represents asymptotic symmetries. The theory of asymptotic symmetries concerns symmetries and the related conserved charges of any system that has an asymptotic boundary. An important example of such asymptotic symmetries is the BMS supertranslations of asymptotically flat spacetimes explored by Bondi, Van der Burg, Metznar and Sachs (BMS) [5, 6] in the 1960's.

The top corner represents the memory effect, first introduced by Zel'dovich and Polnarev [7] in the context of gravity, where they studied gravitational scattering and later generalised to gauge theories [8, 9]. Gravitational-wave memory is the permanent changes in the relative position of the pair of inertial detectors in space due to the passing of a gravitational wave. The permanent displacement carries information about the net gravitational flux detected over some time interval and maybe potentially be detected in LIGO experiments in the future.

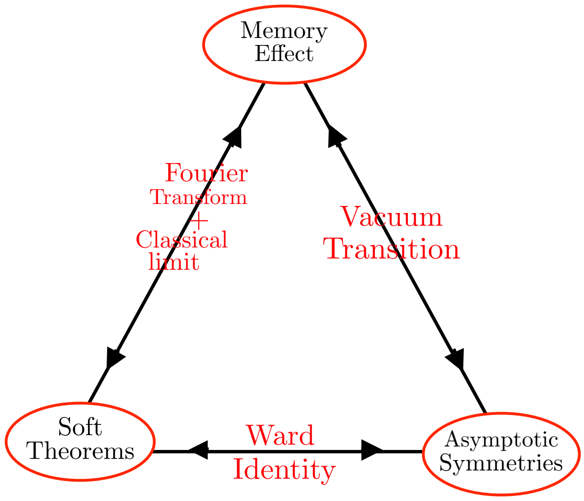

FIGURE 1.1: The infrared triangle [1].

The three corners of the triangle are connected to one another by mathematical relations. The soft graviton theorem is equivalent to the Ward identity associated with supertranslations; the gravitational memory is the Fourier transform of the classical limit of the soft graviton factor, and the gravitational memory effect can be understood as a vacuum transition of two inequivalent vacua of the gravitational field that are related by specific supertranslation.

In this thesis, we focus on certain technical problems at the three corners of the infrared triangle. We attempt to make the corners and the connections between the corners of the IR triangle richer and larger. In chapters 2 and 3, we explore the asymptotic symmetry corner. We study the BMS supertranslation symmetry at timelike infinity in chapter 2, and the large gauge symmetries for a U(1) gauge field in a specific spacetime with horizon in chapter 3. In chapter 4, we explore the soft theorem and the memory effect corner and connection between these corners. The connection to the asymptotic symmetries corner is not the subject of this thesis. A more detailed synopsis of the three chapters is as follows.

Supertranslations at timelike infinity

In electromagnetic theory, we study the properties of isolated charge contributions by studying the asymptotic fall-off conditions of the charge density j^μ and the electromagnetic field tensor $F^{\mu\nu}$ at large values of the spatial coordinate r. More precisely, in the stationary case, the coefficients of the multipole expansion of the electromagnetic field at large distances determines the structure of the charge-current distribution. In the non-stationary case, the multipole expansion coefficients determines the energy radiated to infinity. One aims at obtaining similar results for an isolated system in general relativity.

Unlike electrodynamics, the study of isolated systems in general relativity is not straightforward as the metric $g_{\alpha\beta}$ is not flat everywhere in space. Thus, one cannot specify the fall-off rates of the "curvature" of spacetime in terms of a preferred coordinate r as the inertial coordinate system is not "global".

The motivation of the first chapter starts from the simple question that was asked by Bondi, Metznar and Sachs in 1960's: Is the symmetry group of asymptotically flat spacetimes Poincaré? As it turns out, the answer that they found was something more than a Poincaré which they themselves found somewhat surprising. The asymptotic symmetries group they found in known as the Bondi-Metznar-Sachs (BMS) symmetries. These symmetries are exact symmetries of general relativity, in the sense that they leave the action invariant up to a surface term. This strongly suggests that they should be visible in any description, in particular at spatial infinity and timelike infinity, provided boundary conditions at spatial and timelike infinity are compatible with boundary conditions at null infinity.

In chapter 2 of the thesis, we propose a definition of asymptotic flatness at timelike infinity in four spacetime dimensions. We present a detailed study of the asymptotic equations of motion and the action of supertranslations on asymptotic fields. We show that the Lee-Wald symplectic form $\Omega(g, \delta_1 g, \delta_2 g)$ does not get contributions from future timelike infinity with our boundary conditions. As a result, the "future charges" can be computed on any two-dimensional surface surrounding the sources at timelike infinity. We present expressions for supertranslation and Lorentz charges.

Horizon hair from inversion symmetry

The asymptotic symmetries of quantum electrodynamics unlike the gravity have been discussed systematically only recently by Strominger and collaborators [10]. In general, when we study a gauge theory in some spacetime and impose boundary conditions that specify the behaviour of fields near the boundary, the asymptotic symmetry group (ASG) is defined as,

$$\mathbf{ASG} = \frac{\text{allowed gauge symmetries}}{\text{trivial gauge symmetries}}. \tag{1.0.1}$$

The "allowed gauge symmetries" are the ones that respect the boundary conditions and "trivial gauge symmetries" are the ones with vanishing gauge charges.

In Minkowski space electrodynamics, we are interested in boundary conditions at $\mathcal{J}^+$ or $\mathcal{J}^-$. The asymptotic behaviour is then derived from the fall-off conditions and the field equations. If we consider a sphere at large r, its surface area grows like

r^2, so for the energy flux at any moment to be finite, T_{uu} must fall off as

$$T_{uu} = \mathcal{O}(1/r^2).$$

It follows from the energy flux condition that the components of the field fall-off in a certain way near $\mathcal{I}^+$,

$$F_{uz} \sim \mathcal{O}(1), \qquad F_{ur}, F_{zr} \sim \mathcal{O}(1/r^2), \tag{1.0.2}$$

which in turn suggests that the boundary fall-off conditions for the gauge fields to be

$$A_z \sim \mathcal{O}(1), \qquad A_r \sim \mathcal{O}(1/r^2), \qquad A_u \sim \mathcal{O}(1/r). \tag{1.0.3}$$

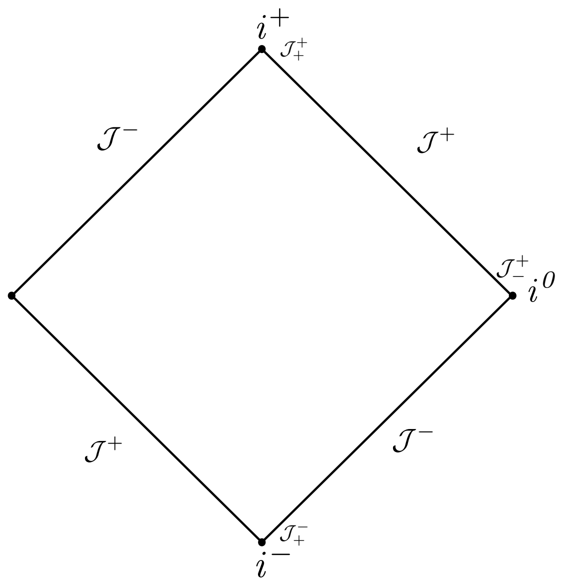

FIGURE 1.2: Penrose diagram of flat spacetime as discussed in [1].

In recent years, it has been shown that there are infinite-dimensional symmetries consisting of large gauge transformations for quantum electrodynamics (QED) in Minkowski spacetime [10]. Roughly speaking, the boundary conditions Eq. (1.0.3) are consistent with the gauge parameter,

$$\varepsilon = \varepsilon(z, \bar{z}) + \mathcal{O}\left(\frac{1}{r}\right). \tag{1.0.4}$$

The transformations related to gauge parameter of the form Eq. (1.0.4) are the exact analog of the BMS transformations that were perviously well known for gravity.

The charges related to these symmetries are soft electric charges (soft electric hair). Recently Hawking, Perry, and Strominger [11] argued that non-extremal stationary black holes also exhibit infinite-dimensional supertranslation symmetries in the near horizon region, and can carry soft supertranslation hair. It is believed that global charges associated with supertranslations receive contributions from the horizon as well as from null infinity. Thus, a complete discussion of conservation laws associated with supertranslations requires a detailed understanding of how the symmetries at the horizon relate to the symmetries at null infinity. It is natural to wonder if these ideas be made more explicit in the context of electrodynamics on a black hole background?

We take some preliminary steps in this direction in the chapter 3. We work with a probe Maxwell field in the background of an extreme Reissner-Nordström (ERN) black hole spacetime. The ERN spacetime has a discrete conformal isometry known as Couch-Torrence (CT) inversion isometry which maps the future event horizon to future null infinity and vice versa. We use this symmetry to study the dynamics of probe Maxwell field on ERN background metric. We study the dynamics of a probe Maxwell field on the extreme Reissner-Nordström solution in light of this symmetry. We present a gauge fixing that is compatible with the inversion symmetry. The gauge fixing allows us to relate the gauge parameter at the future horizon to future null infinity, which further allows us to study global charges for large gauge symmetries in the exterior of the extreme Reissner-Nordström black hole. Along the way, we construct Newman-Penrose and Aretakis like conserved quantities along future null infinity and the future event horizon, respectively, and relate them via the Couch-Torrence inversion symmetry.

Spin memory effect

The observation of gravitational waves is one of the most important event in the history of physics. The first direct observation of gravitational waves [12] was made in 2015, when a signal generated by the merger of two black holes was received by the LIGO gravitational wave detectors in Livingston and in Hanford. In gravitational-wave astronomy, observations of gravitational waves are used to infer data about the sources of gravitational waves.

Gravitational-wave memory effects are predicted persistent changes in the relative position of pairs of points in space due to the passing of a gravitational wave (GW). Detection of gravitational memory effects has been suggested as a way of validating Einstein's Theory of General Relativity. These memory effects can be "linear" or "non-linear" depending upon the system. The linear memory is *linear* in the sense that it is present in the gravitational waveform obtained from solving linearised Einstein's equations. It is generated for the GW signal produced by unbounded sources that escape to infinity; e.g. the hyperbolic binary encounter of compact objects, core collapse supernovae etc.

It was then shown by Christodoulou [13] that the gravitational waveform accompanies a permanent strain offset. It can be realised as the non-linear effects that is produced by the stress-energy tensor of the linearised theory while acting as a source to the spacetime metric perturbations. One of the most important systems that generate non-linear memory is the compact-binary-coalescences (CBCs) that are observed by LIGO/VIRGO. Apart from these memory effects, there has been a proposal by Laddha and Sen [14] of another kind of gravitational memory, known as the "tail memory" which describes how the gravitational-wave memory builds up, i.e., how the mirrors of gravity wave detector behaves at large retarded time before reaching to their permanent displaced position. This kind of memory is realised for scattering systems where some of the final state light particles (mass less than the scatterer's mass) are massive, e.g. in the merger of two neutron stars.

The late and early time gravitational waveforms responsible for the tail memory effects can be explicitly derived for a classical scattering process [15, 16], also known as the classical soft graviton theorem . The waveforms are also related to the low frequency GW radiation via a Fourier transformation. In chapter 4, we study tail memory effects for such low frequency gravitational radiation. We mainly focus on the gravitational tail memory generated from the scattering of spinning particles. We show that the leading spin dependent tail memory appears at second post-Minkowskian (2PM) order $\mathcal{O}(G^2)$ and behaves as u^{-2} for large retarded time u. At this iterative order, the spin dependent memory can be predicted from classical soft graviton theorem at order $\omega \ln \omega$ as a low frequency expansion of gravitational waveform with frequency ω. In consecutive sections of this chapter, we predict the low frequency gravitational waveform at the order $\omega \ln \omega$ by taking classical limit to quantum soft graviton theorem up to sub-subleading order in the soft expansion. Then, we derive the gravitational waveform for a classical scattering process without any consideration to the soft graviton theorem, and show that it matches with the predicted one from soft graviton theorem at sub-subleading order for spinning particles.

CHAPTER 2

Supertranslations at timelike infinity

2.1 Introduction

The asymptotic properties of gravity have been studied for decades [17–20] in the context of asymptotically flat spacetimes at null infinity, see [1, 21–25] for recent reviews. Much of these studies are rightly motivated by the need to understand the intricate nature of gravitational radiation. One remarkable outcome of these studies was the discovery of the infinite-dimensional Bondi-Metzner-Sachs (BMS) group of asymptotic symmetries in asymptotically flat spacetimes at null infinity. Recent works have shown that the BMS group is related to the infrared properties of gravity, namely soft-theorems and memory effects [1]. See references [26–28] for earlier works on these issues and the reviews [1,24] for further references. The exploration of the connections between the BMS group, soft-theorems, and memory effects have led to enormous activity. Further enlargements of the BMS group [25, 29–33] have also been argued to be relevant.

The BMS symmetries are exact symmetries of General Relativity, in the sense that they leave the action invariant up to a surface term. This strongly suggests that they should be visible in any description, in particular at spatial infinity and timelike infinity, provided boundary conditions at spatial and timelike infinity are compatible with boundary conditions at null infinity. This poses the dynamical question: how to relate boundary conditions at null, spatial, and timelike infinity? The answer to this question remains poorly understood, and therein lies the key to many unresolved issues. The importance of understanding these issues has been stressed by Friedrich in a recent article [34].

On the specific question of BMS symmetries at spatial infinity there has been a lot of progress in recent years, motivated in part by the need to understand the relation between the BMS group, soft-theorems, and memory effects. Earlier investigations of the asymptotic symmetries at spatial infinity [35–38] successfully found boundary conditions that gave Poincaré group as asymptotic symmetries.

This situation was exhilarating on one hand and disappointing on the other. Exhilarating because at least at spatial infinity there are consistent boundary conditions that lead to Poincaré group as asymptotic symmetries whereas at null-infinity this does not seem desirable. Disappointing because the lack of understanding of the BMS symmetries at spatial infinity means that the relation between boundary conditions at null and spatial infinity is incomplete. This had remained a deep puzzle for many years.

Henneaux and Troessaert [39–41] in a series of paper have resolved this tension, both in the cylindrical representation and the hyperbolic representation of spatial infinity. They have proposed boundary conditions at spatial infinity that are invariant under BMS symmetries. The BMS symmetries have non-trivial action on the fields and have generically non-zero charges. They have also related BMS generators at spatial infinity to BMS generators at past and future null infinity. Other works in this direction include [42, 43].

The situation at timelike infinity remains much less developed. Following works at spatial infinity [37, 38], earlier work [44–46] had proposed boundary conditions that gave Poincaré group as asymptotic symmetries. To the best of our knowledge, no attempt has been made to realise BMS symmetries at timelike infinity in the non-linear theory. Motivated by the relation between the BMS group and soft-theorems, these issues were addressed in the linearised gravity in [47, 48], though the main focus in these papers is somewhat different. The main aim of this chapter is to present boundary conditions in non-linear general relativity at timelike infinity that realise BMS symmetries in the sense that BMS symmetries have non-trivial action on the fields and have generically non-zero charges. Our work in motivated by the corresponding developments at spatial infinity [39–43].

Such a study is important for several reasons. Over the last two decades, it has been argued in a variety of contexts that stationary black holes also possess an infinite number of symmetries in the near horizon region [11, 49–53].[1] Typically a class of these symmetries is similar to supertranslations at null infinity. It is believed that global charges associated with supertranslations receive contributions from the horizon as well as from null infinity. Clearly, a complete discussion of conservation laws associated with supertranslations requires a detailed understanding of how the

[1]The symmetry groups in these papers do not coincide. This is so because different authors preserve different structures: some prefer to preserve a particular geometric structure on the null surface, whereas others preserve the near horizon geometry.

symmetries at the horizon relate to the symmetries at null infinity. However, this has not been understood.[2] Toy model studies include [54, 55]. It has been suggested by several authors, specifically by Chandrasekaran, Flanagan, and Prabhu in [53] that timelike infinity can be used to relate symmetries at the horizon to symmetries at null infinity.

In this chapter we only focus on timelike infinity with perhaps the simplest boundary conditions that allow for the BMS symmetries. The dynamical questions on the relationship of our boundary conditions to null and spatial infinity is beyond the scope of this work. Issues related to further enlargement of BMS symmetries [29–31, 33] are also beyond the scope of this work. We hope to return to these questions in future works.

The rest of the chapter is organised as follows. In section 2.3, we introduce our notion of asymptotic flatness at timelike infinity. Many of the calculations here are direct translations of those at spatial infinity. Having said so, we must add that the literature at spatial infinity is fairly large and confusing. Therefore, it is absolutely essential to work-out things from the start to the end for timelike infinity separately. In section 2.4, a detailed study of the asymptotic equations of motion is presented. In section 2.5, expressions for supertranslation and Lorentz charges are proposed. In section 2.6, the Schwarzschild solution is written in a form such that it manifestly satisfies our boundary conditions. In section 2.7, some general remarks on supertranslations are made. We close with a brief discussion in section 2.8. Dynamical questions regarding the non-triviality of our construction, i.e., whether non-trivial radiative spacetimes exist that satisfy our boundary conditions at timelike infinity requires a separate investigation.

2.2 Notations and Conventions

In this chapter, we use the sign convention $(-, +, +, +)$ throughout. We list some of the conventions related to this chapter below

- Tensors indices on 4 dimensional spacetimes are denoted by lowercase Greek letters μ, ν etc.
- Tensors indices on constant time three-dimensional hypersurfaces are denoted by lowercase Roman indices a, b, c etc. The Christoffel symbol Γ^a_{bc} is constructed from the induced metric h_{ab} on constant time three-dimensional hypersurfaces.
- The Ricci tensor, R_{ab} is defined by $R_{ab} = R^c{}_{acb}$.

[2] To some extent these issues were explored in [11, 51]. In these references, advanced Bondi coordinates are used; however, since advanced Bondi coordinates do not cover future null infinity, the relation between symmetries at future null infinity and the future horizon remains unexplored. These points were recently emphasised in [54, 55]. More broadly, in recent years several studies of null boundaries have advanced our knowledge of fluxes along null surfaces [56–60].

- We have used boldface quantities e.g., $\boldsymbol{\omega}$ to denote differential forms in this chapter.

2.3 Asymptotic flatness at timelike infinity

In this section we introduce our notion of asymptotic flatness at timelike infinity. It is based on the corresponding notion introduced by Beig and Schmidt [61, 62] at spatial infinity, which has been extensively studied over the years [63–67]. We work with a coordinate based definition. If needed, our results can be readily translated to geometric frameworks. A notion of asymptotic flatness at timelike infinity in the geometrical framework of Ashtekar-Hansen [37] was introduced by Cutler [44] and Porrill [45]. A closely related notion in the geometrical framework of Ashtekar-Romano [38] was discussed by Gen and Shiromizu [46]. Our notion is different from all these previous works, as we allow a class of spi-supertranslations to act as asymptotic symmetries at timelike infinity.

2.3.1 Asymptotic metric

To introduce our notion of asymptotic flatness at timelike infinity we start by introducing a set of "polar coordinates" for Minkowski spacetime $\{\tau, \rho, \theta, \varphi\}$ as follows

$$\eta_{\mu\nu}x^\mu x^\nu = -\tau^2, \qquad \frac{r}{t} = \frac{\rho}{\sqrt{1+\rho^2}}, \tag{2.3.1}$$

where $\eta_{\mu\nu} = \texttt{diag}\{-1, 1, 1, 1\}$ and x^μ are a standard set of cartesian coordinates and $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. In these coordinates flat spacetime metric takes the form

$$\begin{aligned} ds^2 &= -d\tau^2 + \tau^2 \left(\frac{d\rho^2}{1+\rho^2} + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2) \right), & (2.3.2)\\ &\equiv -d\tau^2 + \tau^2 h^{(0)}_{ab} d\phi^a d\phi^b & (2.3.3)\end{aligned}$$

where we denote coordinates $\{\rho, \theta, \varphi\}$ collectively as ϕ^a. Metric $h^{(0)}_{ab}$ is the unit metric on Euclidean AdS_3 hyperboloid $\mathcal{H}$.

We start by considering a general class of spacetime admitting an expansion at timelike infinity of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n=1}^{m} \frac{\ell^{(n)}_{\mu\nu}}{\tau^n} + o(\tau^{-m}), \tag{2.3.4}$$

where $\ell^{(n)}_{\mu\nu}$, for each n, is a function of $\frac{x^\sigma}{\tau}$. Following Beig and Schmidt [62], this metric can be put in the following more convenient form

$$ds^2 = -N^2 d\tau^2 + h_{ab} d\phi^a d\phi^b, \tag{2.3.5}$$

where

$$N = 1 + \frac{\sigma(\phi^a)}{\tau}, \tag{2.3.6}$$

$$h_{ab} = \tau^2 \left[h^{(0)}_{ab}(\phi^c) + \frac{1}{\tau} h^{(1)}_{ab}(\phi^c) + \frac{1}{\tau^2} h^{(2)}_{ab}(\phi^c) + \mathcal{O}\left(\frac{1}{\tau^3}\right)\right]. \tag{2.3.7}$$

A derivation of the form of the metric Eq. (2.3.5)–Eq. (2.3.7) starting from Eq. (2.3.4) is given in appendix 2.A. We define asymptotically flat spacetimes at timelike infinity as spacetimes admitting an asymptotic expansion as in Eq. (2.3.5)–Eq. (2.3.7). Further boundary conditions will be specified below. We will comment on the smoothness of fields σ, $h^{(1)}_{ab}$, $h^{(2)}_{ab}$, etc. on $EAdS_3$ hyperboloid $\mathcal{H}$ at a later stage.

2.3.2 Supertranslation at timelike infinity

A natural question to ask is what is the set of diffeomorphisms preserving the form of the metric Eq. (2.3.5)–Eq. (2.3.7).[3] In particular, if supertranslations are genuine symmetries of general relativity then they should also be realisable at timelike infinity. In order to spell out our boundary conditions explicitly, we start by looking at the action of supertranslations on asymptotic fields.

First order

As shown in detail in appendix 2.B, the diffeomorphism

$$\tau = \bar{\tau} - \omega(\bar{\phi}^a) + \mathcal{O}\left(\frac{1}{\bar{\tau}}\right), \tag{2.3.8}$$

$$\phi^a = \bar{\phi}^a + \frac{1}{\bar{\tau}} h^{(0)ab} \partial_b \omega(\bar{\phi}^c) + \mathcal{O}\left(\frac{1}{\bar{\tau}^2}\right), \tag{2.3.9}$$

preserves the asymptotic form of the metric to order $\frac{1}{\tau}$. Here $\omega(\phi^a)$ is an arbitrary function on $\mathcal{H}$ that determines the higher order terms in the diffeomorphism uniquely. When $\omega(\phi^a)$ is in the four-parameter class of solutions of

$$\mathcal{D}_a \mathcal{D}_b \, \omega - \omega h^{(0)}_{ab} = 0, \tag{2.3.10}$$

with $\mathcal{D}_a$ being the covariant derivative on $\mathcal{H}$ compatible with metric $h^{(0)}_{ab}$, the transformation Eq. (2.3.8)–Eq. (2.3.9) correspond to a translation. More generally, the above diffeomorphism corresponds to a supertranslation. The four functions satisfying (Eq. (2.3.10)) are

$$\left\{ \sqrt{1+\rho^2}, \rho \cos\theta, \rho \sin\theta \sin\phi, \rho \sin\theta \cos\phi \right\}, \tag{2.3.11}$$

[3] In the context of spatial infinity this question has been analysed in great detail by many authors over the years; see [64, 65] for a concise summary of the earlier results.

representing respectively, the time-translation and three-spatial translations.

Under general supertranslation Eq. (2.3.8)–Eq. (2.3.9), the zeroth order field $h^{(0)}_{ab}$ does not transform,

$$h^{(0)}_{ab} \to h^{(0)}_{ab}, \tag{2.3.12}$$

whereas the first order fields transform as,

$$\sigma \to \sigma, \tag{2.3.13}$$

$$h^{(1)}_{ab} \to h^{(1)}_{ab} + 2\mathcal{D}_a\mathcal{D}_b\,\omega - 2\omega h^{(0)}_{ab}. \tag{2.3.14}$$

We define,

$$k_{ab} := h^{(1)}_{ab} + 2\sigma h^{(0)}_{ab}. \tag{2.3.15}$$

It follows from Eq. (2.3.14) that under general supertranslation,

$$k_{ab} \to k_{ab} + 2\mathcal{D}_a\mathcal{D}_b\omega - 2\omega h^{(0)}_{ab}. \tag{2.3.16}$$

Now, there are two natural set of boundary conditions to consider. First, one can dispose of all the supertranslations by demanding

$$k_{ab} = 0. \tag{2.3.17}$$

These are the boundary conditions used in [44–46]. As is clear from Eq. (2.3.16) that with these boundary conditions, supertranslations are not allowed asymptotic symmetries. In the class of diffeomorphisms Eq. (2.3.8)–Eq. (2.3.9) only translations (cf. Eq. (2.3.10)) are allowed asymptotic symmetries.

Second, motivated by the work on spatial infinity [66] and [39,40] one can choose,

$$k := h^{(0)ab}k_{ab} = 0. \tag{2.3.18}$$

The requirement that the trace of k_{ab} vanishes should be invariant under supertranslations. From Eq. (2.3.16) we therefore deduce that the following differential equation for the function ω,

$$(\Box - 3)\,\omega = 0. \tag{2.3.19}$$

This is the class of supertranslations we work with in this chapter. Here $\Box$ is the Laplacian on $\mathcal{H}$: $\Box = \mathcal{D}_a\mathcal{D}^a$.

There can be other classes of transformations with appropriately modified notions of asymptotic flatness, e.g., logarithmic translations, superrotations, more general supertranslations, that one can explore. We do not study them in this work. Very likely, our considerations can be extended to include a study of logarithmic translations following, say, [66]. However, how superrotations at timelike infinity [47, 48, 68] can feature in such an analysis is not clear to us. Naively, the introduction of superrotations does not look compatible with the zeroth order equations of motion in the $1/\tau$ expansion.[4]

Second order

It is of interest to study the action of supertranslations on the second order fields. At second order the diffeomorphism presented in Eq. (2.3.8)–Eq. (2.3.9) generalises to,

$$\tau = \bar{\tau} - \omega(\bar{\phi}^a) + \frac{1}{\bar{\tau}} F^{(2)}(\bar{\phi}^a) + \mathcal{O}\left(\frac{1}{\bar{\tau}^2}\right) , \tag{2.3.20}$$

$$\phi^a = \bar{\phi}^a + \frac{1}{\bar{\tau}} h^{(0)ab} \partial_b \omega(\bar{\phi}^c) + \frac{1}{\bar{\tau}^2} G^{(2)a}(\bar{\phi}^c) + \mathcal{O}\left(\frac{1}{\bar{\tau}^3}\right) , \tag{2.3.21}$$

where the functions $F^{(2)}(\bar{\phi}^a)$ and $G^{(2)a}(\bar{\phi}^c)$ are uniquely fixed in terms of the function $\omega(\bar{\phi}^a)$ by the requirement that the form of the metric should remain the same to order $\frac{1}{\tau^2}$. We apply the above transformations and expand the metric in Eq. (2.3.5)–Eq. (2.3.7) upto second order. Using the boundary condition $k = 0$, we find,

$$\begin{aligned} h^{(2)}_{ab} \to h^{(2)}_{ab} &- \omega k_{ab} + \omega^c \mathcal{D}_c k_{ab} + \frac{1}{2}\omega^c_b k_{ac} + \frac{1}{2}\omega^c_a k_{bc} - \frac{1}{2}\omega^c (\mathcal{D}_a k_{bc}) - \frac{1}{2}\omega^c (\mathcal{D}_b k_{ac}) \\ &+ 2\sigma\omega h^{(0)}_{ab} + \sigma_{(a}\omega_{b)} - \sigma\omega_{ab} - \sigma_{c(a}\omega^c_{b)} - \sigma_c \omega^c{}_{(ab)} + (\sigma \leftrightarrow \omega) \\ &+ \omega^2 h^{(0)}_{ab} - 2\omega\omega_{ab} + \omega^c_a \omega_{bc} \, . \end{aligned} \tag{2.3.22}$$

Here $\sigma_a = \mathcal{D}_a\sigma, \mathcal{D}_a\mathcal{D}_b\sigma = \sigma_{ba}, \mathcal{D}_a\mathcal{D}_b\mathcal{D}_c\sigma = \sigma_{cba}$ etc. and similarly for ω. A detailed derivation is given in appendix 2.B.

A non-trivial consistency check on this expression is presented in appendix 2.D. There we consider doing a supertranslation on flat spacetime. We begin with (for flat spacetime)

$$\sigma = 0, \quad h^{(1)}_{ab} = 0, \quad h^{(2)}_{ab} = 0. \tag{2.3.23}$$

We note that σ does not change under supertranslations. Thus for the supertranslated spacetime too $\sigma = 0$. From Eq. (2.3.14), it follows that for the supertranslated spacetime

$$h^{(1)}_{ab} = k_{ab} = -2\omega h^{(0)}_{ab} + 2\omega_{ab}, \tag{2.3.24}$$

and from Eq. (2.3.22), it follows that

$$h^{(2)}_{ab} = \omega^2 h^{(0)}_{ab} - 2\omega\omega_{ab} + \omega^c_a \omega_{bc}. \tag{2.3.25}$$

[4] We thank the anonymous referee for suggesting us to add these comments.

In appendix 2.D, we check that the expression in Eq. (2.3.24) for k_{ab} and Eq. (2.3.25) for $h^{(2)}_{ab}$ are consistent with the second order equations of motion.[5]

2.4 Asymptotic expansion of the equation of motion

Einstein's equations can be split into 3+1 form, providing a set of three equations appropriately projected along normal direction to constant τ hypersurface. The split provides the Hamiltonian and momentum constraints, and the evolution equation for the metric on the 3-dimensional $\tau =$ constant hypersurface. These equations read,

$$H \equiv \frac{1}{N}\partial_\tau K + K_{ab}K^{ab} - \frac{1}{N}h^{ab}D_aD_bN = 0, \tag{2.4.1}$$

$$H_a \equiv D_bK^b_a - D_aK = 0, \tag{2.4.2}$$

$$H_{ab} \equiv \mathcal{R}_{ab} + \frac{1}{N}\partial_\tau K_{ab} - 2K_{ac}K^c_b + KK_{ab} - \frac{1}{N}D_aD_bN = 0. \tag{2.4.3}$$

Here D is the covariant derivative compatible with metric h_{ab} and $K_{ab} = \frac{1}{2N}\partial_\tau h_{ab}$ is the extrinsic curvature of the constant τ hypersurface.

These equations can be expanded in inverse powers of τ as,

$$H \equiv \frac{H^{(0)}}{\tau^2} + \frac{H^{(1)}}{\tau^3} + \frac{H^{(2)}}{\tau^4} + \mathcal{O}\left(\frac{1}{\tau^5}\right), \tag{2.4.4}$$

$$H_a \equiv \frac{H^{(0)}_a}{\tau} + \frac{H^{(1)}_a}{\tau^2} + \frac{H^{(2)}_a}{\tau^3} + \mathcal{O}\left(\frac{1}{\tau^4}\right), \tag{2.4.5}$$

$$H_{ab} \equiv H^{(0)}_{ab} + \frac{1}{\tau}H^{(1)}_{ab} + \frac{1}{\tau^2}H^{(2)}_{ab} + \mathcal{O}\left(\frac{1}{\tau^3}\right). \tag{2.4.6}$$

The expansion coefficients at zeroth, first, and second order are summarised in the following subsections. A detailed derivation of these results is given in appendix 2.C.

2.4.1 Zeroth and first order

At zeroth order, the Hamiltonian and the momentum constraints are identically satisfied. The evolution equation implies that the three-dimensional metric $h^{(0)}_{ab}$ on $\mathcal{H}$ must satisfy,

$$H^{(0)}_{ab} = \mathcal{R}^{(0)}_{ab} + 2h^{(0)}_{ab} = 0. \tag{2.4.7}$$

[5] An expression for corresponding transformation of $h^{(2)}_{ab}$ at spatial infinity was reported in equation (4.111) of [66]. All the ω-ω-terms, the analog of the third line in Eq. (2.3.22), are missing there. We note that the action of supertranslations at the second order has not been much discussed in the literature; comments appear in [66, 67], though neither of these papers present any details on this specific calculation. We hope that the reader will find our appendices 2.B and 2.D useful. The action of *translations* was discussed in [62].

This condition implies that $\mathcal{H}$ is maximally symmetric with $\mathcal{R}^{(0)} = -6$ and the Riemann tensor is given by,

$$\mathcal{R}^{(0)}_{abcd} = \frac{\mathcal{R}^{(0)}}{6}\left(h^{(0)}_{ac}h^{(0)}_{bd} - h^{(0)}_{ad}h^{(0)}_{bc}\right) = -h^{(0)}_{ac}h^{(0)}_{bd} + h^{(0)}_{ad}h^{(0)}_{bc}\,. \tag{2.4.8}$$

Thus $\mathcal{H}$ is Euclidean AdS_3 space, as noted earlier.

At first order, the Hamiltonian constraint gives,

$$H^{(1)} = (-\Box + 3)\,\sigma = 0. \tag{2.4.9}$$

The momentum constraint gives,

$$\mathcal{D}^b k_{ab} = \mathcal{D}_a k. \tag{2.4.10}$$

The evolution equations $H^{(1)}_{ab} = 0$ gives,

$$(\Box + 3)\,k_{ab} = \mathcal{D}_a\mathcal{D}_b k + k h^{(0)}_{ab}. \tag{2.4.11}$$

Boundary conditions presented in Eq. (2.3.18) further simplify these equations to

$$\mathcal{D}^b k_{ab} = 0, \qquad\qquad (\Box + 3)\,k_{ab} = 0. \tag{2.4.12}$$

2.4.2 Second order

At second order, the Hamiltonian constraint takes the form,

$$h^{(2)} = 12\sigma^2 + \frac{1}{4}k^{ab}\,k_{ab} - k^{ab}\mathcal{D}_a\,\mathcal{D}_b\sigma - \mathcal{D}_c\sigma\,\mathcal{D}^c\sigma, \tag{2.4.13}$$

where $h^{(2)}$ is the trace of $h^{(2)}_{ab}$, $h^{(2)} = h^{(0)ab}h^{(2)}_{ab}$. In arriving at this equation we have used the boundary condition $k = 0$ cf. Eq. (2.3.18) and the first order equations of motion. The momentum constraint reads,

$$\mathcal{D}_b h^{(2)b}_a = \frac{1}{2}k^{bp}\,(\mathcal{D}_b\,k_{pa}) + \mathcal{D}_a\left(-\frac{1}{8}\,k^{bc}\,k_{bc} + 8\sigma^2 - k^{ab}\mathcal{D}_a\,\mathcal{D}_b\sigma - \mathcal{D}_c\sigma\mathcal{D}^c\sigma\right). \tag{2.4.14}$$

The evolution equation $H^{(2)}_{ab} = 0$ yields,

$$(\Box + 2)\,h^{(2)}_{ab} = S^{(kk)}_{ab} + S^{(k\sigma)}_{ab} + S^{(\sigma\sigma)}_{ab}, \tag{2.4.15}$$

where the non-linear source terms have the following expressions,

$$\begin{aligned} S^{(kk)}_{ab} = {} & \left(\mathcal{D}_c\,k_{d(a}\,\mathcal{D}_{b)}\,k^{cd}\right) - \frac{1}{2}\mathcal{D}_a\,k^{cd}\,\mathcal{D}_b\,k_{cd} + (\mathcal{D}^c\,k_{ad})\left(\mathcal{D}_c k^d_b\right) - (\mathcal{D}^c\,k_{ad})\left(\mathcal{D}^d\,k_{bc}\right) \\ & - k^p_a\,k_{pb} + k^{cd}\left(\mathcal{D}_c\,\mathcal{D}_d\,k_{ab} - \mathcal{D}_c\mathcal{D}_{(a}\,k_{b)d}\right)\,, \end{aligned} \tag{2.4.16}$$

$$
\begin{aligned}
S^{(k\sigma)}_{ab} =& -\mathcal{D}_a\mathcal{D}_b\left(k^{cd}\mathcal{D}_c\,\mathcal{D}_d\sigma\right) + 4\mathcal{D}^c\sigma\left(-\mathcal{D}_c k_{ab} + \mathcal{D}_{(a}\,k_{b)c}\right) - 4\sigma k_{ab} \\
&+ \left(-2h^{(0)}_{ab}k^{cd}\mathcal{D}_c\,\mathcal{D}_d\sigma + 4k^{cd}h^{(0)}_{d(a}\mathcal{D}_{b)}\mathcal{D}_c\sigma\right)\ , \qquad (2.4.17)\\
S^{(\sigma\sigma)}_{ab} =& \mathcal{D}_a\,\mathcal{D}_b\left(5\sigma^2 - \mathcal{D}_c\sigma\mathcal{D}^c\sigma\right) + h^{(0)}_{ab}\left(18\sigma^2 + 4\mathcal{D}^c\sigma\mathcal{D}_c\sigma\right) + 4\sigma\mathcal{D}_a\mathcal{D}_b\sigma\ . \qquad (2.4.18)
\end{aligned}
$$

The second order equations of motion, in the form presented above, with more restrictive boundary condition $k_{ab} = 0$ take a particularly nice form and can be concisely presented in terms of the electric and magnetic parts of the Weyl tensors, as is the case at spatial infinity [65, 69, 70]. These results are presented in appendix 2.E.

2.5 Charges at timelike infinity

Next we would like to understand contributions from timelike infinity to the Iyer-Wald global charges [71,72] (see also [53]) for supertranslations and Lorentz symmetries. To this end, we compute contributions from timelike infinity to the Lee-Wald symplectic form. This computation is presented in section 2.5.1. We find that with our boundary conditions this contribution vanishes. It has been suggested by several authors[6] that this should be the case with appropriate boundary conditions at timelike infinity. As a result, "future charges" can be computed on any two-dimensional topologically-spherical surface surrounding the "sources" at timelike infinity. We present charge expressions in section 2.5.2. Some further properties of these charges are studied in section 2.5.3.

What are these sources at timelike infinity? Note that bound objects (and fields) reach timelike infinity. Fields close to these bound objects do not become weak and cannot be regarded as asymptotic fields in the usual sense. In the $\tau \to \infty$ limit, it is convenient to regard individual bound systems, gravitationally unbound relative to each other, as finite number of points on the timelike infinity hyperboloid $\mathcal{H}$. These points serve as sources for the charge integrals. This picture will become more clear in section 2.6 where we discuss the Schwarzschild solution in the $\tau \to \infty$ limit.

2.5.1 Contributions to the Lee-Wald symplectic form

Consider a spacetime with no horizons. The components of the boundary are the past and future null infinity $\mathcal{J}^-$, $\mathcal{J}^+$ and the points (in the Penrose diagram in Fig. 2.1) past and future timelike infinity i^-, i^+, and spatial infinity i^0. Since the global charge variation is invariant under local deformations of the Cauchy surface Σ, one can deform Σ in the far future to $i^+ \cup \mathcal{J}^+$. Then, the first variation of the

[6] See for example section 7 of [53].

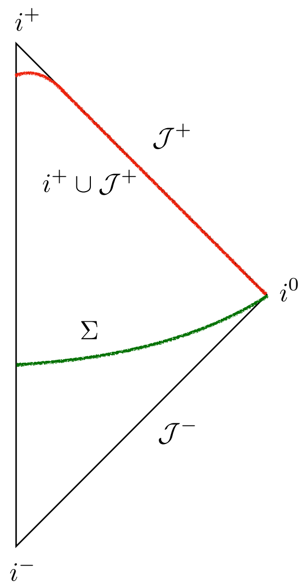

FIGURE 2.1: Consider a spacetime with no horizons. The components of the boundary are $\mathscr{I}^-$, $\mathscr{I}^+$ and the points at infinity i^-, i^0, and i^+. Since the first variation of global charges is invariant under local deformations of the Cauchy surface Σ, one can deform Σ in the far future to $i^+ \cup \mathscr{I}^+$. Then, the first variation of the Iyer-Wald global charges satisfies $\delta Q_\xi(\Sigma) = \delta Q_\xi(\mathscr{I}^+) + \delta Q_\xi(i^+)$. With our boundary conditions $\delta Q_\xi(i^+) = 0$.

Iyer-Wald global charges satisfies

$$\delta Q_\xi(\Sigma) = \delta Q_\xi(\mathscr{I}^+) + \delta Q_\xi(i^+). \tag{2.5.1}$$

With our boundary conditions we now show that $\delta Q_\xi(i^+) = 0$. This is schematically shown in Fig. 2.1. Recall that

$$\delta Q_\xi(\Sigma) = \Omega(g, \delta g, \pounds_\xi g). \tag{2.5.2}$$

The computation proceeds as follows. The Lee-Wald symplectic form [71, 72] is

$$\Omega(g, \delta_1 g, \delta_2 g) = \int_\Sigma \boldsymbol{\omega}(g, \delta_1 g, \delta_2 g) = \int_\Sigma \omega^\gamma n_\gamma \sqrt{h}\, d^3x, \tag{2.5.3}$$

where

$$\begin{aligned} \omega^\gamma &= P^{\gamma\nu\alpha\beta\mu\delta}\left[\delta_2 g_{\nu\alpha} \nabla_\beta \delta_1 g_{\mu\delta} - (1 \leftrightarrow 2)\right], \\ P^{\gamma\nu\alpha\beta\mu\delta} &= g^{\gamma\mu} g^{\delta\nu} g^{\alpha\beta} - \frac{1}{2} g^{\gamma\beta} g^{\nu\mu} g^{\delta\alpha} - \frac{1}{2} g^{\gamma\nu} g^{\alpha\beta} g^{\mu\delta} - \frac{1}{2} g^{\nu\alpha} g^{\gamma\mu} g^{\delta\beta} + \frac{1}{2} g^{\nu\alpha} g^{\gamma\beta} g^{\mu\delta}, \end{aligned} \tag{2.5.4}$$

and where n^γ is the unit normal to the hypersurface Σ,

$$n = -N d\tau, \tag{2.5.5}$$

and ∇_α is the covariant derivative compatible with the spacetime metric $g_{\mu\nu}$. We choose the hypersurface Σ to be a $\tau =$ constant surface. The volume factor $\sqrt{h}\, d^3x$ grows as τ^3 in the $\tau \to \infty$ limit. The aim, therefore, is to determine how $\omega^\gamma n_\gamma$ behaves in the $\tau \to \infty$ limit. On $\tau =$ constant hypersurface,

$$-\omega^\gamma n_\gamma = N\omega^\tau = \omega^\tau \left(1 + \mathcal{O}(1/\tau)\right). \tag{2.5.6}$$

As a result, the problem simply reduces to analysing the behaviour of ω^τ in the $\tau \to \infty$ limit. For our purposes, the ω^γ expression can be written in a more convenient form as follows,

$$\begin{aligned}
\omega^\gamma &= g^{\gamma\mu} g^{\delta\nu} g^{\alpha\beta} \left(\delta_2 g_{\nu\alpha} \nabla_\beta \delta_1 g_{\mu\delta}\right) - \frac{1}{2} g^{\gamma\beta} g^{\nu\mu} g^{\delta\alpha} \left(\delta_2 g_{\nu\alpha} \nabla_\beta \delta_1 g_{\mu\delta}\right) - \frac{1}{2} g^{\gamma\nu} g^{\alpha\beta} g^{\mu\delta} \left(\delta_2 g_{\nu\alpha} \nabla_\beta \delta_1 g_{\mu\delta}\right) \\
&\quad - \frac{1}{2} g^{\nu\alpha} g^{\gamma\mu} g^{\delta\beta} \left(\delta_2 g_{\nu\alpha} \nabla_\beta \delta_1 g_{\mu\delta}\right) + \frac{1}{2} g^{\nu\alpha} g^{\gamma\beta} g^{\mu\delta} \left(\delta_2 g_{\nu\alpha} \nabla_\beta \delta_1 g_{\mu\delta}\right) - (1 \leftrightarrow 2) \\
&= g^{\gamma\mu} \left(g^{\delta\nu} g^{\alpha\beta} \delta_2 g_{\nu\alpha}\right) \nabla_\beta \delta_1 g_{\mu\delta} - \frac{1}{2} \left(g^{\nu\mu} g^{\delta\alpha} \delta_2 g_{\nu\alpha}\right) \nabla^\gamma \delta_1 g_{\mu\delta} - \frac{1}{2} \left(g^{\gamma\nu} g^{\alpha\beta} \delta_2 g_{\nu\alpha}\right) \nabla_\beta \left(g^{\mu\delta} \delta_1 g_{\mu\delta}\right) \\
&\quad - \frac{1}{2} \left(g^{\nu\alpha} \delta_2 g_{\nu\alpha}\right) \left(g^{\gamma\mu} g^{\delta\beta} \nabla_\beta \delta_1 g_{\mu\delta}\right) + \frac{1}{2} g^{\gamma\beta} \left(g^{\nu\alpha} \delta_2 g_{\nu\alpha}\right) \left(g^{\mu\delta} \nabla_\beta \delta_1 g_{\mu\delta}\right) - (1 \leftrightarrow 2) \\
&= \frac{1}{2} \Bigg[\delta_2 g^{\alpha\beta} \left(\nabla^\gamma \delta_1 g_{\alpha\beta}\right) + \delta_2 \ln g \left(\nabla_\beta \delta_1 g^{\gamma\beta}\right) + \delta_2 g^{\gamma\beta} \left(\nabla_\beta \delta_1 \ln g\right) + \delta_2 \ln g \left(\nabla^\gamma \delta_1 \ln g\right) \\
&\quad - 2\delta_2 g_{\alpha\beta} \left(\nabla^\alpha \delta_1 g^{\gamma\beta}\right) - (1 \leftrightarrow 2) \Bigg]
\end{aligned} \tag{2.5.7}$$

where we have simply raised and lowered the indices in a convenient form and have converted some terms to the determinant g of the metric. In this form, each of the terms in ω^τ can be easily evaluated. The following expressions are useful:

$$\delta g_{\tau\tau} = -\frac{2\delta\sigma}{\tau} + o(1/\tau), \qquad \delta g^{\tau\tau} = \frac{2\delta\sigma}{\tau} + o(1/\tau), \tag{2.5.8}$$

$$\delta g_{ab} = \tau \delta h^{(1)}_{ab} + o(\tau), \qquad \delta g^{ab} = -\frac{1}{\tau^3} \delta h^{(1)ab} + o(1/\tau^3), \tag{2.5.9}$$

and for the four-dimensional Christoffel symbols the following expressions are useful:

$$\Gamma^c_{\tau a} = \frac{1}{2} h^{cd} \partial_\tau h_{ad} = \frac{1}{\tau} \delta^c_a + o(1/\tau), \tag{2.5.10}$$

$$\Gamma^\tau_{\tau\tau} = \frac{1}{2} h^{\tau\tau} \partial_\tau h_{\tau\tau} = -\frac{\sigma}{\tau^2} + o(1/\tau^2), \tag{2.5.11}$$

$$\Gamma^\tau_{ab} = -\frac{1}{2} h^{\tau\tau} \partial_\tau h_{ab} = \tau h^{(0)}_{ab} + o(\tau). \tag{2.5.12}$$

Using these expressions, the first term in Eq. (2.5.7) for $\gamma = \tau$ becomes,

$$\delta_2 g^{\alpha\beta} \left(\nabla^\tau \delta_1 g_{\alpha\beta}\right) = \delta_2 g^{\tau\tau} \left(\nabla^\tau \delta_1 g_{\tau\tau}\right) + \delta_2 g^{ab} \left(\nabla^\tau \delta_1 g_{ab}\right)$$

$$
\begin{aligned}
&= -\Big(\frac{2\delta_2\sigma}{\tau} + \cdots\Big)\Big(\frac{2\delta_1\sigma}{\tau^2} + \cdots\Big) - \Big(- \frac{\delta_2 h^{(1)ab}}{\tau^3} + \cdots\Big)\Big(\delta_1 h^{(1)}_{ab} + \cdots\Big) \\
&= \frac{1}{\tau^3}\Big(\delta_2 h^{(1)ab}\delta_1 h^{(1)}_{ab} - 4\delta_1\sigma\delta_2\sigma\Big) + o(1/\tau^3) \\
&= \frac{1}{\tau^3}\Big(\delta_2 k^{ab}\delta_1 k_{ab} - 2\delta_2\sigma\delta_1 k - 2\delta_1\sigma\delta_2 k + 8\delta_1\sigma\delta_2\sigma\Big) + o(1/\tau^3)
\end{aligned}
\tag{2.5.13}
$$

The second term becomes,

$$
\begin{aligned}
\delta_2 \ln g(\nabla_\beta \delta_1 g^{\tau\beta}) &= \Big(\frac{2\delta_2\sigma}{\tau} + \frac{h^{(0)}_{ab}\delta_2 h^{(1)ab}}{\tau} + \cdots\Big)\Big(\frac{4\delta_1\sigma}{\tau^2} - \frac{h^{(0)}_{ab}\delta_1 h^{(1)ab}}{\tau^2} + \cdots\Big) \\
&= \frac{1}{\tau^3}\Big(- 40\delta_1\sigma\delta_2\sigma + 4\delta_1 k\delta_2\sigma + 10\delta_2 k\delta_1\sigma - \delta_1 k\delta_2 k\Big) + o(1/\tau^3)
\end{aligned}
\tag{2.5.14}
$$

The third term becomes,

$$
\begin{aligned}
\delta_2 g^{\tau\beta}(\nabla_\beta \delta_1 \ln g) &= \delta_2 g^{\tau\tau}\ \partial_\tau(\delta_1 \ln g) \\
&= \Big(\frac{2\delta_2\sigma}{\tau} + \cdots\Big)\Big(- \frac{2\delta_1\sigma}{\tau^2} - \frac{h^{(0)}_{ab}\delta_1 h^{(1)ab}}{\tau^2} + \cdots\Big) \\
&= \frac{1}{\tau^3}(8\delta_1\sigma\delta_2\sigma - 2\delta_2\sigma\delta_1 k) + o(1/\tau^3)
\end{aligned}
\tag{2.5.15}
$$

The fourth term becomes

$$
\begin{aligned}
\delta_2 \ln g \nabla^\tau \delta_1 \ln g &= -\Big(\frac{2\delta_2\sigma}{\tau} + \frac{h^{(0)}_{ab}\delta_2 h^{(1)ab}}{\tau} + \cdots\Big)\partial_\tau\Big(\frac{2\delta_1\sigma}{\tau} + \frac{h^{(0)}_{ab}\delta_1 h^{(1)ab}}{\tau} + \cdots\Big) \\
&= \frac{1}{\tau^3}\Big(16\delta_1\sigma\delta_2\sigma - 4\delta_1\sigma\delta_2 k - 4\delta_1 k\delta_2\sigma + \delta_1 k\delta_2 k\Big) + o(1/\tau^3)
\end{aligned}
\tag{2.5.16}
$$

The fifth term becomes

$$
\begin{aligned}
\delta_2 g_{\alpha\beta}\Big(\nabla^\alpha \delta_1 g^{\tau\beta}\Big) &= \delta_2 g_{\tau\tau}\left(\nabla^\tau \delta_1 g^{\tau\tau}\right) + \delta_2 g_{ab}\Big(\nabla^a \delta_1 g^{\tau b}\Big) \\
&= \Big(\frac{2\delta_2\sigma}{\tau} + \cdots\Big)\Big(-\frac{2\delta_1\sigma}{\tau^2} + \cdots\Big) + \frac{2}{\tau^3}\Big(\delta_1\sigma\delta_2 h^{(1)}_{ab} h^{(0)ab} + \cdots\Big) \\
&= \frac{1}{\tau^3}\Big(2\delta_1\sigma\delta_2 k - 16\delta_1\sigma\delta_2\sigma\Big) + o(1/\tau^3)
\end{aligned}
\tag{2.5.17}
$$

Most of these terms cancel out upon $(1 \leftrightarrow 2)$ anti-symmetrisation. The final expression for ω^τ reads,

$$
\omega^\tau = \frac{2}{\tau^3}(\delta_1\sigma\delta_2 k - \delta_1 k\delta_2\sigma) + o(1/\tau^3)
\tag{2.5.18}
$$

Using the boundary condition, $k = 0$, the $\mathcal{O}(1/\tau^3)$ term in Eq. (2.5.18) vanishes. Hence, in the $\tau \to \infty$ limit

$$\Omega(g, \delta_1 g, \delta_2 g) = 0. \tag{2.5.19}$$

This implies,

$$\delta Q_\xi(i^+) = 0. \tag{2.5.20}$$

To summarise: we have shown that with our notion of asymptotic flatness, timelike infinity does not contribute to the Lee-Wald symplectic form. Hence, the contribution to the first variations of the Iyer-Wald charges from timelike infinity is zero. It has been suggested by several authors that this should be the case. The result is entirely expected on physical grounds. Eq. (2.5.1) simplifies to

$$\delta Q_\xi(\Sigma) = \delta Q_\xi(\mathcal{J}^+). \tag{2.5.21}$$

The contribution from null infinity, $\delta Q_\xi(\mathcal{J}^+)$, is well studied; for a review see [21]. One of key ideas in the subject is that the integral over null infinity can be written as the difference of localised charges [53]

$$Q_\xi(\mathcal{J}^+) = Q_\xi^{\text{loc}}(\mathcal{J}^+_-) - Q_\xi^{\text{loc}}(\mathcal{J}^+_+), \tag{2.5.22}$$

where $\mathcal{J}^+_\pm$ are respectively the future and past 2-sphere limits of null infinity $\mathcal{J}^+$. In the following we will be interested in $Q_\xi^{\text{loc}}(\mathcal{J}^+_+)$, which is what we call "future charges". Timelike infinity hyperboloid $\mathcal{H}$ reaches $\mathcal{J}^+_+$ in the $\rho \to \infty$ limit.

2.5.2 Charges

Since the contributions to the Lee-Wald symplectic form from timelike infinity vanishes, it follows that "future charges" can be computed on any two-dimensional topologically-spherical surface surrounding the "sources" at timelike infinity. To keep the notation simple, we denote future charges by simply Q_ξ, instead of $Q_\xi^{\text{loc}}(\mathcal{J}^+_+)$.

Motivated by the corresponding expressions at spatial infinity [66], we *propose* expressions for supertranslation and Lorentz charges at timelike infinity and show appropriate conservation properties. We do not present a first principal derivation for these expressions. Such a derivation can be given, for example, by relating the expressions below to the corresponding expressions to null infinity, but such a calculation is not attempted in this work.

