Homological Invariants of Hibi Rings and Polyominoes

By

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DECLARATION

I declare that the thesis entitled **"Homological Invariants of Hibi Rings and Polyominoes"** submitted by me for the degree of **Doctor of Philosophy in Mathematics** is the record of academic work carried out by me under the guidance of Professor Manoj Kummini and this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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CERTIFICATE

I certify that the thesis entitled **"Homological Invariants of Hibi Rings and Polyominoes"** submitted for the degree of **Doctor of Philosophy in Mathematics** by Dharm Veer is the record of research work carried out by him under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent his independent work in a very substantial measure.

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Manoj Kummini
Thesis Supervisor.

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In Memory of my Grandfather

Abstract

In this thesis, we study the minimal graded free resolution of Hibi rings and the h-polynomial of polyomino algebras.

Green and Lazarsfeld defined property N_p for $p \in \mathbb{N}$ to study the graded minimal free resolution of S/I, where S is a polynomial ring over a field and I is an ideal generated by quadratics. The ring S/I satisfies property N_p if S/I is normal and the graded minimal free resolution of S/I over S is linear up to p-th position. We prove necessary conditions for Hibi rings to satisfy Green-Lazarsfeld property N_p for $p = 2$ and 3. We also show that a Hibi ring satisfies property N_4 if and only if either it is a polynomial ring or it has a linear resolution. In particular, it satisfies property N_p for all p.

Let $\mathcal{P}$ be a polyomino. Qureshi associated a finitely generated graded algebra $K[\mathcal{P}]$ over a field K to $\mathcal{P}$. Rinaldo and Romeo showed that if $\mathcal{P}$ is a simple thin polyomino, then the h-polynomial of $K[\mathcal{P}]$ is the rook polynomial of the polyomino $\mathcal{P}$ and they conjectured that this property characterises thin polyominoes.

In this thesis, we verify the conjecture of Rinaldo and Romeo when $\mathcal{P}$ is a non-thin convex polyomino such that its vertex set is a sublattice of $\mathbb{N}^2$. We also show that the Gorenstein rings associated with simple thin polyominoes satisfy the Charney-Davis conjecture.

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Chapter 1

Introduction

A classical problem in commutative algebra is to study graded minimal free resolutions of graded modules over polynomial rings. One of the fundamental results in this direction is Hilbert's syzygy theorem. Let $S = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field K and let M be a finitely generated graded S-module. Then M has a graded minimal free resolution, which is unique up to isomorphism. Hilbert's syzygy theorem states that the graded minimal free resolution of M has finite length, which is at most n. The length of the graded minimal free resolution of M is called the projective dimension of M. The Auslander-Buchsbaum formula expresses the projective dimension of M in terms of its depth and n.

Let I be a graded S-ideal generated by quadratics. To study the graded minimal free resolution of S/I, Green and Lazarsfeld [GL86] defined property N_p for $p \in \mathbb{N}$. The ring S/I satisfies property N_p if S/I is normal and the graded minimal free resolution of S/I over S is linear upto p-th position. In particular, if S/I satisfies property N_p for all $p \in \mathbb{N}$, then S/I has a linear resolution.

1.1 Aim of the thesis

In this thesis, we study the Green-Lazarsfeld property N_p of Hibi rings and the h-polynomial of polyomino algebras. Both Hibi rings and polyomino algebras are associated to some combinatorial objects, namely finite distributive lattices and polyominoes respectively. We utilize the tools of combinatorics to study the graded minimal free resolution and the Hilbert series of these algebraic objects.

1.2 Green-Lazarsfeld property N_p of Hibi rings

Let P be a finite poset and $\mathcal{I}(P)$ be its ideal lattice. Then $\mathcal{I}(P)$, ordered by inclusion, is a distributive lattice. By Birkhoff's fundamental structure theorem [Bir67, Chapter 9, Theorem 10], every finite distributive lattice occurs in this way.

Let P be a finite poset and $K[\mathcal{I}(P)] = K[\{x_\alpha : \alpha \in \mathcal{I}(P)\}]$ be the polynomial ring over a field K. The *Hibi ideal* associated with $\mathcal{I}(P)$, denoted by $I_{\mathcal{I}(P)}$, is the $K[\mathcal{I}(P)]$-ideal generated by the binomials $x_\alpha x_\beta - x_{\alpha\wedge\beta}x_{\alpha\vee\beta}$ where $\alpha, \beta \in \mathcal{I}(P)$ are incomparable in $\mathcal{I}(P)$. The ring $K[\mathcal{I}(P)]/I_{\mathcal{I}(P)}$ is called the *Hibi ring* associated to $\mathcal{I}(P)$ and denoted by $R[\mathcal{I}(P)]$. These rings were defined by Takayuki Hibi in [Hib87]. He showed that $R[\mathcal{I}(P)]$ is a normal Cohen–Macaulay domain of dimension $\#P + 1$, where $\#P$ is the cardinality of P. He also characterized all posets for which the associated Hibi ring is Gorenstein. In Theorem 6.15, we have characterized all posets for which the associated Hibi ring is a complete intersection.

Hibi rings are normal and Hibi ideals are generated by quadratics. Hence, Hibi rings satisfy property N_1. So it is natural to ask the following question:

Question 1.1. *For $p \in \mathbb{N}$, classify all posets for which the associated Hibi ring satisfies property N_p.*

In [Vee21a, Vee21b], we try to answer the above question for various values of p. In Theorem 3.33, we proved that a Hibi ring satisfies property N_4 if and only if either it is a polynomial ring or it has a linear resolution. In particular, it satisfies property N_p for all p. We also characterize all such Hibi rings combinatorially which gives a different proof of [EQR13, Corollary 10]. In particular, for $p = 3$, we have proved the following:

Theorem 1.2. *(Theorem 3.30) Let P be a connected poset. Assume that P has at least two minimal and maximal elements. Then $R[\mathcal{I}(P)]$ does not satisfy property N_3.*

Answering the above question for $p = 2$ is an extremely difficult task. In this direction, we have proved some necessary conditions for Hibi rings to satisfy property N_2. More precisely,

Theorem 1.3. *(Theorem 3.20) Let P be a poset. Let $\mathcal{S} = \cup_{i=1}^{2}\{p_{i,1}, \dots, p_{i,n_i}\}$ be a subset of the underlying set of P such that*

(i) for all $1 \le i \le 2$, $\{p_{i,1}, \dots, p_{i,n_i}\}$ is a chain in P with $p_{i,1} \lessdot \cdots \lessdot p_{i,n_i}$.

(ii) $p_{1,1} \neq p_{2,1}$*,* $p_{1,n_1} \neq p_{2,n_2}$*.*

(iii) $\{p_{1,1}, p_{2,1}\}$ *and* $\{p_{1,n_1}, p_{2,n_2}\}$ *are antichains in* P*.*
Let P' *be the induced subposet of* P *on the set* $\mathcal{S}$*. If* $R[\mathcal{I}(P')]$ *does not satisfy property* N_2*, then* $R[\mathcal{I}(P)]$ *does not satisfy property* N_2*.*

Under the notations of the above theorem, $\mathcal{I}(P')$ is a planar distributive lattice. Ene [Ene15] characterized all planar distributive lattices for which the associated Hibi ring satisfies property N_2.

Suppose that a poset can be decomposed into a union of three chains and it has three maximal and minimal elements. We prove some necessary conditions regarding when Hibi rings associated to such posets satisfy property N_2.

Theorem 1.4. *(Theorem 3.25) Let* P *be a poset on the set* $\cup_{i=1}^{3}\{p_{i,1}, ..., p_{i,n_i}\}$ *such that*

(i) $p_{1,1}, p_{2,1}, p_{3,1}$ *are distinct and* $p_{1,n_1}, p_{2,n_2}, p_{3,n_3}$ *are distinct,*

(ii) $\{p_{1,1}, p_{2,1}, p_{3,1}\}$ *and* $\{p_{1,n_1}, p_{2,n_2}, p_{3,n_3}\}$ *are the sets of minimal and maximal elements of* P *respectively and*

(iii) for all $1 \leq i \leq 3$*,* $n_i \geq 3$*;* $\{p_{i,1}, \ldots, p_{i,n_i}\}$ *is a chain in* P *with* $p_{i,1} \lessdot \cdots \lessdot p_{i,n_i}$*.*
If P *is connected and none of the minimal elements of* P *is covered by a maximal element, then* $R[\mathcal{I}(P)]$ *does not satisfy property* N_2*.*

The Segre product of two Hibi rings is a Hibi ring. More precisely, let P_1 and P_2 be two posets. Then, $R[\mathcal{I}(P_1)] * R[\mathcal{I}(P_2)] \cong R[\mathcal{I}(P)]$ where $*$ denotes the Segre product and P is the disjoint union of P_1 and P_2. For the Segre product of Hibi rings, we have proved the following result:

Theorem 1.5. *(Corollary 4.3) Let* P *be a poset such that it is a disjoint union of two posets* P_1 *and* P_2*. If* $R[\mathcal{I}(P)]$ *satisfies property* N_p *for some* p*, then so do* $R[\mathcal{I}(P_1)]$ *and* $R[\mathcal{I}(P_2)]$*.*

Since polynomial rings are Hibi rings, the Segre product of polynomial rings may be viewed as a Hibi ring. The property N_p of the Segre product of polynomial rings have been studied by various authors. Let $A = K[x_{1,0}, \ldots, x_{1,n_1}] * \cdots * K[x_{r,0}, \ldots, x_{r,n_r}]$ be the Segre product of r polynomial rings, where $n_i \geq 1$ and $n_i \in \mathbb{N}$ for all i. Sharpe [Sha64] proved that if $r = 2$, then A satisfies property N_2. For $r = 2$, Lascoux [Las78] and Pragacz-Weyman [PW85] proved that A satisfies property N_3 if K contains the rational field $\mathbb{Q}$. Hashimoto [Has90] showed that if $r = 2$, $n_1, n_2 \geq 4$ and characteristic of the field K is 3, then A does not satisfy property N_3. Rubei [Rub02, Rub07] proved that if $r \geq 3$ and $\mathrm{char}(K) = 0$, then A satisfies property N_3 but it does not satisfy property N_4. Based on these results and various examples, we have conjectured the following:

Conjecture 1.6. *Let P_1 and P_2 be two posets and P be their disjoint union.*

1. *If the Hibi ring $R[\mathcal{I}(P_i)]$ satisfies property N_2 for all $i = 1, 2$, then so does $R[\mathcal{I}(P)]$.*
2. *If char$(K) \neq 3$ and the Hibi ring $R[\mathcal{I}(P_i)]$ satisfies property N_3 for all $i = 1, 2$, then so does $R[\mathcal{I}(P)]$.*

If (2) of Conjecture 1.6 is true, then one can completely resolve Question 1.1 for $p = 3$ with the help of Theorem 1.2. If (1) of the Conjecture 1.6 is true, then in order to answer Question 1.1 for $p = 2$, one has to take care of the connected posets only. Generalizing the results of Rubei and giving more evidence in support of Conjecture 1.6, we have proved the following results:

Theorem 1.7. *(Theorem 4.14) Let P be a poset. If $R[\mathcal{I}(P)]$ satisfies property N_2, then so does $R[\mathcal{I}(P)] * K[t_1, \ldots, t_n]$, where $K[t_1, \ldots, t_n]$ is a polynomial ring.*

Theorem 1.8. *(Theorem 4.5) Let P be a poset. If the Hibi ring $R[\mathcal{I}(P)]$ satisfies property N_3, then so does $R[\mathcal{I}(P)] * K[t_1, t_2]$, where $K[t_1, t_2]$ is a polynomial ring.*

Now, suppose that for a poset P, the associated Hibi ring does not satisfy property N_2. Then, the second syzygy module of $R[\mathcal{I}(P)]$, denoted by $Syz_2(R[\mathcal{I}(P)])$, is not generated by linear relations. So one could ask the following question, "Which Koszul relations will be in the minimal generating set of $Syz_2(R[\mathcal{I}(P)])$?". We have partially answered the above question in Theorem 6.13.

Let P be a poset. The comparability graph G_P of P is a graph on the underlying set of P such that $\{x, y\}$ is an edge of G_P if and only if x and y are comparable in P. Hibi and Ohsugi [HO17] characterized chordal comparability graph of posets using toric ideals associated with multichains of poset. Using one of our results [Vee21a, Theorem 5.6] and [Frö90, Theorem 1], we have characterized chordal comparability graph of distributive lattices in terms of the subposet of join-irreducibles of the distributive lattice in Corollary 3.34.

1.3 h-polynomial of Polyomino algebras

In recent joint work with Manoj Kummini [KV23a, KV23b], we have partially resolved the following two conjectures:

1. Charney-Davis conjecture for the Gorenstein toric K-algebras associated to simple thin polyominoes.
2. Rinaldo-Romeo's conjecture concerning characterization of thin polyominoes.

The Charney-Davis conjecture [CD95, Conjecture D] asserts that if $h(t)$ is the h-polynomial of a flag simplicial homology $(d-1)$-sphere, then $(-1)^{\lfloor \frac{d}{2} \rfloor} h(-1) \geq 0$. Stanley [Sta00, Problem 4] extended this conjecture to Gorenstein* flag simplicial complexes. Generalizing it further, Reiner and Welker [RW05, Question 4.4] posed the following:

Question 1.9. *Let K be a field and R a standard graded Gorenstein Koszul K-algebra. Write the Hilbert series of R as $h_R(t)/(1-t)^{\dim(R)}$. Is*

$$(-1)^{\left\lfloor \frac{\deg h_R(t)}{2} \right\rfloor} h_R(-1) \geq 0?$$

We say that a standard graded Gorenstein Koszul K-algebra R is *Charney-Davis (CD)* if it gives an affirmative answer to the above question.

Suppose that, in the notation of Question 1.9, $\deg h_R(t)$ is odd. Then $h_R(-1) = 0$; see, e.g., [BH93, Corollary 4.4.6]. Therefore Question 1.9 is open only when $\deg h_R(t)$ is even. See the bibliography of [RW05] and of [Sta00] for various classes of rings that are CD. A class of CD rings related to the ones we have studied are Gorenstein Hibi rings [Brä06, Corollary 4.3]. Recently, D'Alì and Venturello [DV22] proved that the answer to Question 1.9 is negative in general.

Let K be a field and R be a standard graded finite type K-algebra. The *Hilbert series* $H_R(t)$ of R is the formal power series $\sum_{i \in \mathbb{N}} \dim_K R_i t^i$ where for each i, R_i is the finite-dimensional K-vector-space of the homogeneous elements of R of degree i. There exists a unique polynomial $h_R(t)$ such that

$$H_R(t) = \frac{h_R(t)}{(1-t)^{\dim R}}.$$

The polynomial $h_R(t)$ is called the *h-polynomial* of R.

A *polyomino* is a finite union of unit squares with vertices at lattice points in the plane that is connected and has no finite cut-set [Sta12, 4.7.18]. Qureshi [Qur12] associated a finitely generated graded algebra $K[\mathcal{P}]$ (over a field K) to a polyomino $\mathcal{P}$. Qureshi-Shibuta-Shikama [QSS17, Corollary 2.3] proved that if $\mathcal{P}$ is a simple polyomino, then $K[\mathcal{P}]$ is a Koszul Cohen-Macaulay integral domain.

The S-property of simple thin polyominoes was introduced in [RR21] to characterize such polyominoes $\mathcal{P}$ for which $K[\mathcal{P}]$ is Gorenstein. Therefore it is natural to ask whether $K[\mathcal{P}]$ is CD if $\mathcal{P}$ is a simple thin polyomino with the S-property. In this regard, we showed the following:

Theorem 1.10. *(Theorem 5.17) Let $\mathcal{P}$ be a simple thin polyominoes with the S-property. Then $K[\mathcal{P}]$ is CD.*

In our preprint [KV23b], Manoj Kummini and I have partially proved Rinaldo-Romeo's conjectured characterization of thin polyominoes. For $k \in \mathbb{N}$, a *k-rook configuration* in $\mathcal{P}$ is an arrangement of k rooks in pairwise non-attacking positions. The *rook polynomial* $r_{\mathcal{P}}(t)$ of $\mathcal{P}$ is $\sum_{k\in\mathbb{N}} r_k t^k$ where r_k is the number of k-rook configurations in $\mathcal{P}$.

Rinaldo-Romeo [RR21, Theorem 1.1] showed that if $\mathcal{P}$ is a simple thin polyomino, then $h_{K[\mathcal{P}]}(t) = r_{\mathcal{P}}(t)$ and conjectured [RR21, Conjecture 4.5] that this property characterizes thin polyominoes. We have proved this conjecture in the following case:

Theorem 1.11. *(Theorem 5.19) Let $\mathcal{P}$ be a convex polyomino such that its vertex set $V(\mathcal{P})$ is a sublattice of $\mathbb{N}^2$. Let $h_{K[\mathcal{P}]}(t) = 1 + h_1t + h_2t^2 + \cdots$ be the h-polynomial of $K[\mathcal{P}]$ and $r_{\mathcal{P}}(t) = 1 + r_1t + r_2t^2 + \cdots$ be the rook polynomial of $\mathcal{P}$. If $\mathcal{P}$ is not thin, then $h_2 < r_2$. In particular $h_{K[\mathcal{P}]}(t) \neq r_{\mathcal{P}}(t)$.*

Using results of [EHQR21], we have extended our result to L-convex polyominoes. More precisely,

Theorem 1.12. *(Corollary 5.25) Let $\mathcal{P}$ be an L-convex polyomino that is not thin. Let $h_{K[\mathcal{P}]}(t) = 1+h_1t+h_2t^2+\cdots$ be the h-polynomial of $K[\mathcal{P}]$ and $r_{\mathcal{P}}(t) = 1+r_1t+r_2t^2+\cdots$ be the rook polynomial of $\mathcal{P}$. Then $h_2 < r_2$.*

Though the statements of both of the conjectures are algebraic, our proofs are purely combinatorial. Later on, Qureshi-Rinaldo-Romeo [QRR22] also proved Theorem 1.11 and 1.12.

1.4 Organization of the thesis

In Chapter 2, we discuss the preliminaries and background of commutative algebra and combinatorics required for the thesis. Chapter 3 is about the property N_p of Hibi rings for $p \geq 2$. We prove some sufficient conditions for Hibi rings to not satisfy property N_2 in Sections 3.2 and 3.3. In Section 3.4, we study property N_p of Hibi rings for $p \geq 3$. First, we prove that if a poset is connected and it has at least two minimal and at least two maximal elements, then the associated Hibi ring does not satisfy property N_3. The second main result of this section is about property N_p of Hibi rings for $p \geq 4$. Using this result and [Frö90, Theorem 1], we characterize chordal comparability graph of distributive lattices in terms of the subposet of join-irreducibles of the distributive lattice.

In Chapter 4, we study the property N_p for Segre product of Hibi rings for $p \geq 2$. We prove that if a Hibi ring satisfies property N_2, then its Segre product with a polynomial ring in finitely many variables also satisfies property N_2. When the polynomial ring is in two variables, we prove the above statement for N_3.

In Chapter 5, we study the h-polynomial of Hibi rings and polyomino rings. In particular, we prove the Charney-Davis conjecture for the Gorenstein toric K-algebras associated to simple thin polyominoes and for Gorenstein Hibi rings of regularity 4. Also, we partially prove Rinaldo and Romeo's conjectured characterization of thin polyominoes.

In the last chapter, we study the minimal Koszul syzygies of Hibi ideals and of initial Hibi ideals. We also give a combinatorial characterization of complete intersection Hibi rings.

Chapter 2

Preliminaries

2.1 Basics from commutative algebra

Let K be a field. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring in n variables over K. Set $\deg(x_i) = 1$ for all i. Then a monomial $x_1^{a_1} \cdots x_n^{a_n}$ has degree $\sum_{i=1}^{n} a_i$. For $i \in \mathbb{N}$, we denote by S_i the K-vector space generated by all monomials of degree i. As a K-vector space S has a direct sum decomposition $\bigoplus_{i \in \mathbb{N}} S_i$ such that $S_i S_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{N}$. We refer to this as the *standard grading* of S.

Let M be an S-module. We say that M is *graded* if it has a K-vector space decomposition $\bigoplus_{i \in \mathbb{N}} M_i$ such that $S_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{N}$. For $i \in \mathbb{N}$, M_i is called the i^{th} *homogeneous component* of M. An element of M_i is *homogeneous* of degree i. For graded S-modules M and N, a homomorphism $\phi : M \to N$ is called *graded of degree* r if $\phi(M_i) \subseteq N_{r+i}$ for all $i \in \mathbb{N}$. We write $S(-j)$ for a graded free S-module with a homogeneous generator of degree j. We say that $S(-j)$ is the module S shifted j degrees. For each $i \in \mathbb{N}$, $S(-j)_i = S_{i-j}$ as a K-vector space. A ideal I of S is called *graded* if it is generated by homogeneous elements. The ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ is called the *graded maximal ideal* of S.

Now we define another grading for S which will be used in this thesis. Let $H \subseteq \mathbb{N}^m$ be an affine semigroup with the unique minimal generating set $h_1, \ldots, h_n \in \mathbb{N}^m$. Consider a degree map $\deg : \mathbb{N}^n \to \mathbb{N}^m$ defined by $\deg(e_i) = h_i$. It is easy to see that the degree map is a semigroup homomorphism. A monomial $x_1^{a_1} \cdots x_n^{a_n}$ in S is identified with a vector $(a_1, \ldots, a_n) \in \mathbb{N}^n$. Grading S by H is assigning each monomial $x_1^{a_1} \cdots x_n^{a_n}$ in S to its degree $\deg((a_1, \ldots, a_n)) = \sum_{i=1}^{n} a_i h_i \in H$ where $a_i \in \mathbb{N}$ for all i. We refer to this as H-*grading* of S. For any $h \in H$, the set of homogeneous polynomials $f \in S$

with $\deg(f) = h$ is a K-vector space spanned by the polynomials having degree h in H-grading. As a K-vector space S has the direct sum decomposition $\bigoplus_{h \in H} S_h$ such that $S_h S_{h'} \subseteq S_{h+h'}$ for all $h, h' \in H$. For an S-module M, we say it is *H-graded* if we can write $M = \bigoplus_{h \in H} M_h$ as a K-vector space, such that for all $h, h' \in H$, $S_{h'} M_h \subseteq M_{h'+h}$. For $h \in H$, The module $S(-h)$ is a free S-module of rank one with generator h.

Assume that S is standard graded. Let $I \subset S$ be a graded S-ideal. So $R = S/I$ is a standard graded K-algebra, i.e., R is generated as a K-algebra by homogeneous elements of degree 1. Let M be a finitely generated graded R-module. We say that a homogeneous element $s \in R$ is an *M-regular element* if $(0 :_M s) = 0$. In other words, s is a non-zero divisor on M. A sequence $s_1, \ldots, s_r$ of homogeneous elements of R is called an *M-regular sequence* if the following conditions are satisfied: (i) s_i is $M/(s_1, \ldots, s_{i-1})M$-regular element for all $i = 1, \ldots, r$, and (ii) $M/(s_1, \ldots, s_r)M \neq 0$.

Any two maximal M-regular sequences of M have the same length. The length of a maximal regular sequence is called the *depth* of M and is denoted by $\operatorname{depth}_R(M)$. It is known that $\operatorname{depth}_R(M) \leq \dim(M)$. M is said to be a *Cohen-Macaulay* R-module if $\operatorname{depth}_R(M) = \dim(M)$. If R itself is a Cohen-Macaulay module, then it is called a Cohen-Macaulay ring. We say that R is a *complete intersection* if I is generated by a regular sequence.

2.1.1 Graded free resolution

Let $R = \oplus_{i \geq 0} R_i$ be a finitely generated graded K-algebra with $R_0 = K$ and let $\mathfrak{n} = \oplus_{i \geq 1} R_i$ be the graded maximal ideal of R. A *complex* $\mathbb{F}$ *of R-modules* is a sequence of R-modules F_i and maps $\partial_i : F_i \to F_{i-1}$ such that $\partial_i \partial_{i+1} = 0$ for $i \in \mathbb{Z}$. The i^{th} *homology* of the complex $\mathbb{F}$, denoted by $H_i^R(\mathbb{F})$, is the module $\ker(\partial_i)/\operatorname{im}(\partial_{i+1})$. The complex $\mathbb{F}$ is *exact* if $H_i^R(\mathbb{F}) = 0$ for all i. Let M be an R-module. A *free resolution* of M over R is a complex

$$\mathbb{F} : \cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0$$

of free R-modules such that $\mathbb{F}$ is exact and $\operatorname{coker}(\partial_1) \cong M$. The image of the map ∂_i is called the i^{th} syzygy module of M, denoted by $\operatorname{Syz}_i^R(M)$.

When M is finitely generated, we may take F_i to be of finite rank. We say that $\mathbb{F}$ is *minimal* if $\operatorname{im}(\partial_i) \subseteq \mathfrak{n} F_{i-1}$ for all i. Assume that M is graded. Then a free resolution $\mathbb{F}$

of M is said to be *graded free resolution* if the module F_i are graded free modules, and the maps ∂_i are homogeneous maps of degree 0. If for some $m \in \mathbb{N}$, we have $F_{n+1} = 0$ but $F_i \neq 0$ for all $0 \leq i \leq n$, then we say that $\mathbb{F}$ is *finite free resolution of length n*.

Let M and N be graded R-modules. Let $\mathbb{F}$ (resp. $\mathbb{G}$) be the minimal graded free resolution of M (resp. N) over R. Define

$$Tor_i^R(M, N) := H_i(\mathbb{F} \otimes_R N) \cong H_i(M \otimes_R \mathbb{G}).$$

The $Tor_i^R(M, N)$ is a graded R-module and it is independent of choice of the resolutions of M and N.

Let M be a finitely generated graded S-module, where S is the polynomial ring $K[x_1, \ldots, x_n]$. Then M has a graded minimal free resolution, which is unique up to isomorphism. By the Hilbert's syzygy theorem, the graded minimal free resolution of M is finite and has length $\leq n$. We define the *graded Betti numbers* of M in standard grading and in H-grading, where H is a affine semigroup. First assume that M is a graded in standard grading. Then, the standard graded Betti numbers $\beta_{i,j}(M) = \dim_K Tor_i^S(M, K)_j$ for all $i, j \in \mathbb{N}$. Similarly, assume that S is H-graded and M is an H-graded S-module. Then for any $h \in H$, the H-graded Betti number $\beta_{i,h}(M) = \dim_K Tor_i^S(M, K)_h$ for all $i \in \mathbb{N}$.

In conclusion, the graded minimal free resolution $\mathbb{F}$ of a standard graded S-module M has the following form:

$$\mathbb{F} : 0 \to \bigoplus_j S(-j)^{\beta_{r,j}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1,j}} \to \bigoplus_j S(-j)^{\beta_{0,j}} \quad \text{where } r \leq n.$$

The *Betti table* of M is numerical data consisting of the minimal number of generators in each degree in the minimal generating set of each syzygy module of M. More precisely, the Betti table of M is an array with columns indexed by homological degrees i having the entry $\beta_{i,i+j}$ in the row indexed j. Table 2.1 displays the Betti table of M.

Hilbert's syzygy theorem implies that there are only finitely many pairs (i, j) for which $\beta_{i,j} \neq 0$. The size of a Betti table is given by the projective dimension and the regularity. One defines the *projective dimension* of M as

$$\operatorname{proj\,dim}_S(M) = \max\{i : \beta_{i,j}(M) \neq 0 \text{ for some } j\}$$

-	0	1	2	3	$\cdots$	i	$i+1$	$\cdots$
0	$\beta_{0,0}$	$\beta_{1,0}$	$\beta_{2,0}$	$\beta_{3,0}$	$\cdots$	$\beta_{i,0}$	$\beta_{i+1,0}$	$\cdots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\beta_{3,4}$	$\cdots$	$\beta_{i,i+1}$	$\beta_{i+1,i+1+1}$	$\cdots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\beta_{3,5}$	$\cdots$	$\beta_{i,i+2}$	$\beta_{i+1,i+1+2}$	$\cdots$
3	$\beta_{0,3}$	$\beta_{1,4}$	$\beta_{2,5}$	$\beta_{3,6}$	$\cdots$	$\beta_{i,i+3}$	$\beta_{i+1,i+1+3}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	$\beta_{0,j}$	$\beta_{1,1+j}$	$\beta_{2,2+j}$	$\beta_{3,3+j}$	$\cdots$	$\beta_{i,i+j}$	$\beta_{i+1,i+1+j}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 2.1: Betti table of M

and the *(Castelnuovo-Mumford) regularity* of M as

$$\operatorname{reg}_S(M) = \max\{j - i : \beta_{i,j}(M) \neq 0 \text{ for some } j\}.$$

The next proposition relates the projective dimension of a graded module over a polynomial ring with its depth.

Proposition 2.1. *(Auslander-Buchsbaum formula)[HHO18, Theorem 2.15] Let M be a finitely generated graded S-module. Then*

$$\operatorname{proj}\dim_S(M) + \operatorname{depth}_S(M) = \dim(S) = n.$$

A immediate consequence of the proposition is that M is Cohen-Macaulay if and and only if $\operatorname{proj}\dim_S(M) = \dim(S) - \dim_S(M)$.

Let $I = (f_1, \dots, f_m)$ be a graded S-ideal. Let $\{e_1, \dots, e_m\}$ be a basis of the free S-module S^m. Define a map $\varphi : S^m \to S$ by $\varphi(e_i) = f_i$. Then, $\ker \varphi$ is the second syzygy module of S/I, denoted by $Syz_2(S/I)$. Let f_i and f_j be two distinct generators of I. Then the Koszul relation $f_i e_j - f_j e_i$ belongs to $Syz_2(S/I)$. We say f_i, f_j a *Koszul relation pair* if $f_i e_j - f_j e_i$ is a minimal generator of $Syz_2(S/I)$.

Let I be a graded S-ideal generated by elements of degree d. Then I said to have a *linear resolution* if $\beta_{i,j}(I) = 0$ for $j \neq i + d$. We say that the ring S/I has a linear resolution over S if I has a linear resolution.

Let I be a graded S-ideal with $I \subseteq \mathfrak{m}^2$ and let $R = S/I$. Let $\mathfrak{n}$ be the graded maximal ideal of R. The graded minimal free resolution of $R/\mathfrak{n}$ over R is infinite if and only if $I \neq 0$. One could still ask about the linearity of the graded minimal free resolution of $R/\mathfrak{n}$.

Definition 2.2. *A standard graded K-algebra R is said to be* Koszul *if $R/\mathfrak{n}$ has a linear resolution, i.e., $Tor_i^R(R/\mathfrak{n}, R/\mathfrak{n})_j = 0$ for all i and all $j \neq i$.*

A consequence of the Koszul algebras is the following:

Proposition 2.3. *[Kem90, Lemma 4] Let $R = S/I$ be a Koszul algebra. Then $\beta_{i,j}(R) = 0$ for all $j > 2i$.*

2.1.2 Initial Ideals

A *monomial order* $<$ on S is a total order on the set of monomials of S such that

1. $1 < g$ for all monomial g with $g \neq 1$;
2. if g, g' are two monomials with $g < g'$, then $fg < fg'$ for all monomials f.

Let $f \in S$ be a polynomial. The *initial term* of f with respect to $<$, denoted by $\operatorname{in}_<(f)$, is the largest monomial that appears with a non-zero coefficient in f. Let I be an ideal of S. The ideal generated by the monomials $\{\operatorname{in}_<(f) : f \in I\}$ is called the *initial ideal* of I with respect to $<$, and is denoted by $\operatorname{in}_<(I)$.

The following result provides a comparison between S/I and $S/\operatorname{in}_<(I)$.

Theorem 2.4. *[HHO18, Theorem 2.19] Let I be a graded S-ideal, and let $<$ be a monomial order on S. Then the following holds:*

(a) $\beta_{ij}(S/I) \leq \beta_{ij}(S/\operatorname{in}_<(I))$ *for all i and j;*

(b) $\dim S/I = \dim S/\operatorname{in}_<(I)$, $\operatorname{depth} S/\operatorname{in}_<(I) \leq \operatorname{depth} S/I$ *and* $\operatorname{reg} S/I \leq \operatorname{reg} S/\operatorname{in}_<(I)$;

(c) *if $S/\operatorname{in}_<(I)$ is Cohen-Macaulay, then S/I is Cohen-Macaulay;*

(d) *if $S/\operatorname{in}_<(I)$ is Gorenstein, then S/I is Gorenstein.*

Recently, Conca and Varbaro [CV20, Corollary 2.7] proved that if $\operatorname{in}_<(I)$ is squarefree, then $\operatorname{depth} S/\operatorname{in}_<(I) = \operatorname{depth} S/I$ and $\operatorname{reg} S/I = \operatorname{reg} S/\operatorname{in}_<(I)$. Consequently, if S/I is Cohen-Macaulay, then so is $S/\operatorname{in}_<(I)$.

A *Gröbner basis* for I is a set of polynomials $\{g_1, \ldots, g_r\} \subset I$ such that $\text{in}_<(I) = (in_<(g_1), \ldots, in_<(g_r))$. There exists a finite subset $\mathcal{G}$ of I such that $\mathcal{G}$ is a Gröbner basis of I with respect to $<$ (see [HHO18, Theorem 1.25]). If $\{g_1, \ldots, g_r\}$ is a Gröbner basis of I, then $I = (g_1, \ldots, g_r)$ [HHO18, Theorem 1.16]. Assume that I is graded and it has a quadratic Gröbner basis under some monomial order $<$. Then S/I is Koszul [HHO18, Section 2.4].

2.1.3 Hilbert Series

Let $R = S/I$ be a finitely generated K-algebra. Assume that R is standard graded, i.e., R is generated as a K-algebra by homogeneous elements of degree 1. So one can write R as $\bigoplus_{n\in\mathbb{N}} R_n$ where $R_0 = K$ and for each $n \geq 1$, R_n is the finite-dimensional K-vector space of the homogeneous elements of R of degree n.

The *Hilbert series* $H_R(t)$ of R is the formal power series $\sum_{n\in\mathbb{N}} \dim_K(R_n)t^n$. There exists a unique polynomial $h_R(t)$ [BH93, Corollary 4.1.8] such that

$$H_R(t) = \frac{h_R(t)}{(1-t)^{\dim R}}.$$

The polynomial $h_R(t)$ is called the *h-polynomial* of R. Write $h_R(t) = h_0 + h_1 t + \cdots + h_r t^r$ with $h_r \neq 0$. If R is Cohen-Macaulay, then $\deg h_R(t)$ is the regularity of R [HHO18, Corollary 2.18] and $h_i \geq 0$ for all i [BH93, Corollary 4.1.10]. If R is Gorenstein, then $h_i = h_{r-i}$ for all $0 \leq i \leq r$. When R is a domain, then the converse of the previous statement also holds, i.e, if $h_i = h_{r-i}$ for all $0 \leq i \leq r$, then R is Gorenstein [BH93, Corollary 4.4.6].

For a graded S-ideal I, the Hilbert series of S/I can be reduced to the case when I is a monomial ideal. More precisely,

Proposition 2.5. *[HHO18, Proposition 2.6] Let $<$ be a monomial order on S, and let I be a graded S-ideal. Then*

$$H_{S/I}(t) = H_{S/\operatorname{in}_<(I)}(t).$$

2.2 Basics from poset theory

We start by defining some basic notions of posets and distributive lattices. For more details and examples, we refer the reader to [Sta12, Chapter 3] and [Bir67]. Throughout this thesis, all posets and distributive lattices will be finite.

A *partially ordered set* P (or *poset* in brief) is a set, together with a binary relation $\leq$, satisfying the following axioms:

1. reflexive : $x \leq x$ for all $x \in P$;
2. antisymmetric: for any $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$;
3. transitive: for any $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

We use the notation $x \geq y$ to mean $y \leq x$, $x < y$ to mean $x \leq y$ and $x \neq y$. We say that two elements x and y of P are *comparable* if $x \leq y$ or $y \leq x$; otherwise x and y are *incomparable*.

Let P be a poset. For $x, y \in P$, we say that y *covers* x if $x < y$ and there is no $z \in P$ with $x < z < y$. We denote it by $x \lessdot y$. A poset is completely determined by its cover relations. The *Hasse diagram* of poset P is the graph whose vertices are elements of P, whose edges are cover relations, and such that if $x < y$ then y is "above" x (i.e. with a higher vertical coordinate). In this thesis, we use the Hasse diagrams to represent posets. A *subposet* of P is a subset Q with a partial order such that for $x, y \in Q$ we have $x \leq y$ in Q if and only if $x \leq y$ in P.

A *chain* C of P is a totally ordered subset of P, that is, any two elements of C are comparable in P. The *length* of a chain C of P is $\#C - 1$. The *rank* of P, denoted by $\mathrm{rank}(P)$, is the maximum of the lengths of chains in P. A poset is called *pure* if its all maximal chains have the same length. For $x \in P$, $\mathrm{height}(x)$ denotes the rank of the subposet of P which consists of all $y \in P$ with $y \leq x$.

Definition 2.6. Let P and Q be two posets.

1. A nonempty subset S of P is an *antichain* in P if any two distinct elements of S are incomparable. An antichain with n elements is said to have *width* n. Define $width(P) := max\{\#S : S \subseteq P,\ S \text{ is an antichain in } P\}$.

2. A poset P is called *simple* if there is no $p \in P$ with the property that all elements of P are comparable to p.

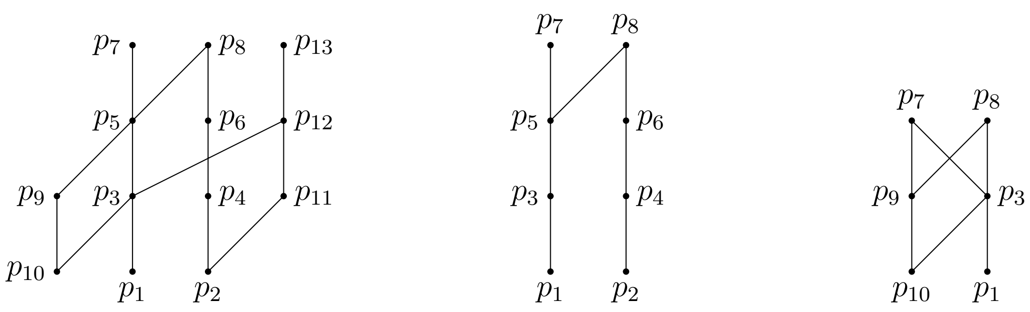
3. The *ordinal sum* $P \oplus Q$ of the disjoint posets P and Q is the poset on the set $P \cup Q$ with the following order: if $x, y \in P \oplus Q$, then $x \leq y$ if either $x, y \in P$ and $x \leq y$ in P or $x, y \in Q$ and $x \leq y$ in Q or $x \in P$ and $y \in Q$.

