Aspects of holography: mainly $nAdS_2$ holography from dimensional reduction and non-relativistic holography for hyperscaling violating Lifshtiz theories

by

Kedar S. Kolekar

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

to

Chennai Mathematical Institute

August 2020

Chennai Mathematical Institute,
Plot H1, SIPCOT IT Park, Siruseri, Kelambakkam,
Chennai, Tamil Nadu, India 603103

Advisor:

Dr. K. Narayan, *Chennai Mathematical Institute (CMI).*

Doctoral Committee Members:

1. Dr. Alok Laddha, *Chennai Mathematical Institute (CMI).*
2. Dr. Suresh Govindarajan, *Indian Institute of Technology Madras (IITM).*

Declaration

This thesis titled *Aspects of holography: mainly* $nAdS_2$ *holography from dimensional reduction and non-relativistic holography for hyperscaling violating Lifshtiz theories* is a presentation of my original research work, carried out with my collaborators and under the guidance of Dr. K. Narayan at Chennai Mathematical Institute. This work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in Chennai Mathematical Institute or any other university or institution of higher education.

Kedar S. Kolekar
August 2020

In my capacity as the supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Dr. K. Narayan
August 2020

Acknowledgments

I am grateful to my advisor K. Narayan for his constant guidance, support and encouragement during my Ph.D. I have learned a good deal of physics through our almost daily meetings and many lengthy discussions. I thank him for constantly monitoring my progress and his continuous feedback which were very helpful.

I would like to thank Alok Laddha at CMI and Suresh Govindarajan at IITM, Chennai for being my doctoral committee members. I had good discussions with them during our doctoral committee meetings and their feedback was very helpful.

I thank Dileep Jatkar at HRI for helpful discussions during our collaboration on one of the projects.

I thank Alok Laddha for being a friend and teacher. I had many discussions with him on physics and other topics. His support and guidance on various things was very helpful.

I thank my friend and collaborator Debangshu for his support. We had both good physics discussions and chats on different topics.

I thank all my friends at CMI and IMSc for being with me during the Ph.D. days. Their love and support kept me going during my Ph.D. I thank my roommate at CMI Deepak, and my flat-mates Vishu, Manu, Ramadas, Sourav, Shivam, Chanakya for making my stay enjoyable. I thank Manu and Aneesh for our almost weekly gatherings in my flat during final years of my Ph.D., which helped me get through some of the stressful times.

I also thank my friends at other institutions for chatting and discussions during our meetings at various schools, workshops and conferences.

I thank my parents, brothers and family members for supporting my decision to pursue career in basic sciences.

Kedar S. Kolekar
August 2020

Dedicated to my mother and father

List of publications

This thesis is based on the following publications.

- Kedar S. Kolekar, D. Mukherjee and K. Narayan, *Notes on hyperscaling violating Lifshitz and shear diffusion*, Phys. Rev. D **96** (2017) no.2, 026003; [arXiv:1612.05950 [hep-th]].

- Kedar S. Kolekar and K. Narayan, *AdS_2 dilaton gravity from reductions of some nonrelativistic theories*, Phys. Rev. D **98** (2018) no.4, 046012; [arXiv:1803.06827 [hep-th]].

- Kedar S. Kolekar and K. Narayan, *On AdS_2 holography from redux, renormalization group flows and c-functions*, J. High Energ. Phys. 1902 (2019) 039; [arXiv:1810.12528 [hep-th]].

- Dileep P. Jatkar, Kedar S. Kolekar and K. Narayan, *N-level ghost-spins and entanglement*, Phys. Rev. D **99**, no. 10, 106003 (2019); [arXiv:1812.07925 [hep-th]].

Other relevant publication not included in this thesis :

- Kedar S. Kolekar, D. Mukherjee and K. Narayan, *Hyperscaling violation and the shear diffusion constant*, Phys. Lett. B **760** (2016) 86-93; [arXiv:1604.05092 [hep-th]].

Contents

Chapter 1

Introduction

The Bekenstein-Hawking entropy formula [1, 2] states that the entropy of a black hole is proportional to area of the horizon rather than its volume. This motivated the *holographic principle*, proposed by 't Hooft [3] and Susskind [4], which suggests that the degrees of freedom in a theory of quantum gravity in a given spacetime region live on the boundary of that region. The first concrete realization of the holographic principle was given by Maldacena [5] in string theory : the AdS/CFT correspondence [5, 6, 7]. AdS/CFT is a duality between string theory in Anti de Sitter (AdS) spacetime and a conformal field theory (CFT), which lives on the boundary of AdS spacetime. Though AdS/CFT was conjectured by Maldacena, its correctness is supported by lot of evidence. Some reviews discussing the correspondence and its evidence are [9, 10, 11, 12, 13, 14]. It was subsequently generalized to various theories, including non-conformal theories [8]. Later, the AdS/CFT correspondence was generalized to non-relativistic holography in the context of Schrödinger, Lifshitz and hyperscaling violating theories (see *e.g.* [15] for a review). Broadly, the AdS/CFT correspondence and its generalizations constitute a class of dualities referred to as *gauge/gravity* dualities.

The AdS/CFT correspondence and more generally, gauge/gravity dualities have constituted an inseparable part of research in theoretical physics. The strong-weak coupling nature of gauge/gravity dualities allows us to probe previously inaccessible aspects of strongly coupled gauge theories, by doing computations in the weakly coupled classical gravity duals. This approach has met with remarkable success in understanding various properties of finite temperature quantum field theories *e.g.* hydrodynamic properties including the viscosity bound [16], holographic superconductors and other interesting aspects of condensed matter theories. Some reviews discussing these developments are [17, 18, 19, 20, 21, 22]. Research in the opposite direction to gain deeper understanding of spacetime geometry and gravity using the dual CFT is also under active pursuit (see [23, 24, 25, 26] for some reviews on these developments.

Holography in 2-dimensions in the context of AdS_2 dilaton-gravity theories has been actively explored, recently, in [27, 28, 29, 30, 31, 32]. 2-dimensional theories of gravity, in general, offer a simpler setup to study problems in quantum gravity, in particular, black hole physics. AdS_2 arises generically in the near-horizon geometry of extremal black holes and branes in string theory and some properties of the effective AdS_2 theories dovetail with the properties of these higher dimensional systems. While 2-dimensional theories offer simpler technical setup for studying some aspects, AdS_2 holography turns out to be harder. This is mainly because of the strong backreaction leading to non-existence of finite energy excitations in theories of pure gravity in AdS_2 [33, 34]. Recently, research in AdS_2 dilaton-gravity theories was renewed, beginning with [35, 36, 37, 38, 39] mainly to analyze the backreaction problem systematically in a tractable setup and also partly motivated by studies in the Sachdev-Ye-Kitaev (SYK) model [40, 41]. The particular setup being actively investigated is dilaton-gravity (coupled to matter) with a varying dilaton, which constitutes the dynamics of *nearly*-AdS_2 theories. This has led to the ongoing research towards developing the *nearly*-AdS_2/*nearly*-CFT_1 ($nAdS_2/nCFT_1$) correspondence.

The lightning introduction above touches various aspects of gauge/gravity dualities and we give detailed discussions on some of these aspects in the following sections. In sec. 1.1, we begin with a review of the AdS/CFT correspondence followed by holographic entanglement entropy and holographic renormalization group flows. Then we briefly discuss non-relativistic holography in the context of hyperscaling violating Lifshitz theories in sec. 1.2. In sec. 1.3, we review some recent developments in the $nAdS_2$ holography.

In this thesis, we are broadly interested in studying some of the aspects of holography mentioned above. The main theme is the investigations on $nAdS_2$ holography in certain theories of dilaton-gravity, obtained from dimensional reductions of higher dimensional systems. We also study aspects of non-relativistic holography for hyperscaling violating Lifshtiz theories.

We begin with a study of certain hydrodynamic properties of the hyperscaling violating Lifshitz (hvLif) theories, in chapter 2. In particular, we study the shear diffusion and compute the ratio of shear viscosity to entropy density for the bulk uncharged hvLif spacetimes. The uncharged hvLif black branes are solutions to Einstein-Maxwell-scalar theories and a charge can be added through an additional

$U(1)$ gauge field. In the extremal limit, the near-horizon regions of these charged hvLif black branes acquire an $AdS_2 \times X$ geometry. Compactification of the transverse space X, then leads to an effective description in terms of 2-dimensional dilaton-gravity coupled to matter. In chapter 3, we study the dynamics of $nAdS_2$ in these theories of dilaton-gravity obtained by dimensional reduction of the extremal charged (non-relativistic) black branes in hvLif theories. We also study the reduction of extremal charged (relativistic) black branes in Einstein-Maxwell theories, which is a simple subcase. In chapter 4, we consider a generalized class of 2-dimensional dilaton-gravity-scalar theories arising from the reductions of higher dimensional gravity-scalar theories. We study holographic renormalization group flows, in these 2-dimensional theories, which end at an AdS_2 fixed point in the IR. We prove the holographic c-theorem for a holographic c-function defined as the dilaton. We also adapt the radial Hamiltonian formulation of holographic RG [42] and analyze the RG flow equations and β functions.

In chapter 5, we present an independent study, not related to the above investigations in the $nAdS_2$ holography and non-relativistic holography. This is broadly motivated by gauge/gravity duality for de Sitter space *i.e.* the dS/CFT correspondence [43, 44, 45] and we give some details on this in the chapter. We construct various generalizations of the 2-level ghost-spins in [46, 47, 48] to N-levels. We build on these earlier studies and analyze entanglement properties of the N-level ghost-spins.

1.1 AdS/CFT correspondence

The AdS_5/CFT_4 correspondence proposed by Maldacena [5] states that a *type IIB string theory on* $AdS_5 \times S^5$ *is dual to a* $\mathcal{N} = 4$ *supersymmetric Yang-Mills theory with* $SU(N)$ *gauge group.*

To understand the origins of this duality, consider a stack of N parallel $D3$ branes. At energies lower than the string scale $1/l_s$, where l_s is the string length, only the massless string states can be excited. The low energy effective action describing these massless states can be written as

$$S_{eff} = S_{bulk} + S_{brane} + S_{int} \ , \tag{1.1}$$

where S_{bulk} is a supergravity action (plus higher derivative terms) and describes perturbations of the 10-dimensional background through closed string excitations. S_{brane} is an action for $\mathcal{N} = 4$ super Yang-Mills on the 4-dimensional brane worldvolume (plus higher derivative terms) and describes excitations of $D3$ branes through open strings. S_{int} describes the interactions of the bulk and the brane modes as open-closed string interactions. In the low energy limit *i.e.* taking $l_s \to 0$ ($\alpha' \to 0$) while keeping the energies and the dimensionless parameters like string coupling g_s and N fixed, the interaction action S_{int} and the higher derivative terms in S_{bulk}, S_{brane} can be omitted in the total effective action S_{eff}. Then in this low energy limit, we get two decoupled theories: a free supergravity in the bulk and a pure $\mathcal{N} = 4$ super-Yang-Mills with $U(N)$ gauge group on the 4-dimensional worldvolume of the branes.

To see the appearance of AdS_5, let us look at the stack of N parallel $D3$ branes from another point of view. The gravitational backreaction of these $D3$ branes depends on the 't Hooft coupling constant $\lambda \equiv g_{YM}^2 N \sim g_s N$, where g_{YM} is the Yang-Mills coupling constant. In the limit $\lambda \gg 1$, the strong gravitational backreaction leads to the collapse of the stack of $D3$ branes to form a black brane. The near-horizon geometry of this black brane is $AdS_5 \times S^5$. For an observer at infinity, the low energy sector consists of two types of decoupled excitations. The massless excitations in the bulk whose wavelength, in the low energy limit for large λ, becomes bigger than the typical size of the brane. These excitations decouple from those closer to the stack of branes, which form the near horizon AdS_5 geometry. Thus in the low energy limit, we get two decoupled theories: a free bulk supergravity and the near horizon $AdS_5 \times S^5$ geometry. In the opposite limit $\lambda \ll 1$, the gravitational backreaction of the $D3$ branes is negligible and the description is in terms of $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensions with $U(N)$ gauge group.

Comparing the effective action and the supergravity descriptions of the stack of $D3$ branes, we see that both have a common decoupled free bulk supergravity in the low energy sector. This leads us to identify the other decoupled theories in the low energy sector. Thus, we see that a type IIB string theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ super-Yang-Mills in 4-dimensions with $U(N)$ gauge group.

Strong-weak duality:
The perturbative analysis in Yang-Mills theory is valid when

$$\lambda = g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4} \ll 1 \tag{1.2}$$

and the classical supergravity description is reliable when the radius of curvature is large compared to the string length *i.e.*

$$\frac{R^4}{l_s^4} \sim \lambda \gg 1 \; . \tag{1.3}$$

We see that the domain of validity of the classical supergravity *i.e.* $\lambda \gg 1$ corresponds to a strong coupling regime of the Yang-Mills theory. Thus, AdS/CFT is a *strong-weak* duality. This strong-weak nature of the duality is very useful practically where we can do computations in the classical gravity and then map the results to corresponding quantities in the strongly coupled field theory.

Symmetries : One of the arguments in support of AdS/CFT is the matching of symmetries. The isometry group of AdS_5 is $SO(2,4)$ which is also the conformal group in 4-dimensions[1]. Also the $SO(6)$ symmetry of the transverse sphere S^5 matches the $SU(4)$ R-symmetry of the $\mathcal{N}=4$ super Yang Mills theory.

1.1.1 AdS/CFT dictionary

The duality between a string theory on the AdS_5 bulk and a boundary CFT_4 means that there is a one-to-one correspondence between fields in the bulk and operators in the boundary CFT_4. The dictionary between these bulk fields and boundary operators was first formulated by Gubser, Klebanov, Polyakov [6] and Witten [7], which (generalized to AdS_{d+1}/CFT_d) states that

$$Z_{CFT}[h_0(x)] = Z_{String}[h(x, r \to 0) = h_0(x)] \ , \tag{1.4}$$

where $x = (t, \vec{x})$ are coordinates in the boundary, r is the bulk radial coordinate and we choose coordinates such that $r \to 0$ is the boundary of AdS. $h(x,r)$ denotes a generic bulk field *e.g.* scalar field, vector field or bulk metric and $h_0(x)$ is the boundary value of the bulk field $h(x,r)$, which acts as the source for the dual operator in boundary CFT. $Z_{CFT}[h_0(x)]$ is the generating functional for correlation functions of operators $\mathcal{O}(x)$ in the boundary CFT given by

$$Z_{CFT}[h_0(x)] = \left\langle e^{i\int d^dx\, h_0(x)\, \mathcal{O}(x)} \right\rangle \ . \tag{1.5}$$

$Z_{String}[h(x, r \to 0) = h_0(x)]$ is the string partition function evaluated at the boundary of the AdS bulk. In the large N, large λ limit, where the classical supergravity description is valid, the string partition function is dominated by the classical supergravity action,

$$Z_{String}[h(x, r \to 0)] \approx e^{-S_{cl}[h(x,r\to 0) = h_0(x)]} \ . \tag{1.6}$$

[1]More generally, the isometry group of AdS_{d+1} and the conformal group in d-dimensions are $SO(2,d)$.

Then using the relation (1.4), we can compute the correlation functions of boundary operators from the bulk partition function as

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = \frac{\delta^n}{\delta h_0(x_1)\cdots\delta h_0(x_n)} e^{-S_{cl}[h(x,r\to 0)=h_0(x)]}\Big|_{h_0=0} \,. \tag{1.7}$$

Scalar field in AdS

To see the dictionary more explicitly, let us consider a massive scalar field in the AdS_{d+1} background

$$S_\chi = \int d^{d+1}x\sqrt{-g}\Big(-\frac{1}{2}\partial_M\chi\partial^M\chi - \frac{m^2\chi^2}{2}\Big) \,, \tag{1.8}$$

where the AdS_{d+1} metric in Poincaré coordinates is

$$ds^2 = g_{rr}dr^2 + g_{ab}dx^a dx^b = \frac{R^2}{r^2}dr^2 + \frac{R^2}{r^2}\Big(-dt^2 + \sum_{i=1}^{d-1} dx_i^2\Big) \,, \tag{1.9}$$

with indices $a,b = 0,1,\ldots,(d-1)$ denoting the boundary coordinates. We substitute $\chi(x,r) = \tilde{\chi}(x)\varrho(r)$, with $\varrho(r) \sim r^\Delta$ as $r \to 0$, in the equation of motion $(\nabla^2 - m^2)\chi = 0$ and obtain

$$\Delta(\Delta-d) - m^2R^2 = 0 \quad \Longrightarrow \quad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2R^2} \,, \qquad \Delta_+ + \Delta_- = d \,. \tag{1.10}$$

Thus the asymptotic behaviour of the scalar field is

$$\chi(x,r) = r^{\Delta_-}\chi_0(x) + r^{\Delta_+}\chi_1(x) + \textit{subleading terms} \,. \tag{1.11}$$

The reality of $\Delta_\pm$ implies a bound for the mass of the scalar field χ, called the *Breitenlohner-Freedman* bound:

$$m^2R^2 \geq -\frac{d^2}{4} \,. \tag{1.12}$$

Now with $\Delta_+ > \Delta_-$, we see that as $r \to 0$, r^{Δ_-} dominates over r^{Δ_+}. Thus, $\chi_1(x)$ is the normalizable mode and $\chi_0(x)$ is the non-normalizable mode, which acts as a source for the dual operator in the boundary CFT, $\chi_0(x) = \lim_{r\to 0} r^{-\Delta_-}\chi(x,r)$.

Under the scaling $\{x,r\} \to \lambda\{x,r\}$, invariance of $\chi(x,r)$ implies that the mode $\chi_0(x)$ transforms as $\chi_0(x) \to \lambda^{-\Delta_-}\chi_0(\lambda x)$. In the boundary CFT, the invariance of the coupling term $\int d^dx\, \chi_0\, \mathcal{O}$ gives the transformation of the operator as

$$\mathcal{O}(x) \to \lambda^{\Delta_- - d}\mathcal{O}(\lambda x) = \lambda^{-\Delta_+}\mathcal{O}(\lambda x) \tag{1.13}$$

showing that the conformal dimension of $\mathcal{O}$ is Δ_+.

2-point function :

Let us compute the 2-point function $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle$ for the dual operator $\mathcal{O}$ in the boundary CFT, using the holographic prescription of [49] (reviewed in [20]).[2]

Owing to the translational symmetry of the boundary, we can Fourier expand the scalar field χ as

$$\chi(x,r) = \int \frac{d^dk}{(2\pi)^d} e^{ikx} f_k(r)\tilde{\chi}_0(k) \ ; \qquad \chi(x,r_c) = \int \frac{d^dk}{(2\pi)^d} e^{ikx}\tilde{\chi}_0(k) \ , \tag{1.16}$$

where $kx = \eta_{ab}k^a x^b$ and $r_c(\to 0)$ is the boundary. We have imposed the boundary condition that $f_k(r_c) = 1$ and the reality of χ implies $f_k^* = f_{-k}$. Substituting $\chi(x,r)$ in the wave equation $(\nabla^2 - m^2)\chi = 0$, we get

$$\frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}\partial_r f_k) - (g^{ab}k_a k_b + m^2)f_k = 0 \ . \tag{1.17}$$

Substituting the Fourier expansion of χ and using (1.17), the on-shell boundary action is

$$S_{bdy} = -\frac{1}{2}\int_{bdy} d^dx\sqrt{-\gamma}\, n_a \chi \partial^a \chi = \int \frac{d^dk}{(2\pi)^d}\tilde{\chi}_0(-k)\mathcal{F}(k,r)\tilde{\chi}_0(k)\Big|_{bdy} \ , \tag{1.18}$$

[2] Alternatively, we can compute the 2-point function in the case of Euclidean AdS using the (normalized) bulk-to-boundary propagator [7, 50] :

$$K(x,x',r) = \frac{\Gamma(\Delta_+)}{\pi^{\frac{d}{2}}\Gamma(\Delta_+ - \frac{d}{2})}\left(\frac{r}{r^2 + (x^2 - x'^2)}\right)^{\Delta_+} \ . \tag{1.14}$$

$K(x,x',r)$ solves $(\nabla^2 - m^2)K(x,x',r) = 0$ with the boundary condition $\lim_{r\to 0} r^{-\Delta_-}K(x,x',r) = \delta(x-x')$. Substituting $\chi(x,r) = \int d^dx'\, K(x,x',r)\chi_0(x')$ in the onshell Euclidean action $I_\chi = \int d^dx\sqrt{\gamma}\chi n_a \partial^a \chi$, obtained by integrating (1.8) by parts and using bulk equations of motion, gives

$$I_\chi = \frac{\Delta_+\Gamma(\Delta_+)}{\pi^{\frac{d}{2}}\Gamma(\Delta_+ - \frac{d}{2})}\int d^dx\, d^dx'\, \frac{\chi_0(x)\chi_0(x')}{|x-x'|^{2\Delta_+}} \ , \tag{1.15}$$

from which we get the 2-point function for $\mathcal{O}(x)$ as $\langle\mathcal{O}(x)\mathcal{O}(x')\rangle \sim \frac{1}{|x-x'|^{2\Delta_+}}$ showing that the conformal dimension of $\mathcal{O}(x)$ is indeed Δ_+.

where we have used $n_r = -\sqrt{g_{rr}}$, as the outward pointing normal and the flux factor is defined as

$$\mathcal{F}(k,r) = \frac{1}{2}\sqrt{-g}\, g^{rr} f_{-k}\partial_r f_k \ . \tag{1.19}$$

The mode equation (1.17) with metric (1.9) becomes

$$r^2 f_k'' + (-d+1) r f_k' - (k^2 r^2 + m^2 R^2) f_k = 0 \ . \tag{1.20}$$

Let us solve this equation for different cases for k^2. For $k^2 > 0$, the solution to (1.20) is

$$f_k(r) = c_1 r^{\frac{d}{2}} I_\nu(kr) + c_2 r^{\frac{d}{2}} K_\nu(kr) \ ; \qquad \nu \equiv \sqrt{\frac{d^2}{4} + m^2 R^2} \ , \quad k = \sqrt{k^2} \ . \tag{1.21}$$

As $r \to \infty$, $I_\nu(kr) \sim e^{kr}$ and $K_\nu(kr) \sim e^{-kr}$. So regularity at the horizon ($r \to \infty$) requires $c_1 = 0$. Then the normalized solution such that $f_k(r_c = \epsilon) = 1$, with ϵ the near-boundary cutoff (*i.e.* the UV cutoff in the boundary field theory), is

$$f_k(r) = \frac{r^{\frac{d}{2}} K_\nu(kr)}{\epsilon^{\frac{d}{2}} K_\nu(k\epsilon)} \ . \tag{1.22}$$

For simplicity, let us restrict to integer ν (though this analysis also applies to non-integer ν). Near the boundary $r_c = \epsilon$ *i.e.* for small r, the Bessel function $K_\nu(kr)$, for integer ν, can be expanded as

$$K_\nu(kr) = (kr)^{-\nu}(a_0 + a_1 (kr)^2 + \cdots) + (kr)^\nu \log(kr)(b_0 + b_1 (kr)^2 + \cdots) \ . \tag{1.23}$$

According to the prescription for Minkowski space correlators [49], the retarded Green's function is

$$G^R(k) = -2\mathcal{F}(k,r)|_{r_c=\epsilon} \ . \tag{1.24}$$

Then using (1.22), $f_k^*(r) = f_{-k}(r)$ and the expansion (1.23), we get after dropping the contact terms,

$$G^R(k) = -R^{d-1}(2\nu)\frac{b_0}{a_0} k^{2\nu} \log(k\epsilon)\epsilon^{2\nu-d} \ . \tag{1.25}$$

Now for $k^2 < 0$, defining $q = \sqrt{-k^2}$, the two independent solutions are $r^{\frac{d}{2}} H_\nu^{(1)}(qr)$ and $r^{\frac{d}{2}} H_\nu^{(2)}(qr)$. As $r \to \infty$, $H_\nu^{(1)}(qr) \sim e^{iqr}$ and $H_\nu^{(2)}(qr) \sim e^{-iqr}$. Recall that the exponential factor in (1.16) has $e^{-i\omega t}$. So imposing ingoing boundary condition at the horizon ($r \to \infty$) and $f_k(r_c = \epsilon) = 1$, we get

$$\begin{aligned} f_k(r) &= \frac{r^{\frac{d}{2}} H_\nu^{(1)}(qr)}{\epsilon^{\frac{d}{2}} H_\nu^{(1)}(q\epsilon)} \qquad \text{for} \quad \omega > 0 \\ &= \frac{r^{\frac{d}{2}} H_\nu^{(2)}(qr)}{\epsilon^{\frac{d}{2}} H_\nu^{(2)}(q\epsilon)} \qquad \text{for} \quad \omega < 0 \ . \end{aligned} \tag{1.26}$$

The retarded Green's function is then evaluated to be

$$G^R(q) = -R^{d-1}(2\nu)\frac{b_0}{a_0} q^{2\nu}\epsilon^{2\nu-d}\Big(\log(q\epsilon) - \frac{i\pi}{2} sgn\,\omega\Big) \ . \tag{1.27}$$

Combining the two expressions for $k^2 > 0$ and $k^2 < 0$, we get

$$G^R(k) = -R^{d-1}\nu\frac{b_0}{a_0} k^{2\nu}\epsilon^{2\nu-d}\Big(\log|k\epsilon|^2 - i\pi\theta(-k^2) sgn\,\omega\Big) \ . \tag{1.28}$$

At zero temperature, the Feynman propagator is related to the retarded Green's function as

$$\begin{aligned} G^F(k) &= Re(G^R(k)) + isgn\omega Im(G^R(k)) \\ &= -R^{d-1}\nu\frac{b_0}{a_0} k^{2\nu}\epsilon^{2\nu-d}\Big(\log|k\epsilon|^2 - i\pi\theta(-k^2)\Big) \ . \end{aligned} \tag{1.29}$$

The divergent factor $\epsilon^{2\nu-d}$ in $G^F(k)$ above is due to differentiating the partition function with respect to $\tilde{\chi}_0(k)$, which is the inverse Fourier transform of $\chi(x, r_c)$; see (1.16). However, the correct identification of the source which couples to the boundary operator is $\chi_0(x) = \lim_{r\to r_c=\epsilon} r^{-\Delta_-}\chi(x,r) = \epsilon^{\nu-\frac{d}{2}}\chi(x,r_c)$. The expression for the renormalized Feynman propagator $G^F_r(k)$ obtained by differentiating the partition function with respect to $\chi_0^r(k)$ (the Fourier transform of $\chi_0(x)$) is, then, same as that for $G^F(k)$ above with the factor $\epsilon^{2\nu-d}$ excluded. Then performing the inverse Fourier transform, the 2-point function in position space (with $\Delta_+ = 2\nu + d$) is

$$\langle T\,\mathcal{O}(x)\mathcal{O}(0)\rangle = i\int d^d x e^{ikx} G^F_r(k) \sim \frac{1}{|x|^{2\Delta_+}} \ . \tag{1.30}$$

The above analysis for scalar fields can also be done for vector fields and tensor fields (*e.g.* the metric) in the bulk. Some comprehensive reviews for the AdS/CFT correspondence and its various aspects are [9, 20, 22].

1.1.2 Holographic entanglement entropy

Consider a quantum theory with a Hilbert space separable as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $A \cup B$ is the total quantum system. Let the system be in a state $|\psi\rangle$ with the density matrix $\rho = |\psi\rangle\langle\psi|$. The entanglement entropy of the subsystem A is defined as the von Neumann entropy $S_A = -tr(\rho_A \log \rho_A)$, where ρ_A is the reduced density matrix of A obtained by partial tracing of ρ over the compliment subsystem B. For example, consider a system of two spin-1/2 states with basis $|s_A s_B\rangle \equiv \{|\uparrow_A\uparrow_B\rangle, |\uparrow_A\downarrow_B\rangle, |\downarrow_A\uparrow_B\rangle, |\downarrow_A\downarrow_B\rangle\}$. The basis for subsystem A consisting of a single spin is $|s_A\rangle = \{|\uparrow_A\rangle, |\downarrow_A\rangle\}$ with the usual positive definite norms $\langle\uparrow_A | \uparrow_A\rangle = \langle\downarrow_A | \downarrow_A\rangle = 1$ and $\langle\uparrow_A | \downarrow_A\rangle = \langle\downarrow_A | \uparrow_A\rangle = 0$, and likewise for the subsystem B. For a generic state $|\psi\rangle = \psi^{s_A s_B}|s_A s_B\rangle$, $\rho_A = tr_B\rho = \langle\uparrow_B |\psi\rangle\langle\psi| \uparrow_B\rangle + \langle\downarrow_B |\psi\rangle\langle\psi| \downarrow_B\rangle$. In particular, for a normalized state $|\psi\rangle = c_1|\uparrow_A\uparrow_B\rangle + c_2|\downarrow_A\downarrow_B\rangle$; $|c_1|^2 + |c_2|^2 = 1$, we get $\rho_A = |c_1|^2|\uparrow_A\rangle\langle\uparrow_A| + |c_2|^2|\downarrow_A\rangle\langle\downarrow_A|$, whose eigenvalues are $\rho_A(i) = |c_i|^2 < 1$, $i = 1, 2$. Then the entanglement entropy of A is $S_A = -tr(\rho_A \log \rho_A) = -\sum_i \rho_A(i) \log \rho_A(i) = -|c_1|^2 \log |c_1|^2 - |c_2|^2 \log |c_2|^2 > 0$.

While the computation above for entanglement entropy of spin systems is quite simple, it is not so in general interacting quantum field theories and conformal field theories. In some cases like 2-dimensional CFTs, the replica trick [51, 52] does provide a remarkable technique to compute entanglement entropy. However, computation of entanglement entropy using field theory techniques is difficult for CFTs in higher dimensions and general quantum field theories. In such cases, the holographic entanglement entropy proposal of Ryu and Takayanagi [53, 54] offers a useful machinery for computations of entanglement entropy. Using the AdS/CFT correspondence and motivated by the Bekenstein-Hawking entropy of black holes [1, 2], Ryu and Takayanagi [53, 54] proposed that the entanglement entropy of a subsystem A with boundary ∂A in the boundary CFT_d is proportional to the area of a $(d-1)$-dimensional minimal surface protruding in the AdS_{d+1} bulk and anchored to the boundary (of AdS) along ∂A, *i.e.*

$$S_A = \frac{\mathcal{A}(\gamma_A)}{4G_{d+1}} \ , \tag{1.31}$$

where $\mathcal{A}$ is the area of the minimal surface γ_A. To illustrate this, consider a strip subsystem on a constant time slice in the boundary CFT_d, for $d > 2$, with the width along the x_1-direction and infinitely extended in the remaining spatial directions,

$$x_1 \in \Big[-\frac{l}{2}, \frac{l}{2}\Big] \ , \qquad x_i \in (-\infty, \infty) \ , \quad i = 2, \ldots, d-1 \ . \tag{1.32}$$

Let us regulate the infinite length in each of the directions $x_2, \dots, x_{d-1}$ as L_0, such that $l \ll L_0$. Then the area of the entangling surface γ_A in the AdS_{d+1} bulk with Poincaré metric (1.9), described by $x_1 = x_1(r)$, and anchored to the boundary (of AdS) along ∂A is given by

$$\mathcal{A} = 2R^{d-1}L_0^{d-2}\int_{\epsilon}^{r_*}\frac{dr}{r^{d-1}}\sqrt{1+\dot{x}_1^2}\ ; \qquad \dot{x}_1 = \frac{dx_1}{dr}\ , \tag{1.33}$$

where ϵ is the near-boundary (*i.e.* the UV) cutoff. Considering $\mathcal{A} = \mathcal{A}(x_1, \dot{x}_1; r)$ like a Lagrangian, the absence of x_1 in $\mathcal{A}$ gives a conserved quantity

$$\frac{\delta\mathcal{A}}{\delta\dot{x}_1} \propto \frac{\dot{x}_1}{r^{d-1}\sqrt{1+\dot{x}_1^2}} \equiv B \qquad \Longrightarrow \qquad \dot{x}_1^2 = \frac{B^2 r^{2d-2}}{1-B^2r^{2d-2}}\ . \tag{1.34}$$

The surface γ_A has a turning point r_* in the bulk at which $\frac{dx_1}{dr}\big|_{r_*} \to \infty$ (*i.e.* $\frac{dr}{dx_1}\big|_{r_*} = 0$). At the turning point, $1 - B^2 r_*^{2d-2} = 0 \implies B = \frac{1}{r_*^{d-1}}$ and we can write $\mathcal{A}$ as

$$\mathcal{A} = 2R^{d-1}L_0^{d-2}r_*^{2-d}\int_{\epsilon/r_*}^{1}\frac{du}{u^{d-1}}\frac{1}{\sqrt{1-u^{2d-2}}}\ ; \qquad u = \frac{r}{r_*}\ . \tag{1.35}$$

The width of the subsystem along the x_1-direction is

$$l = 2r_*\int_0^1 du\frac{u^{d-1}}{\sqrt{1-u^{2d-2}}} = 2r_*\sqrt{\pi}\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}\ . \tag{1.36}$$

Then using the Ryu-Takayanagi prescription (1.31), the holographic entanglement entropy of A is

$$S_A = \frac{1}{4G_{d+1}}\Big[\frac{2R^{d-1}}{d-2}\Big(\frac{L_0}{\epsilon}\Big)^{d-2} - \frac{2^{d-1}\pi^{\frac{d-1}{2}}R^{d-1}}{d-2}\Big(\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}\Big)^{d-1}\Big(\frac{L_0}{l}\Big)^{d-2}\Big]\ . \tag{1.37}$$

The coefficient of the divergent term is proportional to L_0^{d-2} *i.e.* the area of the boundary ∂A of A. This is the known area law of entanglement entropy [55, 56]. The second term is finite, independent of the UV cutoff and encodes a size-dependent measure of the entanglement.

For $d = 2$, there is only one spatial direction in the CFT_2. We take the subsystem A to be an interval of length l along the x-direction *i.e.* $x \in [-\frac{l}{2}, \frac{l}{2}]$. Then the area of the minimal surface in the AdS_3 bulk gives the holographic entanglement entropy

$$S_A = \frac{R}{2G_3}\log\frac{l}{\epsilon} + \cdots = \frac{c}{3}\log\frac{l}{\epsilon} + \cdots\ , \tag{1.38}$$

where we have used that the central charge c of the CFT_2 is related to bulk quantities as $c = \frac{3R}{2G_3}$ [57]. The logarithmic divergence is consistent with the known result in CFT_2 [58, 51].

The original Ryu-Takayanagi prescription for the holographic entanglement entropy was proposed for subsystems on constant time slices in the boundary CFT. It was later generalized to covariant holographic entanglement entropy [59], which also applies to time-dependent subsystems.

1.1.3 Holographic renormalization group flow

In the AdS_{d+1}/CFT_d duality, the CFT lives in one lower dimension. The extra radial direction in the bulk corresponds to the energy scale in the boundary theory [60, 61]. Far away region in the bulk corresponds to the UV and near horizon region corresponds to the IR energy scale of the boundary theory. This radial direction essentially captures the renormalization group (RG) flow of the boundary theory. Following the AdS/CFT correspondence, versions of holographic renormalization and holographic renormalization group flow were formulated, beginning with *e.g.* [62, 63, 64, 65, 66, 67, 42, 68, 69, 70, 71].

In a (2-dimensional) quantum field theory, renormalization group flow is also characterized by the Zamolodchikov c-theorem [72]. The c-theorem states that there exist a positive definite function of the couplings, called a c-function, which monotonically decreases with the energy (*i.e.* along the RG flow) and is constant at the fixed points taking values equal to the central charges of the fixed point CFTs. The holographic formulations of c-theorems were studied in [73, 74, 75, 76].

Holographic c-theorem

We briefly review the holographic formulation of c-theorem by Freedman, Gubser, Pilch, Warner [73]. The holographic c-theorem states that a holographic c-function defined as a function of the radial coordinate, $\mathcal{C}(r)$ decreases monotonically from the asymptotic region to the interior region of the bulk.

In [73], it was shown that the monotonicity of $\mathcal{C}(r)$ follows from the null energy condition. To see this explicitly, consider an ansatz for the metric in $(d+1)$-dimensions

$$ds^2 = e^{2A(\tilde{r})}(\eta_{ab}dx^a dx^b) + d\tilde{r}^2 \ , \qquad (a, b = 0, \ldots, d-1) \ , \tag{1.39}$$

such that $\tilde{r}$ increases from the asymptotic region (UV) to the interior (IR)[3]. The non-zero components of the Ricci tensor are

$$\mathcal{R}_{ab} = e^{2A(\tilde{r})}(A'' + d(A')^2)\eta_{ab} \ , \qquad \mathcal{R}_{\tilde{r}\tilde{r}} = -d(A'' + (A')^2) \ , \tag{1.40}$$

The null energy condition $T_{MN}\xi^M\xi^N \geq 0$ for a null vector $\xi^M = (e^{-A}, \vec{0}, 1)$, $\xi^2 = 0$, using Einstein equations $G_{MN} = 8\pi G_{d+1}T_{MN}$ gives

$$\mathcal{R}_{MN}\xi^M\xi^N = -(d-1)A'' > 0 \quad \Longrightarrow \quad A'' < 0 \ . \tag{1.41}$$

$A''(\tilde{r}) < 0$ implies that $A'(\tilde{r})$ decreases as $\tilde{r}$ increases *i.e.* as we move towards the interior (IR). Then a holographic c-function defined as

$$\mathcal{C}(\tilde{r}) = \frac{\mathcal{C}_0}{(A')^{d-1}} \ ; \qquad \mathcal{C}_0 = constant \ , \tag{1.42}$$

is monotonically decreasing as we move towards the interior *i.e.* along an RG flow from UV to IR. The constant $\mathcal{C}_0$ is determined by equating $C(r)$ to the central charges of the CFTs at the fixed points. Thus we see that the null energy condition leads to a holographic c-theorem.

A more general local, covariant expression for the holographic c-function was proposed by Sahakian in [75] using Bousso's light-sheet construction [74]. In 4-dimensions, the light-sheet of a 2-dimensional spatial surface $\mathcal{B}$ is the congruence of null geodesics emanating from $\mathcal{B}$ such that the expansion of the congruence is non-positive. In the holographic context, taking the spatial surface $\mathcal{B}'$ to be a $(d-1)$-dimensional surface in the boundary, on a constant radial slice and at a constant time, with the light-sheet being the congruence of null geodesics emanating from $\mathcal{B}'$, Sahakian proposed a holographic c-function. Using the criterion for convergence of the null geodesics and the null energy condition $\mathcal{R}_{MN}\xi^M\xi^N \geq 0$ for a null vector ξ^M with components along (t, r)-directions, the proposed c-function was shown to be monotonic, hence proving the c-theorem. For the metric ansatz (1.39), this covariant c-function reduces to (1.42). It also applies to Dp-brane geometries and correctly interpolates between known asymptotics [75].

A holographic RG formulation

We briefly review a formulation of holographic RG flow by de Boer, Verlinde and Verlinde [42], which was further adapted in [77] to study RG flow in D-brane geometries.

[3]The ansatz (1.39) describes AdS metric in Poincaré coordinates (1.9) asymptotically with the coordinate transformation $r = Re^{\frac{\tilde{r}}{R}}$, which shows that $\tilde{r} \to -\infty$ as $r \to 0$ (boundary) .

This formulation is not Wilsonian but provides useful insights into the renormalization group flow and structure of β-functions. Wilsonian holographic renormalization group flow was formulated in [70, 71].

Consider Einstein gravity in 5-dimensions with a negative cosmological constant and coupled to scalar fields Ψ^I with a potential $V(\Psi^I)$ described by the action

$$S = \alpha_G\Big(\int d^5x\sqrt{-g}(\mathcal{R}-2\Lambda) + 2\int d^4x\sqrt{-\gamma}K\Big) \\ -\alpha_m\int d^5x\sqrt{-g}\Big(\frac{1}{2}h_{IJ}\partial_A\Psi^I\partial^A\Psi^J - V(\Psi^I)\Big)\ , \tag{1.43}$$

where we have defined $\alpha_G = \frac{1}{16\pi G_5}$ for notational convenience. The radial decomposition of the metric

$$ds^2 = N^2dr^2 + \gamma_{ab}(N^a dr + dx^a)(N^b dr + dx^b)\ , \qquad (a,b = 0,1,2,3) \tag{1.44}$$

gives a radial Lagrangian L on a constant r slice. The Legendre transformation then gives the radial Hamiltonian $H = \int d^4x\sqrt{-\gamma}(\mathcal{H}N + N^a\mathcal{H}_a)$. N and N^a being non-dynamical give the constraints $\mathcal{H}=0$ and $\mathcal{H}_a = 0$, where we refer to $\mathcal{H}=0$ as the radial Hamiltonian constraint. Defining the conjugate momenta π_{ab}, π_I as

$$\pi_{ab} = \frac{1}{\alpha_G\sqrt{-\gamma}}\frac{\delta L}{\delta\dot{\gamma}^{ab}}\ , \qquad \pi_I = \frac{1}{\alpha_m\sqrt{-\gamma}}\frac{\delta L}{\delta\dot{\Psi}^I}\ , \tag{1.45}$$

we can write the Hamiltonian constraint $\mathcal{H}=0$ as

$$\alpha_G\Big(\pi_{ab}\pi^{ab} - \frac{\pi^2}{3}\Big) + \alpha_m\Big(\frac{\pi^I\pi_I}{2} - \frac{h_{IJ}}{2}\partial_a\Psi^I\partial^a\Psi^J\Big) + \alpha_G(\tilde{\mathcal{R}}-2\Lambda) + \alpha_m V = 0\ , \tag{1.46}$$

where $\pi^{ab} \equiv \gamma^{ac}\gamma^{bd}\pi_{cd}$, $\pi^I = h^{IJ}\pi_J$ and $\tilde{\mathcal{R}}$ is the boundary Ricci scalar for the induced boundary metric γ_{ab}. The boundary action on a radial slice as a function of the boundary fields allows us to write the conjugate momenta π^{ab} and π_I as derivates of the boundary action with respect to γ_{ab} and Ψ^I respectively.

$$\pi_{ab} = \frac{1}{\alpha_G\sqrt{-\gamma}}\frac{\delta S_{bdy}}{\delta\gamma^{ab}}\ , \qquad \pi_I = \frac{1}{\alpha_m\sqrt{-\gamma}}\frac{\delta S_{bdy}}{\delta\Psi^I}\ . \tag{1.47}$$

Using these momenta, we can write the Hamiltonian constraint as

$$\frac{1}{\alpha_G(\sqrt{-\gamma})^2}\Big[\frac{\delta S_{bdy}}{\delta\gamma^{ab}}\frac{\delta S_{bdy}}{\delta\gamma_{ab}} - \frac{1}{3}\Big(\gamma^{ab}\frac{\delta S_{bdy}}{\delta\gamma^{ab}}\Big)^2\Big] + \frac{1}{\alpha_m(\sqrt{-\gamma})^2}\frac{h^{IJ}}{2}\frac{\delta S_{bdy}}{\delta\Psi^I}\frac{\delta S_{bdy}}{\delta\Psi^J} \\ +\alpha_G(\tilde{\mathcal{R}}-2\Lambda) - \frac{\alpha_m h^{IJ}}{2}\partial_a\Psi^I\partial^a\Psi^J + \alpha_m V = 0\ . \tag{1.48}$$

At some low energy scale $\mu << \mu_c$, where μ_c is a UV cutoff, we segregate the boundary action into a local part and a non-local part

$$S_{bdy}[\gamma_{ab}, \Psi^I] = S_{loc}[\gamma_{ab}, \Psi^I] + \Gamma[\gamma_{ab}, \Psi^I] \ , \tag{1.49}$$

where S_{loc} contains no more than second derivatives of the fields,

$$S_{loc} = \int d^4x\sqrt{-\gamma}\Big(W(\Psi^I) + f(\Psi^I)\tilde{\mathcal{R}} + \frac{M_{IJ}(\Psi^I)}{2}\partial_a\Psi^I\partial^a\Psi^J\Big) \ . \tag{1.50}$$

The boundary quantities $f(\Psi^I)$, $M_{IJ}(\Psi^I)$ and the boundary potential $W(\Psi^I)$ are local functions of the couplings $\Psi^I(x, r_c)$ *i.e.* the values of Ψ^I on the boundary slice $r = r_c$. $\Gamma[\gamma_{ab}, \Psi^I]$ contains all higher derivative and non-local terms.

We substitute the boundary action (1.49) in the Hamiltonian constraint (1.48) and expand in the derivatives. Keeping terms upto second order in derivatives, we get relations between the local boundary quantities f, M_{IJ}, W and bulk quantities V, h_{IJ}. In particular, the zeroth order terms *i.e.* the terms with no derivatives give a relation between the bulk and the boundary potential

$$-2\alpha_G^2\Lambda + \alpha_G\alpha_m V = \frac{W^2}{3} - \frac{\alpha_G}{\alpha_m}\frac{h^{IJ}}{2}\partial_I W\partial_J W \ , \tag{1.51}$$

where $\partial_I = \frac{\partial}{\partial\Psi^I}$. The 2-derivative terms give one relation among h^{IJ} and M^{IJ} and a second relation between h^{IJ}, W and f. We will discuss the 4-derivative and higher derivative terms later.

Let us first consider the local part and choose a Fefferman-Graham gauge: $N = 1$, $N^a = 0$. The radial coordinate r corresponds to the RG parameter in the boundary field theory. So we pull the r dependence in the metric as $\gamma_{ab} = a^2\hat{\gamma}_{ab}$, where $a(r)$ is a function of r only and $\hat{\gamma}_{ab}(x)$ is r independent. The function $a(r)$ parametrizes the radial evolution (*i.e.* RG flow) of the bulk quantities. Then from the flow equations (1.45) and using the conjugate momenta in terms of the boundary potential (1.47), we define β-functions for the couplings Ψ^I as

$$\beta^I(\Psi) \equiv a\frac{d}{da}\Psi^I = \frac{6}{W}h^{IJ}\frac{\partial W}{\partial\Psi^J} \ . \tag{1.52}$$

Callan-Symmanzik equation :

Let us consider the higher derivative terms now. After substituting S_{bdy} in the Hamiltonian constraint, the 4-derivative terms give

$$\frac{1}{\sqrt{-\gamma}}\Big[\gamma^{ab}\frac{\delta\Gamma}{\delta\gamma^{ab}} - \beta^J\frac{\delta\Gamma}{\delta\Psi^J}\Big] = 4 - derivative\ terms\ . \tag{1.53}$$

We vary this equation with respect to Ψ^I, then put the fields to their constant values given by the couplings of the gauge theory. Integrating the resulting expression over all space, replacing functional derivatives by partial derivatives *i.e.* $\int \gamma^{ab}\frac{\delta}{\delta\gamma^{ab}} = a\frac{d}{da}$ and $\int \frac{\delta}{\delta\Psi^I} = \frac{\partial}{\partial\Psi^I}$ and using $\frac{1}{\sqrt{-\gamma}}\frac{\partial^n\Gamma}{\partial\Psi^{I_1}\ldots\partial\Psi^{I_n}} \equiv \langle O_{I_1}(x_1)\ldots O_{I_n}(x_n)\rangle$, we get the standard form of the Callan-Symanzik equation. This equation is derived at a finite UV cutoff. We renormalize the metric, scalar fields and Γ and get the Callan-Symmanzik equation for the renormalized n-point functions.

1.2 Non-relativistic holography

Generalization of AdS/CFT to nonrelativistic holography or gauge/gravity duality has been under active exploration over the last few years. In particular, spacetimes conformal to Lifshitz [78, 79], referred to as hyperscaling violating spacetimes arise in effective Einstein-Maxwell-scalar theories[4] *e.g.* [80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91]. The metric of the bulk hyperscaling violating spacetime in $(d_i + 2)$ dimensions is

$$ds^2 = r^{2\theta/d_i}\Big(- \frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{\sum_{i=1}^{d_i} dx_i^2}{r^2}\Big)\ , \tag{1.54}$$

where d_i is the boundary spatial dimension and z is the Lifshitz dynamical exponent with θ the hyperscaling violation exponent. The metric in the parenthesis is the Lifshitz metric in (d_i+2)-dimensions and it is invariant under the scale transformations

$$t \to \lambda^z t\ , \qquad x_i \to \lambda x_i\ , \qquad r \to \lambda r\ . \tag{1.55}$$

The hyperscaling violating metric (1.54) is not scale invariant but transforms as $ds^2 \to \lambda^{\theta/d_i} ds^2$ under the scale transformations (1.55).

Various aspects of Lifshitz and hyperscaling violating holography are developed in *e.g.* [89, 92, 93, 94, 95, 96, 97, 15]. Certain families of hyperscaling violating theories exhibit novel scaling for entanglement entropy *e.g.* [87, 88, 89]. In particular, the $\theta = d_i - 1$ family exhibits logarithmic scaling for entanglement entropy. To see this explicitly, let us use the holographic entanglement entropy proposal of Ryu and Takayanagi [53, 54] (reviewed in sec. 1.1.2).

[4] In the AdS/CMT literature, these are referred to as Einstein-Maxwell-Dilaton theories. We use "dilaton" to refer to the dilaton in 2-dimensional dilaton-gravity-matter theories discussed in Sec. 1.3, Chap. 3 and Chap. 4. This dilaton is distinct from the scalar in the hyperscaling violating Lifshitz theories.

Consider a strip subsystem on a constant time slice in the (d_i+1)-dimensional boundary of the hyperscaling violating bulk, with the width along the x_1-direction and infinitely extended in the remaining spatial directions,

$$x_1 \in \left[-\frac{l}{2}, \frac{l}{2}\right] , \qquad x_i \in \left[-\frac{L_0}{2}, \frac{L_0}{2}\right] \quad i = 2, \ldots, d_i \; ; \qquad l \ll L_0 \; , \tag{1.56}$$

where L_0 is the regulated length of the strip in each of the directions $x_2, \ldots, x_{d-1}$. The area of the entangling surface in the hyperscaling violating bulk with the metric (1.54), which is anchored to the boundary along the edges of the strip, is given by

$$\mathcal{A} = 2L_0^{d_i-1} \int_{\epsilon}^{r_*} dr\, r^{\theta-d_i} \sqrt{1+\left(\frac{dx_1}{dr}\right)^2} = 2L_0^{d_i-1} r_*^{1+\theta-d_i} \int_{\epsilon/r_*}^{1} du \frac{u^{\theta-d_i}}{\sqrt{1-u^{2(d_i-\theta)}}} \; , \tag{1.57}$$

where $u = \frac{r}{r_*}$, r_* is the turning point at which $\frac{dx_1}{dr} \to \infty$ $\left(i.e.\ \frac{dr}{dx_1} = 0\right)$ and $r = \epsilon$ is the UV cutoff. The width of the strip along the x_1-direction is

$$l = 2r_* \int_0^1 du \frac{u^{d_i-\theta}}{\sqrt{1-u^{2(d_i-\theta)}}} \; . \tag{1.58}$$

Now for a family of hyperscaling violating theories with $\theta = d_i - 1$, the above formulae give $l = 2r_*$ and $\mathcal{A} = 2L_0^{d_i-1} \log\left(\frac{l}{\epsilon}\right) + \cdots$. Then using the Ryu-Takayanagi prescription (1.31), we get

$$S_{EE} = \frac{\mathcal{A}}{4G_{d_i+2}} = \frac{L_0^{d_i-1}}{2G_{d_i+2}} \log\left(\frac{l}{\epsilon}\right) + \;\cdots \tag{1.59}$$

showing that the leading term in the entanglement entropy has a logarithmic behaviour. This logarithmic scaling of entanglement entropy was obtained in AdS plane waves for a strip subsystem on a $x^- = constant$ null slice, with the width along x_1-direction [98]. These AdS plane waves upon null x^+-reductions give lower dimensional hyperscaling violating theories with x^- as the time coordinate and lying in the $\theta = d_i - 1$ family [99, 100]. Null x^+-reductions of AdS plane waves [99, 100], which are large boost, low temperature limits [101] of boosted black branes [102] provide certain gauge/string realizations of hyperscaling violating theories. Some of these exhibit novel scaling for entanglement entropy with the string realizations above reflecting this [98, 103, 104, 105], suggesting corresponding regimes in the gauge theory duals exhibiting this scaling.

In the null reductions of AdS plane waves discussed above, the compactification in the bulk is done along the x^+ direction: but this is actually a spacelike direction in the bulk since $g_{++} > 0$. Technically this implies that completing squares to perform the Kaluza-Klein reduction is legitimate thereby implying that the bulk compactification has no subtlety. Holographically $g_{++} >$ implies a nonzero energy momentum tensor component $T_{++} > 0$: these are thus excited states in the dual CFT with macroscopic lightcone momentum density turned on. The effective boundary metric on a cutoff surface $r = \epsilon$ has a nonzero g_{++} component, reflecting this. The usual concerns of DLCQ and zero modes are associated with the ground state in lightfront quantization. The bulk reduction thus suggests that the boundary theory also has interesting and nonsingular behaviour, which would be nice to understand more explicitly with regard to the reduction. Some discussion on these aspects and in particular ultralocality properties appears in [105]. For instance it was argued that while correlation functions in free scalar field theory in lightcone quantization (on null surfaces) vanish consistent with ultralocality, excited states yield nonzero correlators.

1.3 AdS_2 holography

We now turn to a discussion on holography in (bulk) 2-dimensions. AdS_2/CFT_1 correspondence has been investigated extensively earlier with a vast literature: *e.g.* [27, 28, 29, 30, 31, 32, 106, 107, 108, 109, 34, 110, 111]. Our focus in this thesis is on the recent investigations, beginning with [35, 36, 37, 38, 39], in *nearly* AdS_2 ($nAdS_2$) holography in theories of dilaton-gravity coupled to matter.

Gravity in two dimensions, trivial as such, is rendered dynamical in the presence of a dilaton scalar and additional matter. Such dilaton-gravity theories arise generically under dimensional reduction from higher dimensional theories of gravity coupled to matter. There is interesting interplay with AdS_2 holography, which arises in the context of extremal black holes and branes: the near horizon regions typically acquire an $AdS_2 \times X$ geometry, and a 2-dimensional description arises after compactifying the transverse space X. Almheiri and Polchinski [35] considered toy models of 2-dimensional dilaton-gravity of this sort, with backgrounds involving AdS_2 with a varying dilaton. Analyzing the backreaction of a minimally coupled scalar perturbation on the AdS_2 background reveals nontrivial scaling of boundary 4-point correlation functions thereby indicating the breaking of AdS_2 isometries in the deep IR. This breaking amounts to breaking of local reparametrizations of the boundary time coordinate (modulo global $SL(2)$ symmetries), which would have been preserved in the presence of exact conformal symmetry. In [36], as well as [37, 38, 39], it was argued that the leading effects describing such *nearly*-AdS_2

($nAdS_2$) theories are captured universally by a Schwarzian derivative action governing boundary time reparametrizations modulo $SL(2)$, which arises from keeping the leading non-constant dilaton behaviour. The bulk action which reduces to the effective Schwarzian action is the Jackiw-Teitelboim theory of dilaton-gravity [112, 113]. This picture dovetails with the absence of finite energy excitations in AdS_2 discussed previously in [33, 34]. We review some of these developments in this section.

AdS_2 arises quite generally in the near-horizon throat region of extremal black holes and branes in higher dimensions. The Jackiw-Teitelboim (JT) model universally describes these near-horizon AdS_2 regions in a large class of dilaton-gravity theories obtained upon dimensional reductions of higher dimensional theories: *e.g.* [39, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124]. In this thesis, we consider the dimensional reductions of extremal charged black branes in Einstein-Maxwell and Einstein-Maxwell-scalar theories. In the resulting 2-dimensional theories, we study the dynamics of $nAdS_2$ and obtain the universal Schwarzian effective action. Then in a generalized class of 2-dimensional dilaton-gravity-scalar theories, we study holographic renormalization group flows ending at an AdS_2 fixed point in the IR. We discuss our findings, which are reported in [125, 126], in chap. 3 and chap. 4.

1.3.1 Backreaction in AdS_2 dilaton-gravity

Pure AdS_2 doesn't support finite energy excitations owing to the strong backreaction [33, 34]. To investigate this backreaction, Almheiri and Polchinski [35] considered toy models of dilaton-gravity theory in 2-dimensions in which the backreaction is solvable. In the infrared (IR), these models flow to $AdS_2 \times X$, while the nontrivial dilaton profile adjusts these models to flow to a scale invariant theory in the ultraviolet (UV), thereby regulating the backreaction. Turning on a minimally coupled scalar perturbation on the AdS_2 background reveals nontrivial scaling of boundary 4-point correlation functions, which indicates that the backreaction makes the conformal symmetry (*i.e.* the isometry of AdS_2) in the deep IR anomalous.

Let us see this backreaction analysis in some detail. The dilaton-gravity model of Almheiri-Polchinski is

$$S_{grav} = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\,[\,\Phi^2\mathcal{R} - 2(1-\Phi^2)\,] + \frac{1}{8\pi G_2}\int dt\sqrt{-\gamma}\,\Phi^2_{_{bdy}}\,K\ , \quad (1.60)$$

where $\Phi^2_{_{bdy}}$ is the boundary value of the dilaton. The dilaton potential is $U(\Phi) = 2(1-\Phi^2)$, where we have set the AdS_2 radius to unity. The dilaton and Einstein's equations (derived in Appendix B.3) are, respectively,

$$\mathcal{R} + 2 = 0 \,, \tag{1.61}$$

$$g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}(1-\Phi^2) = 0 \,. \tag{1.62}$$

The dilaton equation gives that the background is AdS_2. In the conformal gauge, $ds^2 = e^{2\omega(x^+,x^-)}(-dx^+dx^-) = e^{2\omega(t,\rho)}(-dt^2+d\rho^2)$; $x^\pm = t\pm\rho$, the Ricci scalar is $\mathcal{R} = 2e^{-2\omega}(\partial_t^2 - \partial_\rho^2)\omega = 8e^{-2\omega}\partial_+\partial_-\omega$. Then the Poincaré AdS_2 metric and the solution for dilaton are

$$e^{2\omega} = \frac{1}{\rho^2} = \frac{4}{(x^+-x^-)^2} \,, \qquad \Phi^2 = 1 + \frac{a}{2\rho} = 1 + \frac{a}{x^+-x^-} \,, \tag{1.63}$$

where $\rho\to 0$ is the boundary. This solution interpolates from constant dilaton AdS_2 in the IR to a conformal Lifshitz geometry in the UV [35] *i.e.*

$$\text{IR}: \quad \Phi^2 \sim 1 \quad \text{as} \quad \rho\to\infty \,; \qquad \text{UV}: \quad \Phi^2 \sim \frac{a}{2\rho} \quad \text{as} \quad \rho \to 0 \,. \tag{1.64}$$

Let us turn on a massless scalar perturbation $f(t,\rho)$ coupled minimally to (1.60),

$$S = S_{grav} + S_f \,; \qquad S_f = -\frac{1}{16\pi G_2}\int d^2x \frac{\sqrt{-g}}{2}\partial_\mu f \partial^\mu f \,. \tag{1.65}$$

Due to the minimal coupling of f, we see that the dilaton equation (1.61) remains unchanged giving the background to be the AdS_2. However, the Einstein's equation (1.62) gets modified to

$$g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}(1-\Phi^2) = \frac{1}{2}\Big(\partial_\mu f\partial_\nu f - \frac{g_{\mu\nu}}{2}(\partial f)^2\Big) \,, \tag{1.66}$$

whose $\mu\nu = ++$ and $\mu\nu = --$ components, in the conformal gauge, are

$$- e^{2\omega}\partial_\pm(e^{-2\omega}\partial_\pm\Phi^2) = \frac{1}{2}\partial_\pm f\partial_\pm f \,. \tag{1.67}$$

Integrating (1.67), we get the solution for dilaton as

$$\Phi^2 = \frac{M}{(x^+-x^-)} \,, \qquad M = M_0 + I^+ - I^- \,, \tag{1.68}$$

where M_0 is any vacuum ($f=0$) solution, in particular, $M_0 = a + x^+ - x^-$ in (1.63) and

$$I^\pm(x^+,x^-) = \frac{1}{2}\int dx'^\pm(x'^\pm - x^\mp)(x'^\pm - x^\pm)\partial_\pm f\partial_\pm f \,. \tag{1.69}$$

Also the equation of motion for f *i.e.* $\nabla^2 f = 0$ shows that f is a free scalar field propagating in the AdS_2 background.

Let us study some peculiarities of this solution *i.e.* the AdS_2 background and the dilaton (1.68), with the freely propagating scalar perturbation f on the AdS_2 background. Naively, it appears that the scalar perturbation does not backreact on the AdS_2 geometry, allowing for finite energy excitations on the AdS_2 background, contrary to our understanding of AdS_2 [33, 34]. However, now in S_{grav} in (1.60), the geometry *i.e.* the gravity sector is not just the metric but characterized by both the metric and dilaton together. The modification (1.68) to the dilaton, indeed, reflects the backreaction of the perturbation f on the geometry. More favourably, in this setup, the metric remains to be the AdS_2 and the effects of backreaction due to f are contained entirely in Φ^2, thus, making the backreaction tractable.