We begin by observing some elementary properties of the $1/\tau$ expansion of the Weyl tensor projected on $\tau =$ constant hypersurface. In four spacetime dimensions, the Weyl tensor expressed in terms of the Riemann tensor, Ricci tensor and Ricci scalar takes the form,

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \frac{1}{2}\left(g_{\alpha\mu}R_{\beta\nu} + R_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}R_{\beta\mu} - R_{\alpha\nu}g_{\beta\mu}\right) + \frac{R}{6}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right). \tag{2.5.23}$$

Let (τ, ϕ^a) be the four-dimensional spacetime coordinates associated to the 3+1 split. Then, for a general spacetime coordinates $x^\mu = x^\mu(\tau, \phi^a)$ we define

$$e^\mu_a = \frac{\partial x^\mu}{\partial \phi^a}. \tag{2.5.24}$$

The vectors e^μ_a with $\{a = 1, 2, 3\}$ are tangent to $\tau =$ constant hypersurface. The projected electric part of the Weyl tensor on $\tau =$ constant hypersurface is defined as,

$$E_{ab} = W_{\alpha\beta\mu\nu} e^\alpha_a n^\beta e^\mu_b n^\nu. \tag{2.5.25}$$

For vacuum spacetimes, with $R_{\alpha\beta} = 0 = R$, Gauss–Codazzi equations give,

$$E_{ab} = R_{\alpha\beta\mu\nu} e^\alpha_a n^\beta e^\mu_b n^\nu = -\pounds_n K_{ab} + K_{ac}K^c_b + N^{-1} D_a D_b N\ , \tag{2.5.26}$$

where $\pounds_n$ is the Lie-derivative with respect to the unit normal n^μ.

Given the expansions for the extrinsic curvature components and the lapse function N in powers of $1/\tau$, we can obtain the expansion of the electric part of the Weyl tensor. A calculation gives,

$$E_{ab} \equiv \frac{1}{\tau}E^{(1)}_{ab} + \frac{1}{\tau^2}E^{(2)}_{ab} + \cdots\ , \tag{2.5.27}$$

where the zeroth order expansion coefficient identically vanishes and the first order expansion coefficient is,

$$E^{(1)}_{ab} = \sigma_{ab} - \sigma h^{(0)}_{ab}. \tag{2.5.28}$$

The first order electric part of the Weyl tensor satisfies the following properties on $\mathcal{H}$,

$$E^{(1)}_{ab} = E^{(1)}_{ba}, \qquad \text{(symmetric)} \tag{2.5.29}$$

$$E^{(1)a}_a = \Box\sigma - 3\sigma = 0, \qquad \text{(traceless)} \tag{2.5.30}$$

$$\mathcal{D}_b E^{(1)b}_a = 0, \qquad \text{(divergence-free)} \tag{2.5.31}$$

upon using the first order equations of motion. It then follows that for conformal Killing vectors ξ^a on $\mathcal{H}$, $E^{(1)}_{ab}\xi^a$ is a conserved current. The four translations induce four conformal Killing vectors $\xi^a = \mathcal{D}^a\omega$ on $\mathcal{H}$ (recall when ω represents a translation for $\omega_{ab} - h^{(0)}_{ab}\omega = 0$), and this conserved current can be used to construct "future charges" [45, 46],

$$Q_\xi = -\frac{1}{8\pi G}\int_C \sqrt{q}\, d^2x\, E^{(1)}_{ab}\xi^a r^b \tag{2.5.32}$$

where C is a two dimensional topologically-spherical surface surrounding sources on $\mathcal{H}$. The induced metric on C is q_{ab} and r^a is the unit outward normal to C in $\mathcal{H}$. These charges are "conserved" in the sense that the integral can be done on any topologically-spherical surface C of $\mathcal{H}$ surrounding the sources, and the answer is independent of the choice of C.

Clearly for supertranslations, such a construction *does not* work as $\mathcal{D}^a\omega$ is not a conformal Killing vector on $\mathcal{H}$. Fortunately, a slight modification of this construction works [66]. We have,

$$E^{(1)}_{ab}\xi^a = E^{(1)}_{ab}\omega^a = \sigma_{ab}\omega^a - \sigma\omega_b. \tag{2.5.33}$$

Next consider $2\mathcal{D}^a(\omega_{[a}\sigma_{b]})$ for translations $\omega_{ab} - h^{(0)}_{ab}\omega = 0$, i.e.,

$$\begin{aligned} 2\mathcal{D}^a(\omega_{[a}\sigma_{b]}) &= \mathcal{D}^a(\omega_a\sigma_b) - \mathcal{D}^a(\omega_b\sigma_a) = 3\omega\sigma_b + \omega_a\sigma_b{}^a - \omega_b{}^a\sigma_a - \omega_b(3\sigma) \\ &= 3\omega\sigma_b + \omega^a\sigma_{ab} - \omega\sigma_b - 3\omega_b\sigma \\ &= \sigma_{ab}\omega^a + 2\omega\sigma_b - 3\sigma\omega_b \end{aligned} \tag{2.5.34}$$

Hence, for translations,

$$E^{(1)}_{ab}\xi^a - 2\mathcal{D}^a(\omega_{[a}\sigma_{b]}) = 2(\sigma\omega_b - \omega\sigma_b). \tag{2.5.35}$$

The key point is that the term $\mathcal{D}^a(\omega_{[a}\sigma_{b]})\xi^a r^b$ when integrated over C only contributes a total divergence and therefore is zero. Hence,

$$\int_C \sqrt{q}\, d^2x\, (E^{(1)}_{ab}\xi^a - 2\mathcal{D}^a(\omega_{[a}\sigma_{b]}))\xi^a r^b = \int_C \sqrt{q}\, d^2x\, E^{(1)}_{ab}\xi^a r^b = 2\int_C \sqrt{q}\, d^2x\, (\sigma\omega_b - \omega\sigma_b) r^b. \tag{2.5.36}$$

This last expression admits generalisation for supertranslations. The current $(\sigma\omega_b - \omega\sigma_b)$ is conserved for supertranslations as well, since $(\Box - 3)\omega = 0$, implying

$$\mathcal{D}^b(\sigma\omega_b - \omega\sigma_b) = 0. \tag{2.5.37}$$

Hence, we can define a charge for supertranslation ω as

$$Q_\omega = -\frac{1}{4\pi G}\int_C \sqrt{q}\, d^2x\, (\sigma\omega_b - \omega\sigma_b) r^b. \tag{2.5.38}$$

For translations this expression reduces to the previous expressions [45, 46].

Expression for Lorentz charges is relativity easier to propose. One of the second order equation of motion, namely Eq. (2.4.14), automatically gives a conserved tensor,

$$J_{ab} = -h^{(2)}_{ab} + \frac{1}{2}k^c_a k_{bc} + h^{(0)}_{ab}\left(-\frac{1}{8}k_{cd}k^{cd} + 8\sigma^2 - k_{cd}\mathcal{D}^c\mathcal{D}^d\sigma - \mathcal{D}_c\sigma\mathcal{D}^c\sigma\right) \tag{2.5.39}$$

with $\mathcal{D}^a J_{ab} = 0$. For a Killing vector ξ^a on $\mathcal{H}$ representing a four-dimensional rotation or boost we define,

$$Q_\xi = \frac{1}{8\pi G}\int_C \sqrt{q} d^2x\, J_{ab}\xi^a r^b. \tag{2.5.40}$$

These charges match with [45, 46] upon setting $k_{ab} = 0$ and noting the fact that the second order magnetic part of the Weyl tensor is related to J_{ab} by the curl operation defined in appendix 2.E.

2.5.3 Commutator of charges

In the previous section, we wrote expressions for supertranslation and Lorentz charges. The Poisson bracket between two charges is defined as (see e.g., [53, 73]),

$$\{Q_\chi, Q_{\chi'}\} = -\delta_\chi Q_{\chi'} \tag{2.5.41}$$

where the variation δ_χ acts on the fields as the transformation induced by the asymptotic symmetry. Supertranslation charges defined in Eq. (2.5.38) can also be written as

$$Q_\omega = \frac{1}{4\pi G}\int_C d^2x\,\sqrt{q}\,(\sigma_a\omega - \sigma\omega_a)\, r^a = \frac{1}{4\pi G}\int_{\mathcal{V}} d^3x\,\partial_a\left[\sqrt{-h^{(0)}}\,(\sigma^a\omega - \sigma\omega^a)\right], \tag{2.5.42}$$

where $\mathcal{V}$ is the part of $\mathcal{H}$ surrounded by C. Now, we wish to compute the Poisson bracket between Lorentz charges and supertranslation charges. Identifying $\chi = \xi$ (a Lorentz transformation) and $\chi' = \omega$ (a supertranslation), Eq. (2.5.41) becomes,

$$\{Q_\xi, Q_\omega\} = -\delta_\xi Q_\omega\ . \tag{2.5.43}$$

Using which the Poisson bracket becomes,

$$\begin{aligned}
\{Q_\xi, Q_\omega\} &= -\frac{1}{4\pi G}\int_{\mathcal{V}} d^3x\,\partial_a\left\{\delta_\xi\left[\sqrt{-h^{(0)}}\,(\sigma^a\omega - \sigma\omega^a)\right]\right\}\\
&= -\frac{1}{4\pi G}\int_{\mathcal{V}} d^3x\,\partial_a\left[\sqrt{-h^{(0)}}\left\{\xi^b\mathcal{D}_b\,(\sigma^a\omega - \sigma\omega^a) - \left(\sigma^b\omega - \sigma\omega^b\right)\mathcal{D}_b\xi^a\right\}\right]\\
&= -\frac{1}{4\pi G}\int_{\mathcal{V}} d^3x\,\partial_a\Big[\sqrt{-h^{(0)}}\Big\{\sigma^a\left(\xi^b\mathcal{D}_b\omega\right) + \omega\xi^b\mathcal{D}_b\sigma^a - \left(\xi^b\mathcal{D}_b\sigma\right)\omega^a - \sigma\left(\xi^b\mathcal{D}_b\omega^a\right)\\
&\qquad - \left(\sigma^b\omega - \sigma\omega^b\right)\mathcal{D}_b\xi^a\Big\}\Big]\\
&= \frac{1}{4\pi G}\int_{\mathcal{V}} d^3x\,\partial_a\Big[\sqrt{-h^{(0)}}\Big\{\sigma^a\left(-\xi^b\mathcal{D}_b\omega\right) - \sigma\left(-\xi^b\mathcal{D}_b\omega^a - \omega^b\mathcal{D}_a\xi_b\right)\Big\}\Big]
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{4\pi G}\int_{\mathcal{V}} d^3x\; \partial_a\Big[\sqrt{-h^{(0)}}\Big\{\omega\xi^b\mathcal{D}_b\sigma^a - \left(\xi^b\mathcal{D}_b\sigma\right)\omega^a - \sigma^b\omega\mathcal{D}_b\xi^a\Big\}\Big] \\
&= \frac{1}{4\pi G}\int_{\mathcal{V}} d^3x\; \partial_a\Big[\sqrt{-h^{(0)}}\Big\{\sigma^a\left(\pounds_{-\xi}\omega\right) - \sigma\mathcal{D}^a\left(\pounds_{-\xi}\omega\right)\Big\}\Big] \\
&\quad - \frac{1}{4\pi G}\int_{\mathcal{V}} d^3x\; \partial_a\Big[\sqrt{-h^{(0)}}\Big\{\omega\xi^b\mathcal{D}_b\sigma^a - \left(\xi^b\mathcal{D}_b\sigma\right)\omega^a - \sigma^b\omega\mathcal{D}_b\xi^a\Big\}\Big] \\
&= Q_{\omega'} + \frac{1}{4\pi G}\int_{\mathcal{V}} d^3x\; \sqrt{-h^{(0)}}\mathcal{D}_a\Big[\omega\xi^b\mathcal{D}_b\sigma^a - \left(\xi^b\mathcal{D}_b\sigma\right)\omega^a - \sigma^b\omega\mathcal{D}_b\xi^a\Big] \\
&= Q_{\omega'} + \frac{1}{4\pi G}\int_{\mathcal{V}} d^3x\; \sqrt{-h^{(0)}}\Big[\omega\xi^b\mathcal{D}_a\mathcal{D}_b\sigma^a - \left(\xi^b\mathcal{D}_b\sigma\right)\Box\omega - \sigma^b\omega\mathcal{D}_a\mathcal{D}_b\xi^a\Big] \\
&= Q_{\omega'} + \frac{1}{4\pi G}\int_{\mathcal{V}} d^3x\; \sqrt{-h^{(0)}}\Big[\omega\xi^b[\mathcal{D}_a,\mathcal{D}_b]\sigma^a + 3\omega\xi^b\sigma_b - 3\left(\xi^b\mathcal{D}_b\sigma\right)\omega - \sigma^b\omega[\mathcal{D}_a,\mathcal{D}_b]\xi^a\Big] \\
&= Q_{\omega} + \frac{1}{4\pi G}\int_{\mathcal{V}} d^3x\; \sqrt{-h^{(0)}}\Big[\omega\xi^b R^{(0)}_{ab}\sigma^a + 3\omega\xi^b\sigma_b - 3\left(\xi^b\mathcal{D}_b\sigma\right)\omega - \sigma^b\omega R^{(0)}_{ab}\xi^a\Big] \\
&= Q_{\omega'}
\end{aligned}
\tag{2.5.44}
$$

where $\omega' = \pounds_{-\xi}\omega$. We have used the result, $\delta_\xi\sqrt{-h^{(0)}} = (1/2)\sqrt{-h^{(0)}}h^{(0)ab}\delta_\xi h^{(0)}_{ab} = 0$.

One could attempt the calculation the other way round, i.e., identifying $\chi' = \xi$ (a Lorentz transformation) and $\chi = \omega$ (a supertranslation). That calculation is more involved. We expect to recover $Q_{\omega'}$ possibly with terms that only contribute to a total divergence on C. At spatial infinity the technology for identifying total divergence on the cuts of de Sitter hyperboloid is fairly well developed, see e.g., [70]; at timelike infinity some further technical work is required.

2.6 The Schwarzschild solution near timelike infinity

In this section, we write the Schwarzschild solution near timelike infinity in the Beig-Schmidt form Eq. (2.3.5)–Eq. (2.3.7). The Schwarzschild metric in standard static coordinates takes the form

$$
ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{2.6.1}
$$

where $d\Omega^2 = (d\theta^2 + \sin^2\theta d\varphi^2)$ is the round metric on the unit two-sphere. We begin by introducing (τ_0, ρ_0) coordinates defined as follows:

$$
t = \tau_0\sqrt{1+\rho_0^2}, \tag{2.6.2}
$$

$$
r = \rho_0\tau_0. \tag{2.6.3}
$$

These coordinates do not bring the Schwarzschild solution near timelike infinity in the Beig-Schmidt form as in Eq. (2.3.5)–Eq. (2.3.7). A chain of further coordinate transformations outlined in appendix 2.A are required (as expected). In coordinates (τ_0, ρ_0) the non-zero components of the metric takes the form to leading order in $1/\tau_0$:

$$
g_{\tau_0\tau_0} = -1 + (2GM)\left(\rho_0^{-1} + 2\rho_0\right)\frac{1}{\tau_0} + \mathcal{O}(\tau_0^{-2}) \tag{2.6.4}
$$

$$
g_{\rho_0\tau_0} = 4GM + \mathcal{O}(\tau_0^{-1}) \tag{2.6.5}
$$

$$
g_{\rho_0\rho_0} = \tau_0^2(1+\rho_0^2)^{-1} + (2GM)(1+\rho_0^2)^{-1}\left(\rho_0^{-1} + 2\rho_0\right)\tau_0 + \mathcal{O}(1) \tag{2.6.6}
$$

$$
g_{\theta\theta} = \rho_0^2\tau_0^2 + \mathcal{O}(1) \tag{2.6.7}
$$

$$
g_{\varphi\varphi} = \rho_0^2\tau_0^2\sin^2\theta + \mathcal{O}(1). \tag{2.6.8}
$$

Since $g_{\rho_0\tau_0}$ term does not fall-off as $\mathcal{O}(\tau_0^{-1})$, the metric is not in the Beig-Schmidt form at $\mathcal{O}(\tau_0^{-1})$. To fix this, following appendix 2.A we do the transformation,

$$\rho_0 = \rho_1 + \frac{G(\rho_1)}{\tau_1}, \tag{2.6.9}$$

$$G(\rho_1) = 4GM(1+\rho_1^2), \tag{2.6.10}$$

$$\tau_0 = \tau_1. \tag{2.6.11}$$

In the new coordinates (τ_1, ρ_1) the non-zero metric components take the form,

$$g_{\tau_1\tau_1} = -1 + (2GM)\left(\rho_1^{-1} + 2\rho_1\right)\frac{1}{\tau_1} + \mathcal{O}(\tau_1^{-2}) \tag{2.6.12}$$

$$g_{\rho_1\tau_1} = \mathcal{O}(\tau_1^{-1}) \tag{2.6.13}$$

$$g_{\rho_1\rho_1} = \tau_1^2(1+\rho_1^2)^{-1} + (2GM)(1+\rho_1^2)^{-1}\left(\rho_1^{-1} + 6\rho_1\right)\tau_1 + \mathcal{O}(1) \tag{2.6.14}$$

$$g_{\theta\theta} = \rho_1^2\tau_1^2 + 8GM\rho_1(1+\rho_1^2)\tau_1 + \mathcal{O}(1) \tag{2.6.15}$$

$$g_{\varphi\varphi} = \rho_1^2\tau_1^2\sin^2\theta + 8GM\rho_1(1+\rho_1^2)\sin^2\theta\tau_1 + \mathcal{O}(1). \tag{2.6.16}$$

The above metric is in the Beig-Schmidt form, though it does not satisfy our boundary condition $k=0$. To bring the metric in the requisite form, we do a general supertranslation and call the final coordinates (τ, ρ):

$$\tau_1 = \tau - F(\rho), \tag{2.6.17}$$

$$\rho_1 = \rho + \frac{1+\rho^2}{\tau}\partial_\rho F(\rho), \tag{2.6.18}$$

$$F(\rho) = -GM\left(\rho + 2\sqrt{1+\rho^2}\sinh^{-1}\rho\right). \tag{2.6.19}$$

$F(\rho)$ *does not* satisfy $\Box F = 3F$. The resulting metric is in the requisite Beig-Schmidt form at first order in the expansion in inverse powers of τ, and

$$h^{(1)}_{ab} = -2\sigma h^{(0)}_{ab}. \tag{2.6.20}$$

That is, not only $k=0$, but the full k_{ab} is zero. The field σ takes the value,

$$\sigma = -(GM)\left(\rho^{-1} + 2\rho\right), \qquad \Box\sigma = 3\sigma. \tag{2.6.21}$$

From these transformations, we see that as τ goes to ∞ for fixed r, ρ goes to 0. Thus, the horizon $r = 2GM$ intersects the timelike infinity hyperboloid $\mathcal{H}$ at the origin $\rho = 0$. Note that the function σ is singular at $\rho = 0$.

The four functions satisfying

$$\mathcal{D}_a\mathcal{D}_b\omega - h^{(0)}_{ab}\omega = 0, \tag{2.6.22}$$

are $\left\{\sqrt{1+\rho^2}, \rho\cos\theta, \rho\sin\theta\sin\phi, \rho\sin\theta\cos\phi\right\}$ representing respectively, the time-translation and three-spatial translations. The charge integral

$$Q_\omega = -\frac{1}{4\pi G}\int_C \sqrt{q}\, d^2x\, (\sigma\omega_b - \omega\sigma_b) r^b, \tag{2.6.23}$$

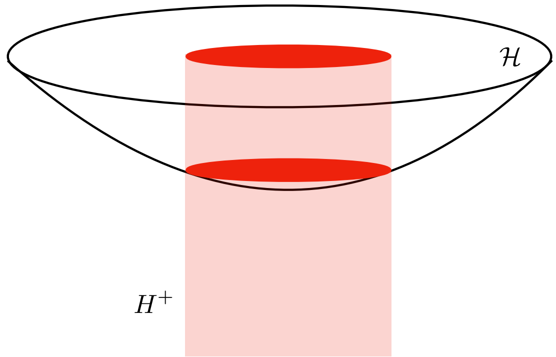

FIGURE 2.2: Horizon H^+ intersecting the timelike infinity hyperboloid $\mathcal{H}$. In the limit $\tau \to \infty$ the intersection shrinks to a point.

on $\rho =$ constant spherical surface C for time-translation $\omega = \sqrt{1+\rho^2}$ gives M.

2.7 Some final remarks

In the previous section we saw that for the Schwarzschild solution the fields σ and $h^{(1)}_{ab} = -2\sigma h^{(0)}_{ab}$ are singular at $\rho = 0$. The singularity is such that the charge integral is finite even on a $\rho = \epsilon$ surface C. Thus, for the region $r > 2GM$ of the Schwarzschild solution, timelike infinity is the hyperboloid $\mathcal{H}$ minus the origin. This indicates that for a system composed of individually bound systems, gravitationally unbound relative to each other, timelike infinity for the spacetime region describing outside the world-tubes of these system can be taken to be $\mathcal{H}$ minus one point each for the individually bound system. These points act as sources for the charge integrals.

For simplicity we focus on only one bound system, represented as a black hole, and take the horizon to intersect the timelike infinity hyperboloid $\mathcal{H}$ at the origin $\rho = 0$. We excise the point $\rho = 0$: $i^+ = \mathcal{H}\backslash\{\rho = 0\}$. The horizon is a blow up of the point $\rho = 0$ as schematically shown in Fig. 2.2. Having excised the point $\rho = 0$, the fields are all smooth at timelike infinity. The considerations of section 2.5 can be carried over. The first variation of the Iyer-Wald charges at timelike infinity vanishes

$$\delta Q_\xi(i^+) = 0. \tag{2.7.1}$$

This is schematically shown in figure Fig. 2.3.

Let us comment on the general form of the solutions for ω, σ and relate it to the Green's function discussion of [68]. The supertranslation function ω and the field σ both satisfy the equation $(\Box - 3)f = 0$. Expanding in spherical harmonics, we have

$$f(\rho,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l(\rho) Y_{lm}(\theta,\varphi). \tag{2.7.2}$$

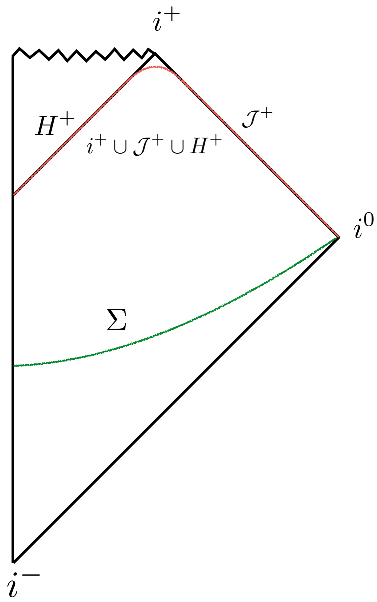

FIGURE 2.3: For a black hole formed by gravitational collapse, components of the boundary are $\mathcal{J}^-$, $\mathcal{J}^+$, H^+ and the points at infinity i^-, i^0, and i^+. Since the first variation of global charges is invariant under local deformations of the Cauchy surface Σ, one can deform Σ to $\mathcal{J}^+ \cup i^+ \cup H^+$ in the far future. Then, $\delta Q_\xi(\Sigma) = \delta Q_\xi(\mathcal{J}^+) + \delta Q_\xi(i^+) + \delta Q_\xi(H^+)$. With our boundary conditions $\delta Q_\xi(i^+) = 0$.

The equation for functions $f_l(\rho)$ admits two classes of solutions. The first set takes the form,

$$f_l^{(I)}(\rho) = \frac{\rho^l}{c_l}\, {}_2F_1\left(\frac{l-1}{2}, \frac{3+l}{2}; \frac{3}{2}+l; -\rho^2\right), \tag{2.7.3}$$

where ${}_2F_1$ is the standard hypergeometric function where $c_l = \frac{\Gamma(l+\frac{3}{2})}{\Gamma(2+\frac{l}{2})\Gamma(\frac{3+l}{2})}$ is a convenient normalisation. In the $\rho \to 0$ limit these solutions go as $f_l^{(I)}(\rho) \sim \frac{1}{c_l}\rho^l$. In the $\rho \to \infty$ limit they behave as $f_l^{(I)}(\rho) \sim \rho$. These functions correspond to supertranslations:

$$\omega(\rho, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} f_l^{(I)}(\rho) Y_{lm}(\theta, \varphi). \tag{2.7.4}$$

This can be seen as follows. For Minkowski space, in outgoing coordinates $(u, r, \theta, \varphi) = (t - r, r, \theta, \varphi)$, the time-translation takes the form,

$$\partial_u = \partial_t = \frac{\partial \tau}{\partial t}\partial_\tau + \frac{\partial \rho}{\partial t}\partial_\rho = \sqrt{1+\rho^2}\,\partial_\tau - \frac{\rho\sqrt{1+\rho^2}}{\tau}\,\partial_\rho. \tag{2.7.5}$$

In the $\tau \to \infty$ limit and then $\rho \to \infty$ limit, $\partial_u \sim \rho\,\partial_\tau$. Thus the expected behaviour of $f(\theta, \varphi)\partial_u$ is indeed the one captured by the supertranslations Eq. (2.3.8)–Eq. (2.3.9) with $\omega(\rho, \theta, \varphi)$ given in Eq. (2.7.4). A general null infinity supertranslation $f(\theta, \varphi)\partial_u$ correspond to

$$f(\theta, \varphi)\partial_u = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta, \varphi)\partial_u. \tag{2.7.6}$$

This construction, from the function $f(\theta,\varphi)$ to $\omega(\rho,\theta,\varphi)$ via Eq. (2.7.4), is the same as the Green's function construction of reference [68].[7]

The second independent set of solutions for the functions $f_l(\rho)$ takes the form

$$f_l^{(II)}(\rho) = \rho^{-l-1}\,{}_2F_1\left(-1+\frac{l}{2}, 1-\frac{l}{2}; \frac{1}{2}-l; -\rho^2\right). \tag{2.7.7}$$

In the $\rho \to 0$ limit these solutions go as $f_l^{(II)}(\rho) \sim \rho^{-l-1}$. Explicitly first few of these functions are

$$\begin{aligned} f_0^{(II)}(\rho) &= \rho^{-1} + 2\rho && (2.7.8) \\ f_1^{(II)}(\rho) &= \rho^{-2}(1-2\rho^2)\sqrt{1+\rho^2}, && (2.7.9) \\ f_2^{(II)}(\rho) &= \rho^{-3}, && (2.7.10) \end{aligned}$$

etc. For $l > 2$, in the $\rho \to \infty$ limit they behave as $f_l^{(II)}(\rho) \sim$ const ρ^{-3}. Our σ for the Schwarzschild solution matches with $f_0^{(II)}(\rho)$. Motivated by the corresponding discussion at spatial infinity, it is natural to speculate that the most general σ consists of the linear sum of the functions $f_l^{(II)}(\rho) Y_{lm}(\theta,\varphi)$

$$\sigma(\rho,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} d_{lm} f_l^{(II)}(\rho) Y_{lm}(\theta,\varphi). \tag{2.7.11}$$

Note that such a σ is singular at $\rho = 0$.

2.8 Conclusions

In this chapter, we have initiated the study of supertranslations at timelike infinity. Largely developing on the previous works at spatial infinity, we have proposed a definition of asymptotic flatness at timelike infinity in four spacetime dimensions. We presented a thorough study of the asymptotic equations of motion and the action of supertranslations on asymptotic fields. We showed that the Lee-Wald symplectic form $\Omega(g, \delta_1 g, \delta_2 g)$ does not get contributions from the future timelike infinity with our boundary conditions. As a result, the "future charges" can be computed on any two-dimensional surface surrounding the sources at timelike infinity. We presented expressions for supertranslation and Lorentz charges. For general spacetimes we expect

$$\text{future charges} \xleftarrow{u\to+\infty} \text{Bondi charges at } \mathcal{J}^+ \xrightarrow{u\to-\infty} \text{spatial infinity charges.} \tag{2.8.1}$$

Whether radiative spacetimes with non-trivial supertranslation charges exist that satisfy this hierarchy is open to argument [24].

[7] A proof can be explicitly written using the addition theorem of spherical harmonics.

2.A Asymptotic form of the metric

We begin by considering a general class of spacetimes admitting an expansion at timelike infinity of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n=1}^{m} \ell^{(n)}_{\mu\nu}\left(\frac{x^\sigma}{\tau}\right)\frac{1}{\tau^n} + \cdots \ , \tag{2.A.1}$$

where

$$\tau^2 = -\eta_{\mu\nu}x^\mu x^\nu, \tag{2.A.2}$$

and where x^μ are a set of Cartesian coordinates on flat spacetime at infinity. This class of the spacetimes can be put in a more convenient form as in Eq. (2.3.5)–Eq. (2.3.7). In this appendix we do so explicitly, following Beig and Schmidt [61]. The form Eq. (2.3.5)–Eq. (2.3.7) is our starting point for defining asymptotically flat spacetimes at timelike infinity.

The ten functions in $\ell^{(n)}_{\mu\nu}$ at any given order n are functions of the dimensionless coordinate (x^σ/τ). To avoid cumbersome notation, henceforth in all the expressions we shall simply write $\ell^{(n)}_{\mu\nu}$ without mentioning its dependence on (x^σ/τ).

Instead of the Cartesian coordinates x^μ, it is more convenient to use (τ, ϕ^a) as a new set of coordinates, with τ defined in Eq. (2.A.2) and ϕ^a are coordinates on hyperboloid $\mathcal{H}$. For any set of ϕ^a we define functions $\omega^\mu(\phi^a)$, such that,

$$\omega^\mu(\phi^a) = \frac{x^\mu}{\tau}. \tag{2.A.3}$$

Using this relation we get,

$$dx^\mu = \omega^\mu d\tau + \tau(\partial_a\omega^\mu)d\phi^a. \tag{2.A.4}$$

Inserting the above equation in Eq. (2.A.1) we obtain the following expression for the line element,

$$\begin{aligned}
ds^2 = g_{\mu\nu}dx^\mu dx^\nu &= \left[\eta_{\mu\nu} + \sum_{n=1}^{m}\ell^{(n)}_{\mu\nu}\frac{1}{\tau^n} + \cdots\right]dx^\mu dx^\nu \\
&= \left[\eta_{\mu\nu} + \sum_{n=1}^{m}\ell^{(n)}_{\mu\nu}\frac{1}{\tau^n} + \cdots\right]\left(\omega^\mu d\tau + \tau(\partial_a\omega^\mu)d\phi^a\right)\left(\omega^\nu d\tau + \tau(\partial_b\omega^\nu)d\phi^b\right) \\
&= \left[\eta_{\mu\nu} + \sum_{n=1}^{m}\ell^{(n)}_{\mu\nu}\frac{1}{\tau^n} + \cdots\right] \\
&\qquad \times\left[\omega^\mu\omega^\nu d\tau^2 + \tau\omega^\nu(\partial_a\omega^\mu)d\phi^a d\tau + \tau\omega^\mu(\partial_a\omega^\nu)d\phi^a d\tau + \tau^2(\partial_a\omega^\mu)(\partial_b\omega^\nu)d\phi^a d\phi^b\right] \\
&= -\left[-\eta_{\mu\nu}\omega^\mu\omega^\nu - \sum_{n=1}^{m}\ell^{(n)}_{\mu\nu}\omega^\mu\omega^\nu\frac{1}{\tau^n} + \cdots\right]d\tau^2 \\
&\quad + 2\tau\left[\eta_{\mu\nu}\omega^\mu(\partial_a\omega^\nu) + \sum_{n=1}^{m}\ell^{(n)}_{\mu\nu}\omega^\mu(\partial_a\omega^\nu)\frac{1}{\tau^n} + \cdots\right]d\tau d\phi^a \\
&\quad + \tau^2\left[\eta_{\mu\nu}(\partial_a\omega^\mu)(\partial_b\omega^\nu) + \sum_{n=1}^{m}\ell^{(n)}_{\mu\nu}(\partial_a\omega^\mu)(\partial_b\omega^\nu)\frac{1}{\tau^n} + \cdots\right]d\phi^a d\phi^b.
\end{aligned} \tag{2.A.5}$$

Using $\eta_{\mu\nu}\omega^\mu\omega^\nu = -1$ and $\eta_{\mu\nu}\omega^\mu(\partial_a\omega^\nu) = (1/2)\partial_a(\omega_\mu\omega^\mu) = 0$, along with the following definitions,

$$\bar{\sigma}^{(n)}(\phi^c) \equiv -\ell^{(n)}_{\mu\nu}\omega^\mu\omega^\nu \ , \tag{2.A.6}$$

$$A^{(n)}_a(\phi^c) \equiv \ell^{(n)}_{\mu\nu}\omega^\mu(\partial_a\omega^\nu) \ , \tag{2.A.7}$$

$$h^{(n)}_{ab}(\phi^c) \equiv \ell^{(n)}_{\mu\nu}(\partial_a\omega^\mu)(\partial_b\omega^\nu) \tag{2.A.8}$$

$$h^{(0)}_{ab}(\phi^c) \equiv \eta_{\mu\nu}(\partial_a\omega^\mu)(\partial_b\omega^\nu) \ , \tag{2.A.9}$$

the asymptotic form of the line element at timelike infinity takes the form,

$$\begin{aligned} ds^2 = &- \left[1 + \sum_{n=1}^{m}\frac{\bar{\sigma}^{(n)}(\phi^c)}{\tau^n} + \mathcal{O}\left(\tau^{-m-1}\right)\right] d\tau^2 + 2\tau\left[\sum_{n=1}^{m}\frac{A^{(n)}_a(\phi^c)}{\tau^n} + \mathcal{O}\left(\tau^{-m-1}\right)\right] d\tau d\phi^a \\ &+ \tau^2\left[h^{(0)}_{ab} + \sum_{n=1}^{m}\frac{h^{(n)}_{ab}(\phi^c)}{\tau^n} + \mathcal{O}\left(\tau^{-m-1}\right)\right] d\phi^a d\phi^b \ , \end{aligned} \tag{2.A.10}$$

$$\begin{aligned} = &- \left[1 + \sum_{n=1}^{m}\frac{\sigma^{(n)}(\phi^c)}{\tau^n} + \mathcal{O}\left(\tau^{-m-1}\right)\right]^2 d\tau^2 + 2\tau\left[\sum_{n=1}^{m}\frac{A^{(n)}_a(\phi^c)}{\tau^n} + \mathcal{O}\left(\tau^{-m-1}\right)\right] d\tau d\phi^a \\ &+ \tau^2\left[h^{(0)}_{ab} + \sum_{n=1}^{m}\frac{h^{(n)}_{ab}(\phi^c)}{\tau^n} + \mathcal{O}\left(\tau^{-m-1}\right)\right] d\phi^a d\phi^b \ . \end{aligned} \tag{2.A.11}$$

Here, $\sigma^{(n)}(\phi^c)$ are functions of $\bar{\sigma}^{(n)}(\phi^c)$, e.g., $\sigma^{(1)} = (\bar{\sigma}^{(1)}/2)$.

Next we show that there exist a coordinate transformation that brings the metric in Eq. (2.A.11) to a form where,

$$\sigma^{(n)}(\phi^c) = 0 \ , \quad \text{for } n \geq 2, \tag{2.A.12}$$

$$A^{(n)}_a(\phi^c) = 0, \quad \text{for } n \geq 1. \tag{2.A.13}$$

We achieve this order by order. At first order, we take

$$\phi^a = \bar{\phi}^a + \frac{G^{(1)a}(\bar{\phi}^b)}{\bar{\tau}}, \tag{2.A.14}$$

$$\tau = \bar{\tau} \ . \tag{2.A.15}$$

This yields,

$$d\phi^a = d\bar{\phi}^a - \frac{G^{(1)a}}{\bar{\tau}^2}d\bar{\tau} + \frac{1}{\bar{\tau}}\left(\partial_b G^{(1)a}\right) d\bar{\phi}^b, \tag{2.A.16}$$

and

$$\sigma^{(n)}(\phi^c) = \sigma^{(n)}(\bar{\phi}^c) + \frac{G^{(1)a}(\bar{\phi}^b)}{\bar{\tau}}\partial_a\sigma^{(n)} + \mathcal{O}(\bar{\tau}^{-2}). \tag{2.A.17}$$

In these new coordinates, line element Eq. (2.A.11) takes the form (keeping track of all the first order terms),

$$\begin{aligned} ds^2 = &- \left[1 + \frac{2\sigma^{(1)}(\bar{\phi}^c)}{\bar{\tau}} + \mathcal{O}\left(\bar{\tau}^{-2}\right)\right] d\bar{\tau}^2 + 2\bar{\tau}\left[\frac{A^{(1)}_a(\bar{\phi}^c)}{\bar{\tau}} + \mathcal{O}\left(\bar{\tau}^{-2}\right)\right] d\bar{\tau}d\bar{\phi}^a \\ &+ \bar{\tau}^2\left[h^{(0)}_{ab} + \frac{h^{(1)}_{ab}(\bar{\phi}^c)}{\bar{\tau}} + \mathcal{O}\left(\tau^{-2}\right)\right] d\bar{\phi}^a d\bar{\phi}^b - 2\bar{\tau}^2\left[h^{(0)}_{ab} + \frac{h^{(1)}_{ab}(\bar{\phi}^c)}{\bar{\tau}} + \mathcal{O}\left(\tau^{-2}\right)\right]\frac{G^{(1)a}}{\bar{\tau}^2}d\bar{\tau}d\bar{\phi}^b \\ &+ 2\bar{\tau}^2\left[h^{(0)}_{ab} + \frac{h^{(1)}_{ab}(\bar{\phi}^c)}{\bar{\tau}} + \mathcal{O}\left(\tau^{-2}\right)\right]\frac{1}{\bar{\tau}}\left(\partial_c G^{(1)a}\right) d\bar{\phi}^c d\bar{\phi}^b \end{aligned}$$

$$= -\left[1 + \frac{2\sigma^{(1)}(\bar{\phi}^c)}{\bar{\tau}} + \mathcal{O}\left(\bar{\tau}^{-2}\right)\right] d\bar{\tau}^2 + 2\left[A_a^{(1)}(\bar{\phi}^c) - h_{ab}^{(0)} G^{(1)b} + \mathcal{O}\left(\bar{\tau}^{-1}\right)\right] d\bar{\tau} d\bar{\phi}^a$$
$$+ \bar{\tau}^2 \left[h_{ab}^{(0)} + \frac{h_{ab}^{(1)}(\bar{\phi}^c)}{\bar{\tau}} + \frac{2}{\bar{\tau}} h_{cb}^{(0)} \left(\partial_a G^{(1)c}\right) + \mathcal{O}\left(\tau^{-2}\right)\right] d\bar{\phi}^a d\bar{\phi}^b \ . \quad (2.A.18)$$

Thus setting,

$$A_a^{(1)} = h_{ab}^{(0)} G^{(1)b} \ , \quad (2.A.19)$$

the line element takes the requisite form at the first order in the inverse powers of τ.

Keeping track of the second order terms, we have

$$ds^2 = -\left[1 + \frac{2\sigma^{(1)}(\phi^c)}{\tau} + \frac{(\sigma^1)^2 + 2\sigma^{(2)}}{\tau^2} + \mathcal{O}\left(\tau^{-3}\right)\right] d\tau^2 + 2\tau \left[\frac{A_a^{(2)}(\phi^c)}{\tau^2} + \mathcal{O}\left(\tau^{-3}\right)\right] d\tau d\phi^a$$
$$+ \tau^2 \left[h_{ab}^{(0)} + \frac{h_{ab}^{(1)}(\phi^c)}{\tau} + \frac{h_{ab}^{(2)}(\phi^c)}{\tau^2} + \mathcal{O}\left(\tau^{-3}\right)\right] d\phi^a d\phi^b \ . \quad (2.A.20)$$

The following coordinate transformation,

$$\tau = \bar{\tau} + \frac{F^{(2)}(\phi^c)}{\bar{\tau}}, \quad (2.A.21)$$
$$\bar{\phi}^a = \phi^a, \quad (2.A.22)$$

yields

$$ds^2 = -\left[1 + \frac{2\sigma^{(1)}}{\bar{\tau}} + \frac{(\sigma^1)^2 + 2\sigma^{(2)}}{\bar{\tau}^2} - \frac{2F^{(2)}}{\bar{\tau}^2} + \mathcal{O}\left(\bar{\tau}^{-3}\right)\right] d\bar{\tau}^2 + 2\bar{\tau}\left[\frac{A_a^{(2)} - \partial_a F^{(2)}}{\bar{\tau}^2} + \mathcal{O}\left(\bar{\tau}^{-3}\right)\right] d\bar{\tau} d\phi^a$$
$$+ \bar{\tau}^2 \left[h_{ab}^{(0)} + \frac{h_{ab}^{(1)}}{\bar{\tau}} + \frac{h_{ab}^{(2)}}{\bar{\tau}^2} + \frac{2F^{(2)}}{\bar{\tau}^2} h_{ab}^{(0)} + \mathcal{O}\left(\bar{\tau}^{-3}\right)\right] d\phi^a d\phi^b \ . \quad (2.A.23)$$

Thus, no $(1/\bar{\tau})$ term has been generated in the coefficient of the $d\bar{\tau} d\phi^a$ term and hence the condition $A_a^{(1)} = 0$ continues to hold. Furthermore, if we choose,

$$2F^{(2)} = 2\sigma^{(2)} \ , \quad (2.A.24)$$

then the $(1/\bar{\tau}^2)$ term in the coefficient of the $d\bar{\tau}^2$ in the metric can be set to $(\sigma^1)^2$. Thus the modified metric has only $[(\sigma^1)^2/\bar{\tau}^2]$ term in the coefficient of the $d\bar{\tau}^2$ and no $(1/\bar{\tau})$ term in the coefficient of the $d\bar{\tau} d\phi^a$. We can now use,

$$\phi^a = \bar{\phi}^a + \frac{G^{(2)a}(\bar{\phi}^b)}{\bar{\tau}^2}, \quad (2.A.25)$$
$$\tau = \bar{\tau}, \quad (2.A.26)$$

and choose the function $G^{(2)a}(\bar{\phi}^b)$, such that $A_a^{(2)} h^{(0)ab} = G^{(2)b}$ and hence the $(1/\bar{\tau}^2)$ term in the coefficient of the $d\bar{\tau} d\phi^a$ can be made to vanishes. Next, setting

$$\tau = \bar{\tau} + \frac{F^{(3)}(\phi^c)}{\bar{\tau}^2} \quad (2.A.27)$$

we can eliminate $(1/\bar{\tau}^3)$ term in the coefficient of the $d\bar{\tau}^2$.

Proceeding in an identical manner, we can eliminate all terms in the coefficient of $d\bar{\tau}d\phi^a$ and all terms beyond $\bar{\tau}^{-2}$ in the coefficient of $d\bar{\tau}^2$. Thus, metric Eq. (2.A.11) can be reduced to the one satisfying conditions Eq. (2.A.12)–Eq. (2.A.13). Thereby, we arrive at our final form Eq. (2.3.5)–Eq. (2.3.7).

2.B Action of supertranslation on asymptotic fields

We apply the following transformation,

$$\tau = \bar{\tau} - \omega(\bar{\phi}^a) + \frac{1}{\bar{\tau}} F^{(2)}(\bar{\phi}^a) + \mathcal{O}\left(\frac{1}{\bar{\tau}^2}\right) , \tag{2.B.1}$$

$$\phi^a = \bar{\phi}^a + \frac{1}{\bar{\tau}} h^{(0)ab}\partial_b\omega(\bar{\phi}^c) + \frac{1}{\bar{\tau}^2} G^{(2)\ a}(\bar{\phi}^c) + \mathcal{O}\left(\frac{1}{\bar{\tau}^3}\right) . \tag{2.B.2}$$

We obtain,

$$d\tau = \left[1 - \frac{1}{\bar{\tau}^2}F^{(2)} + \mathcal{O}\left(\frac{1}{\bar{\tau}^3}\right)\right] d\bar{\tau} + \left[-\partial_a\omega + \frac{1}{\bar{\tau}}\partial_a F^{(2)} + \mathcal{O}\left(\frac{1}{\bar{\tau}^2}\right)\right] d\bar{\phi}^a , \tag{2.B.3}$$

$$\begin{aligned} d\phi^a = & \left[\delta^a_c + \frac{1}{\bar{\tau}}\partial_c\left(h^{(0)ab}\partial_b\omega\right) + \frac{1}{\bar{\tau}^2}\left(\partial_c G^{(2)a}\right) + \mathcal{O}\left(\frac{1}{\bar{\tau}^3}\right)\right] d\bar{\phi}^c \\ & + \left[-\frac{1}{\bar{\tau}^2}\left(h^{(0)ab}\partial_b\omega\right) - \frac{2}{\bar{\tau}^3}G^{(2)a} + \mathcal{O}\left(\frac{1}{\bar{\tau}^4}\right)\right] d\bar{\tau} . \end{aligned} \tag{2.B.4}$$

The following relations are also obtained,

$$\sigma(\phi^a) = \sigma(\bar{\phi}^a) + \frac{1}{\bar{\tau}} h^{(0)ab}\partial_b\omega\partial_a\sigma + \mathcal{O}\left(\frac{1}{\bar{\tau}^2}\right) , \tag{2.B.5}$$

$$\begin{aligned} h^{(0)}_{ab}(\phi^c) = & h^{(0)}_{ab}(\bar{\phi}^c) + \frac{1}{\bar{\tau}} h^{(0)cd}\partial_d\omega\left(\partial_c h^{(0)}_{ab}\right) \\ & + \frac{1}{\bar{\tau}^2}\left[G^{(2)\ c}\left(\partial_c h^{(0)}_{ab}\right) + \frac{1}{2}\left(h^{(0)cp}\partial_p\omega\right)\left(h^{(0)dq}\partial_q\omega\right)\partial_c\partial_d h^{(0)}_{ab}\right] + \mathcal{O}\left(\frac{1}{\bar{\tau}^3}\right) , \end{aligned} \tag{2.B.6}$$

$$h^{(1)}_{ab}(\phi^c) = h^{(1)}_{ab}(\bar{\phi}^c) + \frac{1}{\bar{\tau}} h^{(0)cd}\partial_d\omega\left(\partial_c h^{(1)}_{ab}\right) + \mathcal{O}\left(\frac{1}{\bar{\tau}^2}\right) . \tag{2.B.7}$$

Inserting these expressions in the full metric we can read the changes in the first order fields,

$$\sigma \to \sigma \tag{2.B.8}$$

$$h^{(1)}_{ab} \to h^{(1)}_{ab} - 2\omega h^{(0)}_{ab} + h^{(0)cd}\partial_d\omega\left(\partial_c h^{(0)}_{ab}\right) + h^{(0)}_{pb}\partial_a\left(h^{(0)pq}\partial_q\omega\right) + h^{(0)}_{pa}\partial_b\left(h^{(0)pq}\partial_q\omega\right) . \tag{2.B.9}$$

This last expression can be more conveniently written as,

$$h^{(1)}_{ab} \to h^{(1)}_{ab} + 2\mathcal{D}_a\mathcal{D}_b\omega - 2\omega h^{(0)}_{ab} . \tag{2.B.10}$$

In order to preserve the original form of the metric, we must choose,

$$F^{(2)} = \sigma\omega + h^{(0)ab}\partial_b\omega\partial_a\sigma - \frac{1}{2}h^{(0)}_{ab}\left(h^{(0)ac}\partial_c\omega\right)\left(h^{(0)bd}\partial_d\omega\right) , \tag{2.B.11}$$

$$2G_a^{(2)} = -\partial_a F^{(2)} + 2\sigma\partial_a\omega - (\partial_b\omega)\,\partial_a\left(h^{(0)bp}\partial_p\omega\right) + 2\omega\partial_a\omega - h^{(1)}_{ab}\partial^b\omega - \partial^b\omega\partial^c\omega\left(\partial_c h^{(0)}_{ab}\right). \tag{2.B.12}$$

This results in the transformation for $h^{(2)}_{ab}$ as

$$\begin{aligned}h^{(2)}_{ab} \to h^{(2)}_{ab} &- \mathcal{D}_a\omega\mathcal{D}_b\omega + \left[-\omega h^{(1)}_{ab} + \mathcal{D}^c\omega\mathcal{D}_c h^{(1)}_{ab} + 2h^{(1)c}_{(a}\mathcal{D}_{b)}\mathcal{D}_c\omega\right] + \left(2F^{(2)} + \omega^2\right)h^{(0)}_{ab}\\ &+ 2\mathcal{D}_{(a}G^{(2)}_{b)} - 4\omega\;\mathcal{D}_a\mathcal{D}_b\omega + \mathcal{D}_a\mathcal{D}^c\omega\;\mathcal{D}_b\mathcal{D}_c\omega + 2\mathcal{D}_{(a}\mathcal{D}^c\omega\;\mathcal{D}^d\omega\;\Gamma^{(0)}_{b)cd}\\ &+ \mathcal{D}^c\omega\;\mathcal{D}^d\omega\left(\mathcal{D}_c\Gamma^{(0)}_{(ba)d} + \Gamma^{(0)p}_{cd}\Gamma^{(0)}_{(ba)p} - \Gamma^{(0)p}_{(ac}\Gamma^{(0)}_{b)dp}\right),\end{aligned} \tag{2.B.13}$$

where we have used the following notation

$$\mathcal{D}_c\Gamma^{(0)}_{pqr} := \partial_c\Gamma^{(0)}_{pqr} - \Gamma^{(0)i}_{cp}\Gamma^{(0)}_{iqr} - \Gamma^{(0)i}_{cq}\Gamma^{(0)}_{pir} - \Gamma^{(0)i}_{cr}\Gamma^{(0)}_{pqi}\,. \tag{2.B.14}$$

and the following results

$$\Gamma^{(0)}_{qab} + \Gamma^{(0)}_{aqb} = \partial_b h^{(0)}_{qa}, \tag{2.B.15}$$

$$\Gamma^{(0)}_{cad} + \Gamma^{(0)}_{dca} = \partial_a h^{(0)}_{cd}, \tag{2.B.16}$$

$$\partial_p\omega\;\partial_a h^{(0)cp} + h^{(0)cp}\partial_a\partial_p\omega = \mathcal{D}_a\mathcal{D}^c\omega - \mathcal{D}^d\omega\;\Gamma^{c\;(0)}_{ad}. \tag{2.B.17}$$

Next we simplify Eq. (2.B.13). Defining $\phi_a = \mathcal{D}_a\phi$, for any scalar function ϕ, we obtain,

$$\begin{aligned}\delta_\omega h^{(2)}_{ab} =& -\omega_a\omega_b - \omega k_{ab} + 2\sigma\omega h^{(0)}_{ab} + \omega^c\mathcal{D}_c\,k_{ab} + \omega^c_b\,k_{ac} + \omega^c_a\,k_{bc} - 2\sigma_c\omega^c h^{(0)}_{ab} - 4\sigma\omega_{ab}\\ &+ \left(2\sigma\omega + 2\omega_c\sigma^c - \omega_c\omega^c + \omega^2\right)h^{(0)}_{ab} - 4\omega\omega_{ab} + \omega^c_a\omega_{bc} + \omega^c_a\omega^d\Gamma^{(0)}_{bcd} + \omega^c_b\omega^d\Gamma^{(0)}_{acd}\\ &- \frac{1}{2}\left(\mathcal{D}_a\mathcal{D}_b + \mathcal{D}_b\mathcal{D}_a\right)F^{(2)} + \sigma_a\omega_b + 2\sigma\omega_{ab} + \sigma_b\omega_a\\ &- \frac{1}{2}\omega^c\left(\mathcal{D}_a\,k_{bc}\right) + \sigma_a\omega_b - \frac{1}{2}k_{bc}\omega^c_a + 2\sigma\omega_{ab} - \frac{1}{2}\omega^c\left(\mathcal{D}_b\,k_{ac}\right) + \sigma_b\omega_a - \frac{1}{2}\,k_{ac}\omega^c_b\\ &- \omega_{ac}\omega^c_b - \frac{1}{2}\omega_c\left(\mathcal{D}_a\mathcal{D}_b + \mathcal{D}_b\mathcal{D}_a\right)\omega^c + 2\omega_a\omega_b + 2\omega\omega_{ab}\\ &- \frac{1}{2}\omega^p\omega^q\left(\mathcal{D}_a\Gamma^{(0)}_{bpq} + \mathcal{D}_b\Gamma^{(0)}_{apq}\right) - \omega^p_a\omega^q\Gamma^{(0)}_{bpq} - \omega^p_b\omega^q\Gamma^{(0)}_{apq}\\ &+ \omega^c\omega^d\left(\mathcal{D}_c\Gamma^{(0)}_{(ba)d} + \Gamma^{(0)\;p}_{cd}\,\Gamma^{(0)}_{(ba)p} - \Gamma^{(0)\;p}_{(ac}\,\Gamma^{(0)}_{b)dp}\right).\end{aligned} \tag{2.B.18}$$

We further obtain,

$$\begin{aligned}\left(\mathcal{D}_a\mathcal{D}_b + \mathcal{D}_b\mathcal{D}_a\right)F^{(2)} =\;& 2\sigma_{ab}\omega + 2\omega_a\sigma_b + 2\sigma_a\omega_b + 2\sigma\omega_{ab}\\ &+ \omega_c\left(\mathcal{D}_a\mathcal{D}_b + \mathcal{D}_b\mathcal{D}_a\right)\sigma^c + 2\sigma_{ca}\omega^c_b + 2\sigma_{cb}\omega^c_a + \sigma_c\left(\mathcal{D}_a\mathcal{D}_b + \mathcal{D}_b\mathcal{D}_a\right)\omega^c\\ &- \omega_{ac}\omega^c_b - \omega_{bc}\omega^c_a - \omega_c\left(\mathcal{D}_a\mathcal{D}_b + \mathcal{D}_b\mathcal{D}_a\right)\omega^c.\end{aligned} \tag{2.B.19}$$

Combining these expressions we obtain,

$$\begin{aligned}\delta_\omega h^{(2)}_{ab} =& -\omega\,k_{ab} + \omega^c\mathcal{D}_c\,k_{ab} + \frac{1}{2}\omega^c_b k_{ac} + \frac{1}{2}\omega^c_a\,k_{bc} - \frac{1}{2}\omega^c\left(\mathcal{D}_a\,k_{bc}\right) - \frac{1}{2}\omega^c\left(\mathcal{D}_b\,k_{ac}\right)\\ &+ 2\sigma\omega h^{(0)}_{ab} + \sigma_{(a}\omega_{b)} - \sigma\omega_{ab} - \sigma_{c(a}\omega^c_{b)} - \sigma_c\omega^c_{(ab)} + (\sigma \leftrightarrow \omega)\\ &+ \omega_a\omega_b + \left(-\omega_c\omega^c + \omega^2\right)h^{(0)}_{ab} - 2\omega\omega_{ab} + \omega^c_a\omega_{bc}\\ &+ \frac{1}{2}\omega^c\omega^d\Big[\left(\mathcal{D}_c\Gamma^{(0)}_{bad} - \mathcal{D}_a\Gamma^{(0)}_{bcd} - \Gamma^{(0)\,p}_{ac}\Gamma^{(0)}_{bdp} + \Gamma^{(0)\,p}_{cd}\Gamma^{(0)}_{bap}\right)\end{aligned}$$

$$+\left(\mathcal{D}_c\Gamma^{(0)}_{abd}-\mathcal{D}_b\Gamma^{(0)}_{acd}+\Gamma^{(0)\,p}_{cd}\,\Gamma^{(0)}_{abp}-\Gamma^{(0)\,p}_{bc}\,\Gamma^{(0)}_{adp}\right)\Big]. \tag{2.B.20}$$

Using

$$\mathcal{D}_c\Gamma^{(0)}_{bad}-\mathcal{D}_a\Gamma^{(0)}_{bcd}-\Gamma^{(0)\,p}_{ac}\,\Gamma^{(0)}_{bdp}+\Gamma^{(0)\,p}_{cd}\,\Gamma^{(0)}_{bap}=R^{(0)}_{bdca}=-h^{(0)}_{bc}h^{(0)}_{da}+h^{(0)}_{ba}h^{(0)}_{dc}\,, \tag{2.B.21}$$
$$\mathcal{D}_c\Gamma^{(0)}_{abd}-\mathcal{D}_b\Gamma^{(0)}_{acd}+\Gamma^{(0)\,p}_{cd}\,\Gamma^{(0)}_{abp}-\Gamma^{(0)\,p}_{bc}\,\Gamma^{(0)}_{adp}=R^{(0)}_{adcb}=-h^{(0)}_{ac}h^{(0)}_{bd}+h^{(0)}_{ab}h^{(0)}_{cd}\,, \tag{2.B.22}$$

we obtain our final form for $\delta_\omega h^{(2)}_{ab}$,

$$\begin{aligned}\delta_\omega h^{(2)}_{ab}=&-\omega\,k_{ab}+\omega^c\mathcal{D}_c\,k_{ab}+\frac{1}{2}\omega^c_b\,k_{ac}+\frac{1}{2}\omega^c_a\,k_{bc}-\frac{1}{2}\omega^c\left(\mathcal{D}_a\,k_{bc}\right)-\frac{1}{2}\omega^c\left(\mathcal{D}_b\,k_{ac}\right)\\&+2\sigma\omega h^{(0)}_{ab}+\sigma_{(a}\omega_{b)}-\sigma\omega_{ab}-\sigma_{c(a}\omega^c_{b)}-\sigma_c\omega^c{}_{(ab)}+(\sigma\leftrightarrow\omega)\\&+\omega^2h^{(0)}_{ab}-2\omega\omega_{ab}+\omega^c_a\omega_{bc}.\end{aligned} \tag{2.B.23}$$

2.C Expansion of the equations of motion

Given the previous series of coordinate transformations, we arrive at the following form of the asymptotic metric, near timelike infinity,

$$ds^2=-N^2d\tau^2+h_{ab}d\phi^ad\phi^b, \tag{2.C.1}$$

where

$$N=1+\frac{\sigma(\phi^a)}{\tau}, \tag{2.C.2}$$
$$h_{ab}=\tau^2\left[h^{(0)}_{ab}(\phi^c)+\frac{1}{\tau}h^{(1)}_{ab}(\phi^c)+\frac{1}{\tau^2}h^{(2)}_{ab}(\phi^c)+\mathcal{O}\left(\frac{1}{\tau^3}\right)\right]. \tag{2.C.3}$$

The future directed unit normal vector to a $\tau=$ constant surface is,

$$n_\mu=-N\nabla_\mu\tau,\qquad n^\mu=\frac{1}{N}\delta^\mu_\tau\ . \tag{2.C.4}$$

The induced metric on $\tau=$ constant hypersurface is h_{ab}, while the inverse spatial metric has the following expansion,

$$h^{ab}=\frac{1}{\tau^2}h^{(0)ab}-\frac{1}{\tau^3}h^{(1)ab}-\frac{1}{\tau^4}\left(h^{(2)ab}-h^{(1)a}_c h^{(1)cb}\right)+\mathcal{O}\left(\frac{1}{\tau^5}\right)\ . \tag{2.C.5}$$

For any spatial tensor $T^{(n)}_{ab}$ at order n in the expansion, we raise and lower indices with $h^{(0)}_{ab}$, for example,

$$T^{(n)ab}=h^{(0)ac}h^{(0)bd}T^{(n)}_{cd}. \tag{2.C.6}$$

For a general spatial tensor T_{ab}, we have $T^{ab}=h^{ac}h^{bd}T_{cd}$.