4. Let P, Q be two posets on disjoint sets. The *disjoint union* of posets P and Q is the poset $P + Q$ on the set $P \cup Q$ with the following order: if $x, y \in P + Q$, then $x \leq y$ if either $x, y \in P$ and $x \leq y$ in P or $x, y \in Q$ and $x \leq y$ in Q. A poset P which can be written as disjoint union of two posets is called *disconnected*. Otherwise, P is *connected*.

5. P and Q are said to be *isomorphic*, denoted by $P \cong Q$, if there exists an order-preserving bijection $\varphi : P \to Q$ whose inverse is order preserving.

6. A subposet P' of P is said to be a *cover-preserving subposet* of P if for every $x, y \in P'$ with $x \lessdot y$ in P', we have $x \lessdot y$ in P.

Example 2.7. Let P be the poset as shown in Figure 2.1a. Let P' and P'' be the subposets of P as shown in Figure 2.1b and Figure 2.1c respectively. It is easy to see that P' is a cover-preserving subposet of P but P'' is not a cover-preserving subposet of P since $p_3 \lessdot p_7$ in P'' but not in P.



(a) (b) (c)

FIGURE 2.1

Let P be a poset and $x, y \in P$. An *upper bound* of x and y is an element $z \in P$ satisfying $x \leq z$ and $y \leq z$. A *least upper bound* (or *join*) of x and y is a least element of the set $\{z \in P : \ z \text{ is an upper bound of } x \text{ and } y\}$. If a least upper bound of x and y exists, then it is unique and denoted by $x \vee y$. Dually one can define *greatest upper bound* (or *meet*) of x and y, when it exists. It is denoted by $x \wedge y$.

A *lattice* L is a poset for which every pair of elements has a least upper bound and greatest lower bound. A lattice L is said to be a *distributive* if it satisfies one the following equivalent conditions:

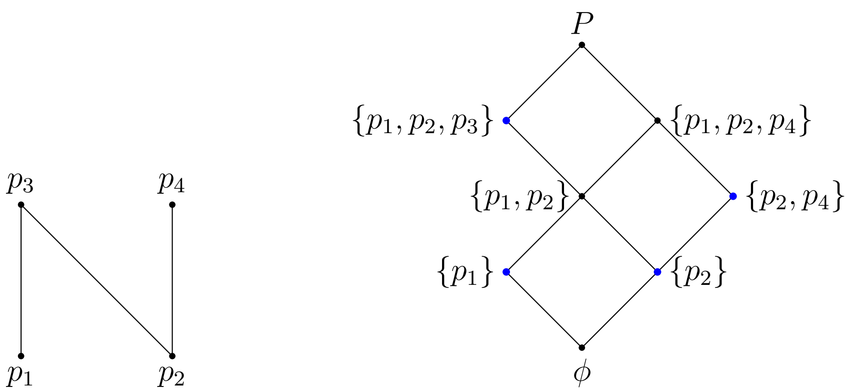
1. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$;
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any $x, y, z \in L$.

Let L be a lattice. An element $x \in L$ is called *join-irreducible* if x is not the minimal element of L and whenever $x = y \vee z$ for some $y, z \in L$, we have either $x = y$ or $x = z$.

Let P be a poset. A subset α of P is called an *order ideal* of P if it satisfies the following condition: for any $x \in \alpha$ and $y \in P$, if $y \leq x$, then $y \in \alpha$. Define $\mathcal{I}(P) := \{\alpha \subseteq P : \alpha \text{ is an order ideal of } P\}$. It is easy to see that $\mathcal{I}(P)$, ordered by inclusion, is a distributive lattice under union and intersection. $\mathcal{I}(P)$ is called the *ideal lattice* of the poset P.

Theorem 2.8. *(Birkhoff)[Bir67, Chapter 9, Theorem 10][Sta12, Theorem 3.4.1] Let L be a distributive lattice. Then there is a unique poset P, up to isomorphism, for which $L \cong \mathcal{I}(P)$.*

Example 2.9. In this example, we illustrate Birkhoff's theorem. Let P be a poset given by Figure 2.2a. Then $\mathcal{I}(P)$ is as shown in Figure 2.2b. The join-irreducible elements of $\mathcal{I}(P)$ are highlighted in blue. One can check that the poset of join-irreducible elements of $\mathcal{I}(P)$ is isomorphic to P.



(a) P (b) $\mathcal{I}(P)$

FIGURE 2.2

2.3 Simplicial complexes

Let V be a non-empty finite set. A *simplicial complex* Δ on V is a collection of subsets of V such that $\{v\} \in \Delta$ for all $v \in V$ and $F \in \Delta$ whenever $F \subseteq G$ for some $G \in \Delta$.

The elements of Δ are called *faces*, and the *dimension* of a face F, denoted by $\dim F$, is the number $\#F - 1$. The *dimension* of the simplicial complex Δ is $\dim \Delta = \max\{\dim F : F \in \Delta\}$. A 0-dimensional face of Δ is called a *vertex* of Δ. We denote the vertex set of Δ by $V(\Delta)$. A *facets* of Δ is a face that is maximal under inclusion. Note that the empty set $\emptyset$ is a face of dimension -1 of Δ.

A *subcomplex* of the simplicial complex Δ is a simplicial complex whose faces are contained in Δ. For $n \geq 0$, the *n-skeleton* of the simplicial complex Δ is the collection of all those faces of Δ whose dimension is at most n. We denote the *n-skeleton* of Δ by $sk^n(\Delta)$.

Let K be a field and $V = \{v_1, \ldots, v_r\}$. For $-1 \leq n \leq \dim \Delta$, let $\widetilde{\Delta}_n$ be the K-vector space of the n-dimensional faces of Δ. A *boundary map* $\widetilde{\partial}_n : \widetilde{\Delta}_n \to \widetilde{\Delta}_{n-1}$ is given by

$$\widetilde{\partial}_n(\{v_0, \ldots, v_n\}) = \sum_{i=0}^{n} (-1)^i \{v_0, \ldots, \widehat{v_i}, \ldots, v_n\}.$$

Then,

$$0 \to \widetilde{\Delta}_{\dim \Delta} \xrightarrow{\partial_{\dim \Delta}} \widetilde{\Delta}_{\dim \Delta - 1} \to \cdots \to \widetilde{\Delta}_1 \xrightarrow{\partial_1} \widetilde{\Delta}_0 \xrightarrow{\partial_0} \widetilde{\Delta}_{-1} \to 0$$

is a complex of finite dimensional K-vector spaces. The n^{th} *reduced homology* of the simplicial complex Δ with scalars in K, denoted by $\widetilde{H}_n(\Delta, K)$, is the K-vector space $\ker(\partial_n)/\operatorname{im}(\partial_{n+1})$. Elements of $\ker(\partial_n)$ are called *cycles* and elements of $\operatorname{im}(\partial_{n+1})$ are called *boundaries.* Two cycles representing the same homology class are said to be *homologous.* This means that their difference is a boundary.

Let Δ be a simplicial complex on a vertex set V. The *support* of a simplex σ in Δ is the set of all vertices $v \in V$ such that $v \in \sigma$. Let $\alpha = \sum_i a_i \sigma_i$ where $c_i \in \mathbb{Z}$, be a chain in Δ. The *support* of α, denoted by $\operatorname{supp}(\alpha)$, is the union of the support of the simplexes σ_i.

2.4 Graph theory

Let G be a simple graph on the vertex set $[n]$. The *clique complex* (or *flag complex*) $\Delta(G)$ associated to G is a simplicial complex defined in the following way: $\Delta(G)$ has same vertices as G and the simplices of $\Delta(G)$ are exactly the subsets F of $[n]$ for which every pair in F is an edge of G. A graph G is called *chordal* if every induced cycle in G of length ≥ 4 has a chord, i.e., there is an edge in G connecting two nonconsecutive vertices of the cycle. Let Δ be a simplicial complex. The Stanley-Reisner ideal I_Δ generated by quadratics has linear resolution if and only if $\Delta = \Delta(G)$ for some chordal graph G [Frö90, Theorem 1].

Let P be a poset. The *comparability graph* G_P of P is a graph on the underlying set of P such that $\{x, y\}$ is an edge of G_P if and only if x and y are comparable in P. The *order complex* $\Delta(P)$ of a poset P is the simplicial complex whose i-faces are exactly the chains $u_0 < u_1 < \cdots < u_i$ in P. It is known and easy to verify that $\Delta(P) = \Delta(G_P)$.

2.5 Polyominoes and polyomino ideals

A *cell* in $\mathbb{R}^2$ is a set of the form $\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq a+1, b \leq y \leq b+1\}$ where $(a, b) \in \mathbb{Z}^2$. Let $\mathcal{P}$ be a finite collection of cells. Then $\mathcal{P}$ determines a unique topological subspace $\mathrm{sp}(\mathcal{P}) := \cup_{C\in\mathcal{P}} C$ of $\mathbb{R}^2$. By abuse of terminology, we assign the topological attributes to $\mathcal{P}$ that $\mathrm{sp}(\mathcal{P})$ has. We identify the cells of $\mathcal{P}$ by their top-right corners: For $v \in \mathbb{Z}^2$, $C(v)$ is the cell whose top-right corner is v. We say that $\mathcal{P}$ is a *polyomino* if $\mathcal{P}$ is connected and does not have a finite cut-set [Sta12, 4.7.18] (i.e., $\mathrm{sp}(\mathcal{P})$ has these properties).

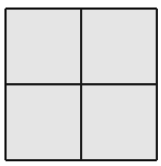


FIGURE 2.3

We say that a polyomino $\mathcal{P}$ is *simple* if $\mathrm{sp}(\mathcal{P})$ is simply connected; it is *thin* if it does not have a 2×2 square such as the one shown in Figure 2.3. We say that a polyomino $\mathcal{P}$ is *horizontally convex* if for every line segment ℓ parallel to the x-axis with end-points in $\mathcal{P}$, $\ell \subseteq \mathcal{P}$. Similarly we define *vertically convex* polyominoes. We say that a polyomino $\mathcal{P}$ is *convex* if it is horizontally convex and vertically convex. Figure 2.4 shows three examples of polyominoes that are convex, non-convex simple and non-simple thin respectively. The set of cells of $\mathcal{P}$ is denoted by $C(\mathcal{P})$. The *vertex set* $V(\mathcal{P})$ of $\mathcal{P}$ is $\mathcal{P} \cap \mathbb{Z}^2$. By the *left-boundary vertices* of $\mathcal{P}$, we mean the elements of $\mathbb{Z}^2 \cap \partial\mathcal{P}$ that are top-left vertices of the cells of $\mathcal{P}$; the *bottom-boundary vertices* of $\mathcal{P}$ are the elements of $\mathbb{Z}^2 \cap \partial\mathcal{P}$ that are bottom-right vertices of the cells of $\mathcal{P}$.

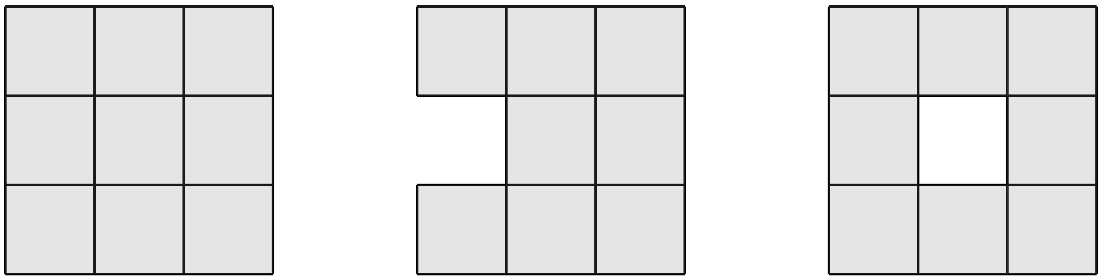


FIGURE 2.4: From left to right: a convex polyomino, a non-convex simple polyomino and a non-simple thin polyomino

Let $\mathcal{P}$ be a finite collection of cells. As mentioned earlier, we treat $\mathcal{P}$ interchangeably with the topological space $\mathrm{sp}(\mathcal{P})$. Qureshi [Qur12] associated a finitely generated graded algebra $K[\mathcal{P}]$ (over a field K) to P. Let $S = K[\{x_{ij} : (i,j) \in \mathcal{P} \cap \mathbb{Z}^2\}]$ be the standard graded polynomial ring in the variables x_{ij}. Let $I_\mathcal{P}$ be the binomial ideal generated by the binomials $x_{ij}x_{kl} - x_{il}x_{kj}$ for all $(i,j),(k,l) \in \mathcal{P} \cap \mathbb{Z}^2$ such that the rectangle with vertices (i,j), (k,l), (k,j) and (i,l) is a subset of $\mathrm{sp}(\mathcal{P})$. Define $K[\mathcal{P}] = S/I_\mathcal{P}$. When $\mathcal{P}$ is a polyomino, $I_\mathcal{P}$ is called a *polyomino ideal*.

Example 2.10. Let $\mathcal{P}$ be the polyomino as shown in Figure 2.5. Then,

$I_\mathcal{P} = (x_{01}x_{12} - x_{02}x_{11},\ x_{01}x_{22} - x_{02}x_{21},\ x_{01}x_{32} - x_{02}x_{31},\ x_{10}x_{21} - x_{11}x_{20},\ x_{10}x_{22} - x_{12}x_{20},\ x_{11}x_{22} - x_{12}x_{21},\ x_{11}x_{32} - x_{12}x_{31},\ x_{10}x_{23} - x_{13}x_{20},\ x_{11}x_{23} - x_{13}x_{21},\ x_{12}x_{23} - x_{13}x_{22},\ x_{21}x_{32} - x_{22}x_{31})$.

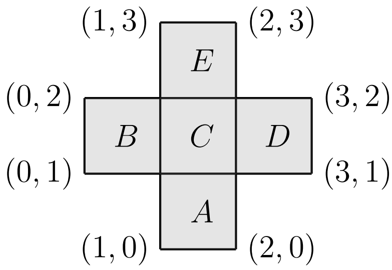


FIGURE 2.5

Theorem 2.11. *[HM14, Corollary 2.2] [QSS17, Corollary 2.3] Let $\mathcal{P}$ be a simple polyomino. Then $K[\mathcal{P}]$ is a Koszul Cohen-Macaulay integral domain.*

The height of unmixed polyomino ideals have a very nice combinatorial interpretation. Qureshi [Qur12] proved that for a convex polyomino $\mathcal{P}$, height of the polyomino ideal $I_{\mathcal{P}}$ is the number of cells of $\mathcal{P}$. Extending this further, Herzog, Hibi and Moradi [HHM22] recently proved that for a finite collection of cells $\mathcal{P}$, if $I_{\mathcal{P}}$ is an unmixed ideal, then the height of the ideal $I_{\mathcal{P}}$ is the number of cells of $\mathcal{P}$.

Let $\mathcal{P}$ be a finite collection of cells. Let $C, D \in \mathcal{P}$. We say that C is a *neighbour* of D if $C \cap D$ is a line segment. A *path* from C to D is a sequence of cells $C = C_0, C_1, \dots, C_m = D$ such that for all $i \neq j$, $C_i \neq C_j$ and for all $1 \leq i \leq m$, C_i is a neighbour of C_{i-1}. If $\mathcal{P}$ is a simple thin polyomino, then for all cells C, D of $\mathcal{P}$, there is a unique path from C to D.

A *inner interval* of $\mathcal{P}$ is a subcollection I of $\mathcal{P}$ such that $\mathrm{sp}(I)$ (which is a subspace of $\mathrm{sp}(\mathcal{P})$) is a rectangle with vertices (i_1, j_1), (i_1+1, j_1), (i_1, j_2) and (i_1+1, j_2) or a rectangle with vertices (i_1, j_1), (i_1, j_1+1), (i_2, j_1) and (i_2, j_1+1) for some $i_1, i_2, j_1, j_2 \in \mathbb{Z}$ with $i_1 < i_2$ and $j_1 < j_2$. An inner interval of $\mathcal{P}$ is *maximal* if it is maximal under inclusion.

For $k \in \mathbb{N}$, a *k-rook configuration* in $\mathcal{P}$ is an arrangement of k rooks in pairwise non-attacking positions. The *rook polynomial* $r_{\mathcal{P}}(t)$ of $\mathcal{P}$ is $\sum_{k \in \mathbb{N}} r_k t^k$ where r_k is the number of k-rook configurations in $\mathcal{P}$. The *rook number* $r(\mathcal{P})$ of $\mathcal{P}$ is the degree of $r_{\mathcal{P}}(t)$, i.e., the largest k such that there is a k-rook configuration in $\mathcal{P}$.

Theorem 2.12. *[RR21, Theorem 1.1] Let $\mathcal{P}$ be a simple thin polyomino. Then $h_{K[\mathcal{P}]}(t) = r_{\mathcal{P}}(t)$.*

Example 2.13. Let $\mathcal{P}$ be as shown in Figure 2.5. Note that $\mathcal{P}$ is simple thin. We write the Hilbert series of the ring $K[\mathcal{P}]$. The height of the polyomino ideal $I_{\mathcal{P}}$ is the number of cells of $\mathcal{P}$, i.e., 5. The dimension of the ring $K[\mathcal{P}]$ is $\#V(P)-$ number of cells of $\mathcal{P}$,i.e., $12 - 5 = 7$. By Theorem 2.12, the h-polynomial of $K[\mathcal{P}]$ is the rook polynomial of the polyomino $\mathcal{P}$. We compute r_k, namely the number of k-rook configurations in $\mathcal{P}$ for $k \geq 0$ as follows:
$(k = 0)$ $\emptyset$;
$(k = 1)$ $\{A\}$, $\{B\}$, $\{C\}$, $\{D\}$, $\{E\}$;
$(k = 2)$ $\{A, B\}$, $\{A, D\}$, $\{B, E\}$, $\{D, E\}$;
$(k \geq 3)$ there is no k-rook configurations in $\mathcal{P}$.

Therefore

$$r_0 = 1,\ r_1 = 5,\ r_2 = 4,\ r_k = 0 \text{ for all } k \geq 3.$$

Hence,

$$H_{K[\mathcal{P}]}(t) = \frac{1 + 5t + 4t^2}{(1-t)^7}.$$

Let $\mathcal{P}$ be a simple thin polyomino. Observe that any cell of $\mathcal{P}$ belongs to at most two maximal inner intervals. A cell C is said to be an *end-cell* of a maximal inner interval I if $C \in I$ and C has exactly one neighbour cell in I. A cell of $\mathcal{P}$ is called *single* if it belongs to exactly one maximal inner interval of $\mathcal{P}$. We say that $\mathcal{P}$ has the *S-property* if every maximal inner interval of $\mathcal{P}$ has exactly one single cell.

Theorem 2.14. *[RR21, Theorem 4.2] Let $\mathcal{P}$ be a simple thin polyomino. Then $K[\mathcal{P}]$ is Gorenstein if and only if $\mathcal{P}$ has the S-property.*

2.6 Hibi rings

Let $L = \mathcal{I}(P)$ be a distributive lattice with $P = \{p_1, \dots, p_n\}$. Let $R = K[t, z_1, \dots, z_n]$ be a polynomial ring in $n+1$ variables over a field K. The *Hibi ring* associated with L, denoted by $R[L]$, is the subring of R generated by the monomials $u_\alpha = t\prod_{p_i \in \alpha} z_i$ where $\alpha \in L$. If we set $\deg(t) = 1$ and $\deg(z_i) = 0$ for all $1 \leq i \leq n$, then $R[L]$ may be viewed as a standard graded algebra over K. Hibi rings were defined by Takayuki Hibi in [Hib87]. He showed that $R[\mathcal{I}(P)]$ is a normal Cohen–Macaulay domain of dimension $\#P + 1$, where $\#P$ is the cardinality of P. In that article, he also proved that the Hibi ring $R[\mathcal{I}(P)]$ is Gorenstein if and only if P is pure.

Let $K[L] = K[\{x_\alpha : \alpha \in L\}]$ be the polynomial ring over K and $\pi : K[L] \to R[L]$ be the K-algebra homomorphism with $x_\alpha \mapsto u_\alpha$. Let $I_L = (x_\alpha x_\beta - x_{\alpha\wedge\beta}x_{\alpha\vee\beta} : \alpha, \beta \in L \text{ and } \alpha, \beta \text{ incomparable})$ be an $K[L]$-ideal. Let $<$ be a total order on the variables of $K[L]$ with the property that one has $x_\alpha < x_\beta$ if $\alpha < \beta$ in L. Consider the graded reverse lexicographic order $<$ on $K[L]$ induced by this order of the variables.

Theorem 2.15. *[HHO18, Theorem 6.19] The generators of I_L described above forms a Gröbner basis of* $\ker(\pi)$ *with respect to $<$. In particular,* $\ker(\pi) = I_L$.

The ideal I_L is called the *Hibi ideal* of L. By Theorem 2.15, it follows that

$$\text{in}_<(I_L) = (x_\alpha x_\beta : \alpha, \beta \in L \text{ and } \alpha, \beta \text{ incomparable}).$$

Example 2.16. Let $P = \{p_1, \ldots, p_4\}$ be a poset as shown in Figure 2.2a. So the polynomial ring $R = K[t, z_1, \ldots, z_4]$. Note that $u_\emptyset = t$, $u_{\{p_1,p_2\}} = tz_1z_2$ and $u_P = tz_1z_2z_3z_4$. The Hibi ring associated to P is

$$R[\mathcal{I}(P)] = K[t, tz_1, tz_2, tz_1z_2, tz_1z_2z_3, tz_2z_4, tz_1z_2z_4, tz_1z_2z_3z_4].$$

Let P be a poset. Then, P is a chain if and only if $R[\mathcal{I}(P)]$ is a polynomial ring. The proof of this statement is elementary. First observe that P is a chain if and only if $\mathcal{I}(P)$ is a chain. Now, suppose that P is a chain. Write $P = \{p_1, \ldots, p_n\}$ with $p_1 \lessdot \cdots \lessdot p_n$. Then $\mathcal{I}(P) = \{\emptyset, \{p_1\}, \{p_1, p_2\}, \ldots, P\}$. So, $R[\mathcal{I}(P)] = K[t, tz_1, tz_1z_2, \ldots, tz_1 \cdots z_n]$ which is a polynomial ring. On the other hand, if $R[\mathcal{I}(P)]$ is a polynomial ring, then the Hibi ideal $I_{\mathcal{I}(P)} = 0$. Therefore, there are no incomparable pairs in $\mathcal{I}(P)$. Hence, $\mathcal{I}(P)$ is a chain.

Let $L = \mathcal{I}(P)$ be a distributive lattice. The Krull-dimension of the Hibi ring $R[L]$ is $\#P + 1$ [HHO18, Theorem 6.38]. $R[L]$ is an affine semigroup rings (see Section 2.8). Since $\operatorname{in}_<(I_L)$ is a square-free monomial ideal, $R[L]$ is normal [EH12, Theorem 5.16]. Normal affine semigroup ring generated by monomials over a field are Cohen-Macaulay [Hoc72, Theorem 1]. Hence $R[L]$ is Cohen-Macaulay. The initial ideal $\operatorname{in}_<(I_L)$ is the Stanley-Reisner ideal of the order complex of L (see Lemma 6.2). By [BH93, Theorem 5.1.12], this complex is shellable; thus $K[L]/\operatorname{in}_<(I_L)$ is Cohen-Macaulay [BH93, Theorem 5.1.13]. Now, we give a maximal regular sequence for $K[L]/I_L$ and $K[L]/\operatorname{in}_<(I_L)$ generated by linear forms.

Lemma 2.17. *Let $L = \mathcal{I}(P)$ be a distributive lattice with $\#P = n$ and $R[L] = K[L]/I_L$ be the Hibi ring associated with L. For all $0 \leq j \leq n$, define $y_j = \sum_{\substack{\alpha \in L \\ \operatorname{height}(\alpha) = j}} x_\alpha$. Let $I_j = I_L + (y_0, \ldots, y_j)$ for all $0 \leq j \leq n$. Then the following hold:*
(a) $x_\alpha \in \sqrt{I_j}$ for all $\alpha \in L$ with $\operatorname{height}(\alpha) \leq j$.
(b) $x_\alpha x_\beta \in \sqrt{I_j}$ for all $\alpha, \beta \in L$ such that α, β are incomparable and $\operatorname{height}(\alpha) = \operatorname{height}(\beta) = j + 1$.

Proof. We proceed by induction on j. Consider the case $j = 0$. $I_0 = I_L + (y_0) = I_L + (x_\emptyset)$. Clearly (a) holds. Let $\alpha, \beta \in L$ be such that α, β are incomparable and $\operatorname{height}(\alpha) = \operatorname{height}(\beta) = 1$. Since α, β are incomparable, $\operatorname{height}(\alpha \wedge \beta) < \operatorname{height}(\alpha)$. Therefore, $\alpha \wedge \beta = \emptyset$. Thus, $x_{\alpha\wedge\beta} x_{\alpha\vee\beta} \in I_0$. Hence $x_\alpha x_\beta \in I_0 \subset \sqrt{I_0}$.

Now, assume that $j > 0$. To prove (a), let $\beta \in L$ with $\operatorname{height}(\beta) = j$. Consider $x_\beta y_j = {x_\beta}^2 + \sum_{\alpha \neq \beta} x_\beta x_\alpha$. From the observation $I_{j-1} \subset I_j$ and by induction hypothesis,

$\sum_{\alpha\neq\beta} x_\beta x_\alpha \in \sqrt{I_j}$. Hence ${x_\beta}^2 \in \sqrt{I_j}$ which implies $x_\beta \in \sqrt{I_j}$. Let $\alpha, \beta \in L$ be such that α, β are incomparable and $\operatorname{height}(\alpha) = \operatorname{height}(\beta) = j+1$. Since α, β are incomparable, $\operatorname{height}(\alpha \wedge \beta) < \operatorname{height}(\alpha)$. Thus, $x_{\alpha\wedge\beta} x_{\alpha\vee\beta} \in \sqrt{I_j}$ from (a). Hence $x_\alpha x_\beta \in \sqrt{I_j}$. □

Proposition 2.18. *Under the hypothesis of Lemma 2.17,*
(1) $y_0, \ldots, y_n$ *is a regular sequence of* $R[L]$.
(2) $y_0, \ldots, y_n$ *is a regular sequence of* $K[L]/\operatorname{in}_<(I_L)$.

Proof. For a Cohen-Macaulay ring, every system of parameter is a regular sequence. So it suffice to show that $\{y_0, \ldots, y_n\}$ forms a system of parameters of $R[L]$ and $K[L]/\operatorname{in}_<(I_L)$. Proof of (1) follows from Lemma 2.17. For (2), define $\mathcal{J}_j = \operatorname{in}_<(I_L) + (y_0, \ldots, y_j)$ for all $0 \leq j \leq n$. Note that (b) of Lemma 2.17 holds for $\mathcal{J}_j$ by the definition of $\operatorname{in}_<(I_L)$. Also, (a) of Lemma 2.17 holds for $\mathcal{J}_j$ by the similar argument. Hence (2) holds. □

We now discuss how Hibi rings behave under the ordinal sum of two posets. Let P_1 and P_2 be two posets and P be the ordinal sum of P_1 and P_2. Let $R[\mathcal{I}(P_1)] = K[\{x_\alpha : \alpha \in \mathcal{I}(P_1)\}]/I_{\mathcal{I}(P_1)}$, $R[\mathcal{I}(P_2)] = K[\{y_\beta : \beta \in \mathcal{I}(P_2)\}]/I_{\mathcal{I}(P_2)}$ and $R[\mathcal{I}(P)] = K[\{z_\gamma : \gamma \in \mathcal{I}(P)\}]/I_{\mathcal{I}(P)}$.

Lemma 2.19. *Let* P_1, P_2 *and* P *be as above. Then*

$$R[\mathcal{I}(P)] \cong (R[\mathcal{I}(P_1)] \otimes_K R[\mathcal{I}(P_2)])/(x_{P_1} - y_\emptyset).$$

Proof. Let $T = K[\{x_\alpha : \alpha \in \mathcal{I}(P_1)\} \cup \{y_\beta : \beta \in \mathcal{I}(P_2)\}]/(x_{P_1} - y_\emptyset)$ and $T' = T/(I_{\mathcal{I}(P_1)}T + I_{\mathcal{I}(P_2)}T)$. Define a map

$$\varphi : K[\mathcal{I}(P)] \to T$$

by

$$\varphi(z_\gamma) = \begin{cases} x_\gamma & \text{if } \ \gamma \subseteq P_1, \\ y_{\gamma'} & \text{if } \ \gamma = P_1 \cup \gamma', \text{ where } \gamma' \subseteq P_2. \end{cases}$$

It is easy to see that φ is an isomorphism. If $\alpha, \beta \in \mathcal{I}(P)$ are incomparable then either $\alpha, \beta \in \mathcal{I}(P_1)$ or $\alpha = P_1 \cup \alpha'$ and $\beta = P_1 \cup \beta'$ where $\alpha', \beta' \in \mathcal{I}(P_2)$ and α', β' incomparable. Let $\pi : T \to T'$ be the natural projection. Thus, $\pi \circ \varphi : K[\mathcal{I}(P)] \to T'$ and $\ker(\pi \circ \varphi) = \varphi^{-1}(I_{\mathcal{I}(P_1)}T + I_{\mathcal{I}(P_2)}T)$.

Thus, it is sufficient to show that $\varphi(I_{\mathcal{I}(P)}) = I_{\mathcal{I}(P_1)}T + I_{\mathcal{I}(P_2)}T$. Let α, β be two incomparable elements of $\mathcal{I}(P)$. If $\alpha, \beta \in \mathcal{I}(P_1)$ then $\varphi(z_\alpha z_\beta - z_{\alpha\cap\beta} z_{\alpha\cup\beta}) = x_\alpha x_\beta - x_{\alpha\cap\beta} x_{\alpha\cup\beta} \in I_{\mathcal{I}(P_1)}T$. If $\alpha = P_1 \cup \alpha'$ and $\beta = P_1 \cup \beta'$ where $\alpha', \beta' \in \mathcal{I}(P_2)$, then $\varphi(z_\alpha z_\beta - z_{\alpha\cap\beta} z_{\alpha\cup\beta}) = y_{\alpha'} y_{\beta'} - y_{\alpha'\cap\beta'} y_{\alpha'\cup\beta'} \in I_{\mathcal{I}(P_2)}T$. Hence, $\varphi(I_{\mathcal{I}(P)}) \subseteq I_{\mathcal{I}(P_1)}T + I_{\mathcal{I}(P_2)}T$. On the other hand, if α, β are two incomparable elements of $\mathcal{I}(P_1)$ then $\varphi(z_\alpha z_\beta - z_{\alpha\cap\beta} z_{\alpha\cup\beta}) = x_\alpha x_\beta - x_{\alpha\cap\beta} x_{\alpha\cup\beta}$ while if α', β' are two incomparable elements of $\mathcal{I}(P_2)$ then $\varphi(z_{P_1\cup\alpha'} z_{P_1\cup\beta'} - z_{(P_1\cup\alpha')\cap(P_1\cup\beta')} z_{(P_1\cup\alpha')\cup(P_1\cup\beta')}) = y_{\alpha'} y_{\beta'} - y_{\alpha'\cap\beta'} y_{\alpha'\cup\beta'}$. Hence the equality. $\square$

Lemma 2.20. *Let $P_1, \{p\}$ and P_2 be posets. Let P be the ordinal sum $P_1 \oplus \{p\} \oplus P_2$. Then*

$$R[\mathcal{I}(P)] \cong R[\mathcal{I}(P_1 \oplus P_2)] \otimes_K K[y] \cong R[\mathcal{I}(P_1)] \otimes_K R[\mathcal{I}(P_2)],$$

where $K[y]$ is a polynomial ring.

Proof. First, we prove that

$$R[\mathcal{I}(P)] \cong R[\mathcal{I}(P_1 \oplus P_2)] \otimes_K K[y].$$

Let $R[\mathcal{I}(P_1 \oplus P_2)] = K[\{u_\beta \ : \beta \in \mathcal{I}(P_1 \oplus P_2)\}]/I_{\mathcal{I}(P_1\oplus P_2)}$ and $R[\mathcal{I}(P)] = K[\{v_\alpha \ : \alpha \in \mathcal{I}(P)\}]/I_{\mathcal{I}(P)}$. Define a map

$$\varphi : K[\{v_\alpha \ : \alpha \in \mathcal{I}(P)\}] \to T := K[y, \{u_\beta \ : \beta \in \mathcal{I}(P_1 \oplus P_2)\}]$$

by

$$\varphi(v_\gamma) = \begin{cases} u_\gamma & \text{if} \quad \gamma \subseteq P_1, \\ y & \text{if} \quad \gamma = P_1 \cup \{p\}, \\ u_{\gamma'} & \text{if} \quad \gamma = P_1 \cup \{p\} \cup \gamma', \ \text{where } \gamma' \subseteq P_2. \end{cases}$$

It is easy to see that φ is an isomorphism. If $\alpha, \beta \in \mathcal{I}(P)$ are incomparable, then either $\alpha, \beta \in \mathcal{I}(P_1)$ or $\alpha = P_1 \cup \{p\} \cup \alpha'$ and $\beta = P_1 \cup \{p\} \cup \beta'$ where $\alpha', \beta' \in \mathcal{I}(P_2)$ and α', β' incomparable. Let $T' = T/(I_{\mathcal{I}(P_1\oplus P_2)}T)$ and $\pi : T \to T'$ be the natural surjection. Thus, $\pi \circ \varphi : K[\mathcal{I}(P)] \to T'$ and $\ker(\pi \circ \varphi) = \varphi^{-1} I_{\mathcal{I}(P_1\oplus P_2)}T$.

It is sufficient to show that $\varphi(I_{\mathcal{I}(P)}) = I_{\mathcal{I}(P_1\oplus P_2)}T$. The proof of this is similar to the proof of Lemma 2.19.

Now, the minimal generating set of the Hibi ideal $I_{\mathcal{I}(P)}$ can be partitioned between two disjoint set of variables $\{v_\alpha \ : \alpha \in \mathcal{I}(P) \text{ and } \alpha \subseteq P_1\}$ and $\{v_\alpha \ : \alpha \in \mathcal{I}(P) \text{ and } P_1 \cup$

$\{p\} \subseteq \alpha\}$. So the Hibi ring $R[\mathcal{I}(P)]$ admits a tensor product decomposition, where one of the rings is isomorphic to $R[\mathcal{I}(P_1)]$ and the other ring is isomorphic to $R[\mathcal{I}(P_2)]$. $\square$

In [Hib87], Hibi proved that $R[\mathcal{I}(P_1) \oplus \mathcal{I}(P_2)] \cong R[\mathcal{I}(P_1)] \otimes_K R[\mathcal{I}(P_2)]$. One can immediately check that the poset of join-irreducibles of $\mathcal{I}(P_1) \oplus \mathcal{I}(P_2)$ is isomorphic to $P_1 \oplus \{p\} \oplus P_2$.

Corollary 2.21. *Let P be a poset and $P' = \{p_{i_1}, ..., p_{i_r}\}$ be the subset of all elements of P which are comparable to every element of P. Let P'' be the induced subposet of P on the set $P \setminus P'$. Then,*

$$R[\mathcal{I}(P)] \cong R[\mathcal{I}(P'')] \otimes_K K[y_1, \dots, y_r],$$

where $K[y_1, \dots, y_r]$ is a polynomial ring.

Proof. Without loss of generality, we may assume that $p_{i_1} < \cdots < p_{i_r}$ in P. Let $P_0 = \{p \in P : p < p_{i_1}\}$, $P_j = \{p \in P : p_{i_j} < p < p_{i_{j+1}}\}$ for $1 < j < r-1$ and $P_r = \{p \in P : p > p_{i_r}\}$. Then P is the ordinal sum $P_0 \oplus \{p_{i_1}\} \oplus P_1 \oplus \cdots \oplus \{p_{i_r}\} \oplus P_r$. Now, the result follows from Lemma 2.20. $\square$

2.7 Algebras with straightening laws (ASL)

In this section, we define algebra with straightening laws (in short ASL) and we prove that Hibi rings are ASL.

Let $\mathcal{A} = \oplus_{i \in \mathbb{N}} \mathcal{A}_i$ be a finite type graded K-algebra and let $\mathcal{H}$ be a finite poset. Assume that an injective map $i : \mathcal{H} \to \mathcal{A}$ is given. We identify the elements of $\mathcal{H}$ with their images. A monomial $\alpha_1 \cdots \alpha_n$ in $\mathcal{A}$ is called a *standard monomial* if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ in $\mathcal{H}$.

Definition 2.22. *We say that $\mathcal{A}$ is an ASL on $\mathcal{H}$ over K if the following conditions are satisfied:*

ASL-1 The set of standard monomials is a K-basis of the algebra $\mathcal{A}$.

ASL-2 If α, β are incomparable elements of $\mathcal{H}$ and if $\alpha\beta = \sum c_i \gamma_{i1} \cdots \gamma_{ik_i}$, where $c_i \in K \backslash \{0\}$ and $\gamma_{i1} \leq \cdots \leq \gamma_{ik_i}$, is the unique expression of $\alpha\beta$ as a linear combination of standard monomials, then $\gamma_{i1} \leq \alpha, \beta$ for all i.

The relations in axiom ASL-2 are called the *straightening relations* of $\mathcal{A}$. It follows from the two axioms that the straightening relations are indeed the defining equations of $\mathcal{A}$ as a quotient of the polynomial ring $K[\mathcal{H}] = K[x_\alpha : \alpha \in \mathcal{H}]$ [DCEP82, Proposition 1.1]. That is, the kernel I of the canonical surjective map $K[\mathcal{H}] \to \mathcal{A}$ of K-algebras induced by the map $i : \mathcal{H} \to \mathcal{A}$ is generated by the straightening relations regarded as elements of $K[\mathcal{H}]$.