This setup also allows us to study the effects of backreaction on boundary correlation functions systematically. To see this, let us consider an operator $\mathcal{O}_f$, dual to the scalar field f, in the boundary theory at $\rho \to 0$. To compute correlators of f ,we continue to the Euclidean time $\tau = it\,;\, x \equiv -i\tau + z = t + z = x^+$ and $\bar{x} \equiv i\tau + z = -(t-z) = -x^-$ and regulate the boundary with a UV cutoff $\rho = \epsilon$. Then using the bulk equations (1.61) and $\nabla^2 f = 0$ in $S^E = S^E_{grav} + S^E_f$, where the superscript E stands for Euclidean, we get the onshell boundary action

$$S^E = -\frac{1}{16\pi G_2}\int_\epsilon d\tau\Big(\frac{2}{\epsilon} - \frac{2\Phi^2_{bdy}}{\epsilon}\Big) + \frac{1}{32\pi G_2}\int_\epsilon d\tau\sqrt{\gamma}\, n_\mu f\partial^\mu f\ . \tag{1.70}$$

Adding the local counter-term $S_{ct} = \frac{1}{8\pi G_2}\int_\epsilon d\tau(\frac{1}{\epsilon} - \frac{\Phi^2_{bdy}}{\epsilon})$ to cancel the divergent terms (as $\epsilon \to 0$) in S^E and using the outward normal $n_\rho = -\frac{1}{\rho}$, we get the renormalized onshell boundary action

$$S_{ren} = -\frac{1}{32\pi G_2}\int d\tau f\partial_\rho f\ . \tag{1.71}$$

Using the bulk-to-boundary propagator $K(\rho;\tau,\tau') = \frac{1}{2\pi}\left(\frac{1}{x+i\tau'} + \frac{1}{\bar{x}-i\tau'}\right)$, we get f in terms of it's non-normalizable mode $j(\tau) = \lim_{\rho\to 0} f(\tau,\rho)$ as $f(\tau,\rho) = \int d\tau'\, K(\rho;\tau,\tau')$ $j(\tau')$. Substituting this $f(\tau,\rho)$ in S_{ren} above gives

$$S_{ren}[j] = -\frac{1}{32\pi G_2}\int d\tau d\tau' \frac{1}{(\tau-\tau')^2} j(\tau) j(\tau')\ . \tag{1.72}$$

This gives the 2-point function $\langle\mathcal{O}_f(\tau)\mathcal{O}_f(\tau')\rangle \sim \dfrac{1}{|\tau-\tau'|^2}$ for the boundary operator $\mathcal{O}_f$ of conformal dimension 1, which couples to $j(\tau)$. This result is correct in the

IR AdS_2 region since a massless scalar field f in the 2-dimensional bulk couples to an operator of dimension 1 in the boundary theory.[5] However, this behavior of the 2-point function continues in the UV regime since f does not couple to the dilaton Φ^2. This cannot be true since the backreaction of f breaks the conformal invariance of the theory. This apparent inconsistency is because τ is not the correct time coordinate in the boundary theory. We define the new boundary time coordinate such that the asymptotic behavior of Φ^2 is normalized to the sourceless asymptotic (1.64). In order to do so, let us first note that in Euclidean coordinates, the corrected solution for Φ^2 due to the backreaction of f, is

$$\Phi^2 = \frac{M}{x+\bar{x}} = \frac{M_0 - (I + I^*)}{x+\bar{x}} \; ; \qquad I(x,\bar{x}) = \frac{1}{2}\int dx'(x'+\bar{x})(x'-x)\partial_{x'} f \partial_{x'} f \; , \tag{1.73}$$

where $M_0 = a + x + \bar{x}$ is a vacuum solution. Now, we define the new time coordinate $\tilde{\tau}$ in the boundary theory such that in the new coordinates ($\tilde{x} = \rho - i\tilde{\tau}$)

$$\frac{M}{x+\bar{x}} \approx \frac{a}{\tilde{x}+\bar{\tilde{x}}} \qquad \text{as} \quad \rho \to 0 \; . \tag{1.74}$$

We take two infinitesimally separated Euclidean times in the boundary τ, τ' and $\tilde{\tau}$, $\tilde{\tau}'$, which give, as $\rho \to 0$, $x + \bar{x}' = -i(\tau - \tau') = -i\Delta\tau$ and $\tilde{x} + \bar{\tilde{x}}' = -i(\tilde{\tau} - \tilde{\tau}') = -i\Delta\tilde{\tau}$. Then the normalization for Φ^2 in (1.74), taken at infinitesimally separated Euclidean times, implies the differential equation

$$\frac{\partial\tilde{\tau}}{\partial\tau} = \frac{a}{M(\tau)} \; . \tag{1.75}$$

To solve this differential equation, let us first write the expression for the asymptotic form of $M(\tau)$. From (1.73), using $M_0 = a + x + \bar{x} = a + 2\rho \to a$ as $\rho \to 0$, we get

$$\begin{aligned} M(\tau) = \lim_{\rho\to 0}(x+\bar{x})\Phi^2 = a - \lim_{\rho\to 0}(I + I^*) \\ = a - \lim_{\rho\to 0}\frac{1}{2}\Big[\int dx'(x'+\bar{x})(x'-x)\partial_{x'} f \partial_{x'} f \, + \\ + \int d\bar{x}'(\bar{x}'+x)(\bar{x}'-\bar{x})\partial_{\bar{x}'} f \partial_{\bar{x}'} f\Big], \end{aligned} \tag{1.76}$$

[5]Using (1.10), the conformal dimension of the boundary operator which couples to a bulk massless scalar field is $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2} = 1$ for $m = 0$ and $d = 1$.

which, upon substituting $f = \int d\tau'\, K(\rho;\tau,\tau') j(\tau')$ with $K(\rho;\tau,\tau') = \frac{1}{2\pi}\left(\frac{1}{x+i\tau'} + \frac{1}{\bar{x}-i\tau'}\right)$ gives

$$M(\tau) = a + 4\int_{-\infty}^{\infty} d\tau_1 d\tau_2 H(\tau,\tau_1,\tau_2)\partial_1 j(\tau_1)\partial_2 j(\tau_2)\ ;$$
$$H(\tau,\tau_1,\tau_2) = \frac{(\tau-\tau_1)^2\theta(\tau-\tau_1) - (\tau-\tau_2)^2\theta(\tau-\tau_2)}{\tau_1-\tau_2}\ , \tag{1.77}$$

where $\partial_1 j(\tau_1) = \frac{dj(\tau_1)}{d\tau_1}$, etc. Substituting $M(\tau)$ in (1.75) and integrating $\int d\tilde{\tau} = \int d\tau \frac{a}{M(\tau)}$, we get

$$\tilde{\tau} = \int d\tau \Big(1 + \frac{4}{a}\int_{-\infty}^{\infty} d\tau_1 d\tau_2 H(\tau,\tau_1,\tau_2)\partial_1 j(\tau_1)\partial_2 j(\tau_2)\Big)^{-1}$$
$$= \int d\tau \Big(1 - \frac{4}{a}\int_{-\infty}^{\infty} d\tau_1 d\tau_2 H(\tau,\tau_1,\tau_2)\partial_1 j(\tau_1)\partial_2 j(\tau_2) + O(j^4)\Big)\ . \tag{1.78}$$

To find the transformation from τ to $\tilde{\tau}$, we invert the above solution $\tilde{\tau}(\tau)$ and get

$$\tau = \tilde{\tau} + \gamma(\tilde{\tau}) + O(\tilde{j}^4)\ ;$$
$$\gamma(\tilde{\tau}) = \frac{1}{6\pi a}\int_{-\infty}^{\infty} d\tilde{\tau}_1 d\tilde{\tau}_2 \frac{(\tilde{\tau}-\tilde{\tau}_1)^3\theta(\tilde{\tau}-\tilde{\tau}_1) - (\tilde{\tau}-\tilde{\tau}_2)^3\theta(\tilde{\tau}-\tilde{\tau}_2)}{\tilde{\tau}_1-\tilde{\tau}_2}\partial_{\tilde{1}}\tilde{j}(\tilde{\tau}_1)\partial_{\tilde{2}}\tilde{j}(\tilde{\tau}_2)$$
$$+O(\tilde{j}^4)\ , \tag{1.79}$$

where we have used that $\tilde{j}(\tilde{\tau}) = j(\tau)$ under $\tau \to \tilde{\tau}$. Substituting (1.79) in (1.72), we get the renormalized action in the new boundary time coordinate $\tilde{\tau}$ as

$$S_{ren}[\tilde{j}] = -\frac{1}{32\pi G_2}\int_{-\infty}^{\infty} d\tilde{\tau} d\tilde{\tau}' \frac{1}{(\tilde{\tau}-\tilde{\tau}')^2}\Big(1 + \partial_{\tilde{\tau}}\gamma(\tilde{\tau}) + \partial_{\tilde{\tau}'}\gamma(\tilde{\tau}') \tag{1.80}$$
$$-2\frac{\gamma(\tilde{\tau})-\gamma(\tilde{\tau}')}{\tilde{\tau}-\tilde{\tau}'}\Big)\tilde{j}(\tilde{\tau})\tilde{j}(\tilde{\tau}') + O(\tilde{j}^6)\ .$$

Using the holographic dictionary[6] $\langle \mathcal{O}_f(\tilde{\tau}_1)\mathcal{O}_f(\tilde{\tau}_2)\mathcal{O}_f(\tilde{\tau}_3)\mathcal{O}_f(\tilde{\tau}_4)\rangle \sim \frac{\delta^4}{\delta\tilde{j}(\tilde{\tau}_1)\delta\tilde{j}(\tilde{\tau}_2)\delta\tilde{j}(\tilde{\tau}_3)\delta\tilde{j}(\tilde{\tau}_4)}$ $\log\langle e^{-S_{ren}[\tilde{j}]}\rangle\Big|_{\tilde{j}=0}$, we get (from the terms $\partial_{\tilde{\tau}}\gamma(\tilde{\tau})$, $\partial_{\tilde{\tau}'}\gamma(\tilde{\tau}')$ and $\frac{\gamma(\tilde{\tau})-\gamma(\tilde{\tau}')}{\tilde{\tau}-\tilde{\tau}'}$ in (1.80)) the

[6] The generating functional for the connected correlation functions is $W[j] = -\log Z[j] = -\log(e^{-S^E_{grav}-S^E_f}) \equiv -\log\langle e^{-S_{ren}[j]}\rangle$.

connected 4-point function

$$\begin{aligned}\langle \mathcal{O}_f(\tilde\tau_1)\mathcal{O}_f(\tilde\tau_2)\mathcal{O}_f(\tilde\tau_3)\mathcal{O}_f(\tilde\tau_4)\rangle \sim \frac{1}{G_2 a}\frac{1}{\tilde\tau_{12}^3\tilde\tau_{34}^3}\Big[&\theta(\tilde\tau_{13})(\tilde\tau_{13}^3 - 3\tilde\tau_{13}^2\tilde\tau_{23} - 3\tilde\tau_{13}\tilde\tau_{23}\tilde\tau_{34})\\ &-\theta(\tilde\tau_{23})(\tilde\tau_{23}^3 - 3\tilde\tau_{23}^2\tilde\tau_{13} - 3\tilde\tau_{23}\tilde\tau_{13}\tilde\tau_{34})\\ &-\theta(\tilde\tau_{14})(\tilde\tau_{14}^3 - 3\tilde\tau_{14}^2\tilde\tau_{24} + 3\tilde\tau_{14}\tilde\tau_{24}\tilde\tau_{34}.)\\ &+\theta(\tilde\tau_{24})(\tilde\tau_{24}^3 - 3\tilde\tau_{24}^2\tilde\tau_{14} + 3\tilde\tau_{24}\tilde\tau_{14}\tilde\tau_{34})\Big],\end{aligned} \tag{1.81}$$

where $\tilde\tau_{ij} = \tilde\tau_i - \tilde\tau_j$. The connected 4-point function scales as $\frac{1}{\tilde\tau_{ij}^3}$ rather than $\frac{1}{\tilde\tau_{ij}^4}$, which would be the case if the theory has full conformal symmetry. Thus, we see the explicit breaking of the conformal invariance due to backreaction of the scalar perturbation f. The connected and disconnected (which scales as $\frac{1}{G_2^2\tilde\tau_{ij}^4}$) pieces are comparable when $\tilde\tau \sim a/G_2 \propto aV/G_D$, determining the scale where the conformal behavior breaks down to be $E_{breaking} \sim G_D/aV$. Here V is volume of the $(D-2)$-dimensional compact space X and G_D is the Newton's constant in D dimensions.

From the above analysis, we see that the AdS_2 time coordinate $t(u)$ becomes a dynamical filed on the boundary. Its dynamics is captured by a Schwarzian derivative action [36, 37, 38], which governs the boundary time reparametrizations modulo global $SL(2)$. We see how the Schwarzian action arises in the following subsection.

1.3.2 Symmetries and Schwarzian

A theory with pure AdS_2 admits the full reparametrization symmetry of the time coordinate. This reparametrization symmetry is an asymptotic symmetry and is spontaneously broken by the AdS_2 geometry, while keeping a global $SL(2)$ unbroken. The leading order correction away from the conformal limit (pure AdS_2) breaks this reparametrization symmetry explicitly. This correction describing the *nearly-AdS_2* ($nAdS_2$) is governed by the Schwarzian derivative action which is invariant under the unbroken $SL(2)$. Let us see this pattern of symmetry breaking in some detail below.

Consider pure (Euclidean) AdS_2 with the Poincaré metric [7]

$$ds^2 = \frac{L^2}{\rho^2}(d\tau^2 + d\rho^2)\ , \tag{1.82}$$

which is a hyperbolic disk, and L is the AdS_2 radius. In the asymptotic region *i.e.* large ρ, we cut off the space along a trajectory given by $(\tau(u), \rho(u))$ and parametrized

[7] In this subsection, we always consider Euclidean AdS_2 with the Euclidean time $\tau = it$.

by the coordinate u. We impose the boundary condition on the metric as

$$L^2\frac{(\tau'^2+\rho'^2)}{\rho^2}=\frac{L^2}{\epsilon^2}=g_{uu}\ , \tag{1.83}$$

where prime denotes derivative w.r.t. u *i.e.* $\tau'=\frac{d\tau}{du}$ and ϵ is the UV cutoff. This boundary condition gives $\rho=\epsilon\tau'+O(\epsilon^3)$ implying that the boundary of the cutoff region is described by $\tau(u)$ alone. Under a reparametrization $u\to\tilde{u}(u)$, the transformation $\tau(u)\to\tilde{\tau}(\tilde{u})$ takes a given cutout geometry to another one, where the two are locally same in the bulk but have distinct boundaries given by $\tau(u)$ and $\tilde{\tau}(\tilde{u})$. However, not all reparametrizations give distinct cutout geometries and the $SL(2,\mathbb{R})$ subgroup of transformations $\tau(u)\to\frac{a\tau(u)+b}{c\tau(u)+d}$, $ad-bc=1$ give the same cutout geometry. Thus, these cutouts of the Euclidean AdS_2 described by $\tau(u)$ spontaneously break the full reparametrization symmetry to $SL(2,\mathbb{R})$ symmetry.

The transformations of $\tau(u)$ above under reparametrizations $u\to\tilde{u}(u)$ form a symmetry of the action

$$I_0=-\frac{\varphi_0}{16\pi G_2}\Big(\int d^2x\sqrt{g}\,\mathcal{R}+2\int_{bdy}\sqrt{\gamma}K\Big)\ , \tag{1.84}$$

where φ_0 is a constant. This action is topological and typically arises as a background term in dimensional reductions from higher dimensional systems. Extremal black holes and branes in these higher dimensional theories have AdS_2 throat with a constant dilaton φ_0 as their near horizon geometry and the action I_0 gives the extremal entropy $S_{BH}=\frac{\varphi_0}{4G_2}$ (see *e.g.* sec. 3.1.1.2, in particular, the discussion around (3.40), where we show this in some detail).

Next we include the leading effects which explicitly break the conformal symmetry slightly. This slight breaking leads to the $nAdS_2$ geometry, which is described by the Jackiw-Teitelboim theory [112, 113]

$$I_{JT}=-\frac{1}{16\pi G_2}\int d^2x\sqrt{g}\,\varphi\Big(\mathcal{R}+\frac{2}{L^2}\Big)-\frac{1}{8\pi G_2}\int_{bdy}\sqrt{\gamma}\,\varphi_{bdy}\,K\ , \tag{1.85}$$

where φ is the dilaton with φ_{bdy} its boundary value and $\varphi\ll\varphi_0$. The equation of motion for the dilaton, $\mathcal{R}+\frac{2}{L^2}=0$ fixes the metric to be AdS_2. In 2-dimensions, the Einstein tensor vanishes identically and Einstein's equations become

$$T^{(\varphi)}_{\mu\nu}=\frac{1}{8\pi G_2}(\nabla_\mu\nabla_\nu\varphi-g_{\mu\nu}\nabla^2\varphi+g_{\mu\nu}\varphi)=0\ , \tag{1.86}$$

whose general solution is $\varphi=\frac{\alpha+\gamma\tau+\delta(\tau^2+\rho^2)}{\rho}$.

Now let us analyze the JT action. We solve only the dilaton equation, which gives that the background is AdS_2. We do not solve the Einstein equation (1.86), but motivated by the general solution for φ above and also from (1.64), we impose the boundary condition on φ as

$$\varphi_{bdy} = \frac{\varphi_r(u)}{\epsilon} \ , \tag{1.87}$$

where $\varphi_r(u)$ is finite at the boundary (as $\epsilon \to 0$). Using the AdS_2 solution, the bulk part of the JT action vanishes and I_{JT} reduces to the Gibbons-Hawking boundary action. We take the boundary to be given by $(\tau(u), \rho(u))$ and impose the boundary conditions (1.83) on the metric and (1.87) on the dilaton, which gives $I_{JT} = -\frac{1}{8\pi G_2}\int du \frac{L}{\epsilon}\frac{\varphi_r(u)}{\epsilon}K$. Evaluating K, expanding in ϵ and dropping the divergent terms (as $\epsilon \to 0$, which can be done by adding a suitable counter-term), we get from the $O(\epsilon^2)$ term

$$I_{JT} \ \to \ I_{Sch} = -\frac{1}{8\pi G_2}\int du\varphi_r(u) Sch(\tau(u), u) \ ; \qquad Sch(\tau(u), u) = \frac{\tau'''}{\tau''} - \frac{3}{2}\frac{(\tau'')^2}{(\tau')^2} \ . \tag{1.88}$$

We have worked out the details of this calculation in Appendix B.2. Thus the leading correction which breaks the exact conformal symmetry slightly, achieved by the taking the non-constant dilaton, is governed by the Schwarzian derivative action for the reparametrizations.

The equation of motion obtained by varying the Schwarzian action I_{Sch} with respect to $\tau(u)$ is

$$\left[\frac{1}{\tau'}\left(\frac{(\tau'\varphi_r)'}{\tau'}\right)'\right]' = 0 \ . \tag{1.89}$$

For constant $\varphi_r = \bar{\varphi}_r$, the Schwarzian equation of motion reduces to $\frac{\bar{\varphi}_r}{\tau'}(Sch(\tau(u), u))'$ $= 0$. So for non-trivial functions $\tau(u)$ *i.e.* $\tau' \neq 0$, the Schwarzian equation of motion becomes $(Sch(\tau(u), u))' = 0$ giving $Sch(\tau(u), u) = constant$.

We take $\varphi_r = \bar{\varphi}_r$ to be constant and perform a transformation $\tau(u) = \tan\frac{\tilde{\tau}(u)}{2}$. $\tilde{\tau}(u) = \frac{2\pi}{\beta}u$ gives a constant Schwarzian and thus is a solution to the Schwarzian equation of motion. It describes thermal AdS_2 with temperature $T = \frac{1}{\beta}$. Evaluating the Schwarzian action on this solution gives $I_{Sch} = -2\pi^2 CT$, with $C \equiv \frac{\bar{\varphi}_r}{8\pi G_2}$ and leads to the near-extremal correction to the entropy $\Delta S = 4\pi^2 CT$.

Coupling to matter and chaos

We couple the JT theory (*i.e.* the Schwarzian action) to matter and compute correlation functions for the dual operators in the boundary theory. We see that the correction to the out of time order 4-point function shows chaotic behaviour [36, 37, 38].

Let us consider a massive scalar field coupled minimally to the JT theory (1.85),

$$I_m = \int d^2x \frac{\sqrt{g}}{2} \left[\partial_\mu \chi \partial^\mu \chi + m^2 \chi^2 \right] . \tag{1.90}$$

Due to the minimal coupling, the dilaton equation is unchanged and gives the background to be AdS_2. However, it modifies the Einstein's equation (1.86) and thus couples to gravity non-trivially :

$$T^{(\varphi)}_{\mu\nu} + T^{(m)}_{\mu\nu} = 0 \; ; \qquad T^{(m)}_{\mu\nu} = \partial_\mu \chi \partial_\nu \chi - \frac{g_{\mu\nu}}{2} (\partial\chi)^2 - \frac{g_{\mu\nu}}{2} m^2 \chi^2 \; . \tag{1.91}$$

Solving the equation of motion $(\nabla^2 - m^2)\chi = 0$, which follows from (1.90), in the AdS_2 background, the asymptotic behaviour of the solution is

$$\chi = \rho^{1-\Delta} \tilde{\chi}_r(\tau) + \cdots \; , \qquad \Delta = \frac{1 + \sqrt{1 + 4m^2 L^2}}{2} \; , \tag{1.92}$$

where the non-normalizable mode $\tilde{\chi}_r(\tau)$ acts as a source for the boundary operator $\mathcal{O}_\chi$ of dimension Δ. The onshell boundary action is

$$I_m = -c_0 \int d\tau_1 d\tau_2 \frac{\tilde{\chi}_r(\tau_1)\tilde{\chi}_r(\tau_2)}{|\tau_1 - \tau_2|^{2\Delta}} \; , \qquad c_0 = \frac{\Delta \Gamma(\Delta)}{\sqrt{\pi}\,\Gamma(\Delta - \frac{1}{2})} \; . \tag{1.93}$$

This is a standard result for a 2-point function of an operator of conformal weight Δ in a CFT, suggesting that the boundary theory has the full conformal symmetry. However, τ is the bulk time coordinate and we should be using the correct boundary time u. Taking the boundary given by $(\tau(u), \rho(u))$ with $\rho = \epsilon \tau'$, the asymptotic behaviour of χ is

$$\chi = \epsilon^{1-\Delta} \chi_r(u) + \cdots \; ; \qquad \chi_r(u) = \tau'(u)^{1-\Delta} \tilde{\chi}_r(\tau) \; . \tag{1.94}$$

Then the onshell boundary action on the boundary given by $(\tau(u), \rho(u))$ is

$$I_m = -c_0 \int du_1 du_2 \left[\frac{\tau'(u_1)\tau'(u_2)}{(\tau(u_1) - \tau(u_2))^2} \right]^{\Delta} \chi_r(u_1) \chi_r(u_2) \; , \tag{1.95}$$

showing corrections to the 2-point function owing to the background being *nearly-AdS_2* with broken conformal symmetry. The boundary partition function to leading order in G_2 is $Z[\chi_r] \approx e^{-I_0 - I_{Sch} - I_m}$.

From the above analysis, we have $I_{Sch} \sim G_2^{-1}$ and $I_m \sim \Delta^{3/2}$ for large Δ. So as long as Δ grows slower than $G_2^{-2/3}$, the backreaction of χ is suppressed and the saddle of $Z[\chi_r]$ *i.e.* the solution $\tau(u)$ is evaluated with the Schwarzian action alone. In the strict $G_2 \to 0$ limit, the only non-zero connected correlator of the operator V dual to χ is

$$\langle V(u_1)V(u_2)\rangle \sim \Big[\frac{\tau'(u_1)\tau'(u_2)}{(\tau(u_1)-\tau(u_2))^2}\Big]^{\Delta} . \tag{1.96}$$

Now let us consider the gravitational loop corrections coming from $\tau(u)$. To compute these corrections, we recourse to perturbation theory by expanding around the thermal saddle (with $\beta = 2\pi$)

$$\tau(u) = \tan\left(\frac{u+\varepsilon(u)}{2}\right) , \tag{1.97}$$

which upon substitution in the Schwarzian action gives $I^{(\varepsilon)}_{Sch} = -\frac{C}{2}\int du(\varepsilon'^2 - \varepsilon'')$ and leads to

$$\langle \varepsilon(u_1)\varepsilon(u_2)\rangle = \frac{1}{2\pi C}\Big[-\frac{(|u_1-u_2|-\pi)^2}{2} + (|u_1-u_2|-\pi)\sin|u_1-u_2| + a + b\cos|u_1-u_2|\Big], \tag{1.98}$$

where a and b are constants[8]. With the perturbed $\tau(u)$ above, we get

$$\Big[\frac{\tau'(u_1)\tau'(u_2)}{(\tau(u_1)-\tau(u_2))^2}\Big]^{\Delta} = \frac{1}{(2\sin\frac{u_{12}}{2})^{2\Delta}}\Big(1 + \mathcal{B}(u_1,u_2) + \mathcal{C}(u_1,u_2) + O(\varepsilon^3)\Big) , \tag{1.99}$$

where $u_{12} = u_1 - u_2$, $\mathcal{C}(u_1,u_2) \sim O(\varepsilon^2)$ and

$$\mathcal{B}(u_1,u_2) = \Delta\left(\varepsilon'(u_1) + \varepsilon'(u_2) - \frac{(\varepsilon(u_1)-\varepsilon(u_2))}{\tan\frac{u_{12}}{2}}\right) . \tag{1.100}$$

Expanding the exponential in the partition function $Z[\chi_r]$ and taking the expectation value in the linearized Schwarzian theory ($I^{(\varepsilon)}_{Sch}$), we get the generator of connected

[8] $a = 1 + \frac{\pi^2}{6}$ and $b = \frac{5}{2}$ [36, 143].

correlators, from the terms quadratic in ε, as[9]

$$\begin{aligned}\log\langle e^{-I_m}\rangle &= c_0\int du_1du_2\frac{\chi_r(u_1)\chi_r(u_2)}{(2\sin\frac{u_{12}}{2})^{2\Delta}}[1+\langle\mathcal{C}(u_1,u_2)\rangle]\\ &+\frac{c_0^2}{2}\int du_1du_2du_3du_4\frac{\chi_r(u_1)\chi_r(u_2)\chi_r(u_3)\chi_r(u_4)}{(2\sin\frac{u_{12}}{2})^{2\Delta}(2\sin\frac{u_{34}}{2})^{2\Delta}}[1+\langle\mathcal{C}(u_1,u_2)\rangle\\ &\qquad+\langle\mathcal{C}(u_3,u_4)\rangle+\langle\mathcal{B}(u_1,u_2)\mathcal{B}(u_3,u_4)\rangle]\\ &+O(G_2^2)\ . \end{aligned}\tag{1.101}$$

With an anticipation of the chaotic behaviour, which is seen through an out-of-time-order correlator of two operators at different times [127, 128], we consider coupling two scalar fields $\chi^{(1)}$ and $\chi^{(2)}$ having the same mass to the JT theory. Then proceeding with the same calculation as above, we get the generator of connected correlators as $\log\langle e^{-I_m^{(1)}-I_m^{(2)}}\rangle = \langle I_m^{(1)}\rangle+\langle I_m^{(2)}\rangle+\frac{\langle(I_m^{(1)}+I_m^{(2)})^2\rangle}{2}+\cdots$. With V and W as operators dual to $\chi^{(1)}$ and $\chi^{(2)}$, we compute the following 4-point function

$$\begin{aligned}F_{V_1V_2W_3W_4} &\equiv \frac{\langle V(u_1)V(u_2)W(u_3)W(u_4)\rangle-\langle V(u_1)V(u_2)\rangle\langle W(u_3)W(u_4)\rangle}{\langle V(u_1)V(u_2)\rangle\langle W(u_3)W(u_4)\rangle}\\ &= \langle\mathcal{B}(u_1,u_2)\mathcal{B}(u_3,u_4)\rangle\ .\end{aligned}\tag{1.102}$$

Using (1.100), we get

$$\begin{aligned}F_{V_1V_2W_3W_4} = \Delta^2\Big[&\langle\varepsilon_1'\varepsilon_3'\rangle+\langle\varepsilon_1'\varepsilon_4'\rangle+\langle\varepsilon_2'\varepsilon_3'\rangle+\langle\varepsilon_2'\varepsilon_4'\rangle\\ &+\frac{1}{\tan\frac{u_{12}}{2}}\Big(-\langle\varepsilon_1\varepsilon_3'\rangle-\langle\varepsilon_1\varepsilon_4'\rangle+\langle\varepsilon_2\varepsilon_3'\rangle+\langle\varepsilon_2\varepsilon_4'\rangle\Big)\\ &+\frac{1}{\tan\frac{u_{34}}{2}}\Big(-\langle\varepsilon_1'\varepsilon_3\rangle+\langle\varepsilon_1'\varepsilon_4\rangle-\langle\varepsilon_2'\varepsilon_3\rangle+\langle\varepsilon_2'\varepsilon_4\rangle\Big)\\ &+\frac{1}{\tan\frac{u_{12}}{2}\tan\frac{u_{34}}{2}}\Big(\langle\varepsilon_1\varepsilon_3\rangle-\langle\varepsilon_1\varepsilon_4\rangle-\langle\varepsilon_2\varepsilon_3\rangle+\langle\varepsilon_2\varepsilon_4\rangle\Big)\Big]\ ,\end{aligned}\tag{1.103}$$

where $\varepsilon_1'\equiv\varepsilon'(u_1)$, etc. Now let us first consider the ordering $u_1>u_2>u_3>u_4$ for the Euclidean times. Using (1.98) followed by a straightforward simplification gives

$$F_{V_1V_2W_3W_4} = \frac{\Delta^2}{2\pi C}\Big(\frac{u_{12}}{\tan\frac{u_{12}}{2}}-2\Big)\Big(\frac{u_{34}}{\tan\frac{u_{34}}{2}}-2\Big)\ .\tag{1.104}$$

Now let us consider a different ordering $u_1>u_3>u_2>u_4$. For this ordering, the only terms that are different in (1.103) than those for the ordering $u_1>u_2>u_3>u_4$ are the terms involving contractions of $\varepsilon(u_2)$ and $\varepsilon(u_3)$, and their derivatives. Then

[9] $Z[\chi_r]\approx e^{-I_0-I_{Sch}^{(\varepsilon)}-I_m}\equiv\langle e^{-I_m}\rangle_{(\varepsilon)}=\langle 1-I_m+\cdots\rangle_{(\varepsilon)}$. The generating functional for connected correlators is $W[\chi_r]=-\log Z[\chi_r]\approx-\log\langle 1-I_m\rangle_{(\varepsilon)}=\langle I_m\rangle_{(\varepsilon)}+\frac{\langle I_m^2\rangle_{(\varepsilon)}}{2}+\cdots$.

adding and subtracting these contractions between $\varepsilon(u_2)$ and $\varepsilon(u_3)$, we get

$$F_{V_1W_3V_2W_4} = F_{V_1V_2W_3W_4} + \Delta^2\Big[\langle\varepsilon_2'\varepsilon_3'\rangle_{(32)} - \langle\varepsilon_2'\varepsilon_3'\rangle_{(23)} + \frac{\langle\varepsilon_2\varepsilon_3'\rangle_{(32)} - \langle\varepsilon_2\varepsilon_3'\rangle_{(23)}}{\tan\frac{u_{12}}{2}} - \frac{\langle\varepsilon_2'\varepsilon_3\rangle_{(32)} - \langle\varepsilon_2'\varepsilon_3\rangle_{(23)}}{\tan\frac{u_{34}}{2}} - \frac{\langle\varepsilon_2\varepsilon_3\rangle_{(32)} - \langle\varepsilon_2\varepsilon_3\rangle_{(23)}}{\tan\frac{u_{12}}{2}\tan\frac{u_{34}}{2}}\Big], \quad (1.105)$$

where the subscripts (23) and (32) denote the orderings $u_2 > u_3$ and $u_3 > u_2$ respectively. Then a straightforward simplification using (1.98) gives

$$F_{V_1W_3V_2W_4} = F_{V_1V_2W_3W_4} + \frac{\Delta^2}{C}\Big[\frac{\sin(\frac{u_{12}+u_{34}}{2}) - \sin(\frac{u_{14}+u_{23}}{2})}{\sin\frac{u_{12}}{2}\sin\frac{u_{34}}{2}} + \frac{u_{23}}{\tan\frac{u_{12}}{2}\tan\frac{u_{34}}{2}}\Big]. \quad (1.106)$$

Now restoring the temperature by rescaling $u_i \to \frac{2\pi}{\beta}u_i$, continuing to the Lorentzian time $u = i\hat{u}$ and choosing $\hat{u}_1 = a$, $\hat{u}_2 = 0$, $\hat{u}_3 = b + \hat{u}$, $\hat{u}_4 = \hat{u}$ such that $\frac{\beta}{2\pi} \ll \hat{u} \ll \frac{\beta}{2\pi}\log\frac{C}{\beta}$, we get the out-of-time-order 4-point function

$$F_{V_{\hat{1}}W_{\hat{3}}V_{\hat{2}}W_{\hat{4}}} \sim \frac{\beta\Delta^2}{C}e^{\frac{2\pi}{\beta}\hat{u}}\,, \quad (1.107)$$

where $a, b \sim \beta$. Thus, we see that the out-of-time-order correlator $F_{V_{\hat{1}}W_{\hat{3}}V_{\hat{2}}W_{\hat{4}}}$ grows exponentially at a rate $\frac{2\pi}{\beta}$. This is a chaotic behaviour with the Lyapunov exponent $\lambda_L = \frac{2\pi}{\beta}$ saturating the chaos bound [127, 128].

The description of $nAdS_2$ by the Schwarzian action and the saturation of the chaos bound discussed above are also obtained in [37], with the effective action as the Schwarzian coupled to a conformal quantum mechanics. Further, it is shown that this effective action is that of a hydrodynamics, using the classification of [129].

1.3.3 The SYK model

The various aspects of $nAdS_2$ discussed above are related to parallel studies in the Sachdev-Ye-Kitaev (SYK) model [40, 41], discussed more recently in *e.g.* [130, 131, 132] and related SYK/tensor models *e.g.* [133, 134, 135, 136, 137, 138, 139, 140, 141, 142]. Here we state a few properties of the SYK model showing connections with the $nAdS_2$ dynamics above. A detailed discussion on various aspects of the SYK and tensor models is given in [143, 144].

The SYK model is a quantum mechanical model of interacting Majorana fermions with the Hamiltonian $H = \sum_{i,j,k,l=1}^{N} J_{ijkl}\psi^i\psi^j\psi^k\psi^l$. J_{ijkl} is an all-to-all random coupling drawn from a Gaussian distribution with mean $\mu = 0$ and variance $\sigma = \frac{\sqrt{3!}J}{N^{3/2}}$.

The generalization to q-body interaction has variance $\sigma \propto \frac{J}{N^{\frac{q-1}{2}}}$. In the large N limit, at low energies, the SYK model exhibits an emergent reparametrization symmetry $t \to f(t)$. This reparametrization symmetry is spontaneously broken by the vacuum to $SL(2, \mathbb{R})$. The deformation away from the strict IR breaks the reparmetrization symmetry explicitly and is governed by the Schwarzian derivative action for $t(u)$. This symmetry breaking pattern with the appearance of the Schwarzian is similar to that in the JT theory describing the $nAdS_2$ dynamics. The SYK model also shows chaotic behaviour in parallel to the JT theory.

Chapter 2

Shear diffusion in hyperscaling violating Lifshitz theories

Understanding hydrodynamic behaviour in the nonrelativistic gauge/gravity dualities, which we briefly reviewed in Sec. 1.2, is of great interest: see *e.g.* [145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160] for previous and recent investigations. In [161], we had studied the shear diffusion constant in certain hyperscaling violating Lifshitz theories by obtaining it as the coefficient of the diffusion equation satisfied by certain near horizon metric perturbations. This approach in [161] to studying hydrodynamics and viscosity has been somewhat different, and based on Kovtun, Son, Starinets [162]. They observed that metric perturbations governing diffusive shear and charge modes in the near horizon region of the dual black branes of relevance simplify allowing a systematic expansion. This results in a diffusion equation for these shear modes on a stretched horizon, with universal behaviour for the diffusion constant, thereby leading to the viscosity bound [16]. This is akin to the membrane paradigm [163] for black branes, the horizon exhibiting diffusive properties. This approach is based simply on the fact that near horizon metric perturbations lead to a diffusion equation: thus it does not rely on any holographic duality per se. It is of course consistent with holographic results *e.g.* [164, 49] (see *e.g.* [17] for a review of these aspects of hydrodynamics).

In [161], we adapted the membrane-paradigm-like analysis of [162] and studied the shear diffusion constant in bulk $(d+1)$-dimensional hyperscaling violating theories (2.7) with z,θ exponents. Specifically the diffusion of shear gravitational modes on a stretched horizon is mapped to charge diffusion in an auxiliary theory obtained by compactifying one of the $d_i = d-1$ boundary spatial dimensions exhibiting translation invariance. This gives a near horizon expansion for perturbations with modifications involving z, θ. For generic exponents with $d - z - \theta > -1$, we found the shear diffusion constant to be

$$\mathcal{D} = \frac{1}{d-z-\theta+1}\Big(\frac{4\pi}{d+z-\theta-1}\Big)^{\frac{z-2}{z}} T^{\frac{z-2}{z}} \ , \tag{2.1}$$

i.e. power-law scaling with the temperature $T \sim r_0^z$. Studying various special cases motivated the guess

$$\frac{\eta}{s} = \frac{(d-z-\theta+1)}{4\pi}\Big(\frac{4\pi}{d+z-\theta-1}\Big)^{\frac{2-z}{z}} T^{\frac{2-z}{z}} = \frac{1}{4\pi} \ , \tag{2.2}$$

suggesting that $\frac{\eta}{s}$ has universal behaviour. The condition $z < 2 + d_i - \theta$ representing this universal sector appears related to requiring standard quantization from the point of view of holography. When the exponents satisfy $d - z - \theta = -1$, the diffusion constant exhibits logarithmic behaviour, suggesting a breakdown of some sort in this analysis. The exponents arising in null reductions of AdS plane waves or highly boosted black branes [99, 101, 100] satisfy this condition, which can be written as $z = 2 + d_{eff}$.

The analysis above arose solely from perturbations in the metric sector. In theories with a gauge field, the near-horizon diffusion equation analysis above must be extended to also include the gauge field sector which mixes with some of the metric perturbations. The resulting story is somewhat more intricate, both calculationally and conceptually, and is the subject of this chapter. To give a flavour of this, it is worth describing the analysis above in a little more detail. Shear gravitational perturbations h_{xy}, h_{ty}, satisfy the diffusion equation in the near-horizon region within certain approximations, as stated earlier: they are mapped to $U(1)$ gauge field modes $\mathcal{A}_x, \mathcal{A}_t$ upon compactifying the y-direction which enjoys translation invariance. Near horizon membrane currents can be appropriately defined in terms of the field strengths for this gauge field $\mathcal{A}_\mu = (\mathcal{A}_t, \mathcal{A}_x)$, which then can be shown to satisfy Fick's law $j^x = -\mathcal{D}\partial_x j^t$, which in turn using current conservation leads to the diffusion equation $\partial_t j^t = \mathcal{D}\partial_x^2 j^t$, valid within a self-consistent set of approximations imposed near horizon. In terms of the original linearized Einstein equations for metric perturbations (without this y-compactification), the diffusion equation stems from one of the Einstein equations, which is essentially a conservation equation schematically of the form $\partial_x(\partial_r(\#h_{xy})) \sim \#\partial_t(\partial_r(\#h_{ty}))$ where the $\#$ are r-dependent factors. The other linearized Einstein equations are coupled second order equations for h_{ty}, h_{xy}. In the case where the hyperscaling violating Lifshitz theory has a background gauge field A_M, it turns out that the h_{ty} metric perturbation mixes with the gauge field component a_y. The resulting Einstein equations along with the gauge field equation are coupled equations for h_{xy}, h_{ty}, a_y (with the other modes decoupling for modes respecting the y-compactification ansatz), and at first sight they do not reveal any such diffusion-equation-type structure.

Towards understanding this better, it is important to note that the hyperscaling violating Lifshitz black branes here are not charged black branes: the gauge field and scalar here simply serve as sources that support the nonrelativistic metric as a solution to the gravity theory. Using intuition from the fluid-gravity correspondence [165], the fact that these are uncharged black branes means that the near-horizon perturbations are effectively characterized simply by local temperature and velocity fluctuations. Thus since charge cannot enter as an extra variable characterizing the near-horizon region, the structure of the diffusion equation and the diffusion constant should not be dramatically altered by the presence of the gauge field. In light of this intuition, a closer look reveals that the relevant component of the Einstein equation is of the form $\partial_x(\partial_r(\#h_{xy})) \sim \#\partial_t(\partial_r(\#h_{ty})) - \partial_t(\#a_y)$. This naively suggests that perhaps the correct field variable in terms of which the Einstein equation can be recast as a diffusion equation is in fact $\tilde{h}_{ty} \equiv h_{ty} - \int \#a_y dr$. Analyzing this in greater detail shows that this essential logic is consistent, and thereby leads to a generalization of the analysis in [161] mapping shear diffusion to charge diffusion after y-compactification. This results in the same expression for the shear diffusion constant but obtained using the leading near-horizon expressions for $\tilde{h}_{xy} \equiv h_{xy}$ and $\tilde{h}_{ty} \equiv h_{ty} - \int \#a_y dr$.

In this chapter, which is based on [166], we analyze the shear diffusion constant in hyperscaling violating Lifshitz spacetime with the metric perturbations coupled to gauge field perturbations. In sec. 2.1, we briefly review the hyperscaling violating Lifshitz background as a solution to the Einstein-Maxwell-scalar action. In sec. 2.2, we discuss the perturbations in the general hyperscaling violating Lifshitz background incorporating the gauge field perturbations as well. We then describe the various modifications in terms of the new field variables leading to the diffusion equation and thereby the shear diffusion constant. Sec. 2.3 has a Discussion. The Appendices A.1 and A.2 provide various technical details.

2.1 Hyperscaling violating Lifshitz spacetime

Consider an Einstein-Maxwell-scalar action in $(d+1)$-dimensions

$$S = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x\sqrt{-g}\left[\mathcal{R} - \frac{1}{2}\partial_M\Psi\partial^M\Psi - \frac{Z(\Psi)}{4}F_{MN}F^{MN} + V(\Psi)\right] ,$$
$$Z(\Psi) = e^{\lambda\Psi} , \qquad V(\Psi) = V_0 e^{-\gamma\Psi} , \tag{2.3}$$

whose variations with respect to the fields g_{MN}, Ψ and A_M give the field equations

$$\mathcal{R}_{MN} - \frac{1}{2}\partial_M\Psi\partial_N\Psi + g_{MN}\frac{V(\Psi)}{d-1} - \frac{Z(\Psi)}{2}\Big(F_{MP}F_N^{\ P} - \frac{g_{MN}}{2(d-1)}F^2\Big) = 0 , \tag{2.4}$$
$$\nabla_M(Z(\Psi)F^{MN}) = 0 , \tag{2.5}$$
$$\frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g}g^{MN}\partial_N\Psi) + \frac{\partial V(\Psi)}{\partial\Psi} - \frac{1}{4}\frac{\partial Z(\Psi)}{\partial\Psi}F^2 = 0 , \tag{2.6}$$

where $F^2 = F_{MN}F^{MN} = g^{MP}g^{NQ}F_{MN}F_{NQ}$ and $F_{MP}F_N^{\ P} = g^{PQ}F_{MP}F_{NQ}$. These equations admit a hyperscaling violating Lifshitz (hvLif) solution at a finite temperature given by

$$ds^2 = r^{2\theta/d_i}\Big(-\frac{f(r)}{r^{2z}}dt^2 + \frac{dr^2}{r^2 f(r)} + \frac{\sum_{i=1}^{d_i}dx_i^2}{r^2}\Big), \qquad f(r) = 1-(r_0 r)^{d+z-\theta-1} ,$$
$$e^{\Psi} = r^{\sqrt{2(d_i-\theta)(z-\theta/d_i-1)}} , \tag{2.7}$$
$$A_t = \frac{\alpha f(r)}{r^{d_i+z-\theta}} , \qquad F_{rt} = \frac{-\alpha(d_i+z-\theta)}{r^{d_i+z-\theta+1}} ,$$

where

$$\gamma = \frac{2\theta/d_i}{\sqrt{2(d_i-\theta)(z-\theta/d_i-1)}} , \qquad V_0 = (d_i+z-\theta)(d_i+z-\theta-1) ,$$
$$\alpha = -\sqrt{\frac{2(z-1)}{d_i+z-\theta}} , \qquad \lambda = \frac{2\theta/d_i + 2d_i - 2\theta}{\sqrt{2(d_i-\theta)(z-\theta/d_i-1)}} \tag{2.8}$$

and our coordinates are such that $r \to 0$ is the boundary and $r = \frac{1}{r_0}$, the horizon. The temperature of the dual field theory (*i.e.* the Hawking temperature of the hvLif black brane) is

$$T = \frac{(d+z-\theta-1)}{4\pi}\ r_0^z . \tag{2.9}$$

Here $d_i = d-1$ is the boundary spatial dimension while $d_{eff} = d_i - \theta$ is the effective spatial dimension governing various properties of these theories, for instance the entropy density $s \sim T^{d_{eff}/z}$. The null energy conditions following from (2.7), for two null vectors with one having components along (t, r) directions and the other having components along (t, x_i) directions, constrain the exponents, giving

$$(d-1-\theta)\big((d-1)(z-1)-\theta\big) \geq 0 \ , \qquad (z-1)(d-1+z-\theta) \geq 0 \ . \tag{2.10}$$

The second null energy condition above naively suggests that $d+z-\theta-1=0$ is allowed. However, the exponents satisfying $d+z-\theta-1=0$ are not physically allowed as they do not correspond to stable hvLif black brane solutions. This can be seen, for example, through the violation of positivity of the specific heat, which requires $\frac{d-1-\theta}{z} \geq 0$. For $d+z-\theta-1=0$, we get $\frac{d-1-\theta}{z} = \frac{-z}{z} = -1$ violating the positivity of the specific heat. Thus the second null energy condition in (2.10) implies $z \geq 1$ and $d+z-\theta-1>0$, giving a finite Hawking temperature (2.9).

2.2 Perturbations to hyperscaling violating spacetime

We turn on generalized gravitational, gauge field and scalar field perturbations $h_{MN}(x^a, r)$, $a_M(x^a, r)$ and $\psi(x^a, r)$ where $x^a = (t, x_i)$ for $a = 0, 1, \ldots, d_i$ denotes all the boundary coordinates collectively. Later, we will make a certain gauge choice (radial gauge) for the perturbations in order to simplify our calculations. At the linearized level, the Einstein's equations (2.4) are given by

$$\begin{aligned}\mathcal{R}^{(1)}_{MN} = &\frac{1}{2}\partial_M\Psi\partial_N\psi + \frac{1}{2}\partial_M\psi\partial_N\Psi - \frac{V}{2}(h_{MN} - g_{MN}\,\gamma\,\psi) \\ &+\frac{Z}{2}\left[g^{PQ}F_{MP}f_{NQ} + g^{PQ}f_{MP}F_{NQ} - h^{PQ}F_{MP}F_{NQ} + \lambda\psi F_{MP}F_N{}^P\right] \\ &-Z\left[\frac{1}{4}g_{MN}(F_{PQ}f^{PQ} - g^{P\alpha}h^{Q\beta}F_{PQ}F_{\alpha\beta}) + \frac{1}{8}h_{MN}F^2 + \frac{1}{8}\lambda\psi g_{MN}F^2\right] ,\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}&\mathcal{R}^{(1)}_{MN} = \frac{1}{2}[\nabla_P\nabla_N h^P_M + \nabla_P\nabla_M h^P_N - \nabla_P\nabla^P h_{MN} - \nabla_N\nabla_M h] \ ; \\ &f_{MN} = \partial_M a_N - \partial_N a_M \ ; \qquad h = g^{MN}h_{MN} \ .\end{aligned} \tag{2.12}$$

Similarly, the Maxwell's equations (2.5) upto linearized order give

$$\begin{aligned}\nabla_M(Z\,f^{MN}) - \nabla_M(Z\,h^{MP}F_P{}^N) - Z(\nabla_M h^{NQ})F^M{}_Q \\ +\frac{1}{2}(\nabla_M h)Z\,F^{MN} + \lambda\,Z\,F^{MN}\partial_M\psi = 0 \ .\end{aligned} \tag{2.13}$$

Finally, the linearized scalar field equation is

$$\frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g}g^{MN}\partial_N\psi) - \frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g}h^{MN}\partial_N\Psi) + \frac{1}{2}g^{MN}\partial_N\Psi\partial_M h + V\gamma^2\psi - \frac{\lambda Z}{4}(2F_{MN}f^{MN} - 2g^{MP}h^{NQ}F_{MN}F_{PQ} + \lambda\psi F^2) = 0 \,. \tag{2.14}$$

In the linearized field equations (2.11), (2.12), (2.13), (2.14), all indices are raised with respect to the background metric (2.7). For the sake of simplicity our subsequent analysis will be for $d = 3$ (*i.e.* $d_i = 2$) but we expect this procedure can be generalized for higher dimensions.

2.2.1 Perturbations to hyperscaling violating spacetime in 4 dimensions ($d = 3$)

In the presence of a background gauge field, the perturbations in the metric sector h_{xy} and h_{ty} couple to the perturbation to the background gauge field a_y. In the radial gauge (*i.e.* $h_{Mr} = 0$) assuming perturbations of the form $h_{MN} = e^{-i\omega t+iqx}h_{MN}(r)$, the coupled set of equations governing h_{ty}, h_{xy} and a_y become

$$\partial_r(r^{5-z-\theta}f\partial_r a_y) + \frac{\omega^2}{f}r^{3+z-\theta}a_y - q^2r^{5-z-\theta}a_y - k\partial_r(r^{2-\theta}h_{ty}) = 0 \,, \tag{2.15}$$

$$\partial_r(r^{z+\theta-3}\partial_r(r^{2-\theta}h_{ty})) - \frac{r^{z+\theta-3}}{f}q(\omega r^{2-\theta}h_{xy} + qr^{2-\theta}h_{ty}) - k\partial_r a_y = 0 \,, \tag{2.16}$$

$$\partial_r(r^{-1-z+\theta}f\partial_r(r^{2-\theta}h_{xy})) + \frac{r^{z+\theta-3}}{f}\omega(\omega r^{2-\theta}h_{xy} + qr^{2-\theta}h_{ty}) = 0 \,, \tag{2.17}$$

$$q\partial_r(r^{2-\theta}h_{xy}) + \frac{\omega}{f}r^{2z-2}\partial_r(r^{2-\theta}h_{ty}) - k\frac{\omega}{f}r^{z-\theta+1}a_y = 0 \,, \tag{2.18}$$

where

$$k = (2+z-\theta)\alpha \,, \qquad \alpha = -\sqrt{\frac{2(z-1)}{2+z-\theta}} \,. \tag{2.19}$$

Note that the last equation (2.18) is a constraint equation in r which we will eventually use to map to Fick's Law. Now we will further assume that the solutions to the perturbations h_{xy}, h_{ty} and a_y can be expanded as a series in $\frac{q^2}{T^{2/z}}$ which we schematically write as

$$\begin{pmatrix} h_{ty}(t,x,r) \\ h_{xy}(t,x,r) \\ a_y(t,x,r) \end{pmatrix} \equiv \begin{pmatrix} h^{(0)}_{ty}(t,x,r) \\ h^{(0)}_{xy}(t,x,r) \\ a^{(0)}_y(t,x,r) \end{pmatrix} + \begin{pmatrix} h^{(1)}_{ty}(t,x,r) \\ h^{(1)}_{xy}(t,x,r) \\ a^{(1)}_y(t,x,r) \end{pmatrix} + \cdots \ ,$$
$$\begin{pmatrix} h^{(1)}_{ty}(t,x,r) \\ h^{(1)}_{xy}(t,x,r) \\ a^{(1)}_y(t,x,r) \end{pmatrix} = O\left(\frac{q^2}{T^{2/z}}\right) \ . \tag{2.20}$$

Subsequently we will show that this formalism is indeed consistent with the proposed series ansatz. Compactifying the y-direction, the fields in the effective 3-dimensional theory and the 4-dimensional hyperscaling violating theory are related as

$$\mathcal{A}_t = r^{2-\theta} h_{ty} \ , \qquad \mathcal{A}_x = r^{2-\theta} h_{xy} \ , \qquad \chi = a_y \ , \qquad \tilde{g}_{\mu\nu} = r^{\theta-2} g_{\mu\nu} \ , \tag{2.21}$$

and

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu, \qquad Z = r^{4-\theta}, \qquad e^{2\sigma} = g_{yy} = r^{\theta-2} \ , \tag{2.22}$$

where the indices μ, ν take values in (t, x_i, r) excluding y and $\tilde{g}_{\mu\nu}$ is the 3-dimensional metric. In terms of the fields defined above, (2.16), (2.17) and (2.18) take the form

$$\sqrt{-\tilde{g}} e^{4\sigma} \tilde{g}^{tt} \tilde{g}^{xx} \partial_x \mathcal{F}_{tx} + \partial_r(\sqrt{-\tilde{g}} e^{4\sigma} \tilde{g}^{rr} \tilde{g}^{tt} \mathcal{F}_{tr}) = \sqrt{-\tilde{g}} e^{2\sigma} Z F^{rt} \partial_r \chi \ , \tag{2.23}$$
$$\sqrt{-\tilde{g}} e^{4\sigma} \tilde{g}^{tt} \tilde{g}^{xx} \partial_t \mathcal{F}_{tx} + \partial_r(\sqrt{-\tilde{g}} e^{4\sigma} \tilde{g}^{rr} \tilde{g}^{xx} \mathcal{F}_{rx}) = 0 \ , \tag{2.24}$$
$$\tilde{g}^{tt} \partial_t \mathcal{F}_{tr} + \tilde{g}^{xx} \partial_x \mathcal{F}_{xr} = e^{-2\sigma} \tilde{g}_{rr} Z F^{rt} \partial_t \chi \ . \tag{2.25}$$

The perturbation to the background gauge field a_y becomes an effective scalar field χ in the lower dimensional theory whose equation of motion is given by

$$-\frac{r^{3+z-\theta}}{f} \partial_t^2 \chi + r^{5-z-\theta} \partial_x^2 \chi + \partial_r(r^{5-z-\theta} f \, \partial_r \chi) - k \mathcal{F}_{rt} = 0 \ . \tag{2.26}$$

The equations (2.23), (2.24), (2.25) and (2.26) can be derived by compactifying (2.3) along y and varying the effective lower dimensional action with respect $\mathcal{A}_\mu$ and χ as detailed in A.1. The field strengths also satisfy the Bianchi identity

$$\partial_t \mathcal{F}_{rx} + \partial_x \mathcal{F}_{tr} - \partial_r \mathcal{F}_{tx} = 0 \ , \tag{2.27}$$

which is a trivial relation in the higher dimensional theory.

Like the case of gravity-scalar theory with no gauge field analyzed in [161], we could define the horizon currents as $j^\nu = n_\mu \mathcal{F}^{\mu\nu}$. However, unlike the case of gravity-scalar theory where the Einstein's equation corresponding to (2.18) with $a_y = 0$ was mapped to Fick's Law in the y-compactified theory, we do not observe such a structure for (2.18). In the presence of a background gauge field, the behaviour of the perturbations h_{ty} and a_y is expected to be different than that in the gravity-scalar theory in [161] since even in the $q = \Gamma = 0$ sector, they are coupled. The equations governing them follow from (2.15) and (2.16),

$$\partial_r(r^{5-z-\theta} f \partial_r a_y) - k\partial_r(r^{2-\theta} h_{ty}) = 0 \ , \qquad \partial_r(r^{z+\theta-3}\partial_r(r^{2-\theta}h_{ty})) - k\partial_r a_y = 0 \ . \quad (2.28)$$

From the expression for the diffusion constant (2.67) *i.e.* $\mathcal{D} \sim r_0^{z-1} \frac{\tilde{\mathcal{A}}_t}{\tilde{\mathcal{F}}_{rt}}|_{r\sim r_h}$ (we will derive this shortly), one might expect that the detailed solutions to (2.28) is required to compute $\mathcal{D}$. This is a system of two second-order coupled differential equations: eliminating a_y gives a 3rd order differential equation for h_{ty}, and likewise eliminating h_{ty} leads to a 3rd order equation for a_y. Thus we have 3 independent solutions for each of the functions h_{ty} and a_y. These solutions can be found explicitly but we relegate discussing them in detail to Appendix A.2, since it turns out interestingly that the diffusion analysis that follows does not depend in detail on them.

In this regard, it is important to note that the hyperscaling violating Lifshitz black branes here are not charged: the gauge field and scalar here simply serve as sources that support the nonrelativistic metric as a solution to the gravity theory. Using intuition from the fluid-gravity correspondence [165], the fact that these are uncharged black branes means that the near-horizon perturbations must effectively be characterized simply by local temperature and velocity fluctuations. Charge cannot enter as an extra variable characterizing the near-horizon region. Thus the structure of the diffusion equation and the diffusion constant should not be dramatically altered by the presence of the gauge field, although the gauge field perturbation a_y is not "subleading" to the h_{ty} perturbation in any sense, from (2.28), and also the linearized Einstein equations (2.15)-(2.18).

Armed with this intuition, looking closer, we see that we can rearrange (2.18) to write

$$q\partial_r(r^{2-\theta}h_{xy}) + \frac{\omega}{f} r^{2z-2}\partial_r \left(r^{2-\theta}h_{ty} - k\int_{r_c}^{r} ds\ s^{3-z-\theta} a_y \right) = 0 \ . \qquad (2.29)$$

In terms of a new field variable

$$r^{2-\theta}\tilde{h}_{ty} = r^{2-\theta}h_{ty} - k\int_{r_c}^{r} ds\ s^{3-z-\theta}a_y\ , \tag{2.30}$$

the above Einstein equation can be written as

$$q\partial_r(r^{2-\theta}h_{xy}) + \frac{\omega}{f}r^{2z-2}\partial_r(r^{2-\theta}\tilde{h}_{ty}) = 0\ , \tag{2.31}$$

which can now be mapped to a Fick's law in the y-compactified theory with appropriately defined horizon currents j^{μ}, as we will see in the following analysis. At the boundary $r = r_c \sim 0$, we impose the boundary conditions that these perturbations vanish. This in turn motivates a redefinition to new field variables in the y-compactified theory as

$$\begin{aligned}\tilde{\mathcal{A}}_t &= \mathcal{A}_t - k\int_{r_c}^{r} ds\, s^{3-z-\theta}\chi\ ,\\ \tilde{\mathcal{A}}_x &= \mathcal{A}_x\ .\end{aligned} \tag{2.32}$$

For the new gauge field variables $\tilde{\mathcal{A}}_t$ and $\tilde{\mathcal{A}}_x$, we define the field strengths $\tilde{\mathcal{F}}_{rt}$ and $\tilde{\mathcal{F}}_{tx}$ as (in radial gauge $\tilde{\mathcal{A}}_r = \mathcal{A}_r = 0$)

$$\tilde{\mathcal{F}}_{rt} = \mathcal{F}_{rt} - kr^{3-z-\theta}\chi\ , \qquad \tilde{\mathcal{F}}_{tx} = \partial_t\mathcal{A}_x - \partial_x\tilde{\mathcal{A}}_t\ , \qquad \tilde{\mathcal{F}}_{rx} = \mathcal{F}_{rx} = \partial_r\mathcal{A}_x\ . \tag{2.33}$$

In terms of the newly defined field strengths, the Maxwell's Equations (2.23)-(2.25), Bianchi identity (2.27) and the equation of motion for χ (2.26) become

$$\partial_r(r^{z+\theta-3}\tilde{\mathcal{F}}_{rt}) - \frac{r^{z+\theta-3}}{f}\partial_x\left(\tilde{\mathcal{F}}_{tx} - k\int_{r_c}^{r} ds\, s^{3-z-\theta}\partial_x\chi\right) = 0\ , \tag{2.34}$$

$$\partial_r(r^{-1-z+\theta}f\mathcal{F}_{rx}) - \frac{r^{z+\theta-3}}{f}\partial_t\left(\tilde{\mathcal{F}}_{tx} - k\int_{r_c}^{r} ds\, s^{3-z-\theta}\partial_x\chi\right) = 0\ , \tag{2.35}$$

$$\partial_t\tilde{\mathcal{F}}_{rt} - r^{2-2z}f\partial_x\mathcal{F}_{rx} = 0\ , \tag{2.36}$$

$$\partial_t\mathcal{F}_{rx} + \partial_x\tilde{\mathcal{F}}_{tr} - \partial_r\tilde{\mathcal{F}}_{tx} = 0\ , \tag{2.37}$$

$$\partial_r(r^{5-z-\theta}f\partial_r\chi) - k^2r^{3-z-\theta}\chi - \frac{r^{3+z-\theta}}{f}\partial_t^2\chi + r^{5-z-\theta}\partial_x^2\chi - k\tilde{\mathcal{F}}_{rt} = 0\ . \tag{2.38}$$

Differentiating (2.36) w.r.t. t we can eliminate $\mathcal{F}_{rx}$ using the Bianchi Identity (2.37) to get the following equation

$$\partial_t^2\tilde{\mathcal{F}}_{tr} + r^{2-2z}f\partial_x(-\partial_x\tilde{\mathcal{F}}_{tr} + \partial_r\tilde{\mathcal{F}}_{tx}) = 0\ . \tag{2.39}$$

In the the near horizon region approximating the thermal factor as $f(r) \approx (2 + z -$

$\theta)\frac{(1/r_0)-r}{1/r_0}$ and parametrizing the frequency as $\omega = -i\Gamma$ for some positive Γ so that the perturbations decay in time, (2.39) can be written as

$$\left(1 + (2+z-\theta)r_0^{2z-2}\frac{q^2}{\Gamma^2}\cdot\frac{\frac{1}{r_0}-r}{\frac{1}{r_0}}\right)\tilde{\mathcal{F}}_{tr} \approx -(2+z-\theta)r_0^{2z-2}\frac{iq}{\Gamma^2}\cdot\frac{\frac{1}{r_0}-r}{\frac{1}{r_0}}\partial_r\tilde{\mathcal{F}}_{tx} \,. \tag{2.40}$$

Assuming

$$\frac{\frac{1}{r_0}-r}{\frac{1}{r_0}} \ll \frac{\Gamma^2}{q^2 r_0^{2z-2}} \,, \tag{2.41}$$

we differentiate both sides w.r.t. x and approximate (2.40) further

$$\partial_x\tilde{\mathcal{F}}_{tr} \approx (2+z-\theta)\frac{q^2 r_0^{2z-2}}{\Gamma^2}\cdot\frac{\frac{1}{r_0}-r}{\frac{1}{r_0}}\partial_r\tilde{\mathcal{F}}_{tx} \;\equiv\; \epsilon(2+z-\theta)\partial_r\tilde{\mathcal{F}}_{tx} \,, \tag{2.42}$$

where

$$\epsilon = \frac{q^2}{\Gamma^2}r_0^{2z-2}\cdot\frac{\frac{1}{r_0}-r}{\frac{1}{r_0}} \ll 1 \,, \tag{2.43}$$

which is essentially implied by (2.41). In other words, we have

$$\partial_x\tilde{\mathcal{F}}_{tr} \ll \partial_r\tilde{\mathcal{F}}_{tx} \,, \tag{2.44}$$

which in turn simplifies the Bianchi Identity to

$$\partial_t\mathcal{F}_{rx} = \partial_x\tilde{\mathcal{F}}_{rt} + \partial_r\tilde{\mathcal{F}}_{tx} \;\sim\; \partial_r\tilde{\mathcal{F}}_{tx} \,. \tag{2.45}$$

Differentiating (2.35) w.r.t t we get

$$\partial_r(r^{\theta-z-1}f\partial_t\mathcal{F}_{rx}) - \frac{r^{z+\theta-3}}{f}\partial_t^2\tilde{\mathcal{F}}_{tx} + k\frac{r^{z+\theta-3}}{f}\int_{r_c}^{r} s^{3-z-\theta}\cdot\partial_t^2\partial_x\chi(s)\;ds = 0 \,. \tag{2.46}$$

Using the approximate Bianchi identity (2.45), to substitute for $\mathcal{F}_{rx}$ and then multiplying throughout with $-\frac{f}{r^{z+\theta-3}}$ we get a sourced wave equation for the field strength $\mathcal{F}_{tx}$

$$\partial_t^2\tilde{\mathcal{F}}_{tx} - \nu^2\left(\frac{1}{r_0}-r\right)\partial_r\left(\left(\frac{1}{r_0}-r\right)\partial_r\tilde{\mathcal{F}}_{tx}\right) \approx k\int_{r_c}^{r} s^{3-z-\theta}\cdot\partial_t^2\partial_x\chi(s)ds \,, \tag{2.47}$$

where ν is given by

$$\nu = (2+z-\theta)r_0^z \,. \tag{2.48}$$

Likewise for the scalar equation of motion (2.38), using the approximation (2.41) we can drop the term involving $\partial_x^2\chi$ compared to the other terms: thus in the near horizon regime we obtain that in

$$\partial_t^2\chi - \nu^2\Big(\frac{1}{r_0}-r\Big)\partial_r\Big(\Big(\frac{1}{r_0}-r\Big)\partial_r\chi\Big) + \frac{\nu^2k^2r_0}{2+z-\theta}\Big(\frac{1}{r_0}-r\Big)\chi = -\nu k r_0^{4-\theta}\Big(\frac{1}{r_0}-r\Big)\tilde{\mathcal{F}}_{rt} \ , \tag{2.49}$$

the first term is sub-dominant than the third term by a factor of $\frac{\Gamma^2}{\frac{1}{r_0}-r} \ll 1$. Thus the leading order behaviour for the scalar field χ can simply be estimated as

$$\chi^{(0)} \ \approx \ -\frac{2+z-\theta}{\nu k} r_0^{3-\theta}\tilde{\mathcal{F}}_{rt}^{(0)} \ , \tag{2.50}$$

where the superscript (0) is the leading order behaviour of the field χ at $q = \Gamma = 0$ since we have explicitly dropped the subleading derivative terms. Now, in the near horizon regime, we can use (2.42) to find $\partial_x\chi^{(0)} \sim \partial_x\tilde{\mathcal{F}}_{tr}^{(0)} \sim \epsilon\partial_r\tilde{\mathcal{F}}_{tx}^{(0)}$. Using this, we can estimate the right hand side of (2.47) as

$$k\int_{r_c}^{r} s^{3-z-\theta}\cdot\partial_t^2\partial_x\chi(s)ds \approx k\int_{r_c}^{r} r_0^{z+\theta-3}\partial_t^2\partial_x\chi \sim \partial_t^2\int_{r_c}^{r}\epsilon\ \partial_s\tilde{\mathcal{F}}_{tx}ds \sim \epsilon\cdot\partial_t^2\tilde{\mathcal{F}}_{tx} \ . \tag{2.51}$$

What this means is that while the gauge field perturbation a_y (or χ) is not subleading to h_{ty} (or $\mathcal{A}_t$), once we incorporate its effects in terms of the variable $\tilde{h}_{ty}$ (or $\tilde{\mathcal{A}}_t$) the remaining contributions are in fact subleading, as we see here in (2.51).
The above estimate implies that upto leading order, (2.47) is in fact a source free wave equation whose ingoing solution is

$$\tilde{\mathcal{F}}_{tx} = f_1\left(t + \frac{1}{\nu}\log\left(\frac{1}{r_0}-r\right)\right) \ , \tag{2.52}$$

which further implies

$$\partial_t\tilde{\mathcal{F}}_{tx} + \nu\left(\frac{1}{r_0}-r\right)\partial_r\tilde{\mathcal{F}}_{tx} = 0 \ . \tag{2.53}$$

Using (2.45), we can write the above expression as a total derivative w.r.t. t, *i.e.*

$$\partial_t\left(\tilde{\mathcal{F}}_{tx} + \nu\Big(\frac{1}{r_0}-r\Big)\tilde{\mathcal{F}}_{rx}\right) = 0 \ . \tag{2.54}$$

Imposing the boundary condition that the solutions decay as $t \to \infty$ we end up with the following relation

$$\tilde{\mathcal{F}}_{tx} + \nu\left(\frac{1}{r_0}-r\right)\mathcal{F}_{rx} = 0 \ . \tag{2.55}$$

We can derive this result alternatively arguing as follows. We look for an ingoing condition

$$\partial_u H(t,r,x) = (\partial_t + f\,r^{1-z}\partial_r)H(t,r,x) = 0 \tag{2.56}$$

for a function $H(t,r,x)$ defined as

$$H(t,r,x) = r^{\theta-z-1} f\, \partial_r (r^{2-\theta} h_{xy}) \ , \tag{2.57}$$

with $u = t + r_*$ and $v = t - r_*$, for $r_* = \int dr \frac{r^{z-1}}{f(r)}$, being the outgoing and ingoing null coordinates respectively. We have identified $\tilde{\mathcal{A}}_t$ as the relevant perturbative mode and we can write the newly defined field strengths in terms of h_{ty}, h_{xy} and a_y as

$$\partial_t \tilde{\mathcal{F}}_{tx} = -r^{2-\theta}\omega(\omega h_{xy} + q\tilde{h}_{ty}) \ , \qquad \mathcal{F}_{rx} = \partial_r (r^{2-\theta} h_{xy}) = \frac{r^{z+1-\theta}}{f} H \ . \tag{2.58}$$

From (2.35), we have

$$\partial_r H = \frac{r^{z+\theta-3}}{f} \partial_t (\tilde{\mathcal{F}}_{tx} - k \int_{r_c}^{r} ds\, s^{3-z-\theta} \partial_x a_y) \ . \tag{2.59}$$

From (2.51) cancelling the $\partial_t^2 \equiv \Gamma^2$ factor throughout, it follows that

$$k \int_{r_c}^{r} ds\, s^{3-z-\theta} \partial_x a_y \sim \epsilon \tilde{\mathcal{F}}_{tx} \ . \tag{2.60}$$

Substituting this equation in (2.59), we get

$$\partial_r H \approx \frac{r^{z+\theta-3}}{f} \partial_t \tilde{\mathcal{F}}_{tx} \ . \tag{2.61}$$

Since the shear mode is ingoing, we expect on physical grounds that the function H defined in terms of h_{xy} in (2.57) satisfies the ingoing condition (2.56) : this gives

$$\partial_r H = -r^{\theta-2} \partial_t \mathcal{F}_{rx} \ . \tag{2.62}$$

(In the above equations, we have used the y-compactified variables and higher dimensional ones in the same equations, with the understanding that they are interchangeable from the context.) Equating the two expressions for $\partial_r H$ above and using the near horizon approximation for $f(r) \approx (2+z-\theta)\frac{1/r_0 - r}{1/r_0}$, we recover (2.55). This vindicates our intuition on using the $\tilde{\mathcal{A}}_t$, $\tilde{\mathcal{A}}_x$ field variables to obtain the diffusion equation here with the gauge field.