The extrinsic curvature K_{ab}

The extrinsic curvature of $\tau=$ constant hypersurface takes the form,

$$K_{ab}=\frac{1}{2N}\partial_\tau h_{ab}=\tau h^{(0)}_{ab}+\left(\frac{1}{2}h^{(1)}_{ab}-\sigma h^{(0)}_{ab}\right)+\frac{1}{\tau}\left(\sigma^2h^{(0)}_{ab}-\frac{\sigma}{2}h^{(1)}_{ab}\right)+\mathcal{O}\left(\frac{1}{\tau^2}\right). \tag{2.C.7}$$

Upon raising one and two indices respectively we have

$$K^a_b = h^{ac}K_{cb} = \frac{1}{\tau}\delta^a_b + \frac{1}{\tau^2}\left(-\frac{1}{2}h^{(1)a}_b - \sigma\delta^a_b\right) + \frac{1}{\tau^3}\left(\sigma^2\delta^a_b + \frac{\sigma}{2}h^{(1)a}_b - h^{(2)a}_b + \frac{1}{2}h^{(1)ap}h^{(1)}_{pb}\right) + \mathcal{O}\left(\frac{1}{\tau^4}\right) \tag{2.C.8}$$

$$K^{ab} = h^{ac}K^b_c = \frac{1}{\tau^3}h^{(0)ab} + \frac{1}{\tau^4}\left(-\frac{3}{2}h^{(1)ab} - \sigma h^{(0)ab}\right) + \frac{1}{\tau^5}\left(-2h^{(2)ab} + 2h^{(1)ap}h^{(1)b}_p + \frac{3\sigma}{2}h^{(1)ab} + \sigma^2 h^{(0)}_{ab}\right) + \mathcal{O}\left(\frac{1}{\tau^6}\right) . \tag{2.C.9}$$

The trace of the extrinsic curvature becomes,

$$K = \delta^b_a K^a_b = \frac{3}{\tau} + \frac{1}{\tau^2}\left(-\frac{1}{2}h^{(1)} - 3\sigma\right) + \frac{1}{\tau^3}\left(3\sigma^2 + \frac{\sigma}{2}h^{(1)} - h^{(2)} + \frac{1}{2}h^{(1)ab}h^{(1)}_{ab}\right) + \mathcal{O}\left(\frac{1}{\tau^4}\right) . \tag{2.C.10}$$

Asymptotic expansion of intrinsic geometry

For any perturbed symmetric, spatial tensor $S^{(n)}_{ab}$, we note the following identity

$$-\partial_d S^{(n)}_{bc} + \partial_b S^{(n)}_{dc} + \partial_c S^{(n)}_{bd} = -\mathcal{D}_d S^{(n)}_{bc} + \mathcal{D}_b S^{(n)}_{dc} + \mathcal{D}_c S^{(n)}_{bd} + 2\Gamma^{(0)p}_{bc} S^{(n)}_{pd} , \tag{2.C.11}$$

where $\mathcal{D}$ denotes covariant derivative compatible with $h^{(0)}_{ab}$ on $\mathcal{H}$. Using the above identity, the asymptotic expansion of the Christoffel symbol takes the form,

$$\Gamma^a_{bc} = \frac{1}{2}h^{ad}\left(-\partial_d h_{bc} + \partial_b h_{dc} + \partial_c h_{bd}\right) \tag{2.C.12}$$

$$\equiv \Gamma^{(0)a}_{bc} + \frac{1}{\tau}\Gamma^{(1)a}_{bc} + \frac{1}{\tau^2}\Gamma^{(2)a}_{bc} + \mathcal{O}\left(\frac{1}{\tau^3}\right) , \tag{2.C.13}$$

where

$$\Gamma^{(1)a}_{bc} = -h^{(1)a}_d \Gamma^{(0)d}_{bc} + \frac{1}{2}h^{(0)ad}\left(-\mathcal{D}_d h^{(1)}_{bc} + \mathcal{D}_b h^{(1)}_{dc} + \mathcal{D}_c h^{(1)}_{bd} + 2\Gamma^{(0)p}_{bc} h^{(1)}_{pd}\right) , \tag{2.C.14}$$

$$\Gamma^{(2)a}_{bc} = -\left(h^{(2)a}_d - h^{(1)a}_p h^{(1)p}_d\right)\Gamma^{(0)d}_{bc} + \frac{1}{2}h^{(0)ad}\left(-\mathcal{D}_d h^{(2)}_{bc} + \mathcal{D}_b h^{(2)}_{dc} + \mathcal{D}_c h^{(2)}_{bd} + 2\Gamma^{(0)p}_{bc} h^{(2)}_{pd}\right) - \frac{1}{2}h^{(1)ad}\left(-\mathcal{D}_d h^{(1)}_{bc} + \mathcal{D}_b h^{(1)}_{dc} + \mathcal{D}_c h^{(1)}_{bd} + 2\Gamma^{(0)p}_{bc} h^{(1)}_{pd}\right) . \tag{2.C.15}$$

Here, $\Gamma^{(0)a}_{bc}$ is the non-tensorial Christoffel symbol associated with the zeroth order spatial metric $h^{(0)}_{ab}$. The other expansion coefficients are tensors and have the following simplified expressions,

$$\Gamma^{(1)a}_{bc} = \frac{1}{2}\left(-\mathcal{D}^a h^{(1)}_{bc} + \mathcal{D}_b h^{(1)a}_c + \mathcal{D}_c h^{(1)a}_b\right) , \tag{2.C.16}$$

$$\Gamma^{(2)a}_{bc} = \frac{1}{2}\left(-\mathcal{D}^a h^{(2)}_{bc} + \mathcal{D}_b h^{(2)a}_c + \mathcal{D}_c h^{(2)a}_b\right) - \frac{1}{2}h^{(1)ad}\left(-\mathcal{D}_d h^{(1)}_{bc} + \mathcal{D}_b h^{(1)}_{dc} + \mathcal{D}_c h^{(1)}_{bd}\right) . \tag{2.C.17}$$

The three-dimensional Ricci tensor takes the form,

$$\mathcal{R}_{ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ca} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc}$$

$$\equiv \mathcal{R}^{(0)}_{ab} + \frac{1}{\tau}\mathcal{R}^{(1)}_{ab} + \frac{1}{\tau^2}\mathcal{R}^{(2)}_{ab} + \mathcal{O}\left(\frac{1}{\tau^3}\right). \tag{2.C.18}$$

Here, $\mathcal{R}^{(0)}_{ab}$ is the Ricci tensor associated with the spatial metric $h^{(0)}_{ab}$, while the other two expansion coefficients are,

$$\mathcal{R}^{(1)}_{ab} = \frac{1}{2}\left(\mathcal{D}_c\mathcal{D}_a h^{(1)c}_b + \mathcal{D}_c\mathcal{D}_b h^{(1)c}_a - \mathcal{D}_c\mathcal{D}^c h^{(1)}_{ab} - \mathcal{D}_a\mathcal{D}_b h^{(1)}\right) , \tag{2.C.19}$$

$$\begin{aligned}\mathcal{R}^{(2)}_{ab} &= \frac{1}{2}\left(\mathcal{D}_c\mathcal{D}_a h^{(2)c}_b + \mathcal{D}_c\mathcal{D}_b h^{(2)c}_a - \mathcal{D}_c\mathcal{D}^c h^{(2)}_{ab} - \mathcal{D}_a\mathcal{D}_b h^{(2)}\right) \\ &+ \frac{1}{2}\mathcal{D}_b\left(h^{(1)cd}\mathcal{D}_a h^{(1)}_{cd}\right) - \frac{1}{2}\mathcal{D}_c\left[h^{(1)cd}\left(-\mathcal{D}_d h^{(1)}_{ab} + \mathcal{D}_b h^{(1)}_{da} + \mathcal{D}_a h^{(1)}_{bd}\right)\right] \\ &+ \frac{1}{4}\mathcal{D}_c h^{(1)}\left(-\mathcal{D}^c h^{(1)}_{ba} + \mathcal{D}_b h^{(1)c}_a + \mathcal{D}_a h^{(1)c}_b\right) \\ &- \frac{1}{4}\mathcal{D}_a h^{(1)c}_d\mathcal{D}_b h^{(1)d}_c + \frac{1}{2}\mathcal{D}^c h^{(1)}_{ad}\mathcal{D}_c h^{(1)d}_b - \frac{1}{2}\mathcal{D}^c h^{(1)}_{ad}\mathcal{D}^d h^{(1)}_{bc} \ .\end{aligned} \tag{2.C.20}$$

These expressions will be used extensively in what follows.

The Hamiltonian constraint

The Hamiltonian constraint takes the form,

$$H \equiv \frac{1}{N}\partial_\tau K + K_{ab}K^{ab} - \frac{1}{N}h^{ab}D_aD_bN = 0, \tag{2.C.21}$$

where D_a is the covariant derivative compatible with h_{ab}. Expanding out each of these terms we obtain,

$$H = \frac{H^{(0)}}{\tau^2} + \frac{H^{(1)}}{\tau^3} + \frac{H^{(2)}}{\tau^4} + \mathcal{O}\left(\frac{1}{\tau^5}\right) \ . \tag{2.C.22}$$

where

$$H^{(0)} = 0, \tag{2.C.23}$$

$$H^{(1)} = (-\Box + 3)\,\sigma = 0, \tag{2.C.24}$$

$$H^{(2)} = h^{(2)} - 9\sigma^2 - \frac{1}{4}h^{(1)ab}h^{(1)}_{ab} - \frac{1}{2}\sigma h^{(1)} + h^{(1)ab}\mathcal{D}_a\mathcal{D}_b\sigma + \sigma\Box\sigma + h^{(0)ab}\Gamma^{(1)c}_{ab}\mathcal{D}_c\sigma = 0. \tag{2.C.25}$$

Using, $k_{ab} = h^{(1)}_{ab} + 2\sigma h^{(0)}_{ab}$, cf. Eq. (2.3.15), the second order coefficient can be simplified, yielding,

$$\begin{aligned}H^{(2)} &= h^{(2)} - 12\sigma^2 - \frac{1}{4}k^{ab}k_{ab} + k^{ab}\mathcal{D}_a\mathcal{D}_b\sigma + \mathcal{D}_c\sigma\mathcal{D}^c\sigma \\ &+ \frac{1}{2}\sigma k - \sigma\left(\Box - 3\right)\sigma - \frac{1}{2}\mathcal{D}_c\sigma\left(\mathcal{D}^c k\right) + \mathcal{D}_c\sigma\mathcal{D}_a k^{ac} \ .\end{aligned} \tag{2.C.26}$$

Now upon using our boundary condition $k = 0$ and lower order equations of motion it simplifies to

$$H^{(2)} = h^{(2)} - 12\sigma^2 - \frac{1}{4}k^{ab}k_{ab} + k^{ab}\mathcal{D}_a\mathcal{D}_b\sigma + \mathcal{D}_c\sigma\mathcal{D}^c\sigma = 0. \tag{2.C.27}$$

The momentum constraint

The momentum constraint $H_a = 0$ takes the form,

$$H_a \equiv D_b K_a^b - D_a K = 0. \tag{2.C.28}$$

This can be expanded as,

$$H_a = \frac{1}{\tau} H_a^{(0)} + \frac{1}{\tau^2} H_a^{(1)} + \frac{1}{\tau^3} H_a^{(2)} + \dots \tag{2.C.29}$$

where

$$H_a^{(0)} = 0 \tag{2.C.30}$$

$$H_a^{(1)} = -\frac{1}{2}\mathcal{D}_b \left(k_a^b - k\delta_a^b\right) \tag{2.C.31}$$

$$\begin{aligned} H_a^{(2)} =& -\mathcal{D}_b h_a^{(2)b} + \frac{1}{2} k^{bp} \left(\mathcal{D}_b k_{pa}\right) + \frac{1}{2} k_{pa} \left(\mathcal{D}_b k^{bp}\right) - \frac{3}{2}\sigma \left(\mathcal{D}_b k_a^b\right) - \frac{1}{4} k_a^c \left(\mathcal{D}_c k\right) + \frac{\sigma}{2}\left(\mathcal{D}_a k\right) \\ &+ \mathcal{D}_a \left[h^{(2)} - \frac{3}{8} k^{bc} k_{bc} + \sigma k - 4\sigma^2\right] . \end{aligned} \tag{2.C.32}$$

Using second order Hamiltonian constraint $H^{(2)} = 0$ and boundary condition $k = 0$, together with first order equations of motion, we get

$$H_a^{(2)} = -\mathcal{D}_b h_a^{(2)b} + \frac{1}{2} k^{bp} \left(\mathcal{D}_b k_{pa}\right) + \mathcal{D}_a \left(-\frac{1}{8} k^{bc} k_{bc} + 8\sigma^2 - k^{ab}\mathcal{D}_a \mathcal{D}_b \sigma - \mathcal{D}_c \sigma \mathcal{D}^c \sigma\right) = 0. \tag{2.C.33}$$

Asymptotic expansion of the evolution equation

The evolution equation of the spatial metric h_{ab} takes the following form,

$$H_{ab} := \mathcal{R}_{ab} + \frac{1}{N}\partial_\tau K_{ab} - 2K_{ac}K_b^c + KK_{ab} - \frac{1}{N} D_a D_b N = 0. \tag{2.C.34}$$

Expanding in powers of $\frac{1}{\tau}$ we have,

$$H_{ab} \equiv H_{ab}^{(0)} + \frac{1}{\tau} H_{ab}^{(1)} + \frac{1}{\tau^2} H_{ab}^{(2)} + \dots, \tag{2.C.35}$$

where

$$H_{ab}^{(0)} = \mathcal{R}_{ab}^{(0)} + 2h_{ab}^{(0)} = 0, \tag{2.C.36}$$

$$H_{ab}^{(1)} = -\frac{1}{2}\left(\Box + 3\right) k_{ab} = 0, \tag{2.C.37}$$

$$H_{ab}^{(2)} = -\frac{1}{2}\left(\Box + 2\right) h_{ab}^{(2)} + T_{ab}^{(kk)} + T_{ab}^{(k\sigma)} + T_{ab}^{(\sigma\sigma)} = 0, \tag{2.C.38}$$

and where the non-linear terms are

$$\begin{aligned} T_{ab}^{(kk)} = \frac{1}{2}\Bigg[& \left(\mathcal{D}_c k_{d(a}\mathcal{D}_{b)} k^{cd}\right) - \frac{1}{2}\mathcal{D}_a k^{cd}\mathcal{D}_b k_{cd} + \left(\mathcal{D}^c k_{ad}\right)\left(\mathcal{D}_c k_b^d\right) - \left(\mathcal{D}^c k_{ad}\right)\left(\mathcal{D}^d k_{bc}\right) \\ & - k_a^p k_{pb} + k^{cd}\left(\mathcal{D}_c \mathcal{D}_d k_{ab} - \mathcal{D}_c \mathcal{D}_{(a} k_{b)d}\right)\Bigg], \end{aligned} \tag{2.C.39}$$

$$T^{(k\sigma)}_{ab} = \frac{1}{2}\Bigg[- \mathcal{D}_a\mathcal{D}_b\left(k^{cd}\mathcal{D}_c\mathcal{D}_d\sigma\right) + 4\mathcal{D}^c\sigma\left(-\mathcal{D}_c k_{ab} + \mathcal{D}_{(a}k_{b)c}\right) - 4\sigma k_{ab} + \left(-2h^{(0)}_{ab}k^{cd}\mathcal{D}_c\mathcal{D}_d\sigma + 4k^{cd}h^{(0)}_{d(a}\mathcal{D}_{b)}\mathcal{D}_c\sigma\right)\Bigg], \tag{2.C.40}$$

$$T^{(\sigma\sigma)}_{ab} = \frac{1}{2}\left[\mathcal{D}_a\mathcal{D}_b\left(5\sigma^2 - \mathcal{D}_c\sigma\mathcal{D}^c\sigma\right) + h^{(0)}_{ab}\left(18\sigma^2 + 4\mathcal{D}^c\sigma\mathcal{D}_c\sigma\right) + 4\sigma\mathcal{D}_a\mathcal{D}_b\sigma\right], \tag{2.C.41}$$

where we have used the boundary condition $k = 0$ and the first order equations of motion.

2.D A consistency check

In this appendix we perform a non-trivial consistency check on our asymptotic equations of motion and expression Eq. (2.B.23) for $\delta_\omega h^{(2)}_{ab}$. We consider doing a supertranslation on flat spacetime. Thus to begin with we have (for flat spacetime)

$$\sigma = 0, \quad h^{(1)}_{ab} = 0, \quad h^{(2)}_{ab} = 0. \tag{2.D.1}$$

We note that $\sigma = 0$ does not change under supertranslations. Thus for the supertranslated spacetime too $\sigma = 0$ and from Eq. (2.B.10) it follows that

$$h^{(1)}_{ab} = \delta_\omega h^{(1)}_{ab} = k_{ab} = -2\omega h^{(0)}_{ab} + 2\omega_{ab}. \tag{2.D.2}$$

From Eq. (2.B.23) it follows that

$$h^{(2)}_{ab} = \omega^2 h^{(0)}_{ab} - 2\omega\omega_{ab} + \omega^c_a\omega_{bc}. \tag{2.D.3}$$

We check that expression Eq. (2.D.2) for k_{ab} and Eq. (2.D.3) for $h^{(2)}_{ab}$ are consistent with second order equations of motion.

Recall that $\Box\omega = 3\omega$, and also we note the following useful relation,

$$\begin{aligned}\Box\omega_a &= \mathcal{D}_b\mathcal{D}^b\mathcal{D}_a\omega = \mathcal{D}_b\mathcal{D}_a\mathcal{D}^b\omega = [\mathcal{D}_b,\mathcal{D}_a]\mathcal{D}^b\omega + \mathcal{D}_a\Box\omega \\ &= R^{(0)b}_{\ \ cba}\mathcal{D}^c\omega + 3\omega_a = -2h^{(0)}_{ca}\omega^c + 3\omega_a = \omega_a.\end{aligned} \tag{2.D.4}$$

Hamiltonian constraint

The Hamiltonian constraint Eq. (2.C.27) becomes,

$$h^{(2)} = \frac{1}{4}k_{ab}k^{ab}. \tag{2.D.5}$$

Given expression Eq. (2.D.2) for k_{ab}, we have

$$\frac{1}{4}k_{ab}k^{ab} = 3\omega^2 - 2\omega\Box\omega + \omega_{ab}\omega^{ab} = -3\omega^2 + \omega_{ab}\omega^{ab}, \tag{2.D.6}$$

which matches with the trace of Eq. (2.D.3), viz.,

$$h^{(2)} = 3\omega^2 - 2\omega\Box\omega + \omega^{ab}\omega_{ab} = -3\omega^2 + \omega^{ab}\omega_{ab}. \tag{2.D.7}$$

Momentum constraint

The momentum constraint presented in Eq. (2.C.33) becomes,

$$\mathcal{D}^b h^{(2)}_{ab} = \frac{1}{2} k^{bp}\, \mathcal{D}_b k_{pa} - \frac{1}{4} k^{bc}\, \mathcal{D}_a k_{bc}. \tag{2.D.8}$$

On the one hand, the right hand side of Eq. (2.D.8) is

$$\begin{aligned}\frac{1}{2} k^{bp}\, \mathcal{D}_b k_{pa} - \frac{1}{4} k^{bc}\, \mathcal{D}_a k_{bc} &= -2\omega_b \omega^b_a + \omega^{bc} \mathcal{D}_b \mathcal{D}_c \omega_a + 3\omega\omega_a - \omega^{bc} R^{(0)}_{cdab} \omega^d \\ &= -2\omega_b \omega^b_a + \omega^{bc} \mathcal{D}_b \mathcal{D}_c \omega_a + 3\omega\omega_a - \omega^{bc} \left(-h^{(0)}_{ac} h^{(0)}_{bd} + h^{(0)}_{bc} h^{(0)}_{ad} \right) \omega^d \\ &= -\omega_b \omega^b_a + \omega^{bc} \mathcal{D}_b \mathcal{D}_c \omega_a. \end{aligned} \tag{2.D.9}$$

On the other hand, the divergence of Eq. (2.D.3) yields for the left hand side of Eq. (2.D.8)

$$\begin{aligned}\mathcal{D}^b h^{(2)}_{ab} &= \mathcal{D}^b \left(\omega^2 h^{(0)}_{ab} - 2\omega\omega_{ab} + \omega^c_a \omega_{cb} \right) \\ &= 2\omega\omega_a - 2\omega^b \omega_{ab} - 2\omega \Box \omega_a + {\omega_a}^{cb} \omega_{cb} + \omega^c_a \Box \omega_c \\ &= -\omega^b \omega_{ab} + {\omega_a}^{cb} \omega_{cb}, \end{aligned} \tag{2.D.10}$$

which matches with Eq. (2.D.9).

Evolution equation

The evolution equation as presented in Eq. (2.C.38) is decomposed into several terms,

$$\begin{aligned}(\Box + 2)\, h^{(2)}_{ab} = \frac{1}{2} &\Bigg(\underbrace{\mathcal{D}_c\, k_{da} \mathcal{D}_b\, k^{cd}}_{\text{Term 1}} + \underbrace{\mathcal{D}_c k^{(1)}_{db} \mathcal{D}_a\, k^{cd}}_{\text{Term 2}} \Bigg) \underbrace{- \frac{1}{2} \mathcal{D}_a\, k^{cd} \mathcal{D}_b\, k_{cd}}_{\text{Term 3}} \\ &+ \underbrace{\left(\mathcal{D}^c\, k_{ad}\right) \left(\mathcal{D}_c\, k^d_b\right)}_{\text{Term 4}} - \underbrace{\left(\mathcal{D}^c\, k_{ad}\right) \left(\mathcal{D}^d\, k_{bc}\right)}_{\text{Term 5}} - \underbrace{k^p_a\, k_{pb}}_{\text{Term 6}} \\ &+ \Bigg(\underbrace{k^{cd} \mathcal{D}_c \mathcal{D}_d\, k_{ab}}_{\text{Term 7}} \Bigg) - \Bigg(\frac{1}{2} \underbrace{k^{cd} \mathcal{D}_c \mathcal{D}_a\, k_{bd}}_{\text{Term 8}} + \frac{1}{2} \underbrace{k^{cd} \mathcal{D}_c \mathcal{D}_b\, k_{ad}}_{\text{Term 9}} \Bigg) \quad . \end{aligned} \tag{2.D.11}$$

We first evaluate the right hand side using Eq. (2.D.2) and then evaluate the left hand side using Eq. (2.D.3) and show the match.

We obtain the following expression for various terms on the right hand side. For "Term 1" we have,

$$\begin{aligned}\frac{1}{2} \mathcal{D}_c\, k_{da} \mathcal{D}_b k^{cd} &= 2\mathcal{D}_c \left(-\omega h^{(0)}_{da} + \omega_{da} \right) \mathcal{D}_b \left(-\omega h^{(0)cd} + \omega^{cd} \right) \\ &= -2\omega^c [\mathcal{D}_b, \mathcal{D}_c] \omega_a - 2\omega^c \omega_{abc} + 2\omega_{adc} [\mathcal{D}_b, \mathcal{D}^c] \omega^d + 2\omega_{adc} {\omega_b}^{dc} \\ &= 2h^{(0)}_{ab} \left(\omega_c \omega^c \right) - 4\omega^c \omega_{abc} + 2\omega_{adc} {\omega_b}^{dc} \ , \end{aligned} \tag{2.D.12}$$

while "Term 2" follows from interchange of (a, b) in "Term 1". For "Term 3" we have,

$$\begin{aligned}\frac{1}{2} \mathcal{D}_a k^{cd} \mathcal{D}_b\, k_{cd} &= 2\mathcal{D}_a \left(-\omega h^{(0)cd} + \omega^{cd} \right) \mathcal{D}_b \left(-\omega h^{(0)}_{cd} + \omega_{cd} \right) \\ &= -6\omega_a \omega_b + 2 \left\{ [\mathcal{D}_a, \mathcal{D}^c] \omega^d + {\omega_a}^{dc} \right\} \left\{ [\mathcal{D}_b, \mathcal{D}_c] \omega_d + \omega_{bdc} \right\} \end{aligned}$$

$$= 2\omega_c\omega^c h^{(0)}_{ab} - 4\omega^c\omega_{abc} + 2\omega_{adc}\omega_b{}^{dc} \; . \tag{2.D.13}$$

For "Term 4" we have,

$$\begin{aligned}\left(\mathcal{D}^c k_{ad}\right)\left(\mathcal{D}_c k^d_b\right) &= 4\mathcal{D}^c\left(-\omega h^{(0)}_{ad} + \omega_{ad}\right) D^{(0)}_c\left(-\omega\delta^d_b + \omega_b{}^d\right)\\ &= 4\omega_c\omega^c h^{(0)}_{ab} - 8\omega^c\omega_{abc} + 4\omega_{adc}\omega_b{}^{dc}.\end{aligned} \tag{2.D.14}$$

For "Term 5" we have,

$$\begin{aligned}\left(\mathcal{D}^c\, k_{ad}\right)\left(\mathcal{D}^d\, k_{bc}\right) &= 4\mathcal{D}^c\left(-\omega h^{(0)}_{ad} + \omega_{ad}\right)\mathcal{D}^d\left(-\omega h^{(0)}_{bc} + \omega_{bc}\right)\\ &= 4\omega_a\omega_b - 4\omega^c[\mathcal{D}_b,\mathcal{D}_c]\omega_a - 8\omega^c\omega_{abc} - 4\omega^c[\mathcal{D}_a,\mathcal{D}_c]\omega_b + 4[\mathcal{D}_c,\mathcal{D}_d]\omega_a\omega_b{}^{cd} + 4\omega_{acd}\omega_b{}^{cd}\\ &= -4\omega_a\omega_b + 8\omega_c\omega^c h^{(0)}_{ab} - 12\omega^c\omega_{abc} + 4\omega_{acd}\omega_b{}^{cd} + 4\omega^c\omega_{bca}\\ &= -4\omega_a\omega_b + 8\omega_c\omega^c h^{(0)}_{ab} - 8\omega^c\omega_{abc} + 4\omega_{acd}\omega_b{}^{cd} + 4\omega^c[\mathcal{D}_a,\mathcal{D}_c]\omega_b\\ &= 4\omega_c\omega^c h^{(0)}_{ab} - 8\omega^c\omega_{abc} + 4\omega_{acd}\omega_b{}^{cd} \; .\end{aligned} \tag{2.D.15}$$

For "Term 6" we have,

$$\begin{aligned}k^p_a\, k_{pb} &= 4\left(-\omega\delta^p_a + \omega^p_a\right)\left(-\omega h^{(0)}_{pb} + \omega_{pb}\right)\\ &= 4\omega^2 h^{(0)}_{ab} - 8\omega\omega_{ab} + 4\omega^p_a\omega_{pb} \; .\end{aligned} \tag{2.D.16}$$

For "Term 7" we have,

$$\begin{aligned}k^{cd}\,\mathcal{D}_c\,\mathcal{D}_d\, k_{ab} &= 4\left(-\omega h^{(0)cd} + \omega^{cd}\right)\mathcal{D}_c\mathcal{D}_d\left(-\omega h^{(0)}_{ab} + \omega_{ab}\right)\\ &= 12\omega^2 h^{(0)}_{ab} - 4\omega^{cd}\omega_{cd}h^{(0)}_{ab} - 4\omega\Box\omega_{ab} + 4\omega^{cd}\omega_{abdc} \; ,\end{aligned} \tag{2.D.17}$$

and finally for "Term 8" we have,

$$\begin{aligned}&\frac{1}{2}k^{cd}\,\mathcal{D}_c\,\mathcal{D}_a\, k_{bd} = 2\left(-\omega h^{(0)cd} + \omega^{cd}\right)\mathcal{D}_c\,\mathcal{D}_a\left(-\omega h^{(0)}_{db} + \omega_{bd}\right)\\ &= 2\omega\omega_{ab} - 2\omega_{ac}\omega^c_b - 2\omega h^{(0)cd}\mathcal{D}_c\left\{[\mathcal{D}_a,\mathcal{D}_d]\omega_b\right\} - 2\omega\Box\omega_{ab} + 2\omega^{cd}\omega_{badc} + 2\omega^{cd}\mathcal{D}_c\left\{[\mathcal{D}_a,\mathcal{D}_d]\omega_b\right\}\\ &= 6\omega^2 h^{(0)}_{ab} - 2\omega\Box\omega_{ab} + 2\omega^{cd}\omega_{badc} - 2\omega_{cd}\omega^{cd}h^{(0)}_{ab} \; .\end{aligned} \tag{2.D.18}$$

"Term 9" is obtained by interchanging (a, b) in "Term 8". Collecting all these expressions we get,

$$(\Box + 2)\, h^{(2)}_{ab} = -4\omega^2 h^{(0)}_{ab} + 8\omega\omega_{ab} + 2h^{(0)}_{ab}\left(\omega_c\omega^c\right) - 4\omega^c_a\omega_{cb} - 4\omega^c\omega_{abc} + 2\omega_{adc}\omega_b{}^{dc} \tag{2.D.19}$$

On the other hand, using expression Eq. (2.D.3) for $h^{(2)}_{ab}$ we obtain,

$$\begin{aligned}(\Box + 2)\, h^{(2)}_{ab} &= 2\omega^2 h^{(0)}_{ab} - 4\omega\omega_{ab} + 2\omega^c_a\omega_{cb} + h^{(0)}_{ab}\mathcal{D}^c\,\mathcal{D}_c\omega^2 - 2\mathcal{D}^c\,\mathcal{D}_c\left(\omega\omega_{ab}\right) + \mathcal{D}^c\,\mathcal{D}_c\left(\omega^d_a\omega_{db}\right)\\ &= 8\omega^2 h^{(0)}_{ab} - 10\omega\omega_{ab} + 2h^{(0)}_{ab}\left(\omega_c\omega^c\right) + 2\omega^c_a\omega_{cb} - 4\omega^c\omega_{abc} + 2\omega_{acd}\omega_b{}^{cd}\\ &\quad + \omega_{cb}\Box\omega^c_a + \omega^c_a\Box\omega_{cb} - 2\omega\Box\omega_{ab} \; .\end{aligned} \tag{2.D.20}$$

We now have the following identity,

$$\Box\omega_{ab} = \mathcal{D}^c\,\mathcal{D}_c\,\mathcal{D}_a\,\mathcal{D}_b\omega$$

$$
\begin{aligned}
&= \mathcal{D}^c \left(R^{(0)}_{bpca} \omega^p \right) + \mathcal{D}_c \mathcal{D}_a \mathcal{D}_b \omega^c \\
&= \left(-h^{(0)}_{bc} h^{(0)}_{pa} + h^{(0)}_{ba} h^{(0)}_{pc} \right) \omega^{pc} + [\mathcal{D}_c, \mathcal{D}_a] \mathcal{D}_b \omega^c + \mathcal{D}_a \mathcal{D}_c \mathcal{D}_b \omega^c \\
&= -\omega_{ab} + h^{(0)}_{ab} \Box\omega + R^{(0)}_{bpca} \omega^{cp} + R^{(0)}_{pa} \omega^p_b + \mathcal{D}_a [\mathcal{D}_c, \mathcal{D}_b] \omega^c + \mathcal{D}_a \mathcal{D}_b \Box\omega \\
&= -\omega_{ab} + 3\omega h^{(0)}_{ab} + \left(-h^{(0)}_{bc} h^{(0)}_{pa} + h^{(0)}_{ba} h^{(0)}_{pc} \right) \omega^{cp} - 2\omega_{ab} + R^{(0)}_{pb} \omega^p_a + 3\omega_{ab} \\
&= -3\omega_{ab} + 6\omega h^{(0)}_{ab} \ .
\end{aligned}
\tag{2.D.21}
$$

Thus we obtain,

$$
\begin{aligned}
(\Box + 2)\, h^{(2)}_{ab} &= 8\omega^2 h^{(0)}_{ab} - 10\omega\omega_{ab} + 2h^{(0)}_{ab} (\omega_c \omega^c) + 2\omega^c_a \omega_{cb} - 4\omega^c \omega_{abc} + 2\omega_{acd} \omega_b{}^{cd} \\
&\quad + \omega_{cb} \left(-3\omega^c_a + 6\omega \delta^c_a \right) + \omega^c_a \left(-3\omega_{cb} + 6\omega h^{(0)}_{cb} \right) - 2\omega \left(-3\omega_{ab} + 6\omega h^{(0)}_{ab} \right) \\
&= \ -4\omega^2 h^{(0)}_{ab} + 8\omega\omega_{ab} + 2h^{(0)}_{ab} (\omega_c \omega^c) - 4\omega^c_a \omega_{cb} - 4\omega^c \omega_{abc} + 2\omega_{acd} \omega_b{}^{cd} \ ,
\end{aligned}
\tag{2.D.22}
$$

which matches with Eq. (2.D.19). A similar calculation at spatial infinity was done in [67].

2.E Expansion of the Weyl tensor

In four spacetime dimensions, the Weyl tensor expressed in terms of the Riemann tensor, Ricci tensor and Ricci scalar takes the form,

$$
W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \frac{1}{2} \left(g_{\alpha\mu} R_{\beta\nu} + R_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} R_{\beta\mu} - R_{\alpha\nu} g_{\beta\mu} \right) + \frac{R}{6} \left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} \right) \ .
\tag{2.E.1}
$$

Let (τ, ϕ^a) be the four-dimensional spacetime coordinates associated to the 3+1 split. Then, for a general set of spacetime coordinates $x^\mu = x^\mu(\tau, \phi^a)$ we define

$$
e^\mu_a = \frac{\partial x^\mu}{\partial \phi^a}.
\tag{2.E.2}
$$

The vectors e^μ_a with $\{a = 1, 2, 3\}$ are tangent to the $\tau =$ constant hypersurface. The projected electric part of the Weyl tensor on $\tau =$ constant hypersurface is defined as,

$$
E_{ab} = W_{\alpha\beta\mu\nu} e^\alpha_a n^\beta e^\mu_b n^\nu.
\tag{2.E.3}
$$

For vacuum spacetimes, with $R_{\alpha\beta} = 0 = R$, it simplifies to,

$$
E_{ab} = R_{\alpha\beta\mu\nu} e^\alpha_a n^\beta e^\mu_b n^\nu = -\pounds_n K_{ab} + K_{ac} K^c_b + N^{-1} D_a D_b N \ ,
\tag{2.E.4}
$$

where $\pounds_n$ is the Lie-derivative with respect to the unit normal Eq. (2.C.4). We have used the fact that $\tau =$ constant surface is spacelike.

The projected magnetic part of the Weyl tensor is defined as,

$$
\begin{aligned}
B_{ab} &= \frac{1}{2} \left(\epsilon_{\alpha\beta\rho\sigma} W^{\rho\sigma}{}_{\mu\nu} \right) e^\alpha_a n^\beta e^\mu_b n^\nu \\
&= \frac{1}{2} \left(e^\alpha_a n^\beta \epsilon_{\alpha\beta\rho\sigma} \right) g^{\rho\gamma} g^{\sigma\delta} W_{\gamma\delta\mu\nu} e^\mu_b n^\nu
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\left(e^{\alpha}_{a} n^{\beta}\epsilon_{\alpha\beta\rho\sigma}\right)\left(h^{\rho\gamma}+\epsilon n^{\rho}n^{\gamma}\right)\left(h^{\sigma\delta}+\epsilon n^{\sigma}n^{\delta}\right)W_{\gamma\delta\mu\nu}e^{\mu}_{b}n^{\nu} \\
&= \frac{1}{2}\left(e^{\alpha}_{a} n^{\beta}\epsilon_{\alpha\beta\rho\sigma}\right)\left(h^{\rho\gamma}h^{\sigma\delta}W_{\gamma\delta\mu\nu}e^{\mu}_{b}n^{\nu}\right). \qquad (2.E.5)
\end{aligned}
$$

For vacuum spacetimes,

$$
B_{ab} = \frac{1}{2}\left(e^{\alpha}_{a} n^{\beta}\epsilon_{\alpha\beta\rho\sigma}\right)\left(h^{\rho\gamma}h^{\sigma\delta}R_{\gamma\delta\mu\nu}e^{\mu}_{b}n^{\nu}\right) = -\frac{1}{2}\epsilon_{acd}\left(D^{c}K^{d}_{b} - D^{d}K^{c}_{b}\right) = -\epsilon_{acd}D^{c}K^{d}_{b}\ . \qquad (2.E.6)
$$

Note that we have used the result, $\epsilon_{\rho\alpha\beta\mu}n^{\rho} = \epsilon_{abc}e^{a}_{\alpha}e^{b}_{\beta}e^{c}_{\mu}$, where ϵ_{abc} is the three-dimensional Levi-Civita tensor. In what follows we will expand both the electric and magnetic parts of the Weyl tensor.

Expansion of the electric part of the Weyl tensor

Given the expansions for the extrinsic curvature components and the lapse function N in powers of $1/\tau$, we can obtain the expansion of the electric part of the Weyl tensor. A calculation gives,

$$
E_{ab} \equiv \frac{1}{\tau}E^{(1)}_{ab} + \frac{1}{\tau^2}E^{(2)}_{ab} + \cdots \qquad (2.E.7)
$$

where the zeroth order expansion coefficient identically vanishes and the first order expansion coefficient is,

$$
E^{(1)}_{ab} = -\sigma h^{(0)}_{ab} + \mathcal{D}_a\mathcal{D}_b\sigma, \qquad (2.E.8)
$$

while the second order expansion coefficient is,

$$
\begin{aligned}
E^{(2)}_{ab} &= 3\sigma^2 h^{(0)}_{ab} - h^{(2)}_{ab} + \frac{1}{4}h^{(1)p}_{a}h^{(1)}_{pb} - \sigma\mathcal{D}_a\mathcal{D}_b\sigma - \Gamma^{(1)c}_{ab}\mathcal{D}_c\sigma - \frac{\sigma}{2}h^{(1)}_{ab} \\
&= -h^{(2)}_{ab} + 5\sigma^2 h^{(0)}_{ab} + \frac{1}{4}k^{p}_{a}k_{pb} - \sigma k_{ab} - \frac{\sigma}{2}k_{ab} - \sigma\mathcal{D}_a\mathcal{D}_b\sigma \\
&\quad - \frac{1}{2}\left(-\mathcal{D}^c k_{ab} + \mathcal{D}_a k^{c}_{b} + \mathcal{D}_b k^{c}_{a}\right)\mathcal{D}_c\sigma + 2D_a\sigma\mathcal{D}_b\sigma - h^{(0)}_{ab}\mathcal{D}_c\sigma\mathcal{D}^c\sigma. \qquad (2.E.9)
\end{aligned}
$$

For the first order term, we have the following properties,

$$
\begin{aligned}
&E^{(1)}_{ab} = E^{(1)}_{ba}, && \text{(symmetric)} && (2.E.10) \\
&E^{(1)a}_{a} = -3\sigma + \Box\sigma = 0, && \text{(traceless)} && (2.E.11) \\
&\mathcal{D}_b E^{(1)b}_{a} = 0. && \text{(divergence-free)} && (2.E.12)
\end{aligned}
$$

We consider the following combination at the second order

$$
\begin{aligned}
E^{(2)}_{ab} - \sigma E^{(1)}_{ab} = &-h^{(2)}_{ab} + 6\sigma^2 h^{(0)}_{ab} - 2\sigma\mathcal{D}_a\mathcal{D}_b\sigma + 2\mathcal{D}_a\sigma\mathcal{D}_b\sigma - h^{(0)}_{ab}\mathcal{D}_c\sigma\mathcal{D}^c\sigma \\
&- \frac{1}{2}\left(-\mathcal{D}^c k_{ab} + \mathcal{D}_a k^{c}_{b} + \mathcal{D}_b k^{c}_{a}\right)\mathcal{D}_c\sigma + \frac{1}{4}k^{p}_{a}k_{pb} - \frac{3\sigma}{2}k_{ab} \qquad (2.E.13)
\end{aligned}
$$

For $k_{ab} = 0$,

$$
E^{(2)}_{ab} - \sigma E^{(1)}_{ab} = -h^{(2)}_{ab} + 6\sigma^2 h^{(0)}_{ab} - 2\sigma\mathcal{D}_a\mathcal{D}_b\sigma + 2\mathcal{D}_a\sigma\mathcal{D}_b\sigma - h^{(0)}_{ab}\mathcal{D}_c\sigma\mathcal{D}^c\sigma\ , \qquad (2.E.14)
$$

is also symmetric, traceless, and divergence free upon using second order equations of motion. The trace and divergence equations for $h^{(2)}_{ab}$ can equivalently be thought of as tracefree and divergence free conditions for $E^{(2)}_{ab} - \sigma E^{(1)}_{ab}$.

Expansion of the magnetic part of the Weyl tensor

We now compute the expansion of the magnetic part of the Weyl tensor starting from Eq. (2.E.6),

$$B_{ab} = -\epsilon_{acd} h^{cm} D_m K^d_b \equiv \frac{1}{\tau} B^{(1)}_{ab} + \frac{1}{\tau^2} B^{(2)}_{ab} + \cdots \,. \tag{2.E.15}$$

The first order expansion coefficient is,

$$\begin{aligned} B^{(1)}_{ab} &= \epsilon^{(0)}_{acd} \left(\frac{1}{2} \mathcal{D}^c h^{(1)d}_b + \delta^d_b \mathcal{D}^c \sigma \right) \\ &= \epsilon^{(0)}_{acd} \left[\frac{1}{2} \mathcal{D}^c \left(k^d_b - 2\sigma \delta^d_b \right) + \delta^d_b \mathcal{D}^c \sigma \right] = \frac{1}{2} \epsilon^{(0)}_{acd} \left(\mathcal{D}^c k^d_b \right), \end{aligned} \tag{2.E.16}$$

while the second order expansion coefficient is,

$$\begin{aligned} B^{(2)}_{ab} = \epsilon^{(0)}_{acd} \Bigg\{ & \left[\mathcal{D}^c h^{(2)d}_b - 2\delta^d_b \mathcal{D}^c \left(\sigma^2\right) \right] - \frac{1}{2} \left(k^c_m + \sigma \delta^c_m \right) \mathcal{D}^m k^d_b + \frac{1}{2} h^{(0)cm} \Gamma^{(1)d}_{mp} k^p_b \\ & - \frac{1}{2} h^{(0)cm} \Gamma^{(1)p}_{mb} k^d_p - \mathcal{D}^c \left(\frac{\sigma}{2} k^d_b \right) - \mathcal{D}^c \left(\frac{1}{2} k^{dp} k_{pb} \right) + \mathcal{D}^c \left(2\sigma k^d_b \right) \Bigg\} \end{aligned} \tag{2.E.17}$$

where, we have used the result, $h^{(1)dp} h^{(1)}_{pb} = (k^{dp} - 2\sigma h^{(0)dp})(k_{pb} - 2\sigma h^{(0)}_{pb}) = k^{dp} k_{pb} - 4\sigma k^d_b + 4\sigma^2 \delta^d_b$. These expressions become much simpler for $k_{ab} = 0$, in which case, we have,

$$B^{(1)}_{ab} = 0, \tag{2.E.18}$$

$$B^{(2)}_{ab} = \epsilon^{(0)}_{acd} \mathcal{D}^c \left(h^{(2)d}_b - 2\delta^d_b \sigma^2 \right). \tag{2.E.19}$$

$B^{(2)}_{ab}$ in Eq. (2.E.19) is symmetric,

$$\epsilon^{(0)abp} B^{(2)}_{ab} = \epsilon^{(0)abp} \epsilon^{(0)}_{acd} \mathcal{D}^c \left(h^{(2)d}_b - 2\delta^d_b \sigma^2 \right) \tag{2.E.20}$$

$$\begin{aligned} &= \left(\delta^b_c \delta^p_d - \delta^b_d \delta^p_c \right) \mathcal{D}^c \left(h^{(2)d}_b - 2\delta^d_b \sigma^2 \right) \\ &= \left(\mathcal{D}^b h^{(2)p}_b \right) + 4 \left(\mathcal{D}^b \sigma^2 \right) - \mathcal{D}^p \left(h^{(2)} \right) \\ &= \mathcal{D}^p \left(8\sigma^2 - \mathcal{D}_c \sigma \mathcal{D}^c \sigma \right) + 4 \left(\mathcal{D}^b \sigma^2 \right) - \mathcal{D}^p \left(12\sigma^2 - \mathcal{D}_c \sigma \mathcal{D}^c \sigma \right) = 0, \end{aligned} \tag{2.E.21}$$

where we have used the second order equations of motion. $B^{(2)}_{ab}$ is traceless,

$$B^{(2)a}_a = \epsilon^{(0)}_{acd} \mathcal{D}^c \left(h^{(2)ad} - 2h^{(0)ad} \sigma^2 \right) = 0, \tag{2.E.22}$$

furthermore $B^{(2)}_{ab}$ is divergence-free,

$$\begin{aligned} \mathcal{D}_a B^{(2)a}_b &= \epsilon^{(0)acd} \mathcal{D}_a \left[\mathcal{D}_c \left(h^{(2)}_{bd} - 2h^{(0)}_{bd} \sigma^2 \right) \right] \\ &= \epsilon^{(0)acd} \mathcal{D}_a \left[\mathcal{D}_c \left(h^{(2)}_{bd} - 2h^{(0)}_{bd} \sigma^2 \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\epsilon^{(0)acd}[\mathcal{D}_a,\mathcal{D}_c]h^{(2)}_{bd} - 2h^{(0)}_{bd}\,\epsilon^{(0)acd}\,\mathcal{D}_a\mathcal{D}_c\sigma^2 \\
&= \frac{1}{2}\epsilon^{(0)acd}\left(R_{bpac}h^{(2)p}_{d} + R_{dpac}h^{(2)p}_{b}\right) \\
&= \frac{1}{2}\epsilon^{(0)acd}\left[\left(-h^{(0)}_{ab}h^{(0)}_{pc} + h^{(0)}_{bc}h^{(0)}_{pa}\right)h^{(2)p}_{d} + \left(-h^{(0)}_{da}h^{(0)}_{pc} + h^{(0)}_{dc}h^{(0)}_{pa}\right)h^{(2)p}_{b}\right] = 0.
\end{aligned} \tag{2.E.23}
$$

The trace and divergence equations for $h^{(2)}_{ab}$ can equivalently be thought of as tracefree and divergence free conditions for $B^{(2)}_{ab}$.

Evolution of the electric and magnetic parts of the Weyl tensor

Here we describe the evolution equation for the electric and magnetic parts of the Weyl tensor. It will be advantageous to define,

$$
\text{curl } T_{ab} = \epsilon^{(0)}_{acd}\mathcal{D}^c T^{d}_{b}\ . \tag{2.E.24}
$$

It follows that,

$$
\begin{aligned}
\text{curl}\,(\text{curl } T_{ab}) &= \epsilon^{(0)}_{acd}\,\mathcal{D}^c\left(\epsilon^{(0)dpq}\,\mathcal{D}_p T_{qb}\right) \\
&= \epsilon^{(0)}_{dac}\epsilon^{(0)dpq}\,\mathcal{D}^c\mathcal{D}_p T_{qb} = \left(\delta^p_a\delta^q_c - \delta^q_a\delta^p_c\right)\mathcal{D}^c\mathcal{D}_p T_{qb} \\
&= \mathcal{D}^c\mathcal{D}_a T_{cb} - \Box^{(3)}T_{ab} \\
&= [\mathcal{D}^c,\mathcal{D}_a]T_{cb} + \mathcal{D}_a\left(\mathcal{D}^c T_{cb}\right) - \Box^{(3)}T_{ab} \\
&= R^{(0)}{}_{c}{}^{mc}{}_{a}\,T_{mb} + R^{(0)}{}_{b}{}^{mc}{}_{a}\,T_{cm} + \mathcal{D}_a\left(\mathcal{D}^c T_{cb}\right) - \Box^{(3)}T_{ab} \\
&= -2T_{ab} + \left(-\delta^c_b\delta^m_a + h^{(0)}_{ab}h^{(0)mc}\right)T_{cm} + \mathcal{D}_a\left(\mathcal{D}^c T_{cb}\right) - \Box^{(3)}T_{ab} \\
&= -\left(\Box + 3\right)T_{ab} + \mathcal{D}_a\left(\mathcal{D}^c T_{cb}\right) + h^{(0)}_{ab}T^{c}_{c}\ .
\end{aligned} \tag{2.E.25}
$$

Thus, if the tensor T_{ab} is traceless and divergence free, the above expression yields,

$$
\text{curl}\,(\text{curl } T_{ab}) = -\left(\Box + 3\right)T_{ab}\ . \tag{2.E.26}
$$

Since the combination $(E^{(2)}_{ab} - \sigma E^{(1)}_{ab})$ and $B^{(2)}_{ab}$ are both traceless and divergence free, both of them satisfy the above identity. A calculation then shows that

$$
\text{curl}\ \left(E^{(2)}_{ab} - \sigma E^{(1)}_{ab}\right) = -B^{(2)}_{ab} - 4\epsilon^{(0)}{}_{(a}{}^{cd}\left(\mathcal{D}_c\sigma\right)E^{(1)}_{b)d}\ . \tag{2.E.27}
$$

On the other hand,

$$
\begin{aligned}
\text{curl } B^{(2)}_{ab} &= \epsilon^{(0)}{}_{a}{}^{cd}\mathcal{D}_c B^{(2)}_{db} \\
&= \epsilon^{(0)}{}_{a}{}^{cd}\mathcal{D}_c\left[\epsilon^{(0)}_{dpq}\,\mathcal{D}^p\left(h^{(2)q}_{b} - 2\delta^q_b\sigma^2\right)\right] \\
&= -h^{(2)}_{ab} + 6\sigma^2 h^{(0)}_{ab} - h^{(0)}_{ab}\left(\mathcal{D}_c\sigma\mathcal{D}^c\sigma\right) + 2\mathcal{D}_a\sigma\mathcal{D}_b\sigma - 2\sigma\mathcal{D}_a\,\mathcal{D}_b\sigma \\
&= E^{(2)}_{ab} - \sigma E^{(1)}_{ab}.
\end{aligned} \tag{2.E.28}
$$

As a result, the evolution equation in terms of the electric part of the Weyl tensor takes the form

$$\left(\Box + 2\right)\left(E^{(2)}_{ab} - \sigma E^{(1)}_{ab}\right) = 4\ \text{curl}\left[\epsilon^{(0)}{}_{(a}{}^{cd}\left(\mathcal{D}_c\sigma\right)E^{(1)}_{b)d}\right]. \tag{2.E.29}$$

and equivalently in terms of the magnetic part of the Weyl tensor takes the form

$$(\Box + 2)\, B^{(2)}_{ab} = 4\epsilon^{(0)}_{cd(a}E^{(1)d}_{b)}\mathcal{D}^c\sigma. \tag{2.E.30}$$

Eq. (2.C.38) can equivalently be thought of as Eq. (2.E.29) or Eq. (2.E.30). In terms of the electric and magnetic parts of the Weyl tensor, the second order equations take much simpler forms. The above analysis is inspired by the corresponding results at spacelike infinity [65, 69, 70].

CHAPTER 3

Horizon hair from Inversion symmetry

3.1 Introduction

In general relativity, diffeomorphisms that preserve fall-off conditions near null infinity give rise to the infinite-dimensional BMS group [17, 18, 74]. In recent years, it has been shown that there are closely related infinite-dimensional symmetries consisting of large gauge transformations for quantum electrodynamics (QED) [10, 75] in Minkowski spacetime. The gauge parameter is an arbitrary function on the sphere $\epsilon_{\mathcal{I}^+}(z, \bar{z})$ at future null infinity. These symmetries enable one to view soft photon theorems in QED as associated Ward identities [1].