Let $L = \mathcal{I}(P)$ be a distributive lattice with $P = \{p_1, \ldots, p_n\}$. Let $i : L \to R = K[t, z_1, \ldots, z_n]$ be given by $i(\alpha) = t \prod_{p_i \in \alpha} z_i$, where $\alpha \in L$. Note that L is embedded into the polynomial ring R by the injective map i. Also, note that for all $\alpha, \beta \in L$,

$$i(\alpha)i(\beta) = i(\alpha \vee \beta)i(\alpha \wedge \beta) \tag{2.1}$$

Now, we shall show that $R[L]$ is an ASL on L over K. The proof follows the argument of [Ene15, Page 14]. It follows from Equation 2.1 that the Hibi ring $R[L]$ satisfies axiom ASL-2. For ASL-1, it suffices to show that the standard monomials are distinct because they are monomials of the polynomial ring $S = K[t, z_1, \ldots, z_n]$. To prove that, it is enough to show that for any two chains $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_s$ in L, we have $i(\alpha_1) \cdots i(\alpha_r) = i(\beta_1) \cdots i(\beta_s)$ if and only if $r = s$ and $\alpha_l = \beta_l$ for all l.

The proof of 'if' direction is immediate. To prove the 'only if' direction, let $i(\alpha_1) \cdots i(\alpha_r) = i(\beta_1) \cdots i(\beta_s)$. Equivalently, we have

$$t^r \prod_{l=1}^{r} \Big(\prod_{p_j \in \beta_l} z_j \Big) = t^s \prod_{l=1}^{s} \Big(\prod_{p_j \in \beta_l} z_j \Big).$$

Clearly, $r = s$. Also, we have $\prod_{l=1}^{r} (\prod_{p_j \in \beta_l} z_j) = \prod_{l=1}^{r} (\prod_{p_j \in \beta_l} z_j)$. Therefore,

$$\Big(\prod_{p_j \in \alpha_1} z_j\Big)^r \Big(\prod_{p_j \in \alpha_2 \setminus \alpha_1} z_j\Big)^{r-1} \cdots \Big(\prod_{p_j \in \alpha_r \setminus \alpha_{r-1}} z_j\Big) = \Big(\prod_{p_j \in \beta_1} z_j\Big)^r \Big(\prod_{p_j \in \beta_2 \setminus \beta_1} z_j\Big)^{r-1} \cdots \Big(\prod_{p_j \in \beta_r \setminus \beta_{r-1}} z_j\Big).$$

Thus, we have $\alpha_l = \beta_l$ for all l. Hence the proof.

Since the straightening relations generate the defining ideal of $R[L]$, We get that $R[L] \cong K[L]/I_L$, where $K[L] = K[x_\alpha : \alpha \in L]$ and $I_L = (x_\alpha x_\beta - x_{\alpha\wedge\beta} x_{\alpha\vee\beta} : \alpha, \beta \in L$ and α, β incomparable). This is what we have proved in Section 2.6.

Let $A = \oplus_{i\geq 0} A_i$ and $B = \oplus_{i\geq 0} B_i$ be two graded K-algebras. Then the *Segre product* of A and B is the graded K-algebra

$$A * B = \bigoplus_{i\geq 0} (A_i \otimes_K B_i).$$

Let P_1 and P_2 be two posets and P be their disjoint union. It was observed in [HHR00] that $R[\mathcal{I}(P)] \cong R[\mathcal{I}(P_1)] * R[\mathcal{I}(P_2)]$, where $*$ denotes the Segre product. Observe that $\mathcal{I}(P) = \{(\alpha, \beta) : \alpha \in \mathcal{I}(P_1) \text{ and } \beta \in \mathcal{I}(P_2)\}$.

2.8 Semigroup rings

Let $H \subset \mathbb{N}^n$ be an affine semigroup. Suppose that $h_1, \ldots, h_m \in \mathbb{N}^n$ is the unique minimal set of generators of H. We consider the polynomial ring $T = K[t_1, \ldots, t_n]$ in n variables. Then, the semigroup ring attached to H, denoted by $K[H]$, is the subring of T generated by the monomials $u_i = \prod_{j=1}^{n} t_j^{h_i(j)}$ for $1 \leq i \leq m$, where $h_i(j)$ denotes the jth component of the integer vector h_i. In the following example, we discuss a semigroup ring structure of Hibi rings.

Example 2.23. Let $L = \mathcal{I}(P)$ be a distributive lattice with $P = \{p_1, \ldots, p_n\}$. For $\alpha \in L$, define a $(n+1)$-tuple h_α such that for $1 \leq i \leq n$,

$$\begin{cases} 1 & \text{at the } 1^{st} \text{ position,} \\ 1 & \text{at } (i+1)^{th} \text{ position if} \quad p_i \in \alpha, \\ 0 & \text{at } (i+1)^{th} \text{ position if} \quad p_i \notin \alpha. \end{cases}$$

Let H be the affine semigroup generated by $\{h_\alpha : \alpha \in L\}$. Then, we have $K[H] = R[L]$.

Let $S = K[x_1, \ldots, x_m]$ be a polynomial ring over K. Consider a K-algebra map $S \to K[H]$ defined by $x_i \mapsto u_i$ for all $i = 1, \ldots, m$. Let I_H be the kernel of this K-algebra map. Set $\deg x_i = h_i$ to assign a $\mathbb{Z}^n$-graded ring structure to S. Let $\mathfrak{m}$ be the graded maximal S-ideal. Then $K[H]$ become $\mathbb{Z}^n$-graded S-module. Thus, $K[H]$ admits a minimal $\mathbb{Z}^n$-graded S-resolution $\mathbb{F}$.

Given $h \in H$, we define the *squarefree divisor complex* Δ_h as follows:

$$\Delta_h := \{F \subseteq [m] : \prod_{i\in F} u_i \text{ divides } t_1^{h(1)} \cdots t_n^{h(n)} \text{ in } K[H]\}.$$

Equivalently,

$$\Delta_h := \{F \subseteq [m] : h - \sum_{i \in F} h_i \in H\}.$$

Clearly, Δ_h is a simplicial complex. We denote the i^{th} reduced simplicial homology of a simplicial complex Δ with coefficients in K by $\widetilde{H}_i(\Delta, K)$.

Proposition 2.24. *[BH97, Proposition 1.1], [Stu96, Theorem 12.12] With the notation and assumptions introduced one has* $\operatorname{Tor}_i(K[H], K)_h \cong \widetilde{H}_{i-1}(\Delta_h, K)$. *In particular,*

$$\beta_{ih}(K[H]) = \dim_K \widetilde{H}_{i-1}(\Delta_h, K).$$

Let H' be a subsemigroup of H generated by a subset $\mathcal{X}$ of $\{h_1, \ldots, h_m\}$, and let $S' = K[\{x_i : h_i \in \mathcal{X}\}] \subseteq S$. Furthermore, let $\mathbb{F}'$ be the $\mathbb{Z}^n$-graded free S'-resolution of $K[H']$. Then, since S is a flat S'-module, $\mathbb{F}' \otimes_{S'} S$ is a $\mathbb{Z}^n$-graded free S-resolution of $S/I_{H'}S$. The inclusion $(S'/I_{H'}S) \otimes_{S'} S \to S/I_H S$ induces a $\mathbb{Z}^n$-graded S-module complex homomorphism $\mathbb{F}' \otimes_{S'} S \to \mathbb{F}$. Applying $\_ \otimes_S K$ on this complex homomorphism with $K = S/\mathfrak{m}$, we obtain the following sequence of isomorphisms and natural maps of $\mathbb{Z}^n$-graded K-modules

$$\operatorname{Tor}_i^{S'}(K[H'], K) \cong H_i(\mathbb{F}' \otimes_{S'} K) \cong H_i(\mathbb{F}' \otimes_{S'} S) \otimes_S K) \to$$
$$H_i(\mathbb{F} \otimes_S K) \cong \operatorname{Tor}_i^S(K[H], K).$$

Corollary 2.25. *[EHH15, Corollary 3.3] With the notation and assumptions introduced, let h be an element of H' with the property that $h_i \in A$ whenever $h - h_i \in H$. Then the natural K-vector space homomorphism* $\operatorname{Tor}_i^{S'}(K[H'], K)_h \to \operatorname{Tor}_i^S(K[H], K)_h$ *is an isomorphism for all i.*

Proof. Let Δ'_h be the squarefree divisor complex of h where h is viewed as an element of H'. Then we obtain the following commutative diagram

$$\begin{array}{ccc} \operatorname{Tor}_i(K[H'], K)_h & \longrightarrow & \operatorname{Tor}_i(K[H], K)_h \\ \downarrow & & \downarrow \\ \tilde{H}_{i-1}(\Delta'_h, K) & \longrightarrow & \tilde{H}_{i-1}(\Delta_h, K). \end{array}$$

The vertical maps and the lower horizontal map are isomorphisms, simply because $\Delta'_h = \Delta_h$, due to assumptions on h. This yields the desired conclusion. $\square$

Definition 2.26. Let $H \subset \mathbb{N}^n$ be an affine semigroup generated by $h_1, \ldots, h_m$. An affine subsemigroup $H' \subseteq H$ generated by a subset of $\{h_1, \ldots, h_m\}$ will be called a *homologically pure* subsemigroup of H if for all $h \in H'$ and all h_i with $h - h_i \in H$, it follows that $h_i \in H'$.

We need the following proposition several times in this thesis.

Proposition 2.27. *[EHH15, Corollary 3.4] Let H' be a homologically pure subsemigroup of H. If $\mathbb{F}'$ is the minimal $\mathbb{Z}^n$-graded free S'-resolution of $K[H']$ and $\mathbb{F}$ is the minimal $\mathbb{Z}^n$-graded free S-resolution of $K[H]$, then the complex homomorphism $\mathbb{F}' \otimes S \to \mathbb{F}$ induces an injective map $\mathbb{F}' \otimes K \to \mathbb{F} \otimes K$. Hence,*

$$\operatorname{Tor}_i^{S'}(K[H'], K) \to \operatorname{Tor}_i^{S}(K[H], K)$$

is injective for all i. In particular, any minimal set of generators of $\operatorname{Syz}_i(K[H'])$ is part of a minimal set of generators of $\operatorname{Syz}_i(K[H])$. Moreover, $\beta_{ij}(K[H']) \leq \beta_{ij}(K[H])$ for all i and j.

We want to use the above result for Hibi rings. To do that, we give a different semigroup ring structure to Hibi rings which was defined by Herzog and Hibi in [HH05]. Let $L = \mathcal{I}(P)$ be a distributive lattice with $P = \{p_1, \ldots, p_n\}$ and let $S = K[y_1, \ldots, y_n, z_1, \ldots, z_n]$ be a polynomial ring in $2n$ variables over a field K. Let $S[L]$ be the subring of S generated by the monomials $v_\alpha = (\prod_{p_i \in \alpha} y_i)(\prod_{p_i \notin \alpha} z_i)$, where $\alpha \in L$.

Now, we show that $S[L]$ is isomorphic to $R[L]$ as a K-algebra. In order to show that, we prove that $S[L]$ is a ASL on L over K with same straightening relations as $R[L]$. Let $\varphi : L \to S[L]$ be defined by $\alpha \mapsto v_\alpha$. Note that for all $\alpha, \beta \in L$,

$$\varphi(\alpha)\varphi(\beta) = \varphi(\alpha \vee \beta)\varphi(\alpha \wedge \beta)$$

ASL-2 follows from the above equation. For ASL-1, it suffices to show that the standard monomials are distinct because they are monomials of the polynomial ring $S = K[y_1, \ldots, y_n, z_1, \ldots, z_n]$. So it is enough to show that for any two chains $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_s$ in L, we have $\varphi(\alpha_1) \cdots \varphi(\alpha_r) = \varphi(\beta_1) \cdots \varphi(\beta_s)$ if and only if $r = s$ and $\alpha_i = \beta_i$ for all i. The proof of 'if' direction is immediate. To prove the 'only if' direction, let $\varphi(\alpha_1)\cdots\varphi(\alpha_r)=\varphi(\beta_1)\cdots\varphi(\beta_s)$. Equivalently, we have

$$\prod_{i=1}^{r}(\prod_{p_j\in\beta_i} y_j)(\prod_{p_j\notin\beta_i} z_j)=\prod_{i=1}^{s}(\prod_{p_j\in\beta_i} y_j)(\prod_{p_j\notin\beta_i} z_j).$$

In the above monomial, let the exponents of y_1 and z_1 be a_1 and b_1 respectively. Then, $a_1+b_1=r$ and $a_1+b_1=s$. Hence, $r=s$. Also, $\prod_{i=1}^{r}(\prod_{p_j\in\beta_i} y_j)=\prod_{i=1}^{r}(\prod_{p_j\in\beta_i} y_j)$. Therefore,

$$(\prod_{p_j\in\alpha_1} y_j)^r(\prod_{p_j\in\alpha_2\setminus\alpha_1} y_j)^{r-1}\cdots(\prod_{p_j\in\alpha_r\setminus\alpha_{r-1}} y_j)=(\prod_{p_j\in\beta_1} y_j)^r(\prod_{p_j\in\beta_2\setminus\beta_1} y_j)^{r-1}\cdots(\prod_{p_j\in\beta_r\setminus\beta_{r-1}} y_j).$$

Thus, we have $\alpha_i=\beta_i$ for all i. Hence the proof.

For $\alpha\in L$, define a $2n$-tuple h_α such that for $1\le i\le n$,

$$\begin{cases} 1 & \text{at } i^{th} \text{ position if } \quad p_i\in\alpha,\\ 0 & \text{at } i^{th} \text{ position if } \quad p_i\notin\alpha,\\ 0 & \text{at } (n+i)^{th} \text{ position if } \quad p_i\in\alpha,\\ 1 & \text{at } (n+i)^{th} \text{ position if } \quad p_i\notin\alpha. \end{cases}$$

Let H be the affine semigroup generated by $\{h_\alpha:\alpha\in L\}$. Then, we have $K[H]=S[L]$. We will use this semigroup ring structure to conclude the results about the Hibi ring $R[L]$.

Let us now explain how we will use Proposition 2.24 for standard grading. We have $S[L]\cong K[L]/I_L$. In order to use Proposition 2.24, we need to set $\deg(x_\alpha)=h_\alpha$ for all $\alpha\in L$. Note that $\sum_{i=1}^{2n} h_\alpha(i)=n$ for all $\alpha\in L$. For a $h\in H$, we set $\deg(x^h)=(\sum_{i=1}^{2n} h(i))/n$. In particular, $\deg(x_\alpha)=1$ for all $\alpha\in L$.

Chapter 3

Property N_p of Hibi rings

In this chapter, we study Green-Lazarsfeld property N_p for Hibi rings. First, we identify two kinds of homologically pure subsemigroups of an affine semigroup associated to a Hibi ring; see Section 2.8 for the definition of homologically pure subsemigroups. Using these, we prove necessary conditions for Hibi rings to satisfy property N_p for $p = 2$ and 3. We also show that if a Hibi ring satisfies property N_4, then it is a polynomial ring or it has a linear resolution. Therefore, it satisfies property N_p for all $p \geq 4$ as well.

Let $S = K[x_1, \dots, x_n]$ be a standard graded polynomial ring over K and I be a graded S-ideal. Let $\mathbb{F}$ be the graded minimal free resolution of S/I over S:

$$\mathbb{F} : 0 \to \bigoplus_j S(-j)^{\beta_{rj}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1j}} \to \bigoplus_j S(-j)^{\beta_{0j}}.$$

Let $p \in \mathbb{N}$. Under the notations as above, we say that S/I satisfies *Green-Lazarsfeld property* N_p if S/I is normal and $\beta_{ij}(S/I) = 0$ for all $i \neq j+1$ and $1 \leq i \leq p$. Therefore, S/I satisfies property N_0 if and only if it is normal; it satisfies property N_1 if and only if it is normal and I is generated by quadratics; it satisfies property N_2 if and only if it satisfies property N_1 and I is linearly presented and so on. We know that the Hibi rings are normal and the Hibi ideals are generated by quadratics. Hence, the Hibi rings satisfy property N_1.

First, we state a result of Ene which characterizes all simple planar distributive lattices for which the associated Hibi ring satisfies property N_2. We start by defining the notion of planar distributive lattice.

Definition 3.1. *[HHO18, Section 6.4] A finite distributive lattice $L = \mathcal{I}(P)$ is called* planar *if P can be decomposed into a disjoint union $P = \{p_1, \dots, p_m\} \cup \{q_1, \dots, q_n\}$, where $m, n \geq 0$ such that $\{p_1, \dots, p_m\}$ and $\{q_1, \dots, q_n\}$ are chains in P.*

Remark 3.2. Let us consider the infinite distributive lattice $\mathbb{N}^2$ with the partial order defined as $(i, j) \leq (k, l)$ if $i \leq k$ and $j \leq l$. Let $L = \mathcal{I}(P)$ be a finite planar distributive lattice, where $P = \{p_1, \dots, p_m\} \cup \{q_1, \dots, q_n\}$. Assume that $\{p_1, \dots, p_m\}$ and $\{q_1, \dots, q_n\}$ are chains in P with $p_1 \leq \dots \leq p_m$ and $p_1 \leq \dots \leq p_m$. Define a map

$$\varphi : \mathcal{I}(P) \to \mathbb{N}^2$$

by

$$\varphi(\alpha) = \begin{cases} (0,0) & \text{if} \quad \alpha = \emptyset, \\ (i,0) & \text{if} \quad \alpha = \{p \in P : \ p \leq p_i\}, \\ (0,j) & \text{if} \quad \alpha = \{p \in P : \ p \leq q_j\} \ , \\ (i,j) & \text{if} \quad \alpha = \{p \in P : \text{either } p \leq p_i \text{ or } p \leq q_j\}. \end{cases}$$

It is easy to see that φ is an order-preserving injective map. Hence, any finite planar distributive lattice can be embedded into $\mathbb{N}^2$. Also, observe that $[(0,0),(m,n)]$ is the smallest interval of $\mathbb{N}^2$ which contains L. □

Let L be a distributive lattice. If the poset of join-irreducibles of L is a simple poset, then sometimes we abuse the notation and say that L is a *simple distributive lattice*. Now we state Ene's theorem.

Theorem 3.3. *[Ene15, Theorem 3.12] Let $L = \mathcal{I}(P)$ be a simple planar distributive lattice with $\#P = n + m$, $L \subset [(0,0),(m,n)]$ with $m, n \geq 2$. Then $R[\mathcal{I}(P)]$ satisfies property N_2 if and only if the following conditions hold:*

(i) At least one of the vertices $(m, 0)$ and $(0, n)$ belongs to L.

(ii) The vertices $(1, n-1)$ and $(m-1, 1)$ belong to L.

Corollary 3.4. *Let $L = \mathcal{I}(P)$ be a simple planar distributive lattice with $P = \{a_1, \dots, a_m, b_1, \dots, b_n\}$. Let $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_n\}$ be chains in P with $a_1 \lessdot a_2 \lessdot \dots \lessdot a_m$ and $b_1 \lessdot b_2 \lessdot \dots \lessdot b_n$. Assume that $\{a_1, \dots, a_m\}$ is an order ideal of P. If $R[\mathcal{I}(P)]$ satisfies property N_2, then P is one of the posets as shown in Figure 3.1.*

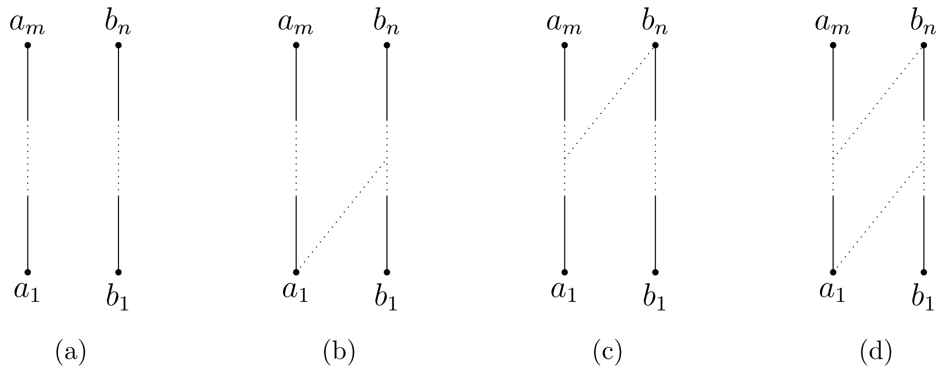


FIGURE 3.1

3.1 Homologically pure subsemigroups

In this section, we identify two kinds of homologically pure subsemigroups of a semigroup associated to a Hibi ring and we use them to conclude results about property N_p of Hibi rings. The first one is the following and the second one is in Notation 3.8.

Let $L = \mathcal{I}(P)$ be a distributive lattice. Let $\beta, \gamma \in L$ such that $\beta \leq \gamma$. Define $L_1 = \{\alpha \in L : \beta \leq \ \alpha \leq \gamma\}$. Clearly, L_1 is a sublattice of L. Let H be the affine semigroup associated to H and let H_1 be the affine subsemigroup of H generated by $\{h_\alpha : \alpha \in L_1\}$.

Proposition 3.5. *Let H and H_1 be as defined above. Then H_1 is a homologically pure subsemigroup of H.*

Proof. We show that if $\alpha \notin L_1$ then $h - h_\alpha \notin H$ for all $h \in H_1$. Suppose that $\alpha \notin L_1$ then either $\alpha \nleq \gamma$ or $\alpha \ngeq \beta$.

If $\alpha \nleq \gamma$, then there exists a $p_i \in \alpha$ such that $p_i \notin \gamma$. So i^{th} entry of h_α is 1 but for any $\alpha' \in L_1$, i^{th} entry of $h_{\alpha'}$ is 0. Hence, $h - h_\alpha \notin H$ for all $h \in H_1$.

If $\alpha \ngeq \beta$, then there exists a $p_j \in \beta$ such that $p_j \notin \alpha$. So $(n+j)^{th}$ entry of h_α is 1 but for any $\alpha' \in L_1$, $(n+j)^{th}$ entry of $h_{\alpha'}$ is 0. Hence, $h - h_\alpha \notin H$ for all $h \in H_1$. $\square$

Proposition 3.6. *Let L and L_1 be as above. Let $\beta = \{p_{a_1}, \dots, p_{a_r}\}$ and $\gamma = \{p_{a_1}, \dots, p_{a_r}, p_{b_1}, \dots, p_{b_s}\}$. Then, the induced subposet P_1 of P on the set $\{p_{b_1}, \dots, p_{b_s}\}$ is isomorphic to the poset of join-irreducible elements of L_1.*

Proof. The idea of the proof is based on the proof of [HHO18, Theorem 6.4]. For finite posets Q and Q', if $\mathcal{I}(Q) \cong \mathcal{I}(Q')$ then $Q \cong Q'$. So it is enough to prove that

$\mathcal{I}(P_1) \cong L_1$. Define a map

$$\varphi : \mathcal{I}(P_1) \to L_1$$

by

$$\varphi(\alpha) = (\bigvee_{i=1}^{r} p_{a_i}) \vee (\bigvee_{p\in\alpha} p).$$

In particular, $\varphi(\emptyset) = \vee_{i=1}^{r} p_{a_i}$. Clearly, φ is order-preserving.

Let α and δ be two order ideals of P_1 with $\alpha \neq \delta$, say $\delta \nleq \alpha$. Let p_0 be a maximal element of δ with $p_0 \notin \alpha$. We show that $\varphi(\alpha) \neq \varphi(\delta)$. Suppose, on the contrary, that $\varphi(\alpha) = \varphi(\delta)$, then

$$(\bigvee_{i=1}^{r} p_{a_i}) \vee (\bigvee_{p\in\alpha} p) = (\bigvee_{i=1}^{r} p_{a_i}) \vee (\bigvee_{q\in\beta} q).$$

Since L_1 is distributive, it follows that

$$((\bigvee_{i=1}^{r} p_{a_i}) \vee (\bigvee_{p\in\alpha} p)) \wedge p_0 = (\bigvee_{i=1}^{r} (p_{a_i} \wedge p_0)) \vee (\bigvee_{p\in\alpha} (p \wedge p_0)).$$

Since p_0 is join-irreducible and for any $p \in P$, $p \wedge p_0 \leq p_0$. It follows that $(\vee_{i=1}^{r} p_{a_i} \vee (\vee_{p\in\alpha} p)) \wedge p_0 < p_0$. However, since $p_0 \in \delta$, $(\vee_{i=1}^{r} p_{a_i} \vee (\vee_{q\in\beta} q)) \wedge p_0 = p_0$. This is a contradiction. Hence, φ is injective.

Since each $a \in L_1$ can be the join of the join-irreducible elements p with $p \leq a$ in L_1, it follows that $\varphi(\alpha) = a$, where α is an order ideal of P_1 consisting of those $p \in P_1$ with $p \leq a$. Thus, φ is surjective.

Now, $\varphi^{-1} : L_1 \to \mathcal{I}(P_1)$ is defined as follows: for $x \in L_1$,

$$\varphi^{-1}(x) = \{p \in L_1 : p \leq x, p \text{ is a join-irreducible}\} \setminus \bigcup_{i=1}^{r} p_{a_i}.$$

Clearly, φ^{-1} is order-preserving. Hence the proof. $\square$

We now try to understand how we are going to use the above propositions. For a distributive lattice $\mathcal{L}$, suppose that we want to prove $\beta_{ij}(R[\mathcal{L}]) \neq 0$ for some i, j. The idea of the proof is to reduce the lattice $\mathcal{L}$ to a suitably chosen sublattice $\mathcal{L}_1$. Therefore, by Propositions 3.5 and 2.27, if $\beta_{ij}(R[\mathcal{L}_1]) \neq 0$, then $\beta_{ij}(R[\mathcal{L}]) \neq 0$. Proposition 3.6 describes the subposet of join-irreducibles of $\mathcal{L}_1$. More precisely,

Discussion 3.7. Let P be a poset. Let B and B' be two antichains of P such that for each $p \in B$ there is a $q \in B'$ such that $p < q$ and for each $q' \in B'$ there is a $p' \in B$ such that $p' < q'$. Furthermore, let $\gamma = \{p \in P : p \leq q \text{ for some } q \in B'\}$ and $\beta' = \{p \in P : p' \leq p \leq q \text{ for some } p' \in B, q \in B'\}$. Let $\beta = \gamma \setminus \beta'$. Note that β, γ are the order ideals of $\mathcal{I}(P)$ and $\beta < \gamma$. Let $L_1 = \{\alpha \in \mathcal{I}(P) : \beta \leq \alpha \leq \gamma\}$. Furthermore, let H_1 be the affine subsemigroup of H generated by $\{h_\alpha : \alpha \in L_1\}$. Then, by Proposition 3.5, H_1 is a homologically pure subsemigroup of H. Also, by Proposition 3.6, the induced subposet P_1 of P on the set $\gamma \setminus \beta$ is isomorphic to the poset of join-irreducible elements of L_1. Furthermore, by Proposition 2.27, $\beta_{ij}(R[L_1]) \leq \beta_{ij}(R[L])$. □

Notation 3.8. For a poset P, let X_P and Y_P be the sets of minimal and maximal elements of P respectively. Define $X'_P = \{q \in P : p \lessdot q \text{ for some } p \in X_P\}$ and $Y'_P = \{p \in P : p \lessdot q \text{ for some } q \in Y_P\}$. When the context is clear, we will omit the subscripts and denote X_P, X'_P, Y_P and Y'_P by X, X', Y and Y' respectively.

Let P be a poset. For $x, y \in P$ with $x < y$, define $L' := \{\alpha \in \mathcal{I}(P) : \text{if } x \in \alpha \text{ then } y \in \alpha\}$. It is easy to see that L' is a sublattice of $\mathcal{I}(P)$. Define a poset P' on the set $P \setminus \{p \in P : x \leq p < y\}$ with the following minimal order relations: if $p, q \in P'$, then $p \leq q$ in P' if either

(1) $p \in P' \setminus \{y\}, q \in P'$ and $p \leq q$ in P or

(2) $p = y$ and there is a $p' \in \{a \in P : x \leq a \leq y\}$ such that $p' \leq q$ in P.

Let H be the semigroup corresponding to $\mathcal{I}(P)$ and H' be the subsemigroup of H corresponding to L'. □

Lemma 3.9. *Let P, P', L', H, H' be as in Notation 3.8. Then $L' \cong \mathcal{I}(P')$ and H' is a homologically pure subsemigroup of H.*

Proof. Define a map
$$\varphi : \mathcal{I}(P') \to L'$$
by
$$\varphi(\alpha) = \begin{cases} \alpha & \text{if } y \notin \alpha, \\ \alpha \cup \{p \in P : x \leq p < y\} & \text{if } y \in \alpha. \end{cases}$$
Clearly, φ is order-preserving. If $\gamma \in L'$, then $\varphi(\gamma') = \gamma$, where $\gamma' = \gamma \setminus \{p \in P : x \leq p < y\}$. Hence, φ is surjective. Now, we claim that, for any $\alpha \in \mathcal{I}(P')$, $\varphi(\alpha) \cap P' = \alpha$. If $y \in \alpha$, then $\varphi(\alpha) \cap P' = (\alpha \cup \{p \in P : x \leq p < y\}) \cap P' = \alpha$ and if $y \notin \alpha$, then $\varphi(\alpha) = \alpha$. Therefore, if $\varphi(\alpha) = \varphi(\beta)$ for any $\alpha, \beta \in \mathcal{I}(P')$ then $\alpha = \beta$. This proves that φ is injective.

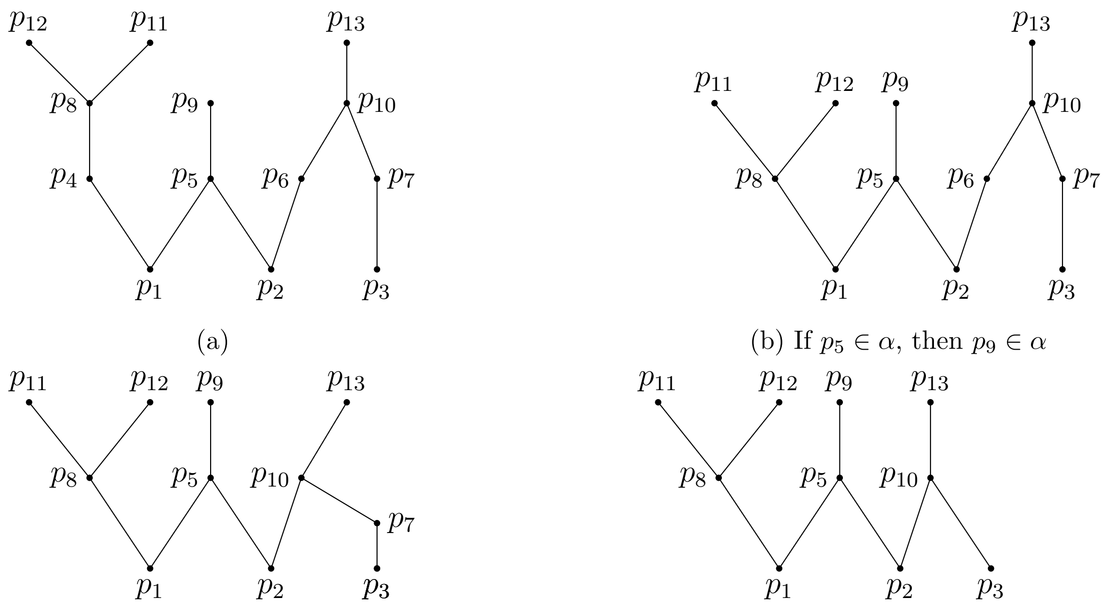
Now, $\varphi^{-1} : L' \to \mathcal{I}(P')$ is defined as follows: for $a \in L'$,

$$\varphi^{-1}(a) = \{p \in L' : p \leq a, p \text{ is a join-irreducible}\} \setminus \{p \in P : x \leq p < y\}.$$

Clearly, φ^{-1} is order-preserving. Hence, φ is an isomorphism.

To prove that H' is a homologically pure subsemigroup of H, we show that if $\alpha \notin L'$ then $h - h_\alpha \notin H$ for all $h \in H'$. Suppose that $\alpha \notin L'$ then $x \in \alpha$ but $y \notin \alpha$. Let $h = \sum_{i=1}^{s} h_{\beta_i} \in H'$ and let the position corresponding to x of h be r. Then the positions corresponding to x and y of $h - h_\alpha$ are $r-1$ and r respectively. Hence, $h - h_\alpha \notin H$. $\square$

Discussion 3.10. For a poset P_0, let X_{P_0}, Y_{P_0}, X'_{P_0} and Y'_{P_0} be as defined in Notation 3.8. If there is an $x \in X'_{P_0}$ and a $y \in Y'_{P_0}$ with $x < y$, reduce P_0 to P_1, using the methods in Notation 3.8. Observe that $y \in X'_{P_1} \cap Y'_{P_1}$, $X_{P_0} = X_{P_1}, Y_{P_0} = Y_{P_1}$ and $\#P_1 = \#(P_0 \setminus \{p \in P_0 : x \leq p < y\}) \leq \#P_0 - 1$. Repeating it, we get a sequence of posets $P_0, \dots, P_n$, where $n \leq \#P_0 - \#X_0 - \#Y_0 - 1$ such that for each $0 \leq i \leq n-1$, there is an $x \in X'_{P_i}$ and $y \in Y'_{P_i}$ with $x < y$ and P_i is reduced to P_{i+1} as in Notation 3.8. Moreover, there is no $x \in X'_{P_n}$ and $y \in Y'_{P_n}$ with the property $x < y$. Here, P_n is a poset defined on the set $X_{P_0} \cup Y_{P_0} \cup Y'_{P_n}$ and $\operatorname{rank}(P_n) \leq 2$. An example of this reduction is given in Figure 3.2. By Lemma 3.9 and Proposition 2.27, if $\beta_{24}(R[\mathcal{I}(P_i)]) \neq 0$ for some $1 \leq i \leq n$, then $\beta_{24}(R[\mathcal{I}(P_0)]) \neq 0$. $\square$

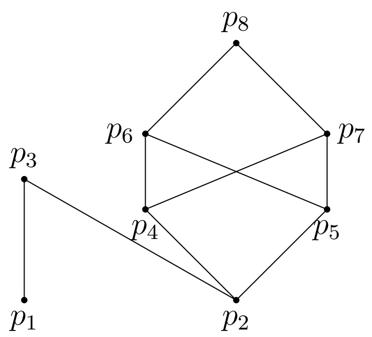


(a) (b) If $p_5 \in \alpha$, then $p_9 \in \alpha$

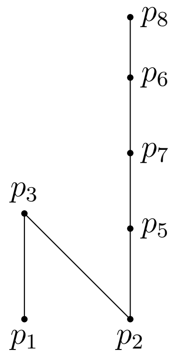
(c) If $p_6 \in \alpha$, then $p_{10} \in \alpha$ (d) If $p_7 \in \alpha$, then $p_{10} \in \alpha$

FIGURE 3.2

Example 3.11. *In this example, we show that the converse of the conclusion in Discussion 3.10 may not be true. Let P be a poset as shown in Figure 3.3a. By Lemma 3.16, $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$. Now, let $x = p_4$ and $y = p_7$. Reduce P to P', using the methods of Notation 3.8. It is easy to see that P' is as shown in Figure 3.3b. By Theorem 3.3, $\beta_{24}(R[\mathcal{I}(P')]) = 0$.*



(a)



(b)

FIGURE 3.3

We now discuss when the converse of Discussion 3.10 holds. Let P be a poset and let $p, q \in P$ with $p < q$. Let L' be as defined in Notation 3.8. Let $K[\mathcal{I}(P)] = K[\{x_\alpha : \alpha \in \mathcal{I}(P)\}]$ and $K[L'] = K[\{x_\alpha : \alpha \in L'\}] \subset K[\mathcal{I}(P)]$. Let $\mathcal{S} = \{\alpha_1, \beta_1, \alpha_2, \beta_2\} \subset L'$ with α_i, β_i are incomparable in $\mathcal{I}(P)$ and L' for all $i = 1, 2$. Then we have

Proposition 3.12. *Under the notations and assumptions as above, if* $x_{\alpha_1}x_{\beta_1} - x_{\alpha_1\wedge\beta_1}x_{\alpha_1\vee\beta_1}$, $x_{\alpha_2}x_{\beta_2} - x_{\alpha_2\wedge\beta_2}x_{\alpha_2\vee\beta_2}$ *is a Koszul relation pair of* $R[\mathcal{I}(P)]$, *then* $x_{\alpha_1}x_{\beta_1} - x_{\alpha_1\wedge\beta_1}x_{\alpha_1\vee\beta_1}$, $x_{\alpha_2}x_{\beta_2} - x_{\alpha_2\wedge\beta_2}x_{\alpha_2\vee\beta_2}$ *is a Koszul relation pair of* $R[L']$.

Proof. Let H be the semigroup associated to $\mathcal{I}(P)$ and H' be the subsemigroup of H corresponding to the sublattice L'. Let $h = h_{\alpha_1} + h_{\alpha_2} + h_{\beta_1} + h_{\beta_2}$. Since $\mathcal{S} \subset L'$, we have $h \in H'$. By Corollary 2.25, we get that $\mathrm{Tor}_i^{K[L']}(R[L'], K)_h \to \mathrm{Tor}_i^{K[\mathcal{I}(P)]}(R[\mathcal{I}(P)], K)_h$ is an isomorphism for all i. This completes the proof. $\square$

3.2 Property N_2 of Hibi rings

In this section, we prove some sufficient conditions regarding when Hibi rings do not satisfy property N_2. The main result of this section is Theorem 3.20. It shows how to reduce checking property N_2 to a planar distributive sublattice. We begin by proving some relevant lemmas.

Lemma 3.13. *[HHO18, Problem 2.16] Let K be a field, $S = K[x_1, \dots, x_n]$ and $T = K[y_1, \dots, y_m]$ be two polynomial rings. Let M be a finitely generated graded S-module and N be a finitely generated graded T-module. Then $M \otimes_K N$ is a finitely generated* *graded $S \otimes_K T$-module and*

$$\beta_{pq}(M \otimes_K N) = \sum \beta_{p_1q_1}(M)\beta_{p_2q_2}(N),$$

where the sum is taken over all p_1 and p_2 with $p_1 + p_2 = p$, and over all q_1 and q_2 with $q_1 + q_2 = q$.