We define the currents in the new tilde variables as

$$j^x = n_r \mathcal{F}^{xr} = \frac{\mathcal{F}_{rx}}{\tilde{g}_{xx}\sqrt{\tilde{g}_{rr}}} \ , \qquad \tilde{j}^t = n_r \tilde{\mathcal{F}}^{tr} = \frac{\tilde{\mathcal{F}}_{rt}}{\tilde{g}_{tt}\sqrt{\tilde{g}_{rr}}} \ , \tag{2.63}$$

since as we have seen, these $\tilde{\mathcal{A}}_\mu$ variables play the role here of the variables $\mathcal{A}_\mu$ in gravity-scalar theory in [161] (it would be interesting to find appropriate modifications of the prescriptions in [167] here). At this point we make another assumption for the $\tilde{\mathcal{A}}_t$, $\tilde{\mathcal{A}}_x$ variables, namely

$$|\partial_t \mathcal{A}_x| \ll |\partial_x \tilde{\mathcal{A}}_t| \ . \tag{2.64}$$

This implies

$$\tilde{\mathcal{F}}_{tx} \approx -\partial_x \tilde{\mathcal{A}}_t \ . \tag{2.65}$$

We can now formulate Fick's Law *i.e.* $j^x = -\mathcal{D}\partial_x \tilde{j}^t$ on the stretched horizon and calculate the diffusion constant as

$$\mathcal{D} \equiv -\frac{j^x}{\partial_x \tilde{j}^t} = -\frac{\tilde{g}_{tt}}{\tilde{g}_{xx}} \frac{\mathcal{F}_{rx}}{\partial_x \tilde{\mathcal{F}}_{rt}} \approx -r_0^{z-1} \frac{\tilde{\mathcal{F}}_{tx}}{\partial_x \tilde{\mathcal{F}}_{rt}} \ , \tag{2.66}$$

where we have used (2.55) to write the third equality. Using (2.65), the diffusion constant at leading order is given by

$$\mathcal{D} = r_0^{z-1} \left. \frac{\tilde{\mathcal{A}}_t}{\tilde{\mathcal{F}}_{rt}} \right|_{r\sim r_h} \ , \tag{2.67}$$

where r_h is the location of the stretched horizon, and the prefactor arises from the metric components.

2.2.1.1 Shear diffusion constant: $z < 4 - \theta$

Making an ansatz of the form (2.20) naturally implies such a series expansion ansatz for the fields $\tilde{\mathcal{A}}_t$, $\tilde{\mathcal{A}}_x$ and χ in the y-compactified theory.

$$\begin{pmatrix} \tilde{\mathcal{A}}_t(t,x,r) \\ \tilde{\mathcal{A}}_x(t,x,r) \\ \chi(t,x,r) \end{pmatrix} \equiv \begin{pmatrix} \tilde{\mathcal{A}}_t^{(0)}(t,x,r) \\ \tilde{\mathcal{A}}_x^{(0)}(t,x,r) \\ \chi^{(0)}(t,x,r) \end{pmatrix} + \begin{pmatrix} \tilde{\mathcal{A}}_t^{(1)}(t,x,r) \\ \tilde{\mathcal{A}}_x^{(1)}(t,x,r) \\ \chi^{(1)}(t,x,r) \end{pmatrix} + \cdots \ ,$$
$$\begin{pmatrix} \tilde{\mathcal{A}}_t^{(1)}(t,x,r) \\ \tilde{\mathcal{A}}_x^{(1)}(t,x,r) \\ \chi^{(1)}(t,x,r) \end{pmatrix} = O\left(\frac{q^2}{T^{2/z}}\right) \ . \tag{2.68}$$

The $q = \Gamma = 0$ sector of (2.34) which is

$$\partial_r (r^{z+\theta-3} \partial_r \tilde{\mathcal{A}}_t) = 0 \ , \tag{2.69}$$

gives us an expression for the leading solution of $\tilde{\mathcal{A}}_t$

$$\tilde{\mathcal{A}}_t^{(0)}(t,x,r) = C\ e^{-\Gamma t+iqx}\int_{r_c}^{r} dr.\ r^{3-z-\theta}\ , \tag{2.70}$$

where C is an arbitrary constant. When $4-z-\theta>0$ the leading solution $\tilde{\mathcal{A}}_t^{(0)}$ has a power-law behaviour

$$\tilde{\mathcal{A}}_t^{(0)}(t,x,r) = e^{-\Gamma t+iqx}\frac{C}{(4-z-\theta)}r^{4-z-\theta} \tag{2.71}$$

It is expected that close to the boundary *i.e.* near $r\approx r_c$ the hyperscaling violating phase breaks down and we require $r_0 r_c \ll 1$. The analogous statement for the boundary field theory will be to assume that the temperature is sufficiently below the UV cut-off. Thus, the condition $z<4-\theta$ arises from the boundary condition that $\tilde{\mathcal{A}}_t^{(0)}\to 0$ as $r\to 0$.

Substituting $\tilde{\mathcal{A}}_t^{(0)}$ in (2.26), the particular solution to the inhomogeneous equation (at $q=0$, $\omega=0$) for χ is

$$\chi^{(0)} = -\frac{C}{k}\ . \tag{2.72}$$

Substituting $\chi^{(0)}=a_y^{(0)}=-C/k$ in (2.30), and considering only the leading order terms we get

$$\tilde{h}_{ty}^{(0)} = h_{ty}^{(0)} + \frac{C}{(4-z-\theta)}r^{2-z}\ . \tag{2.73}$$

Thus, we see that although $h_{ty}=r^{2-z}$ does not satisfy the linearized equations (2.15)-(2.18) at $q=0$, $\omega=0$, the r^{2-z} fall-off appears in the expression for $\tilde{h}_{ty}$ which is indeed the relevant perturbative mode that should be considered. We see that $\tilde{h}_{ty}=\frac{C}{4-z-\theta}r^{2-z}$ and $a_y=-\frac{C}{k}$ indeed satisfy the linearized equations (2.34)-(2.38) at $q=0$, $\omega=0$. Note that this implies that the solutions of interest here in the original variables are $h_{ty}^{(0)}=0$ and $a_y^{(0)}=-\frac{C}{k}$, as can be seen from the form of $\tilde{h}_{ty}$. Thus the solutions of relevance arise entirely from the leading solution to the gauge field perturbation. It is important to note that the solution $a_y^{(0)}=const$ does not change the asymptotic boundary conditions on the background being hyperscaling violating Lifshitz.

The leading solution for $\mathcal{A}_x$ *i.e.* $\mathcal{A}_x^{(0)}$ can be determined by plugging in the series ansatz for $\mathcal{A}_x$ and $\tilde{\mathcal{A}}_t$ in (2.35). The leading order equation is given by

$$\partial_r\mathcal{A}_x^{(0)} = \frac{i\Gamma}{q}\frac{r^{2z-2}}{f}\partial_r\tilde{\mathcal{A}}_t^{(0)}\ . \tag{2.74}$$

Integrating the above and using (2.71) we obtain an expression for $\mathcal{A}_x^{(0)}$ as

$$\mathcal{A}_x^{(0)} = \frac{i\Gamma}{q} \frac{Ce^{-\Gamma t+iqx}}{(2+z-\theta)r_0^{2+z-\theta}} \log(1-(r_0 r)^{2+z-\theta}) \ . \tag{2.75}$$

From (2.70) and the solution derived above, we see that the assumption (2.64) is essentially

$$\frac{\Gamma^2}{q^2} r_0^{2-2z} \log\left(\frac{(1/r_0)}{(1/r_0)-r_h}\right) \ll 1 \ . \tag{2.76}$$

Using $\frac{\Gamma}{q} \sim \frac{q}{r_0^{2-z}}$ and noting that the temperature $T \sim r_0^z$, we can recast this condition as

$$\frac{q^2}{T^{2/z}} \log\left(\frac{(1/r_0)}{(1/r_0)-r_h}\right) \ll 1 \ . \tag{2.77}$$

Physically the above assumption means that we cannot push the stretched horizon located at r_h exponentially close to the horizon $\frac{1}{r_0}$.

Using (2.71) we can now evaluate the shear diffusion constant on the stretched horizon for the hyperscaling violating theory with $4-z-\theta > 0$ as

$$\mathcal{D} = r_0^{z-1} \cdot \frac{1}{(4-z-\theta)r_h} \approx \frac{r_0^{z-2}}{4-z-\theta} + O(q^2) \ . \tag{2.78}$$

The solution for $\tilde{\mathcal{A}}_t^{(0)}$ is evaluated at the stretched horizon r_h: however $r_h \sim \frac{1}{r_0} + O(q^2)$ so to leading order $\mathcal{D}$ is evaluated at the horizon $\frac{1}{r_0}$. It is interesting that the effect of the hyperscaling violating exponent θ cancels in the final expression for $\mathcal{D}$ which is essentially the ratio of $\tilde{\mathcal{A}}_t$ to a field strength $\tilde{\mathcal{F}}_{rt}$ both of which has non-trivial θ-dependence.

Using the expression (2.9) we can express the diffusion constant in terms of the temperature as

$$\mathcal{D} = \frac{1}{4-z-\theta} \left(\frac{4\pi}{2+z-\theta}\right)^{\frac{z-2}{z}} T^{\frac{z-2}{z}} \tag{2.79}$$

which is identical to the one obtained in [161] *i.e.* (2.1) for the case without the gauge field, for $d_i = 2$ boundary spatial dimensions. As discussed there, for pure AdS when $z=1,\ \theta=0$, we recover the standard relation $\mathcal{D} = \frac{1}{4\pi T}$ which further implies $\frac{\eta}{s} = \frac{1}{4\pi}$. Likewise for all theories with $z=1$, it can be seen that θ cancels from the prefactors in $\mathcal{D}$ which becomes $\mathcal{D} = \frac{1}{4\pi T}$. This is in accord with the known behaviour [162] of *e.g.* nonconformal Dp-branes whose dimensional reduction on the transverse sphere S^{8-p} gives rise to hyperscaling violating theories with $z=1,\ \theta \neq 0$ [89]: it would seem reasonable to expect that the sphere should not affect long-wavelength diffusive properties.

2.2.1.2 Shear diffusion constant: $z = 4 - \theta$

Now, we focus on the family of hyperscaling violating solutions where $z = 4 - \theta$. In this case, from (2.70) it follows that the leading solution of $\tilde{\mathcal{A}}_t$ has logarithmic behaviour

$$\tilde{\mathcal{A}}_t^{(0)} = Ce^{-\Gamma t + iqx} \log \frac{r}{r_c} \ , \qquad z = 4 - \theta \ . \tag{2.80}$$

Working further, we can evaluate the diffusion constant upto leading order from (2.67) as

$$\mathcal{D} = r_0^{z-2} \log \frac{1}{r_0 r_c} \ . \tag{2.81}$$

This implies that in the low temperature limit as $r_0 \to 0$, the diffusion constant vanishes if $z > 2$. The new condition on the exponents z and θ, namely $z < 4 - \theta$ appears to be a new constraint which is separate from the null energy conditions

$$(2-\theta)(2(z-1)-\theta) \geq 0 \ , \qquad (z-1)(2+z-\theta) \geq 0 \ . \tag{2.82}$$

The regime of validity for this analysis (equivalently, the "thickness" of the stretched horizon) gets modified in this special case to

$$\exp\left(-\frac{T^{2/z}}{q^2}\frac{1}{\log\frac{1}{r_0 r_c}}\right) \ll \frac{\frac{1}{r_0} - r_h}{\frac{1}{r_0}} \ll \frac{q^2}{T^{2/z}} \log^2 \frac{1}{r_0 r_c} \ . \tag{2.83}$$

However, since we are manifestly in the hydrodynamic regime, it means $r_c \ll \frac{1}{r_0}$ implying $\log \frac{1}{r_0 r_c} \gg 1$. This does not over-constrain the window of the stretched horizon: however the subleading terms contain the logarithmic piece affecting the validity of the series expansion.

The logarithmic scaling necessitates the presence of the UV scale r_c appearing in the diffusion constant in the hydrodynamic description which is manifestly a description at long wavelengths. However from our discussion, it is clear that this is due to the two fall-offs for $\tilde{\mathcal{A}}_t$ coinciding when $z = 4 - \theta$: this leads to the second solution being logarithmic and thence to the scaling above in $\mathcal{D}$. Recall that the parameters z and θ are related precisely in this way when the hyperscaling violating theory is constructed from the x^+-reduction of AdS plane waves (or highly boosted AdS_5 black branes), as well as nonconformal Dp-brane plane waves, as discussed in [161]. (The zero temperature AdS plane waves are structurally similar to the null deformations appearing in the string realizations [92, 93] of $z = 2$ Lifshitz theories, except that the null deformation is normalizable.) As outlined in [161], to gain more insight into the diffusion behaviour, it might be interesting to understand the null reduction of the boosted black brane and its hydrodynamics in greater detail. This might be similar in spirit to nonconformal brane hydrodynamics arising under dimensional reduction of the hydrodynamics of black branes in M-theory [168, 169], although the details are likely to be interestingly different of course. It is also worth noting that in the higher dimensional description, these D-brane plane waves are dual to excited states in the field theory which correspond to anisotropic phases in the boosted frame: the corresponding anisotropic hydrodynamics might be interesting as well (see *e.g.* [170, 171, 172, 173, 174, 175, 176, 177] for previous studies of anisotropic systems and shear viscosity, and *e.g.* [178] for a review of the viscosity bound and violations).

2.2.2 Subleading terms for $z < 4 - \theta$

In this section we will estimate the subleading terms as proposed in (2.68) and explicitly show that $\tilde{\mathcal{A}}_t^{(1)}$, $\tilde{\mathcal{A}}_x^{(1)}$ and $\chi^{(1)}$ (infact all the other terms following it) are subleading compared to the leading order values $\tilde{\mathcal{A}}_t^{(0)}$, $\tilde{\mathcal{A}}_x^{(0)}$ and $\chi^{(0)}$ respectively.

Estimate for $\tilde{\mathcal{A}}_t^{(1)}$

Substituting the series for $\tilde{\mathcal{A}}_t^{(1)}$ from (2.68) in (2.34), we get

$$\partial_r(r^{z+\theta-3}(\partial_r\tilde{\mathcal{A}}_t^{(0)}+\partial_r\tilde{\mathcal{A}}_t^{(1)})+\cdots)-\frac{r^{z+\theta-3}}{f}\partial_x\left(\tilde{\mathcal{F}}_{tx}^{(0)}-k\int_{r_c}^{r}ds\, s^{3-z-\theta}\partial_x\chi^{(0)}+\cdots\right)=0\,. \tag{2.84}$$

The leading term in the above equation is $\partial_r(r^{z+\theta-3}\partial_r\tilde{\mathcal{A}}_t^{(0)})=0$, which is consistent with (2.71). $O(q^2)$ terms in the above equation give

$$\partial_r(r^{z+\theta-3}\partial_r\tilde{\mathcal{A}}_t^{(1)})-\frac{r^{z+\theta-3}}{f}\partial_x\left(\partial_t\mathcal{A}_x^{(0)}-\partial_x\tilde{\mathcal{A}}_t^{(0)}\right)=0\,. \tag{2.85}$$

Here we have neglected $k\int_{r_c}^{r}ds\, s^{3-z-\theta}\partial_x\chi^{(0)}$ since $k\int_{r_c}^{r}ds\, s^{3-z-\theta}\partial_x\chi^{(0)}\ll\tilde{\mathcal{F}}_{tx}^{(0)}$, using the arguments in *e.g.* (2.50), (2.51). Then

$$\partial_r\tilde{\mathcal{A}}_t^{(1)}\sim\frac{1}{r_0}\left(q^2\log\left(\frac{1/r_0}{1/r_0-r}\right)+\frac{\Gamma^2}{r_0^{2(z-1)}}\log^2\left(\frac{1/r_0}{1/r_0-r}\right)^2\right)\tilde{\mathcal{A}}_t^{(0)}\,. \tag{2.86}$$

Using the estimate $\frac{\Gamma}{q}\sim\frac{q}{T^{2/z-1}}$, we can write

$$\partial_r\tilde{\mathcal{A}}_t^{(1)}\sim r_0\left[\frac{q^2}{T^{2/z}}\log\left(\frac{\frac{1}{r_0}}{\frac{1}{r_0}-r}\right)+\frac{q^4}{T^{4/z}}\log^2\left(\frac{\frac{1}{r_0}}{\frac{1}{r_0}-r}\right)\right]\tilde{\mathcal{A}}_t^{(0)}\,. \tag{2.87}$$

Integrating the above equation,

$$\tilde{\mathcal{A}}_t^{(1)} \sim -(1-r_0 r)\Big[\frac{q^2}{T^{2/z}}\Big(1+\log\Big(\frac{\frac{1}{r_0}}{\frac{1}{r_0}-r}\Big)\Big) + \frac{q^4}{T^{4/z}}\Big(1+\log\Big(\frac{\frac{1}{r_0}}{\frac{1}{r_0}-r}\Big)+\log^2\Big(\frac{\frac{1}{r_0}}{\frac{1}{r_0}-r}\Big)\Big)\Big]\tilde{\mathcal{A}}_t^{(0)} \ , \tag{2.88}$$

which implies $\tilde{\mathcal{A}}_t^{(1)} \ll \tilde{\mathcal{A}}_t^{(0)}$.

Estimate for $\mathcal{A}_x^{(1)}$

Substituting the series ansatz for $\mathcal{A}_x$ i.e (2.68) in (2.35) gives

$$\partial_r\mathcal{A}_x^{(0)} + \partial_r\mathcal{A}_x^{(1)} + \cdots = \frac{i\Gamma}{q}\frac{r^{2z-2}}{f}(\partial_r\tilde{\mathcal{A}}_t^{(0)} + \partial_r\tilde{\mathcal{A}}_t^{(1)} + \cdots) \ . \tag{2.89}$$

The leading terms have been derived in (2.75), so we will focus on $O(q^2)$ terms which gives us the equation

$$\partial_r\mathcal{A}_x^{(1)} = \frac{i\Gamma}{q}\frac{r^{2z-2}}{f}\partial_r\tilde{\mathcal{A}}_t^{(1)} \ , \tag{2.90}$$

which give

$$\mathcal{A}_x^{(1)} \sim \Big(\frac{q^2}{T^{2/z}}\log^2\Big(\frac{1/r_0}{1/r_0-r}\Big) + \frac{q^4}{T^{4/z}}\log^3\Big(\frac{1/r_0}{1/r_0-r}\Big)^2\Big)\frac{i\Gamma r_0^{2-2z}}{q}\tilde{\mathcal{A}}_t^{(0)} \ . \tag{2.91}$$

Using $\mathcal{A}_x^{(0)} \sim \frac{i\Gamma r_0^{2-2z}}{q}\log\Big(\frac{1/r_0}{1/r_0-r}\Big)\tilde{\mathcal{A}}_t^{(0)}$,

$$\mathcal{A}_x^{(1)} \sim \Big[\frac{q^2}{T^{2/z}}\log\Big(\frac{1/r_0}{1/r_0-r}\Big) + \frac{q^4}{T^{4/z}}\log^2\Big(\frac{1/r_0}{1/r_0-r}\Big)\Big]\mathcal{A}_x^{(0)} \ , \tag{2.92}$$

which implies $\mathcal{A}_x^{(1)} \ll \mathcal{A}_x^{(0)}$.

Estimate for $\chi^{(1)}$

Finally, substituting the series ansatz for χ i.e (2.68) in (2.38), we get

$$\partial_r(r^{5-z-\theta}f(\partial_r\chi^{(0)}+\partial_r\chi^{(1)})) - k^2 r^{3-z-\theta}(\chi^{(0)}+\chi^{(1)}) - \frac{r^{3+z-\theta}}{f}(\partial_t^2\chi^{(0)}+\partial_t^2\chi^{(1)}) + \cdots = k\partial_r(\tilde{\mathcal{A}}_t^{(0)} + \tilde{\mathcal{A}}_t^{(1)} + \cdots) \ . \tag{2.93}$$

Writing down (2.38) collecting all $O(q^2)$ terms give

$$\partial_r(r^{5-z-\theta}f\partial_r\chi^{(1)}) - k^2r^{3-z-\theta}\chi^{(1)} = \frac{r^{3+z-\theta}}{f}\Gamma^2\chi^{(0)} + k\partial_r\tilde{\mathcal{A}}_t^{(1)} \ . \tag{2.94}$$

To see that $\chi^{(1)}$ is subleading compared to $\chi^{(0)}$ quickly, let us focus on the first term on both sides of the above equation near the horizon;

$$\partial_r\Big(r_0\Big(\frac{1}{r_0} - r\Big)\partial_r\chi^{(1)}\Big) \sim \frac{q^4}{r_0^2}\frac{1}{r_0(\frac{1}{r_0} - r)}\chi^{(0)} \ , \tag{2.95}$$

where we have used $\frac{\Gamma}{q} \sim \frac{q}{r_0^{2-z}}$. Integrating twice, we get

$$\chi^{(1)} \sim \frac{q^4}{r_0^4}\log^2\Big(\frac{1/r_0}{1/r_0 - r}\Big)\chi^{(0)} \ . \tag{2.96}$$

Using (2.77), the above expression shows that $\chi^{(1)} \ll \chi^{(0)}$. This succinct order of magnitude analysis for the subleading nature of $\chi^{(1)}$ can be substantiated through a more detailed analysis as follows. In the near horizon region, (2.94) simplifies to

$$\begin{aligned}\partial_r\Big(\Big(\frac{1}{r_0} - r\Big)\partial_r\chi^{(1)}\Big) &- 2(z-1)r_0\chi^{(1)} \\ &= \frac{r_0^{4-z-\theta}}{2+z-\theta}\left(k\partial_r\tilde{\mathcal{A}}_t^{(1)} + \frac{r_0^{\theta-z-3}}{(2+z-\theta)r_0\Big(\frac{1}{r_0} - r\Big)}\Gamma^2\chi^{(0)}\right) \ .\end{aligned} \tag{2.97}$$

The Green's function for the above equation is effectively the function $G(r,s)$ that satisfies the equation

$$\partial_r\left(\left(\frac{1}{r_0} - r\right)\partial_r G(r,s)\right) - 2(z-1)r_0 \cdot G(r,s) = \delta(r-s) \ . \tag{2.98}$$

The inhomogeneous solution to the Green's function is given by

$$\begin{aligned}G_{in}(r,s) = 2\Theta(r-s)\big[\, &I_0(2\sqrt{2(z-1)}\sqrt{1-r_0s}) \cdot K_0(2\sqrt{2(z-1)}\sqrt{1-r_0r}) \\ &-K_0(2\sqrt{2(z-1)}\sqrt{1-r_0s}) \cdot I_0(2\sqrt{2(z-1)}\sqrt{1-r_0r})\,\big] \ .\end{aligned} \tag{2.99}$$

Correspondingly the inhomogeneous solution to $\chi^{(1)}$ is given by

$$\chi^{(1)} = \int_0^{1/r_0} ds \cdot G_{in}(r,s) \cdot \frac{r_0^{4-z-\theta}}{2+z-\theta}\left(k\partial_r\tilde{\mathcal{A}}_t^{(1)} + \frac{r_0^{\theta-z-3}}{(2+z-\theta)r_0\left(\frac{1}{r_0} - r\right)}\Gamma^2\chi^{(0)}\right) \ , \tag{2.100}$$

where I_0 and K_0 are modified Bessel functions of the first and second kind respectively. Since we are interested only in the near-horizon behaviour for $\chi^{(1)}$. Instead of explicitly performing the integral exactly and then taking the limit $r \to \frac{1}{r_0}$, we will instead approximate the integrand close to $\frac{1}{r_0}$. Upto leading order the modified Bessel functions I_0 and K_0 near $x \approx 0$ are given by

$$I_0(x) \approx 1 \;, \qquad\qquad K_{(0)} \approx -\log x + \log 2 - \gamma \;, \tag{2.101}$$

where γ is the Euler constant. Close to the horizon, we can hence approximate the inhomogeneous part of the Green's function as

$$G_{in}(r,s) = \Theta(r-s) \log\left(\frac{1-r_0 s}{1-r_0 r}\right) \;. \tag{2.102}$$

Hence $\chi^{(1)}$ can be simplified using the above approximation along with (2.87)

$$\begin{aligned}\chi^{(1)} &= \int_0^{\frac{1}{r_0}} ds\, G_{in}(r,s)\, \frac{r_0^{4-z-\theta}}{2+z-\theta}\left(k\partial_r \tilde{\mathcal{A}}_t^{(1)} + \frac{r_0^{\theta-z-3}}{(2+z-\theta) r_0 \left(\frac{1}{r_0}-s\right)}\Gamma^2\chi^{(0)}\right)\\ &\sim \int_0^{\frac{1}{r_0}} ds\, \Theta(r-s)\log\left(\frac{1-r_0 s}{1-r_0 r}\right)\Big[r_0\Big(\frac{q^2}{T^{2/z}}\log\left(\frac{\frac{1}{r_0}}{\frac{1}{r_0}-s}\right)\\ &\quad + \frac{q^4}{T^{4/z}}\log^2\left(\frac{\frac{1}{r_0}}{\frac{1}{r_0}-s}\right)\Big)\tilde{\mathcal{A}}_t^{(0)}\Big] + \int_0^{\frac{1}{r_0}} ds\,\Theta(r-s)\log\left(\frac{1-r_0 s}{1-r_0 r}\right)\frac{q^4}{T^{4/z}}\frac{1}{\frac{1}{r_0}-s}\chi^{(0)}.\end{aligned} \tag{2.103}$$

The above integral can be divided into two parts. One ranging from 0 to r and another from r to $\frac{1}{r_0}$. The Heaviside Theta function is non-zero for $r > s$ only. So, the upper bound in the above integral can simply be replaced with r instead of $1/r_0$. Simplifying and performing the integral over s we get,

$$\begin{aligned}\chi^{(1)} \sim \Big[&\frac{q^2}{T^{2/z}}\Big\{(1-r_0 r) - (1-r_0 r)\log(1-r_0 r) + (1-r_0 r)\log^2(1-r_0 r)\Big\}\\ &+\frac{q^4}{T^{4/z}}\Big\{-(1-r_0 r) + (1-r_0 r)\log(1-r_0 r) - (1-r_0 r)\log^2(1-r_0 r)\\ &+(1-r_0 r)\log^3(1-r_0 r)\Big\}\Big]\tilde{\mathcal{A}}_t^{(0)} + \frac{q^4}{T^{4/z}}\log^2(1-r_0 r)\chi^{(0)} \ll \chi^{(0)}.\end{aligned} \tag{2.104}$$

If we now use the two assumptions mentioned earlier i.e (2.41), (2.77) we explicitly see that $\chi^{(1)} \ll \chi^{(0)}$ thus demonstrating that all subsequent terms in the series are smaller than the leading piece.

Estimate for $h_{ty}^{(1)}$

Note that in (2.20) we proposed the series expansion for the modes h_{ty}, h_{xy} and a_y. From the definition of $\tilde{h}_{ty}$ and the series ansatze (2.20), (2.68), we can write

$$h_{ty}^{(0)}+h_{ty}^{(1)}+\cdots = \Big(\tilde{h}_{ty}^{(0)}+kr^{\theta-2}\int_{r_c}^{r} ds\, s^{3-z-\theta}a_y^{(0)}\Big)+\Big(\tilde{h}_{ty}^{(1)}+kr^{\theta-2}\int_{r_c}^{r} ds\, s^{3-z-\theta}a_y^{(1)}\Big)+\cdots \, . \tag{2.105}$$

From (2.88) and (2.104), we have

$$h_{ty}^{(1)} = \tilde{h}_{ty}^{(1)} + kr^{\theta-2}\int_{r_c}^{r} ds\, s^{3-z-\theta}a_y^{(1)} \sim O\Big(\frac{q^2}{T^{2/z}}\Big)\tilde{h}_{ty}^{(0)} \, . \tag{2.106}$$

Using

$$\tilde{h}_{ty}^{(2)} \sim \frac{q^2}{T^{2/z}}\tilde{h}_{ty}^{(1)} \, , \qquad a_y^{(2)} \sim \frac{q^2}{T^{2/z}}a_y^{(1)} \, , \tag{2.107}$$

we see that

$$\frac{h_{ty}^{(2)}}{h_{ty}^{(1)}} = \frac{\tilde{h}_{ty}^{(2)} + kr^{\theta-2}\int_{r_c}^{r} ds\, s^{3-z-\theta}a_y^{(2)}}{\tilde{h}_{ty}^{(1)} + kr^{\theta-2}\int_{r_c}^{r} ds\, s^{3-z-\theta}a_y^{(1)}} \sim O\Big(\frac{q^2}{T^{2/z}}\Big) \ll 1 \, . \tag{2.108}$$

Thus we see that the mode h_{ty} also admits a series expansion in the parameter $\frac{q^2}{T^{2/z}}$ in the near-horizon region. This is of course expected from the self-consistent series expansions of $\tilde{h}_{ty}, h_{xy}, a_y$.

2.2.2.1 Subleading terms for $z = 4-\theta$

In this case, from the solutions of $\tilde{\mathcal{A}}_t$ (2.80) and $\mathcal{A}_x$ (2.75) we get,

$$\frac{\mathcal{A}_x^{(0)}}{\mathcal{A}_t^{(0)}} \sim \frac{1}{r_0^{2(z-1)}}\frac{\Gamma}{q}\frac{\log(\frac{1/r_0}{1/r_0-r})}{\log(\frac{1}{r_0 r_c})} \, . \tag{2.109}$$

Imposing (2.64) then implies

$$\frac{1}{r_0^{2(z-1)}}\cdot\frac{\Gamma^2}{q^2}\cdot\frac{\log(\frac{1/r_0}{1/r_0-r})}{\log(\frac{1}{r_0 r_c})} \ll 1 \, . \tag{2.110}$$

We can obtain an estimate for $\mathcal{D}$ in this case from the diffusion equation which is $\frac{\Gamma}{q} \sim \frac{q}{T^{2/z-1}}\log(\frac{1}{r_0 r_c})$. Thus, the assumptions in this special case gets modified to (2.83). The subleading term for $\tilde{\mathcal{A}}_t$ now is given by

$$\partial_r\tilde{\mathcal{A}}_t^{(1)} \sim r_0\left[\frac{q^2}{T^{2/z}}\log\left(\frac{\frac{1}{r_0}}{\frac{1}{r_0}-r}\right)+\frac{q^4}{T^{4/z}}\log^2\left(\frac{\frac{1}{r_0}}{\frac{1}{r_0}-r}\right)\log\left(\frac{1}{r_0 r_c}\right)\right]\tilde{\mathcal{A}}_t^{(0)} \, . \tag{2.111}$$

Note that $r_0 r_c \ll 1$ implies that $\log(\frac{1}{r_0 r_c})$ is large which means that the $O(q^4)$ term need not be small even if we are working withing the hydrodynamic regime *i.e.* $\frac{q^2}{T^{2/z}} \ll 1$, suggesting a breakdown of the series expansion. The expression for the the subleading part of χ *i.e.* $\chi^{(1)}$ also changes to

$$\begin{aligned}\chi^{(1)} \sim \Big(\Big[\frac{q^2}{T^{2/z}}\Big\{(1-r_0 r) - (1-r_0 r)\log(1-r_0 r) + (1-r_0 r)\log^2(1-r_0 r)\Big\} \\ +\frac{q^4}{T^{4/z}}\Big\{-(1-r_0 r) + (1-r_0 r)\log(1-r_0 r) - (1-r_0 r)\log^2(1-r_0 r) \\ +(1-r_0 r)\log^3(1-r_0 r)\Big\}\log\Big(\frac{1}{r_0 r_c}\Big)\Big]\tilde{\mathcal{A}}_t^{(0)} \\ +\frac{q^4}{T^{4/z}}\log^2(1-r_0 r)\log^2\Big(\frac{1}{r_0 r_c}\Big)\chi^{(0)}\Big) \quad \ll \quad \chi^{(0)} \,. \qquad (2.112)\end{aligned}$$

From the preceding argument, we see again that the $O(q^4)$ term can be arbitrarily large hinting at a breakdown of the series expansion.

Estimate for $h_{ty}^{(1)}$

In the case when $z = 4-\theta$ (2.105) takes the form

$$h_{ty}^{(0)} + h_{ty}^{(1)} + \cdots = \Big(\tilde{h}_{ty}^{(0)} + kr^{2-z}\int_{r_c}^{r}\frac{ds}{s}a_y^{(0)}\Big) + \Big(\tilde{h}_{ty}^{(1)} + kr^{2-z}\int_{r_c}^{r}\frac{ds}{s}a_y^{(1)}\Big) + \cdots \,. \qquad (2.113)$$

The above further implies that

$$\begin{aligned}h_{ty}^{(1)} &= \tilde{h}_{ty}^{(1)} + kr^{2-z}\int_{r_c}^{r}\frac{ds}{s}a_y^{(1)} \\ &\sim \frac{q^4}{T^{4/z}}(1-r_0 r)\log\Big(\frac{1}{r_0 r_c}\Big)\Big(1+\log\Big(\frac{1/r_0}{1/r_0 - r}\Big) + \log^2\Big(\frac{1/r_0}{1/r_0 - r}\Big)\Big)\tilde{h}_{ty}^{(0)} \\ &\quad +\frac{q^4}{T^{4/z}}(1-r_0 r)\log^2\Big(\frac{1}{r_0 r_c}\Big)\log^2\Big(\frac{1/r_0}{1/r_0 - r}\Big)\chi^{(0)} \,. \qquad (2.114)\end{aligned}$$

The above estimate is written using the estimate $\frac{\Gamma}{q} \sim \frac{q}{T^{2/z-1}}\log(\frac{1}{r_0 r_c})$. Further, the assumptions (2.83) implies that $h_{ty}^{(1)}$ may not be subleading compared to $h_{ty}^{(0)}$ thus suggesting a breakdown of some sort in this analysis.

2.3 Discussion

In this chapter, we have explored in greater detail our investigations of shear diffusion in nonrelativistic hyperscaling violating Lifshitz theories [161], adapting the membrane-paradigm-like analysis [162] of near horizon perturbations. In theories where a gauge field is present as a source for the nonrelativistic metric (along with a scalar), some of the metric perturbations h_{ty}, h_{xy} mix with some of the gauge field perturbations a_y. Since these are uncharged black branes, the near-horizon region should still be characterized by simply temperature and velocity variables, and charge cannot enter. Thus we expect that the gauge field cannot dramatically alter the structure of the near horizon diffusion equation found in [161] without the gauge field. Our analysis in this chapter vindicates this: we find a similar near-horizon analysis can be obtained resulting in a diffusion equation for new field variables $\tilde{h}_{xy} \equiv h_{xy}$ and $\tilde{h}_{ty} \equiv h_{ty} - r^{\theta-2} \int_{r_c}^{r} s^{3-z-\theta} a_y ds$ (for 4 bulk dimensions). Then, as in [161], for $z < 4-\theta$, we obtain universal behaviour for the shear diffusion constant, suggesting that the viscosity bound $\frac{\eta}{s} = \frac{1}{4\pi}$ holds. The regime $z > 4-\theta$ includes *e.g.* hyperscaling violating theories arising from the dimensional reduction of *e.g.* $D6$-branes (giving $d_i = 6, z = 1, \theta = 9$) which do not admit a good gauge/gravity duality (ill-defined asymptotics with gravity not decoupling): however it might be interesting to find and understand reasonable holographic theories whose exponents lie in this window. For $z = 4-\theta$, we find logarithmic behaviour as found previously. The hyperscaling violating Lifshitz theories arising from AdS plane waves (highly boosted black branes) as well as nonconformal brane plane waves [99, 101, 103], fall in this category: this suggests that a null reduction of the hydrodynamics of the boosted black brane might need a closer study to realize this in detail, as we have described. It would be interesting to explore this further.

We have seen the condition $z < 2 + d_i - \theta$ (or $z < 4-\theta$ here, for bulk 4-dims) arising naturally from the perturbations falling off asymptotically (2.70) in our case. We implicitly regard hyperscaling violating theories as infrared phases arising from *e.g.* string realizations in the ultraviolet: however the window $z < 2 + d_i - \theta$ ensures that the ultraviolet structure is essentially unimportant, the diffusion constant arising solely from the near horizon long-wavelength modes. This still needs to be reconciled with a clear holographic calculation: however some preliminary remarks are as follows. We have seen that the $\tilde{h}_{ty}$ mode has asymptotic fall-offs $r^{\theta-2}(\tilde{h}_- + \ldots) + r^{2-z}(\tilde{h}_+ + \ldots)$ in bulk 4-dimensions. For $z < 4-\theta$, the dominant mode near the boundary $r \to r_c \sim 0$ is $r^{\theta-2}$ which is slower, leading to fixed h_- boundary conditions relevant for standard quantization (h_- taken as source). This is the sector that is continuously connected to AdS-like relativistic theories ($z = 1, \theta = 0$), as our perturbation analysis suggests. With the conformal dimensions satisfying $\Delta_- + \Delta_+ = 2 + z - \theta$ [89] (see also [96, 15]), the momentum density operator $\mathcal{P}^i$ has dimension $3-\theta$: so taking $\Delta_+ = 3-\theta$ gives $\Delta_- = z-1$ and $\Delta_- < \Delta_+$ implies $z < 4-\theta$. In a reasonable theory where this is violated, it would seem that the analog of alternative quantization [179] is at work, with fixed h_+ boundary conditions. In this light, $z = 4 - \theta$ is the case where the two fall-offs coincide with $\Delta_- = \Delta_+$, and a logarithmic second solution will arise suggesting logarithmic behaviour in the correlation function as well. This is the case for AdS plane waves (or highly boosted black branes): this may be interesting to explore.

It is worth putting the analysis here leading to (2.1), (2.79), in perspective with the calculation of viscosity via the Kubo formula $\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{xy,xy}(\omega)$, with G^R the retarded Green's function [49], assuming $T_{ij} \sim \eta(\partial_i v_j + \ldots)$ in the dual field theory. The h^x_y perturbation is modeled holographically as a massless scalar leading to the $\langle T_{xy}T_{xy}\rangle$ holographic correlation function (see *e.g.* [145, 150, 151, 154] for various subfamilies in (2.7)). For instance from [154], the appropriate zero momentum $\vec{k} = 0$ solutions to the scalar wave equation eventually lead to $G^R = -i\frac{\omega}{16\pi G}\frac{R^{d_i}}{r^\theta_{hv}}r_0^{d_i-\theta}$ and thereby η: here the metric (2.7) is written as $ds^2 = R^2(\frac{r}{r_{hv}})^{2\theta/d_i}(-f(r)\frac{dt^2}{r^{2z}} + \ldots)$, retaining explicitly the dimensionful factors R and the scale r_{hv} inherent in these theories [89]. Likewise the horizon area gives the entropy density $s = \frac{1}{4G}\frac{R^{d_i}}{r^\theta_{hv}}r_0^{d_i-\theta}$ which leads to $\frac{\eta}{s} = \frac{1}{4\pi}$ in agreement with our analysis. (We have seen that θ disappears from the temperature dependence of $\mathcal{D}$ in (2.1): this is consistent with *e.g.* cases where the hyperscaling violating phase arises from string constructions such as nonconformal branes which are known to have universal $\frac{\eta}{s}$ behaviour.)

In light of the above, note that the Kubo analysis stemming from a zero frequency $\omega \to 0$ limit for the h^x_y mode alone, does not appear to give any insight into where a condition like $z < 2 + d_i - \theta$ could arise from. On the other hand, our analysis here and in [161] in terms of the near-horizon perturbations involves the h_{ty} perturbation as well (as in [162]), which is coupled at nonzero ω to h^x_y, and leads to the diffusion equation. The h_{ty} mode (or $\tilde{h}_{ty}$ here) exhibits this nontrivial behaviour where the normalizable mode can turn around depending on the exponents z, θ, the critical condition being the family $z = 2 + d_i - \theta$ where the two modes coincide. This condition is trivially satisfied for all relativistic theories of interest, with $z = 1,\ \theta = 0$, so the Kubo limit is in perfect agreement with the near horizon diffusion analysis. However in the present nonrelativistic cases, the near horizon perturbations analysis appears to exhibit more structure. It would seem that the structure of these perturbations is straightforward and simply involves analysing gravitational perturbations, not requiring detailed understanding of the holographic dictionary in this case. Therefore assuming that this is reliable, our analysis suggests that the Kubo limit might need to be understood better in theories where $z < 2 + d_i - \theta$ is violated. In the case with a gauge field, the field variable $\tilde{h}_{ty}$ which exhibits this behaviour naively suggests that perhaps a new energy-momentum tensor variable $\tilde{T}_{\mu\nu}$ involving some linear combination of $T_{\mu\nu}$ and the current density j_μ is the relevant hydrodynamic observable that systematically encodes the thermodynamic/hydrodynamic relations between the expansion of the energy-momentum tensor, the shear viscosity η and the diffusion constant $\mathcal{D}$. It would be interesting to explore these issues further.

It is worth mentioning that the analysis of correlation functions and the Kubo formula for computing the shear viscosity was done in a subsequent paper [180]. The authors study the spectrum of quasi-normal modes of shear gravitational perturbations in hvLif black branes. The lowest quasi-normal modes give a dispersion relation from which the shear diffusion constant is obtained. This shear diffusion constant is consistent with our results in this chapter (and [161]). Further, they compute the $\langle T_{xy}T_{xy}\rangle$ correlation function and then using the Kubo formula for η obtain $\frac{\eta}{s} = \frac{1}{4\pi}$ for $z \leq 2 + d_i - \theta$. This includes the special case $z = 2 + d_i - \theta$, where though $\mathcal{D}$ scales logarithmically, $\frac{\eta}{s}$ saturates the viscosity bound.

Chapter 3

AdS_2 dilaton-gravity from reductions of some nonrelativistic theories

AdS_2 throats arise quite generally in the near horizon regions of extremal black holes and black branes, where other fields acquire near constant "attractor" values. This attractor mechanism, first discussed in [181] for BPS black holes in $\mathcal{N} = 2$ theories, arises from extremality rather than supersymmetry, as studied in [182], [183]. In the last several years, this has been ubiquitous in the context of nonrelativistic generalizations of holography: a nice review is [22]. A large family of such theories is obtained by considering Einstein-Maxwell-scalar theories with a negative cosmological constant and potential: the $U(1)$ gauge field and scalar serve to support the nonrelativistic background, typically of the form of a Lifshitz, or hyperscaling violating (conformally Lifshitz) theory. The duals to the bulk uncharged black branes in these hvLif theories capture many features of finite density condensed matter-like systems. Towards studying extremal black branes, we note that charge can be added to these theories by adding an additional $U(1)$ gauge field, as discussed in *e.g.* [184], [185], [186]. Now at extremality, the infrared region approaches an $AdS_2 \times X$ throat, with X typically of the form of an extended transverse plane R^d. The discussion of AdS_2 holography in sec. 1.3 now applies upon compactifying X taken as *e.g.* a torus T^d. This was in fact the broad context for [35]: other recent discussions of reduction from higher dimensional theories appear in *e.g.* [114, 115, 116, 117, 118, 119, 120, 121, 123, 124]; see also [122].

Towards studying such AdS_2 theories arising in this nonrelativistic context, we study effective gravity theories of the above form, with two $U(1)$ gauge fields and a scalar field Ψ with a negative cosmological constant and potential. We focus for concreteness on the charged hyperscaling violating Lifshitz black branes in 4-dimensions described in [185]. In the extremal limit, the near horizon geometry of these charged hyperscaling violating Lifshitz black branes becomes $AdS_2 \times \mathbb{R}^2$. These charged hyperscaling violating Lifshitz attractors arise for certain regimes of the Lifshitz z and hyperscaling violating θ exponents allowed by the energy conditions, with the additional requirement that the theory exhibits hvLif boundary conditions in the ultraviolet: these are perhaps best regarded as intermediate infrared phases themselves in some bigger phase diagram. Then compactifying the two spatial directions as a torus T^2, we dimensionally reduce this charged hvLif extremal black brane to obtain a 2-dimensional dilaton-gravity-matter theory. This theory is equivalent to gravity with a dilaton Φ and an additional scalar Ψ that descends from the hvLif scalar in the higher dimensional theory, along with an interaction potential $U(\Phi, \Psi)$. The interaction potential raises the question of whether the extra scalar destabilizes the AdS_2 regime, possibly in some region of parameter space. Towards understanding this, we study small fluctuations about the extremal AdS_2 background in these theories and argue that these are in fact stable, the stability stemming from the restrictions imposed on z, θ stated above from energy conditions and asymptotic boundary conditions. Studying the action for small fluctuations up to quadratic order, it can be seen that the leading corrections to AdS_2 arise at linear order in $\delta\Phi$ leading again to a Schwarzian derivative action from the Gibbons-Hawking term, although there are subleading coupled quadratic corrections (sec. 3.2). The coefficient of the Schwarzian is proportional to the entropy of the compactified extremal black branes, which being the number of microstates of the background is akin to a central charge of the effective theory. In sec.3.1.1, we first describe in detail the simpler case of the relativistic black brane, which has $z = 1, \theta = 0$, arising in Einstein-Maxwell theory, the extra scalar being absent: at leading order this shows how the Jackiw-Teitelboim theory [112, 113] arises, with subleading terms at quadratic order. We finally study in sec. 3.3 a null reduction of the charged relativistic black brane: this results in charged hvLif black brane backgrounds with specific exponents, but with an extra scalar background profile (for the uncharged case, these coincide with [99]). Sec. 3.4 contains a brief Discussion and Appendices B.1, B.2 and B.3 contain some technical details.

3.1 Einstein-Maxwell theory in 4-dimensions

Einstein-Maxwell theory with a negative cosmological constant is a useful playground for various interesting physics: see *e.g.* [22] for a review. We focus on 4-dimensions for simplicity: as a consistent truncation of M-theory on appropriate 7-manifolds, the bulk gauge field can be taken as the dual to the $U(1)_R$ current. The action is

$$S = \int d^4x\sqrt{-g^{(4)}}\left[\frac{1}{16\pi G_4}\Big(\mathcal{R}^{(4)} - 2\Lambda\Big) - \frac{1}{4}F_{MN}F^{MN}\right] , \tag{3.1}$$

where $\Lambda = -3$ is the cosmological constant in 4-dimensions. The field equations are

$$\mathcal{R}^{(4)}_{MN} - \Lambda g_{MN} - 8\pi G_4\Big(F_{MP}F_N^{\ P} - \frac{g_{MN}}{4}F^2\Big) = 0 , \qquad \partial_M(\sqrt{-g}F^{MN}) = 0 . \tag{3.2}$$

These equations have both electrically and magnetically charged black branes as solutions.

<u>*Magnetic branes:*</u> These are slightly simpler and we discuss them first, mostly reviewing discussions already in the literature. The metric and field strength [187] are

$$ds^2 = -r^2 f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2(dx^2 + dy^2), \qquad f(r) = 1 - \Big(\frac{r_0}{r}\Big)^3 + \frac{Q_m^2}{r^4}\Big(1 - \frac{r}{r_0}\Big),$$
$$F_{xy} = Q_m , \tag{3.3}$$

where Q_m is related to the magnetic charge of the black brane, r_0 is the location of the horizon and $r \to \infty$ is the boundary. In the extremal limit, the Hawking temperature vanishes, fixing the horizon location in relation to the charge,

$$T = \frac{3r_0}{4\pi}\Big(1 - \frac{Q_m^2}{3r_0^4}\Big) = 0 \quad \Longrightarrow \quad Q_m^2 = 3r_0^4 . \tag{3.4}$$

The near horizon geometry of the magnetic black brane becomes $AdS_2 \times \mathbb{R}^2$,

$$ds^2 = -r_0^2 f(r)dt^2 + \frac{dr^2}{r_0^2 f(r)} + r_0^2(dx^2 + dy^2) , \qquad f(r)|_{r\to r_0} \simeq \frac{6}{r_0^2}(r - r_0)^2 . \tag{3.5}$$

We compactify the two spatial dimensions x^i as T^2 and dimensionally reducing with an ansatz for the metric

$$ds^2 = g^{(2)}_{\mu\nu}dx^\mu dx^\nu + \Phi^2(dx^2 + dy^2) , \tag{3.6}$$

with $g^{(2)}_{\mu\nu}$ and Φ being independent of the compact coordinates $x, y \in T^2$. The action (3.1) for the magnetic black brane solution then reduces to

$$S = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g^{(2)}} \left[\Phi^2 \mathcal{R}^{(2)} - 2\Lambda\Phi^2 - \frac{Q_m^2}{2\Phi^2} + 2\partial_\mu\Phi\partial^\mu\Phi \right] , \tag{3.7}$$

where $G_2 = G_4/V_2$ is the dimensionless Newton constant in 2-dimensions. A Weyl transformation $g_{\mu\nu} = \Phi g^{(2)}_{\mu\nu}$ absorbs the kinetic term for the dilaton Φ in the Ricci scalar giving

$$S = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left(\Phi^2 \mathcal{R} - 2\Lambda\Phi - \frac{Q_m^2}{2\Phi^3} \right) \equiv \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left(\Phi^2 \mathcal{R} - U(\Phi) \right) . \tag{3.8}$$

The equations of motion from this action (see *e.g.* Appendix B.3) are

$$U(\Phi) = 2\Lambda\Phi + \frac{Q_m^2}{2\Phi^3} ; \qquad \mathcal{R} - \frac{\partial U}{\partial \Phi^2} = 0 ,$$
$$g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2} U(\Phi) = 0 . \tag{3.9}$$

This 2-dimensional dilaton-gravity theory admits AdS_2 as a solution with a constant dilaton. This constant dilaton, AdS_2 solution is just the near horizon AdS_2 geometry of the extremal magnetic black brane in 4-dimensions (which asymptotically, as $r \to \infty$, is AdS_4).

The purpose of this section was to simply illustrate that the original theory with the gauge field is equivalent to a dilaton-gravity theory with an appropriate dilaton potential: this will be a recurrent theme. A simple toy model capturing many features of 2-dim dilaton gravity is the Jackiw-Teitelboim theory [112, 113]. In the discussion above, we have not been careful with length-scales: in the next section for the relativistic electric brane, we will reinstate various scales.

3.1.1 Relativistic electric black brane, reduction to 2-dimensions

The electric black brane solution to (3.1), (3.2), is

$$ds^2 = -\frac{r^2 f(r)}{R^2} dt^2 + \frac{R^2}{r^2 f(r)} dr^2 + \frac{r^2}{R^2}(dx^2 + dy^2) ,$$
$$f(r) = 1 - \left(\frac{r_0}{r}\right)^3 + \frac{Q_e^2}{r^4}\left(1 - \frac{r}{r_0}\right) ,$$
$$A_t = \frac{Q_e}{2\sqrt{\pi G_4}\, R r_0}\left(1 - \frac{r_0}{r}\right) , \qquad F_{rt} = \frac{Q_e}{2\sqrt{\pi G_4}\, R} \frac{1}{r^2} . \tag{3.10}$$

The gauge field A_t vanishes at the horizon. The charge parameter Q_e is related to the chemical potential μ and the charge density σ of the black brane as

$$\frac{Q_e}{2\sqrt{\pi G_4}\,Rr_0} = \mu\ , \qquad \sigma = \mu\frac{r_0}{R^2} = \frac{Q_e}{2\sqrt{\pi G_4}\,R^3}\ . \tag{3.11}$$

Reinstating the dimensionless gauge coupling e^2 in μ and σ as $\mu \to \frac{\mu}{e}$ and $\sigma \to \sigma e$ and using (3.11), we recover the expressions for the gauge field, field strength and the thermal factor in terms of r_0, μ, σ as given in sec. 4.2.1 in [22]. Note that in (3.10) the charge parameter Q_e has dimensions of charge times length-squared, and the gauge field A_t has mass dimension one. In the extremal limit, the temperature vanishes giving

$$T = \frac{3r_0}{4\pi R^2}\Big(1-\frac{Q_e^2}{3r_0^4}\Big) = 0 \qquad \Longrightarrow \qquad Q_e^2 = 3r_0^4\ . \tag{3.12}$$

The near horizon geometry of the electric black brane becomes $AdS_2 \times \mathbb{R}^2$,

$$ds^2 = -\frac{r_0^2}{R^2}f(r)dt^2 + \frac{R^2}{r_0^2 f(r)}dr^2 + \frac{r_0^2}{R^2}(dx^2+dy^2)\ , \qquad f(r)|_{r\to r_0} \simeq \frac{6}{r_0^2}(r-r_0)^2\ , \tag{3.13}$$

as in the magnetic case. The Bekenstein-Hawking entropy is the horizon area in Planck units

$$S_{BH} = \frac{r_0^2}{R^2}\frac{V_2}{4G_4} = \frac{Q_e/\sqrt{3}}{R^2}\frac{V_2}{4G_4}\ . \tag{3.14}$$

With $V_2 = \int dxdy$ the area, this is finite entropy density for noncompact branes.

It is worth noting that asymptotically, these branes (3.10) give rise to an AdS_4 geometry, with scale R. In the near horizon region, we obtain an AdS_2 throat with scale $\frac{R}{\sqrt{6}}$: this is a well-defined AdS_2 throat in the regime $\frac{r-r_0}{R} \gg 1$ and $\frac{r-r_0}{r_0} \ll 1$. The AdS_2 region is well-separated from the boundary of the AdS_4 geometry at $r \sim r_C \gg r_0$ if $\frac{r-r_0}{r_c} \ll 1$.

Compactifying the two spatial dimensions x^i as T^2 and dimensionally reducing with the metric ansatz (3.6) reduces the action (3.1) for the electric black brane solution to

$$S = \int d^2x\sqrt{-g^{(2)}}\left[\frac{1}{16\pi G_2}(\Phi^2\mathcal{R}^{(2)} - 2\Lambda\Phi^2 + 2\partial_\mu\Phi\partial^\mu\Phi) - \frac{V_2\Phi^2}{4}F_{\mu\nu}F^{\mu\nu}\right]\ , \tag{3.15}$$

and we have suppressed a total derivative term which cancels with a corresponding term arising from the dimensional reduction of the Gibbons-Hawking boundary term (more on this later). Performing a Weyl transformation $g_{\mu\nu} = \Phi g_{\mu\nu}^{(2)}$ to absorb the kinetic term for the dilaton Φ^2 in the Ricci scalar, we get

$$S = \int d^2x\sqrt{-g}\left[\frac{1}{16\pi G_2}(\Phi^2\mathcal{R} - 2\Lambda\Phi) - \frac{V_2\Phi^3}{4}F_{\mu\nu}F^{\mu\nu}\right] . \tag{3.16}$$

The Maxwell equations for the gauge field are

$$\partial_\mu(\sqrt{-g}\,\Phi^3 F^{\mu\nu}) = 0 . \tag{3.17}$$

The two components $b = t, r$ of (3.17), *i.e.* $\partial_t(\sqrt{-g}\Phi^3F^{tr}) = 0 = \partial_r(\sqrt{-g}\Phi^3F^{tr})$, imply

$$\sqrt{-g}\,\Phi^3 F^{tr} = const . \tag{3.18}$$

Using the gauge field solution in (3.10) to fix this constant as $\frac{Q_e}{2\sqrt{\pi G_4}R^3}$, we get

$$F^{\mu\nu} = \frac{Q_e}{2\sqrt{\pi G_4}\,R^3}\frac{1}{\sqrt{-g}\,\Phi^3}\,\varepsilon^{\mu\nu} , \tag{3.19}$$

where $\varepsilon^{\mu\nu}$ is defined as $\varepsilon^{tr} = 1 = -\varepsilon^{rt}$ and $\varepsilon_{\mu\nu} = g_{\mu\rho}g_{\nu\sigma}\varepsilon^{\rho\sigma}$. Substituting $F_{\mu\nu}F^{\mu\nu} = \frac{-Q_e^2}{2\pi G_4 R^6\Phi^6}$ and $F_{\mu\rho}F_\nu{}^\rho = \frac{-Q_e^2}{4\pi G_4 R^6\Phi^6}g_{\mu\nu}$ in eqns.(B.1), we get

$$\begin{aligned} g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}\left(2\Lambda\Phi + \frac{2Q_e^2}{R^6\,\Phi^3}\right) &= 0 , \\ \mathcal{R} - \frac{\Lambda}{\Phi} + \frac{3Q_e^2}{R^6\,\Phi^5} &= 0 . \end{aligned} \tag{3.20}$$

These field equations can be obtained by varying the following equivalent action (see *e.g.* Appendix B.3)

$$S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\left(\Phi^2\mathcal{R} - 2\Lambda\Phi - \frac{2Q_e^2}{R^6\Phi^3}\right) \equiv \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\left(\Phi^2\mathcal{R} - U(\Phi)\right), \tag{3.21}$$

This equivalent action is obtained by substituting the solution for $F^{\mu\nu}$ (in terms of the dilaton Φ^2) in the action (3.16) and changing the sign of the F^2 term which contains a minus sign for electric branes alone, arising from g_{tt} (a similar treatment appears also in *e.g.* [39]). Note that this is also consistent with and expected from electric-magnetic duality $Q_e \to Q_m$, $Q_m \to -Q_e$, which would suggest that the effective dilaton potential for magnetic branes (3.8) is unchanged in going to electric branes. Now for instance the second equation in (3.20) becomes $R - \frac{\partial U}{\partial\Phi^2} = 0$. The constant dilaton, AdS_2 solution to the equations (3.20), consistent with the T^2

compactification of the near horizon geometry in (3.13), is

$$ds^2 = L^2\Big(- \frac{r_0^2}{L^4R^2}(r-r_0)^2dt^2 + \frac{dr^2}{(r-r_0)^2}\Big) , \qquad \Phi = \frac{r_0}{R} ,$$
$$L^2 = \frac{Rr_0}{6} , \qquad Q_e^2 = 3r_0^4 , \tag{3.22}$$

with L the AdS_2 scale. Changing the radial coordinate to $\rho = \frac{R^2}{6(r-r_0)}$, we write the metric in conformal gauge

$$ds^2 = e^{2\omega}(-dt^2 + d\rho^2) = e^{2\omega}(-dx^+dx^-) , \qquad e^{2\omega} = \frac{L^2}{\rho^2} , \tag{3.23}$$

where the lightcone coordinates are $x^\pm = t \pm \rho$. To see that (3.21) admits the above AdS_2 solution, we compute $\frac{\partial U}{\partial \Phi^2}$ for the above solution, which gives

$$\frac{\partial U}{\partial \Phi^2} = -\frac{12}{Rr_0} = -\frac{2}{L^2} \qquad \Longrightarrow \qquad \mathcal{R} = \frac{\partial U}{\partial \Phi^2} = -\frac{2}{L^2} , \tag{3.24}$$

using (3.20) for the Ricci scalar. This constant dilaton, AdS_2 solution (3.22) is just the compactification of the near horizon AdS_2 geometry of the 4-dim extremal electric black brane.

3.1.1.1 Perturbations about the constant dilaton, AdS_2 background

The 4-dimensional theory has a large spectrum of tensor, vector and scalar perturbations, which upon reduction to 2-dimensions give a corresponding spectrum: we will discuss this briefly later, in sec. 3.2.2.3. In this section, we focus on perturbations to only those fields that have nontrivial background profiles in the effective 2-dimensional dilaton-gravity theory: thus we turn on perturbations to the metric and the dilaton

$$\Phi = \Phi_b + \phi(x^+,x^-) , \qquad \omega = \omega_b + \Omega(x^+,x^-) , \tag{3.25}$$

where Φ_b and ω_b denote the background (3.22). We expand the action (3.21) (in conformal gauge) about this background upto quadratic order to get

$$S = \frac{1}{16\pi G_2}\int d^2x \left(4\Phi^2\partial_+\partial_-\omega - \frac{e^{2\omega}}{2}U(\Phi)\right) \equiv S_0 + S_1 + S_2 , \tag{3.26}$$

where

$$S_0 = \frac{1}{16\pi G_2}\int d^2x \left(4\Phi_b^2\partial_+\partial_-\omega_b - \frac{e^{2\omega_b}}{2}U(\Phi_b)\right) \tag{3.27}$$

is the background action and S_1 is linear in perturbations and vanishes by equations of motion. S_2 is quadratic in perturbations given by

$$S_2 = \frac{1}{16\pi G_2} \int d^2x \left(\frac{4r_0^2}{3L^2}\, \phi\, \partial_+ \partial_- \Omega + \frac{1}{(x^+ - x^-)^2} \left(\frac{8r_0^2}{3L^2}\, \Omega\, \phi - 16\phi^2 \right) \right) . \qquad (3.28)$$

Varying this action, we get the linearized equations of motion for the perturbations,

$$\begin{aligned} \partial_+ \partial_- \phi + \frac{2}{(x^+ - x^-)^2}\, \phi &= 0 , \\ \partial_+ \partial_- \Omega + \frac{1}{(x^+ - x^-)^2} \left(2\,\Omega - \frac{24L^2}{r_0^2} \phi \right) &= 0 . \end{aligned} \qquad (3.29)$$

These equations are consistent at linear order with the "constraint" equations for the $++$ and $--$ components of the Einstein equation in (3.20). From these linearized equations, we see that the dilaton fluctuation ϕ is decoupled from the metric fluctuation Ω. Solving the equation for ϕ in (3.29), we get

$$\phi = \frac{a + bt + c(t^2 - \rho^2)}{\rho} , \qquad (3.30)$$

where a, b, c are independent constants. Substituting the solution (3.30) for ϕ in the equation for Ω in (3.29), we can solve for the metric perturbation Ω, which implies that the AdS_2 metric gets corrected at the same order as the dilaton. The on-shell (boundary) action obtained then by using the linearized field equations in (3.28) gives terms at quadratic order in the perturbations,

$$S_2 = \frac{1}{16\pi G_2} \int dt \sqrt{-\gamma}\, n^\mu \left(\frac{2r_0^2}{3L^2} (\Omega\, \partial_\mu \phi - \phi\, \partial_\mu \Omega) \right) , \qquad (3.31)$$

where n^μ is the outward unit normal to the boundary.