Related developments have found that stationary black holes also possess an infinite number of symmetries in the near horizon region [11, 49–53, 76–84][1,2]. Often, the symmetries are diffeomorphisms that preserve a notion of the near horizon geometry or diffeomorphisms that preserve a particular geometric structure on the horizon. Typically, a class of these symmetries is similar to supertranslations at null infinity. It is believed that global charges associated with supertranslations receive contributions from the horizon as well as from null infinity. A complete discussion of conservation laws associated with supertranslations requires a detailed understanding of how the symmetries at the horizon relate to the symmetries at null infinity. However, the precise relation between the horizon and null infinity symmetries has not been sufficiently understood. It is therefore of considerable interest to understand, say, even in a toy model, the relation between the horizon and null infinity symmetries. The aim of this work is to make progress on this issue in the context of the dynamics of a probe Maxwell field on the extreme Reissner-Nordström (ERN) black hole spacetime.

[1] The references are representative of the very large literature on the subject.

[2] The symmetry groups in the referenced papers do not necessarily coincide. This is so because different authors preserve different structures: some prefer to preserve a particular geometric structure on the null surface, whereas others preserve the near horizon geometry. Reference [81] performs the near-horizon asymptotic symmetry analysis for the Reissner-Nordström black hole in the Einstein-Maxwell theory.

To some extent these issues were explored in [1, 11], where electromagnetic soft-hair shockwaves into the Schwarzschild black hole were considered.[3] In these references, the gauge $A_v = 0$ in advanced Bondi coordinates was used. This gauge is natural in analysing how excitations from past null infinity relate to excitations near the future horizon. However, since the advanced Bondi coordinates do not cover future null infinity, the relation between gauge parameters at future null infinity and the future horizon remains unexplored (at best indirect).[4]

The aim of this work is to overcome this shortcoming in a toy model. We hope that the fundamental ideas will find broader applicability. Our toy model is the dynamics of a probe Maxwell field on the exterior of the ERN black hole. The ERN background enjoys a *discrete conformal* symmetry. The symmetry acts as a spatial inversion interchanging the future event horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$: the Couch-Torrence (CT) inversion symmetry [86].[5] This inversion symmetry is the key property of the ERN spacetime used in this work.

The organisation of the rest of the chapter is as follows. The CT symmetry and its action on a probe Maxwell field is presented in section 3.3. In section 3.4 we present a study of a CT invariant gauge fixing for the probe Maxwell field on the ERN background. We show that the CT invariant gauge fixing is closely related to the harmonic gauge in an asymptotic expansion near null infinity. Specifically, we show that in an asymptotic expansion, the harmonic gauge condition is compatible with the CT invariant gauge condition. We then analyse the CT invariant gauge condition near the horizon in an asymptotic expansion. We conclude that at the future horizon too, the gauge parameter is an arbitrary function on the sphere $\epsilon_{\mathcal{H}^+}(z, \bar{z})$ independent of the ingoing Eddington-Finkelstein coordinate v.

How are the two functions $\epsilon_{\mathcal{H}^+}(z, \bar{z})$ and $\epsilon_{\mathcal{I}^+}(z, \bar{z})$ related to each other? The CT invariant gauge condition leads to a fourth order differential equation for the residual gauge parameter. The fourth order equation is difficult to analyse. Motivated by the results of the previous sections, namely the usefulness of the harmonic gauge and conformal transformations, we solve a simpler problem that captures the essential ideas in section 3.5. We study the residual gauge parameter in the harmonic gauge on a spacetime obtained by a conformal rescaling of the ERN spacetime. This spacetime has two asymptotically flat ends (two future null infinities): one representing the future null infinity of the ERN spacetime and the other representing the horizon. The spacetime has the inversion symmetry. We show that the gauge parameter smoothly extends from an arbitrary function on the sphere from one null infinity to the other null infinity in such a way that $\epsilon_{\mathcal{H}^+}(z, \bar{z}) = \epsilon_{\mathcal{I}^+}(z, \bar{z})$.

In section 3.6 we present an expression for the global charge (often called the Iyer-Wald charge) for the probe Maxwell field on the ERN spacetime. The charge integral is well defined when we push the Cauchy surface to $\mathcal{I}^+ \cup \mathcal{H}^+$. The answer is written as a sum of two terms: one at $\mathcal{I}^+$ and the other at $\mathcal{H}^+$. We argue that soft electric hairs on the horizon of the ERN

[3] In a gravitational setting, references [51, 85] consider throwing a soft-hair shockwave into the Schwarzschild black hole; reference [81] considers soft-hair shockwaves in the Reissner-Nordström black hole.

[4] The importance of understanding this relation is emphasised by Chandrasekaran-Flanagan-Prabhu in [53].

[5] The CT inversion also interchanges the past event horizon $\mathcal{H}^-$ and past null infinity $\mathcal{I}^-$. However, for most of the chaper we will be only concerned with the mapping between the future event horizon and future null infinity.

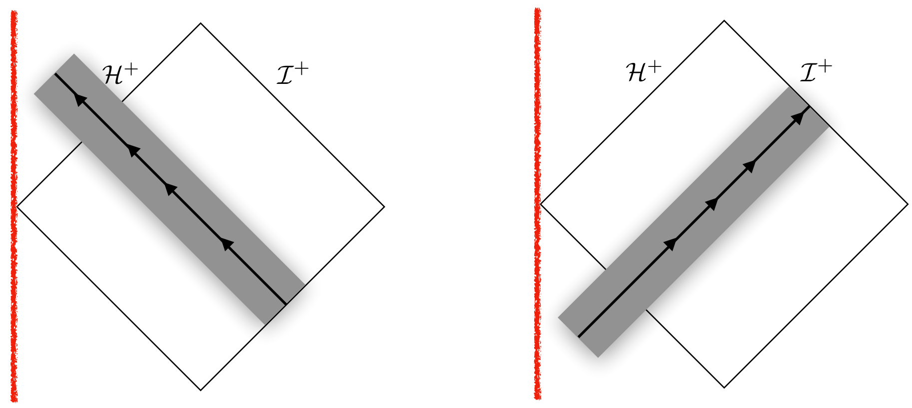

FIGURE 3.1: Under the Couch-Torrence (CT) mapping, a purely ingoing configuration of a probe Maxwell field on the extreme Reissner-Nordström (ERN) background (left) is transformed to a purely outgoing configuration (right) and vice versa. Later in the chapter we show that if the right configuration carries soft charges at future null infinity, the ingoing left configuration carries soft charges at the horizon.

spacetime follows from the CT transformation of configurations that have soft electric hairs at null infinity, and vice versa. This is schematically shown in figure Fig. 3.1.

Finally, in section 3.7 we construct Newman-Penrose and Aretakis like conserved quantities along future null infinity and the future event horizon, respectively. These constants are related via the CT inversion symmetry. This section is an extension of the corresponding mapping understood for the massless scalar field on the ERN spacetime.

3.2 Notations and Conventions

In this chapter we use the sign convention $(-, +, +, +)$ throughout. We list some of the conventions related to this chapter below

- Tensors indices on 4 dimensional spacetimes are denoted by lowercase Roman letters a, b, c etc.
- The pull back of the Couch-Torrence (CT) map will be denoted by $\mathcal{T}_*$. Tildes, like $\tilde{\mathcal{A}}_a$, denote conformally transformed objects.
- The Ricci tensor, R_{ab} is defined by $R_{ab} = R^c{}_{acb}$.

3.3 A probe Maxwell field on the extreme RN background

The ERN solution has a discrete conformal symmetry [86], which acts as a spatial inversion interchanging the future event horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$. The significance of this inversion map for scalar dynamics has been explored by several authors in the physics literature [87–93] and in the mathematical general relativity literature [94]. In this chapter, we are interested in the significance of this inversion map on the dynamics of a probe Maxwell field. We are especially interested in the transformation of the probe Maxwell field under this symmetry.

Let us consider the metric of the 4-dimensional ERN spacetime in static coordinates

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2, \tag{3.3.1}$$

where $d\Omega_2^2$ is the line element of the 2-sphere. Throughout the chapter, we will describe the metric of the 2-sphere in terms of the complex stereographic coordinates $(z, \bar{z})$,

$$d\Omega_2^2 = 2\gamma_{z\bar{z}} dz d\bar{z} = \frac{4}{(1 + z\bar{z})^2} dz d\bar{z}\,. \tag{3.3.2}$$

The ERN metric in static coordinates admits a discrete conformal symmetry under the CT transformation,

$$\mathcal{T} : (t\,, r\,, z\,, \bar{z}) \to \left(t\,, M + \frac{M^2}{r - M}\,, z\,, \bar{z}\right). \tag{3.3.3}$$

The pull-back of this transformation on the metric acts as a conformal transformation

$$\mathcal{T}_*(g) = \omega^2 g \qquad \text{where} \qquad \omega = \frac{M}{r - M}. \tag{3.3.4}$$

The transformation equation (Eq. (3.3.3)) is an involution, i.e. $\mathcal{T}^2 = 1$. On the tortoise coordinate $r_*(r)$, defined by,

$$r_*(r) = r - M - \frac{M^2}{r - M} + 2M \log\left(\frac{|r - M|}{M}\right)\,, \tag{3.3.5}$$

it acts as $\mathcal{T} : r_* \to -r_*$. This, in particular, implies that it interchanges the ingoing and outgoing Eddington-Finkelstein coordinates $v = t + r_*$ and $u = t - r_*$:

$$\mathcal{T} : u \leftrightarrow v. \tag{3.3.6}$$

Hence, it can be concluded that the CT transformation through its action as a spatial inversion interchanges the future event horizon $\mathcal{H}^+$ with future null infinity $\mathcal{I}^+$.

Let us now consider a *probe Maxwell field* on the ERN background. The probe field is different from the Maxwell field under which the ERN black hole is charged. Let us denote the probe field by $\mathcal{A}_a$, with the corresponding field strength written as

$$\mathcal{F}_{ab} = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a. \tag{3.3.7}$$

The field satisfies the source-free Maxwell equations

$$g^{ca} \nabla_c \mathcal{F}_{ab} = 0, \tag{3.3.8}$$

where ∇_a is the covariant derivative. Let us now consider a general conformal transformation of the spacetime metric g_{ab}

$$g_{ab} \to \tilde{g}_{ab} = \Omega^2 g_{ab}\,, \tag{3.3.9}$$

with Ω being the conformal factor. In four spacetime dimensions Maxwell's equations are known to be conformally invariant

$$\tilde{\nabla}^a \mathcal{F}_{ab} \to \Omega^{-2} \nabla^a \mathcal{F}_{ab}, \qquad \tilde{\nabla}_{[a} \mathcal{F}_{bc]} \to \nabla_{[a} \mathcal{F}_{bc]} \tag{3.3.10}$$

with conformal weight zero.

Using equation (Eq. (3.3.4)) in equation (Eq. (3.3.8)), it follows that,

$$0 = \nabla^a \mathcal{F}_{ab} = \mathcal{T}_*(\nabla^a \mathcal{F}_{ab}) = \nabla^a_{\mathcal{T}_*(g)}(\mathcal{T}_* \mathcal{F}_{ab}) = \nabla^a_{\omega^2 g}(\mathcal{T}_* \mathcal{F}_{ab}) = \omega^{-2}(\nabla^a(\mathcal{T}_* \mathcal{F}_{ab})). \tag{3.3.11}$$

Thus, if $\mathcal{F}_{ab}$ is a solution of Maxwell's equations, then so is $\mathcal{T}_* \mathcal{F}_{ab}$. Specifically, in coordinates, if $\mathcal{F}_{ab}(x)$ is a given solution in static coordinates $x^a = \{t, r, z, \bar{z}\}$, then

$$(\mathcal{T}_* \mathcal{F}_{ab})(y) = \left(\frac{\partial x^c}{\partial y^a}\right)\left(\frac{\partial x^d}{\partial y^b}\right) \mathcal{F}_{cd}(x), \tag{3.3.12}$$

is a *different* solution in static coordinates where $y^a = \left\{t, M + \frac{M^2}{r-M}, z, \bar{z}\right\}$. Since the coordinate transformation only changes the radial coordinate, only the radial components of the two-form field pick up additional factors. That is, if $\mathcal{F}_{ir}$ and $\mathcal{F}_{ij}$ for $i = t, z, \bar{z}$ is a solution, then so is

$$\mathcal{F}_{ir}(t, r, z, \bar{z}) = -\frac{M^2}{(r-M)^2} \mathcal{F}_{ir}\left(t, M + \frac{M^2}{r-M}, z, \bar{z}\right), \tag{3.3.13}$$

$$\mathcal{F}_{ij}(t, r, z, \bar{z}) = \mathcal{F}_{ij}\left(t, M + \frac{M^2}{r-M}, z, \bar{z}\right). \tag{3.3.14}$$

In terms of ingoing and outgoing coordinates, if $\mathcal{F}_{vr}(v, r, z, \bar{z})$ and $\mathcal{F}_{z\bar{z}}(v, r, z, \bar{z})$ are components of a given solution in the *ingoing* Eddington-Finkelstein coordinates then,

$$\mathcal{F}_{ur}(u, r, z, \bar{z}) = -\frac{M^2}{(r-M)^2} \mathcal{F}_{vr}\left(u, M + \frac{M^2}{r-M}, z, \bar{z}\right), \tag{3.3.15}$$

and

$$\mathcal{F}_{z\bar{z}}(u, r, z, \bar{z}) = \mathcal{F}_{z\bar{z}}\left(u, M + \frac{M^2}{r-M}, z, \bar{z}\right), \tag{3.3.16}$$

are the components of a *different* solution in *outgoing* Eddington-Finkelstein coordinates.

3.4 Eastwood-Singer Couch-Torrence invariant gauge condition

While solutions of Maxwell's equations are invariant under conformal transformations, this property does not extend to arbitrary gauge fixings of the Maxwell field. For instance, as is well known the harmonic gauge $\nabla^a \mathcal{A}_a = 0$ is not conformally invariant [95]. Under conformal transformations $\tilde{g}_{ab} = \Omega^2 g_{ab}$,

$$\nabla^a \mathcal{A}_a \to \tilde{\nabla}^a \tilde{\mathcal{A}}_a = \Omega^{-2}\left(\nabla^a \mathcal{A}_a + 2\Upsilon^a \mathcal{A}_a\right), \tag{3.4.1}$$

where $\Upsilon_a = \nabla_a \ln \Omega$. Tildes will be used to denote all conformally transformed objects.

A conformally invariant gauge choice for the source-free Maxwell equations was introduced by Eastwood and Singer [96]:

$$\mathcal{D}^a\mathcal{A}_a := \nabla_b\left(\nabla^b\nabla^a - S^{ab}\right)\mathcal{A}_a = 0\,, \tag{3.4.2}$$

where

$$S^{ab} = -2R^{ab} + \frac{2}{3}Rg^{ab}. \tag{3.4.3}$$

The invariance of equation (Eq. (3.4.2)) under conformal transformations for the source-free Maxwell equations can be established via a straightforward, if somewhat, tedious calculation. Consider $\nabla^b\nabla^a\mathcal{A}_a$ and S^{ab} terms separately. The conformal transformation of the tensor S^{ab} defined in equation (Eq. (3.4.3)) is:

$$\tilde{S}^{ab} = \Omega^{-4}\left(S^{ab} + 2\nabla^{(a}\Upsilon^{b)} - 2g^{ab}\nabla^c\Upsilon_c - 4\Upsilon^a\Upsilon^b\right). \tag{3.4.4}$$

The conformal transformation of the term $\nabla^b\nabla^a\mathcal{A}_a$ is:

$$\tilde{\nabla}^b\tilde{\nabla}^a\tilde{\mathcal{A}}_a = \tilde{\nabla}^b\left(\Omega^{-2}\left(\nabla^a + 2\Upsilon^a\right)\tilde{\mathcal{A}}_a\right) = \Omega^{-4}\left(\nabla^b - 2\Upsilon^b\right)\left(\nabla^a + 2\Upsilon^a\right)\mathcal{A}_a \tag{3.4.5}$$

Combining the above two results, the transformation of $\mathcal{D}^a\mathcal{A}_a$ is,

$$\tilde{\mathcal{D}}^a\tilde{\mathcal{A}}_a = \tilde{\nabla}_b\left(\tilde{\nabla}^b\tilde{\nabla}^a - \tilde{S}^{ab}\right)\tilde{\mathcal{A}}_a = \tilde{\nabla}_b\left(\Omega^{-4}\mathcal{V}^b\right), \tag{3.4.6}$$

where

$$\mathcal{V}^b = \left(\nabla^b\nabla^a - S^{ab}\right)\mathcal{A}_a + 2\left[\left(\Upsilon^a\nabla^b - \Upsilon^b\nabla^a\right)\mathcal{A}_a + \mathcal{A}^b\nabla_a\Upsilon^a - \mathcal{A}_a\nabla^a\Upsilon^b\right]. \tag{3.4.7}$$

Next we observe that for an arbitrary vector V^a,

$$\tilde{\nabla}_b\left(\Omega^{-4}V^b\right) = \Omega^{-4}\nabla_b V^b. \tag{3.4.8}$$

To demonstrate the conformal invariance of the Eastwood-Singer gauge, we use equation (Eq. (3.4.8)) in the last expression of equation (Eq. (3.4.6)) to find

$$\tilde{\mathcal{D}}^a\tilde{\mathcal{A}}_a = \Omega^{-4}\nabla_a\mathcal{V}^a = \Omega^{-4}\left(\mathcal{D}^a\mathcal{A}_a + 2\Upsilon^b\nabla^a\mathcal{F}_{ab}\right) = \Omega^{-4}\mathcal{D}^a\mathcal{A}_a. \tag{3.4.9}$$

In arriving at the last equality, we made use of the source-free Maxwell equations $\nabla^a\mathcal{F}_{ab} = 0$. Thus equation (Eq. (3.4.2)) is invariant under conformal transformations.

This gauge condition is also invariant under the CT transformation on the ERN spacetime. Denoting the CT transformed gauge field as $\mathcal{T}_*\mathcal{A}_a$, we have,

$$\mathcal{T}_*\left(\mathcal{D}^a\mathcal{A}_a\right) = \nabla_b^{\mathcal{T}_*(g)}\left(\nabla^b_{\mathcal{T}_*(g)}\nabla^a_{\mathcal{T}_*(g)} - \mathcal{T}_*(S^{ab})\right)\mathcal{T}_*\mathcal{A}_a = \omega^{-4}\mathcal{D}^a\left(\mathcal{T}_*\mathcal{A}_a\right) = 0, \tag{3.4.10}$$

where we have used $\mathcal{T}_*(g) = \omega^2 g$ from equation (Eq. (3.3.4)). Hence, if $\mathcal{A}_a$ is gauge fixed by equation (Eq. (3.4.2)), then the CT transformed gauge field $\mathcal{T}_*\mathcal{A}_a$ also satisfies the same gauge condition.

We are now in position to discuss the residual gauge transformations for the Eastwood-Singer gauge. Under the gauge transformations $\mathcal{A}_a \to \mathcal{A}_a + \nabla_a \epsilon$, the expression in (Eq. (3.4.2)) provides the equation satisfied by the residual gauge parameter ϵ,

$$\mathcal{D}^a \nabla_a \epsilon = \nabla_b \left(\nabla^b \Box + \left(2R^{ab} - \frac{2}{3} R g^{ab} \right) \nabla_a \right) \epsilon = 0 \,, \tag{3.4.11}$$

where $\Box = g^{ab} \nabla_a \nabla_b = g^{ab} \left(\partial_a \partial_b - \Gamma^c_{ab} \partial_c \right)$ is the D'Alembertian operator on curved spacetimes. Equation (Eq. (3.4.11)) simplifies on the ERN spacetime, for which $R = 0$ and $\nabla_a R^{ab} = 0$ (from Einstein's equations). As a result, equation (Eq. (3.4.11)) on the ERN spacetime becomes

$$\left(\Box g^{ab} + 2R^{ab} \right) \nabla_a \nabla_b \epsilon = \Box\Box \epsilon + 2R^{ab} \partial_a \partial_b \epsilon - 2R^{ab} \Gamma^c_{ab} \partial_c \epsilon = 0. \tag{3.4.12}$$

This is a fourth-order equation for ϵ in the exterior of the ERN spacetime. Our interest in residual gauge transformations is largely in the context of soft charges at the asymptotic boundaries of the spacetime, namely at future null infinity $\mathcal{I}^+$ and the future event horizon $\mathcal{H}^+$. We also note that if ϵ is a function satisfying (Eq. (3.4.12)) then so is $\mathcal{T}_* \epsilon$.

To investigate possible solutions for ϵ near $\mathcal{I}^+$ we use the ERN metric in outgoing Eddington-Finkelstein coordinates

$$ds^2 = -\left(1 - \frac{M}{r} \right)^2 du^2 - 2du dr + r^2 d\Omega_2^2. \tag{3.4.13}$$

Let us quickly recall the discussion for the harmonic gauge. Inserting the ansatz [1],

$$\begin{aligned} \epsilon(r,u,z,\bar{z}) \;\; = \;\; & \epsilon^{(0)}(u,z,\bar{z}) + \frac{1}{r} \epsilon^{(1)}(u,z,\bar{z}) + \frac{1}{r} f^{(1)}(u,z,\bar{z}) \log \frac{u}{2r} \\ & + \frac{1}{r^2} \epsilon^{(2)}(u,z,\bar{z}) + \frac{1}{r^2} f^{(2)}(u,z,\bar{z}) \log \frac{u}{2r} + \mathcal{O}(r^{-3}), \end{aligned} \tag{3.4.14}$$

in the scalar equation $\Box \epsilon = 0$, and expanding in powers of large r, we find the following equations order-by-order in inverse powers of r,

$$\partial_u \epsilon^{(0)} = 0, \tag{3.4.15}$$

$$\partial_u f^{(1)} = -\frac{1}{2} D^2 \epsilon^{(0)}, \tag{3.4.16}$$

$$\partial_u f^{(2)} = -\frac{1}{2} D^2 f^{(1)}, \tag{3.4.17}$$

$$\partial_u \epsilon^{(2)} = \frac{1}{2} D^2 f^{(1)} - \frac{1}{2} D^2 \epsilon^{(1)} - \frac{1}{2} f^{(1)} - \frac{1}{u} f^{(2)}, \tag{3.4.18}$$

where D^2 denotes the Laplacian on the 2-sphere. The first of these equations tells us that $\epsilon^{(0)}(u,z,\bar{z})$ is independent of u: $\epsilon^{(0)}(u,z,\bar{z}) =: \epsilon_{\mathcal{I}^+}(z,\bar{z})$.

Using expansion (Eq. (3.4.14)) and equations (Eq. (3.4.15))–(Eq. (3.4.18)), a calculation shows that the gauge condition (Eq. (3.4.12)) is satisfied in an expansion in inverse powers of r. Let us demonstrate how this works. Inserting expansion (Eq. (3.4.14)) in equation (Eq. (3.4.12)), the leading order term is $\mathcal{O}(r^{-3})$, whereas the last two terms of equation (Eq. (3.4.12)) start at $\mathcal{O}(r^{-6})$ and $\mathcal{O}(r^{-5})$ respectively. Hence, the last two terms do not contribute at order $\mathcal{O}(r^{-3})$ and $\mathcal{O}(r^{-4})$. As a result, at these orders, the gauge fixing equation simply becomes

$$\Box\Box\epsilon = 0. \tag{3.4.19}$$

At order $\mathcal{O}(r^{-3})$ equation (Eq. (3.4.19)) gives,

$$\partial_u^2 f^{(1)} = 0, \tag{3.4.20}$$

which is consistent with equations (Eq. (3.4.15))–(Eq. (3.4.18)), in this sense that if those equations are satisfied then (Eq. (3.4.20)) is also satisfied:

$$\partial_u(\partial_u f^{(1)}) = \partial_u\left(-\frac{1}{2}D^2(\epsilon^{(0)})\right) = -\frac{1}{2}D^2(\partial_u\epsilon^{(0)}) = 0. \tag{3.4.21}$$

At order $\mathcal{O}\left(r^{-4}\log\frac{u}{2r}\right)$ equation (Eq. (3.4.12)) or (Eq. (3.4.19)) gives,

$$\partial_u^2 f^{(2)} + \frac{1}{2}D^2(\partial_u f^{(1)}) = 0. \tag{3.4.22}$$

This is also consistent with equations (Eq. (3.4.15))–(Eq. (3.4.18)) as,

$$\partial_u^2 f^{(2)} + \frac{1}{2}D^2(\partial_u f^{(1)}) = \partial_u\left(\partial_u f^{(2)} + \frac{1}{2}D^2 f^{(1)}\right) = 0. \tag{3.4.23}$$

At order $\mathcal{O}\left(r^{-4}\right)$ the details are a little more cumbersome. One finds,

$$\begin{aligned}&D^2D^2\epsilon^{(0)} + 2D^2\epsilon^{(0)} + 4M\partial_u\epsilon^{(0)} + \frac{4}{u}D^2 f^{(1)} + 8\partial_u f^{(1)} + 4D^2(\partial_u f^{(1)}) + 4D^2(\partial_u\epsilon^{(1)})\\ &12\partial_u^2 f^{(2)} + \frac{16}{u}\partial_u f^{(2)} - \frac{8}{u^2}f^{(2)} + 8\partial_u^2\epsilon^{(2)} = 0.\end{aligned} \tag{3.4.24}$$

Again, one can check that this equation is consistent with equations (Eq. (3.4.15))–(Eq. (3.4.18)).[6]

As argued above, if ϵ is a solution to the gauge condition then so is $\mathcal{T}_*\epsilon$. The expansion of

$$\epsilon(r,v,z,\bar{z}) = \mathcal{T}_*(\epsilon(r,u,z,\bar{z})), \tag{3.4.25}$$

near the horizon takes the form, cf. (Eq. (3.3.3)),

$$\begin{aligned}\tilde{\epsilon}(r,v,z,\bar{z}) = \tilde{\epsilon}^{(0)}(v,z,\bar{z}) + \frac{(r-M)}{rM}\left\{\tilde{\epsilon}^{(1)}(v,z,\bar{z}) + \tilde{f}^{(1)}(v,z,\bar{z})\log\left[\frac{v(r-M)}{2rM}\right]\right\}\\ +\frac{(r-M)^2}{r^2M^2}\left\{\tilde{\epsilon}^{(2)}(v,z,\bar{z}) + \tilde{f}^{(2)}(v,z,\bar{z})\left[\frac{v(r-M)}{2rM}\right]\right\} + \mathcal{O}((r-M)^3).\end{aligned} \tag{3.4.26}$$

Using ingoing Eddington-Finkelstein coordinates,

$$ds^2 = -\left(1-\frac{M}{r}\right)^2 dv^2 + 2dvdr + r^2 d\Omega_2^2, \tag{3.4.27}$$

and substituting the series expansion (Eq. (3.4.26)) in gauge condition (Eq. (3.4.12)) we find that the first non-trivial term appears at order $(r-M)^{-1}$. The conditions at orders $(r-M)^{-1}$,

[6] The easiest way to confirm this is to substitute $\partial_u\epsilon^{(2)}$ in (Eq. (3.4.24)) from (Eq. (3.4.18)).

$\log\left[\frac{v(r-M)}{2rM}\right]$, and $\mathcal{O}(1)$ respectively give,

$$\partial_v^2 \tilde{f}^{(1)} = 0, \tag{3.4.28}$$

$$\partial_v^2 \tilde{f}^{(2)} + \frac{1}{2}D^2(\partial_v \tilde{f}^{(1)}) = 0, \tag{3.4.29}$$

and

$$\begin{aligned} &D^2D^2\tilde{\epsilon}^{(0)} + 2D^2\tilde{\epsilon}^{(0)} + 4M\partial_v\tilde{\epsilon}^{(0)} + \frac{4}{v}D^2\tilde{f}^{(1)} + 8\partial_v\tilde{f}^{(1)} + 4D^2(\partial_v\tilde{f}^{(1)}) + 4D^2(\partial_v\tilde{\epsilon}^{(1)}) \\ &12\partial_v^2\tilde{f}^{(2)} + \frac{16}{v}\partial_v\tilde{f}^{(2)} - \frac{8}{v^2}\tilde{f}^{(2)} + 8\partial_v^2\tilde{\epsilon}^{(2)} = 0. \end{aligned} \tag{3.4.30}$$

These equations are identical to the ones obtained earlier in the expansion near null infinity. Thus, we conclude that tilde variables satisfying equations (Eq. (3.4.15))–(Eq. (3.4.18)) (with u replaced with v at all places) determine the gauge parameter for $\tilde{\epsilon}(r, v, z, \bar{z})$ near the horizon as well. At the horizon $\lim_{r\to M} \tilde{\epsilon}(r, v, z, \bar{z}) = \tilde{\epsilon}^{(0)}(z, \bar{z}) =: \epsilon_{\mathcal{H}^+}(z, \bar{z})$: an arbitrary function on the sphere.

To summarise: near null infinity we take the ansatz (Eq. (3.4.14)) for the gauge parameter. The various functions entering the expansion are taken to satisfy equations (Eq. (3.4.15))–(Eq. (3.4.18)), which are those for the harmonic gauge choice at null infinity. Such a choice is consistent with the Eastwood-Singer gauge fixing (Eq. (3.4.12)). Next, using the CT inversion symmetry of the Eastwood-Singer gauge fixing condition, we obtain expansion (Eq. (3.4.26)) near the horizon. The same functions enter the expansion as near null infinity, except that at all places u is replaced by v. We conclude that CT inversion allows us to consider gauge fixing such that at the future horizon too, the gauge parameter is an an arbitrary function on the sphere independent of v. At this stage the two arbitrary functions on the sphere $\epsilon_{\mathcal{H}^+}(z, \bar{z})$ and $\epsilon_{\mathcal{I}^+}(z, \bar{z})$ are not related. In next section we conjecture that it is natural to expect that they are the same.

Note that the gauge parameter in the Eastwood-Singer gauge has conformal weight zero. This is related to it satisfying a specific fourth order differential equation. The gauge parameter in the harmonic gauge on the other hand satisfies a second order differential equation. If we require this equation to be conformally invariant, then the conformal weight of the scalar should be one.

3.5 Hyperboloidal slicing and a toy model problem

In the previous section we showed that large gauge transformations on ERN spacetime consists of the union of two types of transformations. First, where the gauge parameter is an arbitrary function on the sphere independent of v at the horizon. Second, where the gauge parameter is an arbitrary function on the sphere independent of u at null infinity. The two sets are related by the CT inversion symmetry. It is natural to ask if there exists a smooth interpolation between the two.

The Eastwood-Singer residual gauge parameter satisfies a fourth order differential equation. It is difficult to analyse that equation. As we saw in the previous section, in an expansion near null infinity the Eastwood-Singer gauge condition is compatible with the harmonic gauge condition. For this reason, in this section, we study the residual gauge parameter

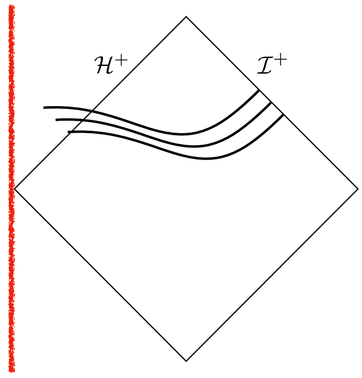

FIGURE 3.2: Schematic drawing of constant s spacelike surfaces in the ERN background. The surfaces reach future null infinity along $u = s$ outgoing null lines. At the future horizon the surfaces intersect at advanced time $v = s$, however, the normal there is timelike.

in the harmonic gauge on a spacetime conformal related to ERN spacetime as a toy model problem. We do so in a convenient 'hyperboloidal' slicing and show that $\epsilon_{\mathcal{H}^+}(z, \bar{z}) = \epsilon_{\mathcal{I}^+}(z, \bar{z})$. We conjecture that the same is true for the Eastwood-Singer gauge parameter in the ERN spacetime.

The hyperboloidal slices [97,98] parameterised by coordinate s (introduced below) intersect future null infinity along an outgoing null line at the retarded time $u = s$. They intersect the future horizon at the advanced time $v = s$, however, the normal to the $s =$ constant surfaces at the future horizon is not null; it is timelike. The key point being that the slices intersect both the future null infinity and the future horizon.

To set up these coordinates, we begin by introducing,

$$\tilde{t} = \frac{t}{4M} \qquad \text{and} \qquad x = \ln\left(\frac{r}{M} - 1\right), \tag{3.5.1}$$

in terms of which the metric takes the form,

$$ds^2 = \frac{16M^2}{(1 + e^{-x})^2}\left(-d\tilde{t}^2 + \cosh^4\left(\frac{x}{2}\right)(dx^2 + d\Omega_2^2)\right). \tag{3.5.2}$$

In contrast to the exterior of the ERN, the spacetime described by the line element in the brackets in equation (Eq. (3.5.2)),

$$d\tilde{s}^2 = -d\tilde{t}^2 + \cosh^4\left(\frac{x}{2}\right)(dx^2 + d\Omega_2^2), \tag{3.5.3}$$

is geodesically complete with $(\tilde{t}, x) \in \mathbb{R}^2$. The Ricci scalar of this spacetime is zero.

This spacetime has two asymptotically flat ends: $x \to \pm\infty$. Asymptotic flatness at these ends can be seen by introducing, say, $\rho = \cosh^2\left(\frac{x}{2}\right)$ and taking $\rho \to \infty$ limit. The reflection symmetry $x \leftrightarrow -x$ is an isometry of the metric $d\tilde{s}^2$. The $x \leftrightarrow -x$ is a conformal symmetry of the metric ds^2. This symmetry is precisely the CT symmetry discussed in the previous sections.

Next we introduce,

$$s = \tilde{t} - \frac{1}{2}\left(\cosh x + \ln(2\cosh x)\right), \tag{3.5.4}$$

and foliate the spacetime by hyperboloidal surfaces of constant s. These surfaces are called hyperboloidal as their asymptotic behavior is similar to standard hyperboloids in Minkowski spacetime [98]. These surfaces are spacelike. The ERN metric in coordinates (s, x, θ, ϕ) takes the form,

$$g_{ss} = -g, \qquad g_{xs} = -\frac{1}{2}g(\sinh x + \tanh x), \tag{3.5.5}$$

$$g_{xx} = g\cosh^4\left(\frac{x}{2}\right)\mathrm{sech}^2(x), \qquad g_{z\bar{z}} = \frac{2g}{(1+z\bar{z})^2}. \tag{3.5.6}$$

where we denote the conformal factor as,

$$g(x) = \frac{16M^2}{(1+e^{-x})^2}. \tag{3.5.7}$$

The inverse metric components take the form

$$g^{ss} = -g^{-1}\mathrm{sech}^2(x), \qquad g^{xs} = -2g^{-1}\mathrm{sech}\, x \tanh\left(\frac{x}{2}\right), \tag{3.5.8}$$

$$g^{xx} = g^{-1}\mathrm{sech}^4\left(\frac{x}{2}\right) \qquad g^{z\bar{z}} = \frac{1}{2}g^{-1}(1+z\bar{z})^2. \tag{3.5.9}$$

From the component g^{ss} it follows that the normal to the constant s surface n_a has the norm,

$$n \cdot n = g^{ss} = -\frac{1}{4M^2}\frac{(1+e^x)^2}{(1+e^{2x})^2}. \tag{3.5.10}$$

The norm goes to zero as $x \to \infty$. Let us calculate the value of u where $s = \text{const}$ hypersurface intersects future null infinity as $x \to \infty$:

$$u = t - r_* = 4M\tau - \left(r - M - \frac{M^2}{r-M} + 2M\ln\left(\frac{r}{M} - 1\right)\right), \tag{3.5.11}$$

$$= 2M\left(2s + e^{-x} + \ln\left[\frac{2\cosh x}{e^x}\right]\right). \tag{3.5.12}$$

Thus,

$$u \to 4Ms \qquad \text{as} \qquad x \to \infty. \tag{3.5.13}$$

The $s = \text{const}$ hypersurface approaches a finite point at null infinity as $x \to \infty$. A similar calculation shows,

$$v = t + r_* = 2M\left(2s + e^x + \ln\left[2\cosh x\, e^x\right]\right). \tag{3.5.14}$$

Thus,

$$v \to 4Ms \qquad \text{as} \qquad x \to -\infty. \tag{3.5.15}$$

i.e., along the $s = \text{const}$ hypersurface we approach a finite point at the future horizon $x \to -\infty$. The norm of the normal $n \cdot n$ approaches $-1/(4M^2)$ as $x \to -\infty$. The slices are schematically shown in figure Fig. 3.2. In the unphysical spacetime $d\tilde{s}^2$ the slices become symmetrical with respect to the left and right null infinities. This is shown in figure Fig. 3.3.

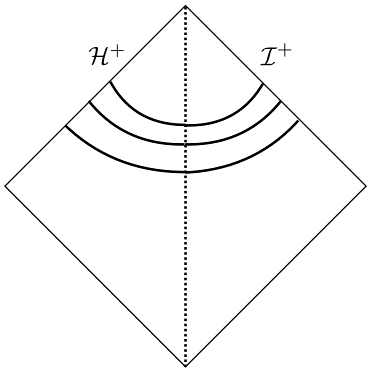

FIGURE 3.3: Schematic drawing of constant s spacelike surfaces in the conformally rescaled ERN background described by the line element (Eq. (3.5.3)). A constant s surface reaches the right future null infinity along the outgoing null line $u = s$ and the left future null infinity along the ingoing null line $v = s$.

We consider the toy model problem (the residual gauge transformations for the harmonic gauge),

$$\widetilde{\Box}\epsilon = 0, \tag{3.5.16}$$

on the spacetime conformally related to the ERN spacetime described by metric (Eq. (3.5.3)). Introducing $\tilde{r} \in (-\infty, \infty)$ such that,

$$\tilde{r} = \int \cosh^2\left(\frac{x}{2}\right) dx = \frac{x}{2} + \frac{\sinh(x)}{2}, \tag{3.5.17}$$

and then $\rho \in (-\infty, \infty)$ such that,

$$\tilde{t} = \tau\sqrt{(1+\rho^2)}, \tag{3.5.18}$$

$$\tilde{r} = \rho\tau, \tag{3.5.19}$$

metric (Eq. (3.5.3)) takes the form,

$$d\tilde{s}^2 = -d\tau^2 + \tau^2\left(\frac{d\rho^2}{1+\rho^2} + \tau^{-2} f(\rho,\tau)^2 d\Omega_2^2\right). \tag{3.5.20}$$

The function $f(\rho, \tau)$ is an implicit function of coordinates ρ, τ,

$$f(\rho,\tau) = \cosh^2\left(\frac{x(\rho,\tau)}{2}\right). \tag{3.5.21}$$

Consider the constant τ slice, say, $\tau = 1$. The metric induced on the slice is

$$ds^2_{\text{slice}} = \frac{d\rho^2}{1+\rho^2} + f(\rho, \tau = 1)^2 d\Omega_2^2. \tag{3.5.22}$$

A graph of $f(\rho, \tau)$ as a function of ρ on the $\tau = 1$ slice is shown in figure Fig. 3.4. In the limit $\rho \to \pm\infty$, function $f(\rho, \tau = 1)$ behaves as $\lim_{\rho\to\pm\infty} f(\rho, \tau = 1) \to |\rho|$. Thus, in the

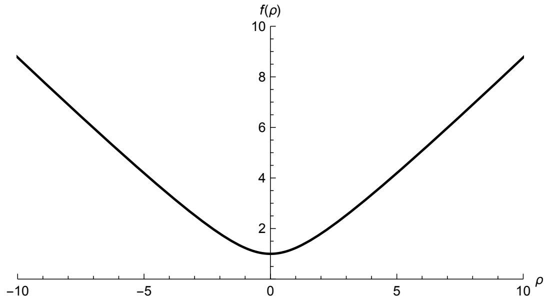

FIGURE 3.4: A graph of $f(\rho,\tau)$ as a function of ρ on the $\tau = 1$ slice.

$\rho \to \pm\infty$ limit the slice is asymptotically Euclidean AdS_3. The slice has two asymptotically AdS_3 ends.[7] On the $\tau = 1$ slice consider

$$\epsilon(\rho,\theta,\phi) = \sum_{l=0}^{\infty} \epsilon_l(\rho) Y_{lm}. \tag{3.5.23}$$

The resulting equation for the function $\epsilon_l(\rho)$ for a fixed l is,

$$(1+\rho^2) f(\rho)^2 \epsilon_l''(\rho) + f(\rho)((2(1+\rho^2) f'(\rho) + \rho f(\rho)))\epsilon_l'(\rho) - l(l+1)\epsilon_l(\rho) = 0. \tag{3.5.24}$$

In the asymptotic region $\rho \to \infty$, the "non-normalisable" solution goes as,

$$\epsilon_l(\rho) \sim \rho^l \, {}_2F_1\left(1 + \frac{1}{2}l, \frac{1}{2}l; l + \frac{3}{2}; -\rho^2\right), \tag{3.5.25}$$

which becomes a constant in the $\rho \to \infty$ limit. With a numerical integration, it can be readily seen that as ρ goes from negative to positive values a symmetric $\epsilon_l(\rho)$ can be found to smoothly interpolate between the same constant values as $\rho \to \pm\infty$. Thus, by summing over spherical harmonics it follows that for this toy model problem, the gauge parameter functions $\epsilon(z,\bar{z})$ at the two null infinities can be taken to be identical. We conjecture that the same can be done for the Eastwood-Singer gauge condition at the future horizon and future null infinity in the ERN spacetime.

Two comments are in order here. First, we have presented the above discussion on the $\tau = 1$ slice, but in fact the discussion is independent of any fixed value of τ. To see this note that for any fixed τ, the induced metric on the constant τ slice is

$$ds_\tau^2 = \tau^2 \left(\frac{d\rho^2}{1+\rho^2} + \tau^{-2} f(\rho,\tau)^2 d\Omega_2^2\right), \tag{3.5.26}$$

where the function $f(\rho,\tau)$ is given in equation (Eq. (3.5.21)). Apart from an overall scaling, the relevant property to understand is the behavior of $\tau^{-1} f(\rho,\tau)$ as a function of ρ. In terms

[7]Additional motivation for working with such a slicing and harmonic gauge fixing comes from the success of such an approach at timelike infinity in flat space [99]. Working in the harmonic gauge, Campiglia and Laddha showed that the angle-dependent large gauge transformations introduced at future null infinity have a natural extension in the interior. The slices then can be pushed to timelike infinity. The gauge transformations at timelike infinity have a well defined action on the asymptotic phase space of massive particles, and the resulting Ward identities are found to be equivalent to Weinberg's soft photon theorem.

of the variable x it is clear that

$$\{\rho, \tau^{-1} f(\rho,\tau)\} = \left\{ \frac{1}{\tau}\left(\frac{x}{2} + \frac{\sinh(x)}{2}\right), \frac{1}{\tau}\cosh^2\left(\frac{x}{2}\right) \right\}. \tag{3.5.27}$$

Figure Fig. 3.4 is precisely the parametric plot for $\tau = 1$ of $\left\{ \frac{1}{\tau}\left(\frac{x}{2} + \frac{\sinh(x)}{2}\right), \frac{1}{\tau}\cosh^2\left(\frac{x}{2}\right) \right\}$. For $\tau \neq 1$ both the x and y axes of the graph get rescaled by a factor of $\frac{1}{\tau}$. In particular, for any τ in the limit $\rho \to \pm\infty$, $\tau^{-1} f(\rho,\tau)$ goes as ρ. Thus, $\tau =$ constant slices all have two asymptotically AdS_3 ends.

Second, in the $\tau \to \infty$ limit the metric

$$ds^2_{i^+} = \frac{d\rho^2}{1+\rho^2} + \tau^{-2} f(\rho,\tau)^2 d\Omega_2^2, \tag{3.5.28}$$

can be thought of as the blow up of the point i^+ for the spacetime described by metric (Eq. (3.5.3)).[8] It can be readily checked that for any finite $\rho \neq 0$, $\tau^{-1} f(\rho,\tau)$ behaves as ρ in the $\tau \to \infty$ limit. In the neighbourhood of $\rho = 0$ it smoothly interpolates between the $\rho > 0$ and the $\rho < 0$ AdS_3 regions.

Studies on interpolating asymptotic dynamics between two different asymptotic regions include [100, 101]. The toy model example studied in this section calls for a corresponding study in four-dimensional asymptotically flat settings.

3.6 Horizon hair and soft charges

For the Maxwell field on flat spacetime, it is now well appreciated that there exists large gauge transformations parametrised by arbitrary non-vanishing gauge parameters on the asymptotic sphere at null infinity [1,10,102]. As a consequence, there exist an infinite number of boundary symmetry charges [53,72]. In this section we present an expression for the global charge (often called the Iyer-Wald charge) for the probe Maxwell field on the ERN spacetime.

For the electromagnetic field, with Lagrangian,

$$L = -\frac{1}{4}\int \sqrt{-g}\mathcal{F}_{ab}\mathcal{F}^{ab} d^4x, \tag{3.6.1}$$

the symplectic form as an integral over an arbitrary Cauchy surface Σ is,

$$\Omega\left(\mathcal{A}, \delta_1\mathcal{A}, \delta_2\mathcal{A}\right) = -\int_\Sigma \left(\delta_1 A^b \delta_2 \mathcal{F}_{ab} - \delta_2 \mathcal{A}^b \delta_1 \mathcal{F}_{ab}\right) n^a \sqrt{h} d^3x, \tag{3.6.2}$$

where n^a is the future pointing unit normal to Σ. The total charge as the generator of gauge transformations is given as,

$$\delta Q_\epsilon = \Omega\left(\mathcal{A}, \delta\mathcal{A}, \delta_\epsilon\mathcal{A}\right). \tag{3.6.3}$$

One finds,

$$Q_\epsilon = \int_\Sigma \nabla^b\left(\epsilon\mathcal{F}_{ab}\right) n^a \sqrt{h} d^3x. \tag{3.6.4}$$

On black hole spacetimes and in the absence of massive particles, a choice of Cauchy surface is $\mathcal{I}^+ \cup \mathcal{H}^+$. The Cauchy surface $\mathcal{I}^+ \cup \mathcal{H}^+$ can be reached by taking the $t \to \infty$ limit

[8]The blow-up in the sense of Ashtekar-Hansen [37]. For a recent discussion of this blow-up in the context of BMS-supertranslations at spacelike infinity see [42].

of constant t Cauchy surfaces Σ_t for the exterior of the ERN black hole. In previous sections, we argued that $\epsilon_{\mathcal{H}^+}(z,\bar{z}) = \epsilon_{\mathcal{I}^+}(z\,\bar{z})$: the gauge parameter is the same function of the sphere coordinates at $\mathcal{I}^+$ and at $\mathcal{H}^+$. As a result, the charge integral (Eq. (3.6.4)) is well defined when we push the Cauchy surface to $\mathcal{I}^+ \cup \mathcal{H}^+$. The total integral splits into two parts,

$$\begin{aligned} Q_\epsilon &:= \lim_{\{\Sigma_t \to \mathcal{I}^+ \cup \mathcal{H}^+\}} \int_\Sigma \nabla^b \left(\epsilon \mathcal{F}_{ab}\right) n^a \sqrt{h} d^3x && (3.6.5)\\ &= \int_{\mathcal{I}^+} \nabla^b \left(\epsilon \mathcal{F}_{ab}\right) n^a \sqrt{h} d^3x + \int_{\mathcal{H}^+} \nabla^b \left(\epsilon \mathcal{F}_{ab}\right) n^a \sqrt{h} d^3x && (3.6.6)\\ &= Q_\epsilon^{\mathcal{I}^+} + Q_\epsilon^{\mathcal{H}^+}. && (3.6.7) \end{aligned}$$

In outgoing Eddington-Finkelstein coordinates (Eq. (3.4.13)), $n^a \sqrt{h} d^3x = \delta^a_u r^2 \gamma_{z\bar{z}} du dz d\bar{z}$. With boundary conditions,

$$\begin{aligned} \mathcal{F}_{ij}\left(u\,, r\,, z\,, \bar{z}\right) &= F_{ij}^{(0)}\left(u\,, z\,, \bar{z}\right) + \mathcal{O}\left(\frac{1}{r}\right), && (3.6.8)\\ \mathcal{F}_{ir}\left(u\,, r\,, z\,, \bar{z}\right) &= \frac{1}{r^2} F_{ir}^{(0)}\left(u\,, z\,, \bar{z}\right) + \mathcal{O}\left(\frac{1}{r^3}\right), && (3.6.9) \end{aligned}$$

where i, j collectively stand for $(u, z, \bar{z})$, the integrand at $\mathcal{I}^+$ becomes

$$\nabla^b \left(\epsilon \mathcal{F}_{ab}\right) n^a \sqrt{h} = -\gamma_{z\bar{z}}\, \epsilon_0\left(z\,, \bar{z}\right) \partial_u F_{ru}^{(0)} + \partial_z \left(\epsilon_0\left(z\,, \bar{z}\right) F_{u\bar{z}}^{(0)}\right) + \partial_{\bar{z}} \left(\epsilon_0\left(z\,, \bar{z}\right) F_{uz}^{(0)}\right). \quad (3.6.10)$$

Hence the $Q_\epsilon^{\mathcal{I}^+}$ contribution in equation (Eq. (3.6.7)) takes the form,

$$Q_\epsilon^{\mathcal{I}^+} = -\int_{\mathcal{I}^+} du dz d\bar{z}\, \gamma_{z\bar{z}}\, \epsilon_0\left(z\,, \bar{z}\right) \partial_u F_{ru}^{(0)}\left(u\,, z\,, \bar{z}\right)\,, \quad (3.6.11)$$

where $\epsilon_0\left(z\,, \bar{z}\right)$ is the limiting value of ϵ on approaching $\mathcal{I}^+$. Equation (Eq. (3.6.11)) is the known expression for the soft charge on asymptotically flat spacetimes in terms of a volume integral over $\mathcal{I}^+$ [1,10].

In ingoing Eddington-Finkelstein coordinates, the integrand of equation (Eq. (3.6.4)) at $\mathcal{H}^+$ $(r = M)$, becomes $n^a \sqrt{h} d^3x = \delta^a_v M^2 \gamma_{z\bar{z}} dv dz d\bar{z}$. With the boundary conditions,

$$\begin{aligned} \mathcal{F}_{ij}\left(v\,, r\,, z\,, \bar{z}\right) &= \bar{F}_{ij}^{(0)}\left(v\,, z\,, \bar{z}\right) + \mathcal{O}(r - M), && (3.6.12)\\ \mathcal{F}_{ir}\left(v\,, r\,, z\,, \bar{z}\right) &= \frac{1}{M^2} \bar{F}_{ir}^{(0)}\left(v\,, z\,, \bar{z}\right) + \mathcal{O}(r - M), && (3.6.13) \end{aligned}$$

where i, j now collectively stand for $(v, z, \bar{z})$, the integrand at $\mathcal{H}^+$ becomes

$$\nabla^b \left(\epsilon \mathcal{F}_{ab}\right) n^a \sqrt{h} = \gamma_{z\bar{z}}\, \epsilon_0\left(z\,, \bar{z}\right) \partial_v \bar{F}_{rv}^{(0)} + \partial_z \left(\epsilon_0\left(z\,, \bar{z}\right) \bar{F}_{v\bar{z}}^{(0)}\right) + \partial_{\bar{z}} \left(\epsilon_0\left(z\,, \bar{z}\right) \bar{F}_{vz}^{(0)}\right). \quad (3.6.14)$$

Hence, $Q_\epsilon^{\mathcal{H}^+}$ contribution to equation (Eq. (3.6.7)) takes the form,

$$Q_\epsilon^{\mathcal{H}^+} = \int_{\mathcal{H}^+} dv dz d\bar{z}\, \gamma_{z\bar{z}}\, \epsilon_0\left(z\,, \bar{z}\right) \partial_v \bar{F}_{rv}^{(0)}\left(v\,, z\,, \bar{z}\right)\,, \quad (3.6.15)$$

in terms of a volume integral over $\mathcal{H}^+$.

Now we can see that the horizon soft charges follow from the CT dual of the null infinity soft charges. We first note from equation (Eq. (3.6.9)) that the CT transformation of

$r^2 \mathcal{F}_{ru}\,(u\,,r\,,z\,,\bar{z})$ is,

$$\mathcal{T}_*\left(r^2 \mathcal{F}_{ru}\,(u\,,r\,,z\,,\bar{z})\right) = \mathcal{T}_*\left(F^{(0)}\,(u\,,z\,,\bar{z}) + \mathcal{O}\left(\frac{1}{r}\right)\right) \tag{3.6.16}$$

At leading order in $(r-M)$, this becomes, via equation (Eq. (3.3.15)),

$$M^2 \mathcal{F}_{rv}\,(v\,,r\,,z\,,\bar{z}) = -F^{(0)}\,(v\,,z\,,\bar{z}) + \mathcal{O}\left(\frac{r-M}{M}\right). \tag{3.6.17}$$

Note that after the transformation, the function $F^{(0)}$ has arguments $(v\,,z\,,\bar{z})$, but it is otherwise the same function. As a result,

$$Q_\epsilon^{\mathcal{H}^+} = -\int_{\mathcal{H}^+} dv dz d\bar{z}\,\gamma_{z\bar{z}}\,\epsilon_0\,(z\,,\bar{z})\,\partial_v F^{(0)}\,(v\,,z\,,\bar{z}) \tag{3.6.18}$$

$$= \mathcal{T}_*\left[\int_{\mathcal{I}^+} du dz d\bar{z}\,\gamma_{z\bar{z}}\,\epsilon_0\,(z\,,\bar{z})\,\partial_u F^{(0)}\,(u\,,z\,,\bar{z})\right] = \mathcal{T}_*\left(Q_\epsilon^{\mathcal{I}^+}\right). \tag{3.6.19}$$

Hence soft electric hair on the horizon of the ERN spacetime follow from the CT transformation on the soft electric hair at null infinity. This is schematically shown in figure Fig. 3.1.

It is natural to expect that the conservation law in the present setting takes the form

$$Q_\epsilon^{\mathcal{I}^+} + Q_\epsilon^{\mathcal{H}^+} = Q_\epsilon^{\mathcal{I}^-} + Q_\epsilon^{\mathcal{H}^-}. \tag{3.6.20}$$

If not on the ERN spacetime, it should be possible to make precise the conservation law following [103] on the spacetime with two asymptotic flat ends considered in section 3.5.

3.7 Aretakis and Newman-Penrose constants

In this section, we write expressions for Aretakis and Newman-Penrose constants for a probe Maxwell field in an ERN background and relate them via the inversion symmetry. This discussion is an extension of the scalar analysis of refs. [88–92]; and is largely independent of the discussion of the previous sections.