Proof. Let $\{a_1, \dots, a_r\}$ (respectively $\{b_1, \dots, b_s\}$) be a minimal generating set of M (respectively N) over S (respectively T). Then $\{a_i \otimes b_j : 1 \leq i \leq r, 1 \leq j \leq s\}$ is a minimal generating set of $M \otimes_K N$ over $S \otimes_K T$.

Let $\mathbb{F}$ (respectively $\mathbb{G}$) be the minimal graded free resolution of M (respectively N) over S (respectively T). Then the total complex of $\mathbb{F} \otimes_K \mathbb{G}$ is the graded minimal free resolution of $M \otimes_K N$ over $S \otimes_K T$. Recall that the total complex of $\mathbb{F} \otimes_K \mathbb{G}$ is the complex whose degree r part is $\bigoplus_{p+q=r} F_p \otimes_K G_q$ and whose differential is given by $\partial(a \otimes b) = (\partial a) \otimes b + (-1)^p a \otimes (\partial b)$ for $a \in F_p$, $b \in G_q$. Its exactness follows from the Künneth formula of complexes (see [Wei94, Theorem 3.6.3]). This implies the relation between the Betti numbers. $\square$

Lemma 3.14. *Let P be a poset and p be an element of P which is comparable to every element of P. Let $P_1 = \{q \in P : q < p\}$ and $P_2 = \{q \in P : q > p\}$ be induced subposets of P. If P_1 and P_2 are not chains, then $R[\mathcal{I}(P)]$ does not satisfy property N_2.*

Proof. Since P_1 and P_2 are not chains, $R[\mathcal{I}(P_1)]$ and $R[\mathcal{I}(P_2)]$ are not polynomial rings. Therefore, $\beta_{12}(R[\mathcal{I}(P_i)]) \neq 0$ for $i = 1, 2$. Note that P is the ordinal sum $P_1 \oplus \{p\} \oplus P_2$. By Lemma 2.20, $R[\mathcal{I}(P)] = R[\mathcal{I}(P_1)] \otimes R[\mathcal{I}(P_2)]$. Hence, $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$ by Lemma 3.13. $\square$

In [Ene15], Ene proved the above lemma for the case when $\mathcal{I}(P)$ is a planar distributive lattice.

Lemma 3.15. *Let P be a simple poset such that $\#P = m + n$. Let $\mathcal{I}(P)$ be a planar distributive such that $\mathcal{I}(P) \subseteq [(0,0),(m,n)]$ with $m, n \geq 2$. On the underlying set of P, let P' be a poset such that every order relation in P is also an order relation in P'. Assume that the set of minimal (respectively maximal) elements of P' coincide with the set of minimal (respectively maximal) elements of P. If $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$, then $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$.*

Proof. If P' is not simple, then there exists an element $p \in P'$ which is comparable to every element of P'. Observe that p is neither a minimal element nor a maximal element. Let $P_1 = \{q \in P : q < p\}$ and $P_2 = \{q \in P : q > p\}$. Since P_1 and P_2 are not chains, $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$ by Lemma 3.14. So we may assume that P' is simple. On the contrary, suppose that $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$. So the conditions (i) and (ii) of Theorem 3.3 hold for $\mathcal{I}(P')$. Since $\mathcal{I}(P') \subseteq \mathcal{I}(P)$, the conditions (i) and (ii) of Theorem 3.3 also hold for $\mathcal{I}(P)$ which is a contradiction. Hence the proof. $\square$

(a) (b) $n \geq 1$

FIGURE 3.4

Lemma 3.16. *Let P be a poset such that the poset $P' = \{p_1, ..., p_4\}$ of Figure 3.4a is a cover-preserving subposet of P. Then $R[\mathcal{I}(P)]$ does not satisfy property N_2.*

Proof. Observe that by Theorem 3.3, $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$. Let $B = \{p_1, p_2\}$, $B' = \{p_3, p_4\}$. By Discussion 3.7, we may replace P by P_1, where P_1 is as defined in Discussion 3.7, and assume that the sets of minimal and maximal elements of P coincide with the sets of minimal and maximal elements of P' respectively.
Now, suppose that there exists an element $p \in P$ such that $p \notin P'$. Then, we have $p_i < p < p_j$ for some $i \in \{1, 2\}$ and $j \in \{3, 4\}$. This contradicts that $p_i \lessdot p_j$. Therefore, $P = P'$. This completes the proof. $\square$

Discussion 3.17. Let P be a poset. For $k \geq 1$, let $\mathcal{S} = \cup_{i=1}^{k}\{p_{i,1}, \ldots, p_{i,n_i}\}$ be a subset of the underlying set of P. Assume that $\{p_{1,1}, \ldots, p_{k,1}\}$ and $\{p_{1,n_1}, \ldots, p_{k,n_k}\}$ are antichains in P. Also, assume that for all $1 \leq i \leq k$, $\{p_{i,1}, \ldots, p_{i,n_i}\}$ is a chain in P with $p_{i,1} \lessdot \cdots \lessdot p_{i,n_i}$. For $q \in P \setminus \mathcal{S}$, define $\mathcal{S}_q^P := \{p \in \mathcal{S} : q \lessdot p\}$. Let $B = \{p_{1,1}, \ldots, p_{k,1}\}$ and $B' = \{p_{1,n_1}, \ldots, p_{k,n_k}\}$. Using Discussion 3.7, reduce P to P_1, where P_1 is as defined in Discussion 3.7. Let $x, y \in P_1 \setminus \mathcal{S}$ with $x \lessdot y$. Reduce P_1 to P_2, using the methods of Notation 3.8. Observe that $\#P_2 = \#P_1 - 1$, $\mathcal{S} \subset P_2$ and B and B' are the sets of minimal and maximal elements of P_2 respectively. Repeating it, we get a sequence $P_0, P_1, \ldots, P_m$, where $m \leq \#P - \#\mathcal{S}$ of posets such that for each $0 \leq i \leq m-1$, there exist $x, y \in P_i \setminus \mathcal{S}$ with $x \lessdot y$ and P_i is reduced to P_{i+1} as in Notation 3.8. Moreover, there are no $x, y \in P_m \setminus \mathcal{S}$ with the property $x \lessdot y$.

Now, we will do more reductions on P_m. Let $q \in P_m \setminus \mathcal{S}$ be such that $\#\mathcal{S}_q^{P_m} = 1$, say $\mathcal{S}_q^{P_m} = \{p\}$. We have $q \lessdot p$ in P_m. Reduce P_m to P_{m+1}, using the methods of Notation 3.8. Under this reduction, $\mathcal{S} \subset P_2$ and B and B' are the sets of minimal and maximal elements of P_{m+1} respectively. Repeating it, we get a sequence $P_m, P_{m+1}, \ldots, P_s$ of posets such that for each $m \leq i \leq s-1$, there exists a $q \in P_i \setminus \mathcal{S}$ with $\#\mathcal{S}_q^{P_i} = 1$ and P_i is reduced to P_{i+1} as in Notation 3.8 and there is no $q \in P_s \setminus \mathcal{S}$ with $\#\mathcal{S}_q^{P_s} = 1$. If $\beta_{ij}(R[\mathcal{I}(P_l)]) \neq 0$ for some i, j and $l \in \{1, \ldots, s\}$, then by Discussion 3.7, Lemma 3.9 and Proposition 2.27, $\beta_{ij}(R[\mathcal{I}(P)]) \neq 0$. $\square$

Lemma 3.18. *Let P be a poset and let the poset $P' = \{p_1, ..., p_4, q_1, ..., q_n\}$ for some $n \geq 1$, as shown in Figure 3.4b is a cover-preserving subposet of P. Then $R[\mathcal{I}(P)]$ does not satisfy property N_2.*

Proof. Note that by Lemma 3.14, $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$. Let

$$\mathcal{S} = \{p_1, q_1, \ldots, q_n, p_3\} \cup \{p_2, q_1, \ldots, q_n, p_4\}.$$

By Discussion 3.17, it suffices to show that $R[\mathcal{I}(P_m)]$ does not satisfy property N_2, where P_m is as defined in Discussion 3.17. Note that $\{p_1, p_2\}$ and $\{p_3, p_4\}$ are the sets of minimal and maximal elements of P_m respectively. If there exists a cover-preserving subposet of P_m as shown in Figure 3.4a then $\beta_{24}(R[\mathcal{I}(P_m)]) \neq 0$. So we may assume that P_m does not contain any cover-preserving subposet as shown in Figure 3.4a. Let $\mathcal{S}_q$ be as defined in Discussion 3.17. There is no $q \in P_m \setminus \mathcal{S}$ with $\mathcal{S}_q = \{p_3, p_4\}$ otherwise P_m will contain a cover-preserving subposet as shown in Figure 3.4a. So we deduce that $\#\mathcal{S}_q = 1$ for all $q \in P_m \setminus \mathcal{S}$. Now, reduce P_m to P_s as in Discussion 3.17. Then $P_s = P'$. This completes the proof. $\square$

Lemma 3.19. *Let $(P, \leq)$ be a poset. Then $\mathcal{I}(P) \cong \mathcal{I}(P^\partial)$, where P^∂ is the* dual poset *of P, that is, $(P^\partial, \preceq)$ is the poset with the same underlying set but its order relation is the opposite of P i.e. $p \leq q$ if and only if $q \preceq p$. Hence, $R[\mathcal{I}(P)] \cong R[\mathcal{I}(P^\partial)]$.*

Theorem 3.20. *Let P be a poset. Let $\mathcal{S} = \cup_{i=1}^2 \{p_{i,1}, \ldots, p_{i,n_i}\}$ be a subset of the underlying set of P such that*

1. *for all $1 \leq i \leq 2$, $\{p_{i,1}, \ldots, p_{i,n_i}\}$ is a chain in P with $p_{i,1} \lessdot \cdots \lessdot p_{i,n_i}$;*

2. $p_{1,1}$ *and* $p_{2,1}$ *are incomparable in* P*;*

3. p_{1,n_1} *and* p_{2,n_2} *are incomparable in* P*.*

Let P' *be the induced subposet of* P *on the set* $\mathcal{S}$*. If* $R[\mathcal{I}(P')]$ *does not satisfy property* N_2 *then so does* $R[\mathcal{I}(P)]$*.*

Proof. For P, let $P_1, \ldots, P_m, P_{m+1}, \ldots, P_s$ be as defined in Discussion 3.17. For $1 \leq i \leq s$, let P_i' be the induced subposet of P_i on the set $\mathcal{S}$. For $1 \leq i \leq s-1$, every order relation between the elements of $\mathcal{S}$ in P_i is also an order relation in P_{i+1}. Also, $\{p_{1,1}, p_{2,1}\}$ and $\{p_{1,n_1}, p_{2,n_2}\}$ are the sets of minimal and maximal elements of P_i respectively, for all $i = 1, \ldots, s$. Therefore, by Lemma 3.15, $\beta_{24}(R[\mathcal{I}(P_i')]) \neq 0$ for all $1 \leq i \leq s$. By Discussion 3.17, it is enough to show that $R[\mathcal{I}(P_s)]$ does not satisfy property N_2. We may replace P by P_s and P' by P_s'.

Let P^∂ be the dual poset of P. If $q \in P \setminus \mathcal{S}$, then $\#\mathcal{S}_q^P \geq 2$. So if there exists a $q \in P^\partial \setminus \mathcal{S}$ with $\#\mathcal{S}_q^{P^\partial} \geq 2$, then P contains a cover-preserving subposet as shown in Figure 3.4b. Thus, by Lemma 3.18, $R[\mathcal{I}(P)]$ does not satisfy property N_2. So we may assume that for all $p \in P^\partial \setminus \mathcal{S}$, $\#\mathcal{S}_p^{P^\partial} = 1$. Repeating the argument of Discussion 3.17, we obtain a poset Q such that there is no $q \in Q \setminus \mathcal{S}$ with $\#\mathcal{S}_q^Q = 1$. Observe that Q is a poset on the set $\mathcal{S}$. By Discussion 3.17, it suffices to prove that $R[\mathcal{I}(Q)]$ does not satisfy property N_2. Note that Q^∂ is a poset on the set $\mathcal{S}$ and all order relations of P' are also the order relations of Q^∂. So by Lemma 3.15, $R[\mathcal{I}(Q^\partial)]$ does not satisfy property N_2. Thus, by Lemma 3.19, $R[\mathcal{I}(Q)]$ does not satisfy property N_2. Hence the proof. $\square$

Remark 3.21. Note that, in the proof of Theorem 3.20, the reduction from the poset P to the poset Q^∂ is independent of the hypothesis that $\mathcal{I}(P')$ is a planar distributive lattice. In fact, we will also use the reduction from P to Q^∂ in Discussion 3.27 where the distributive lattice is not restricted to be planar. We have only used the fact that $\mathcal{I}(P')$ is a planar distributive lattice to conclude that $\beta_{24}(R[\mathcal{I}(Q^\partial)]) \neq 0$ and $\beta_{24}(R[\mathcal{I}(P_i')]) \neq 0$ for $i = 1, \ldots, s$.

3.3 Property N_2 continued

In this section, we prove a result analogous to Ene's result. Suppose that a poset can be decomposed into a union of three chains and it has three maximal and minimal elements. We prove some necessary conditions regarding when Hibi rings associated to such posets satisfy property N_2.

Lemma 3.22. *Let P be a poset on the disjoint union $\cup_{i=1}^{3}\{p_{i,1}, p_{i,2}, p_{i,3}\}$ such that*

1. *for all $1 \leq i \leq 3$, $\{p_{i,1}, p_{i,2}, p_{i,3}\}$ is a chain in P with $p_{i,1} \lessdot p_{i,2} \lessdot p_{i,3}$;*
2. *$\{p_{1,1}, p_{2,1}, p_{3,1}\}$ and $\{p_{1,3}, p_{2,3}, p_{3,3}\}$ are the sets of minimal and maximal elements of P respectively.*

If there exists an element in P such that either it cover three elements or it is covered by three elements and P is not as shown in figure 3.5a, then $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$.

Proof. By Theorem 3.20, we may assume that there is no subposet P' of P, as defined in Theorem 3.20, with $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$. Let $p \in P$ be the element such that either it cover three elements or it is covered by three elements. Then, p is either a maximal element or a minimal element or $p \in \{p_{1,2}, p_{2,2}, p_{3,2}\}$. If p is a maximal element of P, then in P^{∂}, p is a minimal element and it is covered by three elements. In this case by Lemma 3.19, replace P by P^{∂} and we may assume that p is a minimal element of P. So we only have to consider the cases when either p is a minimal element or $p \in \{p_{1,2}, p_{2,2}, p_{3,2}\}$. In most of the subcases of these two cases, we will take $\delta, \gamma \in \mathcal{I}(P)$ and we will show that for the sublattice $L' := \{\alpha \in \mathcal{I}(P) : \delta \leq \alpha \leq \gamma\}$, $\beta_{24}(R[L']) \neq 0$. Hence, by Proposition 3.5 and Proposition 2.27, we conclude that $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$.

Case 1 Assume that p is a minimal element of P. Possibly by relabelling the elements of P, we may assume that $p = p_{1,1}$. We will prove this case in two subcases.

Subcase (a) Consider the subcase when $p_{1,1}$ is covered by $\{p_{1,2}, p_{2,2}, p_{3,2}\}$. Observe that $p_{3,3}$ can not cover $p_{2,2}$ otherwise P will contain a cover-preserving subposet as shown in Figure 3.4b; thus, $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$ by Lemma 3.18. We prove this subcase in two following subsubcases:

(i) Assume that $p_{3,3}$ is covering $p_{1,2}$ and $p_{3,2}$ only. Observe that $\delta = \emptyset$ and $\gamma = P \setminus \{p_{2,3}\}$ are the order ideals of P. By Proposition 3.6, $L' \cong \mathcal{I}(P')$, where P' is the poset as shown in Figure 3.5d. One can use a computer to check that $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$.

(ii) Now, assume that either $p_{3,3}$ is covering $p_{3,2}$ only or $p_{3,3}$ is covering at least $p_{2,1}$ and $p_{3,2}$. Let $\delta = \emptyset$ and $\gamma = P \setminus \{p_{1,3}, p_{2,3}\}$. By Proposition 3.6, $L' \cong \mathcal{I}(P')$, where P' is one of the posets as shown in Figure 3.5e-3.5g. Again, it can be checked by a computer that $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$.

Subcase (b) Consider the subcase when $p_{1,1}$ is not covered by $\{p_{1,2}, p_{2,2}, p_{3,2}\}$. So $p_{1,1}$ is either covered by $\{p_{1,2}, p_{2,3}, p_{3,2}\}$ or $\{p_{1,2}, p_{2,2}, p_{3,3}\}$ or $\{p_{1,2}, p_{2,3}, p_{3,3}\}$. By symmetry, it is enough to consider one of the cases from $\{p_{1,2}, p_{2,3}, p_{3,2}\}$ and $\{p_{1,2}, p_{2,2}, p_{3,3}\}$.

First, consider the subsubcase when $p_{1,1}$ is covered by $\{p_{1,2}, p_{2,3}, p_{3,2}\}$. We have $p_{2,1} \lessdot p_{2,2}$, reduce P to P_1 using the methods of Discussion 3.8. If $p_{2,1}$ is covered by $p_{1,2}$ or $p_{3,2}$ in P, then P_1 will contain a cover-preserving subposet as shown in Figure 3.4a. So we may assume that $p_{2,1}$ is not covered by $p_{1,2}$ and $p_{3,2}$. Observe that P_1 is a poset on the underlying set $P \setminus \{p_{2,1}\}$. Also, $\{p_{1,1}, p_{2,2}, p_{3,1}\}$ and $\{p_{1,3}, p_{2,3}, p_{3,3}\}$ are the sets of minimal and maximal elements of P_1 respectively. Also, $p_{1,1}$ is covered by $\{p_{1,2}, p_{2,3}, p_{3,2}\}$ in P_1. Repeating the argument of the subcase (a), we deduce that the result holds in this subsubcase.

Now, we consider the subsubcase when $p_{1,1}$ is covered by $\{p_{1,2}, p_{2,3}, p_{3,3}\}$. We have $p_{2,1} \lessdot p_{2,2}$, reduce P to P_1 using the methods of Discussion 3.8. If $p_{2,1}$ is covered by $p_{1,2}$ in P, then P_1 will contain a cover-preserving subposet as shown in Figure 3.4a. So we may assume that $p_{2,1}$ is not covered by $p_{1,2}$ in P. Similarly, we may assume that $p_{3,1}$ is not covered by $p_{1,2}$. If either $p_{2,2}$ or $p_{3,2}$ is covered by $p_{1,3}$, then P will contain a subposet P', as defined in Theorem 3.20, with $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$. If either $p_{2,2}$ is covered by $p_{3,3}$ or $p_{3,2}$ is covered by $p_{2,3}$, then we are done by the previous subsubcase. Since P is not as shown in figure 3.5a, the only possibility for P is that P is isomorphic to one of the posets as shown in Figure 3.5h-3.5i. One can use a computer to check that $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$.

Case 2 Assume that $p \in \{p_{1,2}, p_{2,2}, p_{3,2}\}$. Possibly by replacing P with P^∂, we may assume that p is covering all the minimal elements. Possibly by relabelling the elements of P, we may assume that $p = p_{1,2}$. If $p_{1,2}, p_{2,2}$ and $p_{3,2}$ are covered by $p_{1,3}$, then we are done by Case 1. So we may assume that not all elements of $\{p_{1,2}, p_{2,2}, p_{3,2}\}$ are covered by $p_{1,3}$. Let $\delta = \emptyset$ and $\gamma = P \setminus \{p_{2,3}, p_{3,3}\}$. By Proposition 3.6, $L' \cong \mathcal{I}(P')$, where P' is one of the posets as shown in Figure 3.5b-3.5c. Again, one can use a computer to check that $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$.

$\square$

We now give a SageMath [sage] code to compute the β_{24} of a Hibi ring. The code uses the SageMath interface to the Macaulay2 [M2]. One requires to give the cover relation of the poset as an input in the code, then the code returns the Betti table of the Hibi ring up to column 3 as an output.

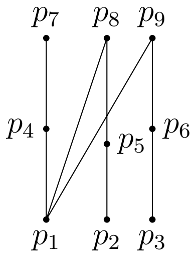
```
def mnml(j):
    return '*'.join(['y0'] + ['y' + str(i) for i in j])

CR = { 1: [4],2:[],3:[]} #Cover relations of the poset
P= Poset(CR)

J = P.order_ideals_lattice();
l=[mnml(j) for j in J]

N = P.cardinality()+1

X = "["+",".join([ "x"+str(i) for i in range(0,len(J))])+"]"
Y = "["+",".join([ "y"+str(i) for i in range(0,N)])+"]"
R = macaulay2.ring('QQ', X, 'GRevLex');
S = macaulay2.ring('QQ',Y, 'GRevLex');

f = macaulay2.map(S,R,l);
I = macaulay2.ker(f) #The Hibi ideal associated to the poset P.
h = macaulay2.res(I,"LengthLimit=>2")
n = macaulay2.betti(h)
```

Listing 3.1: Sagemath code to compute the Betti number of a Hibi ring

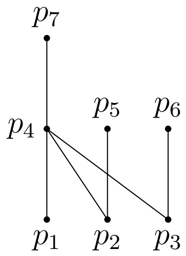
Lemma 3.23. *Let P be as defined in Lemma 3.22. If the induced subposet of P, defined on the underlying set $P \setminus \{p_{1,1}, p_{2,1}, p_{3,1}\}$ or $P \setminus \{p_{1,3}, p_{2,3}, p_{3,3}\}$, is connected. Then $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$.*

Proof. By Theorem 3.20, we may assume that there is no subposet P' of P, as defined in Theorem 3.20, with $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$. Possibly by replacing P with P^{∂}, we may assume that the subposet P' of P defined on the underlying set $P \setminus \{p_{1,3}, p_{2,3}, p_{3,3}\}$ is connected. Observe that P' is isomorphic to one of the posets as shown in Figure 3.6a-3.6b. If P' is as shown in Figure 3.6b, then we are done by Lemma 3.22.

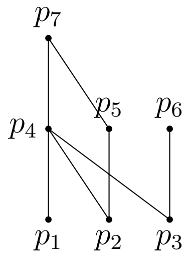
Now, consider the case when P' is as shown in Figure 3.6a. Possibly by relabelling the elements of P, we may assume that $p_{1,2}$ is covering exactly one minimal element of P. If either $p_{1,1} \lessdot p_{3,3}$ or $p_{3,1} \lessdot p_{2,3}$ or $p_{2,2} \lessdot p_{3,3}$ or $p_{3,2} \lessdot p_{2,3}$, then there exists a subposet P' of P, as defined in Theorem 3.20, with $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$. Let $\delta = \emptyset$ and $\gamma = P \setminus \{p_{1,2}, p_{1,3}\}$. Let also $L' = \{\alpha \in \mathcal{I}(P) : \delta \leq \alpha \leq \gamma\}$. By Proposition 3.6, $L' \cong \mathcal{I}(P_1)$, where P_1 is as shown in Figure 3.6c. One can use a computer to check that $\beta_{24}(R[\mathcal{I}(P_1)]) \neq 0$. Hence, by Proposition 3.5 and Proposition 2.27, $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$. $\square$



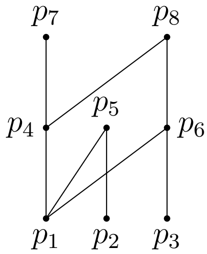
(a)



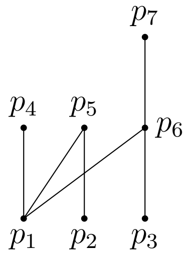
(b) $\beta_{24} = 8$



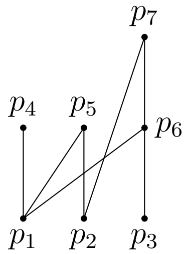
(c) $\beta_{24} = 42$



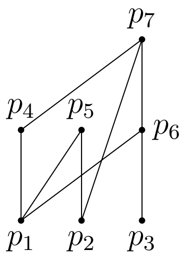
(d) $\beta_{24} = 1$



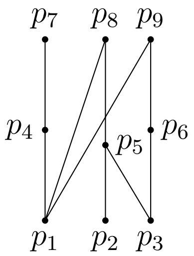
(e) $\beta_{24} = 1$



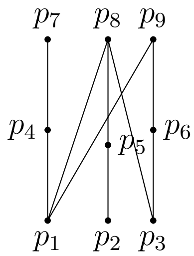
(f) $\beta_{24} = 42$



(g) $\beta_{24} = 28$

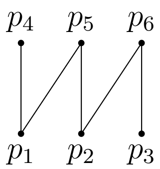


(h) $\beta_{24} = 299$

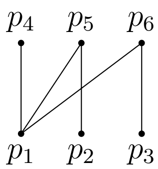


(i) $\beta_{24} = 327$

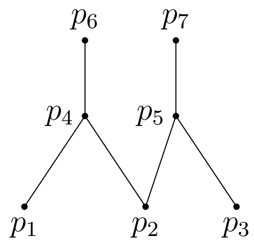
FIGURE 3.5



(a)



(b)

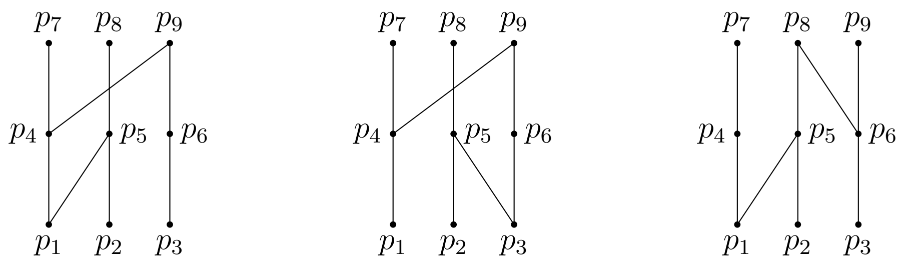


(c) $\beta_{24} = 1$

FIGURE 3.6

Lemma 3.24. *Let P be as defined in Lemma 3.22. If P is pure and connected, then* $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$.

Proof. By Theorem 3.20, we may assume that there is no subposet P' of P, as defined in Theorem 3.20, with $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$. By Lemma 3.22, we may assume that there is no element in P such that either it cover three elements or it is covered by three elements. By Lemma 3.23, we may assume that the subposets of P defined on the



(a) $\beta_{24} = 3$ (b) $\beta_{24} = 1$ (c) $\beta_{24} = 3$

FIGURE 3.7

underlying sets $P \setminus \{p_{1,1}, p_{2,1}, p_{3,1}\}$ and $P \setminus \{p_{1,3}, p_{2,3}, p_{3,3}\}$ are not connected. Then P is isomorphic to one of the posets as shown in Figure 3.7. One can use a computer to check that $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$. This concludes the proof. $\square$

Now we prove the main theorem of this section.

Theorem 3.25. *Let P be a poset on the set $\cup_{i=1}^{3}\{p_{i,1}, ..., p_{i,n_i}\}$ such that*

1. *$p_{1,1}, p_{2,1}, p_{3,1}$ are distinct and $p_{1,n_1}, p_{2,n_2}, p_{3,n_3}$ are distinct;*
2. *$\{p_{1,1}, p_{2,1}, p_{3,1}\}$ and $\{p_{1,n_1}, p_{2,n_2}, p_{3,n_3}\}$ are the sets of minimal and maximal elements of P respectively;*
3. *for all $1 \leq i \leq 3$, $n_i \geq 3$; $\{p_{i,1}, \ldots, p_{i,n_i}\}$ is a chain in P with $p_{i,1} \lessdot \cdots \lessdot p_{i,n_i}$.*

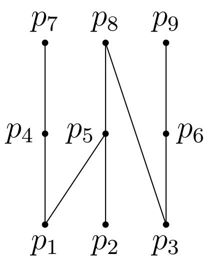
If P is connected and none of the minimal elements of P is covered by a maximal element then $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$.

Proof. Reduce P to P_n, where P_n is as defined in Discussion 3.10. Since P is connected then so is P_n. Since none of the minimal elements of P is covered by a maximal element, we obtain that P_n is pure. So by Discussion 3.10, we may replace P by P_n and assume that P is pure and $n_i = 3$ for all $1 \leq i \leq 3$. Let X' be as defined in Notation 3.8. We will prove the result in the following cases:

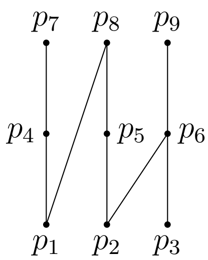
(1) If $\#X' = 1$, then the result follows from Lemma 3.14.

(2) If $\#X' = 2$, then P will contain a cover-preserving subposet as shown in Figure 3.4b. Hence, the result follows from Lemma 3.18.

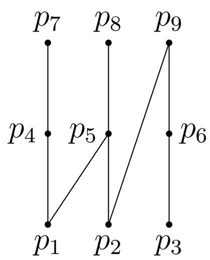
(3) If $\#X' = 3$, then the result follows from Lemma 3.24.



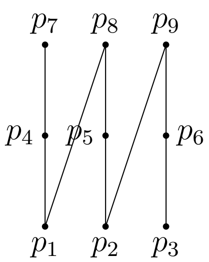
(a) $\beta_{24} = 0$



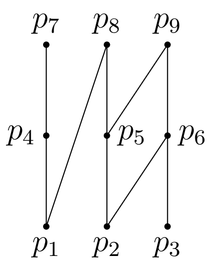
(b) $\beta_{24} = 0$



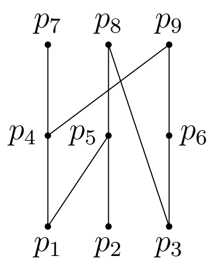
(c) $\beta_{24} = 0$



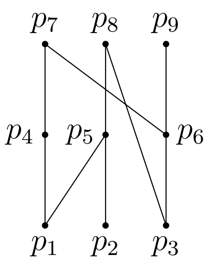
(d) $\beta_{24} = 0$



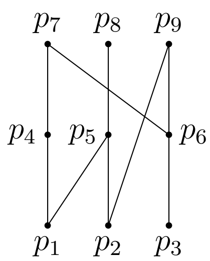
(e) $\beta_{24} = 0$



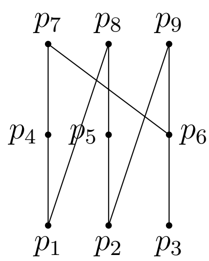
(f) $\beta_{24} = 285$



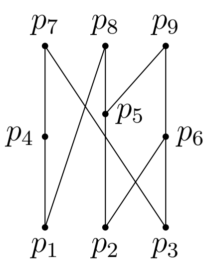
(g) $\beta_{24} = 283$



(h) $\beta_{24} = 246$



(i) $\beta_{24} = 180$



(j) $\beta_{24} = 237$

Figure 3.8

□

Discussion 3.26. Here we answer the following question: what happens if we weaken the hypothesis of Theorem 3.25? Let P be a poset as defined in Theorem 3.25. The case when P is disconnected will be discussed in Corollary 4.15. On the other hand, suppose that P is connected and there exists a minimal element of P which is covered by a maximal element. Using the proof of Theorem 3.25, we may replace the poset P by P_n and assume that $n_i = 3$ for all $1 \leq i \leq 3$. Let X' be as defined in Notation 3.8. Observe that $\#X' \in \{2, 3\}$. If $\#X' = 2$, then we are done by the argument used in the proof of Theorem 3.25.

Now, consider the case when $\#X' = 3$. We know that if P is as shown in figure 3.5a, then $\beta_{24}(R[\mathcal{I}(P)]) = 0$. So we may assume that P is not as shown in figure 3.5a. By Theorem 3.20, we may assume that there is no subposet P' of P, as defined in Theorem 3.20, with $\beta_{24}(R[\mathcal{I}(P')]) \neq 0$. By Lemma 3.22, we may assume that there is no element in P such that either it cover three elements or it is covered by three elements. By Lemma 3.23, we may assume that the subposets of P defined on the underlying sets $P \setminus \{p_{1,1}, p_{2,1}, p_{3,1}\}$ and $P \setminus \{p_{1,3}, p_{2,3}, p_{3,3}\}$ are not connected. Then P is isomorphic to one of the posets as shown in Figure 3.8. One can use a computer to check that if P is isomorphic to one of the posets as shown in Figure 3.8a-3.8e, then $\beta_{24}(R[\mathcal{I}(P)]) = 0$ otherwise $\beta_{24}(R[\mathcal{I}(P)]) \neq 0$. □

Remark 3.27. Let P be a poset. Let $\mathcal{S} = \cup_{i=1}^{3}\{p_{i,1}, \ldots, p_{i,n_i}\}$ be a subset of the underlying set of P such that

1. $p_{1,1}, p_{2,1}, p_{3,1}$ are distinct and $p_{1,n_1}, p_{2,n_2}, p_{3,n_3}$ are distinct;
2. $B := \{p_{1,1}, p_{2,1}, p_{3,1}\}$ and $B' := \{p_{1,n_1}, p_{2,n_2}, p_{3,n_3}\}$ are antichains in P;
3. for all $1 \leq i \leq 3$, $n_i \geq 3$; $\{p_{i,1}, \ldots, p_{i,n_i}\}$ is a chain in P with $p_{i,1} \lessdot \cdots \lessdot p_{i,n_i}$.

Using Discussion 3.17 and the arguments of the proof of Theorem 3.20, we can reduce P to the poset Q^∂, where Q^∂ is a poset on the underlying set $\mathcal{S}$. B and B' are the sets of minimal and maximal elements of Q^∂ respectively. To prove that $R[\mathcal{I}(P)]$ does not satisfy property N_2, it is enough to show that $R[\mathcal{I}(Q^\partial)]$ does not satisfy property N_2 which can be easily checked using Theorem 3.25 and Discussion 3.26. □

3.4 Property N_p of Hibi rings for $p \geq 3$

In this section, we study the property N_p of Hibi rings for $p \geq 3$.

Lemma 3.28. *Let $P_{n,m}$, where $n, m \geq 2$, be the poset as shown in Figure 3.9. Then $R[\mathcal{I}(P_{n,m})]$ does not satisfy N_3.*

Proof. Observe that $R[\mathcal{I}(P_{n,m})]$ satisfies property N_2, by Theorem 3.3. Let $x = p_2$ and $y = p_n$. Reduce $P_{n,m}$ to $P_{2,m}$ using the methods of Notation 3.8. Now in $P_{2,m}$, let $x = q_2$ and $y = q_m$. Reduce $P_{2,m}$ to $P_{2,2}$ using the method discussed in Notation 3.8. For $n, m = 2$, one can use a computer to check that $\beta_{35}(R[\mathcal{I}(P_{n,m})]) \neq 0$. By Lemma 3.9 and Proposition 2.27, we have $\beta_{35}(R[\mathcal{I}(P_{n,m})]) \neq 0$. This completes the proof.

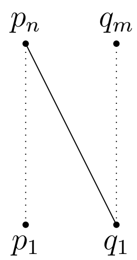


FIGURE 3.9: $P_{n,m}$; $n, m \geq 2$

□

Lemma 3.29. *Let P be a poset such that $\mathcal{I}(P)$ is a planar distributive lattice. Assume that P has two minimal and two maximal elements. If $R([\mathcal{I}(P)])$ satisfies property N_3, then P is a disjoint union of two chains.*

Proof. Suppose that $R([\mathcal{I}(P)])$ satisfies property N_3. Then, it also satisfies property N_2. So P is simple otherwise there exists an element $p \in P$ which is comparable to every element of P. By hypothesis, p is neither a minimal element nor a maximal element. Let $P_1 = \{q \in P : q < p\}$ and $P_2 = \{q \in P : q > p\}$. Since P_1 and P_2 are not chains, $R([\mathcal{I}(P)])$ does not satisfy property N_2 by Lemma 3.14, which is a contradiction. By Corollary 3.4, P is isomorphic to one of the posets as shown in Figure 3.1. If P is not isomorphic to the poset shown in Figure 3.1a, then it will contain a cover-preserving subposet as shown in Figure 3.9, call it P'. Let B and B' be the sets of minimal and maximal elements of P' respectively. Hence, by Discussion 3.7 and Lemma 3.28, $\beta_{35}(R[\mathcal{I}(P)]) \neq 0$. This concludes the proof. □

Now we prove our main theorem about property N_3 of Hibi rings associated to connected posets.

Theorem 3.30. *Let P be a connected poset. Assume that P has at least two minimal and maximal elements. Then $R[\mathcal{I}(P)]$ does not satisfy property N_3.*

Proof. Claim : There exist two maximal chains $C_1 = \{p_1, \dots, p_r\}$ and $C_2 = \{q_1, \dots, q_s\}$ of P such that $p_1 \lessdot \cdots \lessdot p_r$, $q_1 \lessdot \cdots \lessdot q_s$, $p_1 \neq q_1$, $p_r \neq q_s$ and $r, s \geq 2$.

Assume the claim. Let $\mathcal{S} = C_1 \cup C_2$. Using Discussion 3.17 and the proof of Theorem 3.20, we can reduce P to the poset Q^∂, where Q^∂ is a poset on the underlying set $\mathcal{S}$ and it is enough to show that $R[\mathcal{I}(Q^\partial)]$ does not satisfy property N_3. Observe that Q^∂ is connected, $\{p_1, q_1\}$ and $\{p_r, q_s\}$ are the sets of minimal and maximal elements of Q^∂ respectively. By Lemma 3.29, $R[\mathcal{I}(Q^\partial)]$ does not satisfy property N_3. This completes the proof.

Now we prove the claim. Let C be a maximal chain in P with the minimal element p and maximal element q. Fix a maximal element $q' \in P$ where $q' \neq q$. If there exists a maximal chain C' with the maximal element q' and the minimal element not equal to p, then we are done. So we may assume that all maximal chains with the maximal element q' have minimal element p. Fix a minimal element $p' \in P$ where $p' \neq p$. If there exists a maximal chain C'' with the minimal element p' and maximal element not equal to q, then we are done. So we may assume that all maximal chains with the minimal element p' have maximal element q. Then, we can take C_1 to be a maximal chain from p to q' and C_2 to be a maximal chain from p' to q. Hence the proof. $\square$

Recall the notion of graphs from Section 2.4. The following lemma will be needed in the proof of our main theorem about property N_p for $p \geq 4$.