3.1.1.2 The Schwarzian effective action

In this section, we switch to Euclidean time $\tau = it$. The Gibbons-Hawking boundary term in the 2-dimensional theory arises from the reduction of the corresponding term in the higher dimensional theory. The Gibbons-Hawking term on the 3-dimensional boundary of the 4-dimensional theories described by the Euclidean form of the action (3.1) is

$$S_{GH}^{4d} = -\frac{1}{8\pi G_4} \int d^3x \sqrt{\gamma^{(3)}}\, K^{(4)} , \qquad (3.32)$$

where the extrinsic curvature is defined as $K^{(4)}_{AB} = \frac{1}{2}(\nabla_A n_B + \nabla_B n_A)$, n^A being the outward unit normal to the 3-dimensional boundary. Using the ansatz (3.6) for the T^2-compactification, dimensionally reducing and performing the Weyl transformation of the 2-dimensional metric $g_{\mu\nu} = \Phi g^{(2)}_{\mu\nu}$, the Gibbons-Hawking term reduces to[1]

$$S^{4d}_{GH} = -\frac{1}{16\pi G_2}\int d\tau\sqrt{\gamma}\left(2\,\Phi^2 K + \frac{3}{2}\,n_\mu\,\partial^\mu\Phi^2\right) . \tag{3.33}$$

The Ricci scalar term in the bulk 4-dim Euclidean action upon dimensional reduction and after the Weyl transformation becomes

$$-\sqrt{g^{(4)}}\mathcal{R}^{(4)} = -\sqrt{g}\Big(\Phi^2\mathcal{R} - \frac{3}{2}\nabla^2\Phi^2\Big) . \tag{3.34}$$

Note also that $\sqrt{g^{(4)}} = \sqrt{g^{(2)}}\Phi^2$ and $\Phi^2 = g_{xx}$. We write the the total derivative term (the second term) in (3.34) as a boundary term

$$-\frac{1}{16\pi G_2}\int d^2x\sqrt{g}\left(-\frac{3}{2}\nabla^2\Phi^2\right) = \frac{1}{16\pi G_2}\int d\tau\sqrt{\gamma}\left(\frac{3}{2}\,n_\mu\partial^\mu\Phi^2\right) . \tag{3.35}$$

We see that this boundary term which comes from the dimensional reduction of the bulk action in 4-dimensions cancels the second term in (3.33), thereby giving the Gibbons-Hawking term on the boundary of the 2-dimensional theory as (see Appendix C.4.1 for a detailed derivation)

$$S_{GH} = -\frac{1}{8\pi G_2}\int d\tau\sqrt{\gamma}\,\Phi^2 K\ . \tag{3.36}$$

Expanding the Gibbons-Hawking term in the perturbations (3.25) and adding it to the Euclidean form of S_2 (which is $S_2^E = -iS_2$, with $t = -i\tau$ in S_2), the leading term in the total boundary action $I_{bdy} = S_2^E + S_{GH}$ arises at linear order in the dilaton perturbation (with subleading terms at quadratic order). To illustrate this in greater detail, it is important that we define the dilaton perturbation in (3.25) in a physically appropriate manner. Since the background value Φ_b is constant, it is sensible to define the dilaton perturbation as

$$\Phi = \Phi_b\,(1+\tilde{\phi})\ , \qquad \Phi_b = \frac{r_0}{R} \qquad \Rightarrow \qquad \tilde{\phi} = \frac{\Phi - \Phi_b}{\Phi_b} \ll 1\ . \tag{3.37}$$

Thus with this redefinition, the perturbation is reasonable since it automatically satisfies $\tilde{\phi} \ll 1$. In terms of the dilaton background value Φ_b, the entropy (3.14) is

[1] We have $K^{(4)} = \gamma^{(3)\,AB}K^{(4)}_{AB} = \gamma^{(3)\,\tau\tau}K^{(4)}_{\tau\tau} + 2\gamma^{(3)\,xx}K^{(4)}_{xx}$, with $K^{(4)}_{xx} = -\Gamma^r_{xx}n_r = \frac{1}{2}n_r\partial^r\Phi^2 = \frac{1}{2}n_\mu\partial^\mu\Phi^2$ becomes $K^{(4)} = K^{(2)} + \Phi^{-2}n_\mu\partial^\mu\Phi^2$. Then (3.32) gives (3.33) after the Weyl transformation.

simply

$$S_{BH} = \frac{\Phi_b^2 V_2}{4G_4} = \frac{\Phi_b^2}{4G_2} \,. \tag{3.38}$$

This gives

$$S_{GH}^{(1)} = -\frac{2\Phi_b^2}{8\pi G_2}\int d\tau\sqrt{\gamma}\,\tilde{\phi}\,K \ \longrightarrow\ -\frac{\Phi_b^2}{4\pi G_2}\int du\,\phi_r(u)\,\{\tau(u),u\}\,. \tag{3.39}$$

In evaluating the last term, we take the boundary of AdS_2 as a slightly deformed curve $(\tau(u),\rho(u))$ parametrized by the boundary coordinate u, and define $\tilde{\phi} = \frac{\phi_r(u)}{\epsilon}$, as discussed in [36] (reviewed in [143]). Now using the outward unit normal n^μ to the boundary, we expand the extrinsic curvature. Expanding $S_{GH}^{(1)}$ then leads to a Schwarzian derivative action $Sch(\tau(u),u) = \{\tau(u),u\} = \frac{\tau'''}{\tau'} - \frac{3}{2}(\frac{\tau''}{\tau'})^2$; see Appendix B.2 for a derivation. The integral above pertains only to AdS_2 does not contain any further scales besides the AdS_2 scale L which also appears in the extrinsic curvature giving the Schwarzian (also $\sqrt{\gamma} = \frac{L}{\epsilon}$). The various length scales in the original extremal brane have been absorbed into the AdS_2 scale L. Now we note that the coefficient of the Schwarzian is in fact proportional to the entropy (3.38) of the compactified extremal black brane with V_2 finite (the dependence on Φ_b is expected since it controls the transverse area). Since the entropy captures the number of microstates of the unperturbed background, this is akin to a central charge of the effective theory. Similar comments appear in [132] (see also [114], the Schwarzian arising in some cases from the conformal anomaly).

It is worth noting that the coefficient in the Schwarzian term above is proportional to the extremal entropy after the reasonable definition of the perturbation as (3.37) by scaling out Φ_b: apart from this, the Schwarzian term here is as in [36]. As discussed there, we note that the perturbation makes this nearly AdS_2 and contributes to the near-extremal entropy via the Schwarzian. This can be obtained as in the analysis there by a transformation $\tau(u) = \tan\frac{\tilde{\tau}(u)}{2}$ which gives $S_{GH}^{(1)} = -\frac{\Phi_b^2}{4\pi G_2}\bar{\phi}_r\int du\left(\{\tilde{\tau}(u),u\} + \frac{1}{2}\tilde{\tau}'^2\right)$, treating $\bar{\phi}_r$ as constant. Solutions with $\tilde{\tau} = \frac{2\pi}{\beta}u$ have $\tilde{\tau} \sim \tilde{\tau} + 2\pi$, giving the action $S_{GH}^{(1)} = -2\pi^2\frac{\Phi_b^2}{4\pi G_2}\bar{\phi}_r T = -\log Z$, giving the near-extremal correction to the entropy $\Delta S = 4\pi\frac{\Phi_b^2}{4G_2}\bar{\phi}_r T$ (which, being linear in temperature, can also be seen to be the specific heat): this again is proportional to the background entropy with the perturbation defined as (3.37).

The remaining terms in the expansion of S_{GH} and S_2^E are all quadratic in perturbations and thus subleading compared to $S_{GH}^{(1)}$. See also *e.g.* [114, 117, 119, 120], for AdS_2 backgrounds obtained from reductions of higher dimensional theories (see also

[122]). In particular there are parallels with some of the analysis on the reduction of near extremal black holes in [120].

Overall, expanding in the perturbations $\tilde{\phi}, \Omega$, we have $I = S^E + S_{GH} = I_0 + I_1 + I_2 + \ldots$, with

$$I_0 = -\frac{\Phi_b^2}{16\pi G_2}\left(\int d^2x\sqrt{g}\,\mathcal{R} + 2\int_{bndry}\sqrt{\gamma}\,K\right) \tag{3.40}$$

is the background Euclidean action (see (3.27)): it can be checked that $U(\Phi_b) = 0$. The action I_0 is a topological term and gives the extremal entropy $S_{BH} = \frac{\Phi_b^2}{4G_2}$ after regulating this as a near-extremal background[2]. The linear terms are contained in

$$I_1 = -\frac{2\Phi_b^2}{16\pi G_2}\int d^2x\,\sqrt{g}\,\tilde{\phi}\left(\mathcal{R} - \frac{\partial U}{\partial\Phi^2}\right) - \frac{2\Phi_b^2}{8\pi G_2}\int_{bndry}\sqrt{\gamma}\,\tilde{\phi}\,K\ , \tag{3.41}$$

with $\frac{\partial U}{\partial\Phi^2}|_{\Phi_b} = -\frac{2}{L^2}$, which is the Jackiw-Teitelboim theory [112, 113], which serves as a simple model for AdS_2 physics (with parallels with the SYK model). The bulk term vanishes by the $\tilde{\phi}$ equation giving the fixed background AdS_2 geometry, while the boundary term gives the Schwarzian as explained above. The analysis here of the higher dimensional realization serves to recover the background entropy as expected and reveal the various subleading terms beyond the Jackiw-Teitelboim theory emerging from reduction: I_2 is second order in perturbations, from S_2^E (see (3.31)) and the second order terms in the expansion of S_{GH},

$$I_2 = -\frac{1}{16\pi G_2}\int d\tau\sqrt{\gamma}\left[\frac{2r_0^2}{3L^2}\,\Phi_b n^\rho(\Omega\,\partial_\rho\tilde{\phi} - \tilde{\phi}\,\partial_\rho\Omega) + 2\Phi_b^2(\tilde{\phi}^2\,K - 2\tilde{\phi}e^{-\omega_b}\partial_\rho\Omega)\right]\ , \tag{3.42}$$

expanding in conformal gauge.

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[2]Here the Euclidean time periodicity, large for a small near-extremal temperature, precisely cancels the small regularized change in the extremal horizon. In more detail, expanding $f(r)$ in (3.10) about extremality, we have $f(r) \simeq \frac{6(r-r_0)}{r_0^2}(r - r_0 + \frac{r_0}{6}(3 - \frac{Q^2}{r_0^4})) \equiv \frac{6}{r_0^2}(r - r_0' - \frac{\delta}{2})(r - r_0' + \frac{\delta}{2})$ where $\delta = \frac{r_0}{6}(3 - \frac{Q^2}{r_0^4})$ and $r_0' = r_0 - \frac{\delta}{2}$. Then the nearly AdS_2 throat acquires a small horizon with metric $ds^2 \sim \frac{9\delta^2}{R^4}\rho^2 d\tau^2 + d\rho^2$ near the origin: the Euclidean time periodicity then is $\Delta\tau = \beta = \frac{2\pi R^2}{3\delta}$ consistent with (3.12). The horizon contribution to the action gives $I_0 = -\frac{\Phi_b^2}{16\pi G_2}\Delta\tau\,\frac{\delta}{2}(\frac{12}{R^2}) \equiv -\beta F$ and thereby the background extremal entropy $S_{BH} = -I_0$.
The boundary terms in the action above cancel: to elaborate, we have the AdS_2 metric $ds^2 = \frac{L^2}{\rho^2}(d\tau^2 + d\rho^2)$. The boundary at $\rho = \epsilon$ has outward unit normal $n_\rho = -\frac{L}{\rho}$. The extrinsic curvature defined as $K_{\mu\nu} = \frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu)$ gives $K_{\tau\tau} = -\Gamma^\rho_{\tau\tau}n_\rho = \frac{L}{\rho^2}$ and $K = \gamma^{\tau\tau}K_{\tau\tau} = \frac{1}{L}$. Then the terms at the boundary cancel as $-\frac{\Phi_b^2}{16\pi G_2}(\int d\tau\frac{L^2 d\rho}{\rho^2}|_\epsilon^{hrzn}(-\frac{2}{L^2}) + 2\int d\tau\frac{L}{\epsilon}(-\frac{1}{L}))$.

3.2 Charged hyperscaling violating Lifshitz black branes

Over the last several years, nonrelativistic generalizations of holography have been investigated extensively: see *e.g.* [22] for a review of various aspects. A particular family of interesting theories comprises the so-called hyperscaling violating Lifshitz (hvLif) theories, which are conformal to Lifshitz theories. These arise as solutions to Einstein-Maxwell-scalar theories, the $U(1)$ gauge field and dilaton scalar necessary to support the nonrelativistic background. For the most part, we regard these as effective gravity theories: in certain cases these can be shown to arise from gauge/string realizations (see *e.g.* [99]).

These nonrelativistic black branes are uncharged. A minimal way to construct charged black branes is to add an additional $U(1)$ gauge field, which serves to supply charge to the black brane: see *e.g.* [184], [185], [186]. For these latter charged black branes, there exist extremal limits where the near horizon geometry takes the form $AdS_2 \times X$, and contains an AdS_2 throat. Compactifying the transverse space now allows us to study the extremal limits of these theories in the context of a 2-dimensional dilaton gravity theory with additional matter, notably the scalar descending from higher dimensions as well as gauge fields[3].

3.2.1 4-dimensional charged hvLif black brane

Consider Einstein-Maxwell-scalar theory with a further $U(1)$ gauge field, with action [185]

$$S = \int d^4x\sqrt{-g^{(4)}}\left[\frac{1}{16\pi G_4}\Big(\mathcal{R}^{(4)} - \frac{1}{2}\partial_M\Psi\partial^M\Psi + V(\Psi) - \frac{Z_1}{4}F_{1\,MN}F_1^{MN}\Big) - \frac{Z_2}{4}F_{2\,MN}F_2^{MN}\right] , \tag{3.43}$$

where the scalar field dependent couplings and the scalar potential are

$$Z_1 = e^{\lambda_1\Psi} , \qquad Z_2 = e^{\lambda_2\Psi} , \qquad V(\Psi) = V_0 e^{\gamma\Psi} . \tag{3.44}$$

[3]Note that in the AdS/CMT literature, these theories are referred to Einstein-Maxwell-dilaton theories: we here use Einstein-Maxwell-scalar since the 2-dim dilaton Φ here is distinct from the hvLif scalar Ψ.

The field equations following from the above action are

$$\mathcal{R}^{(4)}_{MN} - \frac{1}{2}\partial_M\Psi\partial_N\Psi + g_{MN}\frac{V}{2} - \frac{Z_1}{2}\Big(F_{1\,MP}F_{1N}^{\ \ P} - \frac{g_{MN}}{4}(F_1)^2\Big)$$
$$-8\pi G_4 Z_2\Big(F_{2\,MP}F_{2N}^{\ \ P} - \frac{g_{MN}}{4}(F_2)^2\Big) = 0\ ,$$
$$\nabla_M\nabla^M\Psi + \gamma V - \frac{\lambda_1 Z_1}{4}F_{1\,MN}F_1^{MN} - 4\pi G_4\lambda_2 Z_2 F_{2\,MN}F_2^{MN} = 0\ ,$$
$$\partial_M(\sqrt{-g^{(4)}}Z_1F_1^{MN}) = 0\ , \qquad \partial_M(\sqrt{-g^{(4)}}Z_2F_2^{MN}) = 0\ . \quad (3.45)$$

The charged hvLif black brane solution to these equations is

$$\begin{aligned}
ds^2 &= \Big(\frac{r}{r_{hv}}\Big)^{-\theta}\Big[-\frac{r^{2z}f(r)}{R^{2z}}dt^2 + \frac{R^2}{r^2f(r)}dr^2 + \frac{r^2}{R^2}(dx^2+dy^2)\Big]\ ,\\
f(r) &= 1 - \Big(\frac{r_0}{r}\Big)^{2+z-\theta} + \frac{Q^2}{r^{2(1+z-\theta)}}\Big(1 - \Big(\frac{r}{r_0}\Big)^{z-\theta}\Big)\ ,\\
F_{1rt} &= \sqrt{2(z-1)(2+z-\theta)}\,e^{-\frac{\lambda_1\Psi_0}{2}}\,r_{hv}^2\,R^{\theta-z-4}\,r^{1+z-\theta}\ ,\\
F_{2rt} &= \frac{Q\sqrt{2(2-\theta)(z-\theta)}\,e^{-\frac{\lambda_2\Psi_0}{2}}}{4\sqrt{\pi G_4}}\,R^{z-\theta-2}\,r_{hv}^{-z+\theta+1}\,r^{-(1+z-\theta)}\ ,\\
e^{\Psi} &= e^{\Psi_0}\Big(\frac{r_{hv}\,r}{R^2}\Big)^{\sqrt{(2-\theta)(2z-2-\theta)}}\ ,
\end{aligned} \quad (3.46)$$

being explicit with length scales, and

$$V_0 = \frac{(2+z-\theta)(1+z-\theta)\,e^{-\gamma\Psi_0}}{R^{2-2\theta}\,r_{hv}^{2\theta}}\ , \qquad \gamma = \frac{\theta}{\sqrt{(2-\theta)(2z-2-\theta)}}\ ,$$
$$\lambda_1 = \frac{-4+\theta}{\sqrt{(2-\theta)(2z-2-\theta)}}\ , \qquad \lambda_2 = \sqrt{\frac{2z-2-\theta}{2-\theta}}\ . \quad (3.47)$$

Here r_{hv} is the hyperscaling violating scale arising in the conformal factor in the metric, and the charge parameter Q has dimensions of $r^{1+z-\theta}$: this is equivalent to absorbing factors of r_{hv}, R into Q. For $z=1, \theta=0$, this scaling coincides with that for the relativistic black brane in sec. 3.1.1.

In these charged hyperscaling violating Lifshitz black brane solutions to the action (3.43), the gauge field A_1 and the scalar field Ψ source the hyperscaling violating Lifshitz background while the gauge field A_2 giving charge to the black brane, as mentioned above. This action (3.43) has also been defined by absorbing the Newton constant into the definition of the hyperscaling violating gauge field A_1 and scalar Ψ (which thus makes A_1 and Ψ dimensionless) while retaining the gauge field A_2 in

F_2 as having mass dimension one. Thus the field strength F_{2rt} in (3.46) has mass dimension 2, as for the relativistic brane.

The null energy conditions for the metric follow from the asymptotic hvLif geometry [22] and are given by

$$(z-1)(2+z-\theta) \geq 0\ , \qquad (2-\theta)(2(z-1)-\theta) \geq 0\ . \tag{3.48}$$

In addition, we require the gauge field A_{2t} to vanish at the boundary ($r \to \infty$) so that the theory does not ruin the hvLif boundary conditions we have assumed: this is equivalent to assuming that these charged black branes represent finite temperature charged states in the boundary hvLif theory. The background profile $A_{2t} \sim 1 - (\frac{r_0}{r})^{z-\theta}$ then implies that

$$z - \theta \geq 0\ . \tag{3.49}$$

These conditions together constrain the range of z, θ for these extremal nonrelativistic black brane backgrounds, which will be important in the discussion of perturbations later. Specifically:
(i) First, the last condition (3.49) is specific to the charged case: using this, the first of the null energy conditions (3.48) implies that $z \geq 1$.
(ii) From the second of the conditions (3.48), we have either $2-\theta \geq 0,\ 2z-2-\theta \geq 0$, or $2-\theta < 0,\ 2z-2-\theta < 0$. Considering the second possibility, we obtain $z \geq \theta \geq 2$, but this implies $2z-2-\theta = z-2+z-\theta > 0$, which is a contradiction. This forces $2-\theta \geq 0,\ 2z-2-\theta \geq 0$.
Overall, this gives the conditions

$$z \geq 1\ , \qquad 2z-2-\theta \geq 0\ , \qquad 2-\theta \geq 0\ , \tag{3.50}$$

for the regime of validity of the z, θ exponents of the charged hvLif background above. For the special case of $z=1$, the NEC becomes $(2-\theta)(-\theta) \geq 0$, which forces $\theta \leq 0$ by (3.50).

The relativistic limit of this charged hvLif black brane gives the relativistic electric black brane discussed previously in sec. 3.1.1. From the constraint (3.50), we see that the correct relativistic limit is to take first $\theta = 0$ and then $z = 1$. In this limit, we get

$$\gamma = 0, \qquad \lambda_1 \to -\infty, \qquad \lambda_2 = 0, \qquad V_0 = 6/R^2, \qquad \Psi = \Psi_0. \tag{3.51}$$

With this the Einstein-Maxwell-scalar action (3.43) reduces to the Einstein-Maxwell action (3.1), where F_2 and V_0 in (3.43) are identified with F and -2Λ in (3.1).

3.2.1.1 Extremality and attractors

In the extremal limit,

$$T = \frac{(2+z-\theta)r_0^z}{4\pi R^{z+1}}\Big(1 - \frac{(z-\theta)Q^2 r_0^{-2(1+z-\theta)}}{(2+z-\theta)}\Big) = 0 \quad \Longrightarrow \quad Q^2 = \frac{(2+z-\theta)}{(z-\theta)} r_0^{2(1+z-\theta)}, \tag{3.52}$$

and the near horizon geometry becomes $AdS_2 \times \mathbb{R}^2$,

$$\begin{aligned} ds^2 &= \Big(\frac{r_0}{r_{hv}}\Big)^{-\theta}\Big[-\frac{r_0^{2z} f(r)}{R^{2z}}dt^2 + \frac{R^2}{r_0^2 f(r)}dr^2 + \frac{r_0^2}{R^2}(dx^2+dy^2)\Big] , \\ f(r)|_{r\to r_0} &\simeq \frac{(2+z-\theta)(1+z-\theta)}{r_0^2}(r-r_0)^2 , \end{aligned} \tag{3.53}$$

the AdS_2 scale being $R\,(\frac{r_0}{r_{hv}})^{-\theta/2}$. The Bekenstein-Hawking entropy is the horizon area in Planck units

$$S_{BH} = \Big(\frac{r_0^2}{R^2}\Big)\Big(\frac{r_0}{r_{hv}}\Big)^{-\theta}\frac{V_2}{4G_4} = \Big(\frac{z-\theta}{2+z-\theta}\Big)^{\frac{2-\theta}{2(1+z-\theta)}}\frac{r_{hv}^\theta V_2}{4G_4}\frac{Q^{(2-\theta)/(1+z-\theta)}}{R^2} , \tag{3.54}$$

where $V_2 = \int dxdy$ is the transverse area of the brane. For $z = 1, \theta = 0$, this coincides with the relativistic brane.

It is worth noting that the full metric in (3.46) is asymptotically of hvLif form, for $r \gg r_0$. The boundary of the theory could be taken as $r \sim r_{hv}$, *i.e.* the theory flows to hvLif below this scale, in some bigger phase diagram. The AdS_2 throat, well-defined if $\frac{r-r_0}{r_0} \ll 1$ and $\frac{r-r_0}{R} \gg 1$, is well-separated from the asymptotic hvLif region if $\frac{r-r_0}{r_{hv}} \ll 1$ and the AdS_2 scale satisfies $R\,(\frac{r_0}{r_{hv}})^{-\theta/2} \ll r_{hv}$ *i.e.* $R \ll r_{hv}(\frac{r_0}{r_{hv}})^{\theta/2}$. Note that this is not vacuous since $r_0 \ll r_{hv}$ so that $\frac{r_0}{r_{hv}} \ll 1$ is a small factor.

Along the lines of the attractor mechanism discussion in [182], we would like to convert this theory to a dilatonic gravity theory in 4-dimensions with a potential (and no gauge fields). Towards this end, we integrate Maxwell's equations in (3.45) and use the solutions for field strengths in (3.46) to get

$$\begin{aligned} F_1^{tr} &= \frac{\sqrt{2(z-1)(2+z-\theta)}\; e^{\frac{\lambda_1\Psi_0}{2}}\, r_{hv}^{\theta-2}\, R^{1-\theta}}{\sqrt{-g}\; e^{\lambda_1\Psi}} , \\ F_2^{tr} &= \frac{Q\sqrt{2(2-\theta)(z-\theta)}\; e^{\frac{\lambda_2\Psi_0}{2}}\, r_{hv}^{z-1}}{4\sqrt{\pi G_4}\, R^{2z+1-\theta}\, \sqrt{-g}\; e^{\lambda_2\Psi}} . \end{aligned} \tag{3.55}$$

Substituting (3.55) in (3.45), we obtain equations of motion for the metric and the scalar field Ψ, which can be derived from the following equivalent action

$$S = \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\Big(\mathcal{R} - \frac{1}{2}(\partial\Psi)^2 - V_{eff}(\Psi)\Big) ,$$
$$V_{eff}(\Psi) = -\frac{(2+z-\theta)(1+z-\theta)}{R^{2-2\theta}r_{hv}^{2\theta}}e^{\gamma(\Psi-\Psi_0)} + \frac{1}{g_{xx}^2}\Big(\frac{(z-1)(2+z-\theta)r_{hv}^{2\theta-4}R^{2-2\theta}}{e^{\lambda_1(\Psi-\Psi_0)}} + \frac{(2-\theta)(z-\theta)Q^2 r_{hv}^{2z-2}R^{-4z-2+2\theta}}{e^{\lambda_2(\Psi-\Psi_0)}}\Big) . \tag{3.56}$$

The explicit scales show that the potential term-by-term has mass dimension 2. This equivalent action is obtained by substituting the solutions for F_1^{tr} and F_2^{tr} in the action (3.43) and changing the signs of F_1^2, F_2^2 terms, as earlier. At the critical point (extremality),

$$g_{xx} = \Big(\frac{r_0}{r_{hv}}\Big)^{-\theta}\Big(\frac{r_0}{R}\Big)^2, \qquad e^{\Psi} = e^{\Psi_0}\Big(\frac{r_{hv}\, r_0}{R^2}\Big)^{\sqrt{(2-\theta)(2z-2-\theta)}},$$
$$Q^2 = \frac{(2+z-\theta)}{(z-\theta)}r_0^{2(1+z-\theta)}, \tag{3.57}$$

the first and second derivatives of V_{eff} ((B.2), (B.3)) are

$$\frac{\partial V_{eff}}{\partial\Psi}\Big|_{ext} = 0 , \qquad \frac{\partial^2 V_{eff}}{\partial\Psi^2}\Big|_{ext} = \frac{4(z-1)(2+z-\theta)(1+z-\theta)}{2z-2-\theta}\,\frac{r_0^{\theta}}{r_{hv}^{\theta}R^2} > 0 , \tag{3.58}$$

which imply that the extremal point is stable for all values of z, θ allowed by the conditions (3.50). It is worth mentioning that for $z=1$ and θ nonzero, these and all higher derivatives of V_{eff} in fact vanish (see (B.5)): thus we obtain no insight into the stability of these attractors in this case and we will not discuss this subcase in what follows.

3.2.2 Dimensional reduction to 2-dimensions

Compactifying the two spatial dimensions, x^i as T^2, we dimensionally reduce with the metric ansatz (3.6), taking the lower dimensional fields $g_{\mu\nu}^{(2)}, \Phi, \Psi, A_1, A_2$, to be T^2-independent: then the action (3.43) reduces to (B.6). Performing a Weyl transformation, $g_{\mu\nu} = \Phi g_{\mu\nu}^{(2)}$ to absorb the kinetic term for the dilaton Φ in the Ricci scalar, the 2-dimensional action (B.6) becomes

$$S = \int d^2x\sqrt{-g}\Big[\frac{1}{16\pi G_2}\Big(\Phi^2\mathcal{R} - \frac{\Phi^2}{2}\partial_\mu\Psi\partial^\mu\Psi + V\Phi - \frac{\Phi^3}{4}Z_1F_{1\,\mu\nu}F_1^{\mu\nu}\Big) - \frac{V_2\Phi^3}{4}Z_2F_{2\,\mu\nu}F_2^{\mu\nu}\Big] . \tag{3.59}$$

We only retain fields with nontrivial background profiles: more general comments appear later. The Maxwell equations for the gauge fields are

$$\partial_\mu(\sqrt{-g}\Phi^3Z_1F_1^{\mu\nu}) = 0 , \qquad \partial_\mu(\sqrt{-g}\Phi^3Z_2F_2^{\mu\nu}) = 0 . \tag{3.60}$$

Integrating and using F_{1rt}, F_{2rt} from (3.46) to fix the integration constants gives

$$\begin{aligned} F_1^{\mu\nu} &= \frac{\sqrt{2(z-1)(2+z-\theta)}\, e^{\frac{\lambda_1\Psi_0}{2}}\, r_{hv}^{\theta-2}\, R^{1-\theta}}{\sqrt{-g}\; Z_1\; \Phi^3}\, \varepsilon^{\mu\nu} , \\ F_2^{\mu\nu} &= \frac{Q\sqrt{2(2-\theta)(z-\theta)}\, e^{\frac{\lambda_2\Psi_0}{2}}\, r_{hv}^{z-1}}{4\sqrt{\pi G_4}\, R^{2z+1-\theta}\, \sqrt{-g}\; Z_2\, \Phi^3}\, \varepsilon^{\mu\nu} , \end{aligned} \tag{3.61}$$

where $\varepsilon^{\mu\nu}$ satisfies $\varepsilon^{tr} = 1 = -\varepsilon^{rt}$ and $\varepsilon_{\mu\nu} = g_{\mu\rho}g_{\nu\sigma}\varepsilon^{\rho\sigma}$. We substitute the solutions (3.61) in the remaining field equations obtained by varying the action (3.59) (*i.e.* eq. (B.7)) to obtain

$$\begin{aligned} g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}\Big(\frac{\Phi^2}{2}(\partial\Psi)^2 + U\Big) - \frac{\Phi^2}{2}\partial_\mu\Psi\partial_\nu\Psi &= 0 , \\ \mathcal{R} - \frac{1}{2}(\partial\Psi)^2 - \frac{\partial U}{\partial(\Phi^2)} &= 0 , \\ \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\Phi^2\partial^\mu\Psi) - \frac{\partial U}{\partial\Psi} &= 0 , \end{aligned} \tag{3.62}$$

where $U(\Phi,\Psi)$ is an effective interaction potential. These equations can then be obtained from the following equivalent action (Appendix B.3)

$$S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\,\Big(\Phi^2\mathcal{R} - \frac{\Phi^2}{2}(\partial\Psi)^2 - U(\Phi,\Psi)\Big), \tag{3.63}$$

$$U(\Phi,\Psi) = -\frac{(2+z-\theta)(1+z-\theta)}{R^{2-2\theta}r_{hv}^{2\theta}}e^{\gamma(\Psi-\Psi_0)}\,\Phi + \frac{1}{\Phi^3}\Big(\frac{(z-1)(2+z-\theta)r_{hv}^{2\theta-4}R^{2-2\theta}}{e^{\lambda_1(\Psi-\Psi_0)}} + \frac{(2-\theta)(z-\theta)Q^2r_{hv}^{2z-2}R^{-4z-2+2\theta}}{e^{\lambda_2(\Psi-\Psi_0)}}\Big),$$

where V_0, γ, λ_1, λ_2 are given in (3.47). This equivalent action is obtained by substituting the solutions for $F_1^{\mu\nu}$, $F_2^{\mu\nu}$ in terms of the dilaton Φ^2 and the scalar Ψ in the action (3.59) and changing the signs of F_1^2, F_2^2 terms, as discussed in the case for relativistic electric black brane, sec. 3.1.1. Also note that the relativistic electric black brane is a special case of the dilaton-gravity-matter theory, considered here, for $\theta = 0$ and $z = 1$.

We note that the scalar Ψ that descends from the hyperscaling violating scalar in higher dimensions is not minimally coupled in the 2-dimensional theory. The potential $U(\Phi, \Psi)$ contains nontrivial interactions between the dilaton Φ and the hvLif scalar Ψ. Thus the small fluctuation spectrum of the dilaton and Ψ are coupled, and one might worry about the stability of the 2-dimensional attractor. This is reminiscent of multi-field inflation models, where one scalar field provides a slow-roll phase while another scalar provides a waterfall phase, ending inflation. In the current context, stability would require that no tachyonic modes arise from the interaction induced by $U(\Phi, \Psi)$ between Φ and Ψ. We will address this soon.

The field equations (3.62) admit a constant dilaton, AdS_2 solution as

$$\begin{aligned} ds^2 &= L^2\Big[-\frac{r_0^{2z-3\theta}}{R^{2z} r_{hv}^{-3\theta} L^4}(r-r_0)^2 dt^2 + \frac{dr^2}{(r-r_0)^2}\Big] , \\ \Phi^2 &= \Big(\frac{r_0}{r_{hv}}\Big)^{-\theta}\Big(\frac{r_0}{R}\Big)^2 , \qquad e^{\Psi} = e^{\Psi_0}\Big(\frac{r_{hv}\, r_0}{R^2}\Big)^{\sqrt{(2-\theta)(2z-2-\theta)}} , \\ L^2 &\equiv \frac{R r_0^{1-\frac{3\theta}{2}} r_{hv}^{\frac{3\theta}{2}}}{(2+z-\theta)(1+z-\theta)} , \qquad Q^2 = \frac{(2+z-\theta)}{(z-\theta)} r_0^{2(1+z-\theta)} . \end{aligned} \tag{3.64}$$

Let us choose conformal gauge by doing a coordinate transformation,

$$\rho = \frac{R^{z+1} r_0^{1-z}}{(2+z-\theta)(1+z-\theta)} \frac{1}{(r-r_0)} . \tag{3.65}$$

In conformal gauge, the AdS_2 metric in (3.64) can be written as

$$ds^2 = e^{2\omega}(-dt^2 + d\rho^2) = e^{2\omega}(-dx^+ dx^-), \qquad e^{2\omega} = \frac{L^2}{\rho^2} , \tag{3.66}$$

where the lightcone coordinates are $x^{\pm} = t \pm \rho$ and L is the radius of AdS_2. To see that (3.63) admits the above AdS_2 solution, we compute $\frac{\partial U}{\partial \Phi^2}$ for the above solution, which gives

$$\frac{\partial U}{\partial \Phi^2} = -2\frac{(2+z-\theta)(1+z-\theta)}{R r_0^{1-\frac{3\theta}{2}} r_{hv}^{\frac{3\theta}{2}}} = -\frac{2}{L^2} . \tag{3.67}$$

From (3.62) for $\Psi = constant$ (from (3.64)), we get the Ricci scalar as

$$\mathcal{R} = \frac{\partial U}{\partial \Phi^2} = -\frac{2}{L^2} . \tag{3.68}$$

3.2.2.1 Perturbations about AdS_2

As before, we turn on perturbations to fields with background profiles, *i.e.* to the metric, the dilaton Φ and the scalar field Ψ,

$$\Phi = \Phi_b + \phi(x^+, x^-) , \qquad \omega = \omega_b + \Omega(x^+, x^-) , \qquad \Psi = \Psi_b + \sqrt{2z-2-\theta}\, \psi(x^+, x^-) , \tag{3.69}$$

where Φ_b, ω_b and Ψ_b denote the (3.64) background solution. Expanding the action (3.63) (in conformal gauge) about this background gives

$$S = \frac{1}{16\pi G_2}\int d^2x \left(4\Phi^2\partial_+\partial_-\omega + \Phi^2\partial_+\Psi\partial_-\Psi - \frac{e^{2\omega}}{2}U(\Phi,\Psi)\right) \equiv S_0 + S_1 + S_2 , \tag{3.70}$$

where

$$S_0 = \frac{1}{16\pi G_2}\int d^2x \left(4\Phi_b^2\partial_+\partial_-\omega_b + \Phi_b^2\partial_+\Psi_b\partial_-\Psi_b - \frac{e^{2\omega_b}}{2}U(\Phi_b,\Psi_b)\right) \tag{3.71}$$

is the background action and S_1 vanishes by the equations of motion. S_2 is quadratic in perturbations and is given by

$$\begin{aligned} S_2 = & \frac{1}{16\pi G_2}\int d^2x\, \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2+z-\theta)(1+z-\theta)} \Big[8\,\phi\,\partial_+\partial_-\Omega + \frac{16}{(x^+-x^-)^2}\phi\,\Omega \\ & + \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2+z-\theta)(1+z-\theta)}\Big((2z-2-\theta)\partial_+\psi\partial_-\psi - \frac{4(z-1)}{(x^+-x^-)^2}\psi^2\Big) \\ & + \frac{1}{(x^+-x^-)^2}\Big(- \frac{16L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}}\phi^2 + \frac{8\,\theta}{\sqrt{(2-\theta)}}\psi\phi\Big)\Big] . \end{aligned} \tag{3.72}$$

Varying this action, we get the linearized equations of motion for the perturbations,

$$\partial_+\partial_-\phi + \frac{2}{(x^+-x^-)^2}\,\phi = 0 ,$$

$$(2z-2-\theta)\partial_+\partial_-\psi + \frac{1}{(x^+-x^-)^2}\Big(4(z-1)\psi - \frac{L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}}\frac{4\theta}{\sqrt{(2-\theta)}}\phi\Big) = 0 , \tag{3.73}$$

$$\partial_+\partial_-\Omega + \frac{1}{(x^+-x^-)^2}\left(2\,\Omega - \frac{4L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}}\phi + \frac{\theta}{\sqrt{(2-\theta)}}\psi\right) = 0 .$$

These equations are consistent at linear order with the "constraint" equations for the $\pm\pm$ components of the Einstein equation in (3.62): see Appendix, eq.(B.8)-(B.10). We see that the equation for ψ is coupled to ϕ as well: defining a new field ζ,

$$\zeta = \psi - \frac{2}{\sqrt{2-\theta}} \frac{L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}} \phi \,, \tag{3.74}$$

decouples the equations for ζ and ϕ, which now become

$$\begin{aligned}
\partial_+\partial_-\phi + \frac{2}{(x^+-x^-)^2}\,\phi = 0\,,\\
(2z-2-\theta)\partial_+\partial_-\zeta + 2(z-1)\frac{2}{(x^+-x^-)^2}\,\zeta = 0\,,\\
\partial_+\partial_-\Omega + \frac{1}{(x^+-x^-)^2}\Big(2\Omega + \frac{2(3\theta-4)}{(2-\theta)} \frac{L^2(2+z-\theta)(1+z-\theta)\,\phi}{r_0^{2-2\theta} r_{hv}^{2\theta}} + \frac{\theta\,\zeta}{\sqrt{(2-\theta)}}\Big) = 0\,.
\end{aligned} \tag{3.75}$$

In this form, the perturbations ϕ and ζ are equivalent to scalars with positive mass propagating in a perturbed AdS_2 background, with equation of motion $\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi) - m^2\phi = 0$: in conformal gauge this is $\partial_+\partial_-\phi + \frac{m^2L^2}{(x^+-x^-)^2}\phi = 0$. Let us look at a few special cases here:

- For $z = 1$, $\theta = 0$, we have seen that this system reduces to the relativistic brane case studied earlier (3.29), and the Ψ scalar (the nonrelativistic scalar in higher dimensions) can be then seen to decouple from the system: in particular, the terms containing ψ-perturbations vanish in the action (3.72) for quadratic perturbations. This is expected from the fact that the original action for the higher dimensional nonrelativistic theory reduces to the relativistic brane theory as $z \to 1$, $\theta \to 0$, as discussed after (3.43). In effect, we have defined the ψ-perturbation in (3.69) so that the relativistic brane limit arises smoothly, and the Ψ-scalar freezes out. This is also reflected in the linearized equations for perturbations.

- For $\theta = 0$ and $z > 1$, both ϕ and ζ have positive mass term coefficients, and further ζ decouples entirely from the Ω equation. This means that in fact any linear combination of the fields $A\phi + B\zeta$ also in fact has a positive mass term coefficient in its linearized fluctuation equation, as can be seen by taking that linear combination of the two equations $\partial_+\partial_-(A\phi+B\zeta) + \frac{2}{(x^+-x^-)^2}(A\phi+B\zeta) = 0$. The linear fluctuation analysis thus suggests that the attractor point is in fact perfectly stable for small fluctuations.

- For $\theta \neq 0$ and $z = 1$, we see that the ζ field is a massless mode and further it does not decouple from the Ω equation. This suggests that the linear stability analysis is insufficient to determine stability of the attractor point. However in this case, there is a more basic concern: looking back at the higher dimensional system (3.58), we see that in fact $\frac{\partial^2 V_{eff}}{\partial \Psi^2} = 0$ in this case (in fact all derivatives vanish, (B.5)), so that the higher dimensional theory is also not manifestly a stable attractor. Thus the relevance of the 2-dimensional theory is less clear in this case.

- For generic z, θ values satisfying the energy conditions (3.48), (3.49), (3.50), we see that the mass term coefficients for both ϕ and ζ perturbations are positive. Now a generic linear combination of the fields $A\phi + B\zeta$ satisfies

$$\left(\partial_+\partial_- + \frac{2}{(x^+ - x^-)^2}\right)(A\phi + (2z - 2 - \theta)B\zeta) = -\frac{2}{(x^+ - x^-)^2}\theta\, B\zeta\ . \quad (3.76)$$

This is akin to a scalar field $A\phi+(2z-2-\theta)B\zeta$ with positive mass, driven by the source field ζ. Since ζ is also a positive mass scalar, small fluctuations do not contain any unstable modes growing in time. Thus the general perturbation also is stable. To elaborate a bit further, imagine long-wavelength modes of ϕ, ζ which are spatially uniform, *i.e.* $\phi = \phi(t),\ \zeta = \zeta(t)$. Now the linearized equations are of the form $\ddot{\phi} + m_\phi^2\phi = 0,\ \ddot{\zeta} + m_\zeta^2\zeta = 0$, so that these fields are effectively decoupled harmonic oscillators. Then the general field is a driven oscillator, with the driving force itself executing small oscillations: so there are no unstable modes growing in time. It is important to note that the positivity of the mass term coefficients and the stability they imply stems from the energy conditions and asymptotic boundary conditions, which force $z > 1$ and $2z - 2 - \theta > 0$ for generic z, θ values.

It is worth noting that for fixed ζ, the relative sizes of the dilaton ϕ and hvLif scalar ψ perturbations are $\frac{\psi}{\phi} \sim \frac{L^2}{r_0^2}(\frac{r_0}{r_{hv}})^{2\theta} \ll \frac{L^2}{r_0^2}$ for $\theta > 0$ since $\frac{r_0}{r_{hv}} \ll 1$.

It is worth comparing this analysis with that for the higher dimensional theory discussed earlier in (C.31), (3.58): the scalar Ψ has a canonical kinetic term and the equation governing small fluctuations of Ψ about the attractor point acquires a mass term from $\frac{\partial^2 U}{\partial \Psi^2}$, whose positivity dictates the stability of the attractor point. For a theory with two scalars ϕ_1, ϕ_2 with canonical kinetic terms, the stability of the linearized fluctuations can again be studied by studying the second derivative matrix of the potential $U(\phi_1, \phi_2)$ or the Hessian $[\frac{\partial^2 U}{\partial\phi_i \partial\phi_j}]$. Positivity of the Hessian then translates to stability of the attractor extremum. In the present case however, the effective action is (3.63), and the kinetic terms for Φ, Ψ are not canonical: thus the naive Hessian analysis to study the stability of $U(\Phi, \Psi)$ about the attractor point is not valid. Instead we must analyze perturbations about the attractor point, which are governed by the above equations. From these equations, we see that the mass terms for the decoupled fields ζ and ϕ are positive.

In terms of ϕ and ζ, the quadratic action becomes

$$\begin{aligned} S_2 =& \frac{1}{16\pi G_2} \int d^2x \, \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2+z-\theta)(1+z-\theta)} \Big[8\phi \, \partial_+\partial_-\Omega + \frac{16}{(x^+-x^-)^2}\phi\Omega \\ &+ \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2+z-\theta)(1+z-\theta)} \Big((2z-2-\theta)\partial_+\zeta\partial_-\zeta - \frac{4(z-1)}{(x^+-x^-)^2}\zeta^2 \Big) \\ &+ \frac{L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}} \Big(\frac{4(2z-2-\theta)}{(2-\theta)} \partial_+\phi\partial_-\phi - \frac{16(z+1-2\theta)}{(2-\theta)(x^+-x^-)^2}\phi^2 \Big) \\ &+ 2\sqrt{\frac{2z-2-\theta}{2-\theta}} (\partial_+\zeta\partial_-\phi + \partial_-\zeta\partial_+\phi) - \frac{8(2z-2-\theta)}{\sqrt{2-\theta}} \frac{\zeta\phi}{(x^+-x^-)^2} \Big]. \end{aligned} \tag{3.77}$$

It can be checked that varying this action leads to the linearized equations written in terms of ϕ, ζ above.

3.2.2.2 The Schwarzian

In this section, we switch to Euclidean time $\tau = it$. From the linearized equations (3.73), we see that the dilaton fluctuation ϕ is decoupled from the metric and scalar fluctuations Ω and ψ, as in the case of the relativistic brane. So solving the equation for ϕ (*i.e.* the Euclidean form of (3.73)) gives, as before,

$$\phi = \frac{a + b\tau + c(\tau^2 + \rho^2)}{\rho} \,, \tag{3.78}$$

where a, b, c are independent constants. Substituting ϕ in the equation for ψ in (3.73), we can solve for the scalar perturbation ψ. Using these solutions for ϕ and ψ in the equation for Ω in (3.73), we can solve for the metric perturbation Ω. We see that the AdS_2 metric gets corrected at the same order as the dilaton and the scalar field. The Euclidean on-shell (boundary) action obtained by using linearized field equations in (3.77) and changing to Euclidean time $\tau = it$ is

$$\begin{aligned} S_2^E =& -\frac{1}{16\pi G_2} \int d\tau \sqrt{\gamma} \, n^\mu \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2+z-\theta)(1+z-\theta)} \Big[4(\Omega\partial_\mu\phi - \phi\partial_\mu\Omega) \\ &- \frac{(2z-2-\theta)}{\sqrt{2-\theta}} (\phi\partial_\mu\zeta + \zeta\partial_\mu\phi) - \frac{L^2(2+z-\theta)(1+z-\theta)}{r_0^{2-2\theta} r_{hv}^{2\theta}} \frac{2(2z-2-\theta)}{(2-\theta)} \phi\partial_\mu\phi \\ &- \frac{r_0^{2-2\theta} r_{hv}^{2\theta}}{L^2(2+z-\theta)(1+z-\theta)} (2z-2-\theta)\zeta\partial_\mu\zeta \Big] \,. \end{aligned} \tag{3.79}$$

The discussion of the Gibbons-Hawking term is very similar to that in sec. 3.1.1.2 so we will not be detailed. The Gibbons-Hawking boundary term for the Euclidean form of the bulk action (3.63) is

$$S_{GH} = -\frac{1}{8\pi G_2}\int d\tau\,\sqrt{\gamma}\,\Phi^2\,K\ , \tag{3.80}$$

arising as discussed in the case of the relativistic electric brane earlier. As in sec. 3.1.1.2, we now redefine the dilaton perturbation after rescaling the background value Φ_b out, so that the perturbation satisfies $\frac{\Phi-\Phi_b}{\Phi_b} \equiv \tilde{\phi} \ll 1$. A similar redefinition is appropriate for the hvLif scalar Ψ as well (we have however retained the perturbations in (3.69) without this rescaling simply with a view to not cluttering the resulting expressions). Then the perturbation, the background value (3.64) and the entropy (3.54) are

$$\Phi = \Phi_b\,(1+\tilde{\phi})\ , \qquad \Phi_b^2 = \left(\frac{r_0}{r_{hv}}\right)^{-\theta}\left(\frac{r_0}{R}\right)^2, \qquad S_{BH} = \frac{\Phi_b^2\,V_2}{4G_4} = \frac{\Phi_b^2}{4G_2}\ . \tag{3.81}$$

This gives

$$S_{GH}^{(1)} = -\frac{2\Phi_b^2}{8\pi G_2}\int d\tau\sqrt{\gamma}\ \tilde{\phi}\,K \ \longrightarrow\ -\frac{\Phi_b^2}{4\pi G_2}\int du\,\phi_r(u)\,\{\tau(u),u\}\ . \tag{3.82}$$

In evaluating the last term, we take the boundary of AdS_2 as a slightly deformed curve $(\tau(u),\rho(u))$ parametrized by the boundary coordinate u, as discussed in [36] (reviewed in [143]), and expand the extrinsic curvature using the outward unit normal n^μ to the boundary. Expanding $S_{GH}^{(1)}$ leads to the action above, which contains the Schwarzian derivative $Sch(\tau(u),u) = \{\tau(u),u\} = \frac{\tau'''}{\tau'} - \frac{3}{2}(\frac{\tau''}{\tau'})^2$. The integral above pertains simply to the AdS_2 scale L, into which the various length scales in the nonrelativistic theory have been absorbed. We have also as before defined $\tilde{\phi} = \frac{\phi_r(u)}{\epsilon}$ and $\sqrt{\gamma} = \frac{L}{\epsilon}$.

As for the relativistic brane sec. 3.1.1.2 and (3.39), we note that the coefficient of the Schwarzian effective action is proportional to the entropy (3.54), (3.81) of the compactified black brane, with V_2 finite. As in sec. 3.1.1.2, this coefficient as the entropy arises after making the reasonable definition of the dilaton perturbation as in (3.81), scaling out the background Φ_b. The entropy now contains only Φ_b, which controls the transverse area. Since the entropy captures the number of microstates of the unperturbed background, this is akin to a central charge.

This is the leading term in the total boundary action $I_{bdy} = S_2^E + S_{GH}$. The remaining terms in the expansion of S_{GH} and S_2^E are all quadratic in perturbations and hence are subleading compared to $S_{GH}^{(1)}$ which contains the dilaton perturbation alone at linear order, as for the relativistic brane discussed earlier. This universal behaviour is in accord with the general arguments in *e.g.* [36].

Thus overall, expanding in the perturbations $\tilde{\phi}$, Ω, ψ, we have $I = S^E + S_{GH} = I_0 + I_1 + I_2 + \ldots$, where

$$I_0 = -\frac{\Phi_b^2}{16\pi G_2}\left(\int d^2x\sqrt{g}\,\mathcal{R} + 2\int_{bndry}\sqrt{\gamma}\,K\right) , \tag{3.83}$$

is the background action (see (3.71)): here Ψ_b is constant and it can be checked that $U(\Phi_b, \Psi_b) = 0$. This is a topological term and gives the extremal entropy, very similar to the detailed discussion for the relativistic brane sec. 3.1.1.2. The linear terms are contained in

$$I_1 = -\frac{2\Phi_b^2}{16\pi G_2}\int d^2x\,\sqrt{g}\,\tilde{\phi}\left(\mathcal{R} - \frac{\partial U}{\partial \Phi^2} - \frac{1}{2}(\partial\Psi_b)^2\right) - \frac{2\Phi_b^2}{8\pi G_2}\int_{bndry}\sqrt{\gamma}\,\tilde{\phi}\,K$$
$$-\frac{1}{16\pi G_2}\int d^2x\,\sqrt{g}\Big(-\frac{\Phi_b^2}{2}\partial_\mu\Psi_b\partial^\mu\psi - \psi\frac{\partial U}{\partial \Psi}\Big) . \tag{3.84}$$

On the AdS_2 background with a constant dilaton Φ_b and a constant hvLif scalar field Ψ_b, we get $\frac{\partial U}{\partial \Phi^2}|_{(\Phi_b,\Psi_b)} = -\frac{2}{L^2}$ and the second line in the expression for I_1 above vanishes by the Ψ equation in (3.62). Thus, I_1 reduces to

$$I_1 = -\frac{2\Phi_b^2}{16\pi G_2}\int d^2x\,\sqrt{g}\,\tilde{\phi}\left(\mathcal{R} + \frac{2}{L^2}\right) - \frac{2\Phi_b^2}{8\pi G_2}\int_{bndry}\sqrt{\gamma}\,\tilde{\phi}\,K , \tag{3.85}$$

which is the Jackiw-Teitelboim theory. The fluctuations of the scalar Ψ now propagate on the fixed AdS_2 background at this order. However we see as in sec. 3.1.1.2 that there are various subleading terms at quadratic order ((3.79) and from the Gibbons-Hawking term, see (3.42), as well as possible counterterms), containing the perturbations to the dilaton Φ, metric and scalar Ψ, which all mix (at the same order as the metric): the fluctuation spectrum is stable for physically sensible theories satisfying the energy conditions as we have seen. These encode information about the regularization of the AdS_2 theory by the particular higher dimensional hvLif theory.

3.2.2.3 More general perturbations

In the above analysis we have restricted ourselves to the dimensional reduction of perturbations to only those components of fields (metric, gauge fields, scalar) which have non-trivial background values in the higher dimensional theory. More generally, considering the dimensional reduction of perturbations to all the components of all the fields (some of which have trivial background values) gives

$$h_{MN} \rightarrow h_{\mu\nu}, \quad h_{\mu i}, \quad h_{ij} \; ; \qquad A_M^{(1,2)} \rightarrow A_\mu^{(1,2)}, \quad A_i^{(1,2)} \; ; \qquad \phi \rightarrow \phi \; , \quad (3.86)$$

i.e. tensor, vector and scalar perturbations in the 2-dimensional theory (note that the 2-dim dilaton is g_{xx}). For instance this includes the shear perturbation h_{xy} in the higher dimensional theory as well the spatial components of the gauge fields $A_{1\,i}$, $A_{2\,i}$ for $i = x, y$ which reduce respectively to a non-minimally coupled scalar ($h = g^{(4)\,xx} h_{xy}$) and minimally coupled scalars $A_{1\,i} = \chi_i^{(1)}$, $A_{2\,i} = \chi_i^{(2)}$ in the 2-dimensional theory. The terms in the full 2-dimensional action which govern these perturbations are

$$S = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Big[\cdots - \frac{\Phi^2}{2} (\partial h)^2 - \frac{e^{\lambda_1 \Psi}}{2} (\partial \chi_i^{(1)})^2 - \frac{e^{\lambda_2 \Psi}}{2} (\partial \chi_i^{(2)})^2 \Big] \, . \quad (3.87)$$

The terms involving h_{xy} arise from the higher dimensional Ricci scalar and so contain the overall dilaton factor Φ^2 under reduction to 2-dimensions. The linearized equations for h_{xy} in the higher dimensional theory in *e.g.* [161, 166, 180] can be dimensionally reduced to 2-dimensions: at zero momentum, this is consistent with the Kaluza-Klein ansatz for reduction and the action above. Expanding these terms around the background AdS_2, the leading contributions from these terms appear at quadratic order in perturbations

$$S_2 = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Big[\cdots - \frac{\Phi_b^2}{2} (\partial h)^2 - \frac{e^{\lambda_1 \Psi_b}}{2} (\partial \chi_i^{(1)})^2 - \frac{e^{\lambda_2 \Psi_b}}{2} (\partial \chi_i^{(2)})^2 \Big] \, . \quad (3.88)$$

These are subleading compared to $S_{GH}^{(1)}$ and thus do not contribute to the Schwarzian.

3.3 On a null reduction of the charged AdS_5 black brane

In [99] (see also [101]), it was argued that the null reduction of AdS plane waves, highly boosted limits of uncharged black branes, gives rise to hvLif theories with certain specific z, θ exponents. The lower dimensional hvLif gauge field and scalar arise as the KK gauge field and scalar under x^+-reduction. One might imagine that considering such a null reduction of the charged relativistic black brane might be interesting along these lines. In this section, we describe an attempt to obtain the charged hvLif black branes here by a null x^+-reduction of the charged relativistic black brane in one higher dimension. Unfortunately this turns out to be close, but not quite on the nose: while the charge electric gauge field upstairs does give rise to an electric field in the lower dimensional theory, it also leads to an additional background scalar profile. It would be interesting to understand if this can be refined further.

The action for a charged AdS_5 black brane [22] is[4]

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\Big[\mathcal{R} - 2\Lambda - \frac{2\kappa^2}{e^2}\frac{F^2}{4}\Big] \ . \tag{3.89}$$

The charged AdS_5 black brane metric is

$$ds^2 = \frac{L^2}{r^2}\Big(-f(r)dt^2 + \frac{dr^2}{f(r)} + dx_1^2 + dx_2^2 + dx_3^2\Big) \ , \tag{3.90}$$

$$f(r) = 1 - \Big(1 + \frac{r_0^2\mu^2}{\gamma^2}\Big(1 - \frac{r^2}{r_0^2}\Big)\Big)\Big(\frac{r}{r_0}\Big)^4 \ , \qquad \gamma^2 = \frac{3e^2L^2}{2\kappa^2} \ , \tag{3.91}$$

where the horizon is at $r = r_0$ and the boundary at $r \to 0$. The gauge field A_t, charge density ρ and temperature are

$$A_t = \mu\left(1 - \left(\frac{r}{r_0}\right)^2\right) \ , \qquad \rho = \frac{2L}{e^2r_0^2}\mu \ , \qquad T = \frac{1}{4\pi r_0}\left(4 - 2\frac{r_0^2\mu^2}{\gamma^2}\right) \ . \tag{3.92}$$

Transforming to lightcone coordinates, $x^{\pm} = \frac{t \pm x_3}{\sqrt{2}}$ and performing a boost $x^{\pm} \to \lambda^{\pm}x^{\pm}$, the metric becomes

$$ds^2 = \frac{L^2}{r^2}\Big(-f(r)\Big(\frac{\lambda dx^+ + \lambda^{-1}dx^-}{\sqrt{2}}\Big)^2 + \Big(\frac{\lambda dx^+ - \lambda^{-1}dx^-}{\sqrt{2}}\Big)^2 + \frac{dr^2}{f(r)} + dx_1^2 + dx_2^2\Big) \ . \tag{3.93}$$

Completing squares in dx^+, we get

$$\begin{aligned} ds^2 = &-\frac{2L^2r_0^2f(r)}{\lambda^2r^6\Big(1 + \frac{r_0^2\mu^2}{\gamma^2}\Big(1 - \frac{r^2}{r_0^2}\Big)\Big)}(dx^-)^2 + \frac{L^2}{r^2}\left(\frac{dr^2}{f(r)} + dx_1^2 + dx_2^2\right) \\ &+ \frac{L^2\lambda^2r^2}{2r_0^4}\Big(1 + \frac{r_0^2\mu^2}{\gamma^2}\Big(1 - \frac{r^2}{r_0^2}\Big)\Big)(dx^+ + \mathcal{A}_-dx^-)^2 \ , \end{aligned} \tag{3.94}$$

$$\mathcal{A}_- = \frac{-1 + \frac{r^4}{2r_0^4}\Big(1 + \frac{r_0^2\mu^2}{\gamma^2}\Big(1 - \frac{r^2}{r_0^2}\Big)\Big)}{\frac{\lambda^2r^4}{2r_0^4}\Big(1 + \frac{r_0^2\mu^2}{\gamma^2}\Big(1 - \frac{r^2}{r_0^2}\Big)\Big)} \ . \tag{3.95}$$

The first line in (3.94) after incorporating the conformal factor from x^+-reduction leads approximately to the 4-dim hvLif metric with $z = 3$, $\theta = 1$, in the vicinity of $r \to 0$ and $r \to r_0$. The KK-gauge field becomes the F_1 gauge field in the lower

[4] In this section, $r \to 0$ is the boundary.

dimensional theory: its form becomes that of F_{1rt} only in the vicinity of the horizon $r \to r_0$, giving $A_{1-} \equiv A_{1t} \sim -\frac{1}{(\lambda^2/r_0^4)r^4} + \frac{1}{\lambda^2}$, where we hold $\frac{\lambda^2}{r_0^4}$ fixed which preserves the first term (while the 2nd term dies). This reduction to hvLif is exact if $\mu = 0$, as in [99] for zero temperature (and [101, 161, 166, 180] for finite temperature).

Likewise the $A_t \equiv A_{2t}$ gauge field giving charge becomes in the lower dimensional theory

$$A_{2+} = \lambda A_{2t} , \qquad A_{2-} = \frac{1}{\lambda} A_{2t} \to A_{2t}^{4d} . \tag{3.96}$$

Scaling the chemical potential as $\mu \to \frac{\mu}{\lambda} = fixed$, we obtain precisely the gauge field profile for A_{2t}: however A_{2+} survives as a scalar background in the lower dimensional theory.

It can also be seen that the relativistic brane action (3.1) gives rise upon x^+-reduction to the hvLif action (3.43), upto the extra scalar arising from A_{2+}. It would be interesting look for refinements of the discussion here, towards decoupling this extra scalar.

3.4 Discussion

We have studied dilaton-gravity theories in 2-dimensions obtained by dimensional reduction of certain families of extremal charged hyperscaling violating Lifshitz black branes in Einstein-Maxwell-scalar theories with an extra gauge field in 4-dimensions. We have argued that the near horizon AdS_2 backgrounds here can be obtained in equivalent theories of 2-dim dilaton-gravity with an extra scalar, descending from the higher dimensional scalar, and an interaction potential with the dilaton. A simple subcase is the relativistic black brane with $z = 1, \theta = 0$ (which has no extra scalar), which we have analysed in detail. Studying linearized fluctuations of the metric, dilaton and the extra scalar about these AdS_2 backgrounds suggests stability of the attractor background generically. This is correlated with the requirements imposed by the energy conditions on these backgrounds. From the study of small fluctuations, we have seen that the leading corrections to AdS_2 arise at linear order in the dilaton perturbation resulting in a Schwarzian derivative effective action from the Gibbons-Hawking term, and Jackiw-Teitelboim theory at leading order. We have also seen that the coefficient of the Schwarzian derivative term, (3.39), (3.82), is proportional to the entropy of the (compactified) extremal black branes after defining the perturbations by scaling out the background values (3.37), (3.81): this being the number of microstates of the unperturbed background is thus akin to a central charge. The background entropy arises automatically as a topological term from the compactification. There are of course various subleading terms in the action at quadratic order which mix at the same order as the metric: these encode information on the higher dimensional realization of these AdS_2 backgrounds.

We have explored certain classes of such extremal backgrounds: it would be interesting to understand the space of such AdS_2 theories in a more systematic manner. One might imagine that the parameters in these theories, for instance the dynamical exponents, are reflected in the spectrum of correlation functions, thus distinguishing the specific ultraviolet regularization of the AdS_2 regimes. This requires better understanding of the subleading terms beyond the Schwarzian, which in turn requires a systematic treatment of counterterms and holographic renormalization. We hope to explore these further.

From the point of view of the dual theories, it would seem that the present 2-dim backgrounds are dual to 1-dimensional theories arising from T^2 compactifications of the dual field theories. It would be interesting to understand these better, in part towards possibly exploring parallels with the SYK models [40, 41], discussed more recently in *e.g.* [130, 36, 131, 132] and related SYK/tensor models (see *e.g.* [133, 134, 135, 136, 137, 138, 139, 140, 141, 142]).

Chapter 4

AdS_2 holography from redux, renormalization group flows and c-functions

2-dimensional dilaton gravity is an interesting and relatively simple playground for various physical questions, arising generically from dimensional reduction of higher dimensional gravitational theories, as is well known. In particular the near horizon geometry of extremal black holes and branes in these theories is of the form $AdS_2 \times X$: compactifying the transverse space X gives rise to effective 2-dim dilaton-gravity theories with AdS_2 arising as an attractor point with constant dilaton (which controls the size of X).

Away from the AdS_2 throat region, the 2-dim theory exhibits nontrivial evolution and it is interesting to ask if this can be interpreted as a holographic renormalization group flow. There is a long and rich history of formulating versions of the renormalization group in the holographic context, beginning with *e.g.* [62, 63, 64, 65, 66, 67]. The central feature here is the correspondence between the radial coordinate in the bulk spacetime and the energy scale in the boundary field theory [60, 61]: evolution towards the interior in the bulk corresponds to flowing to lower energies in the boundary theory. In [42, 68, 69], the holographic renormalization group flow was formulated in terms of a radial Hamiltonian evolution, which while not Wilsonian, provides useful insights into the structure of the RG flow and β-functions. The striking Zamolodchikov c-theorem [72] argues that for 2-dim quantum field theories, there exists a positive definite function of couplings that is monotonically decreasing along RG flows, stationary at fixed points and equals the central charge of the corresponding CFT. Holographic versions of c-theorems were discussed in [73, 74, 75, 76]: the monotonicity of the associated c-functions stems ultimately from the null energy conditions which in turn encode the focussing property of null geodesic congruences. Wilsonian versions of the holographic renormalization group were formulated in [70, 71]. Various versions of c-theorems have also been motivated by studies of entanglement entropy: a recent review is [188].