We start with the spherical harmonics decomposition of the Maxwell field $\mathcal{A}_a$ in the outgoing Eddington-Finkelstein coordinates cf. (Eq. (3.4.13)). Expanding various components of the Maxwell field in appropriate scalar and vector spherical harmonics we have both even $(-1)^l$ and odd $(-1)^{l+1}$ parity terms, $\mathcal{A}_a dx^a = \mathcal{A}_a^{\text{odd}} dx^a + \mathcal{A}_a^{\text{even}} dx^a$, as

$$\mathcal{A}_a^{\text{odd}} dx^a = \sum_{lm} \alpha_{lm}\left(\partial_z Y^{lm} dz - \partial_{\bar{z}} Y^{lm} d\bar{z}\right), \tag{3.7.1}$$

$$\mathcal{A}_a^{\text{even}} dx^a = \sum_{lm}\left(f_{lm} Y^{lm} du + h_{lm} Y^{lm} dr + \kappa_{lm}(\partial_z Y^{lm} dz + \partial_{\bar{z}} Y^{lm} d\bar{z})\right), \tag{3.7.2}$$

where Y_{lm} are the standard scalar spherical harmonics satisfying,

$$2\gamma^{z\bar{z}}\partial_z\partial_{\bar{z}} Y^{lm} = -l(l+1)Y^{lm}. \tag{3.7.3}$$

The coefficients in the decomposition, $\alpha_{lm}\,, f_{lm}\,, h_{lm}$ and κ_{lm} are functions of r, u. For now we restrict ourselves to $l \geq 1$, we comment on the $l = 0$ case separately below. Even and odd parity perturbations can be fully described by one gauge invariant variable each. These variables satisfy a decoupled wave equation [104] of the form,

$$2\partial_u\partial_r\psi_l - \partial_r\left(g^{rr}\partial_r\psi_l\right) + \frac{l(l+1)}{r^2}\psi_l = 0\,, \tag{3.7.4}$$

in outgoing Eddington-Finkelstein coordinates, and of the form,

$$2\partial_v\partial_r\psi_l + \partial_r\left(g^{rr}\partial_r\psi_l\right) - \frac{l(l+1)}{r^2}\psi_l = 0\,, \tag{3.7.5}$$

in ingoing Eddington-Finkelstein coordinates. For the odd parity perturbation $\psi_l = \alpha_{lm}$ and for the even parity perturbation $\psi_l = \frac{1}{l(l+1)}r^2(\partial_u h_{lm} - \partial_r f_{lm})$. A small calculation shows that,

$$\begin{aligned} \mathcal{F}^{\text{odd}}_{z\bar{z}} &= -2\alpha_{lm}\partial_z\partial_{\bar{z}}Y^{lm}, & (3.7.6)\\ \mathcal{F}^{\text{even}}_{ur} &= (\partial_u h_{lm} - \partial_r f_{lm})\,Y^{lm}. & (3.7.7)\end{aligned}$$

Thus the ψ_l entering in the above equations are essentially (upto numerical factors) the magnetic component $\mathcal{F}_{z\bar{z}}$ for the odd parity field and r^2 times the electric field component $\mathcal{F}_{ur}$ for the even parity field. Let us denote the even and odd parity fields as ψ_l^+ and ψ_l^-. In situations where the distinction is not relevant, we simply denote the two fields collectively as ψ_l.

We first demonstrate that the wave equation (Eq. (3.7.5)) in ingoing Eddington-Finkelstein coordinates admits an infinite tower of Aretakis constants, one for each l at $\mathcal{H}^+$. Our construction parallels the corresponding discussion in section 6.2 of [89]. Let $f(r)$ be a smooth function that is non-vanishing at the horizon, and without loss of generality we set the function $f(r)\big|_{r=M} = 1$. Multiplying equation (Eq. (3.7.5)) by $r^2 f(r)$ and differentiating l times with respect to r and evaluating at $r = M$ we deduce that,

$$A_l[\psi_l] = \frac{M^{l-1}}{(l+1)!}\left[\partial_r^l(r^2 f(r)\partial_r\psi)\right]\Big|_{r=M}, \tag{3.7.8}$$

is conserved along $\mathcal{H}^+$ for $l > 0$, provided the derivatives of the function $f(r)$ at $r = M$ are related by the following set of equations,

$$f^{(k)}\Big|_{r=M} = -\frac{2(l-k)}{2l+1-k}\left(r^{-1}f\right)^{k-1}\Big|_{r=M}, \tag{3.7.9}$$

for $1 \le k \le l$. Equations (Eq. (3.7.9)) determine the constants $f^{(k)}\big|_{r=M}$ recursively. The Aretakis constants (Eq. (3.7.8)) only depend on these derivatives and are independent of the specific choice of the function $f(r)$. A rich class of configurations can be described by an expansion in powers of $(r - M)$ near the horizon as,

$$\psi_l(v,r) = \sum_{k=0}^{\infty} a_k(v)\frac{(r-M)^k}{M^k}. \tag{3.7.10}$$

We can readily calculate the form of the Aretakis charges for the solution of the form (Eq. (3.7.10)). It gives,

$$A_l = a_l + a_{l+1} \qquad \text{for} \qquad l \geq 1. \tag{3.7.11}$$

We now construct the Newman-Penrose constants. A rich class of configurations can be described as an expansion in inverse powers of r near null infinity in outgoing coordinates as,

$$\psi_l(u,r) = \sum_{k=0}^{\infty} b_k(u) \left(\frac{M}{r}\right)^k . \tag{3.7.12}$$

Inserting this expansion into equation (Eq. (3.7.4)) and looking at successive inverse powers of r gives a set of linear equations. These equations can be expressed concisely in terms of Pascal matrices as first discussed in [92]. We follow the same strategy. For a given l we look at the set of equations in powers of r coming from the first $l+2$ terms in the expansion (Eq. (3.7.12)), i.e., the first $l+1$ equations involving $b_0, b_1, \ldots, b_l, b_{l+1}$. We organise these equations using $(l+1)\times(l+1)$ matrices whose components are labelled by $i = 0, 1, \ldots, l$. We consider the vector $\mathbf{b}$ whose components are b_i, and the vector $\dot{\mathbf{b}}_+$ whose components are $(\dot{\mathbf{b}}_+)_i = \partial_u b_{i+1}$. The equations of motion can then be summarised as

$$M\mathsf{N}_l\dot{\mathbf{b}}_+ = [\tfrac{1}{2}l(l+1) - \mathsf{P}_l]\mathbf{b}, \tag{3.7.13}$$

where N_l is the diagonal matrix $(\mathsf{N}_l)_{ij} = (i+1)\delta_{ij}$ and P_l is a lower triangular matrix with entries $(\mathsf{P}_l)_{ij} = \frac{1}{2}i(i+1)\delta_{i,j} - (i^2-1)\delta_{i,j+1} + \frac{1}{2}(i-2)(i+1)\delta_{i,j+2}$.

The matrix P_l can be diagonalised as

$$\mathsf{P}_l = \mathsf{L}_l\mathsf{T}_l\mathsf{L}_l^{-1}, \tag{3.7.14}$$

where T_l is a diagonal matrix with entries $(\mathsf{T}_l)_{ij} = \frac{1}{2}i(i+1)\delta_{i,j}$. The matrix L_l can be written as a product of two matrices J_l and $\widetilde{\mathsf{L}}_l$, $\mathsf{L}_l = \mathsf{J}_l\widetilde{\mathsf{L}}_l$. The matrix J_l has components

$$(\mathsf{J}_l)_{ij} = (i+1)\delta_{i,j}\,, \tag{3.7.15}$$

while $\widetilde{\mathsf{L}}_l$ has components,

$$\widetilde{\mathsf{L}}_l = \begin{pmatrix} 1 & 0 \\ 0 & {}^{i-1}C_{j-1} \end{pmatrix}, \quad \text{for} \quad 1 \leq i,j \leq l, \tag{3.7.16}$$

where ${}^pC_q$ are the binomial coefficients. The $(\widetilde{\mathsf{L}}_l)_{ij}$ components for $i,j \geq 1$ are those of the Pascal matrices [105]. The inverse of L_l is thus $\mathsf{L}_l^{-1} = (\widetilde{\mathsf{L}}_l)^{-1}\mathsf{J}_l^{-1}$, where J_l^{-1} has components $(\mathsf{J}_l^{-1})_{ij} = \frac{1}{i+1}\delta_{i,j}$, while $(\widetilde{\mathsf{L}}_l)^{-1}$ has components

$$\widetilde{\mathsf{L}}_l = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{i+j-2}\,.\,{}^{i-1}C_{j-1} \end{pmatrix}, \quad \text{for} \quad 1 \leq i,j \leq l, \tag{3.7.17}$$

It follows that

$$M\mathsf{L}_l^{-1}\mathsf{N}_l\dot{\mathbf{b}}_+ = [\tfrac{1}{2}l(l+1) - \mathsf{T}_l]\mathsf{L}_l^{-1}\mathbf{b}. \tag{3.7.18}$$

Since, the last component of the matrix T_l is $\frac{1}{2}l(l+1)$, the right hand side of the last component of this matrix equation is zero. It implies conservation of $(\mathsf{L}_l^{-1}\mathsf{N}_l\mathbf{b}_+)_l$. A short calculation shows that this quantity is,

$$N_l = \sum_{i=1}^{l}(-1)^{l+i-2}\cdot {}^{l-1}C_{i-1}b_{i+1}, \quad \text{for} \quad l \geq 1, \tag{3.7.19}$$
$$\partial_u N_l = 0. \tag{3.7.20}$$

The constants N_l at null infinity are called the Newman-Penrose constants. Newman and Penrose in [106] wrote their expressions as surface integrals over $\mathcal{I}^+$ for arbitrary l. They can be seen to be related to the constants derived above.

Let us now comment on the $l = 0$ mode. For $l = 0$ we only have the even field component,

$$\mathcal{A}_a^{\text{even}} = (f_{00}(u,r), h_{00}(u,r), 0, 0)\,. \tag{3.7.21}$$

This provides only one Maxwell field component,

$$\mathcal{F}_{ur} = (\partial_u h_{00} - \partial_r f_{00}) =: \beta_{00}. \tag{3.7.22}$$

Maxwell's equations simply become,

$$\partial_r\left(r^2\beta_{00}\right) = 0,$$
$$\partial_u\left(\beta_{00}\right) = 0. \tag{3.7.23}$$

These equations have the solution

$$\beta_{00} = \frac{c}{r^2}, \tag{3.7.24}$$

where c is a constant — the electric charge. A similar analysis holds for the ingoing Eddington-Finkelstein coordinates. Thus, the $l = 0$ Aretakis constant and the $l = 0$ Newman-Penrose can be taken to be the electric charge.

We now discuss how the Aretakis charges map to Newman-Penrose charges under the CT transformation. To do so, we recall that a solution of the probe Maxwell field near $\mathcal{I}^+$ can be determined from a known solution at $\mathcal{H}^+$ and vice-versa. The CT transformation on $\mathcal{F}_{ru}$ acts as, cf. (Eq. (3.3.15)),

$$\mathcal{T}_*\left(r^2\mathcal{F}_{ru}drdu\right) \to -\mathcal{T}_*\left(r^2\mathcal{F}_{rv}\right)drdv, \tag{3.7.25}$$

and the other non-radial components of the electromagnetic field are unaffected. As a result, the even parity field ψ_l^+ picks up an additional minus sign and the odd parity field remains the same. With this understanding, let us now work out the CT transformation of the class of configurations described by equation (Eq. (3.7.10)). We have,

$$\psi_l(u,r) = \mathcal{T}_*\psi_l(v,r) = \mathcal{T}_*\left(\sum_{k=0}^{\infty} a_k(v)\left(\frac{r-M}{M}\right)^k\right)$$
$$= \sum_{k=0}^{\infty} a_k(u)\left(\frac{M}{r}\right)^k\left(1-\frac{M}{r}\right)^{-k}$$

$$= \left[a_0 + a_1 \frac{M}{r} + (a_1 + a_2) \left(\frac{M}{r}\right)^2 + (a_1 + 2a_2 + a_3) \left(\frac{M}{r}\right)^3 + \cdots \right]. \tag{3.7.26}$$

Comparing equation (Eq. (3.7.26)) with equation (Eq. (3.7.12)), we find the transformation between coefficients a_i and b_i,

$$\mathbf{b} = \widetilde{\mathsf{L}}_l \mathbf{a}, \tag{3.7.27}$$

which implies,

$$\mathbf{b}_+ = \mathsf{L}_l \mathbf{a}_+, \tag{3.7.28}$$

where L_l is simply the lower triangular Pascal matrix. In the discussion after equation (Eq. (3.7.18)) we noted that the $(l+1)$-th component of the column matrix $\mathsf{L}_l^{-1}\mathsf{N}_l\dot{\mathbf{b}}_+$ provides the Newman-Penrose constants. It then follows that the Newman-Penrose constants for the transformed solutions are the $(l+1)$-th component of the vector,

$$\mathsf{L}_l^{-1}\mathsf{N}_l\mathsf{L}_l\mathbf{a}_+. \tag{3.7.29}$$

A short calculation gives the Newman-Penrose constants for the transformed configuration as

$$N_l = a_l + a_{l+1}, \tag{3.7.30}$$

which are nothing but the Aretakis constants.

Finally, let us discuss the time-independent solutions of the scalar wave equation. We consider the wave equation in $\{t\,,r\,,z\,,\bar{z}\}$ coordinates. For $l \geq 1$, the odd and even parity equations both take the form,

$$\left[-\partial_t^2 + \partial_{r_*}^2\right] \psi_l(t,r) = g^{rr} \frac{l\,(l+1)}{r^2} \psi_l(t,r). \tag{3.7.31}$$

The $l \neq 0$ time independent solutions of equation (Eq. (3.7.31)) are

$$\psi_l(r) = \frac{1}{(r-M)^{l+1}} \left((l+1)\,r - M\right), \tag{3.7.32}$$

$$\psi_l(r) = (r-M)^l (M + lr). \tag{3.7.33}$$

Under the CT transformation, one static solution goes to the other,

$$\mathcal{T}_* \left((r-M)^l (M+lr)\right) = \frac{M^{2l+1}}{(r-M)^{l+1}} \left((l+1)\,r - M\right). \tag{3.7.34}$$

3.8 Conclusions

In this chapter, we have investigated certain properties of solutions of a probe Maxwell field on the exterior of the extreme Reissner-Nordström (ERN) black hole spacetime. We demonstrated in section 3.3 that Maxwell's equations are invariant under the Couch-Torrence (CT) transformation of the spacetime, which maps the future null infinity to the future event horizon and vice versa. This in particular implies that asymptotic solutions of Maxwell's equations at null infinity can be mapped to analogous solutions near the event horizon.

In section 3.4, we showed that the Eastwood-Singer conformally invariant gauge fixing [96] is invariant under the CT symmetry of the spacetime. In Eddington-Finkelstein coordinates, using the known asymptotic solutions of the residual gauge parameters at null infinity [1, 10], we demonstrated that solutions of the residual gauge parameters near the future event horizon have the same exact form as the solutions at null infinity.

This raises the interesting question on whether there exists a smooth interpolation in the bulk between solutions of the residual gauge parameters at the future event horizon and at future null infinity. We argued in the affirmative for the existence of such an interpolation through our analysis in section 3.5. In this section, we mostly studied a toy model problem. It will be interesting to further understand these bulk interpolating solutions. More generally, it will be interesting to understand asymptotic symmetries for spacetimes with two (or more) asymptotically flat ends. Studies on interpolating asymptotic dynamics between two different asymptotic regions in the context of AdS_3 spacetimes include [100, 101]. The toy model example studied in this chapter calls for a corresponding study in four-dimensional asymptotically flat settings.

We investigated conserved charges for the probe Maxwell field on the ERN spacetime in section 3.6. We used the expression for the Iyer-Wald charge to define globally conserved charges for the probe Maxwell field on arbitrary Cauchy slices of the spacetime. One such slice is the union of the future null infinity and the future event horizon. We argued that, soft electric charges on the horizon of the ERN spacetime follow from the CT transformation on the soft electric charges at null infinity. Soft electric charges on the event horizon are often called soft horizon hair. This is schematically shown in figure Fig. 3.1.

Finally, in section 3.7 we constructed Newman-Penrose and Aretakis like constants along future null infinity and the future event horizon, respectively. We showed that these constants are related via the CT inversion symmetry.

CHAPTER 4

Spin dependent Gravitational tail memory in $D = 4$

4.1 Introduction

The observation of the permanent displacement between the mirrors of the gravitational wave detector relative to their initial distance after the passage of full gravitational radiation produced from an astrophysical scattering event is known as gravitational memory. Non-oscillatory sources, e.g. scattering of two un-bounded compact objects in hyperbolic orbit contributes to "linear memory", which can be read off from the matter energy-momentum tensor determined in terms of the trajectories of the scattered objects [7, 107, 108]. In companion with this, there will also be "non-linear memory" due to gravitational radiation from the gravitational waves produced during the scattering process, which can be read off studying gravitational energy-momentum tensor [109–113]. The nonlinear memory effect is always present for any gravitational scattering process whether it is bounded or not. In the recent future there is a hope of direct detection of gravitational memory in the upcoming gravitational wave detectors [114, 115].

In recent years, there has been a proposal of another kind of gravitational memory known as "tail memory", which describes how the mirrors of a gravitational wave detector behaves at large retarded time before reaching their permanent displaced position [116–120]. The late and early time gravitational waveforms responsible for gravitational tail memories are first conjectured from the classical limit of soft graviton theorem with some infrared regulator prescription [117, 121, 122], and then derived explicitly in the name of classical soft graviton theorem [118, 119]. It has also been shown that these late and early time behaviours of gravitational waveform are related to the radiative mode of low frequency gravitational waveform via Fourier transformation in the frequency variable. In [118, 119], the setup considered for the scattering event is schematically described in Fig. 4.1. Consider M number of objects are coming in from asymptotic past with masses $\{m'_a\}$, momenta $\{p'_a\}$, spins $\{\Sigma'_a\}$ going through some unspecified interaction within region $\mathcal{R}$ of spacetime size L and disperse to N number of objects with masses $\{m_a\}$, momenta $\{p_a\}$, spins $\{\Sigma_a\}$. These incoming and outgoing objects can also be massless radiations. We choose the origin of coordinate system (i.e. scattering centre) well inside this region $\mathcal{R}$ and we place the gravitational wave detector at a distance R from this origin, along the direction $\hat{n}$.

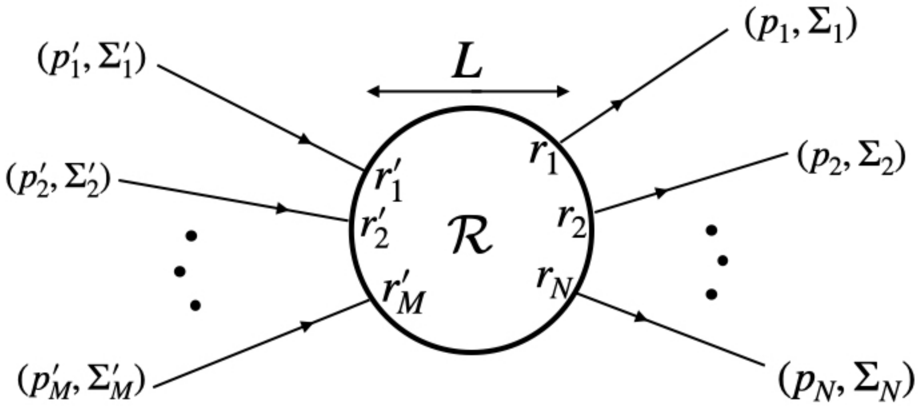

FIGURE 4.1: The setup of the gravitational scattering event.

Now we define the deviation of metric from Minkowski background as:

$$h_{\mu\nu}(x) \equiv \frac{1}{2}\big(g_{\mu\nu}(x) - \eta_{\mu\nu}\big) \qquad , \qquad e_{\mu\nu}(x) \equiv h_{\mu\nu}(x) - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta}(x) \tag{4.1.1}$$

The time Fourier transform of the trace reversed metric is defined as:

$$\widetilde{e}^{\mu\nu}(\omega, \vec{x}) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, e^{\mu\nu}(t, \vec{x}) \tag{4.1.2}$$

With this scattering setup the goal is to determine the radiative mode of $\widetilde{e}^{\mu\nu}(\omega, R\hat{n})$ in the frequency range $R^{-1} << \omega << L^{-1}$. In [118,119] the first three non-analytic contribution in $\omega \to 0$ limit has been evaluated which behave like $\omega^{-1}, \ln\omega$ and $\omega(\ln\omega)^2$ and the coefficients depend only on the incoming and outgoing momenta and $\hat{n}$. These low-frequency gravitational waveforms contribute to DC gravitation memory, u^{-1} tail memory, and $u^{-2}\ln u$ tail memory respectively. To derive these results the authors developed an iterative prescription of solving the geodesic equation of scattered objects and Einstein equation treating gravitational constant G as an iterative parameter. Since $GM\omega$ is a dimensionless quantity in the unit where speed of light is unity, expansion in the power of ω is equivalent to expansion in the power of G, where M represents the mass or momentum of scattered objects. Generalizing this prescription the structure of leading non-analytic contribution in each iterative order has been conjectured in [119], which has been summarized in the second column of the table given in Fig. 4.2.

0-loop (tree) $\mathcal{O}(G)$	ω^{-1} $\theta(u)$	ω^{0} $\delta(u)$	ω $\delta'(u)$	...		
1-loop $\mathcal{O}(G^2)$	$\ln\omega$ u^{-1}	ω^{0} $\delta(u)$	Spin dependent $\omega\ln\omega$ u^{-2}	ω $\delta'(u)$	$\omega^2\ln\omega$ u^{-3}	...
2-loop $\mathcal{O}(G^3)$	$\omega(\ln\omega)^2$ $u^{-2}\ln u$	$\omega\ln\omega$ u^{-2}	ω $\delta'(u)$	$\omega^2(\ln\omega)^2$ $u^{-3}\ln u$	$\omega^2\ln\omega$ u^{-3}	...
n-loop $\mathcal{O}(G^{n+1})$	$\omega^{n-1}(\ln\omega)^n$ $u^{-n}(\ln u)^{n-1}$	$\omega^{n-1}(\ln\omega)^{n-1}$ $u^{-n}(\ln u)^{n-2}$	...			

FIGURE 4.2: This table summarizes different orders of low frequency gravitational waveform $\tilde{e}^{\mu\nu}(\omega, R, \mathbf{n})$ in $\omega \to 0$ limit [written in green] and their relations to post-Minkowskian (PM) expansion. It also describe how the late and early time gravitational waveform $e^{\mu\nu}(u, R\hat{n})$ behaves at large retarded time u [written in blue]. The spin dependent order $\omega\ln\omega$ gravitational waveform indicated inside the blue cell is the primary interest of this chapter.

If the scattered objects carry spin Σ, along with $GM\omega$ there will be another dimensionless quantity $G\Sigma\omega^2$. This simple dimensional analysis tells us that the spin dependence of gravitational waveform at any iterative order in G will carry an extra factor of ω relative to the spin-independent leading non-analytic contribution at that order. With this observation, from the table in Fig. 4.2 we see that at the order G of the gravitational waveform, spin dependence appears at order ω^0. Since the order ω^0 term is analytic in $\omega \to 0$ limit it does not contribute to displacement kind of memory [123]. Now in the next iterative order i.e. in the order G^2 of the gravitational waveform the spin dependence comes at order $\omega\ln\omega$, which is non-analytic in $\omega \to 0$ limit and contributes to order u^{-2} tail memory[1]. The existence of spin-dependent u^{-2} tail memory was first pointed out in the subsection-(5.3) of [119], using some naive analysis of matter and gravitational energy-momentum tensor, indicated within the blue cell in the table of Fig. 4.2. From the table, it is also clear that the spin-dependent order $\omega\ln\omega$ waveform is not exact, but receives corrections from order G^3 as well. The order G^3 correction to $\omega\ln\omega$ gravitational waveform is expected to be spin-independent.

In this chapter, our main goal is to derive the $\omega\ln\omega$ gravitational waveform at order G^2 and the gravitational tail memory it predicts. The main result of the chapter is summarised in Eq. (4.3.9) and Eq. (4.3.10). In section 4.3 we make a conjecture on the $\omega\ln\omega$ gravitational waveform from the classical limit of sub-subleading soft graviton theorem and discuss various applications of the result in different limits. Then in section 4.4 we derive the $\omega\ln\omega$

[1]Here we want to emphasize that the order u^{-2} tail memory is there even for the scattering of non-spinning objects as $G\mathbf{L}\omega^2$ for $\mathbf{L}$ being the orbital angular momentum of the scattered object, is also a dimensionless quantity like $G\Sigma\omega^2$. Hence in our final result of order u^{-2} tail memory, setting the spins of the scattered objects to zero, one can read off the result of the gravitational waveform for non-spinning object's scattering.

gravitational waveform directly for a classical scattering process without any reference to soft graviton theorem. Various appendices discuss intermediate steps required in the analysis of section 4.4. Finally, in section 4.5 we conclude the chapter by discussing the novel features of our result, its theoretical and observational importance, and possibility of re-derivation of our result with other available prescriptions in the literature.

4.2 Notations and Conventions

In this chapter we use the sign convention $(-,+,+,+)$ throughout. We list some of the conventions related to this chapter below

- Tensor indices on 4 dimensional spacetimes are denoted by lowercase Greek letters μ, ν, α, β etc.
- $G_r(k)$ denotes the momentum space retarded Green's function for the scalar d'Alembertian in Minkowski spacetime.

4.3 Prediction of spin dependent gravitational waveform from classical limit of soft graviton theorem

Emboldened by the success of predicting long wavelength gravitational waveform and gravitational tail memory from classical limit of quantum soft graviton theorem at subleading and sub-subleading orders [117, 118], here we proceed to conjecture the leading spin dependent gravitational tail memory in $D = 4$ spacetime dimensions.

In spacetime dimensions $D > 4$, the universal piece of quantum soft graviton operator up to sub-subleading order takes the following form [124]:

$$\mathbf{S}^{gr}_{uni} \quad = \quad \sum_{a=1}^{M+N} \left[\frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} + \frac{\varepsilon_{\mu\nu} p_a^\mu k_\rho \widehat{\mathbf{J}}_a^{\rho\nu}}{p_a.k} + \frac{1}{2} \frac{\varepsilon_{\mu\nu} k_\rho k_\sigma \widehat{\mathbf{J}}_a^{\rho\mu} \widehat{\mathbf{J}}_a^{\sigma\nu}}{p_a.k} \right] \tag{4.3.1}$$

Note that we have not included the non-universal piece in the sub-subleading soft factor which is determined in terms of non-minimal coupling of hard particle field to soft graviton field via Riemann tensor along with the three point 1PI vertex involving two hard particles and a soft graviton [124]. In the above expression of soft graviton operator ε represents the polarization tensor of the soft graviton with momentum $k = \omega \mathbf{n}$, where ω is the energy of the outgoing soft graviton. The scattering process involves M number of ingoing hard particles and N number of outgoing hard particles. Momentum of hard particle-a is denoted by p_a and angular momentum operator is given by:

$$\widehat{\mathbf{J}}_a^{\mu\nu} \equiv p_a^\nu \frac{\partial}{\partial p_{a\mu}} - p_a^\mu \frac{\partial}{\partial p_{a\nu}} + \widehat{\mathbf{\Sigma}}_a^{\mu\nu} \tag{4.3.2}$$

where $\widehat{\mathbf{\Sigma}}_a^{\mu\nu}$ represents the quantum spin generator of Lorentz group for hard particle-a. We are following the convention that the outgoing particles carry positive energy and the incoming particles carry negative energy.

From the classical limit of quantum soft graviton theorem up to sub-subleading order in D=4, we get the following radiative mode of gravitational waveform with frequency ω at distance $R = |\vec{x}|$ from the scattering centre [117, 121, 122]:

$$\tilde{e}^{\mu\nu}(\omega,\vec{x}) = (-i)\,\frac{2G}{R}\,e^{i\omega R}\exp\Big[-2iG\ln\{(\omega+i\epsilon)R\}\sum_{b=1}^{N}p_b.k\Big] \times \sum_{a=1}^{M+N}\left[\frac{p_a^\mu p_a^\nu}{p_a.k} - i\,\frac{p_a^{(\mu}k_\rho \mathbf{J}_a^{\rho\nu)}}{p_a.k} - \frac{1}{2}\frac{\varepsilon_{\mu\nu}k_\rho k_\sigma \mathbf{J}_a^{\rho\mu}\mathbf{J}_a^{\sigma\nu}}{p_a.k}\right] \tag{4.3.3}$$

where $\mathbf{J}_a^{\mu\nu}$ is the classical angular momentum of particle-a expressed in terms of trajectory X_a and classical spin Σ_a of particle-a by the following relation:

$$\mathbf{J}_a^{\mu\nu} = X_a^\mu p_a^\nu - X_a^\nu p_a^\mu + \Sigma_a^{\mu\nu} \tag{4.3.4}$$

In Eq. (4.3.3), the symmetrization convention we are following is: $A^{(\alpha}B^{\beta)} \equiv \frac{1}{2}(A^\alpha B^\beta + A^\beta B^\alpha)$. It is well known that in four spacetime dimensions the asymptotic trajectories of the scattered particles receive logarithmic correction due to long range gravitational force [117, 122] (e.g. if the asymptotic trajectory of particle-a is represented by $X_a^\mu(\sigma) = r_a^\mu + v_a^\mu\sigma + Y_a^\mu(\sigma)$ for proper time σ, then $Y_a(\sigma)$ behaves like $\ln|\sigma|$ for $\sigma\to\pm\infty$). So following the prescription of [117, 122], if we replace $\ln|\sigma|$ by $-\ln(\omega+i\epsilon\eta_a)$ in the classical angular momentum of Eq. (4.3.3), it predicts the correct gravitational waveform, which has been independently verified in [118, 119] up to sub-subleading order non-spinning particle scattering. Here once again using the same prescription, from Eq. (4.3.3) we get,

$$\begin{aligned}
&\tilde{e}^{\mu\nu}(\omega,\vec{x})\\
=\;& (-i)\,\frac{2G}{R}\,e^{i\omega R}\;\exp\Big[-2iG\ln\{(\omega+i\epsilon)R\}\sum_{b=1}^{N}p_b.k\Big]\\
&\times\Bigg[\sum_{a=1}^{M+N}\frac{p_a^\mu p_a^\nu}{p_a.k} + \sum_{a=1}^{M+N}\frac{p_a^{(\mu}k_\rho}{p_a.k}\Bigg\{\Big(p_a^{\nu)}\frac{\partial}{\partial p_{a\rho}} - p_a^\rho\frac{\partial}{\partial p_{a\nu)}}\Big)K_{gr}^{cl} - i\Big(r_a^\rho p_a^{\nu)} - r_a^{\nu)}p_a^\rho + \Sigma_a^{\rho\nu)}\Big)\Bigg\}\\
&+\frac{1}{2}\sum_{a=1}^{M+N}\frac{k_\rho k_\sigma}{p_a.k}\Bigg\{\Big(p_a^\mu\frac{\partial}{\partial p_{a\rho}} - p_a^\rho\frac{\partial}{\partial p_{a\mu}}\Big)K_{gr}^{cl} - i\Big(r_a^\rho p_a^\mu - r_a^\mu p_a^\rho + \Sigma_a^{\rho\mu}\Big)\Bigg\}\\
&\times\Bigg\{\Big(p_a^\nu\frac{\partial}{\partial p_{a\sigma}} - p_a^\sigma\frac{\partial}{\partial p_{a\nu}}\Big)K_{gr}^{cl} - i\Big(r_a^\sigma p_a^\nu - r_a^\nu p_a^\sigma + \Sigma_a^{\sigma\nu}\Big)\Bigg\}\Bigg]
\end{aligned} \tag{4.3.5}$$

where the expression of K_{gr}^{cl} is given by [119]:

$$\begin{aligned}
K_{gr}^{cl} &= -\frac{i}{2}\,(8\pi G)\sum_{\substack{b,c\\ b\neq c}}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\,G_r(\ell)\frac{1}{p_b.\ell+i\epsilon}\frac{1}{p_c.\ell-i\epsilon}\,\Big\{(p_b.p_c)^2 - \frac{1}{2}p_b^2p_c^2\Big\}\\
&= -\frac{i}{2}\,(2G)\sum_{\substack{b,c\\ b\neq c\\ \eta_b\eta_c=1}}\ln\Big\{L(\omega+i\epsilon\eta_b)\Big\}\,\frac{(p_b.p_c)^2 - \frac{1}{2}p_b^2p_c^2}{\sqrt{(p_b.p_c)^2 - p_b^2p_c^2}}
\end{aligned} \tag{4.3.6}$$

In the above expression $\eta_b = +1$ if the particle-b is outgoing and $\eta_b = -1$ if the particle-b is ingoing in the scattering event. In the expression of K_{gr}^{cl}, L^{-1} denotes the UV energy scale and for the value of ω below this UV scale our result for the gravitational waveform can be trusted. Roughly, the length scale $L \simeq |r_a - r_b|$ is of the order of the impact parameters for a scattering event involving un-bounded hyperbolic orbits of compact objects.

Now expanding the exponential in small ω limit we find the following order $\mathcal{O}(\omega \ln \omega)$ contribution of the gravitational waveform at order G^2:

$$\begin{aligned}
\Delta^{(\omega\ln\omega)}_{(G^2)}\ \tilde{e}^{\mu\nu}(\omega,\vec{x}) \quad = \quad & (-i)\ \frac{2G}{R}\ \exp\Big\{ i\omega R - 2iG\ln R\sum_{b=1}^{N} p_b.k\Big\}\Big[-2G\ln\{\omega+i\epsilon\}\sum_{b=1}^{N} p_b.k \\
& \times \sum_{a=1}^{M+N} \frac{p_a^{(\mu}k_\rho}{p_a.k}\Big(r_a^\rho p_a^{\nu)} - r_a^{\nu)}p_a^\rho + \Sigma_a^{\rho\nu)}\Big) \\
& -\frac{i}{2}\sum_{a=1}^{M+N}\frac{k_\rho k_\sigma}{p_a.k}\Big\{ \Big(p_a^\mu\frac{\partial}{\partial p_{a\rho}} - p_a^\rho\frac{\partial}{\partial p_{a\mu}}\Big) K_{gr}^{cl}\times\Big(r_a^\sigma p_a^\nu - r_a^\nu p_a^\sigma + \Sigma_a^{\sigma\nu}\Big) \\
& + \Big(p_a^\nu\frac{\partial}{\partial p_{a\sigma}} - p_a^\sigma\frac{\partial}{\partial p_{a\nu}}\Big) K_{gr}^{cl}\times\Big(r_a^\rho p_a^\mu - r_a^\mu p_a^\rho + \Sigma_a^{\rho\mu}\Big)\Big\}\Big] \qquad (4.3.7)
\end{aligned}$$

Explicitly the above equation can be written in terms of incoming scattering data $\{p'_a, \Sigma'_a, r'_a\}$ and outgoing scattering data $\{p_a, \Sigma_a, r_a\}$ in the following way:

$$\begin{aligned}
& \Delta^{(\omega\ln\omega)}_{(G^2)}\ \tilde{e}^{\mu\nu}(\omega, R, \mathbf{n}) \\
= \quad & \frac{G^2}{R}\exp\Big\{ i\omega R - 2iG\omega\ln R\sum_{b=1}^{N} p_b.\mathbf{n}\Big\} \\
& \times\Big[4i\omega\ln(\omega+i\epsilon)\sum_{b=1}^{N} p_b.\mathbf{n}\Big\{\sum_{a=1}^{N}\frac{p_a^{(\mu}\mathbf{n}_\rho}{p_a.\mathbf{n}}\Big(r_a^\rho p_a^{\nu)} - r_a^{\nu)}p_a^\rho + \Sigma_a^{\rho\nu)}\Big) \\
& -\sum_{a=1}^{M}\frac{p_a^{'(\mu}\mathbf{n}_\rho}{p'_a.\mathbf{n}}\Big(r_a^{'\rho}p_a^{'\nu)} - r_a^{'\nu)}p_a^{'\rho} + \Sigma_a^{'\rho\nu)}\Big)\Big\} \\
& + i\omega\ln(\omega+i\epsilon)\sum_{a=1}^{N}\sum_{\substack{b=1\\ b\neq a}}^{N}\frac{p_a.p_b}{[(p_a.p_b)^2 - p_a^2 p_b^2]^{3/2}}\{2(p_a.p_b)^2 - 3p_a^2p_b^2\}\frac{\mathbf{n}_\rho\mathbf{n}_\sigma}{p_a.\mathbf{n}} \\
& \Big\{ (p_a^\mu p_b^\rho - p_a^\rho p_b^\mu)(r_a^\sigma p_a^\nu - r_a^\nu p_a^\sigma + \Sigma_a^{\sigma\nu}) \ + \ (p_a^\nu p_b^\sigma - p_a^\sigma p_b^\nu)(r_a^\rho p_a^\mu - r_a^\mu p_a^\rho + \Sigma_a^{\rho\mu})\Big\} \\
& + i\omega\ln(\omega-i\epsilon)\sum_{a=1}^{M}\sum_{\substack{b=1\\ b\neq a}}^{M}\frac{p'_a.p'_b}{[(p'_a.p'_b)^2 - p_a^{'2}p_b^{'2}]^{3/2}}\{2(p'_a.p'_b)^2 - 3p_a^{'2}p_b^{'2}\}\frac{\mathbf{n}_\rho\mathbf{n}_\sigma}{p'_a.\mathbf{n}} \\
& \Big\{ (p_a^{'\mu}p_b^{'\rho} - p_a^{'\rho}p_b^{'\mu})(r_a^{'\sigma}p_a^{'\nu} - r_a^{'\nu}p_a^{'\sigma} + \Sigma_a^{'\sigma\nu}) \ + \ (p_a^{'\nu}p_b^{'\sigma} - p_a^{'\sigma}p_b^{'\nu})(r_a^{'\rho}p_a^{'\mu} - r_a^{'\mu}p_a^{'\rho} + \Sigma_a^{'\rho\mu})\Big\}\Big]
\end{aligned}$$
$$(4.3.8)$$

where $k^\mu = \omega\mathbf{n}^\mu = \omega(1,\hat{n})$ with $\hat{n}$ being the unit vector denoting the direction of gravitational radiation. For the scattering of non-spinning objects, we can read off the order $\omega\ln\omega$ gravitational waveform after setting $\Sigma_a = 0$ and $\Sigma'_a = 0$ in the above relation.

4.3.1 Spin dependent tail memory

Performing Fourier transform in ω variable of expression in Eq. (4.3.8), we find the following expressions for late and early time gravitational waveforms:

$$\Delta^{(1/u^2)}_{(G^2)}\ e^{\mu\nu}(u, \vec{x} = R\hat{n})$$

$$= \quad -\frac{G^2}{R}\,\frac{1}{u^2}\Bigg[4\sum_{b=1}^{N} p_b.\mathbf{n}\Bigg\{\sum_{a=1}^{N}\frac{p_a^{(\mu}\mathbf{n}_\rho}{p_a.\mathbf{n}}\Big(r_a^\rho p_a^{\nu)} - r_a^{\nu)}p_a^\rho + \Sigma_a^{\rho\nu)}\Big)$$
$$-\sum_{a=1}^{M}\frac{p_a^{\prime(\mu}\mathbf{n}_\rho}{p'_a.\mathbf{n}}\Big(r_a^{\prime\rho} p_a^{\prime\nu)} - r_a^{\prime\nu)}p_a^{\prime\rho} + \Sigma_a^{\prime\rho\nu)}\Big)\Bigg\}$$
$$+\sum_{a=1}^{N}\sum_{\substack{b=1\\b\neq a}}^{N}\frac{p_a.p_b}{[(p_a.p_b)^2 - p_a^2p_b^2]^{3/2}}\{2(p_a.p_b)^2 - 3p_a^2p_b^2\}\frac{\mathbf{n}_\rho\mathbf{n}_\sigma}{p_a.\mathbf{n}}\Big\{(p_a^\mu p_b^\rho - p_a^\rho p_b^\mu)(r_a^\sigma p_a^\nu - r_a^\nu p_a^\sigma + \Sigma_a^{\sigma\nu})$$
$$+ (p_a^\nu p_b^\sigma - p_a^\sigma p_b^\nu)(r_a^\rho p_a^\mu - r_a^\mu p_a^\rho + \Sigma_a^{\rho\mu})\Big\}\Bigg], \qquad \text{for } u\to+\infty \tag{4.3.9}$$

$$\Delta_{(G^2)}^{(1/u^2)}\; e^{\mu\nu}(u,\vec{x}=R\hat{n})$$
$$= \quad \frac{G^2}{R}\,\frac{1}{u^2}\sum_{a=1}^{M}\sum_{\substack{b=1\\b\neq a}}^{M}\frac{p'_a.p'_b}{[(p'_a.p'_b)^2 - p_a^{\prime 2}p_b^{\prime 2}]^{3/2}}\{2(p'_a.p'_b)^2 - 3p_a^{\prime 2}p_b^{\prime 2}\}\frac{\mathbf{n}_\rho\mathbf{n}_\sigma}{p'_a.\mathbf{n}}$$
$$\Big\{(p_a^{\prime\mu}p_b^{\prime\rho} - p_a^{\prime\rho}p_b^{\prime\mu})(r_a^{\prime\sigma}p_a^{\prime\nu} - r_a^{\prime\nu}p_a^{\prime\sigma} + \Sigma_a^{\prime\sigma\nu}) \;+\; (p_a^{\prime\nu}p_b^{\prime\sigma} - p_a^{\prime\sigma}p_b^{\prime\nu})(r_a^{\prime\rho}p_a^{\prime\mu} - r_a^{\prime\mu}p_a^{\prime\rho} + \Sigma_a^{\prime\rho\mu})\Big\},$$
$$\text{for } u\to-\infty \tag{4.3.10}$$

where retarded time u is given by $u = t - R + 2G\ln R \sum_{b=1}^{N} p_b\cdot\mathbf{n}$. The above expressions predict the order u^{-2} tail memory along with the known order u^{-1} and $u^{-2}\ln u$ tail memories. For non-spinning object scattering the order $\mathcal{O}(u^{-2})$ tail memory is non-vanishing and can be read off by setting $\Sigma_a = 0$ and $\Sigma'_a = 0$ in the above expressions. At this stage the order $\mathcal{O}(u^{-2})$ gravitational tail memory seems to be non-universal even at order G^2, as the result depends not only on the asymptotic data i.e. incoming and outgoing momenta and spins of scattered objects but also on the choice of the region $\mathcal{R}$ through r_a. Let us fix two particular time slices before and after the scattering event such that before and after this time, the value of the kinetic energy of all the particles exceeds the value of their potential energy due to interaction among themselves. This way we can fix the boundary of the region $\mathcal{R}$. In our notation this boundary corresponds to $\sigma = 0$ for the particle's trajectory i.e. $X_a(\sigma = 0) = r_a$. Now if we want to change our definition of region $\mathcal{R}$ to a different time slice say $\sigma = \sigma_0$, it would not affect our result above as under this choice r_a will be shifted by $r_a + \frac{p_a}{m_a}\sigma_0$ and that does not affect $(r_a^\alpha p_a^\beta - r_a^\beta p_a^\alpha)$ combination. Hence this property suggests that in principle, it is possible to write the result only in terms of the asymptotic data. As a piece of evidence, consider a special case of $2\to 2$ scattering event where object-1 is very heavy and object-2 is a probe of small mass and suppose they scatter with a large impact parameter. Now for this process, if we choose the scattering centre at the origin of object-1, then $r_1 = 0$ and r_2 is the impact parameter which can be determined in terms of asymptotic scattering data.

4.3.2 Spin dependent tail memory rewritten

Consider a classical scattering process where out of N outgoing objects $\widetilde{N}$ number of objects are massive and rest are massless radiation including gravitational wave. For this process in [120], it has been shown that the late time gravitational memory at order u^{-1} and $u^{-2}\ln u$ can be fully expressed in terms of the momenta of incoming massive objects, incoming massless radiation and outgoing massive objects only. So the information about outgoing massless radiation is not needed to compute the late time gravitational memory at order u^{-1} and $u^{-2}\ln u$. In this subsection, we show that the late time gravitational memory at order u^{-2}, given in Eq. (4.3.9) can also be rewritten in such a way that it would not carry any information of outgoing massless radiation.

We denote the momenta of outgoing massive objects by $\widetilde{p}_a$, the spins of outgoing massive objects by $\widetilde{\Sigma}_a$ and the coordinates of the intersection points with the boundary of region $\mathcal{R}$ of outgoing massive objects by $\widetilde{r}_a$ for $a = 1, \cdots, \widetilde{N}$. Then after some manipulation using conservation of asymptotic momenta and asymptotic angular momenta the expression in Eq. (4.3.9) can be rewritten as:

$$
\begin{aligned}
&\Delta^{(1/u^2)}_{(G^2)}\, e^{\mu\nu}(u, \vec{x} = R\hat{n}) \\
= \quad & -\frac{G^2}{R}\frac{1}{u^2}\Bigg[4\sum_{b=1}^{\widetilde{N}}\widetilde{p}_b.\mathbf{n}\Bigg\{\sum_{a=1}^{\widetilde{N}}\frac{\widetilde{p}_a^{(\mu}\mathbf{n}_\rho}{\widetilde{p}_a.\mathbf{n}}\Big(\widetilde{r}_a^{\rho}\widetilde{p}_a^{\nu)} - \widetilde{r}_a^{\nu)}\widetilde{p}_a^{\rho} + \widetilde{\Sigma}_a^{\rho\nu)}\Big)\Bigg\} \\
& -4(P'\cdot\mathbf{n})\Bigg\{\sum_{a=1}^{M}\frac{p_a'^{(\mu}\mathbf{n}_\rho}{p'_a.\mathbf{n}}\Big(r_a'^{\rho}p_a'^{\nu)} - r_a'^{\nu)}p_a'^{\rho} + \Sigma_a'^{\rho\nu)}\Big)\Bigg\} \\
& +\sum_{a=1}^{\widetilde{N}}\sum_{\substack{b=1\\ b\neq a}}^{\widetilde{N}}\frac{\widetilde{p}_a.\widetilde{p}_b}{[(\widetilde{p}_a.\widetilde{p}_b)^2 - \widetilde{p}_a^2\widetilde{p}_b^2]^{3/2}}\{2(\widetilde{p}_a.\widetilde{p}_b)^2 - 3\widetilde{p}_a^2\widetilde{p}_b^2\}\frac{\mathbf{n}_\rho\mathbf{n}_\sigma}{\widetilde{p}_a.\mathbf{n}}\Big\{(\widetilde{p}_a^{\mu}\widetilde{p}_b^{\rho} - \widetilde{p}_a^{\rho}\widetilde{p}_b^{\mu})(\widetilde{r}_a^{\sigma}\widetilde{p}_a^{\nu} - \widetilde{r}_a^{\nu}\widetilde{p}_a^{\sigma} + \widetilde{\Sigma}_a^{\sigma\nu}) \\
& + (\widetilde{p}_a^{\nu}\widetilde{p}_b^{\sigma} - \widetilde{p}_a^{\sigma}\widetilde{p}_b^{\nu})(\widetilde{r}_a^{\rho}\widetilde{p}_a^{\mu} - \widetilde{r}_a^{\mu}\widetilde{p}_a^{\rho} + \widetilde{\Sigma}_a^{\rho\mu})\Big\} \\
& -4\widetilde{P}^{(\mu}\,\mathbf{n}_\rho\sum_{a=1}^{\widetilde{N}}\Big(\widetilde{r}_a^{\rho}\widetilde{p}_a^{\nu)} - \widetilde{r}_a^{\nu)}\widetilde{p}_a^{\rho} + \widetilde{\Sigma}_a^{\rho\nu)}\Big) \\
& +4P'^{(\mu}\,\mathbf{n}_\rho\sum_{a=1}^{M}\Big(r_a'^{\rho}p_a'^{\nu)} - r_a'^{\nu)}p_a'^{\rho} + \Sigma_a'^{\rho\nu)}\Big)\Bigg], \qquad \text{for } u\to+\infty
\end{aligned}
\tag{4.3.11}
$$

where $P'^{\mu} = \sum_{a=1}^{M} p_a'^{\mu}$ and $\widetilde{P}^{\mu} = \sum_{a=1}^{\widetilde{N}}\widetilde{p}_a^{\mu}$. The above expression does not carry any information about massless outgoing radiation.

Now consider binary blackhole merger process, where there is only one incoming object (i.e. $M = 1$) which is the bound state of two blackholes and in the final state there are one massive blackhole (i.e. $\widetilde{N} = 1$) and lots of gravitational radiation. Hence for this process the third and fourth lines after the equality vanishes in Eq. (4.3.11). Also the first term within the square bracket cancels the second last term as well as the second term within the square bracket cancels the last term in Eq. (4.3.11). Hence the late time gravitational memory at order G^2u^{-2} given in Eq. (4.3.11) vanishes for blackhole merger process. But as discussed in the introduction, the u^{-2} tail memory receives correction at order G^3, which has not been derived or conjectured yet. So we do not know whether order G^3u^{-2} memory also vanishes or not for blackhole merger process.

Now let us consider a gravitational scattering process where all the incoming and outgoing objects are massless particles/radiation. For this event after using conservation of asymptotic momenta and asymptotic angular momenta, the expression in Eq. (4.3.9) and Eq. (4.3.10) takes the following simple form:

$$
\begin{aligned}
&\Delta^{(1/u^2)}_{(G^2)}\ e^{\mu\nu}(u\to+\infty,\vec{x}=R\hat{n}) \ = \ -\ \Delta^{(1/u^2)}_{(G^2)}\ e^{\mu\nu}(u\to-\infty,\vec{x}=R\hat{n}) \\
&= \ -\frac{4G^2}{R}\frac{1}{u^2}\Bigg[P'^{(\mu}\mathbf{n}_\rho\Big(r'^{\rho}_a p'^{\nu)}_a - r'^{\nu)}_a p'^{\rho}_a + \Sigma'^{\rho\nu)}_a\Big) - (P'\cdot\mathbf{n})\sum_{a=1}^{M}\frac{p'^{(\mu}_a\mathbf{n}_\rho}{p'_a.\mathbf{n}}\Big(r'^{\rho}_a p'^{\nu)}_a - r'^{\nu)}_a p'^{\rho}_a + \Sigma'^{\rho\nu)}_a\Big)\Bigg]
\end{aligned}
\tag{4.3.12}
$$

From the above expression it is clear that for massless particle scattering the order $\mathcal{O}(G^2u^{-2})$ gravitational tail memory is completely determined in terms of the scattering data of ingoing particles only. So the result is independent of scattering angles even if the ingoing massless particles/radiation form a blackhole along with some massless gravitational radiation in the final state. In [120] this feature has been established for u^{-1} and $u^{-2}\ln u$ tail memories as well.

4.4 Derivation of spin dependent gravitational waveform

We consider a classical scattering process where M number of spinning macroscopic objects come in from asymptotic past, go through some complicated process involving fusion, splitting etc. within a finite region of spacetime $\mathcal{R}$, and finally disperse to N number of objects including finite energy radiation flux as shown in Fig. 4.1. For this kind of scattering event, we are interested in determining the late and early time gravitational waveform, which is also related to the radiative mode of low-frequency gravitational waveform via Fourier transformation in the time variable. We choose the region $\mathcal{R}$ to be sufficiently large so that outside this region only long-range gravitational interaction is present. Let us consider the size of the spacetime region $\mathcal{R}$ to be L. Now we choose the scattering centre well inside the region $\mathcal{R}$ and place the gravitational wave detector at a distance R from the scattering centre along the direction $\hat{n}$. With this setup we want to determine the $\frac{1}{R}$ component of the gravitational waveform with frequency ω within the range $R^{-1} << \omega << L^{-1}$.

We proceed by defining the deviation of the metric from flat background and it's trace reversed component in the following way,

$$
h_{\mu\nu}(x) \ \equiv \ \frac{1}{2}\big(g_{\mu\nu}(x)-\eta_{\mu\nu}\big) \qquad , \qquad e_{\mu\nu}(x)\equiv h_{\mu\nu}(x)-\frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta}(x) \tag{4.4.1}
$$

In the above expression η represents the Minkowski metric in mostly positive signature. In several literatures including [118,121,122,125,126] the following relation between the radiative mode of gravitational waveform and the Fourier transform of the total energy-momentum tensor has been derived in four spacetime dimensions:

$$
\widetilde{e}^{\mu\nu}(\omega,R,\hat{n}) \ \simeq \ \frac{2G}{R}\ e^{i\omega R}\ \widehat{T}^{\mu\nu}(k) \tag{4.4.2}
$$

where under $\simeq$ sign, we are neglecting the terms with higher powers of R^{-1}. In the relation above $\vec{x}=R\hat{n}$, $k=\omega(1,\hat{n})\equiv\omega\mathbf{n}$ and the gravitational radiation is considered to be outgoing. The expressions of $\widetilde{e}^{\mu\nu}$ and $\widehat{T}^{\mu\nu}$ are given by,

$$
\widetilde{e}^{\mu\nu}(\omega,\vec{x}) \ = \ \int_{-\infty}^{\infty}dt\ e^{i\omega t}\ e^{\mu\nu}(t,\vec{x}) \tag{4.4.3}
$$

$$\widehat{T}^{\mu\nu}(k) = \int d^4x\, e^{-ik.x}\, T^{\mu\nu}(x) \;+\; \text{boundary terms at } \infty \tag{4.4.4}$$

where $T^{\mu\nu}(x)$ is the total (matter + gravitational) energy-momentum tensor which appears in the RHS of linearised Einstein's equation. In the above relation the Fourier transform of energy-momentum tensor is defined inside the region $|\vec{x}| << R$ or equivalently we may need to add appropriate boundary terms at ∞ to make the integral well defined [118,121,122,126].