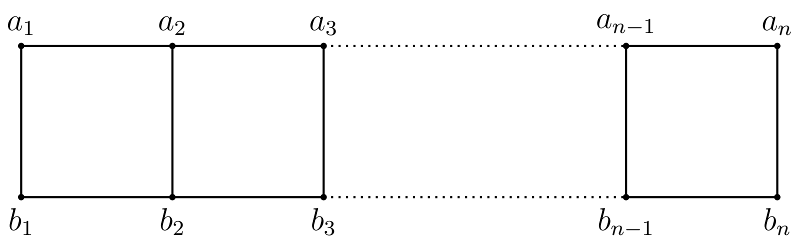


FIGURE 3.10: L

Lemma 3.31. *Let L be a distributive lattice as shown in Figure 3.10. Then the comparability graph G_L of L is chordal.*

Proof. First break the underlying set of L in two disjoint subsets $A_1 = \{a_1, \dots, a_n\}$ and $A_2 = \{b_1, \dots, b_n\}$ (see Figure 3.10 for notational conventions). Let $C = (c_1, \dots, c_r)$ be a induced cycle of G_L of length ≥ 4. If $\{c_1, \dots, c_r\} \cap A_i \geq 3$ for any $i \in \{1, 2\}$, then C has a chord because every pair in A_i is an edge of G_L. So we may assume that $r = 4$ and $\#(\{c_1, \dots, c_r\} \cap A_i) = 2$ for all i. Let $\{c_{i_1}, c_{i_2}\} \subseteq A_1$ and $\{c_{i_3}, c_{i_4}\} \subseteq A_2$. Without loss of generality, we may assume that $c_1 = c_{i_1}$ and $c_1 < c_{i_2}$ in L. Let $c \in \{c_{i_3}, c_{i_4}\}$ be such that $\{c_1, c\}$ is an edge in C. Therefore, c_1 and c are comparable in L; therefore $c < c_1$ because $c_1 \in A_1$ and $c \in A_2$. Therefore $c < c_{i_2}$. Hence (c_1, c, c_{i_2}) is a induced chain in G_L. Thus C has a chord. This completes the proof. $\square$

Example 3.32. Let P_1 be an antichain of cardinality three and P_2 be a poset such that it is a disjoint union of two chains of length 1. By [Hib87, § 3, Corollary], $R[\mathcal{I}(P_i)]$ is a Gorenstein ring for all $i = 1, 2$. For all $i = 1, 2$, the Hibi ring $R[\mathcal{I}(P_i)]$ is Cohen-Macaulay, it is a quotient of a polynomial ring in $\#\mathcal{I}(P_i)$ variables and the Krull-dimension of $R[\mathcal{I}(P_i)]$ is $\#P_i + 1$. So the Auslander-Buchsbaum formula implies that $\operatorname{proj}\dim(R[\mathcal{I}(P_i)]) = \#\mathcal{I}(P_i) - \#P_i - 1$ for $i = 1, 2$. It is easy to see that $\operatorname{proj}\dim(R[\mathcal{I}(P_i)]) = 4$ for all $i = 1, 2$. By self-duality of minimal free resolution of Gorenstein rings, we obtain that $\beta_{4j}(R[\mathcal{I}(P_i)]) \neq 0$ for some $j \geq 6$ and for all $i = 1, 2$ irrespective of the characteristic of the field K.

We are now ready to prove our main theorem about property N_p for $p \geq 4$.

Theorem 3.33. *Let P be a poset and $p \geq 4$. Let $P' = \{p_{i_1}, ..., p_{i_r}\}$ be the subset of all elements of P which are comparable to every element of P. Let P'' be the induced subposet of P on the set $P \setminus P'$. Then the following are equivalent:*

1. *$R[\mathcal{I}(P)]$ satisfies property N_p;*
2. *$R[\mathcal{I}(P)]$ satisfies property N_4;*
3. *Either P is a chain or P'' is a disjoint union of a chain and an isolated element;*
4. *Either $R[\mathcal{I}(P)]$ is a polynomial ring or $K[\mathcal{I}(P'')]/\operatorname{in}_<(I_{\mathcal{I}(P'')})$ has a linear resolution;*
5. *Either $R[\mathcal{I}(P)]$ is a polynomial ring or it has a linear resolution*

Before going to the proof of the theorem, we remark that not all of the equivalent statements are new. For example, (3) $\iff$ (5) was proved in [EQR13, Corollary 10] and (5) $\Rightarrow$ (4) follows from [CV20, Corollary 2.7].

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) If $\text{width}(P) \geq 3$, then there exists an antichain P_1 in P of cardinality three. By Discussion 3.7, $\beta_{ij}(R[\mathcal{I}(P_1)]) \leq \beta_{ij}(R[\mathcal{I}(P)])$ for all i and j. Since $\beta_{4j}(R[\mathcal{I}(P_1)]) \neq 0$ for some $j \geq 6$ by Example 3.32, $\beta_{4j}(R[\mathcal{I}(P)]) \neq 0$. Thus, $R[\mathcal{I}(P)]$ does not satisfy property N_4. So we may assume that $\text{width}(P) \leq 2$. If $\text{width}(P) = 1$, then P is a chain. We now consider $\text{width}(P) = 2$. Observe that P'' is simple. Since $R[\mathcal{I}(P'')]$ satisfies property N_4, it also satisfies property N_3. By Lemma 3.29, P'' is a disjoint union of two chains. Suppose that P'' is a poset on the set $\cup_{i=1}^{2}\{p_{i,1}, \dots, p_{i,n_i}\}$ such that $\{p_{i,1}, \dots, p_{i,n_i}\}$ is a chain in P'' with $p_{i,1} \lessdot \cdots \lessdot p_{i,n_i}$ for all $i = 1, 2$. We have to show that either $n_1 = 1$ or $n_2 = 1$. On the contrary, suppose that $n_i \geq 2$ for all $i = 1, 2$. Let P_2 be the induced subposet of P'' on the set $\cup_{i=1}^{2}\{p_{i,1}, p_{i,2}\}$. Let B and B' be the sets of minimal and maximal elements of P_2 respectively. By Example 3.32 and Discussion 3.7, $\beta_{4j}(R[\mathcal{I}(P)]) \neq 0$ for some $j \geq 6$ which is a contradiction. Hence the proof.

(3) $\Rightarrow$ (4) If P is a chain, then $R[\mathcal{I}(P)]$ is a polynomial ring. Observe that the distributive lattice $\mathcal{I}(P'')$ is as shown in Figure 3.10. The ideal $\text{in}_<(I_{\mathcal{I}(P'')})$ is the Stanley-Reisner ideal of the order complex $\Delta(\mathcal{I}(P''))$ of $I_{\mathcal{I}(P'')}$ (see Section 6.1). It was observed in Subsection 2.4 that $\Delta(\mathcal{I}(P'')) = \Delta(G_{\mathcal{I}(P'')})$ where $G_{\mathcal{I}(P'')}$ is the comparability graph of $\mathcal{I}(P'')$. Now the result follows from Lemma 3.31 and [Frö90, Theorem 1].

(4) $\Rightarrow$ (5) Since the Betti numbers of $K[\mathcal{I}(P'')]/\text{in}_<(I_{\mathcal{I}(P'')})$ over the ring $K[\mathcal{I}(P'')]$ are greater than equal to those of $R[\mathcal{I}(P'')]$ [Pee11, Theorem 22.9], we get that $R[\mathcal{I}(P'')]$ has a linear resolution. Thus, $R[\mathcal{I}(P)]$ has a linear resolution by Corollary 2.21.

(5) $\Rightarrow$ (1) is immediate. $\square$

Hibi and Ohsugi [HO17] characterized chordal comparability graph of posets using toric ideals associated with multichains of poset. We now use Theorem 3.33 and [Frö90, Theorem 1] to characterize comparability graph of distributive lattices which are chordal. It is immediate that for a chain P of length n, G_P is the complete graph on the set $[n+1]$ which is chordal.

Corollary 3.34. *Let $L = \mathcal{I}(P)$ be a distributive lattice and G_L be the comparability graph of L. For P, let P'' be as defined in Theorem 3.33. Then G_L is chordal if and only if P is a chain or P'' is a disjoint union of a chain and an isolated element.*

Chapter 4

Property N_p for Segre product of Hibi rings

In this chapter, we discuss the property N_p of Segre product of Hibi rings for $p \in \{2, 3\}$.

The Segre product of two Hibi rings is a Hibi ring and it was observed in [HHR00]. We sketch a proof here.

Proposition 4.1. *Let P_1 and P_2 be two posets and P be their disjoint union. Then $R[\mathcal{I}(P)] \cong R[\mathcal{I}(P_1)] * R[\mathcal{I}(P_2)]$, where $*$ denotes the Segre product.*

Proof. The idea of the proof is same as a proof in Section 2.8. We show that $R[\mathcal{I}(P_1)] * R[\mathcal{I}(P_2)]$ is a ASL on $\mathcal{I}(P)$ over K with same straightening relations as $R[\mathcal{I}(P)]$. Let

$$R[\mathcal{I}(P_1)] = K[\{u_\alpha = t_1 \prod_{p_i \in \alpha} y_i : \alpha \in \mathcal{I}(P_1)\}] \subseteq K[t_1, \{y_i : p_i \in P_1\}]$$

and

$$R[\mathcal{I}(P_2)] = K[\{v_\beta = t_2 \prod_{q_i \in \beta} z_i : \beta \in \mathcal{I}(P_2)\}] \subseteq K[t_2, \{z_i : q_i \in P_2\}].$$

Then,

$$R[\mathcal{I}(P_1)] * R[\mathcal{I}(P_2)] = K[\{u_\alpha v_\beta : \alpha \in \mathcal{I}(P_1), \beta \in \mathcal{I}(P_2)\}].$$

Let $\varphi : \mathcal{I}(P) \to R[\mathcal{I}(P_1)] * R[\mathcal{I}(P_2)]$ be defined by $(\alpha, \beta) \mapsto u_\alpha v_\beta$. Note that for all $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{I}(P)$,

$$\varphi((\alpha_1, \beta_1))\varphi((\alpha_2, \beta_2)) = \varphi((\alpha_1, \beta_1) \vee (\alpha_2, \beta_2))\varphi((\alpha_1, \beta_1) \wedge (\alpha_2, \beta_2)).$$

ASL-2 follows from the above equation. For ASL-1, it suffices to show that the standard monomials are distinct because they are monomials of the polynomial ring $T = K[t_1, t_2, \{y_i : p_i \in P_1\}, \{z_i : q_i \in P_2\}]$. The proof of this is similar to the proof in Section 2.8. $\square$

From Section 2.8, recall the definition of the semigroup associated to a Hibi ring. For $i \in \{1, 2\}$, let H_i be the affine semigroup generated by $\{h_\alpha : \alpha \in \mathcal{I}(P_i)\}$ and let H be the affine semigroup associated to the Hibi ring $R[\mathcal{I}(P)]$. Since $\mathcal{I}(P) = \{(\alpha, \beta) : \alpha \in \mathcal{I}(P_1) \text{ and } \beta \in \mathcal{I}(P_2)\}$, it is easy to see that, up to isomorphism, H is generated by $\{(h_\alpha, h_\beta) : \alpha \in \mathcal{I}(P_1) \text{ and } \beta \in \mathcal{I}(P_2)\}$.

Theorem 4.2. *Let P_1, P_2, P, H_1, H_2 and H be as above. Then, for each $l \in \{1, 2\}$, H_l is isomorphic to a homologically pure subsemigroup of H. In particular, if $\beta_{ij}(R[\mathcal{I}(P_l)]) \neq 0$ for some $l \in \{1, 2\}$, then $\beta_{ij}(R[\mathcal{I}(P)]) \neq 0$.*

Proof. By symmetry, it suffices to prove the theorem for $l = 1$. Consider the subsemigroup G_1 of H generated by $\{(h_\alpha, h_\emptyset) : \alpha \in \mathcal{I}(P_1)\}$, where $\emptyset$ is the minimal element of $\mathcal{I}(P_2)$. It is easy to see that G_1 is isomorphic to the semigroup H_1. Also, observe that $\delta = (\emptyset, \emptyset)$ and $\gamma = (P_1, \emptyset)$ are the order ideals of H. The subsemigroup G_1 is generated by $\{h_\eta : \delta \leq \eta \leq \gamma\}$. So by Proposition 3.5, G_1 is a homologically pure subsemigroup of H. The second part of the theorem follows from Proposition 2.27. Hence the proof. $\square$

Corollary 4.3. *Let P be a poset such that it is a disjoint union of two posets P_1 and P_2. If $R[\mathcal{I}(P)]$ satisfies property N_p for some $p \geq 2$, then so do $R[\mathcal{I}(P_1)]$ and $R[\mathcal{I}(P_2)]$.*

Proof. The proof follows from Theorem 4.2. $\square$

Lemma 4.4. *Let $R[\mathcal{I}(P)]$ be a Hibi ring associated to a poset P. Then the following statements hold:*
(a) If $\beta_{24}(R[\mathcal{I}(P)]) = 0$, then $R[\mathcal{I}(P)]$ satisfies property N_2.
(b) If $R[\mathcal{I}(P)]$ satisfies property N_2 and $\beta_{35}(R[\mathcal{I}(P)]) = 0$, then it satisfies property N_3.

Proof. (*a*) Since Hibi rings have a quadratic Gröbner basis (see Theorem 2.15), Hibi rings are Koszul. So by Proposition 2.3, $\beta_{2j}(R[\mathcal{I}(P)]) = 0$ for all $j \geq 5$. This concludes the proof.

(*b*) The proof follows from [ACI15, Theorem 6.1]. $\square$

4.1 Segre product with a polynomial ring in two variables

We now wish to study the property N_p of Segre product of Hibi ring and a polynomial ring. The main result of this section, whose proof is postponed to the end of the section, is the following:

Theorem 4.5. *Let P_1 be a poset, $P_2 = \{b\}$ and $p \in \{2, 3\}$. Let P be the disjoint union of P_1 and P_2. If $R[\mathcal{I}(P_1)]$ satisfies property N_p , then so does $R[\mathcal{I}(P)]$.*

The proof of the above theorem follows the argument of Rubei [Rub02]. Let P_1 and P_2 be as in theorem. So $\mathcal{I}(P) = \{(\alpha, \beta) : \alpha \in \mathcal{I}(P_1),\ \beta \in \mathcal{I}(P_2)\}$. Let H be the affine semigroup generated by $\{(h_\alpha, h_\beta) : \alpha \in \mathcal{I}(P_1),\ \beta \in \mathcal{I}(P_2)\}$. In order to prove the above theorem, by Proposition 2.24 and Lemma 4.4, it is enough to show that for $p \in \{2, 3\}$, if $h = (h_1, h_2) \in H$ with $\deg(h) = p + 2$, then $\widetilde{H}_{p-1}(\Delta_h) = 0$. For $i = 1, 2$, let H_i be the affine semigroup generated by $\{h_\alpha : \alpha \in \mathcal{I}(P_i)\}$. Observe that H_2 is generated by two elements $h_\emptyset$ and $h_{\{b\}}$. For simplicity, we denote $h_{\{b\}}$ by h_b.

Before going to the technical details, we refer the reader to Section 2.3 for definitions and notations.

Notation 4.6. Let $g \in H_1$ with $\deg(g) = d$.
(a) Denote $g_\varepsilon = (g, g')$, where $g' = (d - \varepsilon)h_\emptyset + \varepsilon h_b \in H_2$ and $\varepsilon \in \{0, \dots, d\}$.
(b) For $0 \le l \le d - 1$, let

$$F^l(\Delta_g) = \bigcup_{\substack{g_1,\dots,g_d \text{ s.t.} \\ g_1+\dots+g_d = g}} \ \bigcup_{i_0,\dots,i_l \in \{1,\dots,d\}} \big\langle (g_{i_0}, h_\emptyset), \dots, (g_{i_l}, h_\emptyset) \big\rangle.$$

Lemma 4.7. *Under the notations of Notation 4.6.*
(a) For all $i \le l - 1$, $\widetilde{H}_i(F^l(\Delta_g)) \cong \widetilde{H}_i(\Delta_g)$.
(b) For $\varepsilon \in \{1, 2\}$, $F^l(\Delta_g) \subseteq \Delta_{g_\varepsilon}$ if and only if $l \le d - \varepsilon - 1$.

Proof. (a) The proof follows from $F^l(\Delta_g) \cong sk^l(\Delta_g)$.
(b) Let $g_1, \dots, g_d \in H_1$ be such that $\sum_{i=1}^d g_i = g$. Observe that for any $\{i_0, ..., i_l\} \subseteq [d]$, $\{(g_{i_0}, h_\emptyset), \dots, (g_{i_l}, h_\emptyset)\}$ is a simplex in Δ_{g_ε} if and only if $l \le d - \varepsilon - 1$. □

Let $g \in H_1$ with $\deg(g) = d$ and let $\varepsilon \in \{0, \dots, d\}$. Note that $\Delta_{g_\varepsilon} \cong \Delta_g$ for all $\varepsilon \in \{0, d\}$. Also, we have $\Delta_{g_\varepsilon} \cong \Delta_{g_{d-\varepsilon}}$. Thus, to prove the theorem, it suffices to consider the cases $h_2 = (p + 2 - \varepsilon)h_\emptyset + \varepsilon h_b$, where $\varepsilon \in \{1, 2\}$ and $p \in \{2, 3\}$.

Remark 4.8. Let $g \in H_1$ with $\deg(g) = d$ and $\varepsilon \in \{0, \ldots, d\}$. Let $g_1, ..., g_d \in H_1$ be such that $g = \sum_{i=1}^{d} g_i$. Observe that $\sigma = \big\{(g_{i_1}, h_\emptyset), \ldots, (g_{i_{d-\varepsilon+1}}, h_\emptyset)\big\} \notin \Delta_{g_\varepsilon}$ for any $i_1, \ldots, i_{d-\varepsilon+1} \in \{1, \ldots, d\}$. For $l \in \{1, \ldots, d\}$ with $l \neq i_j$, $j \in \{1, \ldots, d-\varepsilon+1\}$, let

$$\sigma' = \sum_{j=1}^{d-\varepsilon+1} (-1)^{j-1}\big\{(g_{i_l}, h_b), (g_{i_1}, h_\emptyset), \ldots, \widehat{(g_{i_j}, h_\emptyset)}, \ldots, (g_{i_{d-\varepsilon+1}}, h_\emptyset)\big\}$$

be a $(d-\varepsilon)$-chain in Δ_{g_ε}. Then $\partial\sigma = \partial\sigma'$.

Definition 4.9. *For any $g \in H_1$ with $\deg(g) = d$ and $\varepsilon \in \{1, \ldots, d\}$, we define $R_{g,\varepsilon}$ to be the following simplicial complex:*

$$\bigcup_{\substack{g_1,\ldots,g_d \in H_1 \\ s.t.\ g_1+\ldots+g_d = g}} \bigcup_{\substack{i_1,\ldots,i_{d-1} \in \{1,\ldots,d\} \\ i_l \neq i_m}} \big\langle (g_{i_1}, h_b), \ldots, (g_{i_{\varepsilon-1}}, h_b), (g_{i_\varepsilon}, h_\emptyset), \ldots, (g_{i_{d-1}}, h_\emptyset)\big\rangle.$$

Lemma 4.10. *Let $g \in H_1$ with $\deg(g) = d$ and $\varepsilon \in \{1, 2\}$. Assume that*

1. *$(i, d) \in \{(0, 3), (1, 4)\}$;*
2. *$\widetilde{H}_i(\Delta_{g_{\varepsilon-1}}) = 0$.*

Then $\widetilde{H}_i(R_{g,\varepsilon}) = 0$.

Proof. Observe that $R_{g,\varepsilon} \subseteq \Delta_{g_{\varepsilon-1}}$. If $\varepsilon = 1$, then $sk^2(\Delta_{g_{\varepsilon-1}}) \subseteq sk^2(R_{g,\varepsilon})$. Thus, $\widetilde{H}_i(\Delta_{g_{\varepsilon-1}}) = \widetilde{H}_i(R_{g,\varepsilon})$ for $i = 0, 1$. So we only have to consider the case $\varepsilon = 2$. Let β be an i-cycle in $R_{g,\varepsilon}$. Since $\widetilde{H}_i(\Delta_{g_{\varepsilon-1}}) = 0$, there exists an $(i+1)$-chain η in $\Delta_{g_{\varepsilon-1}}$ such that $\partial\eta = \beta$. Suppose that $\eta = \sum_j c_j\sigma_j$, where σ_j is an $(i+1)$-simplex in $\Delta_{g_{\varepsilon-1}}$. Now consider an $(i+1)$-chain ψ in $R_{g,\varepsilon}$ such that $\psi = \sum_j c_j\sigma'_j$, where $\sigma'_j = \sigma_j$ if $\sigma_j \in R_{g,\varepsilon}$ else σ'_j is as defined in Remark 4.8 corresponding to σ_j. Then $\partial\psi = \beta$. $\square$

Lemma 4.11. *Let $g \in H_1$ with $\deg(g) = 4$ and $\varepsilon \in \{1, 2\}$. Every 1-cycle γ in Δ_{g_ε} is homologous to an 1-cycle in $F^1(\Delta_g)$ $(\subseteq \Delta_{g_\varepsilon})$.*

Proof. We prove the lemma by induction on the cardinality of $(\mathrm{supp}(\gamma) \cap sk^0(\Delta_{g_\varepsilon})) \setminus F^1(\Delta_g)$.

Let $(f, h_b) \in \mathrm{supp}(\gamma)$. Let $\mathcal{S}_{(f,h_b)}$ be the set of 1-simplexes of γ with vertex (f, h_b). For $\sigma = \{v, (f, h_b)\} \in \mathcal{S}_{(f,h_b)}$, let $\sigma' = \{v, (f, h_\emptyset)\}$. Clearly, σ' is an 1-simplex of Δ_{g_ε}. Let $\alpha = \sum_{\sigma \in \mathcal{S}_{(f,h_b)}} (-\sigma + \sigma')$ be the 1-cycle in Δ_{g_ε}.

Now, we show that for a vertex v in Δ_{g_ε}, if $\langle v,(f,h_b)\rangle \subseteq \Delta_{g_\varepsilon}$, then $v \in R_{g-f,\varepsilon}$. Observe that

$$R_{g-f,1} = \bigcup_{\substack{g_1,g_2,g_3\in H_1 \\ \text{s.t. } g_1+g_2+g_3=g-f}} \bigcup_{\substack{i_1,i_2\in\{1,2,3\} \\ i_1\neq i_2}} \big\langle (g_{i_1},h_\emptyset),(g_{i_2},h_\emptyset)\big\rangle$$

and

$$R_{g-f,2} = \bigcup_{\substack{g_1,g_2,g_3\in H_1 \text{s.t.} \\ g_1+g_2+g_3=g-f}} \bigcup_{\substack{i_1,i_2\in\{1,2,3\} \\ i_1\neq i_2}} \big\langle (g_{i_1},h_b),(g_{i_2},h_\emptyset)\big\rangle.$$

If $\varepsilon=1$, then $v=(f',h_\emptyset)$ for some $f'\in H_1$ such that $g-(f+f')\in H_1$. If $\varepsilon=2$, then either $v=(f',h_\emptyset)$ or $v=(f',h_b)$ for some $f'\in H_1$ such that $g-(f+f')\in H_1$. In both cases, $v\in R_{g-f,\varepsilon}$.

So we obtain that $\mathrm{supp}(\alpha)\subseteq C$, where C is the union of the cones $\langle (f,h_b),R_{g-f,\varepsilon}\rangle$ and $\langle (f,h_\emptyset),R_{g-f,\varepsilon}\rangle$. Notice that $C\subseteq \Delta_{g_\varepsilon}$. Since Hibi rings satisfy property N_1, we have $\widetilde{H}_0(\Delta_{(g-f)_{\varepsilon-1}})=0$. Thus, $\widetilde{H}_0(R_{g-f,\varepsilon})=0$ by Lemma 4.10. Since $\widetilde{H}_i(C)=\widetilde{H}_{i-1}(R_{g-f,\varepsilon})$, we have $\widetilde{H}_1(C)=0$. Thus, α is homologous to 0 which implies that σ is homologous to $\sigma+\alpha$.

Observe that $\#((\mathrm{supp}(\gamma+\alpha)\cap sk^0(\Delta_{g_\varepsilon}))\setminus F^1(\Delta_g)) < \#((\mathrm{supp}(\gamma)\cap sk^0(\Delta_{g_\varepsilon}))\setminus F^1(\Delta_g))$. Hence, we conclude the proof by induction. □

Proof of Theorem 4.5 for N_2. We have to show that if $h=(h_1,h_2)\in H$ with $\deg(h)=4$, then $\widetilde{H}_1(\Delta_h)=0$. We consider the following cases:

(1). Assume that $h_2=3h_\emptyset+h_b$. Let γ be an 1-cycle in Δ_h. By Lemma 4.11, γ is homologous to an 1-cycle γ' of $F^1(\Delta_{h_1})\subset\Delta_h$. In other words, there exists a 2-chain μ in Δ_h such that $\partial\mu=\gamma-\gamma'$. Also, $\widetilde{H}_1(F^2(\Delta_{h_1}))\cong\widetilde{H}_1(\Delta_{h_1})=0$, where the isomorphism is due to Lemma 4.7(a) and the equality is by hypothesis. As $F^1(\Delta_{h_1})\subset F^2(\Delta_{h_1})$, there exists a 2-chain μ' in $F^2(\Delta_{h_1})$ such that $\partial\mu'=\gamma'$. Since $F^2(\Delta_{h_1})\subseteq\Delta_h$, μ' is a 2-chain in Δ_h. Therefore, $[\gamma']=0$ in $\widetilde{H}_1(\Delta_h)$. So $[\gamma]=0$ in $\widetilde{H}_1(\Delta_h)$.

(2). Consider $h_2=2h_\emptyset+2h_b$. Every 1-cycle γ in Δ_h is homologous to an 1-cycle γ' in $F^1(\Delta_{h_1})$ by Lemma 4.11. But in this case, $F^2(\Delta_{h_1})$ is not contained in Δ_h. Since $\widetilde{H}_1(F^2(\Delta_{h_1}))=0$, there exists a 2-chain μ in $F^2(\Delta_{h_1})$ such that $\partial\mu=\gamma'$. Let $\mu=\sum_i c_i\sigma_i$, where σ_i is a 2-simplex in $F^2(\Delta_{h_1})$. Consider a 2-chain ψ in Δ_h such that $\psi=\sum_i c_i\sigma_i'$, where $\sigma_i'=\sigma_i$ if $\sigma_i\in\Delta_h$ else σ_i' is as defined in Remark 4.8

corresponding to σ_i. Then $\partial\psi = \gamma'$. Therefore, $[\gamma'] = 0$ in $\widetilde{H}_1(\Delta_h)$. So $[\gamma] = 0$ in $\widetilde{H}_1(\Delta_h)$. Hence the proof.

$\square$

Lemma 4.12. *Let $g \in H_1$ with $\deg(g) = 5$ and $\varepsilon \in \{1, 2\}$. Every 2-cycle γ in Δ_{g_ε} is homologous to a 2-cycle in $F^2(\Delta_g)$ ($\subset \Delta_{g_\varepsilon}$).*

Proof. We prove the result by induction on the cardinality of $(\operatorname{supp}(\gamma) \cap sk^0(\Delta_{g_\varepsilon})) \setminus F^2(\Delta_g)$.

Let $(f, h_b) \in \operatorname{supp}(\gamma)$. Let $\mathcal{S}_{(f,h_b)}$ be the set of 2-simplexes of γ with vertex (f, h_b). For $\sigma = \{v, u, (f, h_b)\} \in \mathcal{S}_{(f,h_b)}$, let $\sigma' = \{v, u, (f, h_\emptyset)\}$. Clearly, σ' is a 2-simplex of Δ_{g_ε}. Let $\alpha = \sum_{\sigma \in \mathcal{S}_{(f,h_b)}} (-\sigma + \sigma')$ be the 2-cycle in Δ_{g_ε}.

Now, we show that for vertexes v, u in Δ_{g_ε}, if $\langle v, u, (f, h_b)\rangle \subseteq \Delta_{g_\varepsilon}$, then $\langle v, u\rangle \subseteq R_{g-f,\varepsilon}$. Observe that

$$R_{g-f,1} = \bigcup_{\substack{g_1,\ldots,g_4 \in H_1 \\ \text{s.t. } g_1+\ldots+g_4 = g-f}} \bigcup_{\substack{i_1,i_2,i_3 \in \{1,\ldots,4\} \\ i_l \neq i_k}} \big\langle (g_{i_1}, h_\emptyset), (g_{i_2}, h_\emptyset)(g_{i_3}, h_\emptyset)\big\rangle$$

and

$$R_{g-f,2} = \bigcup_{\substack{g_1,\ldots,g_4 \in H_1 \\ \text{s.t. } g_1+\ldots+g_4 = g-f}} \bigcup_{\substack{i_1,i_2,i_3 \in \{1,\ldots,4\} \\ i_l \neq i_k}} \big\langle (g_{i_1}, h_b), (g_{i_2}, h_\emptyset)(g_{i_3}, h_\emptyset)\big\rangle.$$

If $\varepsilon = 1$, then $\langle v, u\rangle = \langle (f_1, h_\emptyset), (f_2, h_\emptyset)\rangle$ for some $f_1, f_2 \in H_1$ such that $g - (f + f_1 + f_2) \in H_1$. If $\varepsilon = 2$, then either $\langle v, u\rangle = \langle (f_1, h_\emptyset), (f_2, h_\emptyset)\rangle$ or $\langle v, u\rangle = \langle (f_1, h_b), (f_2, h_\emptyset)\rangle$ for some $f_1, f_2 \in H_1$ such that $g - (f + f_1 + f_2) \in H_1$. In both cases, $\langle v, u\rangle = \langle (f_1, h_\emptyset), (f_2, h_\emptyset)\rangle \subseteq R_{g-f,\varepsilon}$.

So we obtain that $\operatorname{supp}(\alpha) \subseteq C$, where C is the union of the cones $\langle (f, h_b), R_{g-f,\varepsilon}\rangle$ and $\langle (f, h_\emptyset), R_{g-f,\varepsilon}\rangle$. Notice that $C \subseteq \Delta_{g_\varepsilon}$. Since $R[\mathcal{I}(P)]$ satisfies property N_2, we have $\widetilde{H}_1(\Delta_{(g-f)_{\varepsilon-1}}) = 0$. Thus by Lemma 4.10, $\widetilde{H}_1(R_{g-f,\varepsilon}) = 0$. Since $\widetilde{H}_i(C) = \widetilde{H}_{i-1}(R_{g-f,\varepsilon})$, we have $\widetilde{H}_2(C) = 0$. Thus, α is homologous to 0. So σ is homologous to $\sigma + \alpha$.

Observe that $\#((\operatorname{supp}(\gamma + \alpha) \cap sk^0(\Delta_{g_\varepsilon})) \setminus F^2(\Delta_g)) < \#((\operatorname{supp}(\gamma) \cap sk^0(\Delta_{g_\varepsilon})) \setminus F^2(\Delta_g))$. Hence, we conclude the proof by induction. $\square$

Proof of Theorem 4.5 for N_3. We have to show that if $h = (h_1, h_2) \in H$ with $\deg(h) = 5$, then $\widetilde{H}_2(\Delta_h) = 0$. We consider the following cases:

(1). Consider $h_2 = 4h_\emptyset + h_b$. Let γ be a 2-cycle in Δ_h. By Lemma 4.12, γ is homologous to a 2-cycle γ' of $F^3(\Delta_{h_1}) \subset \Delta_h$. In other words, there exists a 3-chain μ in Δ_h such that $\partial\mu = \gamma - \gamma'$. Also, $\widetilde{H}_2(F^3(\Delta_{h_1})) \cong \widetilde{H}_2(\Delta_{h_1}) = 0$, where the isomorphism is due to Lemma 4.7 and the equality holds because $R[\mathcal{I}(P_1)]$ satisfies property N_3. As $F^2(\Delta_{h_1}) \subset F^3(\Delta_{h_1})$, there exists a 3-chain μ' in $F^3(\Delta_{h_1})$ such that $\partial\mu' = \gamma'$. Since $F^3(\Delta_{h_1}) \subseteq \Delta_h$, μ' is a 3-chain in Δ_h. Therefore, $[\gamma'] = 0$ in $\widetilde{H}_1(\Delta_h)$. So $[\gamma] = 0$ in $\widetilde{H}_1(\Delta_h)$.

(2). Assume $h_2 = 3h_\emptyset + 2h_b$. By Lemma 4.12, every 2-cycle γ in Δ_h is homologous to a 2-cycle γ' in $F^2(\Delta_{h_1})$. But in this case, $F^3(\Delta_{h_1}) \nsubseteq \Delta_h$. Since $\widetilde{H}_2(F^3(\Delta_{h_1})) = 0$, there exists a 3-chain μ in $F^3(\Delta_{h_1})$ such that $\partial\mu = \gamma'$. Let $\mu = \sum_i c_i\sigma_i$, where σ_i is a 3-simplex in $F^3(\Delta_{h_1})$. Consider a 3-chain ψ in Δ_h such that $\psi = \sum_i c_i\sigma'_i$, where $\sigma'_i = \sigma_i$ if $\sigma_i \in \Delta_h$ else σ'_i is as defined in Remark 4.8 corresponding to σ_i. Then $\partial\psi = \gamma'$. Therefore, $[\gamma'] = 0$ in $\widetilde{H}_2(\Delta_h)$. So $[\gamma] = 0$ in $\widetilde{H}_2(\Delta_h)$. Hence the proof.

□

4.2 Segre product with a polynomial ring

In this section, we show that if a Hibi ring satisfies property N_2, then its Segre product with a polynomial ring in finitely many variables also satisfies property N_2.

Proposition 4.13. *Let P be a poset such that it is a disjoint union of a poset P_1 and a chain $P_2 = \{a_1, \ldots, a_n\}$ with $a_1 \lessdot \cdots \lessdot a_n$. Let $\{x\}$ be a poset and P'_2 be the ordinal sum $P_2 \oplus \{x\}$. Let Q be the disjoint union of the posets P_1 and P'_2. If $R[\mathcal{I}(P)]$ satisfies property N_2, then so does $R[\mathcal{I}(Q)]$.*

We now state our main theorem of this section.

Theorem 4.14. *Let P be a poset such that it is a disjoint union of a poset P_1 and a chain P_2. If $R[\mathcal{I}(P_1)]$ satisfies property N_2, then so does $R[\mathcal{I}(P)]$.*

Proof. The proof follows from Theorem 4.5 and Proposition 4.13. □

Corollary 4.15. *Let P be as defined in Theorem 3.25. Assume that P is disconnected. Then $\beta_{24}(R[\mathcal{I}(P)]) = 0$ if and only if P is a disjoint union of two posets P_1 and P_2 such that $\mathcal{I}(P_1)$ is a planar distributive lattice with $\beta_{24}(R[\mathcal{I}(P_1)]) = 0$ and P_2 is a chain.*

Proof. The proof follows from Theorem 4.14 and Corollary 4.3. □

The rest of section is dedicated to the proof of the Proposition 4.13. The proof of Proposition 4.13 is motivated from Rubei [Rub07].

Let P be a poset such that it is a disjoint union of two posets P_1 and P_2. Let $\{x\}$ be a poset and P_2' be the ordinal sum $P_2 \oplus \{x\}$. Let Q be the disjoint union of posets P_1 and P_2'. Let H and H' be the affine semigroups generated by $\{h_\alpha : \alpha \in \mathcal{I}(Q)\}$ and $\{h_\beta : \beta \in \mathcal{I}(P)\}$ respectively. For $i \in \{1,2\}$, let H_i be the affine semigroup generated by $\{h_\alpha : \alpha \in \mathcal{I}(P_i)\}$. For $\alpha \in Q$, the first entry of h_α is 1 if $x \in \alpha$ and the second entry of h_α is 1 if $x \notin \alpha$.

Note : Let $h \in H$ with $\deg(h) = d$. In this subsection, we either denote h by $(\varepsilon, d-\varepsilon, h')$, where $h' \in H'$ or we denote it by $(\varepsilon, d-\varepsilon, h_2, h_1)$, where $h_i \in H_i$ for all $i = 1, 2$.

Let $h \in H$ with $\deg(h) = d$. Then $h = (\varepsilon, d-\varepsilon, h')$, where $h' \in H'$, $\varepsilon \leq d$, $\varepsilon \in \mathbb{N}$. Let X_h be the following simplicial complex:

$$X_h := \Delta_h \cup \Delta_{(\varepsilon-1, d-\varepsilon+1, h')} \cup \ldots \cup \Delta_{(0,d,h')}.$$

Observe that $\Delta_{(0,d,h')} \cong \Delta_{h'} \cong \Delta_{(d,0,h')}$. Let γ be an 1-cycle in X_h. For every vertex $v \in \gamma$, let $\mathcal{S}_{v,\gamma}$ be the set of simplexes of γ with vertex v and $\mu_{v,\gamma}$ be the 0-cycle such that $v * \mu_{v,\gamma} = \sum_{\tau \in \mathcal{S}_{v,\gamma}} \tau$, where $*$ denotes the joining.

Proposition 4.16. *Let $h \in H$ with $\deg(h) = 4$. Let γ be an 1-cycle in Δ_h. Then there exists an 1-cycle γ' in $\Delta_{(0,4,h')}$ such that γ is homologous to γ' in X_h.*

Proof. Let $h_{\alpha_1}, \ldots, h_{\alpha_m}$ be the vertices of γ with non-zero first entry. In other words, these are all the vertices h_α of γ such that $x \in \alpha$. For $1 \leq i \leq m$, let $\beta_i = \alpha_i \setminus \{x\}$. Observe that $\mu_{h_{\alpha_1},\gamma}$ is in $\Delta_{h-h_{\alpha_1}}$ and

$$\widetilde{H}_1(h_{\alpha_1} * \Delta_{h-h_{\alpha_1}} \cup h_{\beta_1} * \Delta_{h-h_{\alpha_1}}) = \widetilde{H}_0(\Delta_{h-h_{\alpha_1}}) = 0,$$

where the last equality holds because $R[\mathcal{I}(Q)]$ satisfies property N_1. So $h_{\alpha_1} * \mu_{h_{\alpha_1},\gamma} - h_{\beta_1} * \mu_{h_{\alpha_1},\gamma}$ is homologous to 0 in X_h. Hence, $\gamma_1 := \gamma - (h_{\alpha_1} * \mu_{h_{\alpha_1},\gamma} - h_{\beta_1} * \mu_{h_{\alpha_1},\gamma})$ is homologous to γ in X_h. Informally speaking, we have got γ_1 from γ by replacing the vertex h_{α_1} with h_{β_1}. Inductively, define

$$\gamma_i := \gamma_{i-1} - (h_{\alpha_i} * \mu_{h_{\alpha_i},\gamma_{i-1}} - h_{\beta_i} * \mu_{h_{\alpha_i},\gamma_{i-1}})$$

for $2 \leq i \leq m$. Since all the vertexes of γ_m have first entry zero, we have $\gamma_m \in \Delta_{(0,d,h')}$. We set $\gamma' = \gamma_m$ and prove that γ_m is homologous to γ in X_h. To prove this, it suffices to show that $h_{\alpha_i} * \mu_{h_{\alpha_i},\gamma_{i-1}} - h_{\beta_i} * \mu_{h_{\alpha_i},\gamma_{i-1}}$ is homologous to 0 for $2 \leq i \leq m$.