In this chapter we formulate a version of holographic renormalization group flows restricting attention to cases where the far infrared bulk geometry acquires an AdS_2 throat, as occurs for extremal black holes and branes. Further restricting to cases where the transverse space is sufficiently symmetric, as *e.g.* for extremal branes that enjoy space/time translational symmetry and spatial rotational symmetry, the transverse part of the bulk spacetime evolves only in terms of its overall size (or warping). Then the essential flow becomes 2-dimensional in the bulk and can be isolated by dimensional reduction to appropriate 2-dim dilaton-gravity-matter theories. (The effect of the gauge fields that gave rise to charge is mimicked by an appropriate potential for the dilaton and other scalars). This investigation was motivated by [125] where the dimensional reduction of extremal black branes in 4-dim (relativistic) Einstein-Maxwell and (nonrelativistic) hyperscaling violating Lifshitz, hvLif, theories was studied to AdS_2 dilaton-gravity(-scalar) theories, as well as the leading departures from AdS_2 (similar embedddings have been studied recently in [116, 114, 117, 118, 119, 120, 122, 123, 124]). Since the bulk flow to the infrared AdS_2 is essentially 2-dimensional, our formulation does not really distinguish whether the higher dimensional completion is relativistic or nonrelativistic. In the far infrared, the AdS_2 fixed point is the very near horizon region of the corresponding compactified extremal black brane and so it is reasonable to take the central charge of the dual CFT_1 to be the extremal entropy of the black brane, which is the number of underlying microstates. The extremal entropy is given by the transverse area $\frac{V_{D-2}\Phi_h^2}{4G_D} = \frac{\Phi_h^2}{4G_2}$ where the 2-dim dilaton $\Phi^2 = g_{ii}^{(D-2)/2}$ controls the size of the transverse space. This suggests formulating a holographic c-function $\mathcal{C}(u) = \frac{\Phi(u)^2}{4G_2}$ away from the AdS_2 region where the bulk has a nontrivial flow. We argue that this c-function monotonically decreases under flow towards the interior (infrared) and satisfies a c-theorem that follows from the null energy conditions and requiring appropriate boundary conditions (that the AdS_2 throat arises in the nonrelativistic hvLif family above; this is a fairly broad family that includes AdS_D, nonconformal branes and so on, but is otherwise not "too generic"). In addition $\mathcal{C}(u) \to S_{BH}$ at the infrared AdS_2 fixed point, which then fixes the precise form of $\mathcal{C}$. This dilatonic c-function has also previously been discussed in [76] in the context of nonsupersymmetric 4-dim black hole attractors [182], which we were in part motivated by: the present context and discussion is however different in detail as will be clear in what follows.

In sec. 4.2, we study the null energy conditions and discuss this c-function, with some explicit analysis in the phase diagram of nonconformal D2-branes (sec. 4.2.3) and nonconformal D4-branes (sec. 4.2.4). In sec. 4.2.5, we compare this dilatonic c-function with the entropic c-function that has been discussed in the context of entanglement. While the entropic c-function c_E scales as the number of local degrees of freedom (this is also the scaling of the c-function in [73]), the dilatonic c-function above is extensive: it scales as the transverse area. Loosely speaking, $\mathcal{C} \sim c_E V_{d_i} w^{d_i}$ where the AdS_2 throat arises after compactification from $AdS_2 \times X^{d_i}$.

In sec. 4.3, we adapt the holographic RG formulation of de Boer, Verlinde, Verlinde [42] to 2-dim dilaton-gravity-scalar theories. In particular, we obtain RG flow equations and β functions for the (scalar) couplings in the 1-dim boundary theory in a derivative expansion. Using this, we compute β-functions for 2-dim bulk theories arising from reductions of conformal and nonconformal branes. This suggests that it is not consistent to place the AdS_2 throat in a bulk region which exhibits nontrivial RG flow (*i.e.* the AdS_2 throat needs to lie within the bulk region corresponding to the RG fixed point), and resolves a concern about apparently massless perturbations found in [125]. This is not Wilsonian: it would be interesting to adapt the holographic Wilsonian RG of [70, 71] to these 2-dim theories and we leave this for future work. Sec. 4.4 contains a Discussion and the Appendices C.1, C.2, C.3 and C.4 contain various technical details.

4.1 The 2-dim theory and the attractor conditions

We consider a general gravity-scalar action in D dimensions

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g^{(D)}} \left(\mathcal{R}^{(D)} - \frac{h_{IJ}}{2} \partial_M \Psi^I \partial^M \Psi^J - V \right) , \qquad (4.1)$$

where $h_{IJ}(\Psi^I)$ is a positive definite, symmetric metric controlling the kinetic terms of the scalars Ψ^I and $V = V(\Psi^I, g)$ is a potential for the scalars Ψ^I which also contains metric data (*i.e.* V is not simply a scalar potential). Such an effective action arises from theories with gravity, scalars and gauge fields after the gauge fields have been replaced with their background profiles (and changing the signs of the F^2 terms for electric profiles): we have seen examples of this sort arising in Einstein-Maxwell and Einstein-Maxwell-dilaton theories in the previous section. For instance, in the Einstein-Maxwell case with no scalars, the term $\int\sqrt{-g}F^2$ gives $\partial_r(\sqrt{-g}F^{tr})=0$ for electric branes: using this F^{tr}-profile gives the term $\frac{g_{tt}g_{rr}}{g}F_0^2$ thus leading to the effective potential $V=-V_0(\Psi^I)+\frac{1}{g_{xx}^{D-2}}V_2(\Psi^I)$, with $V=-V_0(\Psi^I)$ arising from the cosmological constant term in the original theory. The sign of the $V_2(\Psi^I)$-term is fixed by requiring that the gravity-scalar equations are identical with those of the original theory (See Appendix C.1 for details). Note that this sign is also consistent with electric-magnetic duality (for magnetic branes, the F^2 term does not contain a minus sign which only arises from g_{tt} for electric branes). It is worth noting that the gauge fields have not really been "integrated out" and so these gravity-scalar theories are best regarded as equivalent only for certain (classical or semiclassical) purposes as will be clear in what follows.

We now look for 2-dim theories obtained by dimensional reduction of the above theories on a torus T^{D-2} with the ansatz

$$ds^2=g^{(2)}_{\mu\nu}dx^\mu dx^\nu+\Phi^{\frac{4}{D-2}}\sum_{i=1}^{D-2}dx_i^2\ ,\qquad g^{(D)}_{xx}\equiv\Phi^{\frac{4}{D-2}}\ . \tag{4.2}$$

This gives the 2-dim action

$$S=\frac{1}{16\pi G_2}\int d^2x\sqrt{-g^{(2)}}\ \Phi^2\Big(\mathcal{R}^{(2)}+\frac{D-3}{D-2}\frac{(\nabla_{(2)}\Phi^2)^2}{\Phi^4}-\frac{2\nabla^2_{(2)}\Phi^2}{\Phi^2} - \frac{h_{IJ}}{2}\partial_\mu\Psi^I\partial^\mu\Psi^J-\ V\Big)\ , \tag{4.3}$$

where $\nabla_{(2)\,\mu}$ is a covariant derivative w.r.t. $g^{(2)}_{\mu\nu}$. Now performing a Weyl transformation $g_{\mu\nu}=\Phi^{\frac{2(D-3)}{(D-2)}}g^{(2)}_{\mu\nu}$ absorbs the kinetic term[1] for Φ in $\mathcal{R}$. The 2-dim action then becomes

$$S=\frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\Big(\Phi^2\mathcal{R}-\frac{\Phi^2}{2}h_{IJ}\partial_\mu\Psi^I\partial^\mu\Psi^J-U(\Phi,\Psi^I)\Big)\ ,$$
$$U(\Phi,\Psi^I)=V\Phi^{\frac{2}{D-2}}\ . \tag{4.4}$$

We have suppressed a total derivative term $\int d^2x\sqrt{-g}\,[-\frac{(D-1)}{(D-2)}\nabla^2\Phi^2]$ which cancels with a corresponding term arising from the reduction of the Gibbons-Hawking

[1] Using the covariant derivative ∇_μ w.r.t. $g_{\mu\nu}$, and

$$\nabla^2_{(2)}\Phi^2=\Phi^{\frac{2(D-3)}{(D-2)}}\nabla^2\Phi^2,\quad (\nabla_{(2)}\Phi^2)^2=\Phi^{\frac{2(D-3)}{(D-2)}}(\nabla\Phi^2)^2,$$
$$\mathcal{R}^{(2)}=\Phi^{\frac{2(D-3)}{(D-2)}}\left[\mathcal{R}-\frac{D-3}{D-2}\left(\frac{(\nabla\Phi^2)^2}{\Phi^4}-\frac{\nabla^2\Phi^2}{\Phi^2}\right)\right].$$

boundary term. See Appendix C.4 for a detailed derivation of this Kaluza-Klein reduction.

Our choice in (4.2) of the 2-dim dilaton $\Phi^2 = g_{xx}^{(D-2)/2}$ implies that the area of the transverse space is given by Φ^2: also this choice leads to $\int(\Phi^2\mathcal{R}+\ldots)$ uniformly in the Einstein term of the 2-dim action for any higher dimensional theory.

The 2-dim equations of motion then become

$$\begin{aligned} g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}\Big(\frac{\Phi^2}{2}h_{IJ}\partial_\mu\Psi^I\partial^\mu\Psi^J + U\Big) - \frac{\Phi^2}{2}h_{IJ}\partial_\mu\Psi^I\partial_\nu\Psi^J = 0\,, \\ \mathcal{R} - \frac{h_{IJ}}{2}\partial_\mu\Psi^I\partial^\mu\Psi^J - \frac{\partial U}{\partial(\Phi^2)} = 0\,, \\ \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\Phi^2 h_{IJ}\partial^\mu\Psi^J) - \frac{\partial U}{\partial\Psi^I} = 0\,. \end{aligned} \tag{4.5}$$

These equations admit an AdS_2 critical point with constant scalars and dilaton: we have $\Phi,\ \Psi^I = const$, and $\mathcal{R} = -\frac{2}{L^2}$ (with L the AdS_2 scale) which implies

$$U_h = 0\ , \qquad \frac{\partial U}{\partial(\Phi^2)}\Big|_h = \frac{-2}{L^2}\ , \qquad \frac{\partial U}{\partial\Psi^I}\Big|_h = 0\ , \tag{4.6}$$

from the first, second and third equations respectively; the subscript h denotes that the quantity is evaluated at the AdS_2 background (which is the near horizon throat region of the higher dimensional extremal brane). While we focus in this chapter on pure AdS_2 backgrounds with constant dilaton and constant scalars, the field equations (4.5) admit other solutions including a 2-dim black hole (which is locally AdS_2) where the conditions in (4.6) are modified (see *e.g.* [124]). Turning on perturbations,

$$\Phi = \Phi_h + \phi\ , \qquad \Psi^I = \Psi^I_h + \psi^I\ , \qquad \omega = \omega_h + \Omega\ , \tag{4.7}$$

where $ds^2 = e^{2\omega}(-dx^+dx^-)$ (conformal gauge), the linearized field equations for these perturbations are

$$\begin{aligned} &\partial_+\partial_-\phi + \frac{2\phi}{(x^+-x^-)^2} = 0\ , \\ &(h_{IJ}|_h)\partial_+\partial_-\psi^J + \frac{L^2}{(x^+-x^-)^2\Phi_h^2}\Big[\phi\Big(\frac{\partial^2 U}{\partial\Phi\partial\Psi^I}\Big|_h\Big) + \psi^K\Big(\frac{\partial^2 U}{\partial\Psi^K\partial\Psi^I}\Big|_h\Big)\Big] = 0\ , \\ &\partial_+\partial_-\Omega + \frac{1}{(x^+-x^-)^2}\Big[2\Omega - \frac{\phi}{\Phi_h}\Big(1+\frac{L^2}{4}\Big(\frac{\partial^2 U}{\partial\Phi\partial\Phi}\Big|_h\Big)\Big) - \frac{L^2}{4\Phi_h}\Big(\frac{\partial^2 U}{\partial\Phi\partial\Psi^K}\Big|_h\Big)\psi^K\Big] = 0\,. \end{aligned} \tag{4.8}$$

We define new scalar fields $\zeta^I = \psi^I - \beta^I \phi$, where β^I are constants to be determined. Substituting in the linearized equations for ψ^I above, we get decoupled equations for ζ^I

$$(h_{IJ}|_h)\partial_+\partial_-\zeta^J + \frac{L^2}{(x^+ - x^-)^2\Phi_h^2}\Big(\frac{\partial^2 U}{\partial\Psi^K\partial\Psi^I}\Big|_h\Big)\zeta^K = 0 \tag{4.9}$$

provided β^I satisfy

$$\Big[2(h_{IJ}|_h) - \frac{L^2}{\Phi_h^2}\Big(\frac{\partial^2 U}{\partial\Psi^I\partial\Psi^J}\Big|_h\Big)\Big]\beta^J = \frac{L^2}{\Phi_h^2}\Big(\frac{\partial^2 U}{\partial\Phi\partial\Psi^I}\Big|_h\Big) . \tag{4.10}$$

Defining the matrix $H_{IJ} = 2(h_{IJ}|_h) - \frac{L^2}{\Phi_h^2}\Big(\frac{\partial^2 U}{\partial\Psi^I\partial\Psi^J}\Big|_h\Big)$, we can solve for β^J as

$$\beta^I = H^{IJ}\Big[\frac{L^2}{\Phi_h^2}\Big(\frac{\partial^2 U}{\partial\Phi\partial\Psi^J}\Big|_h\Big)\Big] , \tag{4.11}$$

where H^{IJ} is inverse of H_{IJ} (see Appendix C.3 for an example). With h^{IJ} the inverse of h_{IJ}, the condition for a stable AdS_2 critical point with no tachyonic or massless modes is that the eigenvalues m_I^2 of the mass matrix $\frac{(h^{IJ}|_h)}{\Phi_h^2}(\frac{\partial^2 U}{\partial\Psi^J\partial\Psi^K}|_h)$ satisfy the AdS_2 Breitenlohner-Freedman (BF) bound, *i.e.* $m_I^2 L^2 \geq -\frac{1}{4}$. Of course $m_I^2 > 0$ automatically satisfies this, as was the generic case in [125]. For the case with simply one scalar field Ψ, the criteria for a stable AdS_2 critical point are

$$U_h = 0 , \qquad \frac{\partial U}{\partial\Psi}\Big|_h = 0 , \qquad \frac{\partial U}{\partial(\Phi^2)}\Big|_h = \frac{-2}{L^2} , \qquad \frac{\partial^2 U}{\partial\Psi^2}\Big|_h > -\frac{\Phi_h^2}{4L^2} . \tag{4.12}$$

4.2 Null energy conditions and a c-function

We are studying 2-dim dilaton-gravity-matter theories (with a potential) that we regard implicitly as arising from dimensional reduction of higher dimensional gravity-matter theories. Requiring time translations and that the space transverse to the two (t, r)-dimensions is sufficiently symmetric means that the bulk space effectively evolves nontrivially only in the bulk radial direction. For instance, extremal branes enjoy translational and rotational invariance in the spatial directions: these geometries thus flow only in the radial direction. From the dual point of view, with the radial direction taken as encoding energy scales [60, 61], this simply means that the theory has a nontrivial RG flow encoded by the bulk theory in terms of a holographic renormalization group. This has been the subject of much exploration with a large literature over the years *e.g.* [62, 63, 64, 65, 66, 67, 42, 68, 69, 73, 74, 75, 76, 70, 71] (and the recent review [188]).

Focussing on reductions of extremal objects is equivalent to requiring that the 2-dim theories approach an AdS_2 throat in the deep infrared with the dilaton and scalars acquiring fixed point values. The bulk radial flow to the infrared then must terminate at an AdS_2 fixed point: the transverse space symmetries assumed above imply that the bulk flow is effectively just 2-dimensional and the dual theory is effectively encoded by a flow to a one dimensional CFT_1 obtained by the dimensional reduction of the transverse space. The bulk description of this holographic renormalization is consistent with the reduction ansatz we have been discussing with the size of the transverse space controlled by the 2-dimensional dilaton Φ. It is important to note that this effective 2-dimensional flow appears insensitive to whether the higher dimensional theory is relativistic or nonrelativistic. In particular this raises the question of proposing a c-theorem encoding the renormalization group flow in the dual 1-dimensional theory. This is intriguing especially considering that c-theorems and renormalization group flows are not so easily constrained for nonrelativistic theories: if such a c-theorem and associated c-function can be identified for the present context, one may hope that the analysis here may aid progress in understanding c-theorems for higher dimensional nonrelativistic theories away from extremality. Previous investigations on holographic c-theorems in Lifshitz and Schrödinger theories can be found in [189, 190].

From the bulk point of view, the gravitational theory is required to satisfy appropriate energy conditions for being physically well-defined. In particular the null energy conditions require that the energy momentum tensor contracted with any null vector n^μ be non-negative, *i.e.* $T_{\mu\nu}n^\mu n^\nu \geq 0$. From the Einstein equations governing the bulk theory (which is classical in the large N approximation), this imposes $R_{\mu\nu}n^\mu n^\nu \geq 0$, which can be regarded as defining monotonicity relations for bulk metric data. For relativistic theories, there is a single null vector that is independent: for nonrelativistic theories enjoying spatial translation symmetry, there are two independent null vectors. The reduction ansatz we have been discussing suggests a priori two independent null vectors, one with components along the (t, r) directions, the other with components along (t, x^i) directions.

Consider, for simplicity and concreteness, the ansatz for the D-dim metric

$$ds^2 = -B^2 dt^2 + \frac{du^2}{B^2} + \Phi^{\frac{4}{D-2}} \sum_{i=1}^{D-2} dx_i^2 \ , \tag{4.13}$$

where B and Φ depend only on the radial coordinate u for the sufficiently symmetric space we have in mind. We have chosen these coordinates since the null energy conditions then simplify. The components of the Ricci tensor are

$$R_{tt} = B^2\Big[\frac{(B^2)''}{2} + \frac{2BB'\Phi'}{\Phi}\Big]\ , \qquad R_{xx} = -\frac{\Phi^{\frac{4}{D-2}}}{(D-2)}\Big(\frac{4BB'\Phi'}{\Phi} + \frac{B^2(\Phi^2)''}{\Phi^2}\Big)\ ,$$
$$R_{uu} = -\frac{(B^2)''}{2B^2} - \frac{2}{\Phi B}(\Phi'B' + \Phi''B) + \frac{2(D-4)}{(D-2)}\frac{(\Phi')^2}{\Phi^2}\ , \qquad (4.14)$$

where prime ($'$) denotes derivative w.r.t. the radial coordinate u. For the two null vectors,

$$\zeta^M = (\sqrt{-g^{tt}}, \sqrt{g^{uu}}, 0, 0, \dots, 0)\ , \qquad \xi^M = (\sqrt{-g^{tt}}, 0, \sqrt{g^{xx}}, 0, \dots, 0)\ , \qquad (4.15)$$

the null energy conditions give

$$R_{MN}\zeta^M\zeta^N = -2B^2\Big[\frac{\Phi''}{\Phi} - \frac{(D-4)}{(D-2)}\frac{(\Phi')^2}{\Phi^2}\Big] \geq 0\ ,$$
$$R_{MN}\xi^M\xi^N = \frac{B^2}{2}\Big[\frac{(B^2)''}{B^2} - \frac{2}{(D-2)}\frac{(\Phi^2)''}{\Phi^2} + \frac{2(D-4)}{(D-2)}\frac{(B^2)'\Phi'}{B^2\Phi}\Big] \geq 0\ . \qquad (4.16)$$

Note that the first condition is independent of B in the coordinate choice (4.13).

Example: The charged finite temperature D-dim hvLif metric is

$$ds^2 = \Big(\frac{r}{r_{hv}}\Big)^{-\frac{2\theta}{d_i}}\Big[-\frac{r^{2z}}{R^{2z}}f(r)dt^2 + \frac{R^2}{r^2 f(r)}dr^2 + \frac{r^2}{R^2}\sum_{i=1}^{d_i} dx_i^2\Big]\ , \qquad d_i = D-2\ . \quad (4.17)$$

The uncharged zero temperature case ($f(r) = 1$) written in the form (4.13) has

$$B^2(u) = \left(\frac{(z-\frac{2\theta}{d_i})^{2z-\frac{2\theta}{d_i}} R^{\frac{2\theta}{d_i}(z+1)-2z}}{r_{hv}^{\frac{2z\theta}{d_i}}}\right)^{\frac{1}{z-\frac{2\theta}{d_i}}} u^{\frac{2z-\frac{2\theta}{d_i}}{z-\frac{2\theta}{d_i}}}\ ,$$
$$\Phi^2(u) = \left(\frac{(z-\frac{2\theta}{d_i})^{d_i-\theta} R^{3\theta-z\theta-d_i}}{r_{hv}^{2\theta-z\theta}}\right)^{\frac{1}{z-\frac{2\theta}{d_i}}} u^{\frac{d_i-\theta}{z-\frac{2\theta}{d_i}}}\ , \qquad (4.18)$$
$$u = \frac{r_{hv}^{2\theta/d_i}}{(z-2\theta/d_i)R^{z-1}}\, r^{z-\frac{2\theta}{d_i}}\ .$$

Substituting these expressions for B^2 and Φ^2 in (4.16) recovers the familiar null energy conditions for uncharged zero temperature hvLif theories

$$(d_i-\theta)(d_i(z-1)-\theta) \geq 0\ , \qquad (z-1)(d_i+z-\theta) \geq 0\ . \qquad (4.19)$$

4.2.1 A holographic c-function

The existence of a renormalization group flow in the radial direction in the effective 2-dim bulk theory suggests the existence of a c-function that encodes the number of degrees of freedom along the flow. Requiring that the flow terminates at an AdS_2 fixed point implies that the IR fixed point is a nontrivial CFT_1. The fact that the AdS_2 is the very near horizon geometry of the extremal black brane that describes the system suggests that the number of degrees of freedom describing the IR CFT_1 is equal to the entropy of the extremal black brane. The extremal entropy is given by the horizon area

$$S_{BH} = \frac{g_{xx}^{(D-2)/2}|_h\, V_{D-2}}{4G_D} = \frac{\Phi_h^2}{4G_2}\ , \qquad\qquad G_2 = \frac{G_D}{V_{D-2}}\ , \tag{4.20}$$

with Φ_h the value of the dilaton (4.2) in the AdS_2 region and G_2 the 2-dim Newton constant. Note that the dilaton Φ controls the transverse area of the black brane.

This suggests proposing a holographic c-function after reduction of (4.13),

$$\mathcal{C}(u) = \frac{\Phi^2(u)}{4G_2} = \frac{\Phi^2(u)\, V_{D-2}}{4G_D}\ , \qquad\qquad \Phi^2 = g_{xx}^{(D-2)/2}\ , \tag{4.21}$$

describing the number of active degrees of freedom at scale u along the renormalization group flow to the IR AdS_2 fixed point. This was proposed and discussed in [76] in the context of 4-dim nonsupersymmetric black hole attractors: in the present case, our context is different in part but there is overlap in the physics nonetheless.

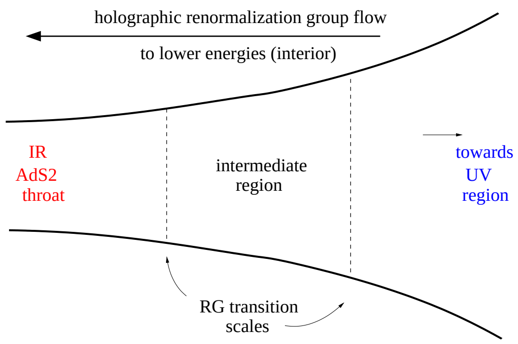

Figure 4.1: A cartoon of the bulk spacetime with the holographic RG flow in the radial direction to the infrared AdS_2 throat region from the far (UV) region through possible intermediate regions (and associated RG transition scales).

To prove the c-theorem for $\mathcal{C}$, we need to prove that $\mathcal{C}(u)$ decreases monotonically as we flow to lower energies u (*i.e.* interior), or equivalently that $\mathcal{C}(u)$ is a monotonically increasing function as u increases (outwards to the boundary). We will do this in two steps.

Step 1: first define $\tilde{\Phi} = \Phi^{2/(D-2)}$. Then the first of the energy conditions (4.16) becomes

$$\tilde{\Phi} = \Phi^{\frac{2}{D-2}} \; ; \quad \tilde{\Phi}'' = \frac{2}{D-2}\left(\frac{\Phi''}{\Phi} - \frac{D-4}{D-2}\frac{(\Phi')^2}{\Phi^2}\right)\Phi^{\frac{2}{D-2}} \quad \Rightarrow \quad -(D-2)\,B^2\,\frac{\tilde{\Phi}''}{\tilde{\Phi}} \geq 0\,, \tag{4.22}$$

so that $\tilde{\Phi}'$ monotonically decreases as u increases towards the boundary. In other words, $\tilde{\Phi}'$ in the interior is larger than $\tilde{\Phi}'$ near the boundary. If we can now argue that $\tilde{\Phi}'$ is positive near the boundary, this would imply that $\tilde{\Phi}' > 0$ everywhere in the bulk as well. This then would imply that $\tilde{\Phi}(u)$ is a monotonically increasing function as u increases and flows towards the boundary. A heuristic picture of the setup appears in Figure 4.1 (see also the discussion on nonconformal D2-branes, sec. 4.2.3, which exemplifies this).

Step 2: Now we proceed to argue that $\tilde{\Phi}'$ is positive near the boundary for suitable boundary conditions, namely that the ultraviolet of the theory belongs in the hvLif family (4.17) that we have been focussing on (which also includes AdS for exponents $z=1, \theta=0$). The extremal branes we are considering here are excited states at finite charge density in these theories: the near boundary region corresponds to the high energy regime of the dual, well above the characteristic scales of the excited states. So it suffices to use the asymptotic (uncharged zero temperature) form of these backgrounds.

Using (4.18), we have $d_i = D-2$ and $\tilde{\Phi} = \Phi^{\frac{2}{d_i}}$. Then retaining only relevant factors, we have

$$\begin{aligned} &\tilde{\Phi} \sim \left(z - \frac{2\theta}{d_i}\right)^{\frac{d_i-\theta}{zd_i-2\theta}} u^{\frac{d_i-\theta}{zd_i-2\theta}}\,, \qquad \tilde{\Phi}' \sim \left(z - \frac{2\theta}{d_i}\right)^{\frac{d_i-\theta}{zd_i-2\theta}} \frac{(d_i-\theta)}{(zd_i-2\theta)} \frac{1}{u^{\frac{d_i(z-1)-\theta}{zd_i-2\theta}}}\,, \\ &\tilde{\Phi}'' \sim -\left(z - \frac{2\theta}{d_i}\right)^{\frac{d_i-\theta}{zd_i-2\theta}} \frac{(d_i-\theta)(d_i(z-1)-\theta)}{(zd_i-2\theta)^2} \frac{1}{u^{\frac{d_i(z-1)-\theta}{zd_i-2\theta}+1}}\,. \end{aligned} \tag{4.23}$$

Then $\frac{\tilde{\Phi}''}{\tilde{\Phi}} \leq 0$ gives the null energy condition $(d_i-\theta)(d_i(z-1)-\theta) \geq 0$. A reasonable dual field theory requires positivity of specific heat if the theory is excited to finite temperature. Since the entropy for these theories scales as $S \sim V_{d_i} T^{\frac{d_i-\theta}{z}}$, the positivity of the corresponding specific heat imposes $\frac{d_i-\theta}{z} \geq 0$. This implies $d_i - \theta \geq 0$ since $z \geq 1$. Alongwith the null energy condition, this leads to $(d_i(z-1)-\theta) \geq 0$. These two conditions together imply

$$zd_i - 2\theta = (d_i(z-1)-\theta) + (d_i-\theta) \geq 0\,. \tag{4.24}$$

Then we see that $\tilde{\Phi}'$ is positive in this near boundary region. Roughly, $\tilde{\Phi} \sim u^n$ and $\tilde{\Phi}' \geq 0$ and $\tilde{\Phi}'' \leq 0$ require $n \geq 0$ and $n(n-1) \leq 0$, *i.e.* $0 \leq n \leq 1$. We have argued that this is true if the null energy conditions and positivity of specific heat are satisfied.

Thus finally, we have shown that for the ultraviolet data we are considering, $\tilde{\Phi}(u) = \Phi^{2/(D-2)}$ is monotonically decreasing as u flows to the interior (lower energies). Since the exponent $\frac{2}{D-2}$ is positive, this implies that $\Phi(u)$ satisfies the same monotonicity property. This proves that the holographic c-function (4.21) we propose in fact satisfies the c-theorem.

At the IR AdS_2 horizon, $\mathcal{C}$ in (4.21) approaches the extremal black hole entropy (4.20), which is the IR number of degrees of freedom controlling the number of black hole microstates, akin to a central charge for this subsector. In fact it is this requirement that $\mathcal{C} \to S_{BH}$ at the IR AdS_2 fixed point which fixes the precise scaling of $\mathcal{C}$ in terms of Φ (else any positive power of Φ is monotonic, from the above arguments). The stationarity of $\mathcal{C}$ at the IR AdS_2 fixed point is implied by the stability of AdS_2 attractor with respect to the fluctuations in (4.7) through stability conditions in (4.12).

It is interesting to note that we have mainly used the first null energy condition in (4.16) in the above arguments. The second null energy condition appears to be more a condition on the matter configurations: for instance, the second condition for hvLif backgrounds (4.17) gives $z \geq 1,\ d_i + z - \theta \geq 0$ in (4.19), which follow from reality of the fluxes supporting the background, and also follows from specific heat positivity. to illustrate the condition in more generality, let us restrict to $D = 4$ for simplicity: then the second condition in (4.16) gives

$$\frac{(\Phi^2)''}{\Phi^2} \leq \frac{(B^2)''}{B^2}\ , \tag{4.25}$$

which says that the dilaton "acceleration" is not greater than that of the 2-dim metric. As we approach the AdS_2 region, we have $B^2 \sim (u-u_0)^2$ so this becomes $\frac{(\Phi^2)''}{\Phi^2} \lesssim \frac{2}{(u-u_0)^2}$ which is trivially satisfied as $u \to u_0$ since the right hand side grows large. Thus the near AdS_2 region does not provide any additional constraint from this energy condition. However the near boundary region gives nontrivial constraints on the exponents defining the theory from this energy condition as we have seen. We will discuss this further later.

One might be concerned that the null energy conditions (and the Einstein equations) are second order equations while renormalization group flow is first order. It is important to note in this regard that the boundary conditions we have imposed is on the first derivative $\tilde{\Phi}'$, which then automatically implies monotonicity. This physical boundary condition has effectively ruled out the other (growing) mode which would likely be singular in the interior.

In explicit examples (*e.g.* nonconformal branes redux, later), we can check this dilatonic c-function in fact has the right behaviour. Consider for instance an extremal brane in an hvLif theory where B^2, Φ^2 near the boundary have the form (4.18) while in the near AdS_2 region, $B^2 \sim (u-u_0)^2$ and $\Phi \sim u^A$ globally, with $A = \frac{d_i-\theta}{zd_i-2\theta}$. Then using the arguments around (4.24), we see that $A \geq 0$ so that $\Phi^2(u)$ can be seen explicitly to monotonically decrease through the bulk as u decreases flowing towards AdS_2. We also see that $A \leq 1$ so that $\Phi'' \leq 0$ in accord with the first energy condition in (4.16). The second energy condition in the near boundary region simply imposes the constraints on the exponents that we have seen, which are required of the theory. In the near AdS_2 region, $B^2 \sim (u-u_0)^2$ and so as described above, the second energy condition is satisfied. This family includes AdS where $z=1, \theta=0$ and $\Phi^2 = u^2$.

From the point of dual 1-dim theories which flow to the CFT_1 dual to the AdS_2 bulk theory, the arguments above suggest that $\mathcal{C}$ in (4.21) is a candidate c-function. While spatial coarse-graining does not make sense in $0+1$-dim (no space!), the renormalization group defined in terms of integrating out high energy modes does make sense, *i.e.* as a flow to lower energies (IR). In the present context, we have defined the holographic c-function $\mathcal{C}$ as essentially inherited from the higher dimensional theory that has been compactified: it would be interesting to understand the c-function from the dual 1-dim point of view.

4.2.2 Null energy conditions from the 2-dim perspective

We have described the null energy conditions $T_{\mu\nu}n^\mu n^\nu \geq 0$ in the higher dimensional theory and recast them in terms of 2-dim bulk variables $g^{(2)}_{\mu\nu}$, Φ. The two independent null vectors give two independent null energy conditions (4.16) as we have seen. However it is interesting to note that only one of the null vectors — $\zeta^M = (\sqrt{-g^{tt}}, \sqrt{g^{uu}})$ — has a natural interpretation intrinsically in the 2-dimensional spacetime. This leads to the first of the energy conditions. The second one appears to have no intrinsic interpretation directly in 2-dimensions: however we can reverse engineer this from the higher dimensional theory and recast it in terms of the potential governing the dilaton and other scalars in the context of the dilaton-gravity-scalar theory (4.4).

The $tr-$ and ii-components, for $i = 1, \ldots, D-2$, of Einstein equations for the gravity-scalar action in D-dimensions (4.1) are

$$\mathcal{R}^{(D)}_{\mu\nu} - \frac{g^{(D)}_{\mu\nu}}{2}\mathcal{R}^{(D)} = \frac{h_{IJ}}{2}\Big(\partial_\mu\Psi^I\partial_\nu\Psi^J - \frac{g^{(D)}_{\mu\nu}}{2}\partial_M\Psi^I\partial^M\Psi^J\Big) - \frac{g^{(D)}_{\mu\nu}}{2}V \ , \tag{4.26}$$

$$\mathcal{R}^{(D)}_{ii} - \frac{g^{(D)}_{ii}}{2}\mathcal{R}^{(D)} = \frac{h_{IJ}}{2}\Big(\partial_i\Psi^I\partial_i\Psi^J - \frac{g^{(D)}_{ii}}{2}\partial_M\Psi^I\partial^M\Psi^J\Big) - \frac{g^{(D)}_{ii}}{2}V + \frac{\partial V}{\partial g^{(D)\,ii}} \ , \tag{4.27}$$

where we have taken the metric to be diagonal in the spatial components *i.e.* $g^{(D)}_{ij} = 0 \ \forall\ i \neq j$ and the potential V in (4.1) to be dependent only on $g^{(D)}_{ii}$ components. These equations $G^{(D)}_{MN} = 8\pi G_D T^{(D)}_{MN}$ give the stress tensor components as

$$8\pi G_D T^{(D)}_{\mu\nu} = \frac{h_{IJ}}{2}\Big(\partial_\mu\Psi^I\partial_\nu\Psi^J - \frac{g^{(D)}_{\mu\nu}}{2}\partial_M\Psi^I\partial^M\Psi^J\Big) - \frac{g^{(D)}_{\mu\nu}}{2}V \ , \tag{4.28}$$

$$8\pi G_D T^{(D)}_{xx} = \frac{h_{IJ}}{2}\Big(\partial_x\Psi^I\partial_x\Psi^J - \frac{g^{(D)}_{xx}}{2}\partial_M\Psi^I\partial^M\Psi^J\Big) - \frac{g^{(D)}_{xx}}{2}V + \frac{\partial V}{\partial g^{(D)\,xx}} \ . \tag{4.29}$$

After dimensional reduction, we obtain the 2-dim action (4.4) and the above equations become the 2-dim Einstein equations and the dilaton equation in (4.5). In particular the higher dimensional $\mu\nu$-components give 2-dim Einstein equations which we write in the form

$$\begin{aligned} &\frac{1}{\Phi^2}[g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2] = 8\pi G_D T^{(D)}_{\mu\nu} \ , \\ &8\pi G_D T^{(D)}_{\mu\nu} = \frac{h_{IJ}}{2}\Big(\partial_\mu\Psi^I\partial_\nu\Psi^J - \frac{g_{\mu\nu}}{2}\partial_M\Psi^I\partial^M\Psi^J\Big) - \frac{g_{\mu\nu}U}{2\Phi^2} \ . \end{aligned} \tag{4.30}$$

The xx-component of the higher dimensional stress tensor can likewise be expressed in terms of the 2-dim potential and its derivative as

$$8\pi G_D T^{(D)}_{xx} = -\frac{\Phi^{\frac{4}{(D-2)}}}{4}h_{IJ}\partial_M\Psi^I\partial^M\Psi^J - \frac{(D-2)}{2}\Phi^{\frac{2}{D-2}+2}\frac{\partial U}{\partial\Phi^2} \ . \tag{4.31}$$

This has no obvious 2-dim origin intrinsically: the null energy condition intrinsic to two dimensions involves only $T_{\mu\nu}$ but not T_{xx}. Further the null vector ξ^M in (4.15) has no intrinsically 2-dimensional meaning. The second null energy condition in (4.16) from the higher dimensional theory can however be recast in 2-dimensional language in terms of the stress tensor components above and it is then interesting to ask what 2-dimensional constraints it leads to. This second NEC $T^{(D)}_{MN}\xi^M\xi^N \geq 0$ for the null vector

$$\xi^M = (\sqrt{-g^{(D)\,tt}}, 0, \sqrt{g^{(D)\,xx}}, 0, \ldots, 0) = (\sqrt{-g^{tt}}\Phi^{\frac{D-3}{D-2}}, 0, \Phi^{\frac{-2}{D-2}}, 0, \ldots, 0) \tag{4.32}$$

becomes

$$8\pi G_D(T_{tt}^{(D)}(\xi^t)^2+T_{xx}^{(D)}(\xi^x)^2) \\ =-\Phi^{\frac{2(D-3)}{(D-2)}}\frac{g^{tt}h_{IJ}}{2}\partial_t\Psi^I\partial_t\Psi^J+\frac{\Phi^{\frac{-2}{D-2}}}{2}\Big(U-(D-2)\Phi^2\frac{\partial U}{\partial\Phi^2}\Big)\geq 0\ . \tag{4.33}$$

For static backgrounds as we have here, $\partial_t\Psi^I=0$: then this second NEC becomes a nontrivial condition on the potential and its derivative

$$U-(D-2)\,\Phi^2\,\frac{\partial U}{\partial\Phi^2}\ \geq\ 0\ . \tag{4.34}$$

In 2-dim dilaton-gravity-matter theories that arise from some higher dimensional reduction, this condition (4.34) is simply recognized as the second NEC in (4.16). However if we regard (4.4) as an intrinsically 2-dim theory, then it appears reasonable to impose such a constraint on the dilaton-matter potential. Note that this constraint is not the same as the stability condition (4.12) : rather (4.34) constraints the space of allowed solutions to the equations of motion (4.5) in an intrinsic 2-dim theory. For example, constant Φ, constant Ψ^I with AdS_2 metric is a solution to (4.5) when (4.6) is satisfied, which is consistent with (4.34).

To illustrate the above constraint, consider first a potential of a form we have seen arising from reduction of 4-dim Einstein-Maxwell theory,

$$U=-V_0\Phi+\frac{V_2}{\Phi^3}\qquad\Rightarrow\qquad U-2\Phi^2\frac{dU}{d\Phi^2}=\frac{4V_2}{\Phi^4}\geq 0\quad\Rightarrow\quad V_2\geq 0\ . \tag{4.35}$$

Of course this can be recognized as the condition $Q^2\geq 0$ in the higher dimensional theory: from the 2-dim point of view, the condition gives positivity constraints on the coefficients that appear in the potential.

The first NEC in D-dimensions, $T^{(D)}_{MN}\zeta^M\zeta^N\geq 0$, or $\mathcal{R}^{(D)}_{MN}\zeta^M\zeta^N\geq 0$ for the null vector $\zeta^M=(\sqrt{-g^{(D)\,tt}},\sqrt{g^{(D)\,rr}},0,0,\ldots,0)=(\sqrt{-g^{tt}}\Phi^{\frac{D-3}{D-2}},\sqrt{g^{rr}}\Phi^{\frac{D-3}{D-2}},0,0,\ldots,0)$ becomes the 2-dim NEC $\mathcal{R}_{\mu\nu}\tilde\zeta^\mu\tilde\zeta^\nu\geq 0$ for the 2-dim null vector $\tilde\zeta^\mu=(\sqrt{-g^{tt}},\sqrt{g^{rr}})$. Using (4.30), this NEC gives

$$-\nabla_\mu\nabla_\nu\Phi^2\tilde\zeta^\mu\tilde\zeta^\nu=g^{tt}\nabla_t\nabla_t\Phi^2-g^{rr}\nabla_r\nabla_r\Phi^2\geq 0\ . \tag{4.36}$$

For static backgrounds, this recovers the condition on the “acceleration” of the dilaton that we have studied earlier in the context of the c-theorem.

4.2.3 The c-function in the $M2$-$D2$ system

Nonconformal Dp-branes upon dimensional reduction on the transverse sphere give rise to hvLif theories with $z = 1$ and nonzero θ [89]. In particular the $D2$-brane supergravity phase upon S^6-reduction give rise to bulk 4-dim hvLif theories with $z = 1,\ \theta = -\frac{1}{3}$. These flow [8] in the infrared to M2-branes, which give rise to AdS_4 upon S^7-reduction (with $z = 1, \theta = 0$). These are all uncharged phases. Adding a $U(1)$ gauge field to this system — which can be taken as the dual to the $U(1)_R$ current — and tuning to extremality gives string realizations for the extremal versions of the above 4-dim theories. We have in mind that the AdS_4 phase eventually terminates in the deep IR at an AdS_2 throat: see Figure 4.1. Since the transition from the D2-phase to the M2-AdS_4 phase occurs at energies well above the IR scale where the AdS_2 emerges, the D2-phase can be essentially regarded as uncharged for the purposes of the discussion below. In the far UV, the D2-branes are described by free 3-dim SYM.

The string frame metric and the dilaton describing N $D2$-branes are

$$ds^2_{st} = \frac{r^{5/2}}{R_2^{5/2}} dx_{\|}^2 + \frac{R_2^{5/2}}{r^{5/2}} dr^2 + \frac{R_2^{5/2}}{r^{1/2}} d\Omega_6^2 \,, \qquad e^{\phi} = g_s \Big(\frac{R_2^5}{r^5}\Big)^{1/4} \,, \tag{4.37}$$

with $e^{\phi_\infty} = g_s$ the asymptotic value of the dilaton, and

$$g^2_{YM} = \frac{g_s}{\sqrt{\alpha'}} \,, \qquad R_2^5 = \alpha'^3 g^2_{YM} N \,, \tag{4.38}$$

where we are ignoring numerical factors (since we will be primarily interested here in the scaling behaviour along the RG flow). The 10-dim Einstein frame metric $ds^2_E = e^{-\frac{1}{2}(\phi-\phi_\infty)} ds^2_{st}$ after dimensional reduction on S^6 gives the Einstein metric of the effective 4-dim hvLif theory with $d_i = 2,\ z = 1,\ \theta = -\frac{1}{3}$,

$$ds^2 = \frac{r^{7/2}}{R_2^{7/2}}(-dt^2 + dx_1^2 + dx_2^2) + \frac{R_2^{3/2}}{r^{3/2}} dr^2 = \Big(\frac{\rho}{R_2}\Big)^{1/3} \Big[\frac{\rho^2}{R_2^2}(-dt^2 + dx_i^2) + \frac{R_2^2}{\rho^2} d\rho^2\Big] \,. \tag{4.39}$$

The second expression is written in coordinates where the hvLif form (4.17) is manifest. We have

$$\rho = \frac{r^{3/2}}{R_2^{1/2}} \,, \qquad w = \frac{r^{3/2}}{R_2^{5/2}} \,, \qquad u = \frac{r^2}{R_2} \,, \tag{4.40}$$

where $w = \frac{r^{(5-p)/2}}{R_p^{(7-p)/2}}$ is the nonconformal Dp-brane supergravity radius/energy variable introduced in [61]. This coordinate has also proved useful in studies of entanglement entropy and its interpretation in the nonconformal brane system [191, 103].

The coordinate u is chosen to cast the metric above in the form (4.13) that we found useful in analysing the c-function in our earlier discussion: in terms of those expressions, we have

$$B^2 = \frac{r^{7/2}}{R_2^{7/2}} = \frac{u^{7/4}}{R_2^{7/4}} \,, \qquad \Phi^2 = \frac{r^{7/2}}{R_2^{7/2}} = w^{7/3}\, R_2^{7/3} \,. \tag{4.41}$$

Then the c-function (4.21) written in terms of the energy variable w in the D2-phase is

$$\mathcal{C}(w) \sim \frac{V_2 \Phi^2}{G_4} = V_2\, w^{7/3}\, N^2 \frac{1}{(g_{YM}^2 N)^{1/3}} \;\; = \;\; V_2\, w^2\, N_{eff}(w) \,, \tag{4.42}$$

after spatial compactification of the D2-branes. Here we have used

$$G_4 \sim \frac{G_{10}}{Vol(S^6)} \sim \frac{g_s^2 \alpha'^4}{R_2^6} \,, \qquad N_{eff}(w) = N^2 \frac{1}{(g_{YM}^2 N/w)^{1/3}} \,, \tag{4.43}$$

and $N_{eff}(w)$ is the scale-dependent number of degrees of freedom for the D2-phase (which also has played useful roles in entanglement studies [191, 103]), while the dimensionless gauge coupling at scale w is $g_{eff} = \frac{g_{YM}^2 N}{w^{3-p}}$.

The M2-phase is given by the $AdS_4 \times S^7$ background (again ignoring numerical factors)

$$ds^2 = \frac{r^2}{R^2} dx_{||}^2 + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_7^2 \,, \qquad R^6 \sim N l_p^6 \,, \qquad G_4 \sim \frac{G_{11}}{Vol(S^7)} \sim \frac{l_p^9}{R^7} \,, \tag{4.44}$$

which after reducing on the S^7 gives AdS_4 (and l_p is the 11-dim Planck length; note that $R^6 \sim N l_P^6 \sim g_s R_2^5 \sqrt{\alpha'}$). This is already in the form (4.13) with $u = r$ and the energy variable $w = \frac{r}{R^2}$ and $\Phi^2 = \frac{r^2}{R^2}$. Then the c-function in this M2-phase after spatial compactification is

$$\mathcal{C}(w) = \frac{V_2}{G_4} \frac{r^2}{R^2} = \frac{R^2}{G_4} V_2\, w^2 = N^{3/2}\, V_2\, w^2 \,, \tag{4.45}$$

using $\frac{R^2}{G_4} = \frac{R^9}{G_{11}} = N^{3/2}$. It is useful to recall [8] that the D2-phase is valid in the regime $g_{YM}^2 N^{1/5} \ll \frac{r}{\alpha'} \ll g_{YM}^2 N$ so that $N^{3/2} \ll N_{eff}(w) \ll N^2$. At the scale $\frac{r}{\alpha'} \sim g_{YM}^2$ the system transits from a smeared (arrayed) M2-phase to the M2-AdS_4 phase. At this scale which corresponds to $w \sim g_{YM}^2 N^{-1/2}$, we have $N_{eff}(w) \sim N^{3/2}$ and the D2-phase c-function can be seen to match that in the M2-phase. The present analysis cannot be applied to the intermediate interpolating phase corresponding to smeared (arrayed) M2-branes.

We have so far discussed uncharged D2-M2 phases. With the AdS_2 emerging in the deep IR (within the AdS_4 region), the transition between the D2- and M2-phase is well approximated by the uncharged system. To see this explicitly, note that the charged hvLif metric arising from D2-redux is

$$\begin{aligned} ds^2 &= \Big(\frac{\rho}{R_2}\Big)^{1/3}\Big[\frac{\rho^2}{R_2^2}(-f(\rho)dt^2+dx_1^2+dx_2^2)+\frac{R_2^2}{\rho^2 f(\rho)}d\rho^2\Big]\ , \\ f(\rho) &= 1-\Big(\frac{\rho_0}{\rho}\Big)^{10/3}+\frac{Q_D^2}{\rho^{14/3}}\Big(1-\Big(\frac{\rho}{\rho_0}\Big)^{4/3}\Big)\ . \end{aligned} \tag{4.46}$$

with $Q_D^2 \sim \rho_0^{7/3}$ at extremality. Since the transition is occurring at a scale $\rho_{trans} \gg \rho_0$, we essentially have $f(\rho) \sim 1$ in that region. In the deep IR, the extremal M2-AdS_4 phase

$$ds^2 = \frac{r^2}{R^2}(-f(r)dt^2+dx_i^2)+\frac{R^2}{r^2 f(r)}dr^2\,, \qquad f(r) = 1-\Big(\frac{r_0}{r}\Big)^3+\frac{Q^2}{r^4}\Big(1-\frac{r}{r_0}\Big)\,, \tag{4.47}$$

(after S^7 reduction) with $Q^2 \sim r_0^4$ develops an AdS_2 throat, with the horizon at $r = r_0$. Then the IR scale at the horizon is $u = \frac{r_0}{R^2}$ and the c-function approaches

$$\mathcal{C} \xrightarrow{AdS_2} N^{3/2}\,\frac{V_2\,Q}{R^4}\ . \tag{4.48}$$

This phase is dual to a doped CFT_3, with dopant density $\sigma_Q \equiv \frac{Q}{R^4}$ which is essentially the number of dopant charge carriers per unit area of the M2-branes: then $\mathcal{C}_{IR}$ is essentially a "central charge" whose $N^{3/2}$ scaling reflects the underlying number of degrees of freedom of the M2-CFT, which has been doped with an additional $V_2\sigma_Q$ number of charge carriers distributed over the volume V_2 of the M2-branes (there are some parallels with the heuristic partonic picture of entanglement for excited AdS plane wave states in [104]). This "central charge" corresponds to the number of microstates of the doped CFT_1 obtained by spatial compactification of the M2-branes: it is essentially dual to the AdS_2 theory describing the extremal black brane with C_{IR} the extremal entropy. In some sense, $w^2 = \frac{Q(w)}{R^4}$ is a scale-dependent dopant density with $w_{IR}^2 = \frac{Q}{R^4}$ the infrared value. String/M-theory realizations of this involve turning on an appropriate G_4-flux in the M2-brane system which after the S^7 reduction gives the additional $U(1)$ gauge field that provides charge [192].

It is clear that the c-function (4.42) in the D2-phase gives a larger number of degrees of freedom than that in the M2-phase (4.45) (noting the regimes for w, which flows to lower energies), in accord with our general discussions of the c-function earlier. This dovetails with the fact that θ is negative in the D2-phase (with $z = 1$). It is also worth noting that the precise N-scalings etc arise from the precise dimensionful factors contained in the dilaton.

4.2.4 c-function in the $M5$-$D4$ system

The M5-D4 brane system flows from an $AdS_7 \times S^4$ phase (dual to the 6-dim $(2,0)$ theory) through the D4-supergravity phase to finally 5-dim SYM in the IR. While this does not admit an AdS_2 region in the deep IR of the phase diagram, it is interesting to study the c-function (4.21) in this case as well. This discussion has parallels with the D2-M2 case so we will be succinct. For the $M5$-AdS_7 phase, we have

$$ds^2 = \frac{r^2}{R^2}dx_{\|}^2 + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_4^2 \,, \qquad R^3 \sim N l_p^3 \,, \qquad G_7 \sim \frac{G_{11}}{Vol(S^4)} \sim \frac{l_p^9}{R^4} \,, \tag{4.49}$$

which after reducing on the S^4 gives AdS_7 (l_p is the 11-dim Planck length). This dovetails with (4.13) with $u = r$ and the energy variable $w = \frac{r}{R^2}$ and $\Phi^2 = \frac{r^2}{R^2}$. The c-function in this M5-phase after spatial T^5-compactification then is

$$\mathcal{C}(w)_{M5} = \frac{V_5}{G_7}\frac{r^2}{R^2} = \frac{R^5}{G_7}V_5\, w^5 = N^3\, V_5\, w^5 \,, \tag{4.50}$$

using $\frac{R^5}{G_7} \sim \frac{R^9}{G_{11}} \sim N^3$ here. Now for the N $D4$-branes phase, the string metric and dilaton are

$$ds_{st}^2 = \frac{r^{3/2}}{R_4^{3/2}}dx_{\|}^2 + \frac{R_2^{3/2}}{r^{3/2}}dr^2 + R_2^{3/2}\, r^{1/2}d\Omega_4^2 \,, \qquad e^{\phi} = g_s\Big(\frac{r}{R_4}\Big)^{3/4} \,,$$
$$g_{YM}^2 \sim g_s\sqrt{\alpha'}, \qquad R_4^3 \sim \alpha' g_{YM}^2 N \,, \tag{4.51}$$

ignoring numerical factors. The 10-dim Einstein frame metric $ds_E^2 = e^{-\frac{1}{2}(\phi-\phi_\infty)}ds_{st}^2$ after S^4-redux gives a 6-dim hvLif theory (4.17) with $d_i = 4$, $z = 1$, $\theta = -1$,

$$ds^2 = \frac{r^{5/4}}{R_4^{5/4}}\Big(-dt^2 + \sum_{i=1}^{4} dx_i^2\Big) + \frac{R_4^{7/4}}{r^{7/4}}dr^2 = \Big(\frac{\rho}{R_4}\Big)^{1/2}\Big[\frac{\rho^2}{R_4^2}(-dt^2 + \sum_{i=1}^{4} dx_i^2) + \frac{R_4^2}{\rho^2}d\rho^2\Big]. \tag{4.52}$$

The D4-brane supergravity radius/energy variable w [61] and the u coordinate in (4.13) are

$$\rho = R_4^{1/2} r^{1/2} \,, \qquad w = \frac{r^{1/2}}{R_4^{3/2}} \,, \qquad u = \frac{r^{3/2}}{R_4^{1/2}} \,. \tag{4.53}$$

With $G_6 \sim G_{10}/Vol(S^4) \sim g_s^2\alpha'^4/R_4^4$, the scale-dependent number of degrees of freedom $N_{eff}(w)$ for the D4-phase [191, 103], and the dilaton $\Phi^2 = g_{xx}^{(D-2)/2} = \frac{r^{5/2}}{R_4^{5/2}}$, the c-function (4.21) is

$$\mathcal{C}(w)_{D4} = \frac{V_4\Phi^2(w)}{4G_6} \sim V_4\, w^4\, N_{eff}(w)\ , \qquad N_{eff}(w) = N^2(g_{YM}^2 Nw)\ , \tag{4.54}$$

after spatial T^4-compactification, and the regime of validity is $1 \ll g_{YM}^2 Nw \ll N^{2/3}$. Noting $R_{11} = g_s\sqrt{\alpha'} \sim g_{YM}^2$ and $V_5 = V_4 R_{11}$, we see that the c-function continuously transits from the M5- to the D4-phase for length scales longer than the 11th circle size R_{11}. This leads to the guess that the c-function in the free 5-dim SYM phase after spatial T^4-compactification is possibly $N^2 V_4 w^4$.

4.2.5 On dilatonic and entropic c-functions

It is interesting to compare the dilatonic c-function we have defined with the entropic c-function [193, 194] that has been studied based on studies of entanglement entropy [195, 196, 197, 198].

Consider the bulk geometry (4.13) with asymptotics being AdS or hvLif, focussing on $D = 4$ dimensions (with no compactification). For a strip subsystem with width along say x, the induced metric on a time slice is $\Phi^2 dy^2 + \frac{du^2}{B^2} + \Phi^2 dx^2$ and the area functional for holographic entanglement [53, 54] is $A = L\int du\, \frac{\Phi}{B}\sqrt{1 + B^2\Phi^2\,(\frac{dx}{du})^2}$ which after extremization gives

$$S = \frac{2L}{4G_4}\int \frac{du}{B}\frac{\Phi^3}{\sqrt{\Phi^4 - \Phi_*^4}}\ , \qquad l = \int \frac{du}{B}\frac{\Phi_*^2}{\Phi\sqrt{\Phi^4 - \Phi_*^4}}\ , \tag{4.55}$$

where $\Phi_* = \Phi(u_*)$ is the value of the dilaton at the turning point u_* of the minimal surface, and L is the size of the (essentially infinitely long) strip in the longitudinal y direction. For instance, for a strip in AdS_4, the area and width integrals are $S = \frac{2L}{4G_4}\int_\epsilon^{u_*}\frac{Rdu}{u}\frac{u^3}{\sqrt{u^4-u_*^4}} \sim \frac{R^4}{G_4}(\frac{L}{\epsilon} - \frac{L}{l})$ and $l \sim \frac{R^2}{u_*}$. The entropic c-function is then defined as

$$c_E = \frac{l^2}{L}\frac{dS}{dl}\ , \tag{4.56}$$

which gives $c_E \sim \frac{R^2}{G_4}$. This is thus a measure of the local number of degrees of freedom, or central charge, in the dual field theory responsible for entanglement. In theories with an RG flow, the entropic c-function is scale dependent and satisfies $\frac{dc_E}{dl} \leq 0$, *i.e.* it monotonically decreases with the width l, and thus plays the role of a c-function based on entanglement entropy. For instance for nonconformal branes, $c_E(l) \sim N_{eff}(l)$.

We will now try to draw comparisons between this entropic c-function and the dilatonic c-function (4.21). Away from the AdS_2 horizon, $u \gg u_0$ and we have $B \sim \frac{u-u_0}{R} \sim \frac{u}{R}$. Since the dilaton monotonically decreases flowing towards the interior, *i.e.* as u decreases in (4.13), we can recast the integrals above as

$$S = \frac{2L}{4G_4}\,\Phi_*\,R\int_{\varphi_\epsilon}^{1}\frac{du}{u}\frac{\varphi^3}{\sqrt{\varphi^4-1}}\ , \qquad l = \frac{2R}{\Phi_*}\int_{\varphi_\epsilon}^{1}\frac{du}{u}\frac{1}{\varphi\sqrt{\varphi^4-1}}\ , \qquad \varphi = \frac{\Phi}{\Phi_*}\ . \tag{4.57}$$

Since the dilaton decreases monotonically, we can redefine the radial variable by φ as $du = \frac{d\varphi}{\varphi'}$, and we note that all the information about the turning point has been scaled out after this recasting to the factors outside the integrals. The entropic c-function receives a nonvanishing contribution simply from the finite part for which the integrals are simply finite numerical factors. Then we see that $S \sim \frac{R}{G_4}\Phi_* L \sim \frac{R^2}{G_4}\frac{L}{l}$ which recovers $c_E \sim \frac{R^2}{G_4}$.

From the discussion of the c-function for M2-branes (4.45), we see that $\mathcal{C}(w) = N^{3/2}V_2 w^2$. Recalling that $\frac{R^2}{G_4} \sim N^{3/2}$, we see that the dilatonic c-function scales as the entropic number of degrees of freedom (*i.e.* c_E), but is in addition extensive: it scales with V_2 and shrinks as w^2 along the flow to the IR. In the 2-dim bulk theory after compactification, c_E cannot be formulated since the spatial directions are compactified but the dilatonic c-function nevertheless encodes the number of degrees of freedom that c_E encodes. Similar comparisons can be drawn for other cases.

It is also interesting to recall the holographic c-function in [73]. For a bulk theory $ds^2 = e^{2A(\varrho)}dx_\parallel^2 + d\varrho^2$ enjoying Lorentz invariance (*i.e.* $z=1$), this c-function is $C_{FGPW}(w) \sim \frac{1}{G_D\,(dA/d\varrho)^{d_i}}$ where d_i is the number of boundary spatial dimensions. For AdS_D, we have $A \sim \log\frac{\varrho}{R}$ and $c_{FGPW} \sim \frac{R^{d_i}}{G_D}$ which gives $c_{FGPW} \sim N^{3/2}$ for M2-AdS_4. For nonconformal Dp-branes ($g_{YM}^2 \sim g_s\alpha'^{(p-3)/2}$) [8]

$$\begin{aligned} ds^2_{st} &= \frac{r^{(7-p)/2}}{R_p^{(7-p)/2}}dx_\parallel^2 + \frac{R_p^{(7-p)/2}}{r^{(7-p)/2}}(dr^2 + r^2 d\Omega_{8-p}^2)\ , \\ e^\Phi &= g_s\Big(\frac{R_p^{7-p}}{r^{7-p}}\Big)^{\frac{3-p}{4}}\ , \qquad R_p^{7-p} \sim g_{YM}^2 N\alpha'^{5-p}\ , \end{aligned} \tag{4.58}$$

upon S^{8-p}-redux give the hvLif metric (4.17) with $z = 1,\ p = d_i$ [89]: in this case, using

$$\theta = p - \frac{9-p}{5-p} = -\frac{(p-3)^2}{5-p}\ , \quad ds^2_{p+2} = \left(\frac{\varrho}{R_p}\right)^{2\left(1-\frac{d_i}{\theta}\right)} dx^2_{\parallel} + d\varrho^2\ , \quad \frac{\varrho}{R_p} = \left(\frac{r}{R_p}\right)^{-\frac{\theta(5-d_i)}{2d_i}}, \tag{4.59}$$

we see that the dilatonic c-function (4.21) we have discussed (after redux to 2-dim) gives

$$\mathcal{C}(w) \sim N_{eff}(w) V_{d_i} w^{d_i}\ , \qquad N_{eff}(w) = N^2 \left(\frac{g^2_{YM} N}{w^{3-p}}\right)^{\frac{p-3}{5-p}}, \quad w = \frac{r^{(5-p)/2}}{R_p^{(7-p)/2}}\ , \tag{4.60}$$

as we have seen earlier in the detailed discussions on the D2-M2 and M5-D4 phases (with w the nonconformal Dp-brane supergravity radius/energy variable [61]). On the other hand, the c-function in [73] mentioned above can be seen to be $c_{FGPW} \sim N_{eff}(w)$, with the same scaling as the entropic c-function c_E: this is a measure of the local degrees of freedom of the higher dimensional theory, while the dilatonic c-function $\mathcal{C}$ has additional extensivity arising from the compactification.

4.3 2-dim radial Hamiltonian formalism and β-functions

A version of the holographic renormalization group was formulated in [42]: using a radial ADM-type split of the bulk spacetime, the radial Hamiltonian constraint gives rise to flow equations for couplings and corresponding β-functions. This is not a Wilsonian formulation since the effective action at the scale corresponding to some radial slice depends on data not just at higher energy scales that have been integrated out: Wilsonian formulations of the holographic renormalization group have been investigated in [70, 71]. Nevertheless this dVV formulation gives useful qualitative insights into the holographic renormalization group. In this section, we will adapt this to obtain renormalization group flow equations and β-functions starting with the 2-dim dilaton-gravity-scalar theory. As in [42], writing the boundary 1-dim action on some radial slice in a radial Hamilton-Jacobi formulation, we separate this at low scales into local and nonlocal parts and then write the local part in a derivative expansion. Taking the leading term to arise from just a "boundary potential" term for the couplings (scalars Φ, Ψ^I), *i.e.* no derivatives, we obtain relations between the original potential and the boundary potential using the Hamiltonian constraint, thereby obtaining β-functions from the flow equations. We will describe this below.