4.4.1 General setup and strategy

Consider the incoming particles have masses $\{m'_a\}$, velocities $\{v'_a\}$, momenta $\{p'_a = m'_a v'_a\}$ and spins $\{\Sigma'_a\}$ at asymptotic past for $a = 1, 2, \cdots, M$ and the outgoing particles have masses $\{m_a\}$, velocities $\{v_a\}$, momenta $\{p_a = m_a v_a\}$ and spins $\{\Sigma_a\}$ at asymptotic future for $a = 1, 2, \cdots, N$. Let $X'_a(\sigma)$ denotes the trajectory of the incoming particles in the affine parameter range $-\infty < \sigma \leq 0$ for $a = 1, 2, \cdots, M$ and $X_a(\sigma)$ denotes the trajectory of the outgoing particles in the affine parameter range $0 \leq \sigma < \infty$ for $a = 1, 2, \cdots, N$. Now to treat incoming and outgoing particles uniformly, we treat the incoming particles as extra outgoing particles under the following identifications:

$$\begin{aligned} m_{N+a} &= m'_a \;,\; v^{\mu}_{N+a} = -v'^{\mu}_a \;,\; p^{\mu}_{N+a} = -p'^{\mu}_a \;,\; \Sigma^{\mu\nu}_{N+a} = -\Sigma'^{\mu\nu}_a \;,\; X^{\mu}_{N+a}(\sigma) = X'^{\mu}_a(-\sigma) \\ &\quad \text{for } a = 1, 2, \cdots, M \text{ and } 0 \leq \sigma < \infty \end{aligned} \tag{4.4.5}$$

The trajectories and spins of the scattered objects satisfy the following boundary conditions,

$$X^{\mu}_a(\sigma = 0) = r^{\mu}_a \;,\; \frac{dX^{\mu}_a(\sigma)}{d\sigma}\bigg|_{\sigma\to\infty} = v^{\mu}_a \;, \tag{4.4.6}$$

$$\text{and} \qquad \Sigma^{\mu\nu}_a(\sigma)\Big|_{\sigma\to\infty} = \Sigma^{\mu\nu}_a \qquad \text{for } a = 1, 2, \cdots, M+N. \tag{4.4.7}$$

with r_a being the coordinate on the boundary of region $\mathcal{R}$, where the trajectory of a'th particle intersects. Now outside the region $\mathcal{R}$, the movement of the scattered objects can be well captured by the following matter energy-momentum tensor [126–137][2],

$$\begin{aligned} T^{X\alpha\beta}(x) &= \sum_{a=1}^{M+N} \int_0^{\infty} d\sigma \left[m_a \frac{dX^{\alpha}_a(\sigma)}{d\sigma} \frac{dX^{\beta}_a(\sigma)}{d\sigma}\, \delta^{(4)}\big(x - X_a(\sigma)\big) \right] \\ &\quad + \sum_{a=1}^{M+N} \sqrt{-det\ g(x)}\, \nabla_{\gamma} \left[\int_0^{\infty} d\sigma\, \frac{dX^{(\alpha}_a(\sigma)}{d\sigma}\, \Sigma^{\beta)\gamma}_a(\sigma)\, \frac{\delta^{(4)}\big(x - X_a(\sigma)\big)}{\sqrt{-det\ g(x)}} \right] \\ &\quad + \cdots \end{aligned} \tag{4.4.9}$$

where unspecified "$\cdots$" terms contain two or more covariant derivatives derivatives operating on the σ integral containing delta function and carry the information of the internal structures of the macroscopic objects in terms of gravitational multipole moments and tidal responses,

[2] We are using the following definition of energy-momentum tensor for the world-line action S_X :

$$T^{X\alpha\beta}(x) = 2\frac{\delta S_X}{\delta g_{\alpha\beta}(x)} \;, \tag{4.4.8}$$

which differs from the canonical definition of energy-momentum tensor by a multiplicative factor of $\sqrt{-det\ g}$.

which will not affect our result to the order we are working in. The symmetrization convention used above is defined as: $A^{(\alpha}B^{\beta)} = \frac{1}{2}\left(A^\alpha B^\beta + A^\beta B^\alpha\right)$. Now using the definition of covariant derivative on a rank-3 tensor and the property $\Gamma^{\gamma}_{\gamma\delta} = \frac{1}{\sqrt{-det\ g}}\partial_\delta\left(\sqrt{-det\ g}\,\right)$, the above matter energy-momentum tensor simplifies to:

$$\begin{aligned} T^{X\alpha\beta}(x) \quad = \quad & \sum_{a=1}^{M+N}\int_0^\infty d\sigma\ \Bigg[\ m_a \frac{dX_a^\alpha(\sigma)}{d\sigma}\frac{dX_a^\beta(\sigma)}{d\sigma}\ \delta^{(4)}\big(x - X_a(\sigma)\big) \\ & + \frac{dX_a^{(\alpha}(\sigma)}{d\sigma}\ \Sigma_a^{\beta)\gamma}(\sigma)\ \partial_\gamma\delta^{(4)}\big(x - X_a(\sigma)\big) \\ & + \Gamma^{(\alpha}_{\gamma\delta}(X_a(\sigma))\frac{dX_a^\delta(\sigma)}{d\sigma}\Sigma_a^{\beta)\gamma}(\sigma)\delta^{(4)}\big(x - X_a(\sigma)\big)\Bigg] \end{aligned} \tag{4.4.10}$$

With the above matter energy-momentum tensor, we have to solve Einstein equation to derive the background metric,

$$\sqrt{-\det g}\left(R^{\alpha\beta} - \frac{1}{2}R\ g^{\alpha\beta}\right) \ = \ 8\pi G\ T^{X\alpha\beta} \tag{4.4.11}$$

We also have to solve geodesic equation and the equation for the time evolution of spin as given below [133, 134, 136, 138–141]:

$$\frac{d^2X_a^\alpha(\sigma)}{d\sigma^2} + \Gamma^\alpha_{\beta\gamma}(X_a(\sigma))\frac{dX_a^\beta(\sigma)}{d\sigma}\frac{dX_a^\gamma(\sigma)}{d\sigma} = -\frac{1}{2m_a}R^\alpha{}_{\nu\rho\sigma}(X_a(\sigma))\Sigma_a^{\rho\sigma}(\sigma)\frac{dX_a^\nu(\sigma)}{d\sigma} \tag{4.4.12}$$

$$\frac{d\Sigma_a^{\mu\nu}(\sigma)}{d\sigma} + \Gamma^\mu_{\alpha\beta}(X_a(\sigma))\Sigma^{\alpha\nu}(\sigma)\frac{dX_a^\beta(\sigma)}{d\sigma} + \Gamma^\nu_{\alpha\beta}(X_a(\sigma))\Sigma^{\mu\alpha}(\sigma)\frac{dX_a^\beta(\sigma)}{d\sigma} = 0 \tag{4.4.13}$$

The above equations follow from Mathisson-Papapetrou equations with some correction terms, which are explicitly derived in appendix-4.A. Also in appendix-4.A the above equations has beed derived demanding the covariant conservation of the canonical version of the matter energy-momentum tensor in Eq. (4.4.10).

Now in terms of trace reversed metric fluctuation the Einstein's equation Eq. (4.4.11) takes the following form:

$$\eta^{\rho\sigma}\partial_\rho\partial_\sigma e^{\alpha\beta}(x) \quad = \quad -8\pi G\ \left(T^{X\alpha\beta}(x)\ +\ T^{h\alpha\beta}(x)\right)\ \equiv\ -8\pi G\ T^{\alpha\beta}(x) \tag{4.4.14}$$

where $T^{h\alpha\beta}(x)$ is the gravitational energy-momentum tensor defined as,

$$T^{h\alpha\beta} \quad \equiv \quad -\frac{1}{8\pi G}\left[\sqrt{-\det g}\left(R^{\alpha\beta} - \frac{1}{2}R\ g^{\alpha\beta}\right)\ +\ \eta^{\rho\sigma}\partial_\rho\partial_\sigma e^{\alpha\beta}\right] \tag{4.4.15}$$

Now we briefly sketch the strategy of our computation:

- We solve the equations Eq. (4.4.14), Eq. (4.4.12) and Eq. (4.4.13) iteratively considering gravitational constant G as an iterative parameter, in post-Minkowskian(PM) sense.
- At zeroth iterative order we set the initial value of the metric fluctuation $e^{\mu\nu}(x) = 0$ and consider the scattered objects travel in asymptotic linearised trajectory $X_a^\mu(\sigma) = r_a^\mu + v_a^\mu\sigma$ with constant value of spin $\Sigma_a^{\mu\nu}(\sigma) = \Sigma_a^{\mu\nu}$.

- Next, the Fourier transform of matter energy-momentum tensor given in Eq. (4.4.10) is computed keeping the terms up to subleading order in ω expansion. We divide the integration region in Eq. (4.4.4) into two parts: one the inside of region $\mathcal{R}$ denoted by $|\vec{x}| \leq L$, where we only need to use the conservation of energy-momentum tensor, and another the outside of region $\mathcal{R}$ denoted by $|\vec{x}| \geq L$, where we only need to use the linearized trajectory approximation of scattered objects [126].

- Now using Eq. (4.4.2) we can read off order $\mathcal{O}(G)$ gravitational waveform up to subleading order in small ω. Here the leading contribution in small ω expansion contributes to gravitational DC memory after Fourier transformation in time variable. But the subleading terms in ω expansion which depends on spin, does not contribute to displacement kind of memory as it is analytic in $\omega \to 0$ limit.

- In the first iterative order we use the $\mathcal{O}(G)$ metric fluctuation as background metric and solve geodesic equation Eq. (4.4.12) with boundary condition Eq. (4.4.6) to find the correction to linearised trajectory. At this order we also need to solve the spin evolution equation Eq. (4.4.13) with boundary condition Eq. (4.4.7).

- Now using this corrected trajectory, spin correction and order $\mathcal{O}(G)$ background metric we compute the first iterative correction to the matter and gravitational energy momentum tensor outside region $\mathcal{R}$[3]. This first iterative correction to energy-momentum tensor contributes to non-analytic terms as $\omega \to 0$ at order $\mathcal{O}(\omega^n \ln \omega)$ for $n \geq 0$ along with analytic terms.

- Using Eq. (4.4.2) with the corrected energy-momentum tensor we get the order $\mathcal{O}(G^2)$ gravitational waveform. At this order the leading non-analytic contribution in small ω expansion behaves like $\ln \omega$ and this has been evaluated in [118], which contributes to u^{-1} tail memory. So in this article our main goal is to systematically evaluate the order $\omega \ln \omega$ contribution to the gravitational waveform which turns out to be spin dependent and after Fourier transform contributes to u^{-2} tail memory as first time pointed out in [119].

- Now if we do the analysis in the next iterative order i.e. at order $\mathcal{O}(G^3)$ of gravitational waveform, we see that it corrects the order $\omega \ln \omega$ gravitational waveform [119]. Possibly there might be corrections also from order $\mathcal{O}(G^n)$ for $n \geq 4$ to order $\omega \ln \omega$ gravitational waveform, which seems to be absent by naive dimensional analysis (e.g. see table in Fig. 4.2). Hence in our analysis we are only able to derive the order $\omega \ln \omega$ contribution of gravitational waveform at order $\mathcal{O}(G^2)$. So our result should be thought of as 2PM contribution of order $\omega \ln \omega$ gravitational waveform, which is also equivalent to 2PM contribution of order u^{-2} gravitational tail memory.

[3]Since at this order we are only interested to extract the non-analytic terms in ω for $\omega^{-1} >> L$, Fourier transform of energy-momentum tensor inside the region $\mathcal{R}$ does not contribute to non-analytic terms in ω.

4.4.2 Order $\mathcal{O}(G)$ gravitational waveform

In this subsection we review the analysis of [126] in four spacetime dimensions[4], which is necessary for our analysis of next subsection. Following the strategy, let $\Delta_{(0)}\widehat{T}^{X}_{<}$ denotes the contribution of Fourier transform of Eq. (4.4.10) inside region $\mathcal{R}$ and $\Delta_{(0)}\widehat{T}^{X}_{>}$ denotes the contribution of Fourier transform of Eq. (4.4.10) outside region $\mathcal{R}$ at order $\mathcal{O}(G^0)$.

After using integration by parts the Fourier transform of Eq. (4.4.10) outside region $\mathcal{R}$ becomes,

$$\begin{aligned}
\widehat{T}^{X\alpha\beta}_{>}(k) \;=\;& \sum_{a=1}^{M+N}\int_0^\infty d\sigma\; e^{-ik.X_a(\sigma)}\Bigg[m_a\frac{dX_a^\alpha(\sigma)}{d\sigma}\frac{dX_a^\beta(\sigma)}{d\sigma}+i\frac{dX_a^{(\alpha}(\sigma)}{d\sigma}\Sigma_a^{\beta)\gamma}k_\gamma \\
&+\Gamma^{(\alpha}_{\gamma\delta}(X_a(\sigma))\frac{dX_a^\delta(\sigma)}{d\sigma}\Sigma_a^{\beta)\gamma}(\sigma)\Bigg] \\
&-\sum_{a=1}^{M+N}\int_0^\infty d\sigma\int d^4x\;\delta(|\vec{x}|-L)\delta^{(4)}(x-X_a(\sigma))\;e^{-ik.X_a(\sigma)} \\
&\times\frac{dX_a^{(\alpha}(\sigma)}{d\sigma}\Sigma_a^{\beta)\gamma}\tilde{n}_\gamma
\end{aligned}\tag{4.4.16}$$

where $\tilde{n}^\alpha\simeq\big(0,\frac{\vec{x}}{|\vec{x}|}\big)$ is the outward unit vector on the boundary $\partial\mathcal{R}$. In the last two lines we get the boundary contribution using $\int_{|x|\geq L}d^4x\;\partial_\gamma A^{\alpha\beta\gamma}(x)=-\int d^4x\;\delta(|\vec{x}|-L)\tilde{n}_\gamma A^{\alpha\beta\gamma}(x)$. Now to evaluate the boundary contribution(last two lines) of the above expression with trajectory $X_a(\sigma)=r_a+v_a\sigma$, we first do the integration over x using the delta function $\delta^{(4)}(x-X_a(\sigma))$, which substitutes $x=r_a+v_a\sigma$ everywhere inside the integrand. Now we do the σ integral using the delta function $\delta(|\vec{r}_a+\vec{v}_a\sigma|-L)$, which contributes to $1/(v_a.\tilde{n}_a)$ where $\tilde{n}_a^\alpha=\big(0,\frac{\vec{r}_a}{|\vec{r}_a|}\big)$. There is no modulus sign in the contribution from delta function as for outgoing particles $v_a.\tilde{n}_a$ is positive. Hence from Eq. (4.4.16) we get[5],

$$\begin{aligned}
\Delta_{(0)}\widehat{T}^{X\alpha\beta}_{>}(k) \;=\;& \sum_{a=1}^{M+N}e^{-ik.r_a}\left[\frac{m_av_a^\alpha v_a^\beta}{i(v_a.k-i\epsilon)}+\frac{v_a^{(\alpha}\Sigma_a^{\beta)\gamma}k_\gamma}{v_a.k-i\epsilon}\right] \\
&-\sum_{a=1}^{M+N}e^{-ik.r_a}\;\frac{1}{\tilde{n}_a.v_a}\;v_a^{(\alpha}\Sigma_a^{\beta)\gamma}\tilde{n}_{a\gamma} \\
=\;&(-i)\sum_{a=1}^{M+N}\left[\frac{p_a^\alpha p_a^\beta}{p_a.k-i\epsilon}+i\frac{p_a^{(\alpha}\Sigma_a^{\beta)\gamma}k_\gamma}{p_a.k-i\epsilon}-ik.r_a\;\frac{p_a^\alpha p_a^\beta}{p_a.k-i\epsilon}\right] \\
&-\sum_{a=1}^{M+N}\frac{1}{\tilde{n}_a.p_a}\;p_a^{(\alpha}\Sigma_a^{\beta)\gamma}\tilde{n}_{a\gamma}\;+\;\mathcal{O}(\omega)
\end{aligned}\tag{4.4.17}$$

where in the last two lines we only kept the terms up to subleading order in ω expansion and used $p_a=m_av_a$. The $i\epsilon$ prescription is fixed demanding the finiteness of $\widehat{T}^X$ from the ∞ range of σ integration.

[4]There are some sign differences relative to [126] due to the fact that we are considering the velocities of outgoing particles to be positive, where as there the velocities of outgoing particles were considered to be negative.

[5]Here we ignore the term containing Christoffel connection inside the square bracket of Eq. (4.4.16), as it will not contribute at order $\mathcal{O}(G^0)$.

Now to derive the Fourier transform of matter energy-momentum tensor inside the region $\mathcal{R}$, let us start with:

$$\begin{aligned} -ik_\alpha \widehat{T}^{X\alpha\beta}_{<}(k) &= \int_{|x|\leq L} d^4x \ \frac{\partial}{\partial x^\alpha}\Big[e^{-ik.x}\Big]\ T^{X\alpha\beta}_{in}(x) \\ &= \int d^4x\ \delta(|\vec{x}|-L)\ e^{-ik.x}\ \tilde{n}_\alpha T^{X\alpha\beta}_{in}(x) \end{aligned} \tag{4.4.18}$$

where $T^{X\alpha\beta}_{in}(x)$ represent the matter energy-momentum tensor inside region $\mathcal{R}$ which is not same as Eq. (4.4.10) in general. To get the second line from the first line above, we use integration by parts and consider the conservation law of matter energy-momentum tensor $\partial_\alpha T^{X\alpha\beta}_{in} = 0$ at linearized order. Since the full contribution is written as just a boundary term on $\partial\mathcal{R}$, T^X_{in} will match with the expression in Eq. (4.4.10) at the boundary. So in the last line above we substitute the expression Eq. (4.4.10) in place of T^X_{in} and then do integration by parts to remove the derivative over the delta function. Finally after performing the x integration using the delta function, we get[6]

$$\begin{aligned} -ik_\alpha \widehat{T}^{X\alpha\beta}_{<}(k) &= \sum_{a=1}^{M+N}\int_0^\infty d\sigma\ e^{-ik.X_a(\sigma)}\ \tilde{n}_\alpha\Bigg[\delta\big(|\vec{X}_a(\sigma)|-L\big)m_a\frac{dX^\alpha_a}{d\sigma}\frac{dX^\beta_a}{d\sigma} \\ &\quad -\tilde{n}_\gamma\delta'\big(|\vec{X}_a(\sigma)|-L\big)\frac{dX^{(\alpha}_a}{d\sigma}\Sigma^{\beta)\gamma}_a + ik_\gamma\delta\big(|\vec{X}_a(\sigma)|-L\big)\frac{dX^{(\alpha}_a}{d\sigma}\Sigma^{\beta)\gamma}_a\Bigg] \end{aligned} \tag{4.4.19}$$

Now we want to evaluate the above expression using asymptotic linearized trajectory $X_a(\sigma) = r_a + v_a\sigma$. To do that we have to use the following two identities of delta function:

$$\delta\big(|\vec{r}_a+\vec{v}_a\sigma|-L\big) = \frac{1}{\tilde{n}_a.v_a}\delta(\sigma)\ ,\ \ \delta'\big(|\vec{r}_a+\vec{v}_a\sigma|-L\big) = \frac{1}{(\tilde{n}_a.v_a)^2}\delta'(\sigma) \tag{4.4.20}$$

Hence using these properties and performing the σ integration we get,

$$\begin{aligned} &-ik_\alpha\Delta_{(0)}\widehat{T}^{X\alpha\beta}_{<}(k) \\ &= \sum_{a=1}^{M+N} e^{-ik.r_a}\Bigg[p^\beta_a - \frac{i}{2}\frac{k.v_a}{\tilde{n}_a.v_a}\Sigma^{\beta\gamma}_a\tilde{n}_{a\gamma} + \frac{i}{2}\Sigma^{\beta\gamma}_a k_\gamma\ + \frac{i}{2}\frac{v^\beta_a}{\tilde{n}_a.v_a}\ \tilde{n}_{a\alpha}\Sigma^{\alpha\gamma}_a k_\gamma\Bigg] \\ &= \sum_{a=1}^{M+N}\Bigg[-ik.r_a\ p^\beta_a - \frac{i}{2}\frac{k.p_a}{\tilde{n}_a.p_a}\Sigma^{\beta\gamma}_a\tilde{n}_{a\gamma} + \frac{i}{2}\Sigma^{\beta\gamma}_a k_\gamma\ + \frac{i}{2}\frac{p^\beta_a}{\tilde{n}_a.p_a}\ \tilde{n}_{a\alpha}\Sigma^{\alpha\gamma}_a k_\gamma\Bigg] + \mathcal{O}(\omega^2) \end{aligned} \tag{4.4.21}$$

Now from the above expression after stripping out the k_α and using total angular momentum conservation relation $\sum_a\Big(\Sigma^{\alpha\beta}_a + r^\alpha_a p^\beta_a - r^\beta_a p^\alpha_a\Big) = 0$ we get[7],

$$\Delta_{(0)}\widehat{T}^{X\alpha\beta}_{<}(k) = \sum_{a=1}^{M+N}\Bigg[r^{(\alpha}_a p^{\beta)}_a + \frac{p^{(\alpha}_a}{\tilde{n}_a.p_a}\Sigma^{\beta)\gamma}_a\tilde{n}_{a\gamma}\Bigg]\ +\ \mathcal{O}(\omega) \tag{4.4.22}$$

[6] Again here we are ignoring the term containing Christoffel connection as it will not contribute at order $\mathcal{O}(G^0)$ in the analysis of $\widehat{T}^{X\alpha\beta}(k)$.

[7] The stripping out of k_α is unique up to this order as illustrated in [126].

Summing over the contributions of Eq. (4.4.17) and Eq. (4.4.22) we find the following order $\mathcal{O}(G^0)$ Fourier transformed energy-momentum tensor:

$$\Delta_{(0)}\widehat{T}^{X\alpha\beta}(k) \quad = \quad (-i)\sum_{a=1}^{M+N}\frac{1}{(p_a.k-i\epsilon)}\Big[p_a^\alpha p_a^\beta \;+\; ip_a^{(\alpha}J_a^{\beta)\gamma}k_\gamma\Big] \;+\; \mathcal{O}(\omega) \quad (4.4.23)$$

where $J_a^{\beta\gamma}=r_a^\beta p_a^\gamma - r_a^\gamma p_a^\beta + \Sigma_a^{\beta\gamma}$ is the total classical angular momentum tensor of particle-a. Here we can see that neither r_a^μ nor $\Sigma_a^{\mu\nu}$ are unambiguously defined as there exists transformation $r_a^\mu \to r_a^\mu + c_a^\mu$ and $\Sigma_a^{\mu\nu}\to\Sigma_a^{\mu\nu} - c_a^\mu p_a^\nu + c_a^\nu p_a^\mu$, under which total angular momentum $J_a^{\mu\nu}$ remains invariant. So to give an unambiguous covariant definition of spin angular momentum we are using $p_{a\mu}\Sigma_a^{\mu\nu}=0$ from the beginning, which is known as Tulczyjew-Dixon spin supplementary condition [131, 142].

Now using the relation Eq. (4.4.2) we get the radiative mode of gravitational waveform at order $\mathcal{O}(G)$ up to subleading order in ω expansion,

$$\Delta_{(0)}\widetilde{e}^{\mu\nu}(\omega,R,\hat{n}) \;\simeq -i\frac{2G}{R}\,e^{i\omega R}\sum_{a=1}^{M+N}\frac{1}{(p_a.k-i\epsilon)}\Big[p_a^\mu p_a^\nu \;+\; ip_a^{(\mu}J_a^{\nu)\rho}k_\rho\Big] \qquad (4.4.24)$$

Now performing Fourier transform in time variable we get,

$$\Delta_{(0)}e^{\mu\nu}(u,R,\hat{n}) \quad\simeq\quad -\frac{2G}{R}\sum_{a=1}^{M+N}\left[\frac{p_a^\mu p_a^\nu}{p_a\cdot\mathbf{n}}\;H(u)\;-\;\frac{p_a^{(\mu}J_a^{\nu)\rho}\mathbf{n}_\rho}{p_a\cdot\mathbf{n}}\;\delta(u)\right] \qquad (4.4.25)$$

where $u=t-R$ is the retarded time and $H(u)$ is the Heaviside theta function. Above the first term within the square bracket contributes to gravitational DC memory [7, 107, 108, 110–113, 116, 118, 123, 143, 144]. On the other hand the second term which depends on spin angular momenta does not contribute to displacement gravitational memory as it is localized near $u=0$[8]. Hence to derive the leading spin dependent memory we need to go to next iterative order and hope to get a spin dependent non-analytic contribution in small ω from the Fourier transform of energy-momentum tensor.

4.4.3 Order $\mathcal{O}(G^2)$ gravitational waveform

Due to the asymptotic linearized trajectory of scattered object-b, the metric fluctuation can be read off from the solution of linearized Einstein equation Eq. (4.4.11) with the momentum space energy-momentum tensor Eq. (4.4.23),

$$\begin{aligned} h_{\alpha\beta}^{(b)}(x) \quad = \quad -(8\pi G)\int\frac{d^4\ell}{(2\pi)^4}\;e^{i\ell\cdot x}\;G_r(\ell)\;\frac{1}{i(p_b.\ell-i\epsilon)}\;\Big[p_{b\alpha}p_{b\beta}-\frac{1}{2}p_b^2\eta_{\alpha\beta} \\ +\; ip_{b(\alpha}J_{b,\beta)\gamma}\ell^\gamma - \frac{i}{2}p_b^\delta J_{b,\delta\gamma}\ell^\gamma\eta_{\alpha\beta}\Big] \qquad (4.4.26)\end{aligned}$$

where $G_r(\ell)$ is the momentum space retarded Greens function given as $G_r(\ell)=\{(\ell^0+i\epsilon)^2-\vec{\ell}^2\}^{-1}$. We should denote the above metric fluctuation as $\Delta_{(0)}h_{\alpha\beta}^{(b)}$ following our convention,

[8] But this angular momenta dependent term can contribute to a different kind of 'spin memory' induced by radiative angular momentum flux [123, 145, 146]. This order $\mathcal{O}(\omega^0)$ term in the gravitational waveform receives corrections from order $\mathcal{O}(G^n)$ for $n\geq 2$ which turns out to be theory dependent and depends on the structures of scattered objects. So the full (all PM order) contribution of order $\mathcal{O}(\omega^0)$ gravitational waveform is non-universal [118, 119].

but just to reduce notational complexity we remove the $\Delta_{(0)}$ piece. The Christoffel connection for the metric above is given by,

$$\begin{aligned}
&\Gamma^{(b)\mu}_{\nu\rho}(x) \\
= \quad & -(8\pi G)\int \frac{d^4\ell}{(2\pi)^4}\, e^{i\ell.x}\, G_r(\ell)\, \frac{1}{p_b.\ell - i\epsilon}\Big[\ell_\nu\Big(p_{b\rho}p_b^\mu - \frac{1}{2}\delta^\mu_\rho p_b^2\Big) + \ell_\rho\Big(p_{b\nu}p_b^\mu - \frac{1}{2}\delta^\mu_\nu p_b^2\Big) \\
& - \ell^\mu\Big(p_{b\nu}p_{b\rho} - \frac{1}{2}p_b^2\eta_{\nu\rho}\Big)\Big] \\
& -i(8\pi G)\int \frac{d^4\ell}{(2\pi)^4}\, e^{i\ell.x}\, G_r(\ell)\, \frac{1}{p_b.\ell - i\epsilon}\Big[\ell_\nu p_{b(\rho}J_b^{\mu)\alpha}\ell_\alpha + \ell_\rho p_{b(\nu}J_b^{\mu)\alpha}\ell_\alpha - \ell^\mu p_{b(\nu}J_{b,\rho)\alpha}\ell^\alpha \\
& -\frac{1}{2}\Big(\ell_\nu\delta^\mu_\rho + \ell_\rho\delta^\mu_\nu - \ell^\mu\eta_{\nu\rho}\Big)p_b^\delta J_{b,\delta\gamma}\ell^\gamma\Big]
\end{aligned} \tag{4.4.27}$$

Now due to the non-flat background metric, consider the leading correction to the asymptotic straight-line trajectory of particle-a outside the region $\mathcal{R}$ to be,

$$X_a^\mu(\sigma) \quad = \quad r_a^\mu + v_a^\mu\sigma + Y_a^\mu(\sigma) \tag{4.4.28}$$

with boundary conditions:

$$Y_a^\mu(\sigma)\Big|_{\sigma=0} = 0 \qquad , \qquad \text{and} \qquad \frac{dY_a^\mu(\sigma)}{d\sigma}\Big|_{\sigma\to\infty} = 0 \tag{4.4.29}$$

Similarly due to the non-flat background metric, there will also be a correction of spin outside the region $\mathcal{R}$, which is given by,

$$\Sigma_a^{\mu\nu}(\sigma) = \Sigma_a^{\mu\nu} + S_a^{\mu\nu}(\sigma) \tag{4.4.30}$$

with boundary condition:

$$S_a^{\mu\nu}(\sigma) = 0\ ,\ \text{ for } |\sigma|\to\infty \tag{4.4.31}$$

Now the order $\mathcal{O}(G)$ contribution of $Y_a^\mu(\sigma)$ and $S_a^{\mu\nu}(\sigma)$ satisfies the equations below, which follows from Eq. (4.4.12) and Eq. (4.4.13).

$$\begin{aligned}
\frac{d^2Y_a^\mu(\sigma)}{d\sigma^2} &= -\Gamma^\mu_{\nu\rho}(r_a + v_a\sigma)v_a^\nu v_a^\rho - \frac{1}{2m_a}\Big(\partial_\rho\Gamma^\mu_{\nu\sigma}(r_a+v_a\sigma) - \partial_\sigma\Gamma^\mu_{\nu\rho}(r_a+v_a\sigma)\Big)\Sigma_a^{\rho\sigma}v_a^\nu \\
\frac{dS_a^{\mu\nu}(\sigma)}{d\sigma} &= -\Gamma^\mu_{\sigma\rho}(r_a+v_a\sigma)\Sigma_a^{\rho\nu}v_a^\sigma + \Gamma^\nu_{\sigma\rho}(r_a+v_a\sigma)\Sigma_a^{\rho\mu}v_a^\sigma
\end{aligned} \tag{4.4.32}$$

where $\Gamma^\mu_{\nu\rho}(r_a + v_a\sigma) = \sum_{\substack{b=1\\ b\neq a}}^{M+N}\Gamma^{(b)\mu}_{\nu\rho}(r_a + v_a\sigma)$. After doing integration with the specified boundary conditions of Eq. (4.4.29) and Eq. (4.4.31), we get

$$\frac{dY_a^\mu(\sigma)}{d\sigma} = \int_\sigma^\infty d\sigma'\,\Big[\Gamma^\mu_{\nu\rho}(r_a+v_a\sigma')v_a^\nu v_a^\rho + \frac{1}{2m_a}\Big(\partial_\rho\Gamma^\mu_{\nu\sigma}(r_a+v_a\sigma') - \partial_\sigma\Gamma^\mu_{\nu\rho}(r_a+v_a\sigma')\Big)\Sigma_a^{\rho\sigma}v_a^\nu\Big] \tag{4.4.33}$$

$$S_a^{\mu\nu}(\sigma) = \int_\sigma^\infty d\sigma'\Big[\Gamma^\mu_{\sigma\rho}(r_a+v_a\sigma')\Sigma_a^{\rho\nu}v_a^\sigma - \Gamma^\nu_{\sigma\rho}(r_a+v_a\sigma')\Sigma_a^{\rho\mu}v_a^\sigma\Big] \tag{4.4.34}$$

Analysis of matter energy-momentum tensor

In this section we compute the order $\mathcal{O}(G)$ correction of the Fourier transformation of matter energy-momentum tensor due to the corrected trajectory and spin of the particle. Since only the non-analytic terms in ω contributes to gravitational memory, it is sufficient to analyze the Fourier transformation of matter energy-momentum tensor outside region $\mathcal{R}$ without the boundary contribution as given in Eq. (4.4.16). Contributions from the boundary terms and inside the region $\mathcal{R}$ always reproduce terms, which are analytic in ω. Hence neglecting the last two lines of Eq. (4.4.16) and substituting the corrected trajectory and spin we get,

$$\begin{aligned}\widehat{T}^{X\mu\nu}_{>}(k) \;\simeq\; & \sum_{a=1}^{M+N}\int_0^\infty d\sigma e^{-ik.(v_a\sigma+r_a)}\Big\{1-ik.Y_a(\sigma)+\cdots\Big\}\bigg[m_a\Big(v_a^\mu v_a^\nu+v_a^\mu\frac{dY_a^\nu(\sigma)}{d\sigma}\\ &+v_a^\nu\frac{dY_a^\mu(\sigma)}{d\sigma}\Big)+i\Big(v_a^{(\mu}\Sigma_a^{\nu)\alpha}+v_a^{(\mu}S_a^{\nu)\alpha}(\sigma)+\frac{dY_a^{(\mu}(\sigma)}{d\sigma}\Sigma_a^{\nu)\alpha}\Big)k_\alpha\\ &+\Gamma^{(\mu}_{\alpha\beta}(r_a+v_a\sigma)\Sigma_a^{\nu)\alpha}v_a^\beta\bigg]\end{aligned} \tag{4.4.35}$$

The order $\mathcal{O}(G)$ correction to the matter energy momentum tensor,

$$\begin{aligned}&\Delta_{(1)}\widehat{T}^{X\mu\nu}(k)\\ =\;&\sum_{a=1}^{M+N}\int_0^\infty d\sigma e^{-ik.(v_a\sigma+r_a)}\bigg[-ik.Y_a(\sigma)\;m_a v_a^\mu v_a^\nu\;+m_a v_a^\mu\frac{dY_a^\nu(\sigma)}{d\sigma}\\ &+m_a v_a^\nu\frac{dY_a^\mu(\sigma)}{d\sigma}\;+i\frac{dY_a^{(\mu}(\sigma)}{d\sigma}\Sigma_a^{\nu)\alpha}k_\alpha+k.Y_a(\sigma)\;v_a^{(\mu}\Sigma_a^{\nu)\alpha}k_\alpha\bigg]\\ &+\sum_{a=1}^{M+N}\int_0^\infty d\sigma e^{-ik.(r_a+v_a\sigma)}\Big[iv_a^{(\mu}S_a^{\nu)\alpha}(\sigma)k_\alpha+\Gamma^{(\mu}_{\alpha\beta}(r_a+v_a\sigma)\Sigma_a^{\nu)\alpha}v_a^\beta\Big]\end{aligned} \tag{4.4.36}$$

Now performing integration by parts and using the boundary conditions of Eq. (4.4.29) we get,

$$\begin{aligned}\Delta_{(1)}\widehat{T}^{X\mu\nu}(k) \;=\; & \sum_{a=1}^{M+N}e^{-ik.r_a}\bigg[-\frac{p_a^\mu p_a^\nu}{p_a.k}k_\beta+p_a^\mu\delta_\beta^\nu+p_a^\nu\delta_\beta^\mu+i\delta_\beta^{(\mu}\Sigma_a^{\nu)\alpha}k_\alpha-i\frac{k_\beta}{k.p_a}p_a^{(\mu}\Sigma_a^{\nu)\alpha}k_\alpha\bigg]\\ &\times\int_0^\infty d\sigma\; e^{-ik.v_a\sigma}\;\frac{dY_a^\beta(\sigma)}{d\sigma}\\ &+\sum_{a=1}^{M+N}\int_0^\infty d\sigma e^{-ik.(r_a+v_a\sigma)}\Big[iv_a^{(\mu}S_a^{\nu)\alpha}(\sigma)k_\alpha+\Gamma^{(\mu}_{\alpha\beta}(r_a+v_a\sigma)\Sigma_a^{\nu)\alpha}v_a^\beta\Big]\end{aligned} \tag{4.4.37}$$

We evaluate the above integrations using the results of Eq. (4.4.33), Eq. (4.4.34) and Eq. (4.4.27) and get,

$$\begin{aligned}I^\beta \;\equiv\; & \int_0^\infty d\sigma\; e^{-ik.v_a\sigma}\;\frac{dY_a^\beta(\sigma)}{d\sigma}\\ =\; & (8\pi G)\sum_{\substack{b=1\\ b\neq a}}^{M+N}\int\frac{d^4\ell}{(2\pi)^4}\;e^{i\ell.r_a}\;G_r(\ell)\frac{1}{p_b.\ell-i\epsilon}\;\frac{1}{p_a.\ell+i\epsilon}\;\frac{1}{p_a.(\ell-k)+i\epsilon}\end{aligned}$$

$$
\begin{aligned}
&\times\Big[2p_a.\ell(p_a.p_b p_b^\beta - \frac{1}{2}p_b^2 p_a^\beta) - \ell^\beta\{(p_a.p_b)^2 - \frac{1}{2}p_a^2 p_b^2\}\Big] \\
&+i(8\pi G)\sum_{\substack{b=1\\ b\neq a}}^{M+N}\int \frac{d^4\ell}{(2\pi)^4}\, e^{i\ell.r_a}\, G_r(\ell)\frac{1}{p_b.\ell - i\epsilon}\,\frac{1}{p_a.\ell + i\epsilon}\,\frac{1}{p_a.(\ell-k)+i\epsilon} \\
&\times\Big[p_a.\ell p_a.p_b J_b^{\beta\alpha}\ell_\alpha + p_a.\ell p_b^\beta p_{a\rho} J_b^{\rho\alpha}\ell_\alpha - \ell^\beta p_a.p_b p_{a\rho} J_b^{\rho\alpha}\ell_\alpha - \frac{1}{2}\{2p_a.\ell p_a^\beta - \ell^\beta p_a^2\} \\
&\times p_b^\delta J_{b,\delta\gamma}\ell^\gamma\Big] \\
&+i(8\pi G)\sum_{\substack{b=1\\ b\neq a}}^{M+N}\int \frac{d^4\ell}{(2\pi)^4} e^{i\ell.r_a} G_r(\ell)\,\frac{1}{p_b.\ell - i\epsilon}\,\frac{1}{p_a.\ell + i\epsilon}\,\frac{1}{p_a.(\ell-k)+i\epsilon} \\
&\Big[p_a.\ell\ \ell_\rho \Sigma_a^{\rho\sigma} p_{b\sigma} p_b^\beta - \frac{1}{2}p_b^2 p_a.\ell \ell_\rho \Sigma_a^{\rho\beta} - \ell^\beta p_a.p_b \ell_\rho \Sigma_a^{\rho\sigma} p_{b\sigma}\Big] \qquad (4.4.38)
\end{aligned}
$$

The leading non-analytic contribution in small ω expansion from Eq. (4.4.37) is of the order of $\ln\omega$ which is determined in [118]. Our goal in this sub-section is to extract the $\mathcal{O}(\omega\ln\omega)$ contribution from Eq. (4.4.37) in the integration region $\omega << |\ell^\mu| << L^{-1}$. In turn this demands that, we have to extract the order $\mathcal{O}(\ln\omega)$ and order $\mathcal{O}(\omega\ln\omega)$ contributions from the integral expression in Eq. (4.4.38) which we denote by I_1^β and I_2^β respectively.

$$
\begin{aligned}
I_1^\beta &= (8\pi G)\sum_{\substack{b=1\\ b\neq a}}^{M+N}\int_\omega^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\, G_r(\ell)\frac{1}{p_b.\ell - i\epsilon}\,\frac{1}{(p_a.\ell + i\epsilon)^2} \\
&\quad\times\Big[2p_a.\ell(p_a.p_b p_b^\beta - \frac{1}{2}p_b^2 p_a^\beta) - \ell^\beta\{(p_a.p_b)^2 - \frac{1}{2}p_a^2 p_b^2\}\Big] \\
&= i\,\frac{\partial}{\partial p_{a\beta}} K_{gr}^{cl} \qquad (4.4.39)
\end{aligned}
$$

where [119],

$$
\begin{aligned}
K_{gr}^{cl} &= -\frac{i}{2}\,(8\pi G)\sum_{\substack{b,c\\ b\neq c}}\int_\omega^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\, G_r(\ell)\frac{1}{p_b.\ell + i\epsilon}\frac{1}{p_c.\ell - i\epsilon}\,\Big\{(p_b.p_c)^2 - \frac{1}{2}p_b^2 p_c^2\Big\} \\
&= -\frac{i}{2}\,(2G)\sum_{\substack{b,c\\ b\neq c\\ \eta_b\eta_c=1}} \ln\Big\{L(\omega + i\epsilon\eta_b)\Big\}\,\frac{(p_b.p_c)^2 - \frac{1}{2}p_b^2 p_c^2}{\sqrt{(p_b.p_c)^2 - p_b^2 p_c^2}}\ . \qquad (4.4.40)
\end{aligned}
$$

$$
\begin{aligned}
I_2^\beta &= i(8\pi G)\,(p_a.k)\sum_{\substack{b=1\\ b\neq a}}^{M+N}\int_\omega^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\ \ell.r_a\ G_r(\ell)\frac{1}{p_b.\ell - i\epsilon}\ \frac{1}{(p_a.\ell + i\epsilon)^3} \\
&\times\Big[2p_a.\ell(p_a.p_b p_b^\beta - \frac{1}{2}p_b^2 p_a^\beta) - \ell^\beta\{(p_a.p_b)^2 - \frac{1}{2}p_a^2 p_b^2\}\Big] \\
&+i(8\pi G)\,(p_a.k)\sum_{\substack{b=1\\ b\neq a}}^{M+N}\int_\omega^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\ G_r(\ell)\frac{1}{p_b.\ell - i\epsilon}\ \frac{1}{(p_a.\ell + i\epsilon)^3} \\
&\times\Big[p_a.\ell p_a.p_b J_b^{\beta\alpha}\ell_\alpha + p_a.\ell p_b^\beta p_{a\rho} J_b^{\rho\alpha}\ell_\alpha - \ell^\beta p_a.p_b p_{a\rho} J_b^{\rho\alpha}\ell_\alpha - \frac{1}{2}\Big\{2p_a.\ell p_a^\beta - \ell^\beta p_a^2\Big\}\ p_b^\delta J_{b,\delta\gamma}\ell^\gamma\Big] \\
&+i(8\pi G)\sum_{\substack{b=1\\ b\neq a}}^{M+N}\int_\omega^{L^{-1}} \frac{d^4\ell}{(2\pi)^4} G_r(\ell)\ \frac{1}{p_b.\ell - i\epsilon}\ \frac{1}{(p_a.\ell + i\epsilon)^3}\quad p_a.k
\end{aligned}
$$

$$\times\Big[p_a.\ell\ \ell_\rho\Sigma_a^{\rho\sigma}p_{b\sigma}p_b^\beta-\frac{1}{2}p_b^2 p_a.\ell\ell_\rho\Sigma_a^{\rho\beta}-\ell^\beta p_a.p_b\ell_\rho\Sigma_a^{\rho\sigma}p_{b\sigma}\Big] \tag{4.4.41}$$

The integral expression of I_2^β above contributes at order $\omega\ln\omega$ and has been explicitly evaluated using contour integration in the complex ℓ^0 plane. The compact form of the result after integration becomes:

$$\begin{aligned}
I_2^\beta \;=\;& iG\,(p_a.k)\ln\{(\omega+i\epsilon\eta_a)L\}\sum_{\substack{b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N}\frac{1}{[(p_a.p_b)^2-p_a^2p_b^2]^{3/2}}\Big[5(p_a.p_b)^2p_b.r_ap_b^\beta-4p_a.p_bp_b^2p_a.r_ap_b^\beta\\
&-2p_b^2p_a.p_bp_b.r_ap_a^\beta+2(p_b^2)^2p_a.r_ap_a^\beta-(p_a.p_b)^2p_b^2r_a^\beta-\frac{1}{2}p_a^2p_b^2p_b.r_ap_b^\beta+\frac{1}{2}p_a^2(p_b^2)^2r_a^\beta\\
&+(p_a.p_b)^2p_b^2r_b^\beta-5(p_a.p_b)^2p_b.r_bp_b^\beta+4p_a.p_bp_b^2p_a.r_bp_b^\beta+2p_a.p_bp_b.r_bp_b^2p_a^\beta-2(p_b^2)^2p_a.r_bp_a^\beta\\
&+\frac{1}{2}p_a^2p_b^2p_b.r_bp_b^\beta-\frac{1}{2}p_a^2(p_b^2)^2r_b^\beta-p_a.p_bp_b^2\Sigma_b^{\beta\alpha}p_{a\alpha}\Big]\\
&-3iG\,(p_a.k)\ \ln\{(\omega+i\epsilon\eta_a)L\}\sum_{\substack{b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N}\frac{1}{[(p_a.p_b)^2-p_a^2p_b^2]^{5/2}}\Big[\Big(p_a.p_bp_b.r_a-p_b^2p_a.r_a\Big)\\
&\times\Big\{(p_a.p_b)^3p_b^\beta-(p_a.p_b)^2p_b^2p_a^\beta-\frac{1}{2}p_a^2p_b^2p_a.p_bp_b^\beta+\frac{1}{2}p_a^2(p_b^2)^2p_a^\beta\Big\}\\
&+(p_a.p_b)^3p_b^2p_a.r_bp_b^\beta+(p_a.p_b)^3p_b^2p_b.r_bp_a^\beta\ -\frac{1}{2}p_a^2(p_b^2)^2p_a.p_bp_a.r_bp_b^\beta-\frac{1}{2}p_a^2(p_b^2)^2p_a.p_bp_b.r_bp_a^\beta\\
&-(p_a.p_b)^4p_b.r_bp_b^\beta+\frac{1}{2}(p_a.p_b)^2p_a^2p_b^2p_b.r_bp_b^\beta-(p_a.p_b)^2(p_b^2)^2p_a.r_bp_a^\beta+\frac{1}{2}p_a^2(p_b^2)^3p_a.r_bp_a^\beta\Big].
\end{aligned} \tag{4.4.42}$$

Above $\eta_b=+1$ if particle-b is outgoing and $\eta_b=-1$ if particle-b is ingoing. We also need to evaluate the last line of Eq. (4.4.37), which contributes at order $\mathcal{O}(\omega\ln\omega)$ in the integration region $\omega<<|\ell^\mu|<<L^{-1}$,

$$\begin{aligned}
&\sum_{a=1}^{M+N}\int_0^\infty d\sigma e^{-ik.(r_a+v_a\sigma)}\Big[iv_a^{(\mu}S_a^{\nu)\alpha}(\sigma)k_\alpha+\Gamma^{(\mu}_{\alpha\beta}(r_a+v_a\sigma)\Sigma_a^{\nu)\alpha}v_a^\beta\Big]\\
\simeq\;& +iG\sum_{a=1}^{M+N}\sum_{\substack{b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N}\ln\big\{(\omega+i\epsilon\eta_a)L\big\}\frac{1}{[(p_a.p_b)^2-p_a^2p_b^2]^{3/2}}\\
&\Big[\{(p_a.p_b)^2-p_a^2p_b^2\}p_{b\rho}\Sigma_a^{\rho\alpha}k_\alpha(p_a^\mu p_b^\nu+p_a^\nu p_b^\mu)-p_b^2\{(p_a.p_b)^2-p_a^2p_b^2\}(p_a^\mu\Sigma_a^{\nu\alpha}k_\alpha+p_a^\nu\Sigma_a^{\mu\alpha}k_\alpha)\\
&-\{(p_a.p_b)^2-p_a^2p_b^2\}p_b.k(p_a^\mu p_{b\rho}\Sigma_a^{\rho\nu}+p_a^\nu p_{b\rho}\Sigma_a^{\rho\mu})+p_a.p_bp_b^2p_{b\rho}\Sigma_a^{\rho\alpha}k_\alpha p_a^\mu p_a^\nu\\
&+(p_a.k)\{(p_a.p_b)^2-p_a^2p_b^2\}\big(p_b^\mu p_{b\alpha}\Sigma_a^{\alpha\nu}+p_b^\nu p_{b\alpha}\Sigma_a^{\alpha\mu}\big)\Big]
\end{aligned} \tag{4.4.43}$$

Now using these results of integrations, the order $\mathcal{O}(\omega\ln\omega)$ contribution to matter energy-momentum tensor from Eq. (4.4.37) becomes,

$$\begin{aligned}
&\Delta^{(\omega\ln\omega)}_{(1)}\widehat{T}^{X\mu\nu}(k)\\
=\;&-\sum_{a=1}^{M+N}(k.r_a)\Big[\frac{p_a^\mu p_a^\nu}{p_a.k}k^\alpha\frac{\partial}{\partial p_a^\alpha}K_{gr}^{cl}-\frac{1}{2}p_a^\mu\frac{\partial}{\partial p_{a\nu}}K_{gr}^{cl}-\frac{1}{2}p_a^\nu\frac{\partial}{\partial p_{a\mu}}K_{gr}^{cl}\Big]\\
&+\frac{1}{2}\sum_{a=1}^{M+N}\Big[\frac{1}{p_a.k}\Big\{p_a^\mu\Sigma_a^{\nu\alpha}k_\alpha+p_a^\nu\Sigma_a^{\mu\alpha}k_\alpha\Big\}k^\beta\frac{\partial}{\partial p_a^\beta}K_{gr}^{cl}-\Sigma_a^{\nu\alpha}k_\alpha\frac{\partial}{\partial p_{a\mu}}K_{gr}^{cl}-\Sigma_a^{\mu\alpha}k_\alpha\frac{\partial}{\partial p_{a\nu}}K_{gr}^{cl}\Big]\\
&-\sum_{a=1}^{M+N}\Big[\frac{p_a^\mu p_a^\nu}{p_a.k}k_\beta I_2^\beta-p_a^\mu I_2^\nu-p_a^\nu I_2^\mu\Big]\ +\sum_{a=1}^{M+N}(k.r_a)\Big[\frac{1}{2}p_a^\mu\frac{\partial}{\partial p_{a\nu}}K_{gr}^{cl}+\frac{1}{2}p_a^\nu\frac{\partial}{\partial p_{a\mu}}K_{gr}^{cl}\Big]
\end{aligned}$$

$$
\begin{aligned}
&+iG\sum_{a=1}^{M+N}\sum_{\substack{b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N}\ln\big\{(\omega+i\epsilon\eta_a)L\big\}\frac{1}{[(p_a.p_b)^2-p_a^2p_b^2]^{3/2}}\\
&\Big[\{(p_a.p_b)^2-p_a^2p_b^2\}p_{b\rho}\Sigma_a^{\rho\alpha}k_\alpha(p_a^\mu p_b^\nu+p_a^\nu p_b^\mu)-p_b^2\{(p_a.p_b)^2-p_a^2p_b^2\}(p_a^\mu\Sigma_a^{\nu\alpha}k_\alpha+p_a^\nu\Sigma_a^{\mu\alpha}k_\alpha)\\
&-\{(p_a.p_b)^2-p_a^2p_b^2\}p_b.k(p_a^\mu p_{b\rho}\Sigma_a^{\rho\nu}+p_a^\nu p_{b\rho}\Sigma_a^{\rho\mu})+p_a.p_bp_b^2p_{b\rho}\Sigma_a^{\rho\alpha}k_\alpha p_a^\mu p_a^\nu\\
&+(p_a.k)\{(p_a.p_b)^2-p_a^2p_b^2\}\big(p_b^\mu p_{b\alpha}\Sigma_a^{\alpha\nu}+p_b^\nu p_{b\alpha}\Sigma_a^{\alpha\mu}\big)\Big]
\end{aligned}
\tag{4.4.44}
$$

Analysis of gravitational energy-momentum tensor

Here we will compute the spin dependent gravitational energy-momentum tensor for the metric fluctuation given in Eq. (4.4.26). Fourier transform of the order G gravitational energy-momentum tensor takes the following form:

$$
\begin{aligned}
\Delta_{(1)}\widehat{T}^{h\mu\nu}(k) \;=\; &-(8\pi G)\sum_{a,b=1}^{M+N}\int\frac{d^4\ell}{(2\pi)^4}\;G_r(k-\ell)G_r(\ell)\;\frac{1}{p_b.\ell-i\epsilon}\;\frac{1}{p_a.(k-\ell)-i\epsilon}\\
&\times\Big\{p_{b\alpha}p_{b\beta}-\frac{1}{2}p_b^2\eta_{\alpha\beta}+ip_{b(\alpha}J_{b,\beta)\gamma}\ell^\gamma-\frac{i}{2}\eta_{\alpha\beta}p_b^\delta J_{b,\delta\gamma}\ell^\gamma\Big\}\;\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)\\
&\times\Big\{p_{a\rho}p_{a\sigma}-\frac{1}{2}p_a^2\eta_{\rho\sigma}+ip_{a(\rho}J_{a,\sigma)\delta}(k-\ell)^\delta-\frac{i}{2}\eta_{\rho\sigma}p_a^\kappa J_{a,\kappa\tau}(k-\ell)^\tau\Big\}
\end{aligned}
\tag{4.4.45}
$$

where [118, 119],

$$
\begin{aligned}
&\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell) \\
= \quad & 2\Big[\frac{1}{2}\ell^{\mu}(k-\ell)^{\nu}\eta^{\alpha\rho}\eta^{\beta\sigma}+(k-\ell)^{\mu}(k-\ell)^{\nu}\eta^{\alpha\rho}\eta^{\beta\sigma}-(k-\ell)^{\nu}(k-\ell)^{\beta}\eta^{\alpha\rho}\eta^{\mu\sigma} \\
&-(k-\ell)^{\mu}(k-\ell)^{\beta}\eta^{\alpha\rho}\eta^{\nu\sigma}+(k-\ell)^{\alpha}(k-\ell)^{\beta}\eta^{\mu\rho}\eta^{\nu\sigma}+(k-\ell).\ell\,\eta^{\nu\beta}\eta^{\alpha\rho}\eta^{\mu\sigma} \\
&-\ell^{\rho}(k-\ell)^{\alpha}\eta^{\nu\beta}\eta^{\mu\sigma}-\frac{1}{2}(k-\ell)^{2}\eta^{\mu\alpha}\eta^{\nu\beta}\eta^{\rho\sigma}\ +\eta^{\mu\alpha}\eta^{\beta\rho}\eta^{\nu\sigma}(k-\ell)^{2} \\
&+\eta^{\nu\alpha}\eta^{\beta\rho}\eta^{\mu\sigma}(k-\ell)^{2}\Big]\ -\eta^{\mu\nu}\Big[\frac{3}{2}(k-\ell).\ell\,\eta^{\alpha\rho}\eta^{\beta\sigma}+2(k-\ell)^{2}\eta^{\alpha\rho}\eta^{\beta\sigma} \\
&-\ell^{\sigma}(k-\ell)^{\alpha}\eta^{\beta\rho}\Big]-\eta^{\alpha\beta}(k-\ell)^{2}\ \eta^{\mu\rho}\eta^{\nu\sigma}+\frac{1}{2}\eta^{\alpha\beta}(k-\ell)^{2}\ \eta^{\rho\sigma}\eta^{\mu\nu} \qquad (4.4.46)
\end{aligned}
$$

follows from the quadratic part of gravitational energy-momentum tensor defined in Eq. (4.4.15). In the integration region $R^{-1} << |\ell^{\mu}| << \omega$, at leading order $\mathcal{F}(k,\ell)$ approximates in the following form up to gauge equivalence,

$$
\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)\simeq -2k^{\beta}k^{\sigma}\eta^{\alpha\rho}\eta^{\mu\nu}+2k^{\alpha}k^{\beta}\eta^{\mu\rho}\eta^{\nu\sigma} \qquad (4.4.47)
$$

Hence in the integration region $R^{-1} << |\ell^{\mu}| << \omega$ the order $\mathcal{O}(\omega\ln\omega)$ contribution from Eq. (4.4.45) turns out to be,

$$
\begin{aligned}
&\Delta^{(\omega\ln\omega)}_{(1)}\widehat{T}^{h\mu\nu}_{1}(k) \\
= \quad & -(8\pi G)\sum_{a,b=1}^{M+N}\frac{1}{p_a.k}\Big\{p_{b\alpha}p_{b\beta}-\frac{1}{2}p_b^2\eta_{\alpha\beta}\Big\}\big[-2k^{\beta}k^{\sigma}\eta^{\alpha\rho}\eta^{\mu\nu}+2k^{\alpha}k^{\beta}\eta^{\mu\rho}\eta^{\nu\sigma}\big] \\
&\times\Big\{ip_{a(\rho}J_{a,\sigma)\delta}k^{\delta}-\frac{i}{2}\eta_{\rho\sigma}p_a^{\kappa}J_{a,\kappa\tau}k^{\tau}\Big\}\int_{R^{-1}}^{\omega}\frac{d^4\ell}{(2\pi)^4}\ G_r(\ell)\frac{1}{p_b.\ell-i\epsilon}\frac{1}{2k.\ell+i\epsilon}
\end{aligned}
\qquad (4.4.48)
$$

Now after using the result of the integration [118]

$$
\int_{R^{-1}}^{\omega}\frac{d^4\ell}{(2\pi)^4}\ G_r(\ell)\frac{1}{p_b.\ell-i\epsilon}\frac{1}{k.\ell+i\epsilon}=\frac{1}{4\pi}\ \delta_{\eta_b,1}\ \frac{1}{p_b.k}\ \ln\big\{(\omega+i\epsilon)R\big\} \qquad (4.4.49)
$$

we get,

$$
\begin{aligned}
&\Delta^{(\omega\ln\omega)}_{(1)}\widehat{T}^{h\mu\nu}_{1}(k) \\
= \quad & -iG\ \ln\big\{(\omega+i\epsilon)R\big\}\sum_{b=1}^{N}\sum_{a=1}^{M+N}\Bigg[-\eta^{\mu\nu}p_b^{\rho}J_{a,\rho\sigma}k^{\sigma}+\frac{p_b.k}{p_a.k}\Big(p_a^{\mu}J_a^{\nu\rho}k_{\rho}+p_a^{\nu}J_a^{\mu\rho}k_{\rho}\Big)\Bigg]
\end{aligned}
\qquad (4.4.50)
$$

After using the total angular momentum conservation relation $\sum_{a=1}^{M+N} J_a^{\rho\sigma}=0$ we get,

$$
\Delta^{(\omega\ln\omega)}_{(1)}\widehat{T}^{h\mu\nu}_{1}(k) \quad = \quad -2iG\ \ln\big\{(\omega+i\epsilon)R\big\}\sum_{b=1}^{N}p_b.k\sum_{a=1}^{M+N}\frac{1}{p_a.k}\ p_a^{(\mu}J_a^{\nu)\alpha}k_{\alpha} \qquad (4.4.51)
$$

Next we analyze the expression in Eq. (4.4.45) in the integration region $\omega << |\ell^{\mu}| << L^{-1}$

and extract the order $\mathcal{O}(\omega\ln\omega)$ contribution. First let us substitute the following identity in Eq. (4.4.45),

$$G_r(k-\ell)G_r(\ell) \quad = \quad G_r(k-\ell)G_r(-\ell) - 2\pi i\delta(\ell^2)\big[H(\ell^0)-H(-\ell^0)\big]G_r(k-\ell) \tag{4.4.52}$$

and focus on the contribution coming from $G_r(k-\ell)G_r(-\ell)$ part. As described in appendix-4.C, the gravitational energy-momentum tensor with $\delta(\ell^2)\big[H(\ell^0)-H(-\ell^0)\big]$ $\times\, G_r(k-\ell)$ part does not contribute to order $\mathcal{O}(\omega\ln\omega)$ in the integration region $\omega << |\ell^\mu| << L^{-1}$. Following the discussion of appendix-B in [118], a part of this contribution can be identified with the soft radiation from the finite energy real gravitational radiation, which is already taken care of in the hard particle sums of the earlier expressions. So in the integration region $\omega << |\ell^\mu| << L^{-1}$, the order $\mathcal{O}(\omega\ln\omega)$ contribution from gravitational energy-momentum tensor with $G_r(k-\ell)G_r(-\ell)$ part turns out to be,

$$\begin{aligned}
&\Delta^{(\omega\ln\omega)}_{(1)}\widehat{T}^{h\mu\nu}_2(k)\\
=\;& (8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\,\Big\{G_r(-\ell)\Big\}^3\,(-2k.\ell)\,\frac{1}{p_b.\ell-i\epsilon}\frac{1}{p_a.\ell+i\epsilon}\\
&\times\Bigg[\Big\{ip_b^{(\alpha}J_b^{\beta)\gamma}\ell_\gamma-\frac{i}{2}\eta^{\alpha\beta}p_{b\kappa}J_b^{\kappa\gamma}\ell_\gamma\Big\}\Delta_{(\ell\ell)}\mathcal{F}^{\mu\nu}{}_{\alpha\beta,\rho\sigma}(k,\ell)\Big\{p_a^\rho p_a^\sigma-\frac{1}{2}p_a^2\eta^{\rho\sigma}\Big\}\\
&+\Big\{p_b^\alpha p_b^\beta-\frac{1}{2}p_b^2\eta^{\alpha\beta}\Big\}\;\Delta_{(\ell\ell)}\mathcal{F}^{\mu\nu}{}_{\alpha\beta,\rho\sigma}(k,\ell)\Big\{-ip_a^{(\rho}J_a^{\sigma)\delta}\ell_\delta+\frac{i}{2}\eta^{\rho\sigma}p_{a\kappa}J_a^{\kappa\delta}\ell_\delta\Big\}\Bigg]\\
&+(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\,\Big\{G_r(-\ell)\Big\}^2\,\frac{1}{p_b.\ell-i\epsilon}\frac{1}{(p_a.\ell+i\epsilon)^2}\,(p_a.k)\\
&\times\Bigg[\Big\{ip_b^{(\alpha}J_b^{\beta)\gamma}\ell_\gamma-\frac{i}{2}\eta^{\alpha\beta}p_{b\kappa}J_b^{\kappa\gamma}\ell_\gamma\Big\}\Delta_{(\ell\ell)}\mathcal{F}^{\mu\nu}{}_{\alpha\beta,\rho\sigma}(k,\ell)\Big\{p_a^\rho p_a^\sigma-\frac{1}{2}p_a^2\eta^{\rho\sigma}\Big\}\\
&+\Big\{p_b^\alpha p_b^\beta-\frac{1}{2}p_b^2\eta^{\alpha\beta}\Big\}\;\Delta_{(\ell\ell)}\mathcal{F}^{\mu\nu}{}_{\alpha\beta,\rho\sigma}(k,\ell)\Big\{-ip_a^{(\rho}J_a^{\sigma)\delta}\ell_\delta+\frac{i}{2}\eta^{\rho\sigma}p_{a\kappa}J_a^{\kappa\delta}\ell_\delta\Big\}\Bigg]\\
&+(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\,\Big\{G_r(-\ell)\Big\}^2\,\frac{1}{p_b.\ell-i\epsilon}\frac{1}{p_a.\ell+i\epsilon}\\
&\times\Bigg[\Big\{ip_b^{(\alpha}J_b^{\beta)\gamma}\ell_\gamma-\frac{i}{2}\eta^{\alpha\beta}p_{b\kappa}J_b^{\kappa\gamma}\ell_\gamma\Big\}\Delta_{(k\ell)}\mathcal{F}^{\mu\nu}{}_{\alpha\beta,\rho\sigma}(k,\ell)\Big\{p_a^\rho p_a^\sigma-\frac{1}{2}p_a^2\eta^{\rho\sigma}\Big\}\\
&+\Big\{p_b^\alpha p_b^\beta-\frac{1}{2}p_b^2\eta^{\alpha\beta}\Big\}\;\Delta_{(k\ell)}\mathcal{F}^{\mu\nu}{}_{\alpha\beta,\rho\sigma}(k,\ell)\Big\{-ip_a^{(\rho}J_a^{\sigma)\delta}\ell_\delta+\frac{i}{2}\eta^{\rho\sigma}p_{a\kappa}J_a^{\kappa\delta}\ell_\delta\Big\}\Bigg]\\
&+(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\,\Big\{G_r(-\ell)\Big\}^2\,\frac{1}{p_b.\ell-i\epsilon}\frac{1}{p_a.\ell+i\epsilon}\\
&\times\Big\{p_b^\alpha p_b^\beta-\frac{1}{2}p_b^2\eta^{\alpha\beta}\Big\}\Delta_{(\ell\ell)}\mathcal{F}^{\mu\nu}{}_{\alpha\beta,\rho\sigma}(k,\ell)\Big\{ip_a^{(\rho}J_a^{\sigma)\delta}k_\delta-\frac{i}{2}\eta^{\rho\sigma}p_{a\kappa}J_a^{\kappa\delta}k_\delta\Big\}
\end{aligned}\tag{4.4.53}$$

In the above expression, $\Delta_{(\ell\ell)}\mathcal{F}(k,\ell)$ and $\Delta_{(k\ell)}\mathcal{F}(k,\ell)$ corresponds to the order $\mathcal{O}(\ell\ell)$ and order $\mathcal{O}(k\ell)$ contributions of $\mathcal{F}(k,\ell)$ respectively, which can be easily extractable from Eq. (4.4.46). The expression in Eq. (4.4.53) is explicitly evaluated in Appendix-4.B and the final result of order $\mathcal{O}(\omega\ln\omega)$ contribution is written in Eq. (4.B.18).