Let θ_0 be the sum of simplexes τ of $\mu_{h_{\alpha_i},\gamma_{i-1}}$ such that τ is a vertex of $\Delta_{h-h_{\alpha_i}}$ and let θ_1 be the sum of simplexes τ of $\mu_{h_{\alpha_i},\gamma_{i-1}}$ such that τ is not a vertex of $\Delta_{h-h_{\alpha_i}}$. Observe that $\mu_{h_{\alpha_i},\gamma_{i-1}} = \theta_0 + \theta_1$ and θ_1 is a 0-cell in $\Delta_{(\varepsilon-1,5-\varepsilon,h')-h_{\alpha_i}}$. Since $\mu_{h_{\alpha_i},\gamma}$ is a 0-cycle, θ_0 is a 0-cycle of $\Delta_{h-h_{\alpha_i}}$ and θ_1 is a 0-cycle of $\Delta_{(\varepsilon-1,5-\varepsilon,h')-h_{\alpha_i}}$. Furthermore, since $R[\mathcal{I}(Q)]$ satisfies property N_1, θ_0 is homologous to 0 in $\Delta_{h-h_{\alpha_i}}$ and θ_1 is homologous to 0 in $\Delta_{(\varepsilon-1,5-\varepsilon,h')-h_{\alpha_i}}$. Thus, they are homologous to 0 in X_h. Therefore, $h_{\alpha_i} * \mu_{h_{\alpha_i},\gamma_{i-1}} - h_{\beta_i} * \mu_{h_{\alpha_i},\gamma_{i-1}}$ is homologous to 0 in X_h. This concludes the proof. $\square$

Lemma 4.17. *Let $\mathcal{P} = \{q_1, \ldots, q_r\}$ be a chain such that $q_1 \lessdot \cdots \lessdot q_r$ and let $\mathcal{H}$ be the semigroup corresponding to $R[\mathcal{I}(\mathcal{P})]$. Let $h = \sum_{i=1}^d h_{\alpha_i} \in \mathcal{H}$. Then*

$$\Delta_h = \langle h_{\alpha_1}, \ldots, h_{\alpha_d} \rangle.$$

Proof. It is enough to show that for some $\alpha \in \mathcal{I}(\mathcal{P})$, if $h_\alpha \notin \{h_{\alpha_1}, \ldots, h_{\alpha_d}\}$, then h_α is not a vertex of Δ_h. If $\alpha = \emptyset$, then the entry corresponding to "$q_1 \notin \alpha$" in $h - h_\alpha$ will be -1. So h_α is not a vertex of Δ_h. If $\alpha_i \leq \alpha$ for all $i \in \{1, \ldots, d\}$, then $h - h_\alpha \notin \mathcal{H}$. Hence, h_α is not a vertex of Δ_h. Now suppose that for all $i \in \{1, \ldots, d\}$, $\alpha_i \nleq \alpha$. Let $\{h_{\alpha_{i_1}}, \ldots, h_{\alpha_{i_m}}\}$ be the subset of $\{h_{\alpha_1}, \ldots, h_{\alpha_d}\}$ such that $h_\alpha < h_{\alpha_{i_j}}$ for all $j = 1, \ldots, m$. Let $\alpha = \{q_1, \ldots, q_s\}$, where $1 \leq s \leq r-1$. Observe that the entries corresponding to q_s and q_{s+1} in h are m. But the entries corresponding q_s and q_{s+1} in $h - h_\alpha$ are $m-1$ and m respectively. Hence, $h - h_\alpha \notin \mathcal{H}$. This completes the proof. $\square$

From now onwards, let P be a poset such that it is a disjoint union of a poset P_1 and a chain $P_2 = \{a_1, \ldots, a_n\}$ with $a_1 \lessdot \cdots \lessdot a_n$. Let P_2' be the ordinal sum $P_2 \oplus \{x\}$. Furthermore, let Q be the disjoint union of posets P_1 and P_2'. Let $H_{P_2'}$ be the semigroup associated to $R[\mathcal{I}(P_2')]$.

Lemma 4.18. *Let $h = \sum_{i=1}^d h_{\alpha_i} \in H$ with $h = (\varepsilon, d-\varepsilon, h')$, $\varepsilon \geq 1$. For $r < d$, assume that there are exactly r number of i's with $\alpha_i = \alpha_i^1 \cup P_2$, where $\alpha_i^1 \in \mathcal{I}(P_1)$. Let $\tau = \{h_{\beta_1}, \ldots, h_{\beta_{m+1}}\}$ be an m-simplex of X_h. Then $\tau \in \Delta_h$ if and only if there are at most r number of β_i's with $\beta_i = \beta_i^1 \cup P_2$, where $\beta_i^1 \in \mathcal{I}(P_1)$.*

Proof. If we write $h = (\varepsilon, d-\varepsilon, h_2, h_1)$, where $h_i \in H_i$ for all $i = 1, 2$, then $(\varepsilon - 1, d - \varepsilon + 1, h') = (\varepsilon - 1, d - \varepsilon + 1, h_2, h_1)$. For $1 \leq i \leq d$, let $\alpha_i = (\alpha_i^2, \alpha_i^1)$, where $\alpha_i^2 \in \mathcal{I}(P_2')$

and $\alpha_i^1 \in \mathcal{I}(P_1)$, So we can write $h_{\alpha_i} = (1, 0, h_{P_2}, h_{\alpha_i^1})$ if $x \in \alpha_i$ and $h_{\alpha_i} = (0, 1, h_{\alpha_i^2}, h_{\alpha_i^1})$ if $x \notin \alpha_i^2$, where $h_{\alpha_i^2} \in H_2$ and $h_{\alpha_i^1} \in H_1$. We have

$$h_1 = \sum_{i=1}^{d} h_{\alpha_i^1}, \quad (\varepsilon, d-\varepsilon, h_2) = \sum_{i=1}^{d} h_{\alpha_i^2}.$$

Let $\tau_1 = \{h_{\beta_1^1}, \ldots, h_{\beta_{m+1}^1}\}$ and $\tau_2 = \{h_{\beta_1^2}, \ldots, h_{\beta_{m+1}^2}\}$. Note that τ_1, τ_2 could be multisets.

Now we show that $\tau \in \Delta_h$ if and only if $\tau_2 \subseteq \{h_{\alpha_1^2}, \ldots, h_{\alpha_d^2}\}$. Observe that if $\tau \in \Delta_h$, then $(\varepsilon, d-\varepsilon, h_2) - \sum_{j=1}^{m+1} h_{\beta_j^2} \in H_{P_2'}$. Hence, $\tau_2 \subseteq \{h_{\alpha_1^2}, \ldots, h_{\alpha_d^2}\}$, by Lemma 4.17. On the other hand, if $\tau_2 \subseteq \{h_{\alpha_1^2}, \ldots, h_{\alpha_d^2}\}$, then $(\varepsilon, d-\varepsilon, h_2) - \sum_{j=1}^{m+1} h_{\beta_j^2} \in H_{P_2'}$. Since $\tau \in X_h$, there exists an $i_0 \in \{0, \ldots, \varepsilon\}$ such that $\tau \in \Delta_{(\varepsilon - i_0, d-\varepsilon+i_0, h_2, h_1)}$. So $(\varepsilon - i_0, d-\varepsilon+i_0, h_2, h_1) - \sum_{j=1}^{m+1} h_{\beta_j} \in H$. Therefore, $h_1 - \sum_{j=1}^{m+1} h_{\beta_j^1} \in H_1$. We obtain $\tau \in \Delta_h$.

The proof of 'only if' part follows from the above claim. To prove 'if', it suffices to show that $\tau_2 \subseteq \{h_{\alpha_1^2}, \ldots, h_{\alpha_d^2}\}$. For $1 \le i \le \varepsilon$, we have

$$(\varepsilon - i, d-\varepsilon+i, h_2) = \sum_{j=1}^{\varepsilon-i} h_{P_2'} + \sum_{j=\varepsilon-i+1}^{\varepsilon} h_{P_2} + \sum_{j=\varepsilon+1}^{\varepsilon+r} h_{P_2} + \sum_{j=\varepsilon-r+1}^{d} h_{\alpha_j^2} \in H_{P_2'}.$$

Let $i_0 \in \{0, \ldots, \varepsilon\}$ be such that $\tau \in \Delta_{(\varepsilon-i_0, d-\varepsilon+i_0, h_2, h_1)}$. Since there are at most r number of β_i's in τ with $\beta_i = \beta_i^1 \cup P_2$, where $\beta_i^1 \in \mathcal{I}(P_1)$, the multiplicity of h_{P_2} in τ_2 is at most r. So by Lemma 4.17,

$$\tau_2 \subseteq \{h_{P_2'}, \ldots, h_{P_2'}, h_{P_2}, \ldots, h_{P_2}, h_{\alpha_{\varepsilon-r+1}^2}, \ldots, h_{\alpha_d^2}\} \subseteq \{h_{\alpha_1^2}, \ldots, h_{\alpha_d^2}\},$$

where the multidegrees of $h_{P_2'}$ and h_{P_2} in the middle set are $\varepsilon - i_0$ and r respectively. Hence, $\tau_2 \subseteq \{h_{\alpha_1^2}, \ldots, h_{\alpha_d^2}\}$. □

Remark 4.19. (1) Let $p \in \{2, 3\}$ and $h = \sum_{i=1}^{p+2} h_{\alpha_i} \in H$. Assume that there is an $\alpha_0^2 \in \mathcal{I}(P_2') \setminus \{\alpha_1^2, \ldots, \alpha_{p+2}^2\}$. Let $\widetilde{h} = \sum_{i=1}^{p+2} h_{\beta_i}$ be an element of H such that

$$\beta_i = \begin{cases} \alpha_i & \text{if } x \notin \alpha_i, \\ \alpha_i^1 \cup \alpha_0^2 & \text{if } x \in \alpha_i. \end{cases}$$

For $\tau = \{h_{\gamma_1}, \ldots, h_{\gamma_m}\}$, define $\tau' := \{h_{\nu_1}, \ldots, h_{\nu_m}\}$, where

$$\nu_j = \begin{cases} \gamma_j & \text{if } x \notin \gamma_j, \\ \gamma_j^1 \cup \alpha_0^2 & \text{if } x \in \gamma_i. \end{cases}$$

Then τ is a simplex of Δ_h if and only if τ' is a simplex of $\Delta_{\widetilde{h}}$. Therefore, $\Delta_h \cong \Delta_{\widetilde{h}}$.

(2) Let $p \in \{2, 3\}$ and $h = \sum_{i=1}^{p+2} h_{\alpha_i} \in H$. Let $\alpha^2, \widetilde{\alpha}^2 \in \{\alpha_1^2, \ldots, \alpha_{p+2}^2\}$ with $\alpha^2 \neq \widetilde{\alpha}^2$. Let $\widetilde{h} = \sum_{i=1}^{p+2} h_{\beta_i}$ be an element of H such that

$$\beta_i = \begin{cases} \alpha_i & \text{if} \quad \alpha_i^2 \neq \alpha^2, \widetilde{\alpha}^2, \\ \alpha_i^1 \cup \widetilde{\alpha}^2 & \text{if} \quad \alpha_i = \alpha_i^1 \cup \alpha^2 \ , \\ \alpha_i^1 \cup \alpha^2 & \text{if} \quad \alpha_i = \alpha_i^1 \cup \widetilde{\alpha}^2. \end{cases}$$

For $\tau = \{h_{\gamma_1}, \ldots, h_{\gamma_m}\}$, define $\tau' := \{h_{\nu_1}, \ldots, h_{\nu_m}\}$, where

$$\nu_j = \begin{cases} \gamma_j & \text{if} \quad \gamma_j^2 \neq \alpha^2, \widetilde{\alpha}^2, \\ \gamma_j^1 \cup \widetilde{\alpha}^2 & \text{if} \quad \gamma_j = \gamma_j^1 \cup \alpha^2 \ , \\ \gamma_j^1 \cup \alpha^2 & \text{if} \quad \gamma_j = \gamma_j^1 \cup \widetilde{\alpha}^2. \end{cases}$$

Observe that τ is a simplex of Δ_h if and only if τ' is a simplex of $\Delta_{\widetilde{h}}$. Therefore, $\Delta_h \cong \Delta_{\widetilde{h}}$. $\square$

Proposition 4.20. *Let $h = (1, 3, h_2, h_1) = \sum_{i=1}^{4} h_{\alpha_i} \in H$. Assume that $\mathcal{I}(P_2') \subseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$ and there are exactly two i's with $\alpha_i^2 = P_2$. Let γ be an 1-cycle in Δ_h. If γ is homologous to 0 in X_h, then it is also homologous to 0 in Δ_h.*

Proof. Let $\eta = \sum c_\sigma \sigma$, where $c_\sigma \in \mathbb{Z}$, be a 2-chain in X_h such that $\partial\eta = \gamma$. We construct an η' in Δ_h such that $\partial\eta' = \gamma$. Let $\{h_{\nu_1}, h_{\nu_2}, h_{\nu_3}\}$ be a simplex in η. By Lemma 4.18, it is not a simplex of Δ_h if and only if $\nu_j^2 = P_2$ for $j = 1, 2, 3$. Let $\sigma = \{h_{\nu_1}, h_{\nu_2}, h_{\nu_3}\}$ be a simplex in η such that it is not a simplex of Δ_h. Note that $(0, 4, h_2, h_1) - \sum_{i=1}^{3} h_{\nu_i} \in H$, call it h_{ν_4}. Observe that $\{h_{\nu_1}, \ldots, h_{\nu_4}\}$ is a face of X_h. Define

$$\sigma' := \sum_{j=1}^{3} (-1)^{j-1} \{h_{\nu_4}, h_{\nu_1}, \ldots, \widehat{h_{\nu_j}}, \ldots, h_{\nu_3}\}.$$

By Lemma 4.18, $\nu_4^2 \neq P_2$. Therefore, σ' is a 2-chain in Δ_h. Observe that $\partial\sigma = \partial\sigma'$. Take $\eta' = \sum_{\sigma \in \eta} c_\sigma \sigma'$, where $\sigma' = \sigma$ if $\sigma \in \Delta_h$ otherwise σ' is as defined above for σ. Then η' is a 2-chain in Δ_h and $\partial\eta' = \gamma$. This completes the proof. $\square$

Remark 4.21. Let Q be a poset such that it is a disjoint union of a poset P_1 and a chain $P_2' = \{a_1, a_2, x\}$, with $a_1 \lessdot a_2 \lessdot x$. Assume that $R[\mathcal{I}(P_1)]$ satisfies property N_2. Let H be the semigroup associated to $R[\mathcal{I}(Q)]$. Let $A = \{\beta_1^1, \ldots, \beta_4^1\}, B = \{\beta_1^1, \beta_2^1, \delta_1^1, \delta_2^1\}$ where $\beta_j^1, \delta_i^1 \in \mathcal{I}(P_1)$, be two multisets with $\{\delta_1^1, \delta_2^1\} \cap \{\beta_3^1, \beta_4^1\} = \emptyset$, $\beta_1^1 \neq \beta_2^1$ and $\sum_{\beta \in A} h_\beta = \sum_{\beta \in B} h_\beta \in H_1$. Let

$$\mathcal{S} = \big\{\{\nu_1, \ldots, \nu_4\} \subseteq \mathcal{I}(Q) : \{\nu_1^2, \ldots, \nu_4^2\} = \mathcal{I}(P_2') \text{ and } \{\nu_1^1, \ldots, \nu_4^1\} \in \{A, B\}\big\}.$$

Let Δ' be the simplicial complex whose facets are $\{h_{\nu_1}, \ldots, h_{\nu_4}\}$, where $\{\nu_1, \ldots, \nu_4\} \in \mathcal{S}$. We use SageMath [sage] to check that $\widetilde{H}_1(\Delta') = 0$ for all choices of A and B.

The choices of A and B, up to isomorphism, are the following:

(a) $\{\beta_1^1, \beta_2^1\} = \{\beta_3^1, \beta_4^1\}$, $\delta_3^1 \neq \delta_4^1$,

```
import itertools;
A= [5,6,5,6];
B= [5,6,7,8];
C = list(itertools.permutations(A));
D = list(itertools.permutations(B));
L1 = [[(0,b1),(1,b2),(2,b3),(3,b4)] for [b1,b2,b3,b4] in C];
L2 = [[(0,b1),(1,b2),(2,b3),(3,b4)] for [b1,b2,b3,b4] in D];
S = L1+L2;
K = SimplicialComplex(S);
print(K.homology())

{0: 0, 1: 0, 2: Z^21, 3: 0}
```

LISTING 4.1: Sagemath code

(b) $\{\beta_1^1, \beta_2^1\} = \{\beta_3^1, \beta_4^1\}$, $\delta_3^1 = \delta_4^1$,

```
A=[5,6,5,6]
B=[5,6,7,7];

{0: 0, 1: 0, 2: Z^11, 3: 0}
```

LISTING 4.2: Sagemath code

(c) $\beta_1^1 = \beta_3^1$, $\beta_2^1 \neq \beta_4^1$, $\delta_3^1 \neq \delta_4^1$ and $\beta_2^1 \notin \{\delta_3^1, \delta_4^1\}$,

```
A=[5,6,5,7]
B=[5,6,8,9];

```

```
{0: 0, 1: 0, 2: Z^25, 3: 0}
```

Listing 4.3: Sagemath code

(d) $\beta_1^1 = \beta_3^1$, $\beta_1^1 \neq \beta_4^1$, $\beta_2^1 \neq \beta_4^1$,δ_3^1 and $\delta_3^1 = \delta_4^1$,

```
A=[5,6,5,7]
B=[5,6,8,8];

{0: 0, 1: 0, 2: Z^15, 3: 0}
```

Listing 4.4: Sagemath code

(e) $\beta_1^1 = \beta_3^1$, $\beta_1^1 \neq \beta_4^1$, $\beta_2^1 \neq \beta_4^1$, $\delta_3^1 \neq \delta_4^1$ and $\beta_2^1 = \delta_3^1$,

```
A=[5,6,5,7]
B=[5,6,6,8];

{0: 0, 1: 0, 2: Z^15, 3: 0}
```

Listing 4.5: Sagemath code

(f) $\beta_1^1 = \beta_3^1$, $\beta_1^1 \neq \beta_4^1$, $\beta_2^1 \neq \beta_4^1$ and $\beta_2^1 = \delta_3^1 = \delta_4^1$,

```
A=[5,6,5,7]
B=[5,6,6,6];

{0: 0, 1: 0, 2: Z^11, 3: 0}
```

Listing 4.6: Sagemath code

(g) $\beta_1^1 = \beta_3^1 = \beta_4^1$, $\delta_3^1 \neq \delta_4^1$ and $\beta_2^1 \notin \{\delta_3^1, \delta_4^1\}$,

```
A=[5,6,5,5]
B=[5,6,7,8];

{0: 0, 1: 0, 2: Z^21, 3: 0}
```

Listing 4.7: Sagemath code

(h) $\beta_1^1 = \beta_3^1 = \beta_4^1$, $\beta_2^1 \neq \delta_3^1$ and $\delta_3^1 = \delta_4^1$,

```
A=[5,6,5,5]
B=[5,6,7,7];

{0: 0, 1: 0, 2: Z^11, 3: 0}
```

Listing 4.8: Sagemath code

(i) $\beta_1^1 = \beta_3^1 = \beta_4^1$, $\delta_3^1 \neq \delta_4^1$ and $\beta_2^1 = \delta_3^1$,

```
A=[5,6,5,5]
B=[5,6,6,7];

{0: 0, 1: 0, 2: Z^11, 3: 0}
```

Listing 4.9: Sagemath code

(j) $\beta_1^1 = \beta_3^1 = \beta_4^1$, $\beta_2^1 = \delta_3^1 = \delta_4^1$,

```
A=[5,6,5,5]
B=[5,6,6,6];

{0: 0, 1: 0, 2: Z^7, 3: 0}
```

Listing 4.10: Sagemath code

(k) $\{\beta_1^1, \beta_2^1\} \cap \{\beta_3^1, \beta_4^1\} = \emptyset$, $\beta_3^1 = \beta_4^1$, $\delta_3^1 = \delta_4^1$,

```
A=[5,6,7,7]
B=[5,6,8,8];

{0: 0, 1: 0, 2: Z^15, 3: 0}
```

Listing 4.11: Sagemath code

(l) each element of A and B appears with multiplicity 1 and $A \cap B = \{\beta_1^1, \beta_2^1\}$.

```
A=[5,6,7,8]
B=[5,6,9,10];

{0: 0, 1: 0, 2: Z^35, 3: 0}
```

Listing 4.12: Sagemath code

□

Proposition 4.22. *Let $h = (1, 3, h_2, h_1) = \sum_{i=1}^{4} h_{\alpha_i} \in H$. Assume that $\{\alpha_1^2, \ldots, \alpha_4^2\} = \mathcal{I}(P_2')$ (as a multiset). Let γ be an 1-cycle in Δ_h. If γ is homologous to 0 in X_h, then it is also homologous to 0 in Δ_h.*

Proof. Let $\eta = \sum c_\sigma \sigma$, where $c_\sigma \in \mathbb{Z}$ be a 2-chain in X_h such that $\partial\eta = \gamma$. Let $\{h_{\nu_1}, h_{\nu_2}, h_{\nu_3}\}$ be a simplex in η. By Lemma 4.18, it is not a simplex of Δ_h if and only if there exist exactly two $j's$ with $\nu_j^2 = P_2$. We prove the proposition by induction on $k^\eta := \sum |c_\sigma|$, where σ is a simplex of η but it is not a simplex of Δ_h.

Let $\sigma_1 = \{h_{\nu_1}, h_{\nu_2}, h_{\nu_3}\}$ be a simplex in η such that it is not a simplex of Δ_h and $\nu_{j_1}, \nu_{j_2} = P_2$. Let a be the sign of the coefficient of σ_1 in η. Observe that $\{h_{\nu_{j_1}}, h_{\nu_{j_2}}\}$ is a simplex in $\partial\sigma_1$ and it is not in Δ_h. Since $\partial\eta$ is in Δ_h, there is another $\{h_{\nu_{j_1}}, h_{\nu_{j_2}}\}$ in $\partial\eta$ with the opposite sign. So there is a simplex σ_2 of η but not a simplex of Δ_h such that $\sigma_1 \neq \sigma_2$ and $\partial(a\sigma_1 + b\sigma_2)$ is an 1-cycle in Δ_h, where b is the sign of the coefficient of σ_2 in η.

Let σ_1, σ_2 be as above. We will define an η_1 such that $k^{\eta_1} < k^\eta$. Suppose that σ_1 and σ_2 are the faces of the same facet, say F. Let σ_3 and σ_4 be other two faces of F. Then $\sigma_3, \sigma_4 \in \Delta_h$, by Lemma 4.18 and there exist $c, d \in \{1, -1\}$ such that $\partial(c\sigma_3 + d\sigma_4) = \partial(a\sigma_1 + b\sigma_2)$. Define

$$\eta_1 := \eta - (a\sigma_1 + b\sigma_2) - (c\sigma_3 + d\sigma_4).$$

Observe that $\partial\eta_1 = \partial\eta$.

On the other hand, suppose that σ_1 and σ_2 are not the faces of the same facet. For $i = 1, 2$, let F_i be the facet of X_h such that σ_i is a face of F_i. Write $F_1 = \{h_{\beta_1}, \ldots, h_{\beta_4}\}$ and $F_2 = \{h_{\beta_1}, h_{\beta_2}, h_{\delta_1}, h_{\delta_2}\}$. Let $A = \{\beta_1^1, \ldots, \beta_4^1\}, B = \{\beta_1^1, \beta_2^1, \delta_1^1, \delta_2^1\}$. For A and B, let Δ' be as defined in Remark 4.21. Observe that Δ' is a subsimplicial complex of Δ_h and $\partial(a\sigma_1 + b\sigma_2)$ is an 1-cycle in Δ'. Since $\widetilde{H}_1(\Delta') = 0$, there exists a 2-chain μ_{σ_1,σ_2} in Δ' such that $\partial\mu_{\sigma_1,\sigma_2} = \partial(a\sigma_1 + b\sigma_2)$. Define

$$\eta_1 := \eta - (a\sigma_1 + b\sigma_2) - \mu_{\sigma_1,\sigma_2}.$$

Observe that $\partial\eta_1 = \partial\eta$. Also, notice that in both cases, $k^{\eta_1} < k^\eta$. Hence the proof. $\square$

Proof of Proposition 4.13. Let H and H' be the semigroup associated to $R[\mathcal{I}(Q)]$ and $R[\mathcal{I}(P)]$ respectively. To prove the theorem, by Proposition 2.24 and Lemma 4.4, it suffices to show that for all $\varepsilon \in \{0, \ldots, 4\}$, if $h = (\varepsilon, 4-\varepsilon, h_2, h_1) = \sum_{i=1}^4 h_{\alpha_i} \in H$ then $\widetilde{H}_1(\Delta_h) = 0$. We prove the theorem in the following two cases: $\mathcal{I}(P_2') \nsubseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$ and $\mathcal{I}(P_2') \subseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$. In particular, if $n \geq 3$, then we always have $\mathcal{I}(P_2') \nsubseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$.

(a) Assume that $\mathcal{I}(P_2') \nsubseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$. If $P_2' \in \{\alpha_1^2, \ldots, \alpha_4^2\}$, then by Remark 4.19(1), $\Delta_h \cong \Delta_{\widetilde{h}}$, where $\widetilde{h}$ is as defined in Remark 4.19(1). Observe that $\widetilde{h} = (0, 4, \widetilde{h}')$, where $\widetilde{h}' \in H'$. We know that $\Delta_{\widetilde{h}} \cong \Delta_{\widetilde{h}'}$. By hypothesis, $\widetilde{H}_1(\Delta_{\widetilde{h}'}) = 0$. Therefore, $\widetilde{H}_1(\Delta_{\widetilde{h}}) = 0$. If $P_2' \notin \{\alpha_1^2, \ldots, \alpha_4^2\}$, then $h = (0, 4, h')$, where $h' \in H'$. Thus, $\widetilde{H}_1(\Delta_h) = 0$.

(*b*) Now we assume that $\mathcal{I}(P_2') \subseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$. We prove this case in two subcases $n = 1$ and $n = 2$. If $n = 1$, then by Remark 4.19(2), it is enough to consider the subcase $h = (1, 3, h_2, h_1)$ and there are exactly two α_i^2's with $\alpha_i^2 = P_2$. Let γ be an 1-cycle in Δ_h. By Proposition 4.16, there exists an 1-cycle γ' of $\Delta_{(0,4,h_2,h_1)}$ such that γ is homologous to γ' in X_h. By hypothesis, γ' is homologous to 0 in Δ_h. Thus, by Proposition 4.20, γ is homologous to 0 in Δ_h. This concludes the proof for $n = 1$. For $n = 2$, if $\mathcal{I}(P_2') \subseteq \{\alpha_1^2, \ldots, \alpha_4^2\}$, then $\mathcal{I}(P_2') = \{\alpha_1^2, \ldots, \alpha_4^2\}$. By the similar argument of the subcase $n = 1$ and Proposition 4.22, we are done in this subcase also. Hence the proof.

□

Chapter 5

h-polynomial of Hibi rings and polyominoes

In this chapter, we partially resolve two conjectures about h-polynomials. More precisely, we prove

1. The Charney-Davis conjecture for the Gorenstein toric K-algebras associated to simple thin polyominoes and for Gorenstein Hibi rings of regularity 4;

2. Rinaldo-Romeo's conjecture concerning characterization of thin polyominoes.

Recall the definitions of Hilbert Series and h-polynomial from Subsection 2.1.3. We start by giving a combinatorial description of h-polynomial of Hibi rings.

Let P be a naturally ordered poset, i.e., P is a poset on a underlying set $\{q_1, \dots, q_n\}$ and $q_i < q_j$ in P implies $i < j$ in $\mathbb{N}$. Let $L = \mathcal{I}(P)$ be a distributive lattice and let $R[L]$ be the Hibi ring associated to L. Assume that P is a poset on the set $[n]$. Let $\Delta(L)$ be the order complex of L and let $K[\Delta(L)]$ be the Stanley-Reisner ring of $\Delta(L)$. It follows from Lemma 6.2 (which is independent of this discussion) and Proposition 2.5 that the h-polynomials of $R[L]$ and of $K[\Delta(L)]$ are the same. We use the results of [BGS82] to relate the h-polynomial of $R[L]$ to the descents in the maximal chains of L.

Discussion 5.1. We follow the discussion of [BGS82, Section 1]. Let $\omega : P \to \{1, \dots, n\}$ be a (fixed) order-preserving map. Let $\mathcal{M}(L)$ be the set of maximal chains of L. Let $\mu \in \mathcal{M}(L)$. We write μ as a chain of order ideals of P: $\hat{0} = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = \hat{1}$. Then $|I_i \setminus I_{i-1}| = \{p_i\}$ for some $p_i \in P$. Define $\omega(\mu) = (\omega(p_1), \dots, \omega(p_n))$. For $1 \leq i \leq n-1$, we say that i is a *descent* of μ if $\omega(p_i) > \omega(p_{i+1})$. The *descent* *set* $\operatorname{Des}(\mu)$ of μ is $\{i \mid 1 \leq i \leq m+n-1,\ i \text{ is a descent of } \mu\}$. For $k \in \mathbb{N}$, define $\mathcal{M}_k(L) = \{\mu \in \mathcal{M}(L) : |\operatorname{Des}(\mu)| = k\}$. $\square$

Proposition 5.1. *Let $R[L]$ be the Hibi ring associated to $L = \mathcal{I}(P)$. Write $h(t) = 1 + h_1 t + h_2 t^2 + \cdots$ for the h-polynomial of $R[L]$. Then $h_i = |\mathcal{M}_i(L)|$.*

Proof. Use [BGS82, Theorems 4.1 and 1.1] with standard grading (i.e. setting $t_i = t$ for all i) to see that the h-polynomial of the Stanley Reisner ring of $\Delta(L)$ is

$$\sum_{i \in \mathbb{N}} |\mathcal{M}_i(L)| t^i.$$

The proposition now follows from Lemma 6.2 and Proposition 2.5. $\square$

5.1 Charney-Davis conjecture for Hibi rings

In this section, we state the Charney-Davis conjecture and we prove it for Gorenstein Hibi rings of regularity 4.

The Charney-Davis conjecture [CD95, Conjecture D] asserts that if $h(t)$ is the h-polynomial of a flag simplicial homology $(d-1)$-sphere, then $(-1)^{\lfloor \frac{d}{2} \rfloor} h(-1) \geq 0$. Stanley [Sta00, Problem 4] extended this conjecture to Gorenstein* flag simplicial complexes. Generalizing it further, Reiner and Welker [RW05, Question 4.4] posed the following:

Question 5.2. *Let K be a field and R a standard graded Gorenstein Koszul K-algebra. Write the Hilbert series of R as $h_R(t)/(1-t)^{\dim(R)}$. Is*

$$(-1)^{\left\lfloor \frac{\deg h_R(t)}{2} \right\rfloor} h_R(-1) \geq 0?$$

We say that a standard graded Gorenstein Koszul K-algebra R is *Charney-Davis (CD)* if it gives an affirmative answer to the above question.

Suppose that, in the notation of Question 5.2, $\deg h_R(t)$ is odd. Then $h_R(-1) = 0$; see, e.g., [BH93, Corollary 4.4.6]. Therefore Question 5.2 is open only when $\deg h_R(t)$ is even. See the bibliography of [RW05] and of [Sta00] for various classes of rings that are CD. Recently, D'Alì and Venturello [DV22] gave an example showing that answer to Question 5.2 is negative, in general.

Let $L = \mathcal{I}(P)$ be a distributive lattice and let $R[L]$ be the Hibi ring associated to L. When P is a antichain it follows from [Pet15, Theorem 4.1] that $R[L]$ is CD. Brändén [Brä06, Corollary 4.3] proved that all Gorenstein Hibi rings are CD. Here, we prove a special case that all Gorenstein Hibi rings of regularity 4 are CD. This work here is done independently of and without the knowledge of Brändén's work.

Recall that the Hibi ring $R[L]$ is Gorenstein if and only if P is pure. It was proved in [EHM15] that the regularity of Hibi rings have very nice combinatorial description, i.e., $\operatorname{reg}(R[L]) = \#P - \operatorname{rank}(P) - 1$. Since Hibi rings are Cohen-Macaulay, we have $\operatorname{reg}(R[L]) = \deg h(t)$.

From now onwards, we only consider pure poset P with $\operatorname{reg}(R[\mathcal{I}(P)]) = 4$. By Corollary 2.21, it suffice to prove the conjecture for simple posets.

Lemma 5.3. *Let P be a simple poset. Then* $\operatorname{rank}(P) \leq 3$.

Proof. By the formula of regularity, we have $\#P - \operatorname{rank}(P) = 5$. If P is simple and $\operatorname{rank}(P) > 3$, then $\#P - \operatorname{rank}(P) > 5$ because P is pure. Which is a contradiction. $\square$

Lemma 5.4. *Let $L = \mathcal{I}(P)$ be a distributive lattice and $R[L] = K[L]/I_L$ be the Hibi ring associated to L. Then the h-polynomial of $R[L]$ has the form $1 + ct + h_2t^2 + ct^3 + t^4$, where c is codimension of I_L and $h_2 = \frac{c+1}{2} - \mu(I_L)$.*

Proof. After applying the additivity property of the Hilbert series to the minimal resolution of $R[L]$, we get

$$H_{R[L]}(t) = \frac{\sum_{i=0}^{c}(-1)^i \sum_j \beta_{ij} t^j}{(1-t)^{\#L}}.$$

So,

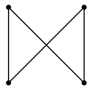
$$\frac{h_0 + h_1 t + h_2 t^2 + h_1 t^3 + h_0 t^4}{(1-t)^{\#P+1}} = \frac{\sum_{i=0}^{c}(-1)^i \sum_j \beta_{ij} t^j}{(1-t)^{\#L}}.$$

This implies,

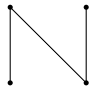
$$(1-t)^c (h_0 + h_1 t + h_2 t^2 + h_1 t^3 + h_0 t^4) = \sum_{i=0}^{c} (-1)^i \sum_j \beta_{ij} t^j.$$

After comparing the coefficients of constant terms, t and t^2 on both sides and using the fact that on RHS, the coefficients of constant term, t and t^2 are $1, 0$ and $\mu(I_L)$ respectively, we get the desired result. $\square$

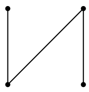
(a) $h(t) = 1 + 4t + t^2$



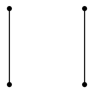
(b) $h(t) = 1 + 2t + t^2$



(c) $h(t) = 1 + 3t + t^2$



(d) $h(t) = 1 + 3t + t^2$



(e) $h(t) = 1 + 4t + t^2$

FIGURE 5.1

Lemma 5.5. *Let P be the ordinal sum of two pure posets P_1 and P_2. Assume that P is simple. If $R[\mathcal{I}(P_1)]$ and $R[\mathcal{I}(P_2)]$ are CD, then so is $R[\mathcal{I}(P)]$.*

Proof. Let $h(t)$ be the h-polynomial of $R[\mathcal{I}(P)]$ and $h_i(t)$ be the h-polynomial of $R[\mathcal{I}(P_i)]$ for $i = 1, 2$. By Lemma 2.19, we have $h(t) = h_1(t)h_2(t)$. We consider the following cases:

Case 1 Either $\deg(h_1(t)) = 3$ or $\deg(h_1(t)) = 1$. Then $h_1(-1) = 0$ since $\deg(h_1(t))$ is odd. Therefore $h(-1) = 0$.

Case 2 $\deg(h_i(t)) = 2$ for all $i = 1, 2$. All simple pure poset P' with $\operatorname{reg}(R[\mathcal{I}(P')]) = 2$ are listed in the Figure 5.1. For all such P', $R[\mathcal{I}(P')]$ is CD. Therefore, $h(-1) = h_1(-1)h_2(-1) \geq 0$.

$\square$

Let P be a pure poset of rank k. Let Q_i be the set of all height i elements of P for $0 \leq i \leq k$. Clearly, Q_i is an antichain of P for all i. Let a_i denotes the width of the antichain Q_i for all $0 \leq i \leq k$. If P is simple, then $a_i \geq 2$ for all i. Label the elements of Q_0 as $1, 2, \ldots, a_0$ and the elements Q_1 as $a_0 + 1, \ldots, a_0 + a_1$. Inductively, label the elements of Q_i as $(\sum_{j=0}^{i-1} a_j) + 1, \ldots, \sum_{j=0}^{i} a_j$ for $2 \leq i \leq k$.

Lemma 5.6. *Let P be a simple poset. Let P' be a pure poset obtained from P by omitting an edge between Q_0 and Q_1. If $R[\mathcal{I}(P)]$ is CD then so is $R[\mathcal{I}(P')]$.*

Proof. Let c and c' denote the codimensions of $R[\mathcal{I}(P)]$ and $R[\mathcal{I}(P')])$ respectively. Let h_2 and h_2' be the coefficients of t^2 in the h-polynomials of $R[\mathcal{I}(P)]$ and $R[\mathcal{I}(P')]$ respectively. Note that the number of order ideals of P' will be greater than or equal to that of P, i.e., $\#\mathcal{I}(P') \geq \#\mathcal{I}(P)$. Therefore, $c' \geq c$. Let $\Delta c = c' - c$ and $\Delta h_2 = h_2' - h_2$. Label the vertices of the omitted edge as a_0 and $a_0 + 1$, where $a_0 \in Q_0$ and $a_0 + 1 \in Q_1$.