Consider the 2-dim gravity-scalar action (4.4) including the Gibbons-Hawking term

$$S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\Big(\Phi^2\mathcal{R} - \frac{\Phi^2}{2}h_{IJ}\partial_\mu\Psi^I\partial^\mu\Psi^J - U(\Phi,\Psi^I)\Big) + \frac{1}{16\pi G_2}\int dt\sqrt{-\gamma}\,\Phi^2\,2\,K\ , \tag{4.61}$$

where $U(\Phi,\Psi^I) = V\Phi^{\frac{2}{D-2}}$. Substituting the radial decomposition of the metric

$$ds^2 = (N^2 + \gamma_{tt}(N^t)^2)dr^2 + 2\gamma_{tt}N^t dtdr + \gamma_{tt}dt^2\ , \tag{4.62}$$

certain boundary terms cancel with the Gibbons-Hawking term: then massaging leads to a radial Lagrangian (in Appendix C.2, we derive this from dimensional reduction of the Hamiltonian formulation of a higher dimensional theory of the sort we have been considering)

$$L = \frac{1}{16\pi G_2}\int dt\sqrt{-\gamma}\,N\Big[-2\Box_t\Phi^2 + \frac{2K}{N}(\partial_r\Phi^2 - N^t\partial_t\Phi^2) - \frac{\Phi^2}{2}h_{IJ}\partial_\mu\Psi^I\partial^\mu\Psi^J - U\Big]\,, \tag{4.63}$$

where the extrinsic curvature and the covariant derivative w.r.t. γ_{tt} are

$$K_{tt} = \frac{1}{2N}(\partial_r\gamma_{tt} - 2D_tN_t)\,,\quad K = \gamma^{tt}K_{tt}\,,\quad D_tN_t = \partial_tN_t - \Gamma^t_{tt}N_t\,,\quad \Box_t \equiv \gamma^{tt}D_tD_t\,, \tag{4.64}$$

where $\gamma^{tt} = (\gamma_{tt})^{-1}$ and $N_t = \gamma_{tt}N^t$. The conjugate momenta for the fields γ_{tt}, Φ and Ψ^I are

$$\begin{aligned}
\pi^{tt} &\equiv \frac{16\pi G_2}{\sqrt{-\gamma}}\frac{\delta L}{\delta\dot\gamma_{tt}} = \frac{\gamma^{tt}}{N}(\partial_r\Phi^2 - N^t\partial_t\Phi^2)\ ,\\
\pi_\Phi &\equiv \frac{16\pi G_2}{\sqrt{-\gamma}}\frac{\delta L}{\delta\dot\Phi} = 4K\Phi = \frac{2\Phi\gamma^{tt}}{N}(\dot\gamma_{tt} - 2D_tN_t)\ ,\\
\pi_I &\equiv \frac{16\pi G_2}{\sqrt{-\gamma}}\frac{\delta L}{\delta\dot\Psi^I} = -\frac{\Phi^2h_{IJ}}{N}(\dot\Psi^J - N^t\partial_t\Psi^J)\ ,
\end{aligned} \tag{4.65}$$

where dot represents the radial derivative,*i.e.* $\dot\Phi = \partial_r\Phi$ and so on. Inverting, we obtain

$$\begin{aligned}
\dot\Phi &= \frac{1}{2\Phi}\Big(\frac{N\pi^{tt}}{\gamma^{tt}} + N^t\partial_t\Phi^2\Big)\ ,\\
\dot\gamma_{tt} &= \frac{N\pi_\Phi}{2\Phi\gamma^{tt}} + 2D_tN_t\ ,\\
\dot\Psi^I &= -\frac{Nh^{IJ}\pi_J}{\Phi^2} + N^t\partial_t\Psi^I\ .
\end{aligned} \tag{4.66}$$

The Hamiltonian is obtained by a Legendre transform of the Lagrangian (4.63) as

$$H = \frac{1}{16\pi G_2}\int dt\sqrt{-\gamma}(\pi^{tt}\dot{\gamma}_{tt} + \pi_\Phi\dot{\Phi} + \pi_I\dot{\Psi}^I) - L = \frac{1}{16\pi G_2}\int dt\sqrt{-\gamma}(N\mathcal{H} + N^t\mathcal{H}_t)\ . \tag{4.67}$$

The fields N and N^t being non-dynamical gives the constraints $\frac{\partial H}{\partial N} = 0$ and $\frac{\partial H}{\partial N^t} = 0$ *i.e.*

$$\mathcal{H} = \frac{\pi^{tt}\pi_\Phi}{2\Phi\gamma^{tt}} + 2\Box_t\Phi^2 + U - \frac{\pi^I\pi_I}{2\Phi^2} + \frac{\Phi^2}{2}h_{IJ}\gamma^{tt}\partial_t\Psi^I\partial_t\Psi^J = 0\ , \tag{4.68}$$

$$\mathcal{H}_t = -2\gamma_{tt}D_t\pi^{tt} + \pi_\Phi\partial_t\Phi + \pi_I\partial_t\Psi^I = 0\ . \tag{4.69}$$

Now as in [42], we imagine that the boundary action on some radial slice can be evaluated as a function of boundary field values at that scale: then thinking of this action in terms of a radial Hamilton-Jacobi formulation allows us to relate the conjugate momenta as derivatives of this action, which we then use in the Hamiltonian constraints in a derivative expansion to relate the bulk and boundary expressions. Towards this, we segregate this boundary action into a local part and a nonlocal part at a low energy scale $\mu \ll \mu_c$ (with μ_c the UV cut-off) in a derivative expansion,

$$S_{bdy} = S_{loc} + \Gamma\ . \tag{4.70}$$

Here Γ contains higher derivative, nonlocal terms which encode the information about correlation functions of the operators in the boundary theory and gives flow equations for these correlation functions, which are the Callan-Symanzik equations (we will not explore that here). A general form of the local action is $S_{loc} = \int dt\sqrt{-\gamma}\,\left(W(\Phi,\Psi^I) + \frac{M_{IJ}}{2}\partial_a\Psi^I\partial^a\Psi^J + \ldots\right)$, with W, M_{IJ} are local functions of the couplings. Approximating the local part of the boundary action in terms of just the leading potential term (ignoring derivatives) as

$$S_{bdy} = \int dt\sqrt{-\gamma}\,W(\Phi,\Psi^I) + \cdots\ , \tag{4.71}$$

we can define the conjugate momenta in terms of the boundary potential $W(\Phi,\Psi^I)$ as

$$\begin{aligned}
\pi^{tt} &\equiv \frac{16\pi G_2}{\sqrt{-\gamma}}\frac{\delta S_{bdy}}{\delta\gamma_{tt}} = 8\pi G_2\gamma^{tt}\,W\ ,\\
\pi_\Phi &\equiv \frac{16\pi G_2}{\sqrt{-\gamma}}\frac{\delta S_{bdy}}{\delta\Phi} = 16\pi G_2\frac{\partial W}{\partial\Phi}\ ,\\
\pi_I &\equiv \frac{16\pi G_2}{\sqrt{-\gamma}}\frac{\delta S_{bdy}}{\delta\Psi^I} = 16\pi G_2\frac{\partial W}{\partial\Psi^I}\ .
\end{aligned} \tag{4.72}$$

Substituting these momenta in the Hamiltonian constraint (4.68) and collecting the potential terms, we get a relation between the bulk potential U and the boundary potential W as

$$\frac{U}{(8\pi G_2)^2} = \frac{2h^{IJ}}{\Phi^2}\frac{\partial W}{\partial \Psi^I}\frac{\partial W}{\partial \Psi^J} - \frac{W}{\Phi}\frac{\partial W}{\partial \Phi} \ . \tag{4.73}$$

Using the momenta (4.72) in terms of W, the flow equations (4.66) can be written as

$$\begin{aligned}
\dot{\Phi} &= \frac{1}{2\Phi}\Big((8\pi G_2)NW + N^t\partial_t\Phi^2\Big) \ , \\
\dot{\gamma}_{tt} &= \frac{(8\pi G_2)N}{\Phi\gamma^{tt}}\frac{\partial W}{\partial \Phi} + 2D_tN_t \ , \\
\dot{\Psi}^I &= -\frac{(16\pi G_2)Nh^{IJ}}{\Phi^2}\frac{\partial W}{\partial \Psi^J} + N^t\partial_t\Psi^I \ .
\end{aligned} \tag{4.74}$$

<u>β-functions:</u> Choosing Fefferman-Graham gauge

$$N = 1 \ , \qquad N^t = 0 \ ; \qquad ds^2 = dr^2 + \gamma_{tt}dt^2 \ , \tag{4.75}$$

the flow equations become

$$\dot{\Phi} = \frac{(4\pi G_2)\,W}{\Phi} \ , \qquad \dot{\gamma}_{tt} = (8\pi G_2)\frac{\gamma_{tt}}{\Phi}\frac{\partial W}{\partial \Phi} \ , \qquad \dot{\Psi}^I = -\frac{(16\pi G_2)h^{IJ}}{\Phi^2}\frac{\partial W}{\partial \Psi^J} \ . \tag{4.76}$$

From the above equation, we see that we can split the radial and time dependence of γ_{tt} as $\gamma_{tt} = a^2\hat{\gamma}_{tt}$, where $a = a(r)$ and $\hat{\gamma}_{tt}$ is independent of r (*i.e.* γ_{tt} simply rescales under RG flow). Then the flow equation for γ_{tt} gives

$$\dot{a} = \frac{(4\pi G_2)}{\Phi}\frac{\partial W}{\partial \Phi}a \ . \tag{4.77}$$

Using this relation, we can write the radial derivatives in terms of $\dot{a}$. In contrast with the higher dimensional cases in [42], note that this brings a factor of $\frac{\partial W}{\partial \Phi}$ in the β-functions, which we define for the RG flow as

$$\beta^I \equiv a\frac{d}{da}\Psi^I = \frac{a\dot{\Psi}}{\dot{a}} = -\frac{4h^{IJ}}{\Phi\,\frac{\partial W}{\partial \Phi}}\frac{\partial W}{\partial \Psi^J} \ , \tag{4.78}$$

$$\beta_\Phi \equiv a\frac{d}{da}\Phi = \frac{a\dot{\Phi}}{\dot{a}} = \frac{W}{\frac{\partial W}{\partial \Phi}} \ . \tag{4.79}$$

We can write the relation (4.73) between U and W in terms of β-functions as

$$\frac{U}{(8\pi G_2)^2} = \frac{W^2 h_{IJ}\beta^I\beta^J}{8\beta_\Phi^2} - \frac{W^2}{\Phi\beta_\Phi} \ . \tag{4.80}$$

4.3.1 β functions for conformal/non-conformal theories

In this subsection, we have set all the scales to unity *i.e.* $R = 1$, $r_{hv} = 1$, $8\pi G_2 = 1$.

β-functions, conformal branes: The effective potential for 2-dim dilaton-gravity theories obtained from *e.g.* reductions of conformal branes is of the form $U(\Phi)$: then from (4.73), the boundary potential is given by

$$\frac{dW^2}{d\Phi^2} = -U \qquad \Longrightarrow \qquad W^2 = -\int_{\Phi_h}^{\Phi} U\, d\Phi^2 \ , \tag{4.81}$$

where we have imposed $W^2(\Phi_h) = 0$. This boundary condition in a sense reflects the fact that the 1-dim background corresponds to zero energy. Expanding U around the critical point,

$$U = \left(\frac{dU}{d\Phi^2}\Big|_h\right)(\Phi^2 - \Phi_h^2) + \cdots \ , \qquad \text{where} \quad U|_h = 0 \ , \tag{4.82}$$

the β-function becomes

$$\beta_\Phi = \frac{W^2}{\Phi\frac{dW^2}{d\Phi^2}} = \frac{\int_{\Phi_h}^{\Phi} d\Phi^2 U}{\Phi U} = \frac{\left(\frac{\Phi^4}{2} - \Phi^2\Phi_h^2\right) + \left(\frac{\Phi_h^4}{2}\right)}{\Phi_h(\Phi^2 - \Phi_h^2)} = \frac{(\Phi^2 - \Phi_h^2)^2}{2\Phi_h(\Phi^2 - \Phi_h^2)} = \frac{(\Phi^2 - \Phi_h^2)}{2\Phi_h} \ . \tag{4.83}$$

At the critical point, $\Phi = \Phi_h$, we see that β_Φ vanishes, consistent with the expectation that the AdS_2 critical point background arises at the fixed point of the RG flow.

β-functions, nonconformal branes: The effective potential for 2-dim dilaton-gravity-scalar theories obtained from reductions of non-conformal branes (the dVV formulation was discussed for nonconformal branes in [77]) is of the form

$$U(\Phi, \Psi) = e^{\gamma\Psi}\tilde{U}(\Phi) \ , \tag{4.84}$$

with *e.g.* $\tilde{U} = -V_0\Phi + \frac{V_2}{\Phi^3}$ for 4-dim theories with $z = 1, \theta \neq 0$, as we have seen. Assuming an ansatz for W, $W = e^{\frac{\gamma\Psi}{2}}\chi(\Phi)$ and substituting in (4.73), we get

$$\frac{d\chi^2}{d\Phi^2} = \frac{\gamma^2\chi^2}{2\Phi^2} - \tilde{U} \ . \tag{4.85}$$

Integrating this equation, the general solution is

$$\chi^2 = \chi_0(\Phi^2)^{\frac{\gamma^2}{2}} - (\Phi^2)^{\frac{\gamma^2}{2}} \int d\Phi^2 \tilde{U}(\Phi^2)^{\frac{-\gamma^2}{2}} \ . \tag{4.86}$$

Then β_Φ can be written as

$$\beta_\Phi = \frac{W^2}{\Phi \frac{\partial W^2}{\partial \Phi^2}} = \frac{\chi^2}{\Phi \frac{d\chi^2}{d\Phi^2}} = \frac{\chi^2}{\Phi(\frac{\gamma^2 \chi^2}{2\Phi^2} - \tilde{U})} \ . \tag{4.87}$$

For arbitrary χ_0 such that $\chi|_h \neq 0$ at the critical point, β_Φ becomes $\beta_\Phi\Big|_h = \frac{2\Phi_h}{\gamma^2} \neq 0$.

Let us consider the case when χ_0 is chosen such that $\chi|_h = 0$: this makes the boundary potential vanish at the critical point, *i.e.* $W|_h = 0$, corresponding to zero energy as in the conformal case above. To study this case, we expand $\tilde{U}$ around the critical point,

$$\tilde{U} = \left(\frac{d\tilde{U}}{d\Phi^2}\Big|_h\right)(\Phi^2 - \Phi_h^2) + \cdots \ , \qquad \text{where} \quad \tilde{U}|_h = 0 \ , \tag{4.88}$$

and substitute the solution for χ in the above expression for β-function to get

$$\beta_\Phi = \frac{2\Phi_h}{\gamma^2} \frac{\left[\chi_0 - (\Phi_h^2)^{1-\frac{\gamma^2}{2}} \left(\frac{d\tilde{U}}{d\Phi^2}\Big|_h\right)(\frac{\Phi^2}{2} - \Phi_h^2)\right]}{\left[\chi_0 - (\Phi_h^2)^{1-\frac{\gamma^2}{2}} \left(\frac{d\tilde{U}}{d\Phi^2}\Big|_h\right)\left(\frac{\Phi^2}{2}\left(1 - \frac{4}{\gamma^2}\right) - \Phi_h^2(1 - \frac{4}{\gamma^2})\right)\right]} \ . \tag{4.89}$$

Choosing $\chi_0 = -(\Phi_h^2)^{1-\frac{\gamma^2}{2}} \left(\frac{d\tilde{U}}{d\Phi^2}\Big|_h\right)\frac{\Phi_h^2}{2}$, which makes $\chi|_h = 0$ (and so $W|_h = 0$), the above expression simplifies to

$$\beta_\Phi = \frac{2\Phi_h}{\gamma^2 - 4} \ . \tag{4.90}$$

For $W = e^{\frac{\gamma\Psi}{2}}\chi(\Phi)$, (4.78) and (4.79) give $\beta_\Psi = \frac{-2\gamma}{\Phi}\beta_\Phi$. We see that both β-functions β_Φ and β_Ψ do not vanish at the AdS_2 critical point for any choice of χ_0. This vindicates the intuition that the AdS_2 critical point can consistently be placed at the fixed point of an RG flow, but not at some intermediate point along the flow.

4.3.2 Examples

β-function, $M2$-phase: The effective 2-dim potential for 4-dim Einstein-Maxwell redux is

$$U = -V_0\Phi + \frac{V_2}{\Phi^3} \ , \quad V_0 = -2\Lambda_{(4)} = 6 \ , \quad V_2 = 2Q^2 \ . \tag{4.91}$$

Then for $W = W(\Phi)$, (4.73) gives

$$W\frac{\partial W}{\partial \Phi} - V_0\Phi^2 + \frac{V_2}{\Phi^2} = 0 \ , \qquad i.e. \qquad \frac{\partial}{\partial \Phi}\Big(\frac{W^2}{2}\Big) = V_0\Phi^2 - \frac{V_2}{\Phi^2} \ . \tag{4.92}$$

Integrating this equation and imposing $W|_h = 0$, we obtain

$$W = -\Big[\frac{2V_0\Phi^3}{3} + \frac{2V_2}{\Phi} + 2\chi_0\Big]^{\frac{1}{2}} \ , \tag{4.93}$$

where the integration constant χ_0 is fixed by our boundary condition (4.81) to be $\chi_0 = -8r_0^3$ using (4.91). The β-function using (4.79) is

$$\beta_\Phi = \frac{\Big[\frac{2V_0\Phi^3}{3} + \frac{2V_2}{\Phi} + 2\chi_0\Big]}{\Big[V_0\Phi^2 - \frac{V_2}{\Phi^2}\Big]} \ . \tag{4.94}$$

As we approach the AdS_2 critical point placed in the $M2$ phase $\Phi_h = r_0$, $Q^2 = 3r_0^4$, we see that β_Φ vanishes, elucidating the general discussion above for conformal branes (note that β_Φ diverges for arbitrary χ_0 so the boundary condition on W is important).

<u>β-function, $D2$-phase:</u> D2-branes after S^6-redux lead to a 4-dim hvLif theory with exponents $z = 1$, $\theta = -\frac{1}{3}$. The effective potential in the 2-dim corresponding theory, again setting dimensionful parameters to unity for convenience, is

$$U = e^{\gamma\Psi}\Big(-V_0\Phi + \frac{V_2}{\Phi^3}\Big) \ , \qquad \gamma = -\frac{-1}{\sqrt{7}} \ , \quad V_0 = \frac{70}{9} \ , \quad V_2 = \frac{28}{9}Q^2 \ . \tag{4.95}$$

Assuming an ansatz $W = e^{\frac{\gamma\Psi}{2}}\chi(\Phi)$ for W and substituting in (4.73) gives

$$\chi\frac{\partial \chi}{\partial \Phi} - \frac{\gamma^2\chi^2}{2\Phi} - V_0\Phi^2 + \frac{V_2}{\Phi^2} = 0 \ , \tag{4.96}$$

whose solution gives

$$W = e^{\frac{\gamma\Psi}{2}}\chi(\Phi) = -e^{\frac{\gamma\Psi}{2}}\Big[\frac{49}{9}\Big(\frac{Q^2}{\Phi} + \Phi^3\Big) + \chi_0\Phi^{\frac{1}{7}}\Big]^{\frac{1}{2}} \ , \tag{4.97}$$

where the integration constant χ_0 is again fixed by the boundary condition $W|_h = 0$

(it will turn out that the precise value of χ_0 drops out in what follows). The β-functions from (4.79), (4.78) become

$$\beta_\Phi = \frac{2\Phi\left[\frac{49}{9}(Q^2+\Phi^4)+\chi_0\Phi^{\frac{8}{7}}\right]}{\left[\frac{49}{9}(-Q^2+3\Phi^4)+\frac{\chi_0}{7}\Phi^{\frac{8}{7}}\right]} \xrightarrow{h} 14 r_0^{\frac{7}{6}} ,$$
$$\beta_\Psi = \frac{4}{\sqrt{7}}\frac{\left[\frac{49}{9}(Q^2+\Phi^4)+\chi_0\Phi^{\frac{8}{7}}\right]}{\left[\frac{49}{9}(-Q^2+3\Phi^4)+\frac{\chi_0}{7}\Phi^{\frac{8}{7}}\right]} \xrightarrow{h} 4\sqrt{7} , \qquad (4.98)$$

where we have evaluated the β-functions at the AdS_2 critical point placed in this $D2$-phase, which has

$$\Phi_h = r_0^{\frac{7}{6}} , \quad e^{\frac{\gamma\Psi_h}{2}} = r_0^{-\frac{1}{6}} , \quad Q^2 = \frac{5}{2}r_0^{\frac{14}{3}} \quad \Rightarrow \quad \Phi^4+Q^2 = \frac{7}{2}r_0^{\frac{14}{3}} , \quad 3\Phi^4 - Q^2 = \frac{1}{2}r_0^{\frac{14}{3}} . \qquad (4.99)$$

These nonvanishing β-functions imply that the theory is still flowing at the AdS_2 critical point which thus is an inconsistency and shows up as the massless scalar mode found previously: the AdS_2 horizon is only consistently placed within the true fixed point region of the RG flow which is the above $M2$-phase in this D2-M2 phase diagram.

β-function, $M5$-$D4$ phases: We can likewise analyse the flow for the M5-D4 system: here the M5-AdS_7 phase (after reducing on the S^4) flows to the D4-supergravity phase obtained by dimensional reduction on the M-theory 11th circle. The AdS_7 phase has $z=1, \theta=0$ while the D4-phase is a 6-dim hvLif theory with $z=1, \theta=-1$ and again the scalar leads to a massless mode if the AdS_2 horizon is placed within this region. Here again, the β-functions can be shown to vanish in the conformal $M5$-phase but not in the $D4$-phase. To obtain extremal branes, we add an additional $U(1)$ gauge field which provides charge: this gives an Einstein-Maxwell or Einstein-Maxwell-scalar theory in the M5- and D4-phases respectively.

The effective 2-dim potential for 7-dim Einstein-Maxwell redux is

$$U = -V_0\Phi^{\frac{2}{5}} + \frac{V_2}{\Phi^{\frac{18}{5}}} = \frac{1}{\Phi^{\frac{3}{5}}}\Big(- V_0\Phi + \frac{V_2}{\Phi^3}\Big) , \quad V_0 = -2\Lambda_{(7)} = 30 , \quad V_2 = 20\,Q^2 . \qquad (4.100)$$

Then (4.73) as for $M2$-branes gives

$$W = -\Big[\frac{5V_0\Phi^{\frac{12}{5}}}{6} + \frac{5V_2}{4\Phi^{\frac{8}{5}}} - \frac{125}{2}r_0^6\Big]^{\frac{1}{2}} , \qquad (4.101)$$

where an integration constant has again been fixed by the boundary condition $W|_h = 0$. Then the β-function using (4.79) is

$$\beta_\Phi = \frac{\left[\frac{5V_0\Phi^{\frac{12}{5}}}{6} + \frac{5V_2}{4\Phi^{\frac{8}{5}}} - \frac{125}{2}r_0^6\right]}{\Phi\left[V_0\Phi^{\frac{2}{5}} - \frac{V_2}{\Phi^{\frac{13}{5}}}\right]} \,, \tag{4.102}$$

which vanishes at the AdS_2 horizon $\Phi_h = r_0^{\frac{5}{2}}$, $Q^2 = \frac{3}{2}r_0^{10}$, if placed in the $M5$ phase.

Now, for the 6-dim hvLif theory with $z = 1$, $\theta = -1$ from D4-redux, the 2-dim effective potential is

$$U = e^{\gamma\Psi}\Big(- V_0\Phi^{\frac{1}{2}} + \frac{V_2}{\Phi^{\frac{7}{2}}}\Big) = e^{\gamma\Psi}\Big(- V_0\Phi + \frac{V_2}{\Phi^3}\Big)\frac{1}{\Phi^{\frac{1}{2}}} \,,$$
$$\gamma = -\frac{-1}{\sqrt{10}} \,, \qquad V_0 = 30 \,, \qquad V_2 = 20\,Q^2 \,. \tag{4.103}$$

As for the D2-case, taking an ansatz $W = e^{\frac{\gamma\Psi}{2}}\chi(\Phi)$ and using (4.73) gives

$$W = e^{\frac{\gamma\Psi}{2}}\chi(\Phi) = -e^{\frac{\gamma\Psi}{2}}\left[\frac{25Q^2}{\Phi^{\frac{3}{2}}} + 25\Phi^{\frac{5}{2}} + \chi_0\Phi^{\frac{1}{10}}\right]^{\frac{1}{2}} \,, \tag{4.104}$$

where the precise value of the integration constant χ_0 will again not play any role. The β-functions using (4.79), (4.78) are

$$\beta_\Phi = 2\Phi\frac{\left[\frac{25Q^2}{\Phi^{\frac{3}{2}}} + 25\Phi^{\frac{5}{2}} + \chi_0\Phi^{\frac{1}{10}}\right]^{\frac{1}{2}}}{\left[\frac{-75Q^2}{2\Phi^{\frac{3}{2}}} + \frac{125}{2}\Phi^{\frac{5}{2}} + \frac{\chi_0}{10}\Phi^{\frac{1}{10}}\right]} \,, \qquad \beta_\Psi = \frac{4}{\sqrt{10}}\frac{\left[\frac{25Q^2}{\Phi^{\frac{3}{2}}} + 25\Phi^{\frac{5}{2}} + \chi_0\Phi^{\frac{1}{10}}\right]^{\frac{1}{2}}}{\left[\frac{-75Q^2}{2\Phi^{\frac{3}{2}}} + \frac{125}{2}\Phi^{\frac{5}{2}} + \frac{\chi_0}{10}\Phi^{\frac{1}{10}}\right]} \,. \tag{4.105}$$

If the AdS_2 critical point is placed in the $D4$ phase, we require

$$\Phi_h = r_0^{\frac{5}{2}} \,, \quad Q^2 = \frac{3}{2}r_0^{10} \quad \Rightarrow \quad \frac{25Q^2}{\Phi^{\frac{3}{2}}} + 25\Phi^{\frac{5}{2}} = \frac{125}{2}r_0^{\frac{25}{4}} \,, \quad \frac{-75Q^2}{2\Phi^{\frac{3}{2}}} + \frac{125}{2}\Phi^{\frac{5}{2}} = \frac{25}{4}r_0^{\frac{25}{4}} \,, \tag{4.106}$$

giving $\beta_\Phi|_h \to 20\,r_0^{\frac{5}{2}}$ and $\beta_\Psi|_h \to 4\sqrt{10}$. As in the D2-case (4.98), these nonvanishing β-functions imply that it is inconsistent to place the AdS_2 critical point in the D4-phase where the theory has a nontrivial RG flow.

It appears nontrivial to carry out this analysis of the flow equations and β-functions for general potential $U(\Phi, \Psi^I)$ as *e.g.* for more general hvLif theories. Since the perturbation analysis in [125] revealed a disconcerting massless mode only for $z = 1$ (which dovetails with our analysis here), it would appear that there would be no problem for the AdS_2 throat to emerge in general hvLif$_{z,\theta}$ theories. It would be interesting to explore this further.

4.4 Discussion

We have formulated a version of the holographic renormalization group flow for 2-dim dilaton-gravity-scalar theories arising from reductions of higher dimensional extremal black branes, as in [125], thereby restricting to 2-dim flows that end at an AdS_2 throat. We have assumed that the transverse space is sufficiently symmetric which then allows this formulation to be insensitive to the higher dimensional branes being relativistic or nonrelativistic. Based on the null energy conditions, we have proposed a holographic c-function in terms of the 2-dim dilaton and given arguments for the corresponding c-theorem (subject to appropriate boundary conditions on the ultraviolet theory): at the IR AdS_2, this becomes the extremal black brane entropy. We have discussed this c-function (essentially inherited from higher dimensions) in detail for nonconformal branes compactified, and compared with other c-functions. Finally, we have adapted the radial Hamiltonian flow formulation of [42] to these 2-dim theories: while this is not Wilsonian, it gives qualitative insight into the flow equations and β-functions.

It would be interesting to understand how general such a holographic RG flow is. For instance, since our formulation has crucially used the sufficiently high symmetry of the transverse space, it is unclear if this directly applies to other situations, involving *e.g.* rotation (see *e.g.* [124]). It is also important to note that unlike a black hole which exhibits a gap, the branes we have considered would contain additional low-lying modes: from our analysis, it would seem that these do not change the essential flow pattern, *i.e.* the c-function does capture the relevant degrees of freedom describing the effective 2-dim physics. This is additionally corroborated by the fact that in the infrared it equals the extremal entropy which is the number of available microstates.

The analysis adapting [42] was motivated by the fact that the scalar perturbation mode in [125] about the AdS_2 background was found to be massless for $z = 1$ hvLif theories: this includes the hvLif family arising from reductions of nonconformal branes. We have seen however that in this case the β-functions do not vanish, whereas they do for reductions of the M2-AdS_4 phase to AdS_2. This suggests that it is consistent for the AdS_2 throat to emerge in a conformal phase of the higher dimensional theory (with AdS_D dual) but not consistent to have the AdS_2 critical point lie within a region encoding nontrivial RG flow. This is exemplified in the D2-M2 phase diagram and is consistent with our discussion of the c-function in sec. 4.2.3. This dVV formulation is not Wilsonian, as discussed in the literature: it would be interesting to adapt the Wilsonian formulations of [70, 71] to the 2-dim context: we hope to report on this in the future. Relatedly it would be interesting to explore holographic renormalization [199, 200, 201] ([168] for uncharged nonconformal branes) in this 2-dim context, perhaps building on [114].

For the 2-dim theories in [125] arising from compactification, the leading departures away from the IR AdS_2 critical point, described by Jackiw-Teitelboim theory, arise from the leading linear term in the dilaton perturbation and are thus governed by the Schwarzian derivative effective action. The dilaton fluctuation in (4.8) has $m^2L^2 = 2$ and so corresponds to an irrelevant operator with dimension $\Delta = 2$ (using $\Delta = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2L^2}$ for a scalar mode φ of mass m with equation of motion $\partial_+\partial_-\varphi + \frac{m^2L^2}{(x^+-x^-)^2}\varphi = 0$). In light of the present work we note that some of the more general 2-dim dilaton-gravity-matter theories (4.4) in the IR AdS_2 region may contain fluctuation modes with masses $-\frac{1}{4} \leq m_I^2L^2 < 2$ corresponding to dual operators with dimension $\Delta < 2$. In such cases, the leading departures from the IR AdS_2 will presumably not be governed by the Schwarzian but some distinct effective theory. It would be interesting to explore this further.

Our analysis here raises the question of understanding renormalization group flow in boundary quantum mechanical theories, which could be interpreted as flowing to lower energies (although not as spatial coarse-graining). The discussions here on *e.g.* nonconformal branes all pertain to large N (highly) supersymmetric theories (although fairly complicated, since in the IR they are dual to the compactified extremal black branes). Although we have not used this, it would seem that the constraints from supersymmetry will be powerful in 1-dimension, just as in higher dimensions as is well-known. It would be interesting to explore this.

Finally it is interesting to ask if the 2-dim dilaton-gravity-scalar theories of the general form (4.4) we have considered admit 2-dim de Sitter space dS_2 as solutions. For simplicity, taking $\Psi^I = 0$ and constant dilaton Φ, the Einstein equations and dilaton equation (4.5) require $U = 0$ and $\frac{\partial U}{\partial \Phi^2} = \mathcal{R} > 0$ at the dS_2 critical point. However this violates the condition (4.34) which we expect must hold if we take the potential U as arising from some higher dimensional reduction as we have discussed (implicitly taking U to have a leading term arising from a negative cosmological constant as in known brane realizations followed by positive flux contributions). Of course there are rolling (time-dependent) scalar solutions, as *e.g.* arises from reductions of dS_4 (say with Poincare metric $ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dw^2 + dx_i^2))$. In 4-dim Einstein gravity with a positive cosmological constant $\Lambda > 0$, the 2-dim potential simply becomes $U = 2\Lambda\Phi > 0$, and the 2-dim dilaton is $\Phi^2 \sim \frac{1}{\tau^2}$. The nature of such solutions (even in this simple classical sense) might be different from our AdS_2 discussions here and might be worth exploring (see *e.g.* [202]).

Chapter 5

N-level ghost-spins and entanglement

5.1 Introduction

Ghost-spin systems [46, 47, 48] and their patterns of entanglement are interesting from multiple points of view. Some of these possible applications involve ghost sectors in theories with gauge symmetry, while others pertain to conjectures involving de Sitter physics and dS/CFT dualities [43, 44, 45, 203, 204, 205, 206, 207, 208, 209, 210] including higher spin versions. Ghost-spins are 2-level spin-like variables but with indefinite norm [46]: thus in some ways, they are best regarded as simple quantum mechanical toy models for theories with negative norm states. Ghost-spin systems have interesting entanglement patterns by virtue of this indefinite norm. If one considers ghost spin systems with even number of ghost spins then it is possible to find subspaces in the Hilbert space where one obtains positive entanglement entropy for positive norm states [46, 47]. The situation is not so fortuitous in the case of odd number of ghost spins. However, several interesting physical systems which contain indefinite norm states seem to admit even numbers of such states. Taking the study of the ghost spins further, it was shown in [48] that appropriate ghost-spin chains in the continuum limit give rise to the bc-ghost CFTs in two dimensions. This suggests that they may be regarded as microscopic building blocks for ghost-CFTs and perhaps more general non-unitary theories.

The explorations of the ghost-spin system so far have used 2-state spin-like variables with indefinite inner product. It is interesting to study generalizations of ghost-spins with flavour degrees of freedom assigned to them. These flavour quantum numbers may be relevant for applications to non-abelian gauge theories. However, the ghost spin system with flavours by themselves is an interesting set up worthy of exploration. Once we allow for flavour symmetry, there are multiple ways in which they can be incorporated in the ghost-spins framework. One way is to assign an index, say A, which takes N values, to a 2-component ghost-spin with either $O(N)$ or $Sp(N)$ symmetry in the flavour index. The inner product in the flavour space is given by δ^{AB} for the $O(N)$ case and Ω^{AB} in the $Sp(N)$ case. More general inner products can also be analyzed.

In this chapter, which is based on [211], we will explore these generalizations of the 2-state ghost-spin system to N-level systems. In sec. 5.2, we will begin with a brief recap of the known results followed by a list of the N-level generalizations of ghost-spins that we discuss in this chapter. We introduce the $O(N)$ and $Sp(N)$ inner products in the flavour indices as well as more general inner products J^{AB} which are symmetric but non-vanishing for $A \neq B$. In sec. 5.3, we look at the $O(N)$ generalization of the 2-level system where we denote the flavoured ghost spins in terms of the 2-component spins carrying an additional index corresponding to the global $O(N)$ symmetry, $|\uparrow^A\rangle$. We write the indefinite inner product between spins by splitting it into indefinite product between $|\uparrow\rangle$ and $|\downarrow\rangle$ and a symmetric product δ_{AB} in the $O(N)$ index A. This as we will see is analogous to the bc-ghost system studied in the flavourless 2-level system. We then consider the ghost-spin chain with $O(N)$ flavour symmetry and show that it leads to the flavoured bc-ghost CFTs (sec. 5.3.1). We then consider correlated ghost-spin states in sec. 5.3.2 and analyse the entanglement pattern in them. We find that the results that we had obtained in the even ghost-spin system in the flavourless case carry over to the case with flavours. We generically find a subspace of correlated ghost-spin states in the Hilbert space with positive norm states having positive entanglement. Finally we briefly comment on more general inner products which lead to spin-glass type couplings in sec. 5.3.3. We then turn our attention in sec. 5.4 to symplectic inner products between certain generalizations of ghost-spins and study entanglement in correlated ghost spin states. Finally in sec. 5.5 we consider a generalization of 2-level ghost-spins to N irreducible levels: this is slightly different from the flavoured generalizations above. We study the entanglement pattern in correlated ghost-spin states here as well. In sec. 5.6, we summarise our results and comment on their applications to the dS/CFT correspondence, in part reviewing the picture in [212] of dS_4 as approximately dual to a thermofield-double type entangled state between two copies of ghost-CFTs.

5.2 Ghost-spins and N-level generalizations

Before getting to N-level generalizations, we first briefly review some essential aspects of ghost-spins. Ghost-spins were defined in [46] as simple toy quantum mechanical models for theories with negative norm states, abstracting from bc-ghost CFTs. These constructions were motivated by the studies [213, 214] on certain complex extremal surfaces in de Sitter space with negative area in dS_4 which amount to analytic continuations of the Ryu-Takayanagi formulations of holographic entanglement entropy [53, 54, 215, 24]. In contrast with a single spin which has $\langle\uparrow|\uparrow\rangle = 1 = \langle\downarrow|\downarrow\rangle$, a single ghost-spin is defined as a 2-state spin variable with indefinite inner product

$$\langle\uparrow|\downarrow\rangle = 1 = \langle\downarrow|\uparrow\rangle\ , \qquad \langle\uparrow|\uparrow\rangle = 0 = \langle\downarrow|\downarrow\rangle\ . \tag{5.1}$$

A general state $\psi^+|+\rangle + \psi^-|-\rangle$ thus has norm $|\psi^+|^2 - |\psi^-|^2$, which is not positive definite. By changing basis, the states $|\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$ have manifestly positive/negative norm, satisfying $\langle\pm|\pm\rangle = \pm 1$. We can then normalize general positive/negative norm states with norm ± 1 respectively. Consider now a state comprising two ghost-spins: this has norm

$$|\psi\rangle = \psi^{\alpha\beta}|\alpha\beta\rangle : \qquad \langle\psi|\psi\rangle = \gamma_{\alpha\kappa}\gamma_{\beta\lambda}\psi^{\alpha\beta}\psi^{\kappa\lambda *}\ , \qquad \gamma_{++} = 1,\ \ \gamma_{--} = \langle -|-\rangle = -1\ , \tag{5.2}$$

where $\gamma_{\alpha\beta}$ is the indefinite metric. Thus although states $|-\rangle$ have negative norm, the state $|-\rangle|-\rangle$ has positive norm. The full density matrix is $\rho = |\psi\rangle\langle\psi| = \sum \psi^{\alpha\beta}\psi^{\kappa\lambda *}|\alpha\beta\rangle\langle\kappa\lambda|$. Tracing over one of the ghost-spins leads to a reduced density matrix $(\rho_A)^{\alpha\kappa} = \gamma_{\beta\lambda}\psi^{\alpha\beta}\psi^{\kappa\lambda *} = \gamma_{\beta\beta}\psi^{\alpha\beta}\psi^{\kappa\beta *}$,

$$\begin{aligned} (\rho_A)^{++} &= |\psi^{++}|^2 - |\psi^{+-}|^2\ , \qquad & (\rho_A)^{+-} &= \psi^{++}\psi^{-+*} - \psi^{+-}\psi^{--*}\ , \\ (\rho_A)^{-+} &= \psi^{-+}\psi^{++*} - \psi^{--}\psi^{+-*}\ , \qquad & (\rho_A)^{--} &= |\psi^{-+}|^2 - |\psi^{--}|^2\ , \end{aligned} \tag{5.3}$$

for the remaining ghost-spin. Then $tr\rho_A = \gamma_{\alpha\kappa}(\rho_A)^{\alpha\kappa} = (\rho_A)^{++} - (\rho_A)^{--}$. Thus the reduced density matrix is normalized to have $tr\rho_A = tr\rho = \pm 1$ depending on whether the state (5.2) is positive or negative norm. The entanglement entropy calculated as the von Neumann entropy of ρ_A is $S_A = -\gamma_{\alpha\beta}(\rho_A \log \rho_A)^{\alpha\beta}$, perhaps best defined using a mixed-index reduced density matrix $(\rho_A)^{\alpha}{}_{\kappa} = \gamma_{\beta\kappa}(\rho_A)^{\alpha\beta}$. This can be illustrated via a simple family of states [46] with a diagonal reduced density matrix: setting $\psi^{-+*} = \psi^{+-}\psi^{--*}/\psi^{++}$ in the states (5.2) gives

$$(\rho_A)^{\alpha\beta}|\alpha\rangle\langle\beta| = \pm x|+\rangle\langle+| \;\mp\; (1-x)|-\rangle\langle-|\,, \quad x = \frac{|\psi^{++}|^2}{|\psi^{++}|^2+|\psi^{--}|^2} \quad [0<x<1]\,,$$
$$(\rho_A)_\alpha^\kappa = \gamma_{\alpha\beta}(\rho_A)^{\beta\kappa}: \qquad (\rho_A)_+^+ = \pm x, \qquad (\rho_A)_-^- = \pm(1-x), \tag{5.4}$$

where the $\pm$ pertain to positive/negative norm states respectively (note that $tr\rho_A = (\rho_A)_+^+ + (\rho_A)_-^- = \pm 1$). The location of the negative eigenvalue is different for positive/negative norm states, leading to different results for the von Neumann entropy. Now $\log\rho_A$ simplifies to $(\log\rho_A)_+^+ = \log(\pm x)$ and $(\log\rho_A)_-^- = \log(\pm(1-x))$. The entanglement entropy defined as $S_A = -\gamma_{\alpha\beta}(\rho_A\log\rho_A)^{\alpha\beta}$ becomes $S_A = -(\rho_A)_+^+(\log\rho_A)_+^+ - (\rho_A)_-^-(\log\rho_A)_-^-$ and so

$$\langle\psi|\psi\rangle \gtrless 0: \qquad S_A = -(\pm x)\log(\pm x) - (\pm(1-x))\log(\pm(1-x))\;. \tag{5.5}$$

For positive norm states, S_A is manifestly positive since $x<1$, just as in an ordinary 2-spin system. Negative norm states give a negative real part for EE since $x<1$ and the logarithms are negative: further there is an imaginary part (the simplest branch has $\log(-1) = i\pi$).

Now consider restricting to the subspace

$$|\psi\rangle = \psi^{++}|++\rangle + \psi^{--}|--\rangle \quad\longrightarrow\quad \langle\psi|\psi\rangle = |\psi^{++}|^2 + |\psi^{--}|^2 > 0\;. \tag{5.6}$$

These states of "correlated ghost-spins" comprise entanglement between two copies of identical states: they can be seen to be strictly positive norm, with a positive reduced density matrix (5.3) and positive entanglement. In [212], a picture of de Sitter space as a thermofield double type state (with de Sitter entropy then emerging as the entanglement entropy) was discussed based on such correlated ghost-spin states in two copies of ghost-CFTs at the future and past boundaries of dS_4 in the static coordinatization. For ensembles with an even number of ghost-spins, such correlated ghost-spin states always exist comprising positive norm subsectors, as argued in [47], where ensembles of ghost-spins were developed further with regard to their entanglement properties. Odd ghost-spins were found to behave differently: for instance, $|\psi\rangle = \psi^{++\cdots}|++\ldots\rangle + \psi^{--\cdots}|--\ldots\rangle$ has norm $\langle\psi|\psi\rangle = |\psi^{++\cdots}|^2 + (-1)^n|\psi^{--\cdots}|^2$ and mixed-index RDM components $(\rho_A)_+^+ = |\psi^{++\cdots}|^2$, $(\rho_A)_-^- = (-1)^n|\psi^{--\cdots}|^2$. This is not positive definite for n odd (even if $\langle\psi|\psi\rangle > 0$). Ensembles of ghost-spins and spins were also found to exhibit interesting entanglement patterns.

In [48], certain 1-dim ghost-spin chains with specific nearest-neighbour interactions were found to yield bc-ghost CFTs in the continuum limit, *i.e.* these ghost-spin chains are in the same universality class as those ghost-CFTs. We will not review this here since this will effectively be encompassed in a related detailed description later.

N-level ghost-spins: In this chapter, we generalize the 2-level ghost-spin reviewed above to N-levels by considering various generalizations as outlined below:

- $O(N)$ symmetry flavour generalization of the bc-ghost system:

$$\langle\downarrow^A \mid \uparrow^B\rangle = \delta^{AB} = \langle\uparrow^A \mid \downarrow^B\rangle\ , \quad \langle\downarrow^A \mid \downarrow^B\rangle = \langle\uparrow^A \mid \uparrow^B\rangle = 0\,, \quad A, B = 1, 2, \ldots, N\,. \tag{5.7}$$

These are essentially N copies of the 2-level ghost-spin system. It is then possible to find appropriate ghost-spin chains which lead to a generalization of the bc-ghost system but with internal $O(N)$ flavour indices. A simple generalization of this case involves the flavours having a spin-glass type interaction with coupling J_{AB} which is in general non-vanishing for $A \neq B$,

$$\langle\downarrow^A \mid \uparrow^B\rangle = J^{AB} = \langle\uparrow^A \mid \downarrow^B\rangle\,; \quad \langle\downarrow^A \mid \downarrow^B\rangle = \langle\uparrow^A \mid \uparrow^B\rangle = 0\,, \quad A, B = 1, 2, \ldots, N\,. \tag{5.8}$$

In flavour space, this thus encodes possibly nonlocal flavour couplings. Taking the J_{AB} matrix to be real and symmetric allows diagonalization and in that diagonal basis, this can be reduced to the above $O(N)$ flavoured case.

- N-levels with symplectic-like structure:

$$\langle\uparrow^A \mid \downarrow^B\rangle = i\,\Omega^{AB}\ , \qquad \langle\downarrow^A \mid \uparrow^B\rangle = i\,\Omega^{AB}\ , \qquad \langle\uparrow^A \mid \uparrow^B\rangle = 0 = \langle\downarrow^A \mid \downarrow^B\rangle\ , \qquad A, B = 1, \ldots, 2N\ . \tag{5.9}$$

These have a symplectic structure built into the inner product, which was in part motivated by 3-dim ghost-CFTs of symplectic fermions [216, 217] that have been discussed in the conjectured duals to higher spin dS_4 [203].

- N irreducible levels, *i.e.* we generalize the two states $|\uparrow\rangle, |\downarrow\rangle$ to $|e_1\rangle, \ldots, |e_N\rangle$ such that

$$\langle e_i|e_i\rangle = 0\ ; \quad \langle e_i|e_j\rangle = 1 \quad \text{for } i \neq j, \quad i, j = 1, 2, \ldots, N\ . \tag{5.10}$$

This case is slightly different from the previous cases in that the elemental ghost-spins are not 2-level anymore (with flavour indices), but irreducibly N-level.

In all these cases representing N-level generalizations of ghost-spin ensembles, we will argue that correlated ghost-spin states exist comprising a uniformly positive norm subspace of states with the interpretation of entanglement between two copies of the state space. This will be the main point of the chapter.

It is possible to find operator realizations consistent with some of these inner products above. The first case essentially comprises N copies of the bc-operator algebra,

$$\{\sigma_b^A, \sigma_c^B\} = \delta^{AB} \ . \tag{5.11}$$

We can then define ghost-spin-chain Hamiltonians with nearest neighbour hopping type interactions but with the flavours decoupled,

$$H = \sum_n \left(\sigma_{bn}^A \sigma_{c(n+1)}^B + \sigma_{b(n+1)}^A \sigma_{cn}^B\right) \delta_{AB} \quad \xrightarrow{?} \quad \int b^A \partial c^A \ . \tag{5.12}$$

Based on the fact that the continuum limit for the single flavour case is the familiar bc-ghost CFT [48], the continuum limit can be argued to be flavoured generalizations of bc-ghost CFTs, with the flavour contractions exhibiting $O(N)$ symmetry. We will discuss this in detail in sec. 3. The symplectic inner products above are consistent with the operator algebra

$$\{\sigma_b^A, \sigma_c^B\} = i\Omega_{AB} \ , \tag{5.13}$$

as we will discuss in sec. 5. The continuum limit is less clear in this case, although there are indications that these may be related to logarithmic CFTs [218, 219, 220, 221, 222, 223, 224].

5.3 N-level ghost-spins with $O(N)$ flavour symmetry

In this section, we consider an N-level generalization of ghost-spins with $O(N)$ symmetry among the N internal flavour indices, defined by (5.7), *i.e.* we have the elemental inner products

$$\begin{gathered}
\langle \uparrow^A | \downarrow^B \rangle = \delta^{AB} = \langle \downarrow^A | \uparrow^B \rangle, \quad \langle \uparrow^A | \uparrow^B \rangle = 0 = \langle \downarrow^A | \downarrow^B \rangle, \qquad A, B = 1, 2, \ldots, N \ , \\
|\pm^A\rangle = \frac{1}{\sqrt{2}}(|\uparrow^A\rangle \pm |\downarrow^A\rangle) \ , \qquad \langle \pm^A | \pm^B \rangle = \pm\delta^{AB} \ , \qquad \langle \pm^A | \mp^B \rangle = 0 \ .
\end{gathered} \tag{5.14}$$

This is essentially N copies of the 2-level ghost-spin reviewed in sec. 2. In the second line, we have defined a convenient basis where the inner product is diagonal: this makes manifest the negative norm basis states.

Consider first a single ghost-spin with N flavours. The general configuration is defined by specifying the simultaneous configurations for each of the N flavours so the general state comprising various basis states $|s_i\rangle$ is

$$|\psi\rangle = \psi^{s_i}|s_i\rangle \ , \qquad |s_i\rangle \equiv \left\{ \left|\begin{matrix}\uparrow^1\\ \uparrow^2\\ \uparrow^3\\ \vdots\end{matrix}\right\rangle , \ \left|\begin{matrix}\downarrow^1\\ \uparrow^2\\ \uparrow^3\\ \vdots\end{matrix}\right\rangle , \ \left|\begin{matrix}\uparrow^1\\ \downarrow^2\\ \uparrow^3\\ \vdots\end{matrix}\right\rangle , \ \ldots \right\} \ , \tag{5.15}$$

i.e. in the first basis state, the first flavour is $\uparrow^1$, the second is $\uparrow^2$, third being $\uparrow^3$ and so on, and likewise for the other basis states. It is important to note that the $|s_i\rangle$ are really direct product states over the various flavour components: although we have written them as column vectors for convenience of notation (especially in light of the discussion later on multiple ghost-spins), the inner products between these states is not a dot product between two column vectors. Instead we define the inner products between the configurations $|s_i\rangle$ as

$$\langle s_i|s_j\rangle = \frac{1}{N!}\sum \epsilon_{A_1A_2\ldots A_N}\epsilon_{B_1B_2\ldots B_N}\langle s_i^{A_1}|s_j^{B_1}\rangle\langle s_i^{A_2}|s_j^{B_2}\rangle\cdots\langle s_i^{A_N}|s_j^{B_N}\rangle \ , \tag{5.16}$$

where $i,j = 1,2,\ldots,2^N$ label the configurations, $A_1, B_1, \cdots = 1,2,\ldots,N$ label the flavours and $\epsilon_{A_1A_2\ldots A_N}$ is the totally symmetric tensor with $\epsilon_{12\ldots N} = 1$ and $\epsilon_{A_1A_2\ldots A_N}$ vanishes if any two labels are the same. In other words, $\epsilon_{A_1A_2\ldots A_N}$ is simply a book-keeping device for ensuring that each elemental state $|s_j^A\rangle$ in $|s_j\rangle$ is paired with another corresponding elemental state in $\langle s_i|$.

To illustrate how this works, let us consider a simple example of a single ghost-spin with $N = 2$ flavours: the distinct configurations of this system are described by the basis states

$$|s_1\rangle = \left|\begin{matrix}\uparrow^1\\ \uparrow^2\end{matrix}\right\rangle \ , \qquad |s_2\rangle = \left|\begin{matrix}\uparrow^1\\ \downarrow^2\end{matrix}\right\rangle \ , \qquad |s_3\rangle = \left|\begin{matrix}\downarrow^1\\ \uparrow^2\end{matrix}\right\rangle \ , \qquad |s_4\rangle = \left|\begin{matrix}\downarrow^1\\ \downarrow^2\end{matrix}\right\rangle \ . \tag{5.17}$$

Then the inner product (5.16) simplifies to

$$\langle s_i|s_j\rangle = \frac{1}{2!}\sum \epsilon_{A_1A_2}\epsilon_{B_1B_2}\langle s_i^{A_1}|s_j^{B_1}\rangle\langle s_i^{A_2}|s_j^{B_2}\rangle = \langle s_i^1|s_j^1\rangle\langle s_i^2|s_j^2\rangle + \langle s_i^1|s_j^2\rangle\langle s_i^2|s_j^1\rangle \ , \tag{5.18}$$

where we have used $\epsilon_{12} = 1 = \epsilon_{21}$. Using the elemental inner products in (5.14) gives

$$\langle s_i|s_j\rangle = \langle s_i^1|s_j^1\rangle\langle s_i^2|s_j^2\rangle \ . \tag{5.19}$$

Writing this out explicitly, we have

$$\begin{aligned} \langle s_1|s_4\rangle &= \langle\uparrow^1|\downarrow^1\rangle\langle\uparrow^2|\downarrow^2\rangle = 1 \ , \qquad & \langle s_2|s_3\rangle &= \langle\uparrow^1|\downarrow^1\rangle\langle\downarrow^2|\uparrow^2\rangle = 1 \ , \\ \langle s_3|s_2\rangle &= \langle\downarrow^1|\uparrow^1\rangle\langle\uparrow^2|\downarrow^2\rangle = 1 \ , \qquad & \langle s_4|s_1\rangle &= \langle\downarrow^1|\uparrow^1\rangle\langle\downarrow^2|\uparrow^2\rangle = 1 \ . \end{aligned} \tag{5.20}$$

The other inner products vanish. Based on these inner products for the basis states, we can write the norm for the generic state as

$$|\psi\rangle = c_i|s_i\rangle \quad \Rightarrow \quad \langle\psi|\psi\rangle = \left(c_j^*\langle s_j|\right)\cdot\left(c_i|s_i\rangle\right) = c_1^*c_4 + c_4^*c_1 + c_2^*c_3 + c_3^*c_2\ . \tag{5.21}$$

Appropriate pairs of states can be used to define a new basis of positive and negative norm states: $s_1 \pm s_4$, $s_2 \pm s_3$ and so on have norm ± 2 respectively. Likewise for $N = 4$ flavours, the $2^4 = 16$ configurations $|s_i\rangle$ are [1]

$$|s_1\rangle = \left|\begin{matrix}\uparrow^1\\ \uparrow^2\\ \uparrow^3\\ \uparrow^4\end{matrix}\right\rangle,\quad |s_2\rangle = \left|\begin{matrix}\uparrow^1\\ \uparrow^2\\ \uparrow^3\\ \downarrow^4\end{matrix}\right\rangle,\quad |s_3\rangle = \left|\begin{matrix}\uparrow^1\\ \uparrow^2\\ \downarrow^3\\ \uparrow^4\end{matrix}\right\rangle,$$
$$\cdots,\quad |s_{15}\rangle = \left|\begin{matrix}\downarrow^1\\ \downarrow^2\\ \downarrow^3\\ \uparrow^4\end{matrix}\right\rangle,\quad |s_{16}\rangle = \left|\begin{matrix}\downarrow^1\\ \downarrow^2\\ \downarrow^3\\ \downarrow^4\end{matrix}\right\rangle \tag{5.22}$$

and the inner product (5.16) is

$$\langle s_i|s_j\rangle = \frac{1}{4!}\sum \epsilon_{A_1A_2A_3A_4}\epsilon_{B_1B_2B_3B_4}\langle s_i^{A_1}|s_j^{B_1}\rangle\langle s_i^{A_2}|s_j^{B_2}\rangle\langle s_i^{A_3}|s_j^{B_3}\rangle\langle s_i^{A_4}|s_j^{B_4}\rangle\ . \tag{5.23}$$

Using (5.14), this simplifies to

$$\begin{aligned}\langle s_i|s_j\rangle &= \frac{1}{4!}\sum_{A_1,A_2,A_3,A_4}\epsilon^2_{A_1A_2A_3A_4}\langle s_i^{A_1}|s_j^{A_1}\rangle\langle s_i^{A_2}|s_j^{A_2}\rangle\langle s_i^{A_3}|s_j^{A_3}\rangle\langle s_i^{A_4}|s_j^{A_4}\rangle\\ &= \langle s_i^1|s_j^1\rangle\langle s_i^2|s_j^2\rangle\langle s_i^3|s_j^3\rangle\langle s_i^4|s_j^4\rangle\ ,\end{aligned} \tag{5.24}$$

noting that there are 4! non-zero $\epsilon_{A_1A_2A_3A_4}$ components, which can succinctly be written as

$$\langle s_i|s_{17-i}\rangle = 1\ , \tag{5.25}$$

[1] These states can be written in terms of a basis which is manifestly $O(N)$ invariant. Namely

$$|t_0\rangle = \prod_{i=1}^{N}|\uparrow^i\rangle\ ,\quad |t_{vec}\rangle = \prod_{j\neq i,i=1}^{N}|\uparrow^i\rangle|\downarrow^j\rangle\ ,\quad \forall j\ ,$$
$$|t_{adj}\rangle = \prod_{j\neq k\neq i,i=1}^{N}|\uparrow^i\rangle|\downarrow^j\rangle\ ,\quad \forall j,k\ ;\cdots$$

Where, 0 is the trivial representation, *vec* corresponds to the vector representation and *adj* is the adjoint(antisymmetric) representation of $O(N)$. To see the relation between the $|t\rangle$ and the $|s\rangle$ bases, consider for example, the states in (5.22): these get organised into representations of $O(4)$ as **1**, **4**, **6**, **4**, **1** with the states $|s_1\rangle$ and $|s_{16}\rangle$ forming two singlets, $|s_2\rangle$ to $|s_5\rangle$ ($|s_{12}\rangle$ to $|s_{15}\rangle$) belonging to the vector representation and $|s_6\rangle$ to $|s_{11}\rangle$ to the adjoint representations of $O(4)$. We will, however, continue using the basis given in (5.15) for convenience.

with other inner products vanishing. Similarly, for general N, the inner product (5.16) is

$$\begin{aligned} \langle s_i|s_j\rangle &= \frac{1}{N!}\sum_{A_1,\ldots,A_N} \epsilon^2_{A_1\ldots A_N}\langle s_i^{A_1}|s_j^{A_1}\rangle\cdots\langle s_i^{A_N}|s_j^{A_N}\rangle \\ &= \langle s_i^1|s_j^1\rangle\cdots\langle s_i^N|s_j^N\rangle = \prod_A\langle s_i^A|s_j^A\rangle\ . \end{aligned} \tag{5.26}$$

Thus two generic states have inner product

$$\langle\psi_1|\psi_2\rangle = (\psi_1^{s_j})^*\psi_2^{s_i}\,\langle s_j|s_i\rangle\ = (\psi_1^{s_j})^*\psi_2^{s_i}\,\prod_A\langle s_j^A|s_i^A\rangle\ . \tag{5.27}$$

5.3.1 Ghost-spin chain for the bc-CFT with $O(N)$ symmetry

We want to now study infinite ghost-spin chains along the lines of those in [48], but with additional flavour structure respecting the $O(N)$ flavour symmetry we have been discussing so far. In effect, this amounts to N flavour copies of ghost-spins at each lattice site. Thus we define two species of N-component commuting spin variables, σ^A_{bn} and σ^A_{cn} at each lattice site n satisfying

$$\{\sigma^A_{bn},\sigma^B_{cn}\} = \delta^{AB}\ , \qquad [\sigma^A_{bn},\sigma^B_{bn'}] = [\sigma^A_{cn},\sigma^B_{cn'}] = [\sigma^A_{bn},\sigma^B_{cn'}] = 0\ . \tag{5.28}$$

These commuting spin-variables are Hermitian $\sigma_b^{\dagger A} = \sigma_b^A,\ \sigma_c^{\dagger A} = \sigma_c^A$ and their action on ghost-spin states is

$$\sigma_b^A|\downarrow^B\rangle = 0,\quad \sigma_b^A|\uparrow^B\rangle = \delta^{AB}|\downarrow^B\rangle,\quad \sigma_c^A|\downarrow^B\rangle = \delta^{AB}|\uparrow^B\rangle,\quad \sigma_c^A|\uparrow^B\rangle = 0, \tag{5.29}$$

where there is no summation over the flavour index B. For multiple ghost-spin states in the chain, the σ^A_{bn} operator acts to lower the A^{th}-flavour state within the ghost-spin configuration at site n, and likewise σ^A_{cn} is the corresponding raising operator. To illustrate this explicitly, consider two ghost-spins with $N=2$ flavours: then

$$\sigma^2_{b1}\big|{}^{\uparrow 1}_{\uparrow 2}\big\rangle\big|{}^{\downarrow 1}_{\downarrow 2}\big\rangle = \big|{}^{\uparrow 1}_{\downarrow 2}\big\rangle\big|{}^{\downarrow 1}_{\downarrow 2}\big\rangle\ ,\quad \sigma^1_{c2}\big|{}^{\uparrow 1}_{\uparrow 2}\big\rangle\big|{}^{\downarrow 1}_{\downarrow 2}\big\rangle = \big|{}^{\uparrow 1}_{\uparrow 2}\big\rangle\big|{}^{\uparrow 1}_{\downarrow 2}\big\rangle\ ,\quad \sigma^2_{b1}\sigma^2_{c2}\big|{}^{\uparrow 1}_{\uparrow 2}\big\rangle\big|{}^{\downarrow 1}_{\downarrow 2}\big\rangle = \big|{}^{\uparrow 1}_{\downarrow 2}\big\rangle\big|{}^{\downarrow 1}_{\uparrow 2}\big\rangle\ ,\quad \ldots \tag{5.30}$$

The ghost-spin chain is then defined by a Hamiltonian encoding interactions between the ghost-spins at various lattice sites. Since the flavours do not mix, the interaction Hamiltonian here is simply a straightforward generalization involving a decoupled sum over various flavours of the single flavour one in [48]. So consider a 1-dim ghost-spin chain with a "hopping" type interaction Hamiltonian

$$H = J\sum_n\sum_{A=1}^{N}(\sigma^A_{bn}\sigma^A_{c(n+1)} + \sigma^A_{bn}\sigma^A_{c(n-1)}) = J\sum_n\sum_{A=1}^{N}(\sigma^A_{bn}\sigma^A_{c(n+1)} + \sigma^A_{b(n+1)}\sigma^A_{cn}) \ , \quad (5.31)$$

where n, $n+1$, $n-1$ label nearest label lattice sites in the chain. The action on a nearest neighbour pair of ghost-spins at lattice sites $(n, n+1)$ is given as

$$\sigma^A_{bn}\sigma^A_{c(n+1)} : \left(\ldots \otimes \left|\begin{matrix}\vdots\\ \uparrow^A\\ \vdots\end{matrix}\right\rangle_n \otimes \left|\begin{matrix}\vdots\\ \downarrow^A\\ \vdots\end{matrix}\right\rangle_{n+1} \otimes \ldots\right) \to J\left(\ldots \otimes \left|\begin{matrix}\vdots\\ \downarrow^A\\ \vdots\end{matrix}\right\rangle_n \otimes \left|\begin{matrix}\vdots\\ \uparrow^A\\ \vdots\end{matrix}\right\rangle_{n+1} \otimes \ldots\right),$$

$$\sigma^A_{b(n+1)}\sigma^A_{cn} : \left(\ldots \otimes \left|\begin{matrix}\vdots\\ \downarrow^A\\ \vdots\end{matrix}\right\rangle_n \otimes \left|\begin{matrix}\vdots\\ \uparrow^A\\ \vdots\end{matrix}\right\rangle_{n+1} \otimes \ldots\right) \to J\left(\ldots \otimes \left|\begin{matrix}\vdots\\ \uparrow^A\\ \vdots\end{matrix}\right\rangle_n \otimes \left|\begin{matrix}\vdots\\ \downarrow^A\\ \vdots\end{matrix}\right\rangle_{n+1} \otimes \ldots\right). \quad (5.32)$$

The N flavours are decoupled and the interaction between ghost-spins at two neighbouring lattice sites is through the same flavour at the two sites. Thus we can follow the analysis in [48] flavour-by-flavour.