Total energy-momentum tensor and gravitational waveform at order $\mathcal{O}(\omega\ln\omega)$

Summing over the contributions of Eq. (4.4.44), Eq. (4.B.18) and Eq. (4.4.51), we get the Fourier transform the total energy-momentum tensor at order $\mathcal{O}(G\,\omega\ln\omega)$:

$$\begin{aligned}\Delta_{(1)}^{(\omega\ln\omega)}\widehat{T}^{\mu\nu}(k) &= -2iG\,\ln\{(\omega+i\epsilon)R\}\sum_{b=1}^{N}p_b.k\sum_{a=1}^{M+N}\frac{1}{p_a.k}\,p_a^{(\mu}J_a^{\nu)\alpha}k_\alpha\\ &\quad-\frac{1}{2}\sum_{a=1}^{M+N}\frac{k_\rho k_\sigma}{p_a.k}\Bigg\{\left(p_a^\mu\frac{\partial}{\partial p_{a\rho}}-p_a^\rho\frac{\partial}{\partial p_{a\mu}}\right)K_{gr}^{cl}\times\left(r_a^\sigma p_a^\nu-r_a^\nu p_a^\sigma+\Sigma_a^{\sigma\nu}\right)\\ &\quad+\left(p_a^\nu\frac{\partial}{\partial p_{a\sigma}}-p_a^\sigma\frac{\partial}{\partial p_{a\nu}}\right)K_{gr}^{cl}\times\left(r_a^\rho p_a^\mu-r_a^\mu p_a^\rho+\Sigma_a^{\rho\mu}\right)\Bigg\} \qquad (4.4.54)\end{aligned}$$

Using the relation in Eq. (4.4.2), the radiative mode of gravitational waveform at order $\mathcal{O}(G^2\omega\ln\omega)$ for the above derived energy-momentum tensor takes the following form:

$$\begin{aligned}&\Delta_{(G^2)}^{(\omega\ln\omega)}\;\widetilde{e}^{\mu\nu}(\omega,\vec{x})\\ &= \frac{2G}{R}\;\exp\{i\omega R\}\Bigg[2iG\ln\{(\omega+i\epsilon)R\}\sum_{b=1}^{N}p_b.k\\ &\quad\times\sum_{a=1}^{M+N}\frac{p_a^{(\mu}k_\rho}{p_a.k}\left(r_a^\rho p_a^{\nu)}-r_a^{\nu)}p_a^\rho+\Sigma_a^{\rho\nu)}\right)\\ &\quad-\frac{1}{2}\sum_{a=1}^{M+N}\frac{k_\rho k_\sigma}{p_a.k}\Bigg\{\left(p_a^\mu\frac{\partial}{\partial p_{a\rho}}-p_a^\rho\frac{\partial}{\partial p_{a\mu}}\right)K_{gr}^{cl}\times\left(r_a^\sigma p_a^\nu-r_a^\nu p_a^\sigma+\Sigma_a^{\sigma\nu}\right)\\ &\quad+\left(p_a^\nu\frac{\partial}{\partial p_{a\sigma}}-p_a^\sigma\frac{\partial}{\partial p_{a\nu}}\right)K_{gr}^{cl}\times\left(r_a^\rho p_a^\mu-r_a^\mu p_a^\rho+\Sigma_a^{\rho\mu}\right)\Bigg\}\Bigg] \qquad (4.4.55)\end{aligned}$$

Now if we compare the above expression with the conjectured result given in Eq. (4.3.7), we observe that our result in the direct derivation completely agrees with the conjectured waveform.

4.5 Conclusion

In this chapter, we derive the leading spin-dependent gravitational tail memory which behaves like u^{-2} for retarded time $u\to\pm\infty$. First, we predict the result from the classical limit of the soft graviton theorem, then derive it for a general gravitational scattering process involving spinning objects. The final result of leading spin-dependent gravitational tail memory has been summarized in sec.4.3.1 and sec.4.3.2. Here we are pointing out the novel features of our result, its theoretical and observational importance and possible way of re-deriving our result with other available prescriptions:

1. Even when the scattered objects do not carry any intrinsic spins still a large part of the order $\mathcal{O}(G^2\omega\ln\omega)$ gravitational waveform is non-vanishing and can be read off from Eq. (4.3.8) by setting $\Sigma_a=\Sigma'_a=0$. This result is fully determined in terms of the asymptotic momenta, asymptotic orbital angular momenta, and the direction cosine and frequency of gravitational wave emission.

2. Our result of order u^{-2} gravitational memory as given in Eq. (4.3.9) and Eq. (4.3.10) has several theoretical and observational importance in the current era of gravitational wave physics. First of all, from the observation of this tail memory one can read off the intrinsic spins of the scattered objects, which is not possible from the earlier results on displacement memory, u^{-1} tail memory, and $u^{-2}\ln u$ tail memory. Secondly, for a black hole binary merger process even if the blackholes carry spin, still in sec.4.3.2 we have shown that gravitational tail memory at this order vanishes, which is another non-trivial prediction from general relativity. In sec.4.3.2 we also have shown that if the scattering event carries some massless particles or radiation along with high-frequency gravitational wave in the final state, still the waveform can be rewritten in such a way that it does not carry any information about the outgoing massless particles or radiation.

4.A Review of geodesic equation and spin evolution of spinning object

In this appendix we review the geodesic equation for spinning object as well as the time evolution of the spin. These equations are known as Mathisson-Papapetrou equations [138, 139] in general relativity and the covariant form of those are discovered by Tulczyjew and Dixon [131, 132]. The Mathisson-Papapetrou-Dixon equations take the following form:

$$\frac{DP^\mu}{D\sigma} = -\frac{1}{2}R^\mu{}_{\nu\rho\sigma}u^\nu\Sigma^{\rho\sigma} \tag{4.A.1}$$

$$\frac{D\Sigma^{\mu\nu}}{D\sigma} = P^\mu u^\nu - P^\nu u^\mu \tag{4.A.2}$$

where P^μ is the kinematical momentum and $\Sigma^{\mu\nu}$ is the spin angular momentum of the object measured along the world line as a function of affine parameter σ. The four velocity is denoted by u^μ and defined as $u^\mu = \frac{dx^\mu}{d\sigma}$, for $x^\mu(\sigma)$ representing the world line and $u^\mu u_\mu = -1$ if σ is the proper time. $\frac{D}{D\sigma}$ represents the covariant derivative along the world line. Let us define the kinematic mass $m \equiv -P\cdot u$, which is in general not a constant of motion [151]. Now contracting with u_ν from Eq. (4.A.2) we get,

$$P^\mu = mu^\mu - \frac{D\Sigma^{\mu\nu}}{D\sigma}u_\nu \tag{4.A.3}$$

To understand how the kinematical mass evolves along the trajectory, let us take derivative over m w.r.t. σ,

$$\frac{dm}{d\sigma} = \frac{Dm}{D\sigma} = -\frac{DP^\mu}{D\sigma}u_\mu - P^\mu\frac{Du_\mu}{D\sigma} = \frac{Du_\mu}{D\sigma}\frac{D\Sigma^{\mu\nu}}{D\sigma}u_\nu \tag{4.A.4}$$

where to get the last expression after equality, we substituted the results of Eq. (4.A.1) and Eq. (4.A.2). Now we substitute the expression of P^μ in Eq. (4.A.1) and use the expressions in Eq. (4.A.4) and Eq. (4.A.2). After this substitutions we simplify Eq. (4.A.1) using $u_\alpha\frac{Du^\alpha}{D\sigma} = 0$ and get the following simplified trajectory equation,

$$m\frac{Du^\mu}{D\sigma} - \frac{D^2\Sigma^{\mu\nu}}{D\sigma^2}u_\nu = -\frac{1}{2}R^\mu{}_{\nu\rho\sigma}u^\nu\Sigma^{\rho\sigma} \tag{4.A.5}$$

Note that to derive the above trajectory equation we do not have to use any spin supplementarity condition(SSC).

Now let us focus on the simplification of the spin evolution equation given in Eq. (4.A.2), using the trajectory equation Eq. (4.A.5) and spin supplementarity condition $u_\mu \Sigma^{\mu\nu} = 0$. We start by substituting the expression of P^μ in Eq. (4.A.2) and get,

$$\frac{D\Sigma^{\mu\nu}}{D\sigma} = -\frac{D\Sigma^{\mu\alpha}}{D\sigma} u_\alpha u^\nu + \frac{D\Sigma^{\nu\alpha}}{D\sigma} u_\alpha u^\mu \tag{4.A.6}$$

Now using the definition of covariant derivative on the spin tensor, moving the σ derivatives from Σ to u and simplifying using spin supplementary condition $u_\mu \Sigma^{\mu\nu} = 0$[9], the above equation takes the following form:

$$\begin{aligned}\frac{d\Sigma^{\mu\nu}}{d\sigma} + \Gamma^\mu_{\alpha\beta}\Sigma^{\alpha\nu}u^\beta + \Gamma^\nu_{\alpha\beta}\Sigma^{\mu\alpha}u^\beta &= \Sigma^{\mu\alpha}\frac{du_\alpha}{d\sigma}u^\nu - \Sigma^{\nu\alpha}\frac{du_\alpha}{d\sigma}u^\mu \\ &\quad -\Gamma^\alpha_{\rho\sigma}\Sigma^{\mu\rho}u^\sigma u_\alpha u^\nu + \Gamma^\alpha_{\rho\sigma}\Sigma^{\nu\rho}u^\sigma u_\alpha u^\mu\end{aligned} \tag{4.A.7}$$

Substituting the trajectory equation, Eq. (4.A.5) in the RHS of the above equation, we get

$$\frac{d\Sigma^{\mu\nu}}{d\sigma} + \Gamma^\mu_{\alpha\beta}\Sigma^{\alpha\nu}u^\beta + \Gamma^\nu_{\alpha\beta}\Sigma^{\mu\alpha}u^\beta = \frac{1}{m}\Big(\Sigma^{\mu\alpha}u^\nu - \Sigma^{\nu\alpha}u^\mu\Big)\left(\frac{D^2\Sigma_{\alpha\beta}}{D\sigma^2}u^\beta - \frac{1}{2}R_{\alpha\beta\rho\sigma}u^\beta\Sigma^{\rho\sigma}\right) \tag{4.A.8}$$

In the above equation the LHS is basically $\frac{D\Sigma^{\mu\nu}}{D\sigma}$, which is equal to terms quadratic in spin as written in RHS. This above result also implies that $\frac{D^2\Sigma^{\mu\nu}}{D\sigma^2}$ in Eq. (4.A.5) is also quadratic in spin. Hence if we are interested in the trajectory and spin evolution equations which are linear in spin[10], Eq. (4.A.5) and Eq. (4.A.8) simplifies to

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\sigma}\frac{dx^\sigma}{d\sigma} = -\frac{1}{2m}R^\mu{}_{\nu\rho\sigma}\frac{dx^\nu}{d\sigma}\Sigma^{\rho\sigma} + \mathcal{O}(\Sigma^2) \tag{4.A.9}$$

and,

$$\frac{d\Sigma^{\mu\nu}}{d\sigma} + \Gamma^\mu_{\alpha\beta}\Sigma^{\alpha\nu}\frac{dx^\beta}{d\sigma} + \Gamma^\nu_{\alpha\beta}\Sigma^{\mu\alpha}\frac{dx^\beta}{d\sigma} = 0 + \mathcal{O}(\Sigma^2) \tag{4.A.10}$$

Now we show that the above equations also follow from the covariant conservation of the canonical version of the matter energy-momentum tensor in Eq. (4.4.10). For a given world-line action S_X of a moving object in gravitational background with metric $g_{\mu\nu}(x)$, the canonical matter energy-momentum tensor is defined by:

$$T_c^{X\alpha\beta}(x) \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_X}{\delta g_{\alpha\beta}(x)} \tag{4.A.11}$$

[9]In the literatures, we find two kinds of spin supplementarity conditions(SSC), which are $u_\mu\Sigma^{\mu\nu} = 0$ and $P_\mu\Sigma^{\mu\nu} = 0$. Different choices of SSC, will modify the $\mathcal{O}(\Sigma^2)$ terms in the RHS of Eq. (4.A.9) and Eq. (4.A.10), which are irrelevant for our analysis.

[10]Here we want emphasize that we are not making any small spin approximation while ignoring terms quadratic in spin. We are ignoring those terms as those contribute at order $\mathcal{O}(G^2)$ to the correction of trajectory and spin in our analysis of 4.4.3 and would not affect our $\mathcal{O}(G^2)$ gravitational waveform.

which satisfies the following conservation equation,

$$\nabla_\alpha T_c^{X\alpha\beta}(x) = 0 \tag{4.A.12}$$

In the above expressions, $g \equiv det(g_{\mu\nu})$ and ∇_α represents the covariant derivative with background metric $g_{\mu\nu}(x)$. Let $\phi_\beta(x)$ be an arbitrary rank-1 tensor with a sufficient fall off at the boundary of spacetime, such that $\left[\sqrt{-g}\, T_c^{X\alpha\beta}(x)\phi_\beta(x)\right]_{|x|\to\infty} = 0$ will be satisfied. Now let us start with the following identity,

$$\begin{aligned}
&\int d^4x\sqrt{-g}\nabla_\alpha\Big(T_c^{X\alpha\beta}(x)\phi_\beta(x)\Big) = \int d^4x\sqrt{-g}\Big[\Big(\nabla_\alpha T_c^{X\alpha\beta}(x)\Big)\phi_\beta(x)\Big) + T_c^{X\alpha\beta}(x)\nabla_{(\alpha}\phi_{\beta)}(x)\Big]\\
&\Rightarrow \int d^4x\sqrt{-g}\; T_c^{X\alpha\beta}(x)\nabla_{(\alpha}\phi_{\beta)}(x) \;=\; 0
\end{aligned} \tag{4.A.13}$$

Above to get the last line we used the fact that LHS of first line is a boundary term which vanishes and the first term in the RHS of the first line vanishes due to the conservation of matter energy-momentum tensor. In our convention, the relation between matter energy-momentum tensor given in Eq. (4.4.10) and the canonical energy-momentum tensor is: $T^{X\alpha\beta}(x) \equiv \sqrt{-g}\; T_c^{X\alpha\beta}(x)$. Now after substituting the matter energy-momentum tensor of Eq. (4.4.10) for a single particle in the above relation, we get

$$\begin{aligned}
&\int d\sigma \int d^4x\Bigg[m\frac{dX^\alpha(\sigma)}{d\sigma}\frac{dX^\beta(\sigma)}{d\sigma}\;\delta^{(4)}\big(x-X(\sigma)\big) \;+\; \frac{dX^{(\alpha}(\sigma)}{d\sigma}\;\Sigma^{\beta)\gamma}(\sigma)\\
&\times\,\partial_\gamma\delta^{(4)}\big(x-X(\sigma)\big) + \Gamma^{(\alpha}_{\gamma\delta}(X)\Sigma^{\beta)\gamma}(\sigma)\frac{dX^\delta(\sigma)}{d\sigma}\delta^{(4)}\big(x-X(\sigma)\big)\Bigg]\\
&\times\Big\{\partial_{(\alpha}\phi_{\beta)}(x) - \Gamma^\rho_{\alpha\beta}(x)\phi_\rho(x)\Big\} \;=\; 0
\end{aligned} \tag{4.A.14}$$

In the above expression, we first perform integration by parts to remove the derivative over delta function and then do the spacetime integration using the delta function. Next, after using the identity $\frac{dX^\alpha}{d\sigma}\frac{\partial}{\partial X^\alpha} = \frac{d}{d\sigma}$ and integration by parts for the derivatives w.r.t. σ we get,

$$\begin{aligned}
&\int d\sigma\;\phi_\rho(X)\Bigg[-m\frac{d^2X^\rho(\sigma)}{d\sigma^2} - m\Gamma^\rho_{\alpha\beta}(X)\frac{dX^\alpha(\sigma)}{d\sigma}\frac{dX^\beta(\sigma)}{d\sigma}\\
&+\partial_\gamma\Gamma^\rho_{\alpha\beta}(X)\frac{dX^\alpha(\sigma)}{d\sigma}\Sigma^{\beta\gamma}(\sigma) - \Gamma^\rho_{\alpha\beta}(X)\Gamma^\alpha_{\gamma\delta}(X)\frac{dX^\delta(\sigma)}{d\sigma}\Sigma^{\beta\gamma}(\sigma)\Bigg]\\
&+\frac{1}{2}\int d\sigma\;\partial_\gamma\phi_\rho(X)\Bigg[\frac{d\Sigma^{\rho\gamma}(\sigma)}{d\sigma} + \Gamma^\rho_{\alpha\beta}(X)\frac{dX^\beta(\sigma)}{d\sigma}\Sigma^{\alpha\gamma}(\sigma) - \Gamma^\gamma_{\alpha\beta}(X)\frac{dX^\beta(\sigma)}{d\sigma}\Sigma^{\alpha\rho}(\sigma)\Bigg]\\
&=\;0
\end{aligned} \tag{4.A.15}$$

Since $\phi_\rho(x)$ is an arbitrary rank-1 tensor, in the above expression the coefficients of $\phi_\rho(X)$ and $\partial_\gamma\phi_\rho(X)$ will vanish individually. Hence replacing $\rho\to\mu$ and $\gamma\to\nu$ and using the antisymmetry property of $\Sigma(\sigma)$ we get the following two equations from the above expression:

$$\begin{aligned}
&\frac{d^2X^\mu(\sigma)}{d\sigma^2} + \Gamma^\mu_{\alpha\beta}(X)\frac{dX^\alpha(\sigma)}{d\sigma}\frac{dX^\beta(\sigma)}{d\sigma} = -\frac{1}{2m}R^\mu{}_{\alpha\beta\gamma}(X)\;\frac{dX^\alpha(\sigma)}{d\sigma}\Sigma^{\beta\gamma}(\sigma)\\
&\frac{d\Sigma^{\mu\nu}(\sigma)}{d\sigma} + \Gamma^\mu_{\alpha\beta}(X)\frac{dX^\alpha(\sigma)}{d\sigma}\Sigma^{\beta\nu}(\sigma) + \Gamma^\nu_{\alpha\beta}(X)\frac{dX^\alpha(\sigma)}{d\sigma}\Sigma^{\mu\beta}(\sigma) \;=\; 0
\end{aligned} \tag{4.A.16}$$

where,

$$R^{\mu}{}_{\alpha\beta\gamma}(X) = \partial_\beta \Gamma^\mu_{\alpha\gamma}(X) - \partial_\gamma \Gamma^\mu_{\alpha\beta}(X) + \Gamma^\delta_{\alpha\gamma}(X)\Gamma^\mu_{\delta\beta}(X) - \Gamma^\delta_{\alpha\beta}(X)\Gamma^\mu_{\delta\gamma}(X) \tag{4.A.17}$$

Now if we compare the above equations with Eq. (4.A.9) and Eq. (4.A.10), we observe that they are identical. This also proves that the geodesic equation and spin evolution equation are consistent with our matter energy-momentum tensor.

4.B Detail analysis of the gravitational energy momentum tensor

To evaluate the expression in Eq. (4.4.53) we need to use the results of the following three integrals:

$$\begin{aligned}
J_1^{\alpha\beta} &= \int_\omega^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\,\big[G_r(-\ell)\big]^2\,\frac{1}{p_a.\ell + i\epsilon}\frac{1}{p_b.\ell - i\epsilon}\,\ell^\alpha \ell^\beta \\
&= -\frac{1}{2}\Big[p_a^\alpha \frac{\partial}{\partial p_{a\beta}} + p_b^\alpha \frac{\partial}{\partial p_{b\beta}} + \eta^{\alpha\beta}\Big] \int_\omega^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\,G_r(-\ell)\,\frac{1}{p_a.\ell + i\epsilon}\frac{1}{p_b.\ell - i\epsilon} \\
&= \frac{1}{8\pi}\delta_{\eta_a\eta_b,1} \ln\big\{L(\omega + i\epsilon\eta_a)\big\}\,\frac{1}{[(p_a.p_b)^2 - p_a^2 p_b^2]^{3/2}}\Big[p_a.p_b\{p_a^\alpha p_b^\beta + p_b^\alpha p_a^\beta\} - p_b^2 p_a^\alpha p_a^\beta \\
&\quad - p_a^2 p_b^\alpha p_b^\beta - \eta^{\alpha\beta}\{(p_a.p_b)^2 - p_a^2 p_b^2\}\Big]
\end{aligned} \tag{4.B.1}$$

$$\begin{aligned}
J_2^{\alpha\beta\gamma} &= \int_\omega^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\,\big[G_r(-\ell)\big]^2\,\frac{1}{p_b.\ell - i\epsilon}\frac{1}{(p_a.\ell + i\epsilon)^2}\,\ell^\alpha \ell^\beta \ell^\gamma \\
&= \frac{1}{2}\bigg[p_b^\gamma \frac{\partial}{\partial p_{b\alpha}}\frac{\partial}{\partial p_{a\beta}} + p_a^\gamma \frac{\partial}{\partial p_{a\alpha}}\frac{\partial}{\partial p_{a\beta}} + \Big(\eta^{\alpha\gamma}\frac{\partial}{\partial p_{a\beta}} + \eta^{\beta\gamma}\frac{\partial}{\partial p_{a\alpha}}\Big)\bigg] \\
&\quad \times \int_\omega^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\,G_r(-\ell)\,\frac{1}{p_a.\ell + i\epsilon}\frac{1}{p_b.\ell - i\epsilon} \\
&= \frac{1}{8\pi}\delta_{\eta_a\eta_b,1} \ln\big\{L(\omega + i\epsilon\eta_a)\big\}\frac{1}{[(p_a.p_b)^2 - p_a^2 p_b^2]^{5/2}}\Big[3(p_a.p_b)^2 p_a^\alpha p_b^\beta p_b^\gamma - 3p_a^2 p_a.p_b p_b^\alpha p_b^\beta p_b^\gamma \\
&\quad - 3p_b^2 p_a.p_b p_a^\alpha p_a^\beta p_b^\gamma + 3p_a^2 p_b^2 p_b^\alpha p_a^\beta p_b^\gamma + 3(p_a.p_b)^2 p_b^\alpha p_b^\beta p_a^\gamma - 3p_b^2 p_a.p_b p_a^\alpha p_b^\beta p_a^\gamma \\
&\quad - 3p_b^2 p_a.p_b p_b^\alpha p_a^\beta p_a^\gamma + 3(p_b^2)^2 p_a^\alpha p_a^\beta p_a^\gamma\Big] \\
&\quad - \frac{1}{8\pi}\delta_{\eta_a\eta_b,1} \ln\big\{L(\omega + i\epsilon\eta_a)\big\}\frac{1}{[(p_a.p_b)^2 - p_a^2 p_b^2]^{3/2}}\Big[p_a^\alpha p_b^\beta p_b^\gamma - 2p_b^\alpha p_a^\beta p_b^\gamma + p_a.p_b \eta^{\alpha\beta} p_b^\gamma \\
&\quad + p_b^\alpha p_b^\beta p_a^\gamma - p_b^2 \eta^{\alpha\beta} p_a^\gamma + p_a.p_b p_b^\beta \eta^{\alpha\gamma} - p_b^2 p_a^\beta \eta^{\alpha\gamma} + p_a.p_b p_b^\alpha \eta^{\beta\gamma} - p_b^2 p_a^\alpha \eta^{\beta\gamma}\Big]
\end{aligned} \tag{4.B.2}$$

$$\begin{aligned}
J_3^{\alpha} &= \int_\omega^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\,G_r(-\ell)\,\frac{1}{p_b.\ell - i\epsilon}\frac{1}{(p_a.\ell + i\epsilon)^2}\,\ell^\alpha \\
&= \frac{1}{4\pi}\delta_{\eta_a\eta_b,1} \ln\big\{L(\omega + i\epsilon\eta_a)\big\}\frac{1}{[(p_a.p_b)^2 - p_a^2 p_b^2]^{3/2}}\Big[p_a.p_b p_b^\alpha - p_b^2 p_a^\alpha\Big]
\end{aligned} \tag{4.B.3}$$

To evaluate the expression in Eq. (4.4.53), we divide it into sum over four integrals:

$$\Delta_{(1)}^{(\omega \ln\omega)}\widehat{T}_2^{h\mu\nu}(k) = \mathcal{L}_1^{\mu\nu} + \mathcal{L}_2^{\mu\nu} + \mathcal{L}_3^{\mu\nu} + \mathcal{L}_4^{\mu\nu} \tag{4.B.4}$$

The first three lines after the equality in Eq. (4.4.53) takes the following form after removing the terms containing $p_a.\ell$ and $p_b.\ell$ in the numerator[11],

$$\begin{aligned}\mathcal{L}_1^{\mu\nu} \;=\;& \frac{i}{2}(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\Big\{G_r(-\ell)\Big\}^3(-2k.\ell)\frac{1}{p_b.\ell-i\epsilon}\frac{1}{p_a.\ell+i\epsilon}\\ &\Big[2\ell^\mu\ell^\nu p_a.p_b p_{a\alpha}J_b^{\alpha\beta}\ell_\beta - p_a^2\ell^\mu\ell^\nu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta + 2\ell^2 p_b^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta + 2\ell^2 p_a.p_b p_a^\nu J_b^{\mu\alpha}\ell_\alpha\\ &-\eta^{\mu\nu}\ell^2 p_a.p_b p_{a\alpha}J_b^{\alpha\beta}\ell_\beta + \frac{1}{2}p_a^2\eta^{\mu\nu}\ell^2 p_{b\alpha}J_b^{\alpha\beta}\ell_\beta - 2\ell^2 p_a^\mu p_a^\nu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta\Big]\\ &-\frac{i}{2}(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\Big\{G_r(-\ell)\Big\}^3(-2k.\ell)\frac{1}{p_b.\ell-i\epsilon}\frac{1}{p_a.\ell+i\epsilon}\\ &\Big[2\ell^\mu\ell^\nu p_a.p_b p_{b\rho}J_a^{\rho\sigma}\ell_\sigma - p_b^2\ell^\mu\ell^\nu p_{a\rho}J_a^{\rho\sigma}\ell_\sigma + \frac{1}{2}p_b^2\ell^2\eta^{\mu\nu}p_{a\rho}J_a^{\rho\sigma}\ell_\sigma - \ell^2 p_b^2 p_a^\mu J_a^{\nu\sigma}\ell_\sigma\\ &-\ell^2 p_b^2 p_a^\nu J_a^{\mu\sigma}\ell_\sigma + 2\ell^2 p_a.p_b p_b^\mu J_a^{\nu\sigma}\ell_\sigma + 2\ell^2 p_a^\nu p_b^\mu p_{b\rho}J_a^{\rho\sigma}\ell_\sigma - \ell^2 p_a.p_b\eta^{\mu\nu}p_{b\rho}J_a^{\rho\sigma}\ell_\sigma\Big]\end{aligned} \tag{4.B.5}$$

In the above expression the particle indices a, b are dummy, so we can exchange $a \leftrightarrow b$ in the second integral above. In doing so we observe that many terms are same within the square bracket of the first and second integrands, but there is relative sign between the two integrals, hence those terms will cancel each other[12]. After cancelling those terms we combine the two integrals into one and get the following expression,

$$\begin{aligned}\mathcal{L}_1^{\mu\nu} \;=\;& -\frac{i}{2}(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\Big\{G_r(-\ell)\Big\}^2(-2k.\ell)\frac{1}{p_b.\ell-i\epsilon}\frac{1}{p_a.\ell+i\epsilon}\\ &\Big[2p_b^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta - 2p_a^\mu p_a^\nu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta + p_a^2 p_b^\mu J_b^{\nu\sigma}\ell_\sigma + p_a^2 p_b^\nu J_b^{\mu\sigma}\ell_\sigma - 2p_b^\nu p_a^\mu p_{a\rho}J_b^{\rho\sigma}\ell_\sigma\\ &+2p_a.p_b p_a^\nu J_b^{\mu\alpha}\ell_\alpha - 2p_a.p_b p_a^\mu J_b^{\nu\alpha}\ell_\alpha\Big]\end{aligned} \tag{4.B.6}$$

Fourth to sixth lines after the equality in Eq. (4.4.53) takes the following form after removing the terms containing $(p_a.\ell)^2$ and $p_b.\ell$ in the numerator,

$$\begin{aligned}\mathcal{L}_2^{\mu\nu} \;=\;& \frac{i}{2}(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\Big\{G_r(-\ell)\Big\}^2(p_a.k)\frac{1}{p_b.\ell-i\epsilon}\frac{1}{(p_a.\ell+i\epsilon)^2}\\ &\Big[2\ell^\mu\ell^\nu p_a.p_b p_{a\alpha}J_b^{\alpha\beta}\ell_\beta - p_a^2\ell^\mu\ell^\nu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta + 2\ell^2 p_b^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta + 2\ell^2 p_a.p_b p_a^\nu J_b^{\mu\alpha}\ell_\alpha\\ &-\eta^{\mu\nu}\ell^2 p_a.p_b p_{a\alpha}J_b^{\alpha\beta}\ell_\beta + \frac{1}{2}p_a^2\eta^{\mu\nu}\ell^2 p_{b\alpha}J_b^{\alpha\beta}\ell_\beta - 2\ell^2 p_a^\mu p_a^\nu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta + 2\ell^\mu p_a^\nu p_a.\ell p_{b\alpha}J_b^{\alpha\beta}\ell_\beta\Big]\\ &-\frac{i}{2}(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\Big\{G_r(-\ell)\Big\}^2(p_a.k)\frac{1}{p_b.\ell-i\epsilon}\frac{1}{(p_a.\ell+i\epsilon)^2}\\ &\Big[2\ell^\mu\ell^\nu p_a.p_b p_{b\rho}J_a^{\rho\sigma}\ell_\sigma - p_b^2\ell^\mu\ell^\nu p_{a\rho}J_a^{\rho\sigma}\ell_\sigma + \frac{1}{2}p_b^2\ell^2\eta^{\mu\nu}p_{a\rho}J_a^{\rho\sigma}\ell_\sigma - \ell^2 p_b^2 p_a^\mu J_a^{\nu\sigma}\ell_\sigma\\ &-\ell^2 p_b^2 p_a^\nu J_a^{\mu\sigma}\ell_\sigma + 2\ell^2 p_a.p_b p_b^\mu J_a^{\nu\sigma}\ell_\sigma + 2\ell^2 p_a^\nu p_b^\mu p_{b\rho}J_a^{\rho\sigma}\ell_\sigma - \ell^2 p_a.p_b\eta^{\mu\nu}p_{b\rho}J_a^{\rho\sigma}\ell_\sigma\\ &+p_b^2\ell^\mu p_a.\ell J_a^{\nu\alpha}\ell_\alpha\Big]\end{aligned} \tag{4.B.7}$$

[11]We do not keep the terms containing $p_a.\ell$ or $p_b.\ell$ in the numerator as they cancel with the denominator of the integrand and then it can be shown that the integration result for those terms vanishes after the ℓ^0 contour integration.

[12]Obviously for the cancelled terms between the two integrals there will be a sign difference in front of $i\epsilon$ in $\{p_a.\ell \pm i\epsilon\}^{-1}$ and $\{p_b.\ell \pm i\epsilon\}^{-1}$ but the contribution turns out to be the same as only the relative signs in front of $i\epsilon$ between the two denominator matter.

Seventh to nineth lines after the equality in Eq. (4.4.53) takes the following form after removing the terms containing $p_a.\ell$ and $p_b.\ell$ in the numerator,

$$
\begin{aligned}
\mathcal{L}_3^{\mu\nu} \;=\;& \frac{i}{2}(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\Big\{G_r(-\ell)\Big\}^2\frac{1}{p_b.\ell-i\epsilon}\frac{1}{p_a.\ell+i\epsilon}\\
&\Big[-2\ell^\mu k^\nu p_a.p_b p_{a\alpha}J_b^{\alpha\beta}\ell_\beta+p_a^2k^\nu\ell^\mu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta-4k^\mu\ell^\nu p_a.p_b p_{a\alpha}J_b^{\alpha\beta}\ell_\beta\\
&+2p_a^2k^\mu\ell^\nu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta+2p_a.p_b p_a^\mu\ell^\nu k_\alpha J_b^{\alpha\beta}\ell_\beta+2p_b.kp_a^\mu\ell^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta\\
&-2p_a.kp_a^\mu\ell^\nu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta-p_a^2p_b^\mu\ell^\nu k_\alpha J_b^{\alpha\beta}\ell_\beta-p_a^2p_b.k\ell^\nu J_b^{\mu\alpha}\ell_\alpha\\
&+2p_a.p_b p_a^\nu\ell^\mu k_\alpha J_b^{\alpha\beta}\ell_\beta+2p_b.kp_a^\nu\ell^\mu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta-2p_a.kp_a^\nu\ell^\mu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta\\
&+6k.\ell p_a^\mu p_a^\nu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta-\frac{5}{2}k.\ell p_a^2\eta^{\mu\nu}p_{b\alpha}J_b^{\alpha\beta}\ell_\beta-2k.\ell p_a.p_b p_a^\mu J_b^{\nu\alpha}\ell_\alpha\\
&-2k.\ell p_b^\nu p_a^\mu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta+k.\ell p_a^2p_b^\mu J_b^{\nu\alpha}\ell_\alpha+k.\ell p_a^2p_b^\nu J_b^{\mu\alpha}\ell_\alpha\\
&-4k.\ell p_b^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta-4k.\ell p_a.p_b p_a^\nu J_b^{\mu\alpha}\ell_\alpha+5k.\ell\eta^{\mu\nu}p_a.p_b p_{a\alpha}J_b^{\alpha\beta}\ell_\beta\Big]\\
&-\frac{i}{2}(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\Big\{G_r(-\ell)\Big\}^2\frac{1}{p_b.\ell-i\epsilon}\frac{1}{p_a.\ell+i\epsilon}\\
&\Big[-2p_a.p_b\ell^\mu k^\nu p_{b\alpha}J_a^{\alpha\beta}\ell_\beta+p_b^2\ell^\mu k^\nu p_{a\alpha}J_a^{\alpha\beta}\ell_\beta-4k^\mu\ell^\nu p_a.p_b p_{b\alpha}J_a^{\alpha\beta}\ell_\beta\\
&+2p_b^2k^\mu\ell^\nu p_{a\alpha}J_a^{\alpha\beta}\ell_\beta+2p_b.kp_a.p_b\ell^\nu J_a^{\mu\alpha}\ell_\alpha+2p_b.kp_a^\mu\ell^\nu p_{b\alpha}J_a^{\alpha\beta}\ell_\beta\\
&-2p_b.kp_b^\mu\ell^\nu p_{a\alpha}J_a^{\alpha\beta}\ell_\beta-p_b^2p_a.k\ell^\nu J_a^{\mu\alpha}\ell_\alpha-p_b^2\ell^\nu p_a^\mu k_\alpha J_a^{\alpha\beta}\ell_\beta\\
&+2p_a.p_bp_b.k\ell^\mu J_a^{\nu\alpha}\ell_\alpha+2p_a^\nu\ell^\mu p_b.kp_{b\alpha}J_a^{\alpha\beta}\ell_\beta-p_b^2p_a.k\ell^\mu J_a^{\nu\alpha}\ell_\alpha\\
&-p_b^2\ell^\mu p_a^\nu k_\alpha J_a^{\alpha\beta}\ell_\beta+2p_b^\mu p_b^\nu k.\ell p_{a\alpha}J_a^{\alpha\beta}\ell_\beta-\frac{5}{2}p_b^2k.\ell\eta^{\mu\nu}p_{a\alpha}J_a^{\alpha\beta}\ell_\beta\\
&-4k.\ell p_a.p_b p_b^\mu J_a^{\nu\alpha}\ell_\alpha-4k.\ell p_b^\mu p_a^\nu p_{b\alpha}J_a^{\alpha\beta}\ell_\beta-2k.\ell p_b^\nu p_a.p_b J_a^{\mu\alpha}\ell_\alpha\\
&-2k.\ell p_b^\nu p_a^\mu p_{b\alpha}J_a^{\alpha\beta}\ell_\beta+3k.\ell p_b^2p_a^\nu J_a^{\mu\alpha}\ell_\alpha+3k.\ell p_b^2p_a^\mu J_a^{\nu\alpha}\ell_\alpha\\
&+5\eta^{\mu\nu}p_a.p_bk.\ell p_{b\alpha}J_a^{\alpha\beta}\ell_\beta\Big]
\end{aligned}
\tag{4.B.8}
$$

In the above expression the particle indices a, b are dummy, so we can exchange $a \leftrightarrow b$ in the second integral above. In doing so we observe that many terms are same within the square bracket of the first and second integrands, but there is relative sign between the two integrals, hence those terms will cancel each other and we get[13],

$$
\begin{aligned}
\mathcal{L}_3^{\mu\nu} \;=\;& \frac{i}{2}(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\Big\{G_r(-\ell)\Big\}^2\frac{1}{p_b.\ell-i\epsilon}\frac{1}{p_a.\ell+i\epsilon}\\
&\Big[2p_a.p_bp_a^\mu\ell^\nu k_\alpha J_b^{\alpha\beta}\ell_\beta+2p_b.kp_a^\mu\ell^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta+2p_a.p_bp_a^\nu\ell^\mu k_\alpha J_b^{\alpha\beta}\ell_\beta+2p_b.kp_a^\nu\ell^\mu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta\\
&-2p_a.kp_a^\nu\ell^\mu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta+4k.\ell p_a^\mu p_a^\nu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta+2k.\ell p_a.p_bp_a^\mu J_b^{\nu\alpha}\ell_\alpha+2k.\ell p_b^\nu p_a^\mu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta\\
&-2k.\ell p_a^2p_b^\mu J_b^{\nu\alpha}\ell_\alpha-2k.\ell p_a^2p_b^\nu J_b^{\mu\alpha}\ell_\alpha-2k.\ell p_b^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta-2k.\ell p_a.p_bp_a^\nu J_b^{\mu\alpha}\ell_\alpha\\
&-2p_a.kp_a.p_b\ell^\nu J_b^{\mu\alpha}\ell_\alpha-2p_a.kp_b^\mu\ell^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta-2p_a.kp_a.p_b\ell^\mu J_b^{\nu\alpha}\ell_\alpha-2p_b^\nu\ell^\mu p_a.kp_{a\alpha}J_b^{\alpha\beta}\ell_\beta\\
&+p_a^2p_b.k\ell^\mu J_b^{\nu\alpha}\ell_\alpha+p_a^2\ell^\mu p_b^\nu k_\alpha J_b^{\alpha\beta}\ell_\beta\Big]
\end{aligned}
\tag{4.B.9}
$$

The last two lines in Eq. (4.4.53) represents $\mathcal{L}_4^{\mu\nu}$, which takes the following form after contracting the indices of the last line of Eq. (4.4.53) and removing the terms containing $p_a.\ell$ and $p_b.\ell$ in the numerator, as they give vanishing contribution when we perform the ℓ integration.

$$
\mathcal{L}_4^{\mu\nu} \;=\; \frac{i}{2}(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\Big\{G_r(-\ell)\Big\}^2\frac{1}{p_b.\ell-i\epsilon}\frac{1}{p_a.\ell+i\epsilon}
$$

[13]Here also the same logic holds as described in footnote-(12).

$$\Big[2\ell^\mu\ell^\nu p_a.p_b p_{b\rho}J_a^{\rho\sigma}k_\sigma + p_b^2 p_a^\nu \ell^\mu \ell_\rho J_a^{\rho\sigma}k_\sigma - p_b^2\ell^\mu\ell^\nu p_{a\rho}J_a^{\rho\sigma}k_\sigma + \frac{1}{2}p_b^2\ell^2\eta^{\mu\nu}p_{a\rho}J_a^{\rho\sigma}k_\sigma$$
$$-\ell^2 p_b^2 p_a^\mu J_a^{\nu\sigma}k_\sigma - \ell^2 p_b^2 p_a^\nu J_a^{\mu\sigma}k_\sigma + 2\ell^2 p_a.p_b p_b^\mu J_a^{\nu\sigma}k_\sigma + 2\ell^2 p_a^\nu p_b^\mu p_{b\rho}J_a^{\rho\sigma}k_\sigma$$
$$-\ell^2 p_a.p_b\eta^{\mu\nu}p_{b\rho}J_a^{\rho\sigma}k_\sigma\Big] \tag{4.B.10}$$

Let us first evaluate the sum of the contributions in Eq. (4.B.6) and Eq. (4.B.9) ,

$$\mathcal{L}_1^{\mu\nu} + \mathcal{L}_3^{\mu\nu}$$
$$= \frac{i}{2}(8\pi G)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\Big\{G_r(-\ell)\Big\}^2\frac{1}{p_b.\ell - i\epsilon}\frac{1}{p_a.\ell + i\epsilon}$$
$$\Big[2k.\ell p_b^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta - 2k.\ell p_b^\nu p_a^\mu p_{a\rho}J_b^{\rho\sigma}\ell_\sigma + 2p_a.p_b p_a^\mu \ell^\nu k_\alpha J_b^{\alpha\beta}\ell_\beta + 2p_b.kp_a^\mu\ell^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta$$
$$+2p_a.p_b p_a^\nu\ell^\mu k_\alpha J_b^{\alpha\beta}\ell_\beta + 2p_b.kp_a^\nu\ell^\mu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta - 2p_a.kp_a^\nu\ell^\mu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta - 2k.\ell p_a.p_b p_a^\mu J_b^{\nu\alpha}\ell_\alpha$$
$$+2k.\ell p_a.p_b p_a^\nu J_b^{\mu\alpha}\ell_\alpha - 2p_a.kp_a.p_b\ell^\nu J_b^{\mu\alpha}\ell_\alpha - 2p_a.kp_b^\mu\ell^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta - 2p_a.kp_a.p_b\ell^\mu J_b^{\nu\alpha}\ell_\alpha$$
$$-2p_b^\nu\ell^\mu p_a.kp_{a\alpha}J_b^{\alpha\beta}\ell_\beta + p_a^2 p_b.k\ell^\mu J_b^{\nu\alpha}\ell_\alpha + p_a^2\ell^\mu p_b^\nu k_\alpha J_b^{\alpha\beta}\ell_\beta\Big] \tag{4.B.11}$$

Now using the result of the integral given in Eq. (4.B.1) we get,

$$\mathcal{L}_1^{\mu\nu} + \mathcal{L}_3^{\mu\nu}$$
$$= \frac{iG}{2}\sum_{\substack{a,b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N}\frac{1}{\big[(p_a.p_b)^2 - p_a^2 p_b^2\big]^{3/2}}\ \ln\big\{L(\omega + i\epsilon\eta_a)\big\}$$
$$\Big[2p_a.p_b p_a.kp_b^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}p_{b\beta} - 4p_a.p_b p_a.kp_b^\nu p_a^\mu p_{a\alpha}J_b^{\alpha\beta}p_{b\beta}$$
$$-4p_a^2 p_b.kp_b^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}p_{b\beta} - 4(p_a.p_b)^2 p_b^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}k_\beta + 2p_a^2 p_b^2 p_b^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}k_\beta$$
$$-p_a^2 p_b^2 p_a^\mu p_b^\nu p_{a\alpha}J_b^{\alpha\beta}k_\beta - 4(p_a.p_b)^2 p_a^\mu p_a^\nu p_{b\alpha}J_b^{\alpha\beta}k_\beta + 4p_a.p_b p_b^2 p_a^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}k_\beta$$
$$+p_a.p_b p_a^2 p_a^\mu p_b^\nu p_{b\alpha}J_b^{\alpha\beta}k_\beta + 2p_a.p_b p_a^2 p_b^\mu p_a^\nu p_{b\alpha}J_b^{\alpha\beta}k_\beta + 4(p_a.p_b)^2 p_a.kp_a^\mu p_{b\alpha}J_b^{\alpha\nu}$$
$$+4p_a.p_b p_b.kp_a^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}p_{b\beta} + 3p_b.kp_a^2 p_b^2 p_a^\mu p_{a\alpha}J_b^{\alpha\nu} - 4p_b.k(p_a.p_b)^2 p_a^\nu p_{a\alpha}J_b^{\alpha\mu}$$
$$+2p_b.kp_a^2 p_b^2 p_a^\nu p_{a\alpha}J_b^{\alpha\mu} + 2p_a.kp_b^2 p_a^\mu p_a^\nu p_{b\alpha}J_b^{\alpha\beta}p_{a\beta} - 3p_a^2 p_a.p_b p_b.kp_a^\mu p_{b\alpha}J_b^{\alpha\nu}$$
$$+2p_a^2 p_a.p_b p_b.kp_a^\nu p_{b\alpha}J_b^{\alpha\mu} - 4p_a.p_b p_b^2 p_a.kp_a^\mu p_{a\alpha}J_b^{\alpha\nu} + 4p_a.p_b\{(p_a.p_b)^2 - p_a^2 p_b^2\}p_a^\mu J_b^{\nu\alpha}k_\alpha$$
$$+4p_a.k(p_a.p_b)^2 p_b^\nu p_{a\alpha}J_b^{\alpha\mu} - 2p_a.kp_a^2 p_b^2 p_b^\nu p_{a\alpha}J_b^{\alpha\mu} - 2p_a.kp_a.p_b p_a^2 p_b^\nu p_{b\alpha}J_b^{\alpha\mu}$$
$$+4p_a.k(p_a.p_b)^2 p_b^\mu p_{a\alpha}J_b^{\alpha\nu} - 2p_a.kp_a^2 p_b^2 p_b^\mu p_{a\alpha}J_b^{\alpha\nu} - 2p_a.kp_a.p_b p_a^2 p_b^\mu p_{b\alpha}J_b^{\alpha\nu}$$
$$+4p_a.kp_a^2 p_b^\mu p_b^\nu p_{a\alpha}J_b^{\alpha\beta}p_{b\beta} - p_a^2 p_a.p_b p_b.kp_b^\mu p_{a\alpha}J_b^{\alpha\nu} + (p_a^2)^2 p_b.kp_b^\mu p_{b\alpha}J_b^{\alpha\nu}$$
$$+p_a^2 p_b.k\{(p_a.p_b)^2 - p_a^2 p_b^2\}J_b^{\mu\nu} + p_a^2\{(p_a.p_b)^2 - p_a^2 p_b^2\}\ p_b^\nu J_b^{\mu\rho}k_\rho$$
$$-p_a^2 p_a.p_b p_b^\mu p_b^\nu p_{a\alpha}J_b^{\alpha\beta}k_\beta + (p_a^2)^2 p_b^\mu p_b^\nu p_{b\alpha}J_b^{\alpha\beta}k_\beta + 2p_a.k\{(p_a.p_b)^2 - p_a^2 p_b^2\}p_a^\nu p_{b\alpha}J_b^{\alpha\mu}\Big] \tag{4.B.12}$$