To prove that $R[\mathcal{I}(P')]$ is CD, by Lemma 5.4 it suffice to show that $\Delta h_2 \geq 2\Delta c$. Equivalently, by Proposition 5.1, it is enough to show that if there are Δc new distinct 1-descents of P', then there will be at least $2\Delta c$ new distinct 2-descents of P'. Observe that $\operatorname{rank}(P) \leq 3$ by Lemma 5.3 and $\operatorname{rank}(P') \in \{1, \ldots, \operatorname{rank}(P)\}$. We prove the lemma individually for each possible rank of P'.

Case 1 If rank $P' = 1$, then possibly by replacing P with P^{∂}, it is enough to consider the two subcases $(a_0, a_1) = (4, 2)$ and $(a_0, a_1) = (3, 3)$.

When $(a_0, a_1) = (4, 2)$, the possible new 1-descents will be 123546, $\pi_1\pi_2 5\pi_3 46$ where π_1, π_2 and π_3 are permutations of $\{1, 2, 3\}$ with $\pi_1 < \pi_2$ and $\rho_1 5\rho_2\rho_3 46$ where ρ_1, ρ_2 and ρ_3 are permutations of $\{1, 2, 3\}$ with $\rho_2 < \rho_3$. Then 132546, 312546, 231546, $\pi_2\pi_1 5\pi_3 46$, $\pi_1\pi_2 54\pi_3 6$, $\rho_1 5\rho_3\rho_2 46$ and $\rho_1\rho_3 5\rho_2 46$ are some distinct new 2-descents of P', which is more than twice the number of new 1-descents of P'.

When $(a_0, a_1) = (3, 3)$, the possible new 1-descents will be 124356, 124536, $\pi_1 4\pi_2 356$ where π_1 and π_2 are permutations of $\{1, 2\}$ and $\rho_1 45\rho_2 36$ where ρ_1 and ρ_2 are permutations of $\{1, 2\}$. Then 214356, 124365, 214536,125436, $\pi_1 43\pi_2 56$, $\pi_1 4\pi_2 365$, $\rho_1 54\rho_2 36$ and $\rho_1 453\rho_2 6$ are some distinct new 2-descents of P', which is more than twice the number of new 1-descents of P'.

Case 2 If rank $P' = 2$, then possibly by replacing P with P^{∂}, it is enough to consider the two subcases $(a_0, a_1, a_2) = (3, 2, 2)$ and $(a_0, a_1, a_2) = (2, 3, 2)$.

When $(a_0, a_1, a_2) = (3, 2, 2)$, the possible new 1-descents will be 1243567 and $\pi_1 4\pi_2 3567$ where π_1 and π_2 are permutations of $\{1, 2\}$. Then 2143567, 1243576 $\pi_1 4\pi_2 3576$ and $\pi_1 43\pi_2 567$ are some distinct new 2-descents of P', which is more than twice the number of new 1-descents of P'.

When $(a_0, a_1, a_2) = (2, 3, 2)$, the possible new 1-descents will be 1324567, $13\pi_1 2\pi_2 67$ where π_1 and π_2 are permutations of $\{4, 5\}$. Then 1325467, 1324576, $13\pi_1 2\pi_2 76$ and $1\pi_1 32\pi_2 67$ are some distinct new 2-descents of P', which is more than twice the number of new 1-descents of P'.

Case 3 If rank $P' = 3$, then $(a_0, a_1, a_2, a_3) = (2, 2, 2, 2)$ is the only subcase. The possible new 1-descent will be 13245678. Then 13245687 and 13246578 are some distinct new 2-descents of P'.

$\square$

Theorem 5.7. *Let $L = \mathcal{I}(P)$ be a simple distributive lattice and $R[L]$ be the Hibi ring associated to L. Then $R[L]$ is CD.*

Proof. If $\text{rank}(P) = 0$, then the result follows from [Pet15, Theorem 4.1]. If $\text{rank}(P) \geq 1$, then the proof follows from Lemma 5.5 and Lemma 5.6. $\square$

5.2 Charney-Davis conjecture for simple thin polyominoes

From Section 2.5, recall the definition of polyomino algebra and how its algebraic features are largely dictated by the combinatorics and topology of the polyomino. For example, if $\mathcal{P}$ is simple then $K[\mathcal{P}]$ is a Koszul and the S-property of simple thin polyominoes characterises such polyominoes $\mathcal{P}$ for which $K[\mathcal{P}]$ is Gorenstein algebra. Moreover, if $\mathcal{P}$ is a simple thin polyomino, then $h_{K[\mathcal{P}]}(t) = r_{\mathcal{P}}(t)$, where $r_{\mathcal{P}}(t)$ is the rook polynomial of the polyomino $\mathcal{P}$.

We begin with an observation about how Hilbert series and rook polynomials behave in disjoint unions of polyominoes.

Note that if $\mathcal{P}_1, \ldots, \mathcal{P}_m$ are the connected components of $\mathcal{P}$, then $K[\mathcal{P}] \simeq K[\mathcal{P}_1] \otimes_K \cdots \otimes_K K[\mathcal{P}_m]$. Therefore $K[\mathcal{P}]$ Gorenstein (respectively, Koszul) if and only if $K[\mathcal{P}_i]$ is Gorenstein (respectively, Koszul) for each i.

Proposition 5.8. *Let $\mathcal{P}$ be a finite collection of cells. Write $\mathcal{P}_1, \ldots, \mathcal{P}_m$ for the connected components. Then:*

$$h_{K[\mathcal{P}]}(t) = \prod_{i=1}^{m} h_{K[\mathcal{P}_i]}(t) \text{ and } r_{\mathcal{P}}(t) = \prod_{i=1}^{m} r_{\mathcal{P}_i}(t).$$

In particular, if $\mathcal{P}_i$ is a simple thin polyomino for each i, then $h_{K[\mathcal{P}]}(t) = r_{\mathcal{P}}(t)$.

Proof. Vertices of the $\mathcal{P}_i$ are disjoint, so $K[\mathcal{P}] \simeq K[\mathcal{P}_1] \otimes_K \cdots \otimes_K K[\mathcal{P}_m]$. Hence $H_{K[\mathcal{P}]}(t) = \prod_{i=1}^{m} H_{K[\mathcal{P}_i]}(t)$, from which it follows that $h_{K[\mathcal{P}]}(t) = \prod_{i=1}^{m} h_{K[\mathcal{P}_i]}(t)$. Let $k \in \mathbb{N}$. Then k-rook configurations in $\mathcal{P}$ corresponds to independent choices of k_i-rook configurations in $\mathcal{P}_i$ for each $1 \leq i \leq m$ and for each tuple $(k_1, \ldots, k_m) \in \mathbb{N}^m$ with

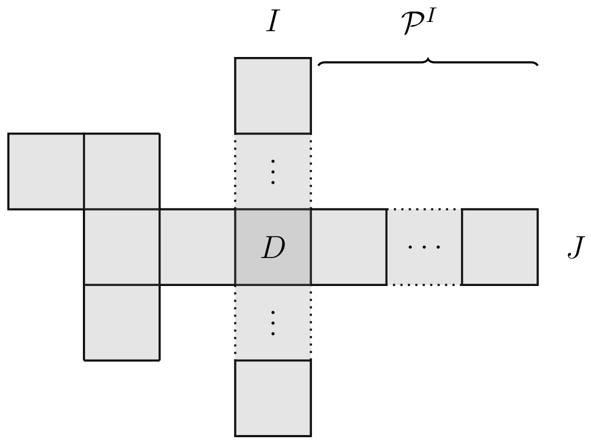


FIGURE 5.2: Collapse datum (*cf.* Definition 5.9)

$\sum_i k_i = k$. Hence $r_{\mathcal{P}}(t) = \prod_{i=1}^m r_{\mathcal{P}_i}(t)$. The final assertion now follows from noting that for each i, $h_{K[\mathcal{P}_i]}(t) = r_{\mathcal{P}_i}(t)$ since $\mathcal{P}_i$ is a simple thin polyomino [see Theorem 2.12]. $\square$

Let $\mathcal{P}$ be a simple thin polyomino. In [RR21, Definition 3.4], Rinaldo and Romeo introduced a notion of collapsing $\mathcal{P}$ in a maximal inner interval, and showed that if $\mathcal{P}$ has at least two maximal inner intervals, then there exists a maximal inner interval in which $\mathcal{P}$ is collapsible [RR21, Proposition 3.7]. We need a refinement of this result for simple thin polyominoes with the S-property, for which we rephrase [RR21, Definition 3.4] in a slightly different way.

Definition 5.9. *Let $\mathcal{P}$ be a simple thin polyomino. A* collapse datum *on $\mathcal{P}$ is a tuple $(I, J, \mathcal{P}^I)$, where I and J are maximal inner intervals and $\mathcal{P}^I$ is a sub-polyomino of $\mathcal{P}$ satisfying the following conditions:*

1. *J is the only maximal inner interval of $\mathcal{P}$ such that $I \cap J$ is a cell;*
2. *$\mathcal{P}^I \subseteq J$ and $I \cap J \not\subset \mathcal{P}^I$.*
3. *$\mathcal{P} \setminus (I \cup \mathcal{P}^I)$ is a non-empty sub-polyomino of $\mathcal{P}$.*

Figure 5.2 gives an example of a collapse datum. Note that since $\mathcal{P}^I$ is a sub-polyomino of $\mathcal{P}$ and $\mathcal{P}^I \subseteq J$, it is an inner interval if it is non-empty. When $\mathcal{P}$ has at least two maximal inner intervals, the maximal inner intervals I and J defined in the Definition 5.9 exist by [RR21, Lemma 3.6].

Discussion 5.10. Let $\mathcal{P}$ be a simple thin polyomino with S-property. Assume that $\mathcal{P}$ is not a cell. Then it has a collapse datum $(I, J, \mathcal{P}^I)$. Since, additionally, $\mathcal{P}$ has the

S-property, I has exactly two cells and $\mathcal{P}^I$ is either empty or a cell. Write $I = \{C, D\}$ with C denoting the single cell of I and $\{D\} = I \cap J$. Denote the single cell of J by E. Let $C_1, \ldots, C_k$ be the cells of J different from D and E. For $1 \leq i \leq k$, let B_i be the cell in $\mathcal{P}$ such that $B_i \notin J$ and C_i is a neighbour cell of B_i; such a B_i must exist, since C_i is not a single cell. We now consider the various cases.

- $\mathcal{P}^I \neq \varnothing$. Equivalently, $\mathcal{P}^I = \{E\}$. Then we may assume that C_k is an end-cell of J and for all $i \in \{1, \ldots, k-1\}$, C_i and C_{i+1} are neighbours. So B_i and B_{i+1} can not be neighbours, since $\mathcal{P}$ is thin. Hence for all $i \in \{1, \ldots, k\}$ whether B_i is above C_i or is below C_i determined by whether i is even or odd. Therefore in the neighbourhood of J, $\mathcal{P}$ is as shown in Figure 5.3.

- $\mathcal{P}^I = \varnothing$, C_k is an end-cell of J and E is a neighbour cell of C_k. We may assume that for all $i \in \{1, \ldots, k-2\}$, C_i and C_{i+1} are neighbours, so B_i and B_{i+1} cannot be neighbours. Therefore, using the same considerations as in the above case, we see that $\mathcal{P}$ is as shown in Figure 5.4 in the neighbourhood of J.

- $\mathcal{P}^I = \varnothing$ and E is an end-cell of J. Then, in the neighbourhood of J, $\mathcal{P}$ is as shown in Figure 5.5.

- $\mathcal{P}^I = \varnothing$, C_k is an end-cell of J and E is not a neighbour cell of C_k. Then, in the neighbourhood of J, $\mathcal{P}$ is one of the figures in Figure 5.6.

□

Discussion 5.11. By the *first end-cell* of J, we mean

$$\begin{cases} E, & \text{if } \mathcal{P}^I = \{E\}; \\ I \cap J, & \text{if } \mathcal{P}^I = \varnothing. \end{cases}$$

(Note that in both of the above cases, the cell in question is an end-cell of J.) We call the other end-cell of J the *second end-cell* of J. If E is the second end-cell of J, then E has exactly one neighbour cell. If C_k is the second end-cell of J and E is not a neighbour cell of C_k, then C_k has exactly two neighbour cells. If C_k is the second end-cell of J and E is a neighbour cell of C_k, then C_k has two or three neighbour cells; see Figures 5.3, 5.4, 5.5 and 5.6. □

The next lemma shows that simple thin polyominoes with the S-property have a special collapse datum. See Figure 5.7 for an example of a simple thin polyomino $\mathcal{P}$

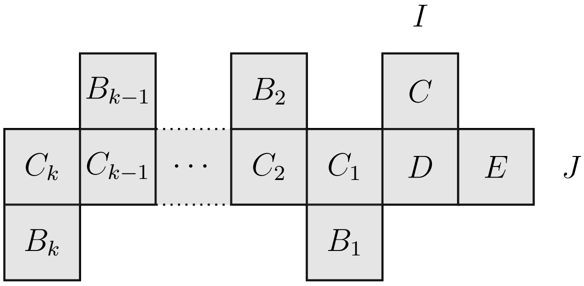


FIGURE 5.3: $\mathcal{P}^I$ non-empty

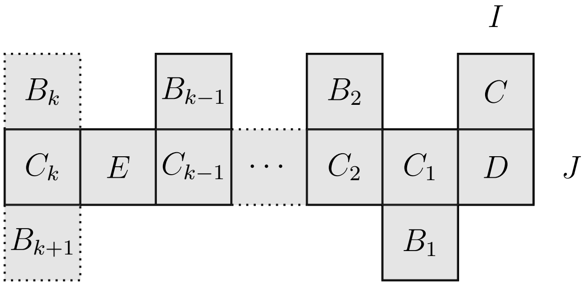


FIGURE 5.4: $\mathcal{P}^I$ is empty, E is a neighbour of C_k but not an end-cell of J

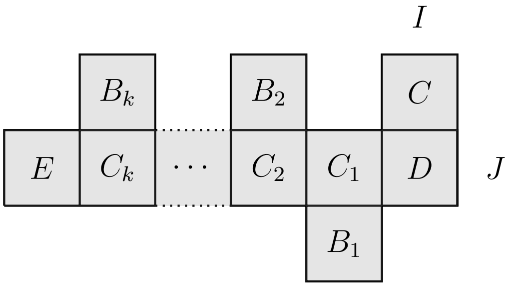


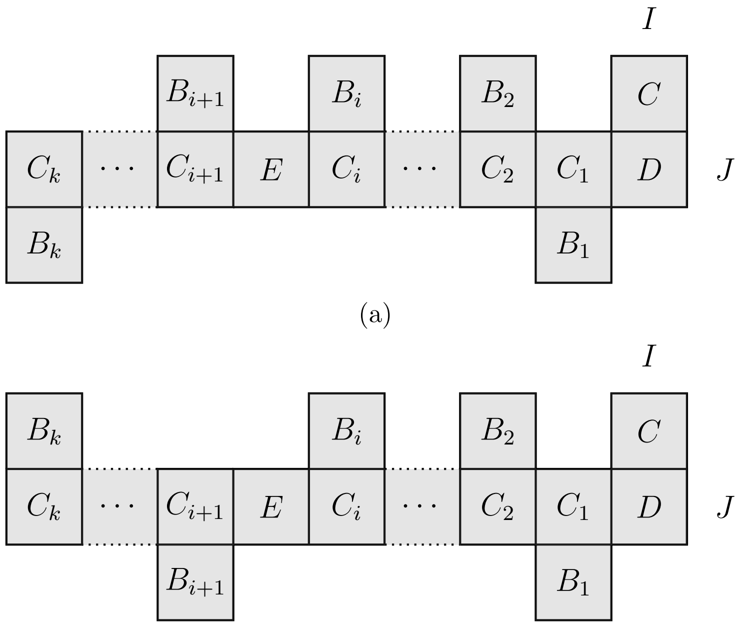
FIGURE 5.5: $\mathcal{P}^I$ is empty, E is an end-cell of J

that does not have any collapse datum $(I, J, \mathcal{P}^I)$ in which the second end-cell of J has two or fewer neighbours.

Lemma 5.12. *Let $\mathcal{P}$ be a simple thin polyomino with S-property. Assume that $\mathcal{P}$ is not a cell. Then there exists a collapse datum $(I, J, \mathcal{P}^I)$ of $\mathcal{P}$ such that one of the following holds:*

1. *The second end-cell of J has at most two neighbour cells.*
2. *If the second end-cell of J has three neighbour cells, then one of its neighbour cells is both a single cell and an end-cell of the maximal inner interval containing it.*

Proof. By way of contradiction, suppose that there exists a simple thin polyomino $\mathcal{P}$ with the S-property for which there does not exist a collapse datum satisfying (1) or (2).



(a)

(b)

FIGURE 5.6: $\mathcal{P}^I$ is empty, E is not a neighbour of C_k

We may assume that $\mathcal{P}$ has the least number of cells, among the polyominoes for which the assertion does not hold.

Let $(I, J, \mathcal{P}^I)$ be a collapse datum of $\mathcal{P}$. If the neighbourhood of J in $\mathcal{P}$ looks like the ones given in Figures 5.3, 5.5 or 5.6, then (1) holds. Therefore we are in the situation of Figure 5.4. Let E be the single cell of J, and C_k the second end-cell of J. We may assume that B_k and B_{k+1} as marked in Figure 5.4 exist, for otherwise (1) would hold.

We may assume that either B_k is not a single cell or it is not an end-cell of the maximal inner interval that contains $\{B_k, C_k, B_{k+1}\}$; for, otherwise, (2) would hold. Similarly for B_{k+1}. Let

$$\mathcal{P}' = (\mathcal{P} \setminus \{A \in \mathcal{P} : \text{the (unique) path between } A \text{ and } E \text{ does not contain } C_k\}) \cup \{E\}.$$

(E.g., in Figure 5.4, $\mathcal{P}'$ is the sub-polyomino comprising E and the cells reachable from E through C_k.) Observe that $\mathcal{P}'$ is a simple thin polyomino. We first show that $\mathcal{P}'$ has the S-property. Let L be a maximal inner interval of $\mathcal{P}'$. Then L is a maximal inner interval of $\mathcal{P}$ or $L = \{C_k, E\}$. In both cases, L has a unique single cell. Thus, $\mathcal{P}'$ has the S-property. Also note that $\mathcal{P}'$ is not a cell. The number of cells in $\mathcal{P}'$ are strictly less than the number of cells in $\mathcal{P}$. Hence $\mathcal{P}'$ has a collapse datum $(I', J', \mathcal{P}'^{I'})$ satisfying the assertions of the lemma.

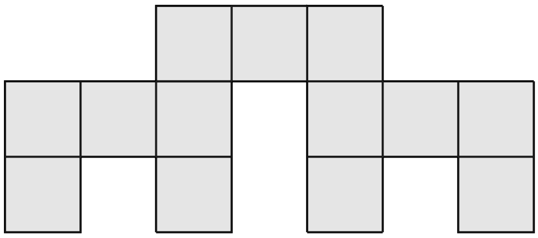


FIGURE 5.7

Note that $I' \neq \{C_k, E\}$ and $J' \neq \{C_k, E\}$; therefore I' and J' are maximal inner intervals of $\mathcal{P}$.

Let J_1 be a maximal inner interval of $\mathcal{P}$ such that $J_1 \cap I'$ is a cell. Since $\mathcal{P}$ is simple, $J_1 \subset \mathcal{P}'$, so J_1 is a maximal inner interval of $\mathcal{P}'$. Hence $J_1 = J'$. Note that $\mathcal{P}'^{I'} \subseteq J'$ and that $I' \cap J' \not\subset \mathcal{P}'^{I'}$. Moreover, since $J' \neq \{C_k, E\}$, it follows that $\{C_k, E\} \subseteq \mathcal{P}' \setminus (I' \cup \mathcal{P}'^{I'})$; hence $\mathcal{P} \setminus (I' \cup \mathcal{P}'^{I'})$ which equals

$$(\mathcal{P}' \setminus (I' \cup \mathcal{P}'^{I'})) \cup \{A \in \mathcal{P} : \text{the path between } A \text{ and } E \text{ does not contain } C_k\}$$

is a non-empty sub-polyomino of $\mathcal{P}$. Hence $(I', J', \mathcal{P}'^{I'})$ is a collapse datum of $\mathcal{P}$ that satisfies the assertion of the lemma for $\mathcal{P}$. This contradicts the assumption on $\mathcal{P}$, and completes the proof of the lemma. $\square$

Discussion 5.13. Let $\mathcal{P}$ be a simple thin polyomino and C be a single cell in $\mathcal{P}$. Let $r_{\mathcal{P},C}(t)$ be the polynomial $\sum_{k\in\mathbb{N}} r_k t^k$, where r_k is the number of k-rook configurations in $\mathcal{P}$ that have a rook at C. Let $r_{\mathcal{P},\widehat{C}}(t)$ be the polynomial $\sum_{k\in\mathbb{N}} r_k t^k$, where r_k is the number of k-rook configurations in $\mathcal{P}$ that have no rook at C. Then,

$$r_{\mathcal{P}}(t) = r_{\mathcal{P},\widehat{C}}(t) + r_{\mathcal{P},C}(t).$$

Let I be the maximal inner interval of $\mathcal{P}$ such that $C \in I$. Let $r_{\mathcal{P},\widehat{I}}(t)$ be the polynomial $\sum_{k\in\mathbb{N}} r_k t^k$, where r_k is the number of k-rook configurations in $\mathcal{P}$ that has no rook at any cell of I. Note that $r_{\mathcal{P},C}(t) = r_{\mathcal{P},\widehat{I}}(t)t$. Hence,

$$r_{\mathcal{P}}(t) = r_{\mathcal{P},\widehat{C}}(t) + r_{\mathcal{P},\widehat{I}}(t)t. \tag{5.1}$$

$\square$

Example 5.14. We illustrate the above definitions now. Let $\mathcal{P}$ be the polyomino as shown in the Figure 5.8. Note that C is a single cell in $\mathcal{P}$. The polynomials $r_{\mathcal{P},\widehat{C}}(t)$ and $r_{\mathcal{P},C}(t)$ are calculated in Table 5.1. The unique maximal inner interval I containing C

k	k-rook configurations that have a rook at C	number	k-rook configurations that do not have a rook at C	number
0	There are no 0-rook configurations that have a rook at C	0	$\varnothing$	1
1	$\{C\}$	1	$\{A\}, \{B\}$	2
2	$\{C, A\}$	1	none	0
$k \geq 3$	none	0	none	0
	$r_{P,C}(t)$	$t + t^2$	$r_{P,\widehat{C}}(t)$	$1 + 2t$

TABLE 5.1: Calculation of $r_{P,C}(t)$ and $r_{P,\widehat{C}}(t)$

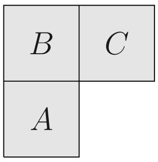


FIGURE 5.8

is $\{B, C\}$. Hence $r_{\mathcal{P},\widehat{I}}(t) = 1 + t$, since this is the rook polynomial of the polyomino consisting of just the cell A. On the other hand, the number of k-rook configurations in $\mathcal{P}$ for $k = 0, 1, 2$ are, respectively, $1, 3, 1$; hence $r_{\mathcal{P}}(t) = 1 + 3t + t^2$. We thus see that

$$r_{\mathcal{P}}(t) = r_{\mathcal{P},\widehat{C}}(t) + r_{\mathcal{P},C}(t) = r_{\mathcal{P},\widehat{C}}(t) + r_{\mathcal{P},\widehat{I}}(t)t. \qquad \square$$

We now wish to express $r_{\mathcal{P},\widehat{C}}(t)$ and $r_{\mathcal{P},\widehat{I}}(t)$ as the rook polynomials of polyominoes when $\mathcal{P}$ has the S-property.

Discussion 5.15. Let $\mathcal{P}$ be a simple thin polyomino that has the S-property. Let $(I, J, \mathcal{P}^I)$ be a collapse datum of $\mathcal{P}$ satisfying the conclusion of Lemma 5.12. Let C and D be the cells of I, with C being the single cell. Let E be the single cell of J. Let $r_{\mathcal{P},\widehat{C}}(t)$ and $r_{\mathcal{P},\widehat{I}}(t)$ be as defined in Discussion 5.13.

Write $\mathcal{Q} = \mathcal{P} \setminus \{C\}$. Then $r_{\mathcal{P},\widehat{C}}(t) = r_{\mathcal{Q}}(t)$. If $\mathcal{P}^I$ is empty, define $\mathcal{R}$ to be the polyomino $\mathcal{P} \setminus I$. Otherwise, i.e. if $\mathcal{P}^I$ is a cell E, define $\mathcal{R}$ to be the polyomino $\mathcal{P} \setminus \{C, E\}$. Then $r_{\mathcal{P},\widehat{I}}(t) = r_{\mathcal{R}}(t)$. Thus (5.1) becomes

$$r_{\mathcal{P}}(t) = r_{\mathcal{Q}}(t) + r_{\mathcal{R}}(t)t. \tag{5.2}$$

$\square$

Note that $\mathcal{Q}$ does not have the S-property, so we cannot use an inductive argument to prove Theorem 5.17 directly. Hence we need to rewrite $r_{\mathcal{Q}}(t)$ in terms of smaller polyominoes. To this end, we observe that D is a single cell of the maximal inner interval J inside $\mathcal{Q}$. Therefore, by (5.1),

$$r_{\mathcal{Q}}(t) = r_{\mathcal{Q},\widehat{D}}(t) + r_{\mathcal{Q},\widehat{J}}(t)t. \tag{5.3}$$

We note that

$$r_{\mathcal{Q},\widehat{D}}(t) = r_{\mathcal{R}}(t). \tag{5.4}$$

We now find an expression for $r_{\mathcal{Q},\widehat{J}}(t)$. Let $E, C_1, \dots, C_k$ be the other cells of J in $\mathcal{P}$. E denotes the single cell of J in $\mathcal{P}$. For $1 \leq i \leq k-1$, let B_i be the cell in $\mathcal{P}$ such that $B_i \notin J$ and C_i is a neighbour cell of B_i. (See Discussion 5.10 and Figures 5.3, 5.4, 5.5, and 5.6 for notational conventions.) When E is the second end-cell or C_k is an end-cell with two neighbour cells, let B_k be the cell in $\mathcal{P}$ such that $B_k \notin J$ and C_k is a neighbour cell of B_k. When C_k is an end-cell with three neighbour cells, let B_k and B_{k+1} be the cells in $\mathcal{P}$ such that $B_k, B_{k+1} \notin J$ and C_k is a neighbour cell of B_k and B_{k+1}. In the case when C_k has three neighbour cells, by Lemma 5.12, we may assume that B_{k+1} is both a single cell and an end-cell of the maximal inner interval containing it.

Now for all $1 \leq i \leq k-1$, define

$$\mathcal{Q}_i := \{A \in \mathcal{Q} : \text{the path between } A \text{ and } B_i \text{ does not contain } C_i\}.$$

Also, define

$$\widetilde{\mathcal{Q}_k} := \{A \in \mathcal{Q} : \text{the path between } A \text{ and } B_k \text{ does not contain } C_k\}.$$

When E is the second end-cell or C_k is an end-cell with two neighbour cells, define $\mathcal{Q}_k = \widetilde{\mathcal{Q}_k}$. When C_k is an end-cell with three neighbour cells, let $\{a, b, a', b'\}$ be the vertices of C_k where $a, b \in V(B_k)$ and $a', b' \in V(B_{k+1})$. We define $\mathcal{Q}_k$ as the polyomino obtained from $\widetilde{\mathcal{Q}_k} \cup \{B_{k+1}\}$ by the identification of the vertices a and b of $V(B_k)$ with the vertices a' and b' of $V(B_{k+1})$, respectively, by translating the cell B_{k+1}.

Lemma 5.16. *With notation as above, we have the following:*

1. *$\mathcal{Q}_1, \dots, \mathcal{Q}_k$ are precisely the connected components of $\mathcal{Q} \setminus J$.*

2. *For each $1 \leq i \leq k$, $\mathcal{Q}_i$ is a simple thin polyomino with the S-property.*

3. $r_{\mathcal{Q},\widehat{J}}(t) = \prod_{i=1}^{k} r_{\mathcal{Q}_i}(t)$ *and* $\sum_{i=1}^{k} r(\mathcal{Q}_i) = r(\mathcal{P}) - 2$.

Proof. (1): Each $\mathcal{Q}_i$ is connected, $\mathcal{Q}_i \cap \mathcal{Q}_j = \varnothing$ for all $i \neq j$ (since P is simple) and $\mathcal{Q} \setminus J = \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_k$.

(2): Since $\mathcal{Q}$ is simple thin, so is $\mathcal{Q}_i$. Let L be a maximal inner interval of $\mathcal{Q}_i$. Then, either L is a maximal inner interval of $\mathcal{P}$ or $L = L' \setminus \{C_i\}$, where L' is a maximal inner interval of $\mathcal{P}$. In both cases, L has a unique single cell. Hence $\mathcal{Q}_i$ has S-property.

(3): By (1) $r_{\mathcal{Q},\widehat{J}}(t)$ is the rook polynomial of $\mathcal{Q} \setminus J$. Thus, by Proposition 5.8, $r_{\mathcal{Q},\widehat{J}}(t) = \prod_{i=1}^{k} r_{\mathcal{Q}_i}(t)$. For any k-rook configuration α of $\mathcal{Q} \setminus J$, we note that $\alpha \cup \{C, E\}$ is a $(k+2)$-rook configuration of $\mathcal{P}$. Hence $\sum_{i=1}^{k} r(\mathcal{Q}_i) \leq r(\mathcal{P}) - 2$. On the other hand, let β be a $r(\mathcal{P})$-rook configuration of $\mathcal{P}$. Since $\mathcal{P}$ has S-property, β is the only $r(\mathcal{P})$-rook configuration of $\mathcal{P}$ and β is the collection of all single cells of $\mathcal{P}$. Then, $\beta \setminus \{C, E\}$ is a rook configuration of $\mathcal{Q} \setminus J$. Therefore $\sum_{i=1}^{k} r(\mathcal{Q}_i) \geq r(\mathcal{P}) - 2$. □

We are now ready to state and prove our main theorem.

Theorem 5.17. *Let $\mathcal{P}$ be a collection of cells such that its connected components are simple thin polyominoes with the S-property. Then $K[\mathcal{P}]$ is CD.*

Proof. By Proposition 5.8, we may assume that $\mathcal{P}$ is a simple thin polyomino with the S-property. Let $h_{K[\mathcal{P}]}/(1-t)^{\dim(K[\mathcal{P}])}$ be the Hilbert series of $K[\mathcal{P}]$. By Theorem 2.12, $h_{K[\mathcal{P}]}(t) = r_{\mathcal{P}}(t)$. We proceed by induction on the rook number $r(\mathcal{P})$. If $r(\mathcal{P})$ is odd (in particular if $r(\mathcal{P}) = 1$), then $r_{\mathcal{P}}(-1) = 0$. Hence we may assume that $r(\mathcal{P})$ is even. Let $(I, J, \mathcal{P}^I)$ be a collapse datum of $\mathcal{P}$ satisfying the conclusion of Lemma 5.12. Apply Discussion 5.15, adopting its notation. Let $\mathcal{Q}$ and $\mathcal{R}$ be as in Discussion 5.15. Then, by (5.2)

$$r_{\mathcal{P}}(t) = r_{\mathcal{Q}}(t) + r_{\mathcal{R}}(t)t.$$

Now apply Discussion 5.13 to the single cell D of the maximal inner interval J of $\mathcal{Q}$. By (5.3), (5.4) and Lemma 5.16, we see that

$$r_{\mathcal{P}}(t) = (1+t)r_{\mathcal{R}}(t) + t\prod_{i=1}^{k} r_{\mathcal{Q}_i}(t).$$

By Lemma 5.16 and induction hypothesis, $K[\mathcal{Q}_i]$ is CD for all $1 \leq i \leq k$. If $r(\mathcal{Q}_i)$ is odd for some i, then $r_{\mathcal{P}}(-1) = 0$. Therefore we may assume that $r(\mathcal{Q}_i)$ is even for all i.

$$\begin{aligned}
(-1)^{\lfloor \frac{r(\mathcal{P})}{2} \rfloor} r_{\mathcal{P}}(-1) &= (-1)^{\frac{r(\mathcal{P})}{2}+1} \prod_{i=1}^{k} r_{\mathcal{Q}_i}(-1) \\
&= (-1)^{\frac{r(\mathcal{P})-2}{2}} \prod_{i=1}^{k} r_{\mathcal{Q}_i}(-1) \\
&= \prod_{i=1}^{k} (-1)^{\frac{r(\mathcal{Q}_i)}{2}} r_{\mathcal{Q}_i}(-1) && \text{by Lemma 5.16} \\
&\geq 0 && \text{by induction.}
\end{aligned}$$

This completes the proof of the theorem. □

5.3 Rinaldo and Romeo's conjecture

As stated in Theorem 2.12, Rinaldo and Romeo showed that if $\mathcal{P}$ is a simple thin polyomino, then $h_{K[\mathcal{P}]}(t) = r_{\mathcal{P}}(t)$, where $h_{K[\mathcal{P}]}(t)$ is the h-polynomial of $K[\mathcal{P}]$ and $r_{\mathcal{P}}(t)$ is the rook polynomial of the polyomino $\mathcal{P}$. They conjectured the following

Conjecture 5.18. *[RR21, Conjecture 4.5] Let $\mathcal{P}$ be a polyomino. Then $\mathcal{P}$ is thin if and only if $h_{K[\mathcal{P}]}(t) = r_{\mathcal{P}}(t)$.*

Recently, the conjecture is confirmed for a class of closed path polyominoes [CNU22, Theorem 5.5]. A closed path polyomino is a non-simple thin polyomino. In this section, we partially confirm this conjectured characterization and prove the following

Theorem 5.19. *Let $\mathcal{P}$ be a convex polyomino such that its vertex set $V(\mathcal{P})$ is a sublattice of $\mathbb{N}^2$. If $\mathcal{P}$ is not thin, then $h_2 < r_2$. In particular $h_{K[\mathcal{P}]}(t) \neq r_{\mathcal{P}}(t)$.*

The proof proceeds as follows: we first observe that $K[\mathcal{P}]$ is the Hibi ring of the distributive lattice $V(\mathcal{P})$. We then use Proposition 5.1 to relate the h-polynomial to descents in maximal chains of $V(\mathcal{P})$, and find an injective map from the set of maximal chains of $V(\mathcal{P})$ to the rook configurations in $\mathcal{P}$, to conclude that $h_k \leq r_k$ in general. We then show that if $\mathcal{P}$ is not thin, this map is not surjective to show that $h_2 < r_2$. We now set up some notations.

Setup 5.5. Let $\mathcal{P}$ be a convex polyomino such that $V(\mathcal{P})$ is a sublattice of $\mathbb{N}^2$. Let $\operatorname{JI}(\mathcal{P})$ be the poset of join-irreducible elements of $V(\mathcal{P})$. After a suitable translation, if necessary, we assume that $(0,0)$ and (m,n) are the elements $\hat{0}$ and $\hat{1}$ of $V(\mathcal{P})$. Hence $|\operatorname{JI}(\mathcal{P})| = m+n$.

When $\mathcal{P}$ is as in Setup 5.5, the polyomino ring $K[\mathcal{P}]$ is the Hibi ring $K[V(\mathcal{P})]$. Hence we are interested in the h-polynomial of the Hibi ring of a distributive lattice.

Discussion 5.20. We continue the Discussion 5.1 for $V(\mathcal{P})$ and $\operatorname{JI}(\mathcal{P})$. Let $\omega : \operatorname{JI}(\mathcal{P}) \to \{1,\dots,m+n\}$ be a (fixed) order-preserving map. Let $\mathcal{M}(\mathcal{P})$ be the set of maximal chains of $V(\mathcal{P})$. Let $\mu \in \mathcal{M}(\mathcal{P})$. We think of μ as a lattice path from $(0,0)$ to (m,n) consisting of horizontal and vertical edges. Label the vertices of μ as $(0,0) = \mu_0, \mu_1, \dots, \mu_{m+n} = (m,n)$, with $\mu_i - \mu_{i-1}$ a unit vector (when we think of these as elements of $\mathbb{R}^2$) pointing to the right or upwards. Then, if $i \in \operatorname{Des}(\mu)$, then the direction of μ changes at μ_i, i.e, the vectors $\mu_i - \mu_{i-1}$ and $\mu_{i+1} - \mu_i$ are perpendicular to each other. Hence μ_{i-1} and μ_{i+1} are the bottom-left and top-right vertices of a cell (the cell $C(\mu_{i+1})$ in our notation, see Section 2.5) of $\mathcal{P}$. Thus we get a function

$$\psi : \mathcal{M}(\mathcal{P}) \to \operatorname{Pow}(C(\mathcal{P})), \qquad \mu \mapsto \{C(\mu_{i+1}) \in C(\mathcal{P}) \mid i \in \operatorname{Des}(\mu)\}. \tag{5.6}$$

□

Discussion 5.21. Let $\mathcal{P}$ be as in Setup 5.5. Left-boundary vertices and bottom-boundary vertices are join-irreducible. Let $p \in V(\mathcal{P})$; assume that p is not a left-boundary vertex or a bottom-boundary vertex. If $p \notin \partial X$ then it is the top-right vertex of a cell in $\mathcal{P}$, and hence is not join-irreducible. If $p \in \partial X$ then p is the bottom-left vertex of the unique cell containing it (i.e., the bottom element $\hat{0}$ of $V(\mathcal{P})$) or the top-right vertex of the unique cell containing it (i.e., the top element $\hat{1}$ of $V(\mathcal{P})$); hence $p \notin \operatorname{JI}(\mathcal{P})$. Thus we have established that $\operatorname{JI}(\mathcal{P})$ is the union of the set of the left-boundary vertices and of the set of the bottom-boundary vertices. The sets of the left-boundary vertices and of the bottom-boundary vertices are totally ordered in $V(\mathcal{P})$. Therefore if (p, p') is a pair of incomparable elements of $\operatorname{JI}(\mathcal{P})$, then one of them is a left-boundary vertex and the other is a bottom-boundary vertex. □

Proposition 5.22. *Let $\mu \in \mathcal{M}(\mathcal{P})$ and $i \in \operatorname{Des}(\mu)$. Write μ as a chain of order ideals $\hat{0} = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{m+n} = \hat{1}$ and $|I_i \setminus I_{i-1}| = \{p_i\}$ with $p_i \in \operatorname{JI}(\mathcal{P})$. Then*

1. *p_i and p_{i+1} are incomparable;*

2. *$i+1 \notin \operatorname{Des}(\mu)$.*

Proof. (1): Assume, by way of contradiction, that they are comparable. Then $p_i < p_{i+1}$. Hence $\omega(p_i) < \omega(p_{i+1})$, contradicting the hypothesis that $i \in \mathrm{Des}(\mu)$.