Towards constructing the continuum limit of the ghost-spin chain (5.31), we note that the $\sigma^A_{b,c}$ operators commute at neighbouring lattice sites as in the single flavour case. Thus we define two species of N-component fermionic operators satisfying the anti-commutation relations

$$\{a^A_{bi}, a^B_{cj}\} = \delta_{ij}\delta^{AB} \ , \qquad \{a^A_{bi}, a^B_{bj}\} = \{a^A_{ci}, a^B_{cj}\} = 0 \ , \quad (5.33)$$

which anti-commute at different lattice sites i, j also. The action of these fermionic operators on ghost-spin states is

$$\begin{aligned} &a^A_b|\downarrow^B\rangle = 0 \ , \quad a^A_b|\uparrow^B\rangle = \delta^{AB}|\downarrow^B\rangle \ , \quad a^A_c|\downarrow^B\rangle = \delta^{AB}|\uparrow^B\rangle \ , \quad a^A_c|\uparrow^B\rangle = 0 \ , \\ &\langle\downarrow^B|a^A_b = 0 \ , \quad \langle\uparrow^B|a^A_b = \delta^{AB}\langle\downarrow^B| \ , \quad \langle\downarrow^B|a^A_c = \delta^{AB}\langle\uparrow^B| \ , \quad \langle\uparrow^B|a^A_c = 0 \ . \end{aligned} \quad (5.34)$$

This is obtained by constructing a flavoured generalization of the Jordan-Wigner transformation in [48] for the commuting spin variables (σ^A_b, σ^A_c) as

$$\begin{aligned} &\sigma^A_{b1} = a^A_{b1} \ , \qquad \sigma^A_{c1} = a^A_{c1} \ , \\ &\sigma^A_{b2} = i(1 - 2a^A_{c1}a^A_{b1})a^A_{b2} \ , \qquad \sigma^A_{c2} = -i(1 - 2a^A_{c1}a^A_{b1})a^A_{c2} \ , \quad \ldots \ , \\ &\sigma^A_{bn} = i(1 - 2a^A_{c1}a^A_{b1})i(1 - 2a^A_{c2}a^A_{b2})\ldots i(1 - 2a^A_{c(n-1)}a^A_{b(n-1)})a^A_{bn} \ , \\ &\sigma^A_{cn} = (-i)(1 - 2a^A_{c1}a^A_{b1})(-i)(1 - 2a^A_{c2}a^A_{b2})\ldots(-i)(1 - 2a^A_{c(n-1)}a^A_{b(n-1)})a^A_{cn} \ , \quad \ldots \end{aligned} \quad (5.35)$$

for each flavour index A independently (*i.e.* the transformations are decoupled for distinct flavours). The inverse transformations for the fermionic ghost-spin variables $(a_b^A,\, a_c^A)$ are

$$\begin{aligned}
&a_{b1}^A = \sigma_{b1}^A\ , \qquad a_{c1}^A = \sigma_{c1}^A\ , \\
&a_{b2}^A = i(1-2\sigma_{c1}^A\sigma_{b1}^A)\sigma_{b2}^A\ , \quad a_{c2}^A = -i(1-2\sigma_{c1}^A\sigma_{b1}^A)\sigma_{c2}^A\ , \quad \ldots\ , \\
&a_{bn}^A = i(1-2\sigma_{c1}^A\sigma_{b1}^A)i(1-2\sigma_{c2}^A\sigma_{b2}^A)\ldots i(1-2\sigma_{c(n-1)}^A\sigma_{b(n-1)}^A)\sigma_{bn}^A\ , \\
&a_{cn}^A = (-i)(1-2\sigma_{c1}^A\sigma_{b1}^A)(-i)(1-2\sigma_{c2}^A\sigma_{b2}^A)\ldots(-i)(1-2\sigma_{c(n-1)}^A\sigma_{b(n-1)}^A)\sigma_{cn}^A\ , \quad \ldots
\end{aligned} \tag{5.36}$$

for each flavour index A independently. The factor $(1-2\sigma_{ci}^A\sigma_{bi}^A)$ is -1 or $+1$ depending on whether the i-th location is occupied by $(\uparrow^A)$ or not $(\downarrow^A)$, which means $(1-2\sigma_{ci}^A\sigma_{bi}^A)^2 = 1$. Using

$$[\pm i(1-2\sigma_{ci}^A\sigma_{bi}^A)]^\dagger = \pm i(1-2\sigma_{ci}^A\sigma_{bi}^A)\ , \tag{5.37}$$

we can check that the operators a_{bn}^A, a_{cn}^A are hermitian. Now substituting the Jordan-Wigner transformation (5.35) in the ghost-spin Hamiltonian in the commuting spin variables (5.31) gives

$$\begin{aligned}
H &= J\sum_n\sum_{A=1}^N(\sigma_{bn}^A\sigma_{c(n+1)}^A + \sigma_{bn}^A\sigma_{c(n-1)}^A)\ , \\
&= J\sum_n\sum_{A=1}^N\big(i^{n-1}[1]^A[2]^A\ldots[n-1]^A a_{bn}^A(-i)^n[1]^A[2]^A\ldots[n]^A a_{c(n+1)}^A \\
&\qquad\qquad + i^{n-1}[1]^A[2]^A\ldots[n-1]^A a_{bn}^A(-i)^{n-2}[1]^A[2]^A\ldots[n-2]^A a_{c(n-1)}^A\big)\ ,
\end{aligned} \tag{5.38}$$

where $[k]^A = (1-2a_{ck}^A a_{bk}^A)$. Commuting the various $[k]^A$ factors gives

$$\begin{aligned}
H &= J\sum_n\sum_{A=1}^N\big((-i)a_{bn}^A(1-2a_{cn}^A a_{bn}^A)a_{c(n+1)}^A + i(1-2a_{c(n-1)}^A a_{b(n-1)}^A)a_{bn}^A a_{c(n-1)}^A\big)\ , \\
&= iJ\sum_n\sum_{A=1}^N a_{bn}^A(a_{c(n+1)}^A - a_{c(n-1)}^A)\ .
\end{aligned} \tag{5.39}$$

We see that this Hamiltonian for the 1-dimensional chain of N-level ghost-spins with $O(N)$ symmetry breaks up as a decoupled sum of N copies of the Hamiltonian for 2-level ghost-spins in [48]. Then following the analysis there for each flavour independently and taking the continuum limit, we can show that we obtain N copies of bc-ghost CFTs with $O(N)$ flavour symmetry. This finally gives

$$H = \sum_{A=1}^{N} \sum_{k} k b_{-k}^{A} c_{k}^{A} = \sum_{A=1}^{N} \sum_{k>0} k \left(b_{-k}^{A} c_{k}^{A} + c_{-k}^{A} b_{k}^{A} \right) \ , \tag{5.40}$$

which is essentially the operator L_0 for a bc-ghost CFT enjoying $O(N)$-flavour symmetry, with action

$$S = \int d^2 z \sum_{A=1}^{N} b^A \partial c^A \ , \tag{5.41}$$

and a corresponding anti-holomorphic part. Further details are similar to [48], except with multiple flavours.

5.3.2 Correlated ghost-spin states and entanglement

We now return to ghost-spin ensembles and their entanglement properties. Along the lines in (5.15) for enumerating states in the $\uparrow, \downarrow$-basis, we can clearly use the $|\pm^i\rangle$-basis to define the basis states $|s_i\rangle$ there: the advantage in the $|\pm^i\rangle$-basis is that positive/negative norm states are easier to identify manifestly. For a single ghost-spin with $N = 2$ flavours, we have then

$$|s_1\rangle = \left|{}^{+1}_{+2}\right\rangle \ , \qquad |s_2\rangle = \left|{}^{+1}_{-2}\right\rangle \ , \qquad |s_3\rangle = \left|{}^{-1}_{+2}\right\rangle \ , \qquad |s_4\rangle = \left|{}^{-1}_{-2}\right\rangle \ . \tag{5.42}$$

We remind the reader that although we are using column vectors for notational convenience, these are really direct product states: the inner product (5.16) here gives

$$\begin{aligned} \langle s_1|s_1\rangle &= \langle +^1|+^1\rangle\langle +^2|+^2\rangle = 1, \qquad & \langle s_4|s_4\rangle &= \langle -^1|-^1\rangle\langle -^2|-^2\rangle = 1, \\ \langle s_2|s_2\rangle &= \langle +^1|+^1\rangle\langle -^2|-^2\rangle = -1, \qquad & \langle s_3|s_3\rangle &= \langle -^1|-^1\rangle\langle +^2|+^2\rangle = -1, \end{aligned} \tag{5.43}$$

and this is an orthonormal basis of positive and negative norm states. A generic state then has norm

$$|\psi\rangle = c_i|s_i\rangle \quad \Rightarrow \quad \langle\psi|\psi\rangle = |c_1|^2 + |c_4|^2 - |c_2|^2 - |c_3|^2 \ . \tag{5.44}$$

Thus states made from $|s_2\rangle$, $|s_3\rangle$ alone have negative norm. It is straightforward to write down similar basis states for arbitrary N flavours. For N flavours, there are 2^N basis states $|s_i\rangle$. The inner products are

$$\langle s_i|s_i\rangle = \prod_A \langle s_i^A|s_i^A\rangle \ . \tag{5.45}$$

Since this is a diagonal basis now, we have $\langle s_i|s_i\rangle = \pm 1$ respectively when there is an even or odd number of $|-^A\rangle$ elemental ghost-spins in $|s_i\rangle$. This is exemplified in the $N=2$ case (5.43) above.

Now let us consider two ghost-spins. The states can be made from the 2^{2N} basis states $|s_i\rangle|s_j\rangle$ obtained by tensor products of the single ghost-spin states. The inner products between them are

$$\langle (\langle s_k|\langle s_l|\,)\cdot(\,|s_i\rangle|s_j\rangle) = \langle s_k|s_i\rangle\,\langle s_l|s_j\rangle \tag{5.46}$$

Then the general state and its norm are

$$|\psi\rangle = \psi^{s_is_j}|s_i\rangle|s_j\rangle\ , \qquad \langle\psi|\psi\rangle = (\psi^{s_ks_l})^*\psi^{s_is_j}\prod_A\langle s_k^A|s_i^A\rangle\prod_B\langle s_l^B|s_j^B\rangle\ , \tag{5.47}$$

with two products over the flavour components of the two basis states. Tracing over say the second ghost-spin in this state leads to a subsystem comprising the single remaining ghost-spin, with reduced density matrix defined as

$$(\rho_A)^{s_k,s_i} = (\psi^{s_ks_l})^*\psi^{s_is_j}\,\langle s_l|s_j\rangle = (\psi^{s_ks_l})^*\psi^{s_is_j}\prod_B\langle s_l^B|s_j^B\rangle\ . \tag{5.48}$$

The entanglement entropy of this reduced density matrix can then be calculated using the formulation in [46, 47, 48]: we will see this below. Restricting attention for simplicity to $N=2$ flavours, we can use the four basis states (5.42). Then the 2 ghost-spin states can be described using the 16 basis states $|s_{i,j}\rangle \equiv |s_i\rangle|s_j\rangle$, or more explicitly,

$$\begin{array}{llll}
\left|{}^{+1}_{+2}\right\rangle\left|{}^{+1}_{+2}\right\rangle, & \left|{}^{+1}_{+2}\right\rangle\left|{}^{+1}_{-2}\right\rangle, & \left|{}^{+1}_{+2}\right\rangle\left|{}^{-1}_{+2}\right\rangle, & \left|{}^{+1}_{+2}\right\rangle\left|{}^{-1}_{-2}\right\rangle, \\
\left|{}^{+1}_{-2}\right\rangle\left|{}^{+1}_{+2}\right\rangle, & \left|{}^{+1}_{-2}\right\rangle\left|{}^{+1}_{-2}\right\rangle, & \left|{}^{+1}_{-2}\right\rangle\left|{}^{-1}_{+2}\right\rangle, & \left|{}^{+1}_{-2}\right\rangle\left|{}^{-1}_{-2}\right\rangle, \\
\left|{}^{-1}_{+2}\right\rangle\left|{}^{+1}_{+2}\right\rangle, & \left|{}^{-1}_{+2}\right\rangle\left|{}^{+1}_{-2}\right\rangle, & \left|{}^{-1}_{+2}\right\rangle\left|{}^{-1}_{+2}\right\rangle, & \left|{}^{-1}_{+2}\right\rangle\left|{}^{-1}_{-2}\right\rangle, \\
\left|{}^{-1}_{-2}\right\rangle\left|{}^{+1}_{+2}\right\rangle, & \left|{}^{-1}_{-2}\right\rangle\left|{}^{+1}_{-2}\right\rangle, & \left|{}^{-1}_{-2}\right\rangle\left|{}^{-1}_{+2}\right\rangle, & \left|{}^{-1}_{-2}\right\rangle\left|{}^{-1}_{-2}\right\rangle.
\end{array} \tag{5.49}$$

The inner products (5.46) can then be seen to give the norms *e.g.*

$$\langle s_{i,i}|s_{i,i}\rangle = (\langle s_i|s_i\rangle)^2 = 1\ , \qquad \langle s_{1,2}|s_{1,2}\rangle = \langle s_1|s_1\rangle\langle s_2|s_2\rangle = -1\ ,$$
$$\langle s_{2,3}|s_{2,3}\rangle = \langle s_2|s_2\rangle\langle s_3|s_3\rangle = 1\ , \qquad \ldots \tag{5.50}$$

and so on, using (5.43). It is clear again that the norms are again ± 1 depending on whether the state $|s_{i,j}\rangle$ contains an even or odd number of $|-^A\rangle$ elemental states.

The general state has norm

$$|\psi\rangle = \sum \psi^{s_i,s_j}|s_i\rangle|s_j\rangle \qquad \langle\psi|\psi\rangle = (\psi^{s_k,s_l})^*\psi^{s_i,s_j}\langle s_k|s_i\rangle\langle s_l|s_j\rangle = |\psi^{s_i,s_j}|^2\langle s_{i,j}|s_{i,j}\rangle \ . \tag{5.51}$$

This is a sum over $|\psi^{s_i,s_j}|^2$ weighted by ± 1 depending on the sign of the norm of $|s_{i,j}\rangle$. A particularly interesting subset of states are what we call "correlated states", generalizing the discussion in [47]. These are of the form

$$|\psi^{corr}\rangle = \sum \psi^{s_i,s_i}|s_i\rangle|s_i\rangle \qquad \langle\psi^{corr}|\psi^{corr}\rangle = \sum_{i=1}^{4} |\psi^{s_j,s_j}|^2 \ > 0 \ , \tag{5.52}$$

and are necessarily positive norm, even though some of the individual basis states are negative norm. The basis states $|s_i\rangle|s_i\rangle$ here are of the form

$$\left|{}^{+1}_{+2}\right\rangle\left|{}^{+1}_{+2}\right\rangle, \quad \left|{}^{+1}_{-2}\right\rangle\left|{}^{+1}_{-2}\right\rangle, \quad \left|{}^{-1}_{+2}\right\rangle\left|{}^{-1}_{+2}\right\rangle, \quad \left|{}^{-1}_{-2}\right\rangle\left|{}^{-1}_{-2}\right\rangle, \tag{5.53}$$

and we see explicitly that this subspace of correlated states is obtained by entangling some configuration for the first ghost-spin with an identical configuration for the second ghost-spin. Thus we have only 4 states which span the correlated ghost-spin subspace. It is clear that these are necessarily positive norm since there is an even number of minus ghost-spins (any odd number in each column is doubled). Note that this is a smaller subspace than that comprising all positive norm states which simply need to have an even number of minus signs: *e.g.* the basis state $\left|{}^{+1}_{+2}\right\rangle\left|{}^{-1}_{-2}\right\rangle$ is positive norm but the two ghost-spins have different configurations. This can be generalized to two ghost-spins with N flavours in a straightforward manner: the general state is again of the form (5.52) but with the $|s_i\rangle$ encoding N flavour ghost-spin configurations. There are 2^N basis states $|s_i\rangle$ so this subspace of correlated states is 2^N-dimensional, somewhat smaller than the 2^{2N}-dimensional space of all states.

These correlated ghost-spin states entangle identical ghost-spins between the two sets of ghost-spins. These states necessarily encode positive entanglement since any sublinear combination of the norm is still positive definite (one way to see this is to note that this can be mapped to an auxiliary system of ordinary spins, which has no minus signs and is entirely positive norm). More explicitly, for a state of the form $|\psi\rangle = \psi^{s_I,s_I}|s_I\rangle|s_I\rangle + \psi^{s_J,s_J}|s_J\rangle|s_J\rangle$ made of the states $|s_{I,I}\rangle, |s_{J,J}\rangle$, the reduced density matrix (5.48) can be taken to construct a mixed-index reduced density matrix as in [46, 47, 48], which then makes explicit the contraction structure with respect to the ghost-spin inner product metric (incorporating the signs for negative norm states). To be explicit, consider $|\psi\rangle = \psi^{s_1,s_1}|s_1\rangle|s_1\rangle + \psi^{s_2,s_2}|s_2\rangle|s_2\rangle$ noting that $|s_1\rangle$ and $|s_2\rangle$ are positive and negative norm respectively. Then

$$(\rho_A)^{s_1}_{s_1} = (\rho_A)^{s_1,s_1} = |\psi^{s_1,s_1}|^2, \qquad (\rho_A)^{s_2}_{s_2} = -(\rho_A)^{s_2,s_2} = |\psi^{s_2,s_2}|^2 \ ;$$
$$\text{tr}\rho_A = \langle\psi|\psi\rangle = 1 \ . \tag{5.54}$$

The mixed-index reduced density matrix is $(\rho_A)^{s_i}_{s_j} = \gamma_{s_j,s_k}(\rho_A)^{s_i,s_k} = \langle s_j|s_k\rangle(\rho_A)^{s_i,s_k}$, where the metric $\gamma_{s_i,s_j} = \langle s_i|s_j\rangle$ is defined by the inner products (5.43). This description can be generalized to all such states, and indeed to all 2-ghost-spin states (5.51): in this case, it can be shown along the lines of the single flavour case that more general positive norm subsectors exist. In general however, the state space has many branches of negative norm states, and the reduced density matrix in general has negative eigenvalues with a complex entanglement entropy correspondingly (as was already the case in the single flavour case).

These states can be generalized to any even number of ghost-spins. For odd numbers of ghost-spins however, this structure does not prevail: there are states that are positive norm but the reduced density matrix continues to have negative eigenvalues so that the entanglement entropy is not positive.

It is now interesting to consider two copies of ghost-spin chains and consider correlated ghost-spin states representing entanglement between the two chains. This is motivated by the discussion and picture in [212] of dS_4 as dual to a thermofield-double type entangled state in two copies $CFT_F \times CFT_P$ of the ghost-CFT at I^+ and I^- (reviewed in the Discussion in sec. 5.6). So let us consider $\mathcal{GC}_1 \times \mathcal{GC}_2$ where each $\mathcal{GC}$ represents a ghost-spin chain whose continuum limit gives a bc-ghost CFT with flavour symmetry as in sec. 5.3.1. Configurations of each $\mathcal{GC}$ can be represented schematically by

$$|\sigma\rangle \ \equiv \ (\ldots|s_n\rangle|s_{n+1}\rangle\ldots) \tag{5.55}$$

Then correlated entangled states in $\mathcal{GC}_1 \times \mathcal{GC}_2$ can be represented as

$$|\psi\rangle = \psi^{\sigma,\sigma}|\sigma\rangle|\sigma\rangle \ , \qquad \langle\psi|\psi\rangle = \sum_{|\sigma\rangle} |\psi^{\sigma,\sigma}|^2 > 0 \ . \tag{5.56}$$

Now the states $|\sigma\rangle$ include the ground state as well as excited states. If we restrict to the ground states alone, then since the flavours are all decoupled from each other, the ground states $|\sigma\rangle$ comprise a 2^N-dimensional subspace noting that each $|s_n\rangle$ at lattice site n in $|\sigma\rangle$ has 2^N possibilities. Thus tracing over the second ghost-spin chain copy $\mathcal{GC}_2$, we obtain the entanglement entropy of $|\psi\rangle$ restricting to ground states $|\sigma_g\rangle$ as

$$S_A = -\sum_{i=1}^{2^N} |\psi^{\sigma_g,\sigma_g}|^2 \log |\psi^{\sigma_g,\sigma_g}|^2 \quad \to \quad -2^N \frac{1}{2^N} \log \frac{1}{2^N} = N \log 2 \ . \tag{5.57}$$

We have used $\sum_g |\psi^{\sigma_g,\sigma_g}|^2 = 1$ from normalization and imposed maximal entanglement, which equates all the coefficients giving $|\psi^{\sigma_g,\sigma_g}|^2 = \frac{1}{2^N}$. Thus the entanglement entropy scale as the number of flavours N.

5.3.3 Symmetric, spin-glass type, inner products

Here we briefly mention a generalization of the $O(N)$ flavoured case but with the various flavours talking to each other, with a spin glass type coupling. We define the elemental inner products (5.8) using a symmetric form J^{AB} and take the inner products between the configurations $|s_i\rangle$ to be (5.16).

Let us consider first $N = 2$ flavours: then the states are as in (5.17) and the inner products are

$$\langle s_i|s_j\rangle = \langle s_i^1|s_j^1\rangle\langle s_i^2|s_j^2\rangle + \langle s_i^1|s_j^2\rangle\langle s_i^2|s_j^1\rangle \ . \tag{5.58}$$

Using the elemental inner products (5.8), the non-zero inner products are

$$\begin{aligned}
&\langle s_1|s_4\rangle = \langle\uparrow^1 | \downarrow^1\rangle\langle\uparrow^2 | \downarrow^2\rangle + \langle\uparrow^1 | \downarrow^2\rangle\langle\uparrow^2 | \downarrow^1\rangle = J^{11}J^{22} + J^{12}J^{21} \ , \\
&\langle s_4|s_1\rangle = \langle\downarrow^1 | \uparrow^1\rangle\langle\downarrow^2 | \uparrow^2\rangle + \langle\downarrow^1 | \uparrow^2\rangle\langle\downarrow^2 | \uparrow^1\rangle = J^{11}J^{22} + J^{12}J^{21} \ , \\
&\langle s_2|s_2\rangle = \langle\uparrow^1 | \downarrow^2\rangle\langle\downarrow^2 | \uparrow^1\rangle = J^{12}J^{21} \ , \qquad \langle s_2|s_3\rangle = \langle\uparrow^1 | \downarrow^1\rangle\langle\downarrow^2 | \uparrow^2\rangle = J^{11}J^{22} \ , \\
&\langle s_3|s_2\rangle = \langle\downarrow^1 | \uparrow^1\rangle\langle\uparrow^2 | \downarrow^2\rangle = J^{11}J^{22} \ , \qquad \langle s_3|s_3\rangle = \langle\downarrow^1 | \uparrow^2\rangle\langle\uparrow^2 | \downarrow^1\rangle = J^{12}J^{21} \ .
\end{aligned} \tag{5.59}$$

The metric in the space of $|s_i\rangle$'s is real, symmetric and its determinant is $(\det J)(J^{11}J^{22} + J^{12}J^{21})^3$, where $\det J = J^{11}J^{22} - J^{12}J^{21}$. For an orthogonal matrix J^{AB}, $\det J \neq 0$ and the metric is non-singular only if $J^{11}J^{22} + J^{12}J^{21} \neq 0$.

We will not dwell more on this, although this may be worth investigating further.

5.4 Symplectic inner products

We want to study "symplectically flavoured" ghost-spins. We introduce the symplectic structure by defining the elemental inner products with an anitsymmetric matrix:

$$\langle\uparrow^A \mid \downarrow^B\rangle = i\,\Omega^{AB}\ , \qquad \langle\downarrow^A \mid \uparrow^B\rangle = i\,\Omega^{AB}\ , \qquad \langle\uparrow^A \mid \uparrow^B\rangle = 0 = \langle\downarrow^A \mid \downarrow^B\rangle\ ;$$
$$A, B = 1, \ldots, 2N\ , \tag{5.60}$$

where Ω^{AB} is a symplectic form, which is antisymmetric, *i.e.* $\Omega^{AB} = -\Omega^{BA}$. We will take the only nonzero elements as

$$\Omega^{12} = 1 = -\Omega^{21}, \qquad \Omega^{34} = 1 = -\Omega^{43}, \qquad \ldots, \qquad \Omega^{2N-1\,2N} = 1 = -\Omega^{2N\,2N-1}\ . \tag{5.61}$$

For a single symplectically flavoured ghost-spin there are 2^{2N} distinct configurations comprising the basis states $|s_1\rangle, \ldots, |s_{2^{2N}}\rangle$ and a generic state is

$$|\psi\rangle = \psi^{s_i}|s_i\rangle \quad \Rightarrow \quad \langle\psi|\psi\rangle = (\psi^{s_j})^*\,\psi^{s_i}\,\langle s_j \mid s_i\rangle\ . \tag{5.62}$$

We define inner products $\langle s_j \mid s_i\rangle$ between the basis states as

$$\langle s_i|s_j\rangle = \frac{1}{(2N)!}\sum \epsilon_{A_1A_2\ldots A_{2N}}\epsilon_{B_1B_2\ldots B_{2N}}\langle s_i^{A_1}|s_j^{B_1}\rangle\langle s_i^{A_2}|s_j^{B_2}\rangle\cdots\langle s_i^{A_{2N}}|s_j^{B_{2N}}\rangle\ , \tag{5.63}$$

where $i, j = 1, 2, \ldots, 2^{2N}$ label the configurations, $A_1, B_1, \cdots = 1, 2, \ldots, 2N$ label the flavours and $\epsilon_{A_1A_2\ldots A_{2N}}$ is the totally symmetric tensor with $\epsilon_{123\ldots 2N} = 1$ and $\epsilon_{A_1A_2\ldots A_{2N}}$ vanishes if any two labels are the same. Thus as in (5.16), $\epsilon_{A_1A_2\ldots A_{2N}}$ ensures that each elemental state $|s_j^A\rangle$ in $|s_j\rangle$ is paired with another corresponding elemental state in $\langle s_i|$.

Let us consider first a single ghost-spin with 2 flavours ($N = 1$): then the distinct configurations comprise the four basis states

$$|s_1\rangle = \left|{}^{\uparrow^1}_{\uparrow^2}\right\rangle\ , \qquad |s_2\rangle = \left|{}^{\uparrow^1}_{\downarrow^2}\right\rangle\ , \qquad |s_3\rangle = \left|{}^{\downarrow^1}_{\uparrow^2}\right\rangle\ , \qquad |s_4\rangle = \left|{}^{\downarrow^1}_{\downarrow^2}\right\rangle\ , \tag{5.64}$$

i.e. in $|s_1\rangle$, the first flavour is $\uparrow^1$ and the second flavour is $\uparrow^2$, and likewise for $|s_2\rangle, |s_3\rangle, |s_4\rangle$. The non-zero elemental inner products in (5.60) are

$$\langle\uparrow^1 \mid \downarrow^2\rangle = i = \langle\downarrow^1 \mid \uparrow^2\rangle\ , \qquad \langle\uparrow^2 \mid \downarrow^1\rangle = -i = \langle\downarrow^2 \mid \uparrow^1\rangle \tag{5.65}$$

and the inners products (5.63) between the configurations simplify to

$$\langle s_i|s_j\rangle = \langle s_i^1|s_j^2\rangle\langle s_i^2|s_j^1\rangle\ . \tag{5.66}$$

Then non-zero inner products for the configurations (5.17) are

$$\begin{aligned} \langle s_1|s_4\rangle &= \langle\uparrow^1|\downarrow^2\rangle\langle\uparrow^2|\downarrow^1\rangle = 1\ , \quad \langle s_4|s_1\rangle = \langle\downarrow^1|\uparrow^2\rangle\langle\downarrow^2|\uparrow^1\rangle = 1\ , \\ \langle s_2|s_2\rangle &= \langle\uparrow^1|\downarrow^2\rangle\langle\downarrow^2|\uparrow^1\rangle = 1\ , \quad \langle s_3|s_3\rangle = \langle\downarrow^1|\uparrow^2\rangle\langle\uparrow^2|\downarrow^1\rangle = 1\ . \end{aligned} \tag{5.67}$$

These give a real, symmetric and non-singular metric in the space of configurations $|s_i\rangle$. Based on these inner products, we can write the norm for the generic state as

$$|\psi\rangle = c_i|s_i\rangle : \qquad \langle\psi|\psi\rangle = c_1^* c_4 + c_4^* c_1 + |c_2|^2 + |c_3|^2\ . \tag{5.68}$$

Likewise for a single ghost-spin with 4 flavours ($N = 2$), there are 16 distinct configurations comprising of basis states (5.22). The non-zero elemental inner products with $\Omega^{12} = 1$ and $\Omega^{34} = 1$ are

$$\begin{aligned} \langle\uparrow^1|\downarrow^2\rangle = i = \langle\downarrow^1|\uparrow^2\rangle\ , \qquad \langle\uparrow^2|\downarrow^1\rangle = -i = \langle\downarrow^2|\uparrow^1\rangle\ , \\ \langle\uparrow^3|\downarrow^4\rangle = i = \langle\downarrow^3|\uparrow^4\rangle\ , \qquad \langle\uparrow^4|\downarrow^3\rangle = -i = \langle\downarrow^4|\uparrow^3\rangle\ . \end{aligned} \tag{5.69}$$

Since these are the only non-zero elemental inner products, the inner products (5.63) for the basis states (5.22) reduce to

$$\langle s_i|s_j\rangle = \frac{1}{4!}\sum \epsilon_{A_1A_2A_3A_4}\epsilon_{\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4}\langle s_i^{A_1}|s_j^{\tilde{A}_1}\rangle\langle s_i^{A_2}|s_j^{\tilde{A}_2}\rangle\langle s_i^{A_3}|s_j^{\tilde{A}_3}\rangle\langle s_i^{A_4}|s_j^{\tilde{A}_4}\rangle\ , \tag{5.70}$$

where $\tilde{A}_k = |\Omega^{\tilde{A}_k A_l}|A_l$, *e.g.* for $A_1 = 1$, $\tilde{A}_1 = 2$, for $A_2 = 4$, $\tilde{A}_2 = 3$, etc. As there are only 4! such non-zero terms in the above inner product, it becomes

$$\langle s_i|s_j\rangle = \epsilon_{1234}\epsilon_{2143}\langle s_i^1|s_j^2\rangle\langle s_i^2|s_j^1\rangle\langle s_i^3|s_j^4\rangle\langle s_i^4|s_j^3\rangle = \langle s_i^1|s_j^2\rangle\langle s_i^2|s_j^1\rangle\langle s_i^3|s_j^4\rangle\langle s_i^4|s_j^3\rangle\ . \tag{5.71}$$

The non-zero inner products between $|s_i\rangle$'s computed using this formula can be written compactly as

$$\langle s_i|\tilde{s}_j\rangle = 1\ , \tag{5.72}$$

where $|\tilde{s}_j\rangle$ is defined such that if the A-th flavour entry in $|s_i\rangle$ is $\uparrow^A$ (or $\downarrow^A$)) then the $\tilde{A}$-th flavour entry in $|\tilde{s}_j\rangle$ is $\downarrow^{\tilde{A}}$ (or $\uparrow^{\tilde{A}}$)) for $\tilde{A} = |\Omega^{\tilde{A}A}|A$. We see that the metric $\langle s_i|s_j\rangle$ is real, symmetric and non-singular.

We can generalize this to $2N$ flavours, where the non-zero elemental inner products are

$$\begin{array}{llll} \langle\uparrow^1 | \downarrow^2\rangle = i\ , & \langle\uparrow^3 | \downarrow^4\rangle = i\ , & \ldots\ldots\ , & \langle\uparrow^{2N-1} | \downarrow^{2N}\rangle = i\ , \\ \langle\downarrow^1 | \uparrow^2\rangle = i\ , & \langle\downarrow^3 | \uparrow^4\rangle = i\ , & \ldots\ldots\ , & \langle\downarrow^{2N-1} | \uparrow^{2N}\rangle = i\ , \\ \langle\uparrow^2 | \downarrow^1\rangle = -i\ , & \langle\uparrow^4 | \downarrow^3\rangle = -i\ , & \ldots\ldots\ , & \langle\uparrow^{2N} | \downarrow^{2N-1}\rangle = -i\ , \\ \langle\downarrow^2 | \uparrow^1\rangle = -i\ , & \langle\downarrow^4 | \uparrow^3\rangle = -i\ , & \ldots\ldots\ , & \langle\downarrow^{2N} | \uparrow^{2N-1}\rangle = -i\ . \end{array} \quad (5.73)$$

Then the inner products (5.63) become

$$\begin{aligned} \langle s_i | s_j\rangle &= \frac{1}{(2N)!}\sum \epsilon_{A_1A_2\ldots A_{2N}}\epsilon_{\tilde{A}_1\tilde{A}_2\ldots\tilde{A}_{2N}}\langle s_i^{A_1}|s_j^{\tilde{A}_1}\rangle\langle s_i^{A_2}|s_j^{\tilde{A}_2}\rangle\cdots\langle s_i^{A_{2N}}|s_j^{\tilde{A}_{2N}}\rangle \\ &= \epsilon_{1234\ldots 2N-1\,2N}\,\epsilon_{2143\ldots 2N\,2N-1}\langle s_i^1|s_j^2\rangle\langle s_i^2|s_j^1\rangle\cdots\langle s_i^{2N-1}|s_j^{2N}\rangle\langle s_i^{2N}|s_j^{2N-1}\rangle \\ &= \langle s_i^1|s_j^2\rangle\langle s_i^2|s_j^1\rangle\cdots\langle s_i^{2N-1}|s_j^{2N}\rangle\langle s_i^{2N}|s_j^{2N-1}\rangle\ , \end{aligned} \quad (5.74)$$

where $\tilde{A}_k = |\Omega^{\tilde{A}_kA_l}|A_l$. Using the elemental inner products, we see that the non-zero inner products are $\langle s_i|\tilde{s_j}\rangle = 1$, with $|\tilde{s_j}\rangle$ as defined earlier.

Thus the norm of a generic state $|\psi\rangle = \psi^{s_i}|s_i\rangle$ is

$$\langle\psi|\psi\rangle = (\psi^{s_j})^*\,\psi^{s_i}\,\langle s_j\,|\,s_i\rangle = (\psi^{s_j})^*\,\psi^{s_i}\langle s_j^1|s_i^2\rangle\langle s_j^2|s_i^1\rangle\cdots\langle s_j^{2N-1}|s_i^{2N}\rangle\langle s_j^{2N}|s_i^{2N-1}\rangle\ . \quad (5.75)$$

Along the same lines, the general inner product between any two states is

$$\langle\psi_1|\psi_2\rangle = (\psi_1^{s_j})^*\,\psi_2^{s_i}\,\langle s_j\,|\,s_i\rangle = (\psi_1^{s_j})^*\,\psi_2^{s_i}\langle s_j^1|s_i^2\rangle\langle s_j^2|s_i^1\rangle\cdots\langle s_j^{2N-1}|s_i^{2N}\rangle\langle s_j^{2N}|s_i^{2N-1}\rangle\ . \quad (5.76)$$

Correlated ghost-spin states: We want to now construct correlated ghost-spin states that are positive norm and positive entanglement, along the lines of the discussion for $O(N)$ flavoured cases in sec. 5.3.2. This is most transparent in a diagonal basis where positive and negative norm states are manifest. For concreteness, let us consider the basis states (5.64) for a single ghost-spin with 2 flavours, *i.e.* $N = 1$. The norm (5.68) for a generic state can be recast using a diagonal basis $|s_\pm\rangle, |s_2\rangle, |s_3\rangle$ as

$$|s_\pm\rangle = \frac{1}{\sqrt{2}}(|s_1\rangle \pm |s_4\rangle):\qquad \langle\psi|\psi\rangle = |c_2|^2 + |c_3|^2 + |c_+|^2 - |c_-|^2\ . \quad (5.77)$$

Thus there are 3 basis states with positive norm and one with negative norm. This can be carried out for more flavours as well. For $N = 2$ for instance, we have 16 basis states which can be recast as 10 positive norm and 6 negative norm states, using (5.72): besides the states with $\langle s_i|s_i\rangle \neq 0$, there are states with off-diagonal inner products like $|s_{1,4}\rangle$ above whose linear combinations then add to the set of diagonal positive norm states. Note that the numbers of positive and negative norm basis states are not equal. With general N, *i.e.* $A,B=1,\ldots,2N$, it can be seen that there are 2^{2N} basis states in all: of these there are $\frac{2^{2N}+2^N}{2}$ positive norm states and $\frac{2^{2N}-2^N}{2}$ negative norm states (this is easily verified for $N=1,2$ above). For large N, we see that the number of positive and negative norm states become asymptotically equal.

In terms of such a diagonal basis, we can consider 2 ghost-spins and explicitly construct correlated ghost-spin states similar structurally to (5.52) in the $O(N)$ flavoured case. Let us label these diagonal basis states for a single ghost-spin as $|s_i\rangle$ (which should not be confused with the earlier nondiagonal $|s_i\rangle$ basis). Then the 2-ghost-spin states can be made from the 2^{4N} basis states $|s_i\rangle|s_j\rangle$ obtained by tensor products of the single ghost-spin states. The general state and its norm are then similar in structure to (5.51) for the $O(N)$ case sec. 5.3.2. Correlated ghost-spin states can then be constructed as in (5.52) giving $|\psi^{corr}\rangle = \sum \psi^{s_i,s_i}|s_i\rangle|s_i\rangle$: it can be seen that these are positive norm and positive entanglement as in (5.52), (5.54). This subspace has dimension 2^{2N}, the number of basis states. Since the details here are very similar to that in sec. 5.3.2, we will not describe them further here.

It is worth noting that the symplectic invariance is at the level of the elemental ghost-spin basis states $\{|\uparrow^A\rangle,\ |\downarrow^A\rangle\}$: generic basis states $|v_i\rangle = v_i^{\uparrow^A}|\uparrow^A\rangle + v_i^{\downarrow^A}|\downarrow^A\rangle$, have inner product

$$\begin{aligned}\langle v_1|v_2\rangle &= (v_1^{\uparrow^A})^* v_2^{\downarrow^B}\langle\uparrow^A|\downarrow^B\rangle + (v_1^{\downarrow^A})^* v_2^{\uparrow^B}\langle\downarrow^A|\uparrow^B\rangle \\ &= i[(v_1^{\uparrow^A})^* v_2^{\downarrow^B}\Omega^{AB} + (v_1^{\downarrow^A})^* v_2^{\uparrow^B}\Omega^{AB}]\ , \end{aligned} \tag{5.78}$$

which is invariant under symplectic transformations. To see this explicitly, consider two flavours ($N=1$) for simplicity. Then a symplectic transformation by a real pseudo-orthogonal matrix $R\in Sp(2)$ is $R^T\boldsymbol{\Omega}R=\boldsymbol{\Omega},\ R\boldsymbol{\Omega}R^T=\boldsymbol{\Omega},\ R^{-1}=-\boldsymbol{\Omega}R^T\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the symplectic form with $\Omega^{12}=1=-\Omega^{21}$, and $\Omega^{-1}=-\Omega$. Thus $(v_1^{\uparrow^A})^* v_2^{\downarrow^B}\Omega^{AB}$ and $(v_1^{\downarrow^A})^* v_2^{\uparrow^B}\Omega^{AB}$ are invariant under $|\uparrow^A\rangle\to R^{AB}|\uparrow^B\rangle$, $|\downarrow^A\rangle\to R^{AB}|\downarrow^B\rangle$, *i.e.*

$$\begin{aligned}(\tilde{v}_1^{\uparrow^A})^*\Omega^{AB}\tilde{v}_2^{\downarrow^B} &= (v_1^{\uparrow^C})^* R^{CA}\Omega^{AB}R^{BD}v_2^{\downarrow^D} = (v_1^{\uparrow^C})^*\Omega^{CD}v_2^{\downarrow^D}\ , \\ (\tilde{v}_1^{\downarrow^A})^*\Omega^{AB}\tilde{v}_2^{\uparrow^B} &= (v_1^{\downarrow^C})^* R^{CA}\Omega^{AB}R^{BD}v_2^{\uparrow^D} = (v_1^{\downarrow^C})^*\Omega^{CD}v_2^{\uparrow^D}\ , \end{aligned} \tag{5.79}$$

where we have used $R^{CA}\Omega^{AB}R^{BD}=R^T\boldsymbol{\Omega}R=\boldsymbol{\Omega}$.

The elemental inner products (5.65) are consistent with (and motivated by) an operator algebra alongwith states, defined as (these arise in theories of symplectic fermions [219])

$$\{\sigma_b^A, \sigma_c^B\} = i\,\Omega^{AB}\hat{K}\ ; \tag{5.80}$$
$$|\uparrow^A\rangle = \sigma_c^A|\downarrow\rangle\ , \quad \langle\downarrow^A| = \langle\uparrow|\sigma_b^A\ , \quad |\downarrow^A\rangle = \sigma_b^A|\uparrow\rangle\ , \quad \langle\uparrow^A| = \langle\downarrow|\sigma_c^A\ ,$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are ghost-spin states with $\langle\uparrow|\downarrow\rangle = 1 = \langle\downarrow|\uparrow\rangle$. The hermiticity of $\{\sigma_b^A, \sigma_c^B\}$ for hermitian σ_b^A, σ_c^B and real Ω^{AB} gives $\hat{K}^\dagger = -\hat{K}$ *i.e.* $\hat{K}$ is anti-hermitian. This anti-Hermitian operator leads

$$(\langle\uparrow|\hat{K}|\downarrow\rangle)^\dagger = \langle\downarrow|\hat{K}^\dagger|\uparrow\rangle = -\langle\downarrow|\hat{K}|\uparrow\rangle\ , \tag{5.81}$$

which implies that for $\langle\uparrow|\hat{K}|\downarrow\rangle = 1$, $\langle\downarrow|\hat{K}|\uparrow\rangle = -1$. Using these we get the elemental inner products as

$$\langle\uparrow^A|\downarrow^B\rangle = \langle\downarrow|\sigma_c^A\sigma_b^B|\uparrow\rangle = i\,\Omega^{BA}\langle\downarrow|\hat{K}|\uparrow\rangle = i\,\Omega^{AB}\ ,$$
$$\langle\downarrow^A|\uparrow^B\rangle = \langle\uparrow|\sigma_b^A\sigma_c^B|\downarrow\rangle = i\,\Omega^{AB}\langle\uparrow|\hat{K}|\downarrow\rangle = i\,\Omega^{AB}\ . \tag{5.82}$$

It is then possible to construct ghost-spin chains with nearest neighbour interactions between operators at neighbouring lattice sites, somewhat similar to the ghost-spin chain for the *bc*-ghost CFTs. However the continuum limit is less clear in this case, in part due to technical difficulties such as the construction of the Jordan-Wigner transformation to obtain fermionic versions of the $\sigma_{b,c}$ operators above which anticommute with each other (the $\sigma_{b,c}$ are bosonic spin-like operators commuting at neighbouring lattice sites while anticommuting at the same site). Note however that the case with $N = 1$ has structure similar to that appearing in the theory of anticommuting scalars: this is a logarithmic CFT in 2-dimensions [218, 219, 220, 221, 222, 223, 224]. So perhaps the continuum limit here gives symplectic fermions $\int \Omega_{AB}\,\partial\phi^A\partial\phi^B$: we hope to explore this further.

5.5 N irreducible levels

In this section, we consider a generalization of ghost-spins that consists of N irreducible levels, defined as

$$\langle e_i|e_i\rangle = 0\ , \qquad \langle e_i|e_j\rangle = 1 \qquad \forall \quad i \neq j\ ; \qquad i,j = 1,2,\ldots,N\ . \tag{5.83}$$

For $N = 2$, the basis states $|e_1\rangle, |e_2\rangle$ are identical to the $|\uparrow\rangle, |\downarrow\rangle$ basis states, and this system reduces to the 2-level ghost-spin reviewed in Sec. 2. Flavoured generalizations can be constructed by adding additional flavour indices to these, along the lines we have described for 2-level ghost-spins in the previous sections: we will not do so here however.

Using the inner products above, it is clear that there are various negative norm states here as well: *e.g.* $|e_i\rangle - |e_j\rangle$ has norm -2. Using a diagonal basis helps as in the 2-level case to identify positive and negative norm states clearly. This can be done using the transformations in Appendix D.1: we can choose an orthonormal basis where the basis states and their inner products are

$$\begin{aligned}
&|\alpha\rangle \equiv \{|+\rangle,\ |2\rangle,\ \ldots,\ |N\rangle\}\ ;\\
&\langle\alpha|\beta\rangle = \eta_{\alpha\beta}\,; \quad \eta_{++}=1\,, \quad \eta_{22}=\eta_{33}=\cdots\cdots=\eta_{NN}=-1\,, \quad \eta_{\alpha\beta}=0 \quad \forall\ \alpha\neq\beta\,,\\
&i.e. \quad \langle+|+\rangle = 1\ , \qquad \langle\alpha|\alpha\rangle=-1, \qquad \alpha=2,...,N\ . \qquad (5.84)
\end{aligned}$$

Then the generic state and its norm in both bases are

$$|\psi\rangle = \psi^i|e_i\rangle; \quad \langle\psi|\psi\rangle = (\psi^i)^*\psi^j\langle e_i|e_j\rangle = \sum_{i\neq j}(\psi^i)^*\psi^j\ , \qquad (5.85)$$

$$|\psi\rangle = \psi^\alpha|\alpha\rangle; \quad \langle\psi|\psi\rangle = (\psi^\alpha)^*\psi^\beta\langle\alpha|\beta\rangle = (\psi^\alpha)^*\psi^\beta\eta_{\alpha\beta} = |\psi^+|^2 - \sum_{\alpha=2}^{N}|\psi^\alpha|^2 . \qquad (5.86)$$

To illustrate this, let us consider $N=3$. The generic state and its norm are

$$\begin{aligned}
|\psi\rangle &= \psi^1|e_1\rangle + \psi^2|e_2\rangle + \psi^3|e_3\rangle = \psi^+|+\rangle + \psi^2|2\rangle + \psi^3|3\rangle\ ,\\
\langle\psi|\psi\rangle &= (\psi^1)^*\psi^2 + (\psi^2)^*\psi^1 + (\psi^1)^*\psi^3 + (\psi^3)^*\psi^1 + (\psi^2)^*\psi^3 + (\psi^3)^*\psi^2\\
&= |\psi^+|^2 - |\psi^2|^2 - |\psi^3|^2\ . \qquad (5.87)
\end{aligned}$$

In some sense, this is a ghost-spin generalization of the N-level spins that arise in the Heisenberg spin chain: perhaps appropriate interaction Hamiltonians for ghost-spin chains on a 1-dim lattice can be studied along those lines.

Correlated ghost-spins and entanglement: We want to construct correlated ghost-spin states analogous to the discussion in sec. 5.3.2. So consider a system of two ghost-spins with N irreducible levels. The orthonormal basis for this system is

$$|u_A u_B\rangle \equiv |\alpha\rangle|\beta\rangle \equiv |\alpha\beta\rangle \qquad \forall \quad \alpha,\beta = +,2,\ldots,N\ , \qquad (5.88)$$

where each $|\alpha\rangle$ is a single ghost-spin basis state in (5.84). A generic state $|\psi\rangle = \psi^{\alpha\beta}|\alpha\beta\rangle$ has a norm $\langle\psi|\psi\rangle = \eta_{\alpha\kappa}\eta_{\beta\lambda}(\psi^{\alpha\beta})^*\psi^{\kappa\lambda}$, which can be expanded as

$$\langle\psi|\psi\rangle = \left(\sum_{\alpha} |\psi^{\alpha\alpha}|^2 + \sum_{\alpha,\beta\neq +} |\psi^{\alpha\beta}|^2\right) - \left(\sum_{\alpha\neq +}(|\psi^{+\alpha}|^2 + \psi^{\alpha +}|^2)\right) . \qquad (5.89)$$

We see a manifest division between the positive and negative norm subspaces. For $N = 3$, we can see this explicitly as

$$\langle\psi|\psi\rangle = (|\psi^{++}|^2+|\psi^{22}|^2+|\psi^{33}|^2+|\psi^{23}|^2+|\psi^{32}|^2)-(|\psi^{+2}|^2+|\psi^{2+}|^2+|\psi^{+3}|^2+|\psi^{3+}|^2) . \qquad (5.90)$$

For general N, by tracing over the second ghost-spin, the reduced density matrix is

$$\rho_A = (\rho_A)^{\alpha\kappa}|\alpha\rangle\langle\kappa| \ ; \qquad (\rho_A)^{\alpha\kappa} = \psi^{\alpha\beta}(\psi^{\kappa\beta})^*\eta_{\beta\beta} . \qquad (5.91)$$

The mixed index reduced density matrix is $(\rho_A)^{\alpha}_{\beta} = \eta_{\beta\kappa}(\rho_A)^{\kappa\alpha} = \eta_{\beta\kappa}\eta_{\lambda\lambda}\psi^{\kappa\lambda}(\psi^{\alpha\lambda})^*$.

Correlated ghost-spins : From the norm above we see that the states $|++\rangle$, $|22\rangle$, $\ldots$, $|NN\rangle$ span the subspace of correlated ghost-spin states, where a generic correlated ghost-spin state is

$$|\psi\rangle = \psi^{\alpha\alpha}|\alpha\alpha\rangle \ ; \qquad \langle\psi|\psi\rangle = |\psi^{++}|^2 + |\psi^{22}|^2 + \cdots + |\psi^{NN}|^2 . \qquad (5.92)$$

Entanglement pattern in a general state : Consider a slightly more general state

$$|\psi\rangle = \sum_{\alpha=+}^{N} \psi^{\alpha\alpha}|\alpha\alpha\rangle + \sum_{\beta=2}^{N}(\psi^{+\beta}| + \beta\rangle + \psi^{\beta +}|\beta +\rangle) , \qquad (5.93)$$

whose norm is

$$\langle\psi|\psi\rangle = \sum_{\alpha=+}^{N} |\psi^{\alpha\alpha}|^2 - \sum_{\beta=2}^{N}(|\psi^{+\beta}|^2 + |\psi^{\beta +}|^2) . \qquad (5.94)$$

The off-diagonal components of the reduced density matrix are

$$\begin{aligned}(\rho_A)^{+\alpha} &= \psi^{++}\psi^{\alpha +^*} - \psi^{+\alpha}\psi^{\alpha\alpha^*} , \qquad && \forall\ \alpha = 2,\ldots,N\ ,\ \alpha \neq + , \\ (\rho_A)^{\alpha\beta} &= \psi^{\alpha +}\psi^{\beta +^*} , && \forall\ \alpha,\beta = 2,\ldots,N\ ,\ \alpha \neq +\ ,\ \beta \neq + . \end{aligned} \qquad (5.95)$$

From $(\rho_A)^{\alpha\beta} = 0$, we see that only one of $\psi^{\alpha +}$ is non-zero, i.e., $\psi^{2+} \neq 0$, $\psi^{\alpha +} = 0$, $\alpha = 3,\ldots,N$. Then $(\rho_A)^{+\alpha} = 0$ gives $\psi^{+2} \neq 0$ and $\psi^{+\alpha} = 0$, $\alpha = 3,\ldots,N$.

So we consider the state

$$|\psi\rangle = \psi^{++}| + +\rangle + \psi^{22}|22\rangle + \cdots + \psi^{NN}|NN\rangle + \psi^{+2}| + 2\rangle + \psi^{2+}|2+\rangle , \qquad (5.96)$$

whose norm is

$$\langle\psi|\psi\rangle = |\psi^{++}|^2 + \cdots + |\psi^{NN}|^2 - |\psi^{+2}|^2 - |\psi^{2+}|^2 \ . \qquad (5.97)$$

The non-zero components of the reduced density matrix are

$$\begin{aligned}(\rho_A)^{++} &= |\psi^{++}|^2 - |\psi^{+2}|^2 \ , \qquad (\rho_A)^{22} = |\psi^{2+}|^2 - |\psi^{22}|^2 \ , \\ (\rho_A)^{+2} &= \psi^{++}\psi^{2+^*} - \psi^{+2}\psi^{22^*} \ , \qquad (\rho_A)^{\alpha\alpha} = -|\psi^{\alpha\alpha}|^2 \ , \quad \alpha = 3, \ldots, N \ .\end{aligned} \qquad (5.98)$$

Choosing $\psi^{2+^*} = \frac{\psi^{+2}\psi^{22^*}}{\psi^{++}}$ and defining $x \equiv |\psi^{++}|^2 - |\psi^{+2}|^2$ and $r \equiv \dfrac{|\psi^{22}|^2}{|\psi^{++}|^2} > 0$, the mixed-index components of ρ_A are

$$(\rho_A)^+_+ = x \ , \quad (\rho_A)^2_2 = xr \ , \quad (\rho_A)^\alpha_\alpha = |\psi^{\alpha\alpha}|^2 \ , \quad \alpha = 3, \ldots, N \qquad (5.99)$$

and

$$\langle\psi|\psi\rangle = x + xr + |\psi^{33}|^2 + \cdots + |\psi^{NN}|^2 = \pm 1 \ . \qquad (5.100)$$

Now depending on if x is positive or negative we have the following three cases.

• If $x > 0$, $|\psi\rangle$ has necessarily positive norm and $\langle\psi|\psi\rangle = 1$ implies $0 < (\rho_A)^\alpha_\alpha < 1$ for all $\alpha = +, 2, \ldots, N$ giving $S_A > 0$.

• If $x < 0$, the norm of $|\psi\rangle$ can be positive or negative.

(i) For positive norm, *i.e.* $\langle\psi|\psi\rangle = -|x| - |x|r + \sum_{\alpha=3}^N |\psi^{\alpha\alpha}|^2 = 1$, we get

$$\begin{aligned} S_A &= |x|\log|x| + |x|r\log|x|r - |\psi^{33}|^2\log(|\psi^{33}|^2) \\ &\quad - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2 \log(|\psi^{\alpha\alpha}|^2) + i\pi|x|(1+r) \\ &= |x|\log|x| + |x|r\log|x|r \\ &\quad - \left(1 + |x| + |x|r - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\right)\log\left(1 + |x| + |x|r - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\right) \\ &\quad - \sum_{\alpha=4}^N |\psi^{\alpha\alpha}|^2\log(|\psi^{\alpha\alpha}|^2) + i\pi|x|(1+r) \ . \end{aligned} \qquad (5.101)$$

We see that $Im(S_A)$ is not constant and $Re(S_A) < 0$ when

$$|x|\log|x| + |x|r\log|x|r \;<\; (1+|x|+|x|r)\log\Big(1+|x|+|x|r-\sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\Big) - \sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\log\Big(1+|x|+|x|r-\sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\Big) + \sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\log(|\psi^{\alpha\alpha}|^2)\ . \tag{5.102}$$

(ii) For a negative norm state $|\psi\rangle$, *i.e.* $\langle\psi|\psi\rangle = -|x| - |x|r + \sum_{\alpha=3}^{N}|\psi^{\alpha\alpha}|^2 = -1$, we get

$$\begin{aligned} S_A &= |x|\log|x| + |x|r\log|x|r - |\psi^{33}|^2\log(|\psi^{33}|^2) - \sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\log(|\psi^{\alpha\alpha}|^2) \\ &\quad + i\pi|x|(1+r) \\ &= |x|\log|x| + |x|r\log|x|r - \sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\log(|\psi^{\alpha\alpha}|^2) + i\pi|x|(1+r) \\ &\quad - \left(-1+|x|+|x|r-\sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\right)\log\left(-1+|x|+|x|r-\sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\right) \end{aligned} \tag{5.103}$$

$Im(S_A)$ is constant if $|x|+|x|r = c$, where c is a constant and $c > 1$ (from the norm). Then $Re(S_A)$ becomes

$$Re(S_A) = |x|\log|x| + (c-|x|)\log(c-|x|) - \sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\log(|\psi^{\alpha\alpha}|^2) - \Big(c-1-\sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\Big)\log\Big(c-1-\sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\Big)\ . \tag{5.104}$$

We see that $Re(S_A) < 0$ for those values of $|x|$, $c > 1$, $\psi^{\alpha\alpha}$ which satisfy

$$|x|\log|x| + (c-|x|)\log(c-|x|) \;<\; \Big(c-1-\sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\Big)\log\Big(c-1-\sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\Big) + \sum_{\alpha=4}^{N}|\psi^{\alpha\alpha}|^2\log(|\psi^{\alpha\alpha}|^2)\ . \tag{5.105}$$

5.6 Discussion

We have constructed N-level generalizations of the 2-level ghost-spins in [46, 47, 48]. These include (i) a flavoured generalization comprising N copies of the ghost-spin system and corresponding ghost-spin chains which lead to 2-dim bc-ghost CFTs with $O(N)$) flavour symmetry, (ii) a spin-glass type coupling in flavour space, (iii) a symplectic generalization involving antisymmetric inner products between the elemental ghost-spins, and (iv) an irreducible ghost-spin system with N internal levels. We have studied entanglement properties in these cases: among other things, these show the existence of positive norm states in two copies of ghost-spin ensembles obtained by entangling identical ghost-spins from each copy: these are akin to the correlated ghost-spin states in [47, 48], and exhibit positive entanglement.

We now describe briefly some of the motivations from dS/CFT, in particular [212], for the studies here. Generalizations of gauge/gravity duality for de Sitter space or dS/CFT involve conjectured dual hypothetical Euclidean non-unitary CFTs living on the future boundary $\mathcal{I}^+$ [43, 44, 45]. Using the dictionary $\Psi_{dS} = Z_{CFT}$ [45], where Ψ_{dS} is the late-time Hartle-Hawking wavefunction of the universe with appropriate boundary conditions and Z_{CFT} the dual CFT partition function, the dual CFT$_d$ energy-momentum tensor correlators reveal central charge coefficients $\mathcal{C}_d \sim i^{1-d}\frac{l^{d-1}}{G_{d+1}}$ in dS_{d+1} (effectively analytic continuations from AdS/CFT). This is real and negative in dS_4, with $\mathcal{C}_3 \sim -\frac{R_{dS}^2}{G_4}$ so that dS_4/CFT_3 is reminiscent of ghost-like non-unitary theories. Bulk expectation values are of the form $\langle\varphi_k\varphi_{k'}\rangle \sim \int D\varphi\ \varphi_k\varphi_{k'}|\Psi_{dS}|^2$. This involves the probability $|\Psi_{dS}|^2 = \Psi_{dS}^*\Psi_{dS}$, which suggests that bulk de Sitter physics involves two copies of the dual CFT — $CFT_F \times CFT_P$ on the future and past boundaries. This is unlike in AdS/CFT where $Z_{bulk} = Z_{bndry}$ implies boundary correlators can be obtained as a limit of bulk ones. In the dS case, while $Z_{CFT} = \Psi_{dS}$ of a single dual CFT copy at I^+ can be used to obtain boundary correlators (*e.g.* $\langle O_k O_{k'}\rangle \sim \frac{\delta^2 Z_{CFT}}{\delta\varphi_k^0\delta\varphi_{k'}^0}$ for operators O_k dual to modes φ_k), bulk observables require $|\Psi_{dS}|^2$, the bulk probability, and so two copies of the dual Z_{CFT}. This dovetails with the structure of extremal surfaces and entanglement as we see below.

In AdS, surfaces anchored at one end of a subsystem dip into the bulk radial direction and then begin to return to the boundary at turning points. In dS, the boundary at I^+ is spacelike and surfaces dip into the time direction which ends up making their structure quite different, as studied in [213]. For instance, considering the dS Poincare slicing $ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dx_i^2)$, a strip subsystem on some boundary Euclidean time $w = const$ slice of I^+ with width along x gives a bulk extremal surface

$x(\tau)$ described by $\dot{x}^2 \equiv (\frac{dx}{d\tau})^2 = \frac{B^2\tau^{2d-2}}{1+B^2\tau^{2d-2}}$ $(B^2 > 0)$. w, x can be taken as any of the x_i (so the boundary Euclidean time slice is not sacrosanct). Compared with the AdS case, the denominator here crucially has a relative minus sign. Thus there is no *real* turning point here where the surface starting at I^+ begins to turn back towards I^+: this requires $|\dot{x}| \to \infty$ while here $|\dot{x}| \leq 1$. There are also complex extremal surfaces however, which exhibit turning points: these end up amounting to analytic continuation from the AdS Ryu-Takayanagi surfaces. While their interpretation is not entirely conclusive, in dS_4 they have *negative* area, consistent with the negative central charge in dS_4/CFT_3 as mentioned above.

Since surfaces starting at I^+ do not turn back, it is then interesting to ask if they could instead stretch all the way to the past boundary I^-. In [212], connected codim-2 extremal surfaces in the static patch coordinatization of de Sitter space were found stretching from I^+ to I^- passing through the vicinity of the bifurcation region with divergent area $\frac{l^2}{4G_4}\frac{1}{\epsilon}$, where $\epsilon = \frac{\epsilon_c}{l}$ is the dimensionless ultraviolet cutoff and the coefficient scales as de Sitter entropy. To elaborate a little, the static patch coordinatization can be recast as $ds^2 = \frac{l^2}{\tau^2}\big(-\frac{d\tau^2}{1-\tau^2} + (1-\tau^2)dw^2 + d\Omega_{d-1}^2\big)$, with the future/past universes F/P parametrized by $0 \leq \tau \leq 1$ with horizons at $\tau = 1$, while the Northern/Southern diamonds N/S have $1 < \tau \leq \infty$. The boundaries at $\tau = 0$ are now of the form $R_w \times S^{d-1}$, resembling the Poincare slicing locally. Setting up the extremization for codim-2 surfaces on boundary Euclidean time slices can be carried out: on S^{d-1} equatorial planes for instance we obtain $\dot{w}^2 = \frac{B^2\tau^{2d-2}}{1-\tau^2+B^2\tau^{2d-2}}$. The minus sign here, reflecting the horizons, makes the structure of these surfaces interesting, drawing parallels with the AdS extremization (we refer to [212] for further details). The limit $B \to 0$ gives surfaces passing through the vicinity of the bifurcation region as stated above, with the width Δw approaching all of $I^\pm$. These connected surfaces stretching between $I^\pm$ are akin to rotated versions of the connected surfaces of Hartman, Maldacena [225] in the AdS black hole. This led to the speculation there that dS_4 is approximately dual to an entangled thermofield-double type state of the form

$$|\psi\rangle = \sum \psi^{i_n^F, i_n^P} |i_n^F\rangle |i_n^P\rangle \tag{5.106}$$

akin to the thermofield double [226] dual to AdS black holes. Here $\psi^{i_n^F,i_n^P}$ are coefficients entangling a generic ghost-spin $|i_n^F\rangle$ from CFT_F at I^+ with an identical one $|i_n^P\rangle$ from CFT_P at I^-. The constituent states are schematically continuum versions of N level ghost-spins, with N related to dS_4 entropy $\frac{l^2}{4G_4}$. Since bulk time evolution maps configurations at I^- to those at I^+ [44], we have the schematic map $|i_n^P\rangle \to S[i_n^P, i_n^F]|i_n^P\rangle \equiv |i_n^F\rangle$ where $S[i_n^P, i_n^F]$ is the operator representing bulk time evolution (note that this is a bulk object that is to be distinguished from operators in the CFT representing boundary Euclidean time evolution). If states $|i^P\rangle$ at I^- map faithfully and completely to states $|i^P\rangle$ at I^+, then S is expected to be a unitary operator (this is also vindicated by the fact that exchanging $I^\pm$ is a bulk symmetry). This suggests that the entangled states (5.106) are unitarily equivalent to similar maximally entangled states $|\psi\rangle = \sum \psi^{i_n^F, i_n^F} |i_n^F\rangle |i_n^F\rangle$ in two CFT_F copies of the ghost-CFT solely at I^+. The state (5.106) is akin to a correlated ghost-spin state with an even number of ghost-spins, as discussed in [47, 48]. It necessarily has positive norm $\sum_{i_n^F, i_n^F} \gamma_{i_n^F, i_n^F} \gamma_{i_n^F, i_n^F} \psi^{i_n^F, i_n^F} (\psi^{i_n^F, i_n^F})^* \rightarrow \sum_{i_n} |\psi^{i_n^F, i_n^F}|^2$, since we are entangling identical states i_n^F and i_n^P: thus it has positive entanglement, as in [47, 48]. Since each constituent state $|i_n^{F,P}\rangle$ is N-level, *i.e.* with N internal degrees of freedom, the entanglement entropy scales as $N \sim \frac{l^2}{G_4}$. The toy models in sec. 5.3.2 of correlated ghost-spin states (5.56) and their entanglement entropy (5.57) are of this form, written explicitly. The state (5.106) is akin to the thermofield double dual to the eternal AdS black hole [226]. This suggests the speculation that 4-dim de Sitter space is perhaps approximately dual to $CFT_F \times CFT_P$ in the entangled state (5.106) and the generalized entanglement entropy of the latter scales as de Sitter entropy. (See [227] for another approach to de Sitter entropy based on the dS/dS correspondence [228].)

The investigations in this chapter on N-level generalizations of ghost-spins are geared towards constructing microscopic ghost-spin states that reflect the N-level internal structure which might ultimately give rise in appropriate continuum limits to theories such as the $Sp(N)$ ghost-CFT dual to higher spin dS_4 (see also the recent work [210]). As we have seen, the N-level generalizations here do admit positive norm subsectors of the form of the correlated ghost-spin states indicated in (5.106).

As mentioned in the Introduction, the ghost-spin system has possible applications in gauge theories. The continuum limit of a d-dim ghost-spin system (as in [48] for the 2-dim case) with flavour quantum numbers may be relevant for studying entanglement in gauge theories in a covariant setting. A better understanding of the ghost-spin system and its coupling to ordinary spin systems as in [47, 48] generalized to d-dimensions would be an ideal sandbox for understanding covariant formulations of subregion entanglement in gauge theories.

We have been thinking of ghost-spins as microscopic building blocks for ghost-like CFTs, and perhaps more general non-unitary CFTs. The discussions in this chapter on ghost-spin chains have recovered 2-dim bc-ghost CFTs with flavour symmetries. The obvious generalization to 3 dimensions of the 2-dim case discussed in sec.3 (and in [48] for the single flavour case) has nearest neighbour hopping-type interactions of the elemental form $h \sim \sum_{nn'} \sigma^A_{b,\vec{n}} \sigma^A_{c,\vec{n}'}$. This contains three σ_b and three σ_c operators at each lattice site, with possible flavour indices reflecting internal flavour symmetries. It would be interesting to study such 3-dim ghost-spin chains towards obtaining 3-dim ghost-CFTs in the continuum limit: so far, we have encountered conceptual difficulties as well as technical ones. We hope to report on this in the future.

Chapter 6

Conclusions

In this thesis, we have focused on certain aspects of holography: mainly $nAdS_2$ holography in certain models of dilaton-gravity theories and non-relativistic holography in the context of hydrodynamics for hyperscaling violating Lifshitz theories. In an independent study, we have also investigated entanglement properties of N-level generalizations of the 2-level ghost-spins. In this chapter, we summarize our investigations and key results, presented in the earlier chapters and also mention some possible future directions.

In chapter 2, we have studied the shear diffusion and the ratio of shear viscosity to entropy density for uncharged finite temperature hyperscaling violating Lifshitz (hvLif) spacetimes. Building upon our earlier investigations [161] adapting the membrane-paradigm like analysis [162], we have analyzed the metric and gauge field perturbations h_{ty}, h_{xy}, a_y in the near horizon region of the hvLif black brane. In the presence of a_y coupled to h_{ty} and h_{xy}, we have seen that $\tilde{h}_{xy} \equiv h_{xy}$ and $\tilde{h}_{ty} \equiv h_{ty} - r^{\theta-2} \int_{r_c}^{r} ds\, s^{3-z-\theta} a_y$ are the correct variables, in which, the relevant component of the linearized Einstein's equation becomes the diffusion equation. We have obtained a formula for the shear diffusion constant $\mathcal{D}$ in terms of the metric components and thus have computed the ratio of shear viscosity to entropy density $\frac{\eta}{s}$. For $z < 4 - \theta$, $\mathcal{D}$ has a power-law scaling with temperature and $\frac{\eta}{s}$ saturates the viscosity bound *i.e.* $\frac{\eta}{s} = \frac{1}{4\pi}$, while for $z = 4 - \theta$, $\mathcal{D}$ scales logarithmically with temperature. The hvLif theories satisfying $z = 4 - \theta$ arise in null reductions of highly boosted black branes (AdS plane waves) and nonconformal brane plane waves. It would be interesting to investigate the null reduction of the hydrodynamics of the boosted black branes.

In chapter 3, we have studied 2-dimensional dilaton-gravity-matter theories arising from reductions of certain families of extremal charged black branes in Einstein-Maxwell and hyperscaling violating Lifshitz (hvLif) theories in 4-dimensions. Charged hvLif black branes are solutions to Einstein-Maxwell-scalar theories with an additional $U(1)$ gauge field. The near horizon geometry of these extremal branes is $AdS_2 \times \mathbb{R}^2$ and compactifying the transverse space as a torus T^2 leads to an effective 2-dimensional theory of dilaton-gravity coupled to the scalar field and the gauge fields. We have argued that this 2-dimensional theory is equivalent to a dilaton-gravity-scalar theory with an interaction potential for the dilaton and scalar field. The perturbation analysis around the AdS_2 background with constant dilaton and constant scalar field reveals that the leading correction is governed by the Jackiw-Teitelboim theory. This occurs at the linear order in the dilaton perturbation resulting in the Schwarzian derivative action from the Gibbons-Hawking term. There are subleading corrections at the quadratic and higher orders, which encode information on the higher dimensional realization of this AdS_2 background. We have also done the above analysis in detail for the reduction of relativistic black branes in Einstein-Maxwell theory to a 2-dimensional dilaton-gravity theory. This is a simple subcase with $z = 1$, $\theta = 0$ and the absence of hvLif scalar field, in this case, simplifies the analysis.

In chapter 4, we have considered a generalized class of 2-dimensional dilaton-gravity-scalar theories, generalizing the reduction from 4-dimensional theories in chapter 3 to higher dimensions. Extremal black branes in these higher dimensional theories upon compactification in the near horizon throat region give rise to AdS_2 dilaton-gravity-scalar theories. Away from the throat region, these background have non-trivial profiles. We have interpreted this as holographic renormalization group (RG) flows that end at an AdS_2 fixed point in the IR. We have defined a holographic c-function in terms of the 2-dimensional dilaton, and have shown its monotonicity using the null energy conditions and appropriate boundary conditions on the ultraviolet theory, thereby proving a holographic c-theorem. At the IR AdS_2 fixed point, this dilatonic c-function becomes the extremal black brane entropy. We have discussed this c-function in detail in compactified conformal and non-conformal branes in $M2-D2$ and $M5-D4$ systems. We have also compared the dilatonic c-function with other holographic c-functions. In particular, comparing with the entropic c-function, which is defined in terms of the entanglement entropy, we have seen that the dilatonic c-function, in addition to scaling as the entropic c-function, is extensive and scales as the transverse area (of the compact space). Reduction of the null energy conditions (NECs) in higher dimensional theories to 2-dimensions shows that one of them reduces to a NEC in 2-dimensions, while the second NEC reduces to a non-trivial constraint on the 2-dimensional effective potential and its derivatives. Finally, we have also adapted the holographic RG formulation of [42] to our 2-dimensional dilaton-gravity-scalar theories. In this radial Hamiltonian formulation, imposing a radial Hamiltonian constraint gives RG flow equations and β functions for the couplings (boundary values of the dilaton and scalar fields) in the 1-dimensional boundary theory in a derivative expansion. Although this formulation is not Wilsonian, it gives qualitative insight into the flow equations and β-functions.

For the 2-dimensional dilaton-gravity theories studied in chap. 3 and chap. 4, it would be interesting to analyze the spectrum of correlation functions. This requires a systematic treatment of subleading terms beyond the Schwarzian, by including suitable counterterms and employing holographic renormalization. Another interesting project would be to adapt the Wilsonian formulations of holographic RG [70, 71] to the 2-dimensional theories discussed above.

In chapter 5, we have constructed generalizations of the 2-level ghost-spin in [46, 47, 48] to N-level ghost-spins. The $O(N)$ flavoured generalization comprises of N copies of the 2-level ghost-spin with $O(N)$ flavour symmetry. We have also considered a spin-glass type coupling in flavour space. The symplectic generalization has antisymmetric inner products between the elemental ghost-spins with a $Sp(N)$ symmetry. Finally, we have constructed an irreducible ghost-spin with N internal levels. We have studied entanglement entropy in these N-level systems. Although the entanglement patterns appear to be complicated, we have found subfamilies of states which have correlated ghost-spins, where positive norm states give positive entanglement entropy. We have also analyzed a one-dimensional chain of $O(N)$ flavoured ghost-spins and in the continuum limit have obtained the 2-dimensional bc-ghost CFT having $O(N)$ flavour symmetry. A similar analysis constructing a chain of the $Sp(N)$ flavoured ghost-spins and their continuum limit is an interesting project to pursue.

Appendix A

Appendix to Chapter 2

A.1 Spatial compactification of the hyperscaling violating Lifshitz theory

The action (2.3) in 4 bulk dimensions (*i.e.* $d = 3$) becomes

$$S = \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\Big(\mathcal{R} - \frac{1}{2}\partial_M\Psi\partial^M\Psi - \frac{Z(\Psi)}{4}F_{MN}F^{MN} + V(\Psi)\Big) \ . \tag{A.1}$$

Compactifying the y-direction (where y is one of the spatial dimensions enjoying translation invariance), we write the perturbed metric in a form suitable for dimensional reduction as

$$ds^2 = g_{MN}dx^M dx^N = \hat{g}_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}(dy + \mathcal{A}_\mu dx^\mu)^2 \ , \tag{A.2}$$

where $\mathcal{A}_\mu \propto h_{\mu y}$. Indices μ, ν run over the 3-dimensional coordinates t, r, x, while the indices M, N run over t, r, x, y. We also decompose the perturbed gauge field A_M as

$$A_M = \begin{bmatrix} A_\mu \\ a_y \equiv \chi \end{bmatrix} . \tag{A.3}$$

The gravity sector under compactification becomes

$$S_{grav} = \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\,\mathcal{R} = \frac{1}{16\pi G_3}\int d^3x\, e^\sigma\sqrt{-\hat{g}}\Big(\hat{\mathcal{R}} - \frac{e^{4\sigma}}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\Big) \ , \tag{A.4}$$

where $\hat{\mathcal{R}}$ is the Ricci scalar for the metric $\hat{g}_{\mu\nu}$. The Maxwell action after the y-compactification can be written as

$$\begin{aligned} S_{gauge} &= \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\Big(-\frac{Z}{4}F_{MN}F^{MN}\Big) \\ &= \frac{1}{16\pi G_3}\int d^3x\left(-\frac{e^{\sigma}Z\sqrt{-\hat{g}}}{4}\right)\left[\hat{g}^{\mu\rho}\hat{g}^{\nu\delta}F_{\mu\nu}F_{\rho\delta}+4\hat{g}^{\nu\rho}F_{\mu\nu}\mathcal{A}^{\mu}(\partial_\rho\chi)\right. \\ &\qquad \left.+2\hat{g}^{\mu\nu}(e^{-2\sigma}+\mathcal{A}_\rho\mathcal{A}^\rho)(\partial_\mu\chi)(\partial_\nu\chi)-2\mathcal{A}^\mu\mathcal{A}^\nu(\partial_\mu\chi)(\partial_\nu\chi)\right] . \end{aligned} \tag{A.5}$$

A Weyl transformation $\tilde{g}_{\mu\nu} = e^{2\sigma}\hat{g}_{\mu\nu}$ enables us to write the gravitational and Maxwell sector of action after compactification in the Einstein frame as

$$\begin{aligned} S_{grav}+S_{Max} &= \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\Big(\mathcal{R}-\frac{Z(\Psi)}{4}F_{MN}F^{MN}\Big) \\ &= \frac{1}{16\pi G_3}\int d^3x\sqrt{-\tilde{g}}\left(\tilde{\mathcal{R}}-\frac{e^{4\sigma}}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right. \\ &\quad + Ze^{2\sigma}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-F_{\mu}{}^{\rho}\mathcal{A}^{\mu}(\partial_\rho\chi)-\frac{e^{-4\sigma}}{2}(\partial_\mu\chi)(\partial^\mu\chi)\right. \\ &\quad \left.\left.-\frac{1}{2}\mathcal{A}_\rho\mathcal{A}^\rho(\partial_\mu\chi)(\partial^\mu\chi)+\frac{1}{2}\mathcal{A}^\mu\mathcal{A}^\nu(\partial_\mu\chi)(\partial_\nu\chi)\right)\right) , \end{aligned} \tag{A.6}$$

where $\tilde{\mathcal{R}}$ is the Ricci scalar of the 3-dimensional bulk metric $\tilde{g}_{\mu\nu}$. The terms appearing in the last line of the above equation will not contribute to the equations of motion at linearized order since they appear at quartic order in the action. Varying the above action w.r.t. the field $\mathcal{A}_\mu$, at linearized level, we get

$$\frac{1}{\sqrt{-\tilde{g}}}\partial_\mu\left(\sqrt{-\tilde{g}}e^{4\sigma}\mathcal{F}^{\mu\nu}\right) = e^{2\sigma}Z\tilde{g}^{\mu\nu}\tilde{g}^{\rho\delta}F_{\mu\delta}(\partial_\rho\chi) , \tag{A.7}$$

which for $\nu = t$, $\nu = x$ and $\nu = r$ gives (2.23), (2.24) and (2.25) respectively.

A.2 Solutions to linearized equations for h_{ty}, a_y, h_{xy} at zero momentum and zero frequency

At $q = 0$, $\omega = 0$, the linearized equations of motion (2.15)-(2.18) reduce to

$$\partial_r(r^{5-z-\theta}f\partial_r a_y) - \alpha(2+z-\theta)\partial_r(r^{2-\theta}h_{ty}) = 0 , \tag{A.8}$$

$$\partial_r(r^{z+\theta-3}\partial_r(r^{2-\theta}h_{ty})) - \alpha(2+z-\theta)\partial_r a_y = 0 , \tag{A.9}$$

$$\partial_r(r^{-1-z+\theta}f\partial_r(r^{2-\theta}h_{xy})) = 0 . \tag{A.10}$$

For the sake of brevity, from now on we will denote the ∂_r operator with a prime "$'$" on the functions. Integrating (A.8) and substituting $\partial_r a_y$ in (A.9) gives

$$\begin{aligned} & f(r)[r^2 h''_{ty} + (1+z-\theta) r h'_{ty} - (\theta-2)(z-2) h_{ty}] \\ & \qquad -2(z-1)(2+z-\theta) h_{ty} = -\alpha^2 (2+z-\theta)^2 c_1 r^{\theta-2} \, , \end{aligned} \tag{A.11}$$

where we have chosen the integration constant as $-\alpha(2+z-\theta)c_1$. This inhomogeneous equation has a particular solution $h_{ty} = c_1 r^{\theta-2}$. The homogeneous part of the above equation,

$$f(r)[r^2 h''_{ty} + (1+z-\theta) r h'_{ty} - (\theta-2)(z-2) h_{ty}] - 2(z-1)(2+z-\theta) h_{ty} = 0 \tag{A.12}$$

can be solved by substituting a series ansatz, $h_{ty} = \sum_{n=0}^{\infty} c_n r^{m+n}$. Along with the two linearly independent homogeneous solutions, the complete solution (including the particular solution) is

$$\begin{aligned} h_{ty} &= c_1 r^{\theta-2} + c_3 r^{\theta-2z} f \\ & \quad + c_4 r^z \Big[1 + \frac{(z-1)(r_0 r)^{2+z-\theta}}{(1+2z-\theta)} \, {}_2F_1\Big(1, \frac{3z-\theta}{2+z-\theta}, \frac{4+5z-3\theta}{2+z-\theta}; (r_0 r)^{2+z-\theta}\Big)\Big]. \end{aligned} \tag{A.13}$$

Substituting h_{ty} from the above expression in (A.9) and integrating, we get

$$\begin{aligned} a_y &= -\frac{C}{k} - \alpha c_3 r^{-(2+z-\theta)} \\ & \quad + c_4 \Big[\frac{r^{2z-2}}{\alpha} + \alpha \frac{r_0^{2+z-\theta} r^{1+3z-\theta}}{2(1+2z-\theta)} \, {}_2F_1{}' \Big(1, \frac{3z-\theta}{2+z-\theta}, \frac{4+5z-3\theta}{2+z-\theta}; (r_0 r)^{2+z-\theta}\Big) \\ & \qquad + \alpha \frac{(2+z-\theta)}{(1+2z-\theta)} r_0^{2+z-\theta} r^{3z-\theta} \, {}_2F_1\Big(1, \frac{3z-\theta}{2+z-\theta}, \frac{4+5z-3\theta}{2+z-\theta}; (r_0 r)^{2+z-\theta}\Big)\Big], \end{aligned} \tag{A.14}$$

where ${}_2F_1{}' = \frac{d}{dr}({}_2F_1)$. ${}_2F_1{}' \left(1, \frac{3z-\theta}{2+z-\theta}, \frac{4+5z-3\theta}{2+z-\theta}; (r_0 r)^{2+z-\theta}\right)$ in the second line above is in fact divergent at the horizon $r = \frac{1}{r_0}$. Integrating (A.10), we get

$$h_{xy} = b_1 r^{\theta-2} \log(1 - (r_0 r)^{2+z-\theta}) + b_2 r^{\theta-2} \, . \tag{A.15}$$

Appendix B

Appendix to chapter 3

B.1 Some details

Relativistic electric black brane: The Einstein equation and the dilaton equation from the action (3.16) are

$$g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}\Big(2\Lambda\Phi + \frac{16\pi G_2 V_2 \Phi^3 F_{\mu\nu}F^{\mu\nu}}{4}\Big) - \frac{16\pi G_2 V_2\Phi^3}{2}F_{\mu\rho}F_\nu{}^\rho = 0\,,$$
$$\mathcal{R} - \frac{\Lambda}{\Phi} - (6\pi G_2)V_2\Phi F_{\mu\nu}F^{\mu\nu} = 0\ . \tag{B.1}$$

<u>**Charged hvLif black brane**</u>

Effective scalar potential in 4-dimensional hvLif black brane and its derivatives: The first and second derivatives of the effective scalar potential in 4-dimensional charged hvLif black brane are

$$\frac{\partial V_{eff}}{\partial\Psi} = -\frac{\gamma(2+z-\theta)(1+z-\theta)e^{\gamma(\Psi-\Psi_0)}}{R^{2-2\theta}r_{hv}^{2\theta}} - \frac{1}{g_{xx}^2}\Big(\frac{\lambda_1(z-1)(2+z-\theta)r_{hv}^{2\theta-4}R^{2-2\theta}}{e^{\lambda_1(\Psi-\Psi_0)}} + \frac{\lambda_2(2-\theta)(z-\theta)Q^2 r_{hv}^{2z-2}R^{-4z-2+2\theta}}{e^{\lambda_2(\Psi-\Psi_0)}}\Big), \tag{B.2}$$

$$\frac{\partial^2 V_{eff}}{\partial\Psi^2} = -\frac{\gamma^2(2+z-\theta)(1+z-\theta)e^{\gamma(\Psi-\Psi_0)}}{R^{2-2\theta}r_{hv}^{2\theta}} + \frac{1}{g_{xx}^2}\Big(\frac{\lambda_1^2(z-1)(2+z-\theta)r_{hv}^{2\theta-4}R^{2-2\theta}}{e^{\lambda_1(\Psi-\Psi_0)}} + \frac{\lambda_2^2(2-\theta)(z-\theta)Q^2 r_{hv}^{2z-2}R^{-4z-2+2\theta}}{e^{\lambda_2(\Psi-\Psi_0)}}\Big). \tag{B.3}$$

Differentiating V_{eff} n times, we get

$$\frac{\partial^n V_{eff}}{\partial \Psi^n} = -\frac{\gamma^n(2+z-\theta)(1+z-\theta)e^{\gamma(\Psi-\Psi_0)}}{R^{2-2\theta}r_{hv}^{2\theta}} + \frac{(-)^n}{g_{xx}^2}\Big(\frac{\lambda_1^n(z-1)(2+z-\theta)r_{hv}^{2\theta-4}R^{2-2\theta}}{e^{\lambda_1(\Psi-\Psi_0)}} + \frac{\lambda_2^n(2-\theta)(z-\theta)Q^2 r_{hv}^{2z-2}R^{-4z-2+2\theta}}{e^{\lambda_2(\Psi-\Psi_0)}}\Big), \tag{B.4}$$

which at the extremal point becomes

$$\frac{\partial^n V_{eff}}{\partial \Psi^n} = \frac{r_0^\theta(2+z-\theta)}{r_{hv}^\theta R^2}\Big[\frac{-\theta^n(1+z-\theta)+(-)^n(\theta-4)^n(z-1)}{(2-\theta)^{\frac{n}{2}}(2z-2-\theta)^{\frac{n}{2}}} + \frac{(-)^n(2z-2-\theta)^{\frac{n}{2}}(2-\theta)}{(2-\theta)^{\frac{n}{2}}}\Big]. \tag{B.5}$$

At $z=1$, $\theta \neq 0$, we see that $\frac{\partial^n V_{eff}}{\partial \Psi^n} = 0 \;\forall\; n$ at the extremal point.

Dimensional reduction to 2-dimensions: The 2-dim action obtained by reducing (3.43) on T^2 is (retaining only fields with background profiles)

$$S = \int d^2x\sqrt{-g^{(2)}}\Big[\frac{1}{16\pi G_2}\Big(\Phi^2\mathcal{R}^{(2)} + 2\partial_\mu\Phi\partial^\mu\Phi - \frac{\Phi^2}{2}\partial_\mu\Psi\partial^\mu\Psi + V\Phi^2 - \frac{\Phi^2}{4}Z_1F_{1\,\mu\nu}F_1^{\mu\nu}\Big) - \frac{V_2\Phi^2}{4}Z_2F_{2\,\mu\nu}F_2^{\mu\nu}\Big]\,, \tag{B.6}$$

Equations of motion from 2-dimensional action (3.59): The equations of motion obtained by varying the action (3.59) are

$$\begin{aligned}
&g_{\mu\nu}\nabla^2\Phi^2 - \nabla_\mu\nabla_\nu\Phi^2 + \frac{g_{\mu\nu}}{2}\Big(\frac{\Phi^2}{2}(\partial\Psi)^2 - V\Phi + \frac{\Phi^3}{4}(Z_1(F_1)^2 + 16\pi G_2V_2Z_2(F_2)^2)\Big) \\
&\quad -\frac{\Phi^2}{2}\partial_\mu\Psi\partial_\nu\Psi - \frac{\Phi^3}{2}(Z_1F_{1\,\mu\rho}F_{1\nu}{}^\rho + 16\pi G_2V_2Z_2F_{2\,\mu\rho}F_{2\nu}{}^\rho) = 0\,, \\
&\mathcal{R} - \frac{1}{2}(\partial\Psi)^2 + \frac{V}{2\Phi} - \frac{3}{8}\Phi(Z_1(F_1)^2 + 16\pi G_2V_2Z_2(F_2)^2) = 0\,, \\
&\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\Phi^2\partial^\mu\Psi) + \gamma V\Phi - \frac{\Phi^3}{4}(\lambda_1Z_1(F_1)^2 + \lambda_2 16\pi G_2V_2Z_2(F_2)^2) = 0\,.
\end{aligned} \tag{B.7}$$

The equations of motion (3.62) in conformal gauge and in lightcone coordinates are

$$\begin{aligned}
-e^{2\omega}\partial_\pm(e^{-2\omega}\partial_\pm\Phi^2) - \frac{\Phi^2}{2}\partial_\pm\Psi\partial_\pm\Psi &= 0 \ , \\
\partial_+\partial_-\Phi^2 - \frac{e^{2\omega}}{4}U &= 0 \ , \\
4\partial_+\partial_-\omega + \partial_+\Psi\partial_-\Psi - \frac{e^{2\omega}}{2}\frac{\partial U}{\partial(\Phi^2)} &= 0 \ , \\
\partial_+(\Phi^2\partial_-\Psi) + \partial_-(\Phi^2\partial_+\Psi) + \frac{e^{2\omega}}{2}\frac{\partial U}{\partial\Psi} &= 0 \ .
\end{aligned} \tag{B.8}$$

Expanding the constraint equations in the first line of (B.8) to linear order in perturbations (3.69) gives

$$\partial_\pm\partial_\pm\phi \pm \frac{2}{(x^+ - x^-)}\partial_\pm\phi = 0 \ , \tag{B.9}$$

the other terms vanishing at linear order. To see that these linearized constraint equations are consistent with the linearized equations (3.73), we differentiate the $++$ constraint equation with respect to x^- to get

$$\partial_+(\partial_+\partial_-\phi) + \frac{2}{(x^+ - x^-)}\partial_+\partial_-\phi + \frac{2}{(x^+ - x^-)^2}\partial_+\phi = 0 \ , \tag{B.10}$$

which is satisfied after using the equation for ϕ in (3.73). Similarly differentiating the $--$ constraint equation with respect to x^+, we can show that the resulting equation is satisfied upon substituting the equation for ϕ in (3.73).

B.2 Extrinsic curvature and the Schwarzian

Consider the boundary of AdS_2 as a slightly deformed curve $(\tau(u), \rho(u))$ parametrized by the boundary coordinate time u, where τ is the Euclidean time. The tangent T^μ and the normal n^μ to this boundary curve are

$$T^\mu = (\tau', \rho') \ , \qquad n^\mu = \frac{\rho}{L\sqrt{\tau'^2 + \rho'^2}}(-\rho', \tau') \ , \tag{B.11}$$

where prime denotes derivative with respect to u *i.e.* $\tau' = \frac{d\tau}{du}$, etc. The extrinsic curvature is given by

$$K = -\frac{T^\mu T^\nu \nabla_\nu n_\mu}{T^\mu T_\mu} \ . \tag{B.12}$$

For the Euclidean AdS_2 metric in Poincaré coordinates, $ds^2 = \frac{L^2}{\rho^2}(d\tau^2 + d\rho^2)$, we have

$$\Gamma^{\tau}_{\tau\rho} = -\frac{1}{\rho} \ , \qquad \Gamma^{\rho}_{\tau\tau} = \frac{1}{\rho} \ , \qquad \Gamma^{\rho}_{\rho\rho} = -\frac{1}{\rho} \ . \tag{B.13}$$

Then using $\partial_u = \tau'\partial_\tau + \rho'\partial_\rho$, we can compute

$$\begin{aligned}
T_\mu T^\nu \nabla_\nu n^\mu &= \frac{L^2}{\rho^2}[\tau'^2\nabla_\tau n^\tau + \tau'\rho'(\nabla_\tau n^\rho + \nabla_\rho n^\tau) + \rho'^2\nabla_\rho n^\rho] \\
&= \frac{L^2}{\rho^2}[\tau'(\tau'\partial_\tau n^\tau + \rho'\partial_\rho n^\tau) + \rho'(\tau'\partial_\tau n^\rho + \rho'\partial_\rho n^\rho) + \tau'^2\Gamma^\tau_{\tau\rho}n^\rho \\
&\quad + \rho'^2\Gamma^\rho_{\rho\rho}n^\rho + \tau'\rho' n^\tau(\Gamma^\rho_{\tau\tau} + \Gamma^\tau_{\rho\tau})] \\
&= \frac{L^2}{\rho^2}[\tau'\partial_u n^\tau + \rho'\partial_u n^\rho - (\tau'^2+\rho'^2)\frac{n^\rho}{\rho}] \\
&= \frac{L}{\rho^2(\tau'^2+\rho'^2)^{\frac{3}{2}}}\Big[-\tau'\rho'^2(\tau'^2+\rho'^2) - \rho\rho''\tau'^3 + \rho\rho'\tau'^2\tau'' \\
&\quad + \tau'\rho'^2(\tau'^2+\rho'^2) + \rho\tau''\rho'^3 - \rho\rho'^2\rho''\tau'\Big] - \frac{(\tau'^2+\rho'^2)\rho\tau'}{\rho^3(\tau'^2+\rho'^2)^{\frac{1}{2}}} \\
&= \frac{L}{\rho^2(\tau'^2+\rho'^2)^{\frac{1}{2}}}[-\rho\rho''\tau' + \rho\tau''\rho' - \tau'(\tau'^2+\rho'^2)]
\end{aligned} \tag{B.14}$$

and the extrinsic curvature becomes

$$K = -\frac{T^\mu T^\nu \nabla_\nu n_\mu}{T^\mu T_\mu} = \frac{\rho^2}{L^2(\tau'^2+\rho'^2)}(-T^\mu T^\nu \nabla_\nu n_\mu) = \frac{\tau'(\tau'^2+\rho'^2+\rho\rho'') - \rho\rho'\tau''}{L(\tau'^2+\rho'^2)^{\frac{3}{2}}} \ . \tag{B.15}$$

The induced metric on the boundary of AdS_2 is given by

$$g_{uu} = \frac{L^2}{\epsilon^2} = L^2\frac{(\tau'^2+\rho'^2)}{\rho^2} \ , \tag{B.16}$$

where ϵ is arbitrarily small. Solving this equation in orders of ϵ gives

$$\rho = \epsilon\tau' + O(\epsilon^3) \ . \tag{B.17}$$

Substituting this expression for ρ in (B.15), we get

$$\begin{aligned}
K &= \frac{[\,\tau'^3 + \tau'(\epsilon^2\tau''^2) + \tau'(\epsilon\tau')(\epsilon\tau''') - \epsilon^2\tau'\tau''^2\,]}{L\,\tau'^3}\Big(1 - \frac{3\rho'^2}{2\tau'^2}\Big) \\
&= \frac{1}{L}\Big[\Big(1 + \epsilon^2\frac{\tau'''}{\tau'}\Big)\Big(1 - \frac{3\,\epsilon^2}{2}\frac{\tau''^2}{\tau'^2}\Big) + O(\epsilon^4)\Big] \\
&= \frac{1}{L}\Big[1 + \epsilon^2\Big(\frac{\tau'''}{\tau'} - \frac{3}{2}\frac{\tau''^2}{\tau'^2}\Big) + O(\epsilon^4)\Big] \\
&= \frac{1}{L}\Big[1 + \epsilon^2\, Sch(\tau(u),u) + O(\epsilon^4)\Big] \ ,
\end{aligned} \tag{B.18}$$

where $Sch(\tau(u),u) \equiv \{\tau(u),u\} = \frac{\tau'''}{\tau'} - \frac{3}{2}\frac{\tau''^2}{\tau'^2}$ is the Schwarzian derivative action.

Substituting the expression for K in the Gibbons-Hawking term, using (3.37) and defining $\tilde{\phi} = \frac{\phi_r(u)}{\epsilon}$, we get

$$\begin{aligned} S_{GH} &= -\frac{1}{8\pi G_2}\int \frac{du}{\epsilon}\frac{2\Phi_b\phi_r}{\epsilon}\Big(1+\epsilon^2\, Sch(\tau(u),u)+O(\epsilon^4)\Big) \\ &= -\frac{2\Phi_b}{8\pi G_2}\int du\frac{\phi_r}{\epsilon^2} - \frac{2\Phi_b}{8\pi G_2}\int du\,\phi_r\, Sch(\tau(u),u)+O(\epsilon^2)\ , \end{aligned} \tag{B.19}$$

where the $\frac{1}{\epsilon^2}$ term cancels with the background term coming from expansion of the bulk action and the second term gives the Schwarzian derivative action at leading linear order in the dilaton perturbation.

B.3 Einstein's equation in 2-dimensions

Consider a dilaton-gravity-scalar action in 2-dimensions

$$S = \frac{1}{16\pi G_2}\left(\int d^2x\sqrt{-g}\left[\Phi^2\mathcal{R} - U(\Phi,\Psi) - \frac{\Phi^2}{2}\partial_\mu\Psi\partial^\mu\Psi\right] + 2\int dt\sqrt{-\gamma}\,\Phi^2\, K\right). \tag{B.20}$$

Varying the action with respect to the metric, we get

$$\begin{aligned} \delta S = \frac{1}{16\pi G_2}&\Bigg(\int d^2x\left(\frac{g\, g_{\mu\nu}}{2\sqrt{-g}}\delta g^{\mu\nu}\left[\Phi^2\mathcal{R} - U - \frac{\Phi^2}{2}\partial_\rho\Psi\partial^\rho\Psi\right]\right. \\ &\left. +\sqrt{-g}\left[\Phi^2\mathcal{R}_{\mu\nu}\delta g^{\mu\nu} + \Phi^2 g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} - \frac{\Phi^2}{2}\partial_\mu\Psi\partial_\nu\Psi\,\delta g^{\mu\nu}\right]\right) \\ &+2\int dt\,\delta(\sqrt{-\gamma}\,K)\Phi^2\Bigg) \\ = \frac{1}{16\pi G_2}&\int d^2x\sqrt{-g}\Big(\Phi^2\mathcal{G}_{\mu\nu} - \frac{\Phi^2}{2}\partial_\mu\Psi\partial_\nu\Psi + \frac{g_{\mu\nu}}{2}\Big[U+\frac{\Phi^2}{2}\partial_\rho\Psi\partial^\rho\Psi\Big]\Big)\delta g^{\mu\nu} \\ +\frac{1}{16\pi G_2}&\Big(\int d^2x\sqrt{-g}\,\Phi^2 g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} + 2\int dt\,\delta(\sqrt{-\gamma}\,K)\Phi^2\Big)\ , \end{aligned} \tag{B.21}$$

where $\mathcal{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{g_{\mu\nu}}{2}\mathcal{R}$ is the Einstein tensor and in 2-dimensions $\mathcal{G}_{\mu\nu} = 0$ identically. Using $g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} = \nabla^\mu[\nabla^\nu(\delta g_{\mu\nu}) - g^{\rho\sigma}\nabla_\mu(\delta g_{\rho\sigma})] \equiv \nabla^\mu v_\mu$, we get

$$\begin{aligned} \int d^2x\sqrt{-g}\,\Phi^2 g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} &= \int d^2x\sqrt{-g}\,\Phi^2\,\nabla^\mu v_\mu \\ &= \int d^2x\sqrt{-g}\,\nabla^\mu(\Phi^2 v_\mu) - \int d^2x\sqrt{-g}(\nabla^\mu\Phi^2)v_\mu \\ &= \int dt\sqrt{-\gamma}\,n^\mu\,\Phi^2 v_\mu - \int d^2x\sqrt{-g}(\nabla^\mu\Phi^2)v_\mu\ . \end{aligned} \tag{B.22}$$

Substituting this expression in (B.21), we see that the boundary term $\int dt\sqrt{-\gamma}\, n^\mu\, \Phi^2 v_\mu$ cancels the variation of the Gibbons-Hawking term $2\int dt\, \delta(\sqrt{-\gamma}\, K)\Phi^2$. Then (B.21) reduces to

$$\begin{aligned}\delta S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\Big(-\frac{\Phi^2}{2}\partial_\mu\Psi\partial_\nu\Psi + \frac{g_{\mu\nu}}{2}\Big[U + \frac{\Phi^2}{2}\partial_\rho\Psi\partial^\rho\Psi\Big]\Big)\delta g^{\mu\nu} \\ -\frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\,(\nabla^\mu\Phi^2)\, v_\mu\ .\end{aligned} \tag{B.23}$$

Let us simplify the term in the second line above. Using the formula for v_μ, we have

$$\int d^2x\sqrt{-g}g^{\mu\nu}(\partial_\mu\Phi^2)v_\nu = \int d^2x\sqrt{-g}g^{\mu\nu}(\partial_\mu\Phi^2)[\nabla^\nu(\delta g_{\mu\nu}) - g^{\rho\sigma}\nabla_\mu(\delta g_{\rho\sigma})]\ . \tag{B.24}$$

To simplify further we use the following expressions.

$$\begin{aligned}\delta g^{\rho\sigma} &= \delta(g_{\mu\nu}g^{\mu\rho}g^{\nu\sigma}) = (\delta g_{\mu\nu})g^{\mu\rho}g^{\nu\sigma} + 2\delta g^{\rho\sigma} \\ \implies \quad \delta g^{\rho\sigma} &= -(\delta g_{\mu\nu})g^{\mu\rho}g^{\nu\sigma}\ .\end{aligned} \tag{B.25}$$

Using the formula for covariant derivative $\nabla_\sigma(\delta g_{\nu\rho}) = \partial_\sigma(\delta g_{\nu\rho}) - \Gamma^\alpha_{\nu\sigma}\delta g_{\alpha\rho} - \Gamma^\alpha_{\rho\sigma}\delta g_{\nu\alpha}$, we can write

$$\begin{aligned}\nabla^\rho(\delta g_{\nu\rho}\nabla^\nu\Phi^2) &= g^{\rho\sigma}\partial_\sigma(\delta g_{\nu\rho}\nabla^\nu\Phi^2) - g^{\rho\sigma}\Gamma^\alpha_{\rho\sigma}\delta g_{\nu\alpha}\nabla^\nu\Phi^2 \\ &= \nabla^\rho(\delta g_{\nu\rho})\nabla^\nu\Phi^2 + \nabla^\rho(\nabla^\nu\Phi^2)\delta g_{\nu\rho}\ ,\end{aligned} \tag{B.26}$$

$$\begin{aligned}&\nabla_\nu(g^{\rho\sigma}\delta g_{\rho\sigma}\nabla^\nu\Phi^2) \\ &\quad = g^{\rho\sigma}\nabla_\nu(\delta g_{\rho\sigma}\nabla^\nu\Phi^2) \\ &\quad = g^{\rho\sigma}[\partial_\nu(\delta g_{\rho\sigma}\nabla^\nu\Phi^2) - \Gamma^\alpha_{\rho\nu}\delta g_{\alpha\sigma}\nabla^\nu\Phi^2 - \Gamma^\alpha_{\sigma\nu}\delta g_{\rho\alpha}\nabla^\nu\Phi^2 + \Gamma^\nu_{\alpha\nu}\delta g_{\rho\sigma}\nabla^\alpha\Phi^2] \\ &\quad = g^{\rho\sigma}\delta g_{\rho\sigma}\nabla_\nu\nabla^\nu\Phi^2 + g^{\rho\sigma}[\partial_\nu(\delta g_{\rho\sigma}) - \Gamma^\alpha_{\rho\nu}\delta g_{\alpha\sigma} - \Gamma^\alpha_{\sigma\nu}\delta g_{\rho\alpha}]\nabla^\alpha\Phi^2 \\ &\quad = g^{\rho\sigma}\delta g_{\rho\sigma}\nabla^2\Phi^2 + g^{\rho\sigma}\nabla_\nu(\delta g_{\rho\sigma})\nabla^\nu\Phi^2\ .\end{aligned} \tag{B.27}$$

Substituting (B.26) and (B.27) in (B.24), we get

$$\begin{aligned}(\partial^\mu\Phi^2)v_\mu &= \nabla^\nu(\delta g_{\mu\nu}\nabla^\mu\Phi^2) - \nabla^\mu\nabla^\nu\Phi^2\,\delta g_{\mu\nu} - \nabla_\mu(g^{\rho\sigma}\delta g_{\rho\sigma}\nabla^\mu\Phi^2) + g^{\rho\sigma}\delta g_{\rho\sigma}\nabla^2\Phi^2 \\ &= -\nabla_\rho(\delta g^{\rho\sigma}\nabla_\sigma\Phi^2) + \nabla_\mu\nabla_\nu\Phi^2\,\delta g^{\mu\nu} + \nabla_\mu(g_{\rho\sigma}\delta g^{\rho\sigma}\nabla^\mu\Phi^2) - g_{\rho\sigma}\delta g^{\rho\sigma}\nabla^2\Phi^2\ ,\end{aligned}$$

where we have used (B.25) in the first term. Substituting this expression in (B.23), we get

$$\delta S = \int \frac{d^2x\sqrt{-g}}{16\pi G_2}\Big(-\frac{\Phi^2}{2}\partial_\mu\Psi\partial_\nu\Psi + \frac{g_{\mu\nu}}{2}\Big[U + \frac{\Phi^2}{2}(\partial\Psi)^2\Big] - \nabla_\mu\nabla_\nu\Phi^2 + g_{\mu\nu}\nabla^2\Phi^2\Big)\delta g^{\mu\nu}$$
$$-\frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\Big[\nabla_\mu(g_{\rho\sigma}\delta g^{\rho\sigma}\nabla^\mu\Phi^2) - \nabla_\rho(\delta g^{\rho\sigma}\nabla_\sigma\Phi^2)\Big] . \quad \text{(B.28)}$$

The terms in the second line above are boundary terms, which vanish by the boundary conditions on the metric $\delta g_{\mu\nu}|_{bdy} = 0$. Then from the first line above, $\frac{\delta S}{\delta g^{\mu\nu}} = 0$ gives the field equations

$$\nabla_\mu\nabla_\nu\Phi^2 - g_{\mu\nu}\nabla^2\Phi^2 + \frac{\Phi^2}{2}\partial_\mu\Psi\partial_\nu\Psi - \frac{g_{\mu\nu}}{2}\Big[U + \frac{\Phi^2}{2}(\partial\Psi)^2\Big] = 0 \ . \quad \text{(B.29)}$$

Appendix C

Appendix to chapter 4

C.1 Effective potential in D-dim gravity-scalar theory

We derive a formula for the effective potential V in gravity scalar action (4.1) starting from gravity-scalar action coupled to $U(1)$ gauge fields. Consider the action

$$S = \frac{1}{16\pi G_D}\int d^D x\sqrt{-g^{(D)}}\Big(\mathcal{R}^{(D)} - \frac{h_{IJ}}{2}\partial_M\Psi^I\partial^M\Psi^J + V_0(\Psi) - \sum_{i=1}^{n}\frac{Z_i(\Psi)}{4}F_i^2\Big)\,, \tag{C.1}$$

where $V_0(\Psi)$ is the scalar potential, $Z_i(\Psi)$ are Ψ dependent couplings and $F_i^2 = F_{i\,MN}F_i^{MN}$. The Einstein's equations are

$$\begin{aligned}\mathcal{R}^{(D)}_{MN} - \frac{g^{(D)}_{MN}}{2}\mathcal{R}^{(D)} = {}& \frac{h_{IJ}}{2}\Big(\partial_M\Psi^I\partial_N\Psi^J - \frac{g^{(D)}_{MN}}{2}\partial_P\Psi^I\partial^P\Psi^J\Big) + \frac{g^{(D)}_{MN}}{2}V_0 \\ & + \sum_i^n \frac{Z_i}{2}\Big(F_{i\,MP}F^{P}_{i\,N} - \frac{g^{(D)}_{MN}F_i^2}{4}\Big)\,.\end{aligned} \tag{C.2}$$

Taking electric ansatz for all gauge fields and solving the Maxwell's equations

$$\partial_M(\sqrt{-g^{(D)}}Z_iF_i^{MN}) = 0 \quad \Longrightarrow \quad F_i^{tr} = \frac{\tilde{c}_i}{\sqrt{-g^{(D)}}Z_i}\,, \qquad F_i^2 = -\frac{2\,\tilde{c}_i^2}{(g^{(D)}_{xx})^{D-2}Z_i^2}\,, \tag{C.3}$$

where $\tilde{c}_i$ is constant and we have restricted to a class of metrics with $g^{(D)}_{ij} = \delta_{ij}g^{(D)}_{xx}$, consistent with the reduction ansatz (4.2). Substituting F_i^{tr} in (C.2), the $t-r$ and xx components become

$$G^{(D)}_{\mu\nu} = \frac{h_{IJ}}{2}\Big(\partial_\mu\Psi^I\partial_\nu\Psi^J - \frac{g^{(D)}_{\mu\nu}}{2}\partial_P\Psi^I\partial^P\Psi^J\Big) + \frac{g^{(D)}_{\mu\nu}}{2}\Big(V_0 - \sum_{i=1}^{n}\frac{\tilde{c}_i^2}{2(g^{(D)}_{xx})^{D-2}Z_i}\Big)\ , \quad \text{(C.4)}$$

$$G^{(D)}_{xx} = \frac{h_{IJ}}{2}\Big(\partial_x\Psi^I\partial_x\Psi^J - \frac{g^{(D)}_{xx}}{2}\partial_P\Psi^I\partial^P\Psi^J\Big) + \frac{g^{(D)}_{xx}}{2}V_0 + \frac{g^{(D)}_{xx}}{4}\frac{\tilde{c}_i^2}{(g^{(D)}_{xx})^{D-2}Z_i}\ , \quad \text{(C.5)}$$

where $G^{(D)}_{MN} = \mathcal{R}^{(D)}_{MN} - \frac{g^{(D)}_{MN}}{2}\mathcal{R}^{(D)}$ is the Einstein tensor. These Einstein equations can be derived from an equivalent gravity-scalar action (4.1), with the effective potential defined as

$$V(\Psi^I, g^{(D)}_{xx}) = -V_0(\Psi) + \sum_{i=1}^{n}\frac{V_i(\Psi)}{(g^{(D)}_{xx})^{D-2}}\ , \quad \text{(C.6)}$$

where $V_i(\Psi) \equiv \frac{\tilde{c}_i^2}{2\,Z_i(\Psi)}$.

C.2 Radial Lagrangian (4.63) from dimensional reduction

Consider the gravity scalar action in D-dimensions

$$S = \frac{1}{16\pi G_D}\Big[\int d^Dx\sqrt{-g^{(D)}}\Big(\mathcal{R}^{(D)} - \frac{h_{IJ}}{2}\partial_M\Psi^I\partial^M\Psi^J - V\Big) + \int d^{D-1}x\sqrt{-\gamma^{(D-1)}}2K^{(D-1)}\Big]\ , \quad \text{(C.7)}$$

where $\gamma^{(D-1)}_{ab}$ is the induced metric and $K^{(D-1)}$ is the extrinsic curvature on the $(D-1)$-dimensional boundary. Foliating the spacetime into surfaces of constant r,

$$ds^2 = g^{(D)}_{MN}dx^Mdx^N = (\tilde{N}^2 + \gamma^{(D-1)}_{ab}N^aN^b)dr^2 + 2\gamma^{(D-1)}_{ab}N^adx^bdr + \gamma^{(D-1)}_{ab}dx^adx^b\ , \quad \text{(C.8)}$$

the D-dim Ricci scalar decomposes as

$$\mathcal{R}^{(D)} = {}^{(D-1)}\mathcal{R} + (K^{(D-1)})^2 - K^{(D-1)}_{ab}K^{(D-1)\,ab} - 2\nabla_A(\tilde{n}^AK^{(D-1)}) + 2\nabla_A(\tilde{n}^B\nabla_B\tilde{n}^A)\ , \quad \text{(C.9)}$$

where the indices M, N take values (t, r, x^i) and a, b take values (t, x^i) for $i = 1, \ldots, D-2$. ${}^{(D-1)}\mathcal{R}$ is the Ricci scalar of the $(D-1)$-dim boundary and $\tilde{n}^A$ is the unit normal to the boundary. The total derivative terms above can be written as

$$\int d^D x\sqrt{-g^{(D)}}[-2\nabla_A(\tilde{n}^A K^{(D-1)}) + 2\nabla_A(\tilde{n}^B\nabla_B\tilde{n}^A)]$$
$$= \int d^{D-1}x\sqrt{-\gamma^{(D-1)}}[-2K^{(D-1)}\tilde{n}^A\tilde{n}_A + \tilde{n}_A\tilde{n}^B\nabla_B\tilde{n}^A]$$
$$= -\int d^{D-1}x\sqrt{-\gamma^{(D-1)}}2K^{(D-1)},$$

where we have used $\tilde{n}^A\tilde{n}_A = 1$ and $\tilde{n}_A\nabla_B\tilde{n}^A = 0$. This boundary term coming from the total derivative terms in (C.9) cancels the Gibbons-Hawking term in (C.7). Then the radial Lagrangian on the $r = constant$ boundary can be written as

$$L = \frac{1}{16\pi G_D}\int d^{D-1}x\sqrt{-\gamma^{(D-1)}}\tilde{N}\Big({}^{(D-1)}\mathcal{R} + (K^{(D-1)})^2 - K^{(D-1)}_{ab}K^{(D-1)\,ab} - \frac{h_{IJ}}{2}\partial_M\Psi^I\partial^M\Psi^J - V\Big)\,, \tag{C.10}$$

where the extrinsic curvature is

$$K^{(D-1)}_{ab} = \frac{1}{2\tilde{N}}\Big(\partial_r\gamma^{(D-1)}_{ab} - D^{(D-1)}_a N^{(D-1)}_b - D^{(D-1)}_b N^{(D-1)}_a\Big)\,,$$
$$N^{(D-1)}_a \equiv \gamma^{(D-1)}_{ab}N^b\,, \qquad D^{(D-1)}_a N^{(D-1)}_b = \partial_a N^{(D-1)}_b - \Gamma^{(D-1)\,c}_{ab}N^{(D-1)}_c\,. \tag{C.11}$$

Radial decomposition of D-dim metric in the KK reduction form:
Expanding the D-dim metric (C.8), into 2-dim (t, r) and transverse components

$$ds^2 = [(\tilde{N}^2 + \gamma^{(D-1)}_{ab}N^aN^b)dr^2 + 2\gamma^{(D-1)}_{tt}N^t dtdr + 2\gamma^{(D-1)}_{ti}N^i dtdr + \gamma^{(D-1)}_{tt}dt^2] + \gamma^{(D-1)}_{ij}dx^idx^j + [2\gamma^{(D-1)}_{ti}N^t dx^idr + 2\gamma^{(D-1)}_{ij}N^i dx^jdr + 2\gamma^{(D-1)}_{ti}dx^idt]\,. \tag{C.12}$$

Imposing the Kaluza-Klein ansatz for the dimensional reduction on T^{D-2}, i.e.,

$$ds^2 = g^{(2)}_{\mu\nu}dx^\mu dx^\nu + \Phi^{\frac{4}{D-2}}\sum_{i=1}^{D-2}dx_i^2\,, \qquad g^{(D)}_{xx} \equiv \Phi^{\frac{4}{D-2}}\,. \tag{C.13}$$

where $g^{(2)}_{\mu\nu}$, Φ depend only on the 2-dim coordinates (t, r), we get

$$\gamma^{(D-1)}_{ij} = \Phi^{\frac{4}{D-2}}\delta_{ij}\,, \quad \gamma^{(D-1)}_{ti} = 0\,, \quad N^i = 0\,, \quad N^{(D-1)}_i = 0\,, \quad \forall\, i, j = 1, \dots, D-1 \tag{C.14}$$

and the components of the 2-dim metric and its inverse are

$$
\begin{aligned}
&g^{(2)}_{rr} = g^{(D)}_{rr} = \tilde{N}^2 + \gamma^{(D-1)}_{tt}(N^t)^2 \,, \qquad g^{(2)}_{tr} = g^{(D)}_{tr} = \gamma^{(D-1)}_{tt} N^t \,,\\
&g^{(2)}_{tt} = g^{(D)}_{tt} = \gamma^{(D-1)}_{tt},\\
&g^{(2)\,rr} = \frac{1}{\tilde{N}^2} \,, \qquad g^{(2)\,tr} = -\frac{N^t}{\tilde{N}^2} \,, \qquad g^{(2)\,tt} = \frac{1}{\gamma^{(D-1)}_{tt}} + \frac{(N^t)^2}{\tilde{N}^2} \,. \qquad \text{(C.15)}
\end{aligned}
$$

Reduction of the radial Lagrangian (C.10)**:**
The induced metric on the $(D-1)$-dim boundary can be written as

$$
ds^2_{(D-1)} = \gamma^{(D-1)}_{ab} dx^a dx^b = \gamma^{(D-1)}_{tt} dt^2 + \gamma^{(D-1)}_{ij} dx^i dx^j = \gamma^{(D-1)}_{tt} dt^2 + \Phi^{\frac{4}{D-2}} \sum_{i=1}^{D-2} dx_i^2 \,. \qquad \text{(C.16)}
$$

The Ricci scalar becomes

$$
{}^{(D-1)}\mathcal{R} = \frac{\partial_t \gamma^{(D-1)}_{tt} \partial_t \Phi^2}{(\gamma^{(D-1)}_{tt})^2 \, \Phi^2} + \frac{(D-3)}{(D-2)} \frac{(\partial_t \Phi^2)^2}{\gamma^{(D-1)}_{tt} \Phi^4} - \frac{2\, \partial_t^2 \Phi^2}{\gamma^{(D-1)}_{tt} \Phi^2} \,. \qquad \text{(C.17)}
$$

The components of the extrinsic curvature are

$$
\begin{aligned}
K^{(D-1)}_{tt} &= \frac{1}{2\tilde{N}} \Big(\partial_r \gamma^{(D-1)}_{tt} - 2 D^{(D-1)}_t N^{(D-1)}_t \Big) = \frac{1}{2\tilde{N}} \Big(\partial_r \gamma^{(1)}_{tt} - 2 D^{(1)}_t N^{(1)}_t \Big) = K^{(1)}_{tt} \,,\\
K^{(D-1)}_{ti} &= \frac{1}{2\tilde{N}} \Big(\partial_r \gamma^{(D-1)}_{ti} - D^{(D-1)}_t N^{(D-1)}_i - D^{(D-1)}_i N^{(D-1)}_t \Big) = 0 \,, \qquad \text{(C.18)}\\
K^{(D-1)}_{ij} &= \frac{1}{2\tilde{N}} \Big(\partial_r \gamma^{(D-1)}_{ij} - D^{(D-1)}_i N^{(D-1)}_j - D^{(D-1)}_j N^{(D-1)}_i \Big) = \delta_{ij} \frac{\Phi^{\frac{4}{D-2}-2}}{(D-2)} \tilde{n}^\mu \partial_\mu \Phi^2 \,,
\end{aligned}
$$

where we have used

$$
\begin{aligned}
&\gamma^{(1)}_{tt} = \gamma^{(D-1)}_{tt} \,, \qquad N^{(1)}_t = \gamma^{(1)}_{tt} N^t = \gamma^{(D-1)}_{tt} N^t = N^{(D-1)}_t \,, \qquad \Gamma^{(D-1)\,t}_{it} = 0 \,,\\
&\Gamma^{(1)\,t}_{tt} = \Gamma^{(D-1)\,t}_{tt} \,, \qquad \tilde{n}_r = \tilde{N} \,, \quad \tilde{n}_t = 0 \,, \quad \tilde{n}^\mu = g^{(2)\,\mu\nu} \tilde{n}_\nu \,, \qquad \text{(C.19)}\\
&D^{(D-1)}_t N^{(D-1)}_t = \partial_t N^{(D-1)}_t - \Gamma^{(D-1)\,t}_{tt} N^{(D-1)}_t = \partial_t N^{(1)}_t - \Gamma^{(1)\,t}_{tt} N^{(1)}_t = D^{(1)}_t N^{(1)}_t \,,
\end{aligned}
$$

where $\gamma^{(1)}_{tt}$ is the induced metric and $K^{(1)}_{tt}$ is the extrinsic curvature on the 1-dim boundary and $\tilde{n}^\mu$ is the outward unit normal to the 1-dim boundary. Then we can compute

$$
\begin{aligned}
K^{(D-1)\,ab} K^{(D-1)}_{ab} &= \gamma^{(D-1)\,ac} \gamma^{(D-1)\,bd} K^{(D-1)}_{ab} K^{(D-1)}_{cd}\\
&= (\gamma^{(D-1)\,tt})^2 (K^{(D-1)}_{tt})^2 + \gamma^{(D-1)\,ik} \gamma^{(D-1)\,jl} K^{(D-1)}_{ij} K^{(D-1)}_{kl}\\
&= K^{(1)\,tt} K^{(1)}_{tt} + \frac{(\tilde{n}^\mu \partial_\mu \Phi^2)^2}{(D-2)\Phi^4} \qquad \text{(C.20)}
\end{aligned}
$$

and

$$K^{(D-1)} = \gamma^{(D-1)\,ab} K^{(D-1)}_{ab} = \gamma^{(D-1)\,tt} K^{(D-1)}_{tt} + \gamma^{(D-1)\,ij} K^{(D-1)}_{ij} = K^{(1)} + \frac{\tilde{n}^\mu \partial_\mu \Phi^2}{\Phi^2} \,, \tag{C.21}$$

where $K^{(1)} = \gamma^{(1)\,tt} K^{(1)}_{tt}$. Substituting these in (C.10), the radial Lagrangian becomes

$$\begin{aligned} L &= \frac{1}{16\pi G_2} \int dt \sqrt{-\gamma^{(1)}_{tt}} \Phi^2 \tilde{N} \Big[{}^{(D-1)}\mathcal{R} + \Big(K^{(1)} + \frac{\tilde{n}^\mu \partial_\mu \Phi^2}{\Phi^2} \Big)^2 - K^{(1)\,tt} K^{(1)}_{tt} \\ &\qquad - \frac{(\tilde{n}^\mu \partial_\mu \Phi^2)^2}{(D-2)\Phi^4} - \frac{h_{IJ}}{2} g^{(2)\,\mu\nu} \partial_\mu \Psi^I \partial_\nu \Psi^J - V \Big] \\ &= \frac{1}{16\pi G_2} \int dt \sqrt{-\gamma^{(1)}_{tt}} \Phi^2 \tilde{N} \Big[{}^{(D-1)}\mathcal{R} + 2K^{(1)} \frac{\tilde{n}^\mu \partial_\mu \Phi^2}{\Phi^2} + \frac{(D-3)}{(D-2)} \frac{(\tilde{n}^\mu \partial_\mu \Phi^2)^2}{\Phi^4} \\ &\qquad - \frac{h_{IJ}}{2} g^{(2)\,\mu\nu} \partial_\mu \Psi^I \partial_\nu \Psi^J - V \Big] \,, \end{aligned} \tag{C.22}$$

where we have used $K^{(1)\,tt} K^{(1)}_{tt} = (K^{(1)})^2$.

Weyl transformation: Performing a Weyl transformation on the 2-dim bulk metric, $g_{\mu\nu} = \Phi^{\frac{2(D-3)}{(D-2)}} g^{(2)}_{\mu\nu}$ which induces a Weyl transformation on the 1-dim boundary metric $\gamma_{tt} = \Phi^{\frac{2(D-3)}{(D-2)}} \gamma^{(1)}_{tt}$, we get

$$\begin{aligned} ds^2 = g_{\mu\nu} dx^\mu dx^\nu &= \Big(\Phi^{\frac{2(D-3)}{(D-2)}} \tilde{N}^2 + \gamma_{tt} (N^t)^2 \Big) dr^2 + 2\gamma_{tt} N^t dt dr + \gamma_{tt} dt^2 \\ &\equiv (N^2 + \gamma_{tt}(N^t)^2) dr^2 + 2\gamma_{tt} N^t dt dr + \gamma_{tt} dt^2 \,, \end{aligned} \tag{C.23}$$

which is same as (4.62) with $N \equiv \Phi^{\frac{D-3}{D-2}} \tilde{N}$. Under the Weyl transformation, we have

$$\begin{aligned} &\Gamma^{(1)\,t}_{tt} = \Gamma^t_{tt} - \frac{(D-3)}{(D-2)} \frac{\partial_t \Phi}{\Phi} \,, \qquad \Gamma^t_{tt} = \frac{\partial_t \gamma_{tt}}{2\gamma_{tt}} \,, \qquad N_t = \gamma_{tt} N^t \,, \\ &D^{(1)}_t N^{(1)}_t = \partial_t (\gamma^{(1)}_{tt} N^t) - \Gamma^{(1)\,t}_{tt} \gamma^{(1)}_{tt} N^t = \Phi^{\frac{-2(D-3)}{(D-2)}} \Big(D_t N_t - \frac{(D-3)}{(D-2)} \frac{N_t \partial_t \Phi}{\Phi^2} \Big) \,. \end{aligned} \tag{C.24}$$

The Ricci scalar (C.17) can be written covariantly as

$$\begin{aligned} {}^{(D-1)}\mathcal{R} &= -2\Phi^{\frac{2(D-3)}{(D-2)} - 2} \Box_t \Phi^2 \,; \\ \Box_t \Phi^2 &= \gamma^{tt} D_t D_t \Phi^2 = \frac{\partial_t (\sqrt{-\gamma_{tt}} \gamma^{tt} \partial_t \Phi^2)}{\sqrt{-\gamma_{tt}}} = \Big[\frac{\partial_t^2 \Phi^2}{\gamma_{tt}} - \frac{\partial_t \gamma_{tt} \partial_t \Phi^2}{2\gamma_{tt}^2} \Big] . \end{aligned} \tag{C.25}$$

The extrinsic curvature becomes

$$K^{(1)}_{tt} = \frac{1}{2\tilde{N}} \Big(\partial_r \gamma^{(1)}_{tt} - 2D^{(1)}_t N^{(1)}_t \Big) = \Phi^{-\frac{(D-3)}{(D-2)}} \Big(K_{tt} - \frac{(D-3)}{2(D-2)} \frac{\gamma_{tt}}{\Phi^2} n^\mu \partial_\mu \Phi^2 \Big) \,, \tag{C.26}$$

$$K^{(1)} = \gamma^{(1)\,tt} K^{(1)}_{tt} = \Phi^{\frac{2(D-3)}{(D-2)}} \gamma^{tt} K^{(1)}_{tt} = \Phi^{\frac{(D-3)}{(D-2)}} \Big(K - \frac{(D-3)}{2(D-2)} \frac{n^\mu \partial_\mu \Phi^2}{\Phi^2} \Big) \,, \tag{C.27}$$

where $K_{tt} = \frac{1}{2N}(\partial_r \gamma_{tt} - 2D_t N_t)$, $K = \gamma^{tt} K_{tt}$, $n_r = N = \Phi^{\frac{D-3}{D-2}} \tilde{N} = \Phi^{\frac{D-3}{D-2}} \tilde{n}_r$ and

$n^\mu\partial_\mu\Phi^2 = g^{\mu\nu}n_\mu\partial_\nu\Phi^2 = \Phi^{-\frac{D-3}{D-2}}\tilde{n}^\mu\partial_\mu\Phi^2$. Substituting these expressions in (C.22), we get

$$L = \frac{1}{16\pi G_2}\int dt\sqrt{-\gamma}N\Big[-2\Box_t\Phi^2 + 2K\,n^\mu\partial_\mu\Phi^2 - \frac{\Phi^2 h_{IJ}}{2}\partial_\mu\Psi^I\partial^\mu\Psi^J - V\Phi^{\frac{2}{D-2}}\Big]\,. \tag{C.28}$$

Simplifying further using $n^\mu\partial_\mu\Phi^2 = \frac{1}{N}(\partial_r\Phi^2 - N^t\partial_t\Phi^2)$ and defining $U \equiv \Phi^{\frac{2}{D-2}}V$, we obtain (4.63).

C.3 The $z = 1, \theta \neq 0$ hvLif family

Setting $z = 1$, $\theta \neq 0$ in the 4-dim Einstein-Maxwell-scalar action (3.43) and the charged hvLif solution (3.46) gives $F_{1\,MN} = 0$ and we get Einstein-scalar theory coupled to an $U(1)$ gauge field $A_{2\,M}$, with

$$V_0 = \frac{(3-\theta)(2-\theta)\,e^{-\gamma\Psi_0}}{R^{2-2\theta}\,r_{hv}^{2\theta}}\,, \qquad \gamma = \frac{\theta}{\sqrt{(2-\theta)(-\theta)}}\,, \qquad \lambda_2 = \sqrt{\frac{-\theta}{2-\theta}}\,. \tag{C.29}$$

Note that the energy conditions (3.50) simplify in this case to give

$$(2-\theta) \geq 0\,, \qquad -\theta \geq 0 \qquad \Rightarrow \qquad \gamma = -\lambda_2\,. \tag{C.30}$$

Substituting the gauge field solution in terms of the scalar field and the metric component $g_{xx}^{(4)}$, gives an effective gravity-scalar theory (4.1) (in 4-dim with one scalar field) with an effective potential

$$V_{eff} = -\frac{(3-\theta)(2-\theta)}{R^{2-2\theta}r_{hv}^{2\theta}}e^{\gamma(\Psi-\Psi_0)} + \frac{1}{(g_{xx}^{(4)})^2}\frac{(2-\theta)(1-\theta)Q^2R^{-6+2\theta}}{e^{\lambda_2(\Psi-\Psi_0)}}\,. \tag{C.31}$$

Using the extremality condition $Q^2 = (\frac{3-\theta}{1-\theta})\,r_0^{2(2-\theta)}$ and $\gamma = -\lambda_2$ simplifies the effective potential (3.63) which is of the form (4.95) with $U = \Phi V_{eff}$ and $\Phi = g_{xx}^{(4)}$. The factors of Φ arise, as seen in Chap.-3, from the T^2-compactification followed by a Weyl transformation of the 2-dim metric. Thus the potential $U(\Phi,\Psi)$ has factorized in this case: the piece inside the brackets is structurally similar to that for the reduction of the M2-AdS_4 case, with an overall Ψ factor dressing outside.

The linear fluctuations ϕ, ψ, Ω to the dilaton, scalar field and the metric respectively are governed by the quadratic action (Sec.-3.2.2.1 in Chap. 3), which gives the linearized equation $\partial_+\partial_-\zeta = 0$ for the fluctuation, $\zeta = \psi - \frac{2}{\sqrt{2-\theta}}\,\frac{L^2(3-\theta)(2-\theta)}{r_0^{2-2\theta}r_{hv}^{2\theta}}\,\phi$. Thus the ζ scalar is massless at linear order.

C.4 Dimensional reduction on T^{D-2}

Consider an ansatz for the D-dimensional metric suitable for Kaluza-Klein (KK) reduction on a torus T^{D-2},

$$ds^2 = g^{(D)}_{MN} dx^M dx^N = g^{(2)}_{\mu\nu} dx^\mu dx^\mu dx^\nu + g^{(D)}_{xx} \sum_{i=1}^{D-2} dx_i^2 \ , \tag{C.32}$$

where the 2-dimensional metric $g^{(2)}_{\mu\nu}$ and $g^{(D)}_{xx}$ depend only on the non-compact coordinates (t, r). With this metric ansatz, components of the Christoffel symbol decompose as

$$\begin{aligned} &\Gamma^{(\mathrm{D})\,\mu}_{\nu\rho} = \Gamma^{(2)\,\mu}_{\nu\rho} \ , \qquad \Gamma^{(\mathrm{D})\,\mu}_{ij} = -\frac{1}{2} g^{(2)\,\mu\nu} \partial_\nu g^{(D)}_{ij} \ , \qquad \Gamma^{(\mathrm{D})\,i}_{j\mu} = \frac{1}{2} g^{(D)\,ik} \partial_\mu g^{(D)}_{jk} \ , \\ &\Gamma^{(\mathrm{D})\,\mu}_{\nu j} = 0 \ , \qquad \Gamma^{(\mathrm{D})\,i}_{\mu\nu} = 0 \ , \qquad \Gamma^{(\mathrm{D})\,i}_{jk} = 0 \ , \end{aligned} \tag{C.33}$$

which give the decomposition of the components of the Riemann tensor as

$$\begin{aligned} \mathcal{R}^{(\mathrm{D})\,\rho}_{\mu\rho\nu} &= \mathcal{R}^{(2)\,\rho}_{\mu\rho\nu} \ , \\ \mathcal{R}^{(\mathrm{D})\,k}_{\mu k\nu} &= -\partial_\nu \Gamma^{(\mathrm{D})\,k}_{\mu k} + \Gamma^{(\mathrm{D})\,k}_{\sigma k} \Gamma^{(\mathrm{D})\,\sigma}_{\mu\nu} - \Gamma^{(\mathrm{D})\,k}_{j\nu} \Gamma^{(\mathrm{D})\,j}_{\mu k} \ , \\ \mathcal{R}^{(\mathrm{D})\,\rho}_{i\rho j} &= \partial_\rho \Gamma^{(\mathrm{D})\,\rho}_{ij} + \Gamma^{(\mathrm{D})\,\rho}_{\sigma\rho} \Gamma^{(\mathrm{D})\,\sigma}_{ij} - \Gamma^{(\mathrm{D})\,\rho}_{kj} \Gamma^{(\mathrm{D})\,k}_{i\rho} \ , \\ \mathcal{