In the above expression we substitute $J_b^{\mu\nu} = r_b^\mu p_b^\nu - r_b^\nu p_b^\mu + \Sigma_b^{\mu\nu}$ and simplify using SSC, $p_{b\mu}\Sigma_b^{\mu\nu} = 0$. Since a, b are dummy indices we interchange them and then the full expression can be written in terms of r_a and Σ_a. After all these steps finally we get,

$$\mathcal{L}_1^{\mu\nu} + \mathcal{L}_3^{\mu\nu}$$

$$= \frac{iG}{2}\sum_{\substack{a,b=1\\b\neq a\\\eta_a\eta_b=1}}^{M+N}\frac{1}{\left[(p_a.p_b)^2-p_a^2p_b^2\right]^{3/2}}\ \ln\left\{L(\omega+i\epsilon\eta_a)\right\}$$
$$\Big[2p_a.p_bp_b.kp_a^\mu p_b^\nu p_b.r_ap_a^2+8p_b^2p_a.kp_a^\mu p_b^\nu p_a.p_bp_a.r_a-8(p_a.p_b)^2p_a^\mu p_b^\nu p_b.r_ap_a.k$$
$$+4(p_a.p_b)^3p_a^\mu p_b^\nu r_a.k-4p_a^2p_b^2p_a^\mu p_b^\nu p_a.p_br_a.k$$
$$-2p_a^2p_b^2p_b.kp_b^\nu p_a^\mu p_a.r_a-8p_a.p_bp_b.kp_a^\nu p_b^\mu p_b.r_ap_a^2+8(p_a.p_b)^2p_b.kp_a^\nu p_b^\mu p_a.r_a$$
$$+2p_a^2p_b^2p_b^\mu p_a^\nu p_b.r_ap_a.k+4p_a^2p_b^2p_b^\mu p_a^\nu p_a.p_br_a.k-2p_a.p_bp_b^2p_b^\mu p_a^\nu p_a.r_ap_a.k$$
$$-4(p_a.p_b)^3p_b^\mu p_a^\nu r_a.k-8(p_a.p_b)^2p_b^\nu p_b^\mu p_a.r_ap_a.k+8p_a.p_bp_a^2p_b^\mu p_b^\nu p_b.r_ap_a.k$$
$$+2p_b.kp_a^2p_b^\mu p_b^\nu p_a.r_ap_a.p_b-2p_b.k(p_a^2)^2p_b^\mu p_b^\nu p_b.r_a+4p_a.p_b\{(p_a.p_b)^2-p_a^2p_b^2\}p_a.kp_b^\mu r_a^\nu$$
$$+4p_a.k(p_a.p_b)^3p_b^\nu r_a^\mu-4p_a.kp_a^2p_b^2p_a.p_bp_b^\nu r_a^\mu-2p_b.k\{(p_a.p_b)^2-p_a^2p_b^2\}p_a^2p_b^\nu r_a^\mu$$
$$+8p_b.k(p_a.p_b)^2p_a^\nu p_b.r_ap_a^\mu-8p_b.kp_a.p_bp_b^2p_a.r_ap_a^\mu p_a^\nu-2p_b^2p_a.p_bp_a.kp_a^\mu p_a^\nu p_b.r_a$$
$$+2(p_b^2)^2p_a.kp_a.r_ap_a^\mu p_a^\nu-4p_b.k(p_a.p_b)^3p_a^\nu r_a^\mu+4p_b.kp_a^2p_b^2p_a.p_bp_a^\nu r_a^\mu$$
$$+p_b^2\{(p_a.p_b)^2-p_a^2p_b^2\}p_a.kp_a^\nu r_a^\mu-4p_b.k(p_a.p_b)^3p_a^\mu r_a^\nu+4p_b.kp_a^2p_b^2p_a.p_bp_a^\mu r_a^\nu$$
$$+p_b^2\{(p_a.p_b)^2-p_a^2p_b^2\}p_a.kp_a^\mu r_a^\nu-4(p_a.p_b)^2p_a^\mu p_b^\nu p_{b\alpha}\Sigma_a^{\alpha\beta}k_\beta+2p_a^2p_b^2p_a^\mu p_b^\nu p_{b\alpha}\Sigma_a^{\alpha\beta}k_\beta$$
$$-p_a^2p_b^2p_b^\mu p_a^\nu p_{b\alpha}\Sigma_a^{\alpha\beta}k_\beta+4p_a.p_bp_a^2p_b^\mu p_b^\nu p_{b\alpha}\Sigma_a^{\alpha\beta}k_\beta+3p_a.kp_a^2p_b^2p_b^\mu p_{b\alpha}\Sigma_a^{\alpha\nu}$$
$$-4p_a.k(p_a.p_b)^2p_b^\nu p_{b\alpha}\Sigma_a^{\alpha\mu}+2p_a.kp_a^2p_b^2p_b^\nu p_{b\alpha}\Sigma_a^{\alpha\mu}-4p_a.p_bp_a^2p_b.kp_b^\mu p_{b\alpha}\Sigma_a^{\alpha\nu}$$
$$+4p_a.p_b\{(p_a.p_b)^2-p_a^2p_b^2\}p_b^\mu\Sigma_a^{\nu\alpha}k_\alpha+4p_b.k(p_a.p_b)^2p_a^\nu p_{b\alpha}\Sigma_a^{\alpha\mu}-2p_b.kp_a^2p_b^2p_a^\nu p_{b\alpha}\Sigma_a^{\alpha\mu}$$
$$+4p_b.k(p_a.p_b)^2p_a^\mu p_{b\alpha}\Sigma_a^{\alpha\nu}-2p_b.kp_a^2p_b^2p_a^\mu p_{b\alpha}\Sigma_a^{\alpha\nu}-p_b^2p_a.p_bp_a.kp_a^\mu p_{b\alpha}\Sigma_a^{\alpha\nu}$$
$$+p_b^2p_a.k\{(p_a.p_b)^2-p_a^2p_b^2\}J_a^{\mu\nu}+p_b^2\{(p_a.p_b)^2-p_a^2p_b^2\}p_a^\nu\Sigma_a^{\mu\rho}k_\rho-p_b^2p_a.p_bp_a^\mu p_a^\nu p_{b\alpha}\Sigma_a^{\alpha\beta}k_\beta\Big] \tag{4.B.13}$$

From Eq. (4.B.10) using the result of the integration Eq. (4.B.1), we get

$$\mathcal{L}_4^{\mu\nu} = \frac{iG}{2}\sum_{\substack{a,b=1\\b\neq a\\\eta_a\eta_b=1}}^{M+N}\frac{1}{\left[(p_a.p_b)^2-p_a^2p_b^2\right]^{3/2}}\ln\left\{L(\omega+i\epsilon\eta_a)\right\}\Big[2(p_a.p_b)^2p_{b\rho}J_a^{\rho\sigma}k_\sigma(p_a^\mu p_b^\nu+p_a^\nu p_b^\mu)$$
$$-2p_a.p_bp_a^2p_{b\rho}J_a^{\rho\sigma}k_\sigma p_b^\mu p_b^\nu-p_a.p_bp_b^2p_{b\rho}J_a^{\rho\sigma}k_\sigma p_a^\mu p_a^\nu-p_a^2p_b^2p_{b\rho}J_a^{\rho\sigma}k_\sigma p_a^\nu p_b^\mu$$
$$-p_a.p_bp_b^2p_{a\rho}J_a^{\rho\sigma}k_\sigma p_a^\mu p_b^\nu+p_a^2p_b^2p_{a\rho}J_a^{\rho\sigma}k_\sigma p_b^\mu p_b^\nu\Big]$$
$$-(iG)\sum_{\substack{a,b=1\\b\neq a\\\eta_a\eta_b=1}}^{M+N}\frac{1}{\left[(p_a.p_b)^2-p_a^2p_b^2\right]^{1/2}}\ \ln\left\{L(\omega+i\epsilon\eta_a)\right\}\Big[-p_b^2p_a^\mu J_a^{\nu\alpha}k_\alpha-\frac{1}{2}p_b^2p_a^\nu J_a^{\mu\alpha}k_\alpha$$
$$+2p_a.p_bp_b^\mu J_a^{\nu\alpha}k_\alpha+2p_a^\nu p_b^\mu p_{b\rho}J_a^{\rho\sigma}k_\sigma\Big] \tag{4.B.14}$$

Now substituting $J_a^{\mu\nu}=r_a^\mu p_a^\nu-r_a^\nu p_a^\mu+\Sigma_a^{\mu\nu}$ in the above expression and using SSC : $p_{a\mu}\Sigma_a^{\mu\nu}=0$, we get

$$\mathcal{L}_4^{\mu\nu} = \frac{iG}{2}\sum_{\substack{a,b=1\\b\neq a\\\eta_a\eta_b=1}}^{M+N}\frac{1}{\left[(p_a.p_b)^2-p_a^2p_b^2\right]^{3/2}}\ln\left\{L(\omega+i\epsilon\eta_a)\right\}\Big[2(p_a.p_b)^2p_b.r_ap_a.k(p_a^\mu p_b^\nu+p_a^\nu p_b^\mu)$$
$$-2(p_a.p_b)^3r_a.k(p_a^\mu p_b^\nu+p_a^\nu p_b^\mu)-2p_a.p_bp_a^2p_b.r_ap_a.kp_b^\mu p_b^\nu+2(p_a.p_b)^2p_a^2r_a.kp_b^\mu p_b^\nu$$
$$-p_a.p_bp_b^2p_b.r_ap_a.kp_a^\mu p_a^\nu+(p_a.p_b)^2p_b^2r_a.kp_a^\mu p_a^\nu-p_a^2p_b^2p_b.r_ap_a.kp_a^\nu p_b^\mu+p_a^2p_b^2p_a.p_br_a.kp_a^\nu p_b^\mu$$

$$
\begin{aligned}
&-p_a.p_b p_b^2 p_a.r_a p_a.k p_a^\mu p_b^\nu + p_a.p_b p_a^2 p_b^2 r_a.k p_a^\mu p_b^\nu + p_a^2 p_b^2 p_a.r_a p_a.k p_b^\mu p_b^\nu - (p_a^2)^2 p_b^2 r_a.k p_b^\mu p_b^\nu \\
&+2(p_a.p_b)^2 p_{b\rho}\Sigma_a^{\rho\sigma} k_\sigma (p_a^\mu p_b^\nu + p_a^\nu p_b^\mu) - 2p_a.p_b p_a^2 p_{b\rho}\Sigma_a^{\rho\sigma} k_\sigma p_b^\mu p_b^\nu - p_a.p_b p_b^2 p_{b\rho}\Sigma_a^{\rho\sigma} k_\sigma p_a^\mu p_a^\nu \\
&-p_a^2 p_b^2 p_{b\rho}\Sigma_a^{\rho\sigma} k_\sigma p_a^\nu p_b^\mu \Big] \\
&-(iG)\sum_{\substack{a,b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N} \frac{1}{\left[(p_a.p_b)^2 - p_a^2 p_b^2\right]^{1/2}} \ln\left\{L(\omega+i\epsilon\eta_a)\right\}\Big[-p_b^2 p_a^\mu r_a^\nu p_a.k + \frac{3}{2} p_b^2 p_a^\mu p_a^\nu r_a.k \\
&-\frac{1}{2} p_b^2 p_a^\nu r_a^\mu p_a.k + 2p_a.p_b p_b^\mu r_a^\nu p_a.k - 4p_a.p_b p_b^\mu p_a^\nu r_a.k + 2p_a^\nu p_b^\mu p_b.r_a p_a.k \\
&-p_b^2 p_a^\mu \Sigma_a^{\nu\alpha} k_\alpha - \frac{1}{2} p_b^2 p_a^\nu \Sigma_a^{\mu\alpha} k_\alpha + 2p_a.p_b p_b^\mu \Sigma_a^{\nu\alpha} k_\alpha + 2p_a^\nu p_b^\mu p_{b\rho}\Sigma_a^{\rho\sigma} k_\sigma\Big]
\end{aligned}
\tag{4.B.15}
$$

Contribution from Eq. (4.B.7) after using the results of integrations Eq. (4.B.1), Eq. (4.B.2) and Eq. (4.B.3) takes the following form,

$$
\begin{aligned}
\mathcal{L}_2^{\mu\nu} \;=\;& \frac{iG}{2}\sum_{\substack{a,b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N} \frac{p_a.k}{\left[(p_a.p_b)^2 - p_a^2 p_b^2\right]^{5/2}} \ln\left\{L(\omega+i\epsilon\eta_a)\right\} \\
&\Big[6(p_a.p_b)^3 p_a^\mu p_b^\nu p_{a\alpha} J_b^{\alpha\beta} p_{b\beta} - 3(p_a.p_b)^2 p_a^2 p_b^\mu p_b^\nu p_{a\alpha} J_b^{\alpha\beta} p_{b\beta} - 6(p_a.p_b)^2 p_b^2 p_a^\mu p_a^\nu p_{a\alpha} J_b^{\alpha\beta} p_{b\beta} \\
&+3p_a.p_b p_a^2 p_b^2 p_b^\mu p_a^\nu p_{a\alpha} J_b^{\alpha\beta} p_{b\beta} - 3p_a.p_b p_a^2 p_b^2 p_a^\mu p_b^\nu p_{a\alpha} J_b^{\alpha\beta} p_{b\beta} + 3p_a^2 (p_b^2)^2 p_a^\mu p_a^\nu p_{a\alpha} J_b^{\alpha\beta} p_{b\beta}\Big] \\
&-\frac{iG}{2}\sum_{\substack{a,b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N} \frac{p_a.k}{\left[(p_a.p_b)^2 - p_a^2 p_b^2\right]^{5/2}} \ln\left\{L(\omega+i\epsilon\eta_a)\right\} \\
&\Big[3p_b^2 (p_a.p_b)^2 p_a^\mu p_b^\nu p_{a\alpha} J_a^{\alpha\beta} p_{b\beta} + 3p_a^2 p_b^2 p_a.p_b p_b^\mu p_b^\nu p_{a\alpha} J_a^{\alpha\beta} p_{b\beta} - 3(p_b^2)^2 p_a.p_b p_a^\mu p_a^\nu p_{a\alpha} J_a^{\alpha\beta} p_{b\beta} \\
&-3(p_b^2)^2 p_a^2 p_b^\mu p_a^\nu p_{a\alpha} J_a^{\alpha\beta} p_{b\beta} + 6(p_a.p_b)^3 p_b^\mu p_b^\nu p_{b\alpha} J_a^{\alpha\beta} p_{a\beta} - 6(p_a.p_b)^2 p_b^2 p_b^\mu p_a^\nu p_{b\alpha} J_a^{\alpha\beta} p_{a\beta}\Big] \\
&-\frac{iG}{2}\sum_{\substack{a,b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N} \frac{p_a.k}{\left[(p_a.p_b)^2 - p_a^2 p_b^2\right]^{3/2}} \ln\left\{L(\omega+i\epsilon\eta_a)\right\} \Big[2p_a.p_b p_a^\mu p_b^\nu p_{a\alpha} J_b^{\alpha\beta} p_{b\beta} \\
&+p_a^2 p_b^\mu p_b^\nu p_{a\alpha} J_b^{\alpha\beta} p_{b\beta} + 2(p_a.p_b)^2 p_b^\nu p_{a\alpha} J_b^{\alpha\mu} - p_a^2 p_a.p_b p_b^\nu p_{b\alpha} J_b^{\alpha\mu} + 2p_a.p_b p_b^2 p_a^\nu p_{a\alpha} J_b^{\alpha\mu} \\
&-p_a^2 p_b^2 p_a^\nu p_{b\alpha} J_b^{\alpha\mu} + 2(p_a.p_b)^2 p_b^\mu p_{a\alpha} J_b^{\alpha\nu} - p_a^2 p_a.p_b p_b^\mu p_{b\alpha} J_b^{\alpha\nu} - 2p_a.p_b p_b^2 p_a^\mu p_{a\alpha} J_b^{\alpha\nu} \\
&+p_a^2 p_b^2 p_a^\mu p_{b\alpha} J_b^{\alpha\nu} - 2(p_a.p_b)^2 p_a^\nu p_{b\alpha} J_b^{\alpha\mu} - 6p_b^2 p_a^\mu p_a^\nu p_{a\alpha} J_b^{\alpha\beta} p_{b\beta} + 2p_a.p_b p_b^\mu p_a^\nu p_{a\alpha} J_b^{\alpha\beta} p_{b\beta}\Big] \\
&+\frac{iG}{2}\sum_{\substack{a,b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N} \frac{p_a.k}{\left[(p_a.p_b)^2 - p_a^2 p_b^2\right]^{3/2}} \ln\left\{L(\omega+i\epsilon\eta_a)\right\} \Big[-p_b^2 p_a^\mu p_b^\nu p_{a\alpha} J_a^{\alpha\beta} p_{b\beta} \\
&+6p_b^2 p_b^\mu p_a^\nu p_{a\alpha} J_a^{\alpha\beta} p_{b\beta} + 2p_a.p_b p_b^\mu p_b^\nu p_{b\alpha} J_a^{\alpha\beta} p_{a\beta} + 2(p_a.p_b)^2 p_b^\nu p_{b\alpha} J_a^{\alpha\mu} - p_b^2 p_a.p_b p_b^\nu p_{a\alpha} J_a^{\alpha\mu} \\
&-(p_b^2)^2 p_a^\nu p_{a\alpha} J_a^{\alpha\mu} - 2(p_a.p_b)^2 p_b^\mu p_{b\alpha} J_a^{\alpha\nu} + 4p_b^2 p_a.p_b p_b^\mu p_{a\alpha} J_a^{\alpha\nu} - 2(p_b^2)^2 p_a^\mu p_{a\alpha} J_a^{\alpha\nu} \\
&+p_b^2 p_a.p_b p_a^\mu p_{b\alpha} J_a^{\alpha\nu} - p_a^2 p_b^2 p_b^\mu p_{b\alpha} J_a^{\alpha\nu} - p_b^2\left\{(p_a.p_b)^2 - p_a^2 p_b^2\right\} J_a^{\mu\nu}\Big]
\end{aligned}
\tag{4.B.16}
$$

Now in the above expression we substitute total angular momenta in terms of orbital and spin and interchange $a \leftrightarrow b$ in appropriate places to make the result r_a and Σ_a dependent.

Then the above expression finally reduces to,

$$\begin{aligned}
\mathcal{L}_2^{\mu\nu} \;=\;& \frac{iG}{2}\sum_{\substack{a,b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N} \frac{1}{\big[(p_a.p_b)^2-p_a^2p_b^2\big]^{5/2}}\ln\big\{L(\omega+i\epsilon\eta_a)\big\}\\
&\Big[p_b.k\Big\{6(p_a.p_b)^3p_b^\mu p_a^\nu p_b.r_a p_a^2-6(p_a.p_b)^4p_b^\mu p_a^\nu p_a.r_a-3(p_a.p_b)^2p_a^2p_b^2p_a^\mu p_a^\nu p_b.r_a\\
&+3(p_a.p_b)^3p_b^2p_a^\mu p_a^\nu p_a.r_a-6(p_a.p_b)^2(p_a^2)^2p_b^\mu p_b^\nu p_b.r_a+6(p_a.p_b)^3p_a^2p_b^\mu p_b^\nu p_a.r_a\\
&+3p_a.p_b(p_a^2)^2p_b^2p_a^\mu p_b^\nu p_b.r_a-3(p_a.p_b)^2p_a^2p_b^2p_a^\mu p_b^\nu p_a.r_a-3p_a.p_b(p_a^2)^2p_b^2p_b^\mu p_a^\nu p_b.r_a\\
&+3(p_a.p_b)^2p_a^2p_b^2p_b^\mu p_a^\nu p_a.r_a+3p_b^2(p_a^2)^3p_b^\mu p_b^\nu p_b.r_a-3p_b^2(p_a^2)^2p_a.p_bp_b^\mu p_b^\nu p_a.r_a\Big\}\\
&-p_a.k\Big\{3p_b^2(p_a.p_b)^3p_a^\mu p_b^\nu p_a.r_a-3p_b^2(p_a.p_b)^2p_a^\mu p_b^\nu p_a^2p_b.r_a+3p_a^2p_b^2(p_a.p_b)^2p_b^\mu p_b^\nu p_a.r_a\\
&-3(p_a^2)^2p_b^2p_a.p_bp_b^\mu p_b^\nu p_b.r_a-3(p_b^2)^2(p_a.p_b)^2p_a^\mu p_a^\nu p_a.r_a+3(p_b^2)^2p_a^2p_a.p_bp_a^\mu p_a^\nu p_b.r_a\\
&-3(p_b^2)^2p_a^2p_a.p_bp_b^\mu p_a^\nu p_a.r_a+3(p_a^2)^2(p_b^2)^2p_b^\mu p_b^\nu p_b.r_a+6(p_a.p_b)^3p_b^\mu p_b^\nu p_a^2p_b.r_a\\
&-6(p_a.p_b)^4p_b^\mu p_b^\nu p_a.r_a-6(p_a.p_b)^2p_a^2p_b^2p_b^\mu p_a^\nu p_b.r_a+6(p_a.p_b)^3p_b^2p_b^\mu p_a^\nu p_a.r_a\Big\}\Big]\\
&-\frac{iG}{2}\sum_{\substack{a,b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N} \frac{1}{\big[(p_a.p_b)^2-p_a^2p_b^2\big]^{3/2}}\ln\big\{L(\omega+i\epsilon\eta_a)\big\}\\
&\Big[p_b.k\Big\{-2(p_a.p_b)^2p_b^\mu p_a^\nu p_a.r_a+p_a^2p_b^2p_a^\mu p_a^\nu p_b.r_a-3p_b^2p_a.p_bp_a^\mu p_a^\nu p_a.r_a\\
&+4(p_a.p_b)^2p_a^\nu p_b.r_ap_a^\mu-2(p_a.p_b)^3p_a^\nu r_a^\mu+p_a^2p_b^2p_a.p_bp_a^\nu r_a^\mu+4p_a.p_bp_a^2p_b^\nu p_b.r_ap_a^\mu\\
&-p_a^2p_b^2p_b^\nu p_a.r_ap_a^\mu+(p_a^2)^2p_b^2p_b^\nu r_a^\mu-2(p_a.p_b)^3p_a^\mu r_a^\nu+p_a^2p_b^2p_a.p_bp_a^\mu r_a^\nu\\
&+2(p_a.p_b)^2p_a^2p_b^\mu r_a^\nu+p_a^2p_b^2p_b^\mu p_a.r_ap_a^\nu-(p_a^2)^2p_b^2p_b^\mu r_a^\nu-4(p_a.p_b)^2p_b^\nu p_a.r_ap_a^\mu\\
&-6(p_a^2)^2p_b^\mu p_b^\nu p_b.r_a+6p_a^2p_b^\mu p_b^\nu p_a.p_bp_a.r_a+2(p_a.p_b)^2p_a^\nu p_{b\alpha}\Sigma_a^{\alpha\mu}\\
&+2p_a.p_bp_a^2p_b^\nu p_{b\alpha}\Sigma_a^{\alpha\mu}+2(p_a.p_b)^2p_a^\mu p_{b\alpha}\Sigma_a^{\alpha\nu}-2p_a.p_bp_a^2p_b^\mu p_{b\alpha}\Sigma_a^{\alpha\nu}\Big\}\\
&-p_a.k\Big\{-2p_b^2p_a^\mu p_b^\nu p_a.r_ap_a.p_b+p_a^2p_b^2p_a^\mu p_b^\nu p_b.r_a+10p_b^2p_b^\mu p_a^\nu p_a.r_ap_a.p_b\\
&-7p_a^2p_b^2p_b^\mu p_a^\nu p_b.r_a+2p_a.p_bp_b^\mu p_b^\nu p_b.r_ap_a^2-2(p_a.p_b)^2p_b^\mu p_b^\nu p_a.r_a+2(p_a.p_b)^2p_b^\nu p_a^\mu p_b.r_a\\
&-2(p_a.p_b)^3p_b^\nu r_a^\mu+p_a^2p_b^2p_a.p_bp_b^\nu r_a^\mu-3(p_b^2)^2p_a.r_ap_a^\mu p_a^\nu+(p_b^2)^2p_a^2p_a^\nu r_a^\mu\\
&-2(p_a.p_b)^2p_b.r_ap_b^\mu p_a^\nu+2(p_a.p_b)^3p_b^\mu r_a^\nu-3p_a^2p_b^2p_a.p_bp_b^\mu r_a^\nu+2p_a^2(p_b^2)^2p_a^\mu r_a^\nu\\
&+p_b^2p_a.p_bp_a^\mu p_b.r_ap_a^\nu-p_b^2(p_a.p_b)^2p_a^\mu r_a^\nu-p_b^2\{(p_a.p_b)^2-p_a^2p_b^2\}J_a^{\mu\nu}+2(p_a.p_b)^2p_b^\nu p_{b\alpha}\Sigma_a^{\alpha\mu}\\
&-2(p_a.p_b)^2p_b^\mu p_{b\alpha}\Sigma_a^{\alpha\nu}+p_b^2p_a.p_bp_a^\mu p_{b\alpha}\Sigma_a^{\alpha\nu}-p_a^2p_b^2p_b^\mu p_{b\alpha}\Sigma_a^{\alpha\nu}\Big\}\Big] \qquad (4.B.17)
\end{aligned}$$

After summing over the contributions of Eq. (4.B.13), Eq. (4.B.17) and Eq. (4.B.15) we find the following order $\mathcal{O}(\omega\ln\omega)$ contribution from the gravitational energy-momentum tensor expression Eq. (4.4.53):

$$\begin{aligned}
&\Delta_{(1)}^{(\omega\ln\omega)}\widehat{T}_2^{h\mu\nu}(k)\\
=\;& \frac{iG}{2}\sum_{\substack{a,b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N} \frac{1}{\big[(p_a.p_b)^2-p_a^2p_b^2\big]^{5/2}}\ln\big\{L(\omega+i\epsilon\eta_a)\big\}\\
&\Big[p_b.k\Big\{6(p_a.p_b)^3p_b^\mu p_a^\nu p_b.r_ap_a^2-6(p_a.p_b)^4p_b^\mu p_a^\nu p_a.r_a-3(p_a.p_b)^2p_a^2p_b^2p_a^\mu p_a^\nu p_b.r_a\\
&+3(p_a.p_b)^3p_b^2p_a^\mu p_a^\nu p_a.r_a-6(p_a.p_b)^2(p_a^2)^2p_b^\mu p_b^\nu p_b.r_a+6(p_a.p_b)^3p_a^2p_b^\mu p_b^\nu p_a.r_a
\end{aligned}$$

$$
\begin{aligned}
&+3p_a.p_b(p_a^2)^2p_b^2p_a^\mu p_b^\nu p_b.r_a - 3(p_a.p_b)^2p_a^2p_b^2p_a^\mu p_b^\nu p_a.r_a - 3p_a.p_b(p_a^2)^2p_b^2p_b^\mu p_a^\nu p_b.r_a \\
&+3(p_a.p_b)^2p_a^2p_b^2p_b^\mu p_a^\nu p_a.r_a + 3p_b^2(p_a^2)^3p_b^\mu p_b^\nu p_b.r_a - 3p_b^2(p_a^2)^2p_a.p_bp_b^\mu p_b^\nu p_a.r_a\Big\} \\
&-p_a.k\Big\{3p_b^2(p_a.p_b)^3p_a^\mu p_b^\nu p_a.r_a - 3p_b^2(p_a.p_b)^2p_a^\mu p_b^\nu p_a^2p_b.r_a + 3p_a^2p_b^2(p_a.p_b)^2p_b^\mu p_b^\nu p_a.r_a \\
&-3(p_a^2)^2p_b^2p_a.p_bp_b^\mu p_b^\nu p_b.r_a - 3(p_b^2)^2(p_a.p_b)^2p_a^\mu p_a^\nu p_a.r_a + 3(p_b^2)^2p_a^2p_a.p_bp_a^\mu p_a^\nu p_b.r_a \\
&-3(p_b^2)^2p_a^2p_a.p_bp_b^\mu p_a^\nu p_a.r_a + 3(p_a^2)^2(p_b^2)^2p_b^\mu p_a^\nu p_b.r_a + 6(p_a.p_b)^3p_b^\mu p_b^\nu p_a^2p_b.r_a \\
&-6(p_a.p_b)^4p_b^\mu p_b^\nu p_a.r_a - 6(p_a.p_b)^2p_a^2p_b^2p_b^\mu p_a^\nu p_b.r_a + 6(p_a.p_b)^3p_b^2p_b^\mu p_a^\nu p_a.r_a\Big\}\Big] \\
&+\frac{iG}{2}\sum_{\substack{a,b=1\\ b\neq a\\ \eta_a\eta_b=1}}^{M+N}\frac{1}{\big[(p_a.p_b)^2-p_a^2p_b^2\big]^{3/2}}\ln\big\{L(\omega+i\epsilon\eta_a)\big\} \\
&\Big[-2p_a.p_bp_b.kp_b.r_ap_a^2p_a^\mu p_b^\nu + 5p_b^2p_a.kp_a.p_bp_a.r_ap_a^\mu p_b^\nu - 4(p_a.p_b)^2p_b.r_ap_a.kp_a^\mu p_b^\nu \\
&+2(p_a.p_b)^3r_a.kp_a^\mu p_b^\nu - 3p_a^2p_b^2p_a.p_br_a.kp_a^\mu p_b^\nu \\
&-p_a^2p_b^2p_b.kp_a.r_ap_a^\mu p_b^\nu - 8p_a.p_bp_b.kp_a^2p_b.r_ap_a^\nu p_b^\mu + 10(p_a.p_b)^2p_b.kp_a.r_ap_a^\nu p_b^\mu \\
&-2p_a^2p_b^2p_b.r_ap_a.kp_a^\nu p_b^\mu - 4(p_a.p_b)^2p_b.r_ap_a.kp_a^\nu p_b^\mu - 3p_a^2p_b^2p_a.p_br_a.kp_a^\nu p_b^\mu \\
&+8p_a.p_bp_b^2p_a.r_ap_a.kp_a^\nu p_b^\mu + 2(p_a.p_b)^3r_a.kp_a^\nu p_b^\mu - 10(p_a.p_b)^2p_a.r_ap_a.kp_b^\mu p_b^\nu \\
&+8p_a.p_bp_a^2p_b.r_ap_a.kp_b^\mu p_b^\nu - 4p_b.kp_a^2p_a.r_ap_a.p_bp_b^\mu p_b^\nu + 4p_b.k(p_a^2)^2p_b.r_ap_b^\mu p_b^\nu \\
&+2(p_a.p_b)^3p_a.kp_b^\mu r_a^\nu - 3p_a^2p_b^2p_a.p_bp_a.kp_b^\mu r_a^\nu + 2p_a.k(p_a.p_b)^3p_b^\nu r_a^\mu \\
&-3p_a.kp_a^2p_b^2p_a.p_bp_b^\nu r_a^\mu - 2p_b.k(p_a.p_b)^2p_a^2p_b^\nu r_a^\mu + (p_a^2)^2p_b^2p_b^\nu r_a^\mu p_b.k \\
&+4p_b.k(p_a.p_b)^2p_b.r_ap_a^\mu p_a^\nu - 5p_b.kp_a.p_bp_b^2p_a.r_ap_a^\mu p_a^\nu - 2p_b^2p_a.p_bp_a.kp_b.r_ap_a^\mu p_a^\nu \\
&-(p_b^2)^2p_a.kp_a.r_ap_a^\mu p_a^\nu - 2p_b.k(p_a.p_b)^3p_a^\nu r_a^\mu + 3p_b.kp_a^2p_b^2p_a.p_bp_a^\nu r_a^\mu \\
&+2p_b^2(p_a.p_b)^2p_a.kp_a^\nu r_a^\mu - p_a^2(p_b^2)^2p_a.kp_a^\nu r_a^\mu - 2p_b.k(p_a.p_b)^3p_a^\mu r_a^\nu \\
&+3p_b.kp_a^2p_b^2p_a.p_bp_a^\mu r_a^\nu + 2p_b^2(p_a.p_b)^2p_a.kp_a^\mu r_a^\nu - p_a^2(p_b^2)^2p_a.kp_a^\mu r_a^\nu \\
&+2(p_a.p_b)^2p_a^2r_a.kp_b^\mu p_b^\nu - 2(p_a.p_b)^2p_b^2r_a.kp_a^\mu p_a^\nu + p_a^2p_b^2p_a.r_ap_a.kp_b^\mu p_b^\nu \\
&-(p_a^2)^2p_b^2r_a.kp_b^\mu p_b^\nu + 3p_a^2(p_b^2)^2r_a.kp_a^\mu p_a^\nu - p_b.kp_a^2p_b^2p_b.r_ap_a^\mu p_a^\nu \\
&-2p_b.k(p_a.p_b)^2p_a^2p_b^\mu r_a^\nu - p_b.kp_a^2p_b^2p_a.r_ap_b^\mu p_a^\nu + p_b.k(p_a^2)^2p_b^2p_b^\mu r_a^\nu \\
&+4p_b.k(p_a.p_b)^2p_a.r_ap_a^\mu p_b^\nu + p_a.kp_a^2p_b^2p_b.r_ap_a^\mu p_b^\nu \\
&-2\{(p_a.p_b)^2-p_a^2p_b^2\}p_a^\mu p_b^\nu p_{b\alpha}\Sigma_a^{\alpha\beta}k_\beta - 2\{(p_a.p_b)^2-p_a^2p_b^2\}p_a^\nu p_b^\mu p_{b\alpha}\Sigma_a^{\alpha\beta}k_\beta \\
&-2\{(p_a.p_b)^2-p_a^2p_b^2\}p_a.kp_b^\mu p_{b\alpha}\Sigma_a^{\alpha\nu} - 2\{(p_a.p_b)^2-p_a^2p_b^2\}p_a.kp_b^\nu p_{b\alpha}\Sigma_a^{\alpha\mu} \\
&+2\{(p_a.p_b)^2-p_a^2p_b^2\}p_b.kp_a^\nu p_{b\alpha}\Sigma_a^{\alpha\mu} + 2\{(p_a.p_b)^2-p_a^2p_b^2\}p_b.kp_a^\mu p_{b\alpha}\Sigma_a^{\alpha\nu} \\
&+2\{(p_a.p_b)^2-p_a^2p_b^2\}p_b^2p_a^\nu\Sigma_a^{\mu\alpha}k_\alpha + 2\{(p_a.p_b)^2-p_a^2p_b^2\}p_b^2p_a^\mu\Sigma_a^{\nu\alpha}k_\alpha \\
&+2p_a.p_bp_a^2p_b^\mu p_b^\nu p_{b\alpha}\Sigma_a^{\alpha\beta}k_\beta - 2p_a.p_bp_a^2p_b.kp_b^\mu p_{b\alpha}\Sigma_a^{\alpha\nu} - 2p_b^2p_a.p_bp_a^\mu p_a^\nu p_{b\alpha}\Sigma_a^{\alpha\beta}k_\beta \\
&-2p_b.kp_a.p_bp_a^2p_b^\nu p_{b\alpha}\Sigma_a^{\alpha\mu}\Big]
\end{aligned}
\tag{4.B.18}
$$

4.C Soft radiation from high frequency gravitational wave emission

Let us first write down the expression of Eq. (4.4.45) with the replacement of $G_r(k-\ell)G_r(\ell)$ by $-2\pi i\delta(\ell^2)\big[H(\ell^0)-H(-\ell^0)\big]G_r(k-\ell)$,

$$\begin{aligned}\widehat{T}^{h\mu\nu}_{extra}(k) \;\equiv\; &(8\pi G)(2\pi i)\sum_{a,b=1}^{M+N}\int \frac{d^4\ell}{(2\pi)^4}\;\delta(\ell^2)\big[H(\ell^0)-H(-\ell^0)\big]G_r(k-\ell)\frac{1}{p_b.\ell-i\epsilon}\;\frac{1}{p_a.(k-\ell)-i\epsilon}\\ &\times\Big\{p_{b\alpha}p_{b\beta}-\frac{1}{2}p_b^2\eta_{\alpha\beta}+ip_{b(\alpha}J_{b,\beta)\gamma}\ell^\gamma-\frac{i}{2}\eta_{\alpha\beta}p_b^\delta J_{b,\delta\gamma}\ell^\gamma\Big\}\;\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)\\ &\times\Big\{p_{a\rho}p_{a\sigma}-\frac{1}{2}p_a^2\eta_{\rho\sigma}+ip_{a(\rho}J_{a,\sigma)\delta}(k-\ell)^\delta-\frac{i}{2}\eta_{\rho\sigma}p_a^\kappa J_{a,\kappa\tau}(k-\ell)^\tau\Big\}\end{aligned}\tag{4.C.1}$$

We want to analyze the above expression in the integration region $\omega << |\ell^\mu| << L^{-1}$. Due to the presence of $\big[H(\ell^0)-H(-\ell^0)\big]$ inside the integrand, the part of the integrand containing even number of ℓ vanishes. On the other hand to produce a $\ln\omega$ factor we need an integrand with four power of ℓ in the denominator, which will vanish due to the presence of $\big[H(\ell^0)-H(-\ell^0)\big]$. Hence just from this argument it is clear that we can not receive any non-vanishing $\ln\omega$ or $\omega\ln\omega$ contribution from the above expression. Still for completeness, let us analysze the non-vanishing contribution of the above expression up to order $\mathcal{O}(\omega^0)$ in the integration range $\omega << |\ell^\mu| << L^{-1}$:

$$\begin{aligned}&\Delta\widehat{T}^{h\mu\nu}_{extra}(k)\\ =\;& -(8\pi G)(2\pi i)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\;\delta(\ell^2)\big[H(\ell^0)-H(-\ell^0)\big]\frac{1}{2k.\ell-i\epsilon}\frac{1}{p_b.\ell-i\epsilon}\;\frac{1}{p_a.\ell+i\epsilon}\\ &\Bigg[\Big\{p_{b\alpha}p_{b\beta}-\frac{1}{2}p_b^2\eta_{\alpha\beta}\Big\}\;\Delta_{(\ell\ell)}\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)\Big\{p_{a\rho}p_{a\sigma}-\frac{1}{2}p_a^2\eta_{\rho\sigma}+ip_{a(\rho}J_{a,\sigma)\delta}k^\delta-\frac{i}{2}\eta_{\rho\sigma}p_a^\kappa J_{a,\kappa\tau}k^\tau\Big\}\\ &-\Big\{p_{b\alpha}p_{b\beta}-\frac{1}{2}p_b^2\eta_{\alpha\beta}\Big\}\;\Delta_{(k\ell)}\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)\Big\{ip_{a(\rho}J_{a,\sigma)\delta}\ell^\delta-\frac{i}{2}\eta_{\rho\sigma}p_a^\kappa J_{a,\kappa\tau}\ell^\tau\Big\}\\ &+\Big\{ip_{b(\alpha}J_{b,\beta)\gamma}\ell^\gamma-\frac{i}{2}\eta_{\alpha\beta}p_b^\delta J_{b,\delta\gamma}\ell^\gamma\Big\}\;\Delta_{(k\ell)}\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)\Big\{p_{a\rho}p_{a\sigma}-\frac{1}{2}p_a^2\eta_{\rho\sigma}\Big\}\Bigg]\\ &-(8\pi G)(2\pi i)\sum_{a,b=1}^{M+N}\int_\omega^{L^{-1}}\frac{d^4\ell}{(2\pi)^4}\;\delta(\ell^2)\big[H(\ell^0)-H(-\ell^0)\big]\frac{1}{2k.\ell-i\epsilon}\frac{1}{p_b.\ell-i\epsilon}\;\frac{p_a.k}{(p_a.\ell+i\epsilon)^2}\\ &\Bigg[-\Big\{p_{b\alpha}p_{b\beta}-\frac{1}{2}p_b^2\eta_{\alpha\beta}\Big\}\;\Delta_{(\ell\ell)}\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)\Big\{ip_{a(\rho}J_{a,\sigma)\delta}\ell^\delta-\frac{i}{2}\eta_{\rho\sigma}p_a^\kappa J_{a,\kappa\tau}\ell^\tau\Big\}\\ &+\Big\{ip_{b(\alpha}J_{b,\beta)\gamma}\ell^\gamma-\frac{i}{2}\eta_{\alpha\beta}p_b^\delta J_{b,\delta\gamma}\ell^\gamma\Big\}\;\Delta_{(\ell\ell)}\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(k,\ell)\Big\{p_{a\rho}p_{a\sigma}-\frac{1}{2}p_a^2\eta_{\rho\sigma}\Big\}\Bigg]\end{aligned}\tag{4.C.2}$$

Now in the above expression after contraction of various terms within the square bracket, we find some terms containing ℓ^2, which will vanish due to the presence of $\delta(\ell^2)$. On the other hand if we get $p_b.\ell$ (or $p_a.\ell$) then it cancels with the denominator $\{p_b.\ell-i\epsilon\}^{-1}$ (or$\{p_a.\ell+i\epsilon\}^{-1}$) and if only one such denominator present then after cancellation of it the rest of the coefficient vanishes after using $\sum\limits_b p_b^\alpha=0$ or $\sum\limits_b J_b^{\alpha\beta}=0$ (or $\sum\limits_a p_a^\alpha=0$ or $\sum\limits_a J_a^{\alpha\beta}=0$). Hence after eliminating those terms and interchanging $a\leftrightarrow b$ in some places[14], the above expression simplifies to:

$$\Delta\widehat{T}^{h\mu\nu}_{extra}(k)$$

[14] Though the signs of $i\epsilon$ are different for denominators between $p_a.\ell$ and $p_b.\ell$, still $a\leftrightarrow b$ interchange makes sense as the integrals have to evaluate with $\delta(\ell^2)$. Hence we can set $\epsilon=0$ from the beginning.

$$
\begin{aligned}
= \;& -(8\pi G)(2\pi i)\sum_{a,b=1}^{M+N}\int_{\omega}^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\,\delta(\ell^2)H(\ell^0)\frac{\ell^\mu\ell^\nu}{k.\ell-i\epsilon}\frac{1}{p_b.\ell-i\epsilon}\,\frac{1}{p_a.\ell+i\epsilon}\Big\{(p_a.p_b)^2-\frac{1}{2}p_a^2p_b^2\Big\}\\
& +(8\pi^2G)\sum_{a,b=1}^{M+N}\int_{\omega}^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\,\delta(\ell^2)H(\ell^0)\frac{1}{k.\ell-i\epsilon}\frac{1}{p_b.\ell-i\epsilon}\,\frac{1}{p_a.\ell+i\epsilon}\,\Big\{2\ell^\mu\ell^\nu p_a.p_bp_{a\rho}J_b^{\rho\sigma}k_\sigma\\
& -p_a^2\ell^\mu\ell^\nu p_{b\rho}J_b^{\rho\sigma}k_\sigma+2p_a.p_bp_a^\mu\ell^\nu k_\alpha J_b^{\alpha\beta}\ell_\beta+2p_b.kp_a^\mu\ell^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta+2p_a.p_bp_a^\nu\ell^\mu k_\alpha J_b^{\alpha\beta}\ell_\beta\\
& +2p_b.kp_a^\nu\ell^\mu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta-2p_a.kp_a.p_b\ell^\nu J_b^{\mu\alpha}\ell_\alpha-2p_a.kp_b^\mu\ell^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta-2p_a.kp_a.p_b\ell^\mu J_b^{\nu\alpha}\ell_\alpha\\
& -2p_b^\nu\ell^\mu p_a.kp_{a\alpha}J_b^{\alpha\beta}\ell_\beta\Big\}\\
& +(8\pi^2G)\sum_{a,b=1}^{M+N}\int_{\omega}^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\,\delta(\ell^2)H(\ell^0)\frac{1}{k.\ell-i\epsilon}\frac{1}{p_b.\ell-i\epsilon}\,\frac{p_a.k}{(p_a.\ell+i\epsilon)^2}\\
& \Big[2\ell^\mu\ell^\nu p_a.p_bp_{a\alpha}J_b^{\alpha\beta}\ell_\beta-p_a^2\ell^\mu\ell^\nu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta-2\ell^\mu\ell^\nu p_a.p_bp_{b\rho}J_a^{\rho\sigma}\ell_\sigma+p_b^2\ell^\mu\ell^\nu p_{a\rho}J_a^{\rho\sigma}\ell_\sigma\Big]\\
& +(8\pi^2G)\sum_{a,b=1}^{M+N}\int_{\omega}^{L^{-1}} \frac{d^4\ell}{(2\pi)^4}\,\delta(\ell^2)H(\ell^0)\frac{1}{p_b.\ell-i\epsilon}\,\frac{1}{p_a.\ell+i\epsilon}\,\Big\{4p_a^\mu p_a^\nu p_{b\alpha}J_b^{\alpha\beta}\ell_\beta+2p_a.p_bp_a^\mu J_b^{\nu\alpha}\ell_\alpha\\
& +2p_b^\nu p_a^\mu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta-2p_a^2p_b^\mu J_b^{\nu\alpha}\ell_\alpha-2p_a^2p_b^\nu J_b^{\mu\alpha}\ell_\alpha-2p_b^\mu p_a^\nu p_{a\alpha}J_b^{\alpha\beta}\ell_\beta-2p_a.p_bp_a^\nu J_b^{\mu\alpha}\ell_\alpha\Big\}
\end{aligned}
\tag{4.C.3}
$$

The first line in the above expression has been identified with leading order soft radiation from real hard gravitational radiation in appendix-B of [118]. So here generalizing appendix-B of [118] to the next order, we try to understand whether the rest of the terms above can be understood as subleading order soft radiation from real hard gravitational radiation. Taking care of gravitational flux up to subleading order, the energy-momentum tensor for soft gravitational radiation with momentum k becomes,

$$
\begin{aligned}
& \widehat{T}_R^{\mu\nu}(k)\\
= \;& \frac{G}{\pi^2}\sum_{a,b=1}^{M+N}\int d^4\ell\;\delta(\ell^2)H(\ell^0)\left[\frac{p_a^\rho p_a^\sigma-ip_a^{(\rho}(k-\ell)_\alpha J_a^{\alpha\sigma)}}{p_a.(\ell-k)+i\epsilon}\right]\left[\frac{p_b^\kappa p_b^\tau-ip_b^{(\kappa}\ell_\beta J_b^{\beta\tau)}}{p_b.\ell-i\epsilon}\right]\\
& \times\frac{\ell^\mu\ell^\nu\sum_r\varepsilon^r_{\rho\sigma}\varepsilon^{r*}_{\kappa\tau}-i\ell^{(\mu}k_\gamma\big[\Sigma_g^{\gamma\nu)}\big]_{\rho\sigma,\kappa\tau}}{i(\ell.k-i\epsilon)}
\end{aligned}
\tag{4.C.4}
$$

where polarisation sum and spin tensor of soft gravitational field are given by,

$$
\sum_r\varepsilon^r_{\rho\sigma}\varepsilon^{r*}_{\kappa\tau} = \frac{1}{2}\Big(\eta_{\rho\kappa}\eta_{\sigma\tau}+\eta_{\rho\tau}\eta_{\sigma\kappa}-\eta_{\rho\sigma}\eta_{\kappa\tau}\Big) \tag{4.C.5}
$$

$$
\big[\Sigma_g^{\gamma\nu)}\big]_{\rho\sigma,\kappa\tau} = -\frac{i}{2}\Big[\eta_{\rho\kappa}(\delta_\sigma^\gamma\delta_\tau^\nu-\delta_\sigma^\nu\delta_\tau^\gamma)+\eta_{\rho\tau}(\delta_\sigma^\gamma\delta_\kappa^\nu-\delta_\sigma^\nu\delta_\kappa^\gamma)+\eta_{\sigma\kappa}(\delta_\rho^\gamma\delta_\tau^\nu-\delta_\rho^\nu\delta_\tau^\gamma)+\eta_{\sigma\tau}(\delta_\rho^\gamma\delta_\kappa^\nu-\delta_\rho^\nu\delta_\kappa^\gamma)\Big] \tag{4.C.6}
$$

Substituting the above results in Eq. (4.C.4) and analyzing in the integration region $\omega << |\ell^\mu| << L^{-1}$, we get the following non-vanishing contribution up to order ω^0:

$$
\begin{aligned}
& \widehat{T}_R^{\mu\nu}(k)\\
= \;& -\frac{iG}{\pi^2}\sum_{a,b=1}^{M+N}\int_{\omega}^{L^{-1}} d^4\ell\;\delta(\ell^2)H(\ell^0)\frac{\ell^\mu\ell^\nu}{k.\ell-i\epsilon}\frac{1}{p_b.\ell-i\epsilon}\,\frac{1}{p_a.\ell+i\epsilon}\Big\{(p_a.p_b)^2-\frac{1}{2}p_a^2p_b^2\Big\}\\
& +\frac{G}{2\pi^2}\sum_{a,b=1}^{M+N}\int_{\omega}^{L^{-1}} d^4\ell\;\delta(\ell^2)H(\ell^0)\frac{1}{k.\ell-i\epsilon}\frac{1}{p_b.\ell-i\epsilon}\,\frac{1}{p_a.\ell+i\epsilon}\,\Big\{2\ell^\mu\ell^\nu p_a.p_bp_{a\rho}J_b^{\rho\sigma}k_\sigma
\end{aligned}
$$

$$-p_a^2 \ell^\mu \ell^\nu p_{b\rho} J_b^{\rho\sigma} k_\sigma + 2p_a.p_b p_a^\mu \ell^\nu k_\alpha J_b^{\alpha\beta} \ell_\beta + 2p_b.k p_a^\mu \ell^\nu p_{a\alpha} J_b^{\alpha\beta} \ell_\beta + 2p_a.p_b p_a^\nu \ell^\mu k_\alpha J_b^{\alpha\beta} \ell_\beta$$
$$+2p_b.k p_a^\nu \ell^\mu p_{a\alpha} J_b^{\alpha\beta} \ell_\beta - 2p_a.k p_a.p_b \ell^\nu J_b^{\mu\alpha} \ell_\alpha - 2p_a.k p_b^\mu \ell^\nu p_{a\alpha} J_b^{\alpha\beta} \ell_\beta - 2p_a.k p_a.p_b \ell^\mu J_b^{\nu\alpha} \ell_\alpha$$
$$-2p_b^\nu \ell^\mu p_a.k p_{a\alpha} J_b^{\alpha\beta} \ell_\beta \Big\}$$
$$+\frac{G}{2\pi^2} \sum_{a,b=1}^{M+N} \int_\omega^{L^{-1}} d^4\ell \; \delta(\ell^2) H(\ell^0) \frac{\ell^\mu \ell^\nu}{k.\ell - i\epsilon} \frac{1}{p_b.\ell - i\epsilon} \; \frac{p_a.k}{(p_a.\ell + i\epsilon)^2}$$
$$\Big[\Big\{2p_a.p_b p_{a\alpha} J_b^{\alpha\beta} \ell_\beta - p_a^2 p_{b\alpha} J_b^{\alpha\beta} \ell_\beta\Big\} - \Big\{2p_a.p_b p_{b\rho} J_a^{\rho\sigma} \ell_\sigma - p_b^2 p_{a\rho} J_a^{\rho\sigma} \ell_\sigma\Big\}\Big] \tag{4.C.7}$$

Now if we compare the above expression with Eq. (4.C.3), we observe that leaving the last two lines of Eq. (4.C.3), rest of the terms matches. But we do not need to worry about these extra terms as they don't contribute at order $\omega \ln \omega$.

CHAPTER 5

Conclusions and Outlook

In this thesis, we have initiated the study of asymptotic symmetries at timelike infinity in non-linear general relativity. We studied the dynamics of a probe Maxwell field on the extreme Reissner-Nordström solution in light of a discrete conformal isometry that maps the future event horizon to future null infinity and vice versa, the Couch-Torrence inversion isometry. Finally, we studied the leading spin-dependent gravitational tail memories. The thesis aims to build a better understanding of the relation between the diverse aspects of infrared physics.

The study of supertranslations at timelike infinity in chapter 2 offers several opportunities for future research. These future directions will help us expand the IR physics dictionary. We list few of them.

It is very much desirable to understand the relation between timelike infinity and null infinity. We expect our charge expressions can be matched with appropriate expressions for supertranslation and Lorentz charges at $\mathcal{I}^+_+$ (the future endpoint of the future null infinity) following [39, 42].

Can our boundary conditions we used to give a prescription for relating supertranslations at future null infinity to supertranslations at the horizon, thereby making the general idea mentioned in section 7 of [53] more precise? Note that this viewpoint differs from that of [51] where global Bondi coordinates were used to link generators at the past null infinity $\mathcal{I}^-$ and the future horizon H^+.

Finally, there are other classes of transformations, e.g., logarithmic translations, super-rotations, more general spi-supertranslations etc. that we have not considered in this work. One would like to understand their action/role at timelike infinity. For much of our non-linear analysis we used the boundary condition $k = 0$. Is it desirable to relax this condition? We hope to return to some of these problems in our future work.

In chapter 3, we have studied the dynamics of a probe Maxwell field in the ERN background in light of the Couch-Torrence (CT) symmetry. A natural question to ask is: Are there other spacetimes which exhibits the CT conformal inversion symmetry? In rotating generalisation, the Kerr-Newman metric does not exhibit a conformal inversion symmetry [86]. However, a study by Cvetič et al. [147] argues that a useful notion of CT inversion can be introduced for a class of rotating extremal 4-charge solutions. It can be interesting to understand this better.

In section 3.4, we have introduced Eastwood-Singer (ES) gauge which remains invariant under the CT transformation. We show that the asymptotic expansion of the gauge parameter is similar to the expansion of harmonic gauge parameter. It is an important question to explore how the solutions for the gauge parameter are consistent with the decay results for scalars in black hole spacetimes. Equally important is to explore the asymptotic dynamics of Maxwell's equation in terms of the gauge field $\mathcal{A}_a$ in the Eastwood-Singer gauge on the ERN spacetime. We leave these questions for future work.

Also in section 3.4 we show that the ES residual gauge parameters have the same form near future null infinity and the future horizon. This motivates us to show section 3.5 for a toy model that there is a smooth interpolation between the residual gauge parameters at the future event horizon and at the future null infinity. The study of these bulk interpolating solutions can be interesting and remains to be further explored. The toy model example studied in section 3.5 calls for a corresponding study in four-dimensional asymptotically flat settings.

Finally in section 3.6 we have made the claim of [11] precise for an ERN black hole by proving that the soft electric charges remain conserved. Our results follow from the CT transformation. If not on the ERN spacetime, it should be possible to make precise the conservation law following [103] on the spacetime with two asymptotic flat ends considered in section 3.5.

In chapter 4, we have derived leading spin-dependent gravitational tail memory which behaves like u^{-2} for retarded time $u \to \pm\infty$. The late and early time gravitational waveforms derived from sub-subleading soft graviton theorem are shown to agree the gravitational waveforms derived from classical scattering process. Other than a baby step towards finding another copy of the IR triangle, our work has theoretical and observational implications. Here we list some of them.

The observation of the leading spin-dependent tail memory will enable us to determine the spins of the scattered objects given the data of scattered momenta of the outgoing particles, impact parameters, and the angular frequency distribution of the emitted gravitational waves.

We have shown in section 4.3.2 that for a black hole binary merger process, even when the black holes carry spin, the gravitational tail memory at u^{-2} order vanishes. This is another non-trivial prediction from general relativity. The u^{-2} tail memory receives correction at order G^3 as shown in Fig. 4.2, which has not been explored yet. Whether order G^3u^{-2} memory also vanishes for binary black hole merger processes is a subject of future research.

In the recent past there has been a lot of progress in deriving various classical observables including gravitational radiation for $2 \to 2$ scattering of spinning bodies under weak gravitational interaction [127, 128, 148, 149]. There are two differences between our analysis and the analysis done in the papers cited above. The first one is about setting the boundary conditions: to derive the low-frequency gravitational waveform, we give both initial and final data for the scattered objects. This has the advantage that we do not have to specify the kind of interaction and strength within the region $\mathcal{R}$. On the other hand in the references [127,128,148,149], one specifies only the initial data for the scattered objects and derive various classical observables assuming weak gravitational interaction responsible for the scattering event. Another difference is that in the references [127, 128, 148, 149] the gravitational radiation has been expressed as an integral form without the evaluation of the integrals in low-frequency limit (soft region), as done here. But recently in [150] an attempt has been taken to relate these two approaches for non-spinning $2 \rightarrow 2$ particle scattering event under weak gravitational interaction. It is possible to generalize this idea for spinning particle scattering and re-derive our result.

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