(2): By way of contradiction, assume that $i+1 \in \mathrm{Des}(\mu)$. Then, by (1), p_{i+1} and p_{i+2} are incomparable. We see from Discussion 5.21 and the definition of the p_i that $p_i < p_{i+2}$. Therefore $\omega(p_i) < \omega(p_{i+2})$ contradicting the hypothesis that $\omega(p_i) > \omega(p_{i+1}) > \omega(p_{i+2})$. □

Proposition 5.23. *The function ψ of (5.6) is injective.*

Proof. Let $\mu, \nu \in \mathcal{M}(\mathcal{P})$ be such that $\psi(\mu) = \psi(\nu)$. As earlier, write μ and ν as chains of order ideals of $\mathrm{JI}(\mathcal{P})$:

$$\mu : \hat{0} = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{m+n} = \hat{1};$$
$$\nu : \hat{0} = I'_0 \subsetneq I'_1 \subsetneq \cdots \subsetneq I'_{m+n} = \hat{1}.$$

For $1 \le i \le m+n$, write $I_i \setminus I_{i-1} = \{p_i\}$ and $I'_i \setminus I'_{i-1} = \{p'_i\}$ with $p_i, p'_i \in \mathrm{JI}(\mathcal{P})$. We will prove by induction on i that $I_i = I'_i$ for all $0 \le i \le m+n$. Since $I_0 = I'_0$, we may assume that $i > 0$ and that $I_j = I'_j$ for all $j < i$.

Assume, by way of contradiction, that $I_i \neq I'_i$. Then I_{i-1} (which equals I'_{i-1}) is the bottom-left vertex of a cell C. Without loss of generality, we may assume that I_i is the top-left vertex of C and that I'_i is the bottom-right vertex of C. (In other words, μ goes up and ν goes to the right from I_{i-1}, or equivalently, p_i is a left-boundary vertex and p'_i is a bottom-boundary vertex.)

Let

$$i_1 = \min\{j > i : p'_i \in I_j\} - 1;$$
$$i_2 = \min\{j > i : p_i \in I'_j\} - 1.$$

Then the edge (I_{i_1-1}, I_{i_1}) is vertical while (I_{i_1}, I_{i_1+1}) is horizontal; this is the first time μ turns horizontal after I_{i-1}. Let C_1 be the cell with I_{i_1-1}, I_{i_1} and I_{i_1+1} as the bottom-left, the top-left and the top-right vertices respectively. Similarly the edge (I'_{i_2-1}, I'_{i_2}) is vertical while (I'_{i_2}, I'_{i_2+1}) is horizontal; this is the first time ν turns vertical after I'_{i-1}. Let C_2 be the cell with I'_{i_2-1}, I'_{i_2} and I'_{i_2+1} as the bottom-left, the bottom-right and the top-right vertices respectively. (The possibility that $C_1 = C$ or $C_2 = C$ has not been ruled out.) See Figure 5.9 for a schematic showing the cells C, C_1 and C_2 and the chains μ and ν.

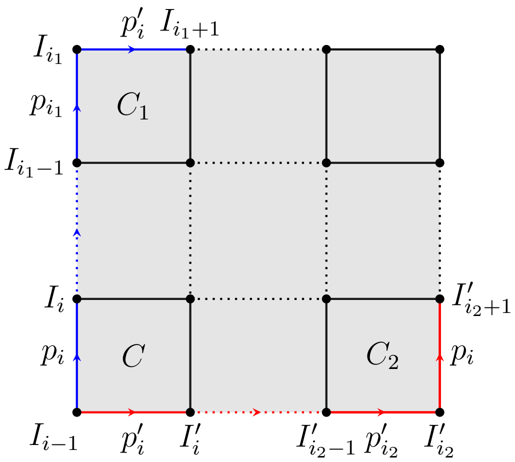


FIGURE 5.9: C, C_1, C_2, μ (blue) and ν (red) from the proof of Proposition 5.23.

We now prove a sequence of statements from which the proposition follows.

1. If $C_1 \notin \psi(\mu)$, then $C_2 \in \psi(\nu)$. Proof: Note that $p_{i_1+1} = p_i'$ and $p'_{i_2+1} = p_i$. Since $C_1 \notin \psi(\mu)$, we see that

$$\omega(p_i') = \omega(p_{i_1+1}) > \omega(p_{i_1}) \geq \omega(p_i),$$

where the last inequality follows from noting that $p_i < \cdots < p_{i_1}$ since they are left-boundary vertices. Therefore, in the chain ν, we have

$$\omega(p'_{i_2}) \geq \omega(p_i') > \omega(p_i) = \omega(p'_{i_2+1}),$$

i.e., $i_2 \in \mathrm{Des}(\nu)$. Hence $C_2 \in \psi(\nu)$.

2. If $C_2 \notin \psi(\nu)$, then $C_1 \in \psi(\mu)$. Immediate from (1).

3. If $C_1 \neq C$ then $C \notin \psi(\mu)$ and $C_1 \notin \psi(\nu)$. Proof: Note that μ does not pass through the top-right vertex of C and that ν does not pass through the bottom-left vertex of C_1.

4. If $C_2 \neq C$ then $C \notin \psi(\nu)$ and $C_2 \notin \psi(\mu)$. Proof: Note that ν does not pass through the top-right vertex of C and that μ does not pass through the bottom-left vertex of C_1.

5. If $C_1 \neq C$, then $\psi(\mu) \neq \psi(\nu)$. Proof: If $C_1 \in \psi(\mu)$, use (3) to see that

$$C_1 \in \psi(\mu) \setminus \psi(\nu).$$

Now assume that $C_1 \notin \psi(\mu)$. Then $C_2 \in \psi(\nu)$ by (1). If $C_2 = C$, then $C_2 \notin \psi(\mu)$ by (3); otherwise, $C_2 \notin \psi(\mu)$ by (4).

6. If $C_2 \neq C$, then $\psi(\mu) \neq \psi(\nu)$. Proof: If $C_2 \in \psi(\nu)$, use (4) to see that

$$C_2 \in \psi(\nu) \setminus \psi(\mu).$$

Now assume that $C_2 \notin \psi(\nu)$. Then $C_1 \in \psi(\mu)$ by (2). If $C_1 = C$, then $C_1 \notin \psi(\nu)$ by (4); otherwise, $C_1 \notin \psi(\nu)$ by (3).

7. C belongs to at most one of $\psi(\mu)$ and $\psi(\nu)$. Proof: Suppose that $C \in \psi(\mu)$. Then $i_1 = i + 1$, $p_{i_1} = p'_i$ and $\omega(p_i) > \omega(p'_i)$. For C to belong to $\psi(\nu)$, we need that $I'_{i+1} = I_{i+1}$ (i.e., μ and ν are the same up to $i + 1$, except at i); for this to hold, it is necessary that $p'_{i+1} = p_i$, but then $i \notin \mathrm{Des}(\nu)$. The other case is proved similarly.

8. If $C_1 = C_2 = C$ then $\psi(\mu) \neq \psi(\nu)$. Proof: By (7), it suffices to show that $C \in \psi(\mu)$ or $C \in \psi(\nu)$. This follows from (1) and (2).

The proposition is proved by (5), (6), and (8). $\square$

Proposition 5.24. *Let $k \in \mathbb{N}$ and $\mu \in \mathcal{M}_k(\mathcal{P})$. Then $\psi(\mu)$ is a k-rook configuration in $\mathcal{P}$.*

Proof. Since $|\psi(\mu)| = k$, it suffices to note that the cells of $\psi(\mu)$ are in distinct rows and columns. This follows from Proposition 5.22(2). $\square$

Proof of Theorem 5.19. For each $i \in \mathbb{N}$, $h_i = |\mathcal{M}_i(\mathcal{P})|$ by Proposition 5.1. By Propositions 5.23 and 5.24 we see that $h_i \leq r_i$ for all i. Since $\mathcal{P}$ is not thin, $\mathcal{P}$ contains a 2-rook configuration as in Figure 5.10. Such a rook configuration cannot be in the image of ψ. Hence $h_2 < r_2$. $\square$

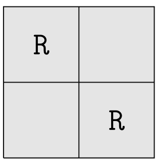


FIGURE 5.10: 2-rook (denoted by R) configuration in a non-thin polyomino.

An another proof of the Theorem 5.19 is given by Qureshi-Rinaldo-Romeo [QRR22]. Using results of [EHQR21], we can extend our result to L-convex polyominoes. First, we define L-convex polyominoes.

Let $\mathcal{C} : C_1, \dots, C_m$ be a path of cells in a polyomino and (i_k, j_k) be the bottom left corner of C_k for $1 \leq k \leq m$. Then $\mathcal{C}$ has change of direction at C_k for some $2 \leq k \leq m-1$ if $i_{k-1} \neq i_{k+1}$ and $j_{k-1} \neq j_{k+1}$. A convex polyomino $\mathcal{P}$ is called *L-convex* if any two cells of $\mathcal{P}$ can be connected by a path of cells in $\mathcal{P}$ with at most one change of direction.

Let $\mathcal{P}$ be an L-convex polyomino. Then there exists a polyomino $\mathcal{P}^*$ (the Ferrer diagram projected by $\mathcal{P}$, in the sense of [EHQR21]) such that

1. $\mathcal{P}^*$ is a convex polyomino such that $V(\mathcal{P}^*)$ is a sublattice of $\mathbb{N}^2$ (since $\mathcal{P}^*$ is a Ferrer diagram);
2. If $\mathcal{P}$ is not thin, then $\mathcal{P}^*$ is not thin;
3. $\mathcal{P}$ and $\mathcal{P}^*$ have the same rook polynomial [EHQR21, Lemma 2.4];
4. $K[\mathcal{P}]$ and $K[\mathcal{P}^*]$ are isomorphic to each other [EHQR21, Theorem 3.1], so they have the same h-polynomial.

Thus we get:

Corollary 5.25. *Let $\mathcal{P}$ be an L-convex polyomino that is not thin. Let $h(t) = 1 + h_1t + h_2t^2 + \cdots$ be the h-polynomial of $K[\mathcal{P}]$ and $r(t) = 1 + r_1t + r_2t^2 + \cdots$ be the rook polynomial of $\mathcal{P}$. Then $h_2 < r_2$.*

Chapter 6

Further results on Hibi rings

In this chapter, we study the Koszul relation pairs of the Hibi ideals and initial Hibi ideals. The term "Koszul relation pair" was defined by Ene, Herzog and Hibi [EHH15], where they studied the Koszul relation pairs of convex polyomino ideals. We start by an observation that the initial Hibi ideal is a Stanley-Reisner ideal of the order complex of the distributive lattice. Then we use Hochster's formula to give a necessary and sufficient condition for the Koszul relation pairs of the initial Hibi ideals. We use this along with Gröbner deformation to give a necessary condition for the Koszul relation pairs of the Hibi ideals. We also characterize complete intersection Hibi rings.

Let $L = \mathcal{I}(P)$ be a distributive lattice. Let $R[L] = K[L]/I_L$ be the Hibi ring associated to L. Let $<$ be a total order on the variables of $K[L]$ with the property that $x_\alpha < x_\beta$ if $\alpha < \beta$ in L. Consider the reverse lexicographic order $<$ on $K[L]$ induced by this order of the variables. Recall from Section 2.6, we have

$$\mathrm{in}_<(I_L) = (x_\alpha x_\beta : \alpha, \beta \in L \text{ and } \alpha, \beta \text{ incomparable}).$$

Let us define $\mathcal{D}_2 := \{(\alpha, \beta) : \alpha, \beta \in L \text{ and } \alpha, \beta \text{ incomparable}\}$.

6.1 Syzygies of initial Hibi ideals

Let K be a field and Δ a simplicial complex on a vertex set $V = \{v_1, ..., v_n\}$. Let $K[\Delta]$ be the Stanley-Reisner ring of the simplicial complex Δ. We know that $K[\Delta] = S/I_\Delta$, where $S = K[x_1, ..., x_n]$ and $I_\Delta = \{x_{i_1} \cdots x_{i_r} : \{v_{i_1}, ..., v_{i_r}\} \notin \Delta\}$. Since $K[\Delta]$ is a

$\mathbb{Z}^n$-graded S-module, it has a minimal $\mathbb{Z}^n$-graded free resolution.

$$\mathbb{F} : 0 \longrightarrow F_p \xrightarrow{\phi_p} F_{p-1} \xrightarrow{\phi_{p-1}} \dots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0,$$

where $F_i = \oplus_{j=1}^{\beta_i} S(-a_{ij})$ for $i = 0, ..., p$ with certain $a_{ij} \in \mathbb{N}^n$, and where the maps ϕ_i are homogeneous of degree 0 with $\phi_i(F_i) \subset (x_1, ..., x_n)F_{i-1}$ for all i. The numbers $\beta_{ia} = \#\{j : a_{ij} = a, a \in \mathbb{Z}^n\}$, are called fine Betti numbers of $K[\Delta]$.
Let $W \subset V$; we set $\Delta_W = \{F \in \Delta : F \subset W\}$. It is clear that Δ_W is again a simplicial complex.

Theorem 6.1. *(Hochster)[BH93, Theorem 5.5.1] Let $H_{T_i}(t) = \sum_{a\in\mathbb{Z}^n} \beta_{ia}t^a$ be the fine Hilbert series of the module $T_i = Tor_i^R(K, K[\Delta])$. Then*

$$H_{T_i}(t) = \sum_{W\subset V} (dim_k \widetilde{H}_{|W|-i-1}(\Delta_W; K)) \prod_{v_j\in W} t_j.$$

Let $L = \mathcal{I}(P)$ be a distributive lattice. Let $\Delta(L)$ be the order complex of L. We have $K[\Delta(L)] = K[L]/I_{\Delta(L)}$, where $K[L] = K[\{x_\alpha : \alpha \in L\}]$ and $I_{\Delta(L)} = (x_{\alpha_1} \dots x_{\alpha_r} : \{\alpha_r, \dots, \alpha_r\} \notin \Delta(L))$.

Lemma 6.2. $I_{\Delta(L)} = \mathrm{in}_<(I_L)$.

Proof. If $\alpha, \beta \in L$ such that α and β are incomparable, then $\{\alpha, \beta\} \notin \Delta(L)$. Hence, $x_\alpha x_\beta \in I_{\Delta(L)}$.

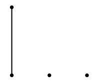
On the other hand, if $x_{\alpha_1} \cdots x_{\alpha_r} \in I_{\Delta(L)}$, then $\{\alpha_1, \dots, \alpha_r\}$ is not a chain. So there exist $\alpha, \beta \in \{\alpha_1, \dots, \alpha_r\}$ such that α and β are incomparable. Hence, $x_{\alpha_1} \cdots x_{\alpha_r} \in (x_\alpha x_\beta) \subseteq I_{\Delta(L)}$. This concludes the proof. □

Theorem 6.3. *Under the above notations, Let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{D}_2$. Then $x_{\alpha_1}x_{\beta_1}, x_{\alpha_2}x_{\beta_2}$ is a Koszul relation pair of $K[\Delta(L)]$ if and only if either $\alpha_2 \vee \beta_2 \leq \alpha_1 \wedge \beta_1$ or $\alpha_1 \vee \beta_1 \leq \alpha_2 \wedge \beta_2$.*

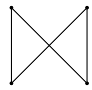
Proof. We are interested in $H_{T_2}(t)$. By Theorem 6.1,

$$H_{T_2}(t) = \sum_{\substack{W\subset L \\ \text{s.t. } \#W=4}} (dim_k \widetilde{H}_{|W|-3}(\Delta_W; K)) \prod_{v_j\in W} t_j.$$

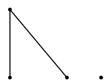
(a) $\widetilde{H}_1(\Delta; K) = 0$



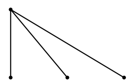
(b) $\widetilde{H}_1(\Delta; K) = 0$



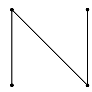
(c) $\widetilde{H}_1(\Delta; K) = K$



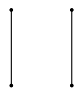
(d) $\widetilde{H}_1(\Delta; K) = 0$



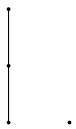
(e) $\widetilde{H}_1(\Delta; K) = 0$



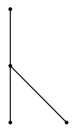
(f) $\widetilde{H}_1(\Delta; K) = 0$



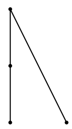
(g) $\widetilde{H}_1(\Delta; K) = 0$



(h) $\widetilde{H}_1(\Delta; K) = 0$



(i) $\widetilde{H}_1(\Delta; K) = 0$



(j) $\widetilde{H}_1(\Delta; K) = 0$

(k) $\widetilde{H}_1(\Delta; K) = 0$

FIGURE 6.1

Suppose that $x_{\alpha_1}x_{\beta_1}$, $x_{\alpha_2}x_{\beta_2}$ is a Koszul relation pair of $K[\Delta(L)]$. Let $W = \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$. Then, by Theorem 6.1, $\widetilde{H}_1(\Delta_W; K) \neq 0$. All possible subsets of L with cardinality 4 are listed in Figure 6.1. For $W' \subset L$ with $\#W' = 4$, one can check that $\widetilde{H}_1(\Delta_{W'}; K) \neq 0$ only if W' is as in Figure 6.1c. Hence, the forward part follows.

For the converse part, suppose that $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in D_2$. Without loss of generality, assume that $\alpha_1 \vee \beta_1 \leq \alpha_2 \wedge \beta_2$. Let $W = \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$. It is easy to see that

$$\widetilde{H}_j(\Delta_W; K) = \begin{cases} K & \text{for} \quad j = 1, \\ 0 & \text{for} \quad j \neq 1. \end{cases}$$

So by Theorem 6.1, $x_{\alpha_1}x_{\beta_1}$, $x_{\alpha_2}x_{\beta_2}$ is a Koszul relation pair of $K[\Delta(L)]$. Hence the proof. $\square$

Note that in Theorem 3.33, we characterized all posets for which the initial Hibi ideal has a linear resolution. Here we prove a result about the (non-)vanishing of β_{24} for the ring $K[\mathcal{I}(P)]/\operatorname{in}_<(I_{\mathcal{I}(P)})$.

Theorem 6.4. *Let P be a poset. Let $P' = \{p_{i_1}, ..., p_{i_r}\}$ be the subset of all elements of P which are comparable to every element of P. Let P'' be the induced subposet of P on the set $P \setminus P'$. Then $\beta_{24}(K[\mathcal{I}(P)]/\operatorname{in}_<(I_{\mathcal{I}(P)})) = 0$ if and only if P is a chain or P''*

is an antichain of three elements or P'' is a disjoint union of a chain and an isolated element.

Proof. First we prove the forward part. If $\text{width}(P) \geq 4$, then there exists an antichain $\{p_1, \dots, p_4\}$ of P. For $i = 1, 2$, define $\alpha_i = \{p \in P : p \leq p_i\}$ and for $i = 3, 4$, define $\alpha_i = \{p \in P : p \leq p_i, p \leq p_1, p \leq p_2\}$. Clearly, α_i is an order ideal of P for all $1 \leq i \leq 4$. Observe that $(\alpha_1, \alpha_2), (\alpha_3, \alpha_4) \in \mathcal{D}_2$ and $\alpha_1 \vee \alpha_2 \leq \alpha_3 \wedge \alpha_4$. Thus, by Theorem 6.3, $\beta_{24}(K[\mathcal{I}(P)]/\operatorname{in}_<(I_{\mathcal{I}(P)})) \neq 0$. So we may assume that $\text{width}(P) \leq 3$. First, observe that P'' is simple. Consider the case $\text{width}(P) = 3$ and $\#P'' \geq 4$. Let $\{p_1, p_2, p_3\}$ be an antichain of P. Possibly by replacing P by P^∂, we may assume that there exists a $p_4 \in P''$ with $p_1 < p_4$. Define $\alpha_i = \{p \in P : p \leq p_i\}$ for $i = 1, 2$, $\alpha_3 = \{p \in P : p \leq p_j \text{ for } 1 \leq j \leq 3\}$ and $\alpha_4 = \{p \in P : p \leq p_4 \text{ and } p \leq p_2\}$. Observe that $(\alpha_1, \alpha_2), (\alpha_3, \alpha_4) \in \mathcal{D}_2$ and $\alpha_1 \vee \alpha_2 \leq \alpha_3 \wedge \alpha_4$. Thus, by Theorem 6.3, $\beta_{24}(K[\mathcal{I}(P)]/\operatorname{in}_<(I_{\mathcal{I}(P)})) \neq 0$.

So we may assume that $\text{width}(P) \leq 2$. If $\text{width}(P) = 1$, then P is a chain. We now consider $\text{width}(P) = 2$. Suppose that P'' is a poset on the set $\cup_{i=1}^{2}\{p_{i,1}, \dots, p_{i,n_i}\}$ such that $\{p_{i,1}, \dots, p_{i,n_i}\}$ is a chain in P'' with $p_{i,1} \lessdot \cdots \lessdot p_{i,n_i}$ for all $i = 1, 2$. We have to show that either $n_1 = 1$ or $n_2 = 1$. Suppose, on the contrary,that $n_i \geq 2$ for all $i = 1, 2$. Let $\alpha_i = \{p_{i,1}\}$ for $i = 1, 2$, $\alpha_3 = \{p_{1,1}, p_{2,1}, p_{1,2}\}$ and $\alpha_4 = \{p_{1,1}, p_{2,1}, p_{2,2}\}$. Observe that $(\alpha_1, \alpha_2), (\alpha_3, \alpha_4) \in \mathcal{D}_2$ and $\alpha_1 \vee \alpha_2 \leq \alpha_3 \wedge \alpha_4$. Thus, by Theorem 6.3, $\beta_{24}(K[\mathcal{I}(P)]/\operatorname{in}_<(I_{\mathcal{I}(P)})) \neq 0$. Hence the proof.

For the converse, if P is a chain, then $R[\mathcal{I}(P)]$ is a polynomial ring; thus $\operatorname{in}_<(I_{\mathcal{I}(P)}) = 0$ we are done. Consider the case when P'' is an antichain of three elements or P'' is a disjoint union of a chain and an isolated element. Let $(\alpha_1, \alpha_2) \in \mathcal{D}_2$. By Theorem 6.3, it suffice to show that $L' = \{\alpha \in \mathcal{I}(P'') : \alpha \geq \alpha_1 \vee \alpha_2\}$ is a chain. When P'' is an antichain of three elements, it is easy to see that either $L' = \{P''\}$ or L' is a chain of two elements.

Now, suppose that P'' is a disjoint union of a chain and an isolated element. Write $P'' = \{p_1, \dots, p_n\} \cup \{q\}$ such that $p_1 \lessdot p_2 \lessdot \cdots \lessdot p_n$ is a chain in P''. Since α_1 is incomparable to α_2, q is in exactly one of them. Thus, $q \in \alpha_1 \vee \alpha_2$. Therefore, L' is a chain. Hence the proof. $\square$

6.2 Syzygies of Hibi ideals

Definition 6.5. *Let $L = \mathcal{I}(P)$ be a distributive lattice with $\#L = n$. Given a weight vector $w = (w_1, ..., w_n)$ with real coordinates, we define a* weight function *w on the monomials of $K[L]$ by*

$$w(x_{\alpha_1}^{a_1} ... x_{\alpha_n}^{a_n}) = w.(a_1, ..., a_n) = \sum_{i=1}^{n} w_i a_i.$$

Define the weight order $<_w$ on the monomials in $K[L]$ by

$$x_{\alpha_1}{}^{a_1} ... x_{\alpha_n}{}^{a_n} \leq_w x_{\alpha_1}{}^{b_1} ... x_{\alpha_n}{}^{b_n} \quad \text{if and only if} \quad \sum_{i=1}^{n} w_i a_i \leq \sum_{i=1}^{n} w_i b_i.$$

This is a partial order.

Theorem 6.6. *[Pee11, Theorem 22.3] Let $<$ be the monomial order on $K[L]$ as defined above. Then, there exist a weight vector w with strictly positive integer coordinates such that $\text{in}_{<_w}(I_L) = \text{in}_{<}(I_L)$.*

Consider the polynomial ring $\widetilde{K[L]} = K[L][t]$ and the weight vector $\widetilde{w} = (w_1, ..., w_n, 1)$. Let $f = \sum_i c_i l_i \in K[L]$, where $c_i \in K \setminus \{0\}$ and l_i is a monomial in $K[L]$. Let l be a monomial in f such that $w(l) = max_i\{w(l_i)\}$. Define $\widetilde{f} = \sum_i t^{w(l)-w(l_i)} c_i l_i$. If we grade $\widetilde{K[L]}$ by $\deg(t) = 1$ and $\deg(x_i) = w_i$ for all i, then $\widetilde{f}$ is homogeneous. Note that the image of $\widetilde{f}$ in $\widetilde{K[L]}/(t-1)$ is f, and its image in $\widetilde{K[L]}/(t)$ is $\text{in}_{<_w}(f)$.

Lemma 6.7. *Let $\widetilde{I_L} = (\widetilde{f} \ |f \in I_L)$. Then $\widetilde{I_L} = (x_\alpha x_\beta - \widetilde{x_{\alpha\cap\beta} x_{\alpha\cup\beta}} : \alpha, \beta \in L$ and α, β incomparable).*

Lemma 6.8. *[Pee11, Theorem 22.8] $\widetilde{K[L]}/\widetilde{I_L}$ is flat as a $K[t]$-module. In particular, $t - c$ is a regular element on $\widetilde{K[L]}/\widetilde{I_L}$ for every $c \in K$.*

Lemma 6.9. *Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ be vectors with non-negative integer coordinates. Suppose that b has positive coordinates. We consider the following two gradings of $S = K[x_1, ..., x_n]$:*

(1) *the a-grading with $deg(x_i) = a_i$ for all i.*

(2) *the b-grading with $deg(x_i) = b_i$ for all i.*

Let I be an ideal in S which is homogeneous with respect to both gradings. Then there exists a minimal free resolution of S/I over the ring S which is both a-graded and b-graded.

Theorem 6.10. *[Pee11, Theorem 22.9] The graded Betti numbers of $K[L]/I_L$ over $K[L]$ are smaller or equal to those of $K[L]/\operatorname{in}_<(I_L)$.*

Proof. Choose a weight vector w with positive integer coordinates such that $\operatorname{in}_{<_w}(I_L) = \operatorname{in}_<(I_L)$. By Lemma 6.9, there exist a minimal free resolution $\widetilde{\mathbb{F}}$ of $\widetilde{K[L]}/\widetilde{I_L}$ over the ring $\widetilde{K[L]}$ which is graded with respect to the following gradings:

(1) $\deg(x_i) = 1$ for all i, and $\deg(t) = 0$.

(2) $\deg(x_i) = w_i$ for all i, and $\deg(t) = 1$.

From now onwards, we grade $\widetilde{K[L]}$ by $\deg(x_i) = 1$ and $\deg(t) = 0$. Let $c \in K$. First consider the case $c = 0$. Since t is a homogeneous non-zero divisor on $\widetilde{K[L]}/\widetilde{I_L}$ and on $\widetilde{K[L]}$, $\widetilde{\mathbb{F}} \otimes \widetilde{K[L]}/(t)$ is a graded free resolution of $K[L]/\operatorname{in}_<(I_L)$ over the ring $K[L]$. The resolution is minimal, since the differential matrices in $\widetilde{\mathbb{F}}$ have entries in $(x_1, ..., x_n, t)$ and after we set $t = 0$ we get that the entries in $(x_1, ..., x_n)$. Therefore, the i'th Betti number of $K[L]/\operatorname{in}_<(I_L)$ is equal to the rank of $\widetilde{F_i}$.

Now, consider the case $c = 1$. Since $t - 1$ is a homogeneous non-zero divisor on $\widetilde{K[L]}/\widetilde{I_L}$ and on $\widetilde{K[L]}$, $\widetilde{\mathbb{F}} \otimes \widetilde{K[L]}/(t-1)$ is a graded free resolution of $K[L]/I_L$ over the ring $K[L]$. This resolution might be non-minimal because we have set $t = 1$ in the matrices of differential. Therefore, the i'th Betti number of $K[L]/\operatorname{in}_<(I_L)$ is less or equal to the rank of $\widetilde{F_i}$. $\square$

Remark 6.11. In the proof of Theorem 6.10, we have also proved that if $x_{\alpha_1}x_{\beta_1} - \widetilde{x_{\alpha_1\cap\beta_1}}x_{\alpha_1\cup\beta_1}$, $x_{\alpha_2}x_{\beta_2} - \widetilde{x_{\alpha_2\cap\beta_2}}x_{\alpha_2\cup\beta_2}$ is a Koszul relation pair of $\widetilde{K[L]}/\widetilde{I_L}$ then $x_{\alpha_1}x_{\beta_1}, x_{\alpha_2}x_{\beta_2}$ is a Koszul relation pair of $K[L]/\operatorname{in}_<(I_L)$. The reason is the following: Let $d_2 : \widetilde{F_2} \to \widetilde{F_1}$ in $\widetilde{\mathbb{F}}$. Fix a basis $\{e_1, ..., e_{\mu(I_L)}\}$ for $\widetilde{F_1}$ and a basis $\{f_1, ..., f_m\}$ for $\widetilde{F_2}$. Then, the map d_2 is given by a matrix A and the map $d_2 \otimes 1_{\widetilde{K[L]}/(t)}$ be given by a matrix B. Since $x_{\alpha_1}x_{\beta_1} - \widetilde{x_{\alpha_1\cap\beta_1}}x_{\alpha_1\cup\beta_1}$, $x_{\alpha_2}x_{\beta_2} - \widetilde{x_{\alpha_2\cap\beta_2}}x_{\alpha_2\cup\beta_2}$ is a Koszul relation pair in $\widetilde{K[L]}/\widetilde{I_L}$. So, there is a column in A, in which the only non-zero entries are $x_{\alpha_1}x_{\beta_1} - \widetilde{x_{\alpha_1\cap\beta_1}}x_{\alpha_1\cup\beta_1}$ and $x_{\alpha_2}x_{\beta_2} - \widetilde{x_{\alpha_2\cap\beta_2}}x_{\alpha_2\cup\beta_2}$. Therefore, there is a column in B, in which the only non-zero entries are $x_{\alpha_1}x_{\beta_1}$ and $x_{\alpha_2}x_{\beta_2}$.

Corollary 6.12. *Let* $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{D}_2$. *If* $x_{\alpha_1}x_{\beta_1} - x_{\alpha_1\cap\beta_1}x_{\alpha_1\cup\beta_1}, x_{\alpha_2}x_{\beta_2} - x_{\alpha_2\cap\beta_2}x_{\alpha_2\cup\beta_2}$ *is a Koszul relation pair of* $R[L]$, *then* $x_{\alpha_1}x_{\beta_1}, x_{\alpha_2}x_{\beta_2}$ *is a Koszul relation pair of* $K[L]/\operatorname{in}_<(I_L)$.

Proof. Suppose that $x_{\alpha_1}x_{\beta_1}, x_{\alpha_2}x_{\beta_2}$ is not a Koszul relation pair of $K[L]/\operatorname{in}_<(I_L)$. Therefore, by Remark 6.11, $\widetilde{x_{\alpha_1}x_{\beta_1} - x_{\alpha_1\cap\beta_1}x_{\alpha_1\cup\beta_1}}, \widetilde{x_{\alpha_2}x_{\beta_2} - x_{\alpha_2\cap\beta_2}x_{\alpha_2\cup\beta_2}}$ is not a Koszul relation pair of $\widetilde{K[L]}/\widetilde{I_L}$. Hence, $x_{\alpha_1}x_{\beta_1} - x_{\alpha_1\cap\beta_1}x_{\alpha_1\cup\beta_1}, x_{\alpha_2}x_{\beta_2} - x_{\alpha_2\cap\beta_2}x_{\alpha_2\cup\beta_2}$ is not a Koszul relation pair of $R[L]$. $\square$

Theorem 6.13. *Let* $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{D}_2$. *If* $x_{\alpha_1}x_{\beta_1} - x_{\alpha_1\wedge\beta_1}x_{\alpha_1\vee\beta_1}$, $x_{\alpha_2}x_{\beta_2} - x_{\alpha_2\wedge\beta_2}x_{\alpha_2\vee\beta_2}$ *is a Koszul relation pair of* $R[L]$, *then either* $\alpha_2 \vee \beta_2 \leq \alpha_1 \wedge \beta_1$ *or* $\alpha_1 \vee \beta_1 \leq \alpha_2 \wedge \beta_2$.

Proof. The proof follows from Corollary 6.12 and Theorem 6.3. $\square$

6.3 Complete intersection Hibi rings

In this section, we will combinatorially characterize complete intersection Hibi rings.

Example 6.14. *Let* P_1 *and* P_2 *be the posets as shown in Figure 6.2b and Figure 6.2c respectively. Then the respective graded Betti table of* $R[\mathcal{I}(P_1)]$ *and* $R[\mathcal{I}(P_2)]$ *are the following:*

	0	1	2	3	4
total:	1	9	16	9	1
0:	1	.	.	.	.
1:	.	9	16	9	.
2:	.	.	.	.	1

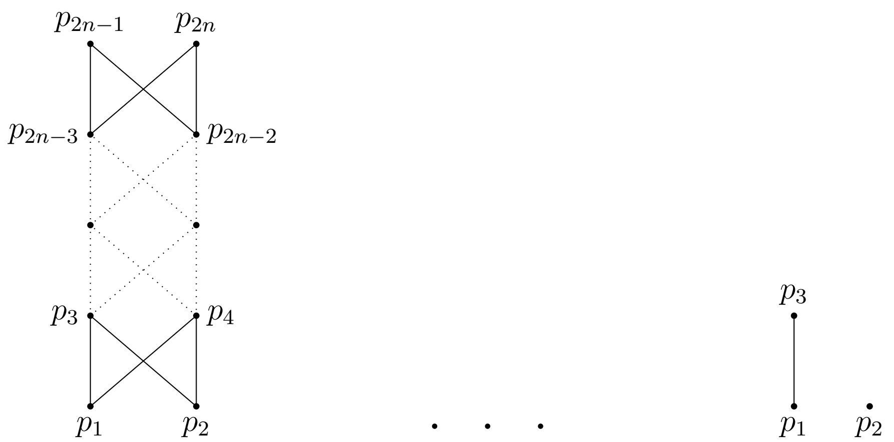
	0	1	2
total:	1	3	2
0:	1	.	.
1:	.	3	2

Now, we prove the main theorem of this section.

Theorem 6.15. *Let* P *be a poset and* $P' = \{p_{i_1}, \ldots, p_{i_r}\}$ *be the subset of all elements of* P *which are comparable to every element of* P. *Let* P'' *be the induced subposet of* P *on the set* $P \setminus P'$. *Then the following are equivalent:*

(*a*) $R[\mathcal{I}(P)]$ *is a complete intersection.*

(*b*) *Either* P *is a chain or* P'' *is as shown in Figure 6.2a.*



(a) (b) $\beta_{23} = 16$ (c) $\beta_{23} = 2$

FIGURE 6.2

Proof. $(a) \Rightarrow (b)$. Suppose that $R[\mathcal{I}(P)]$ is a complete intersection. Then, we have $\beta_{23}(R[\mathcal{I}(P)]) = 0$. Now, we break the proof by width of the poset. If $width(P) \geq 3$, then there exists an antichain $P_1 = \{p_1, p_2, p_3\}$ of P. By Discussion 3.7, $\beta_{23}(R[\mathcal{I}(P_1)]) \leq \beta_{23}(R[\mathcal{I}(P)])$. Since by Example 6.14, $\beta_{23}(R[\mathcal{I}(P_1)]) \neq 0$, we obtain that $\beta_{23}(R[\mathcal{I}(P)]) \neq 0$. So we may assume that $width(P) \leq 2$. If $width(P) = 1$, then P is a chain. Hence, the only case we need to consider is $width(P) = 2$. Now if P'' is not as shown in Figure 6.2a, then P'' will contain the poset as shown in Figure 6.2c as a cover-preserving subposet, call it P_2. Let B and B' be the sets of minimal and maximal elements of P_2 respectively. From Discussion 3.7 and Example 6.14, $\beta_{23}(R[\mathcal{I}(P)]) \neq 0$. This concludes the proof.

$(b) \Rightarrow (a)$. If P is a chain, then $R[\mathcal{I}(P)]$ is a polynomial ring. So we may assume that P is not a chain. Since $R[\mathcal{I}(P)] \cong R[\mathcal{I}(P'')] \otimes_K K[y_1, ..., y_r]$ by Corollary 2.21, it is enough to show that $R[\mathcal{I}(P'')]$ is a complete intersection. For $1 \leq i \leq n$, let $P_i = \{p_{2i-1}, p_{2i}\}$ and $Q_i = \{a \in P'' : a \leq p_{2i-1}\} \cup \{p_{2i}\}$ be the subposets of P. Observe that $Q_n = P''$. For $1 \leq i \leq n-1$, by Lemma 2.19,

$$R[\mathcal{I}(Q_{i+1})] \cong (R[\mathcal{I}(Q_i)] \otimes_K R[\mathcal{I}(P_{i+1})])/(x_{Q_i} - y_{\emptyset}).$$

We prove the theorem by induction on i. It is easy to see that the result holds for $i = 1$. Now assume that the result holds for i. Since $R[\mathcal{I}(P_{i+1})] \cong R[\mathcal{I}(Q_1)]$, we have $R[\mathcal{I}(Q_i)] \otimes_K R[\mathcal{I}(P_{i+1})]$ is a complete intersection. Hence, $R[\mathcal{I}(Q_{i+1})]$ is a complete intersection. $\square$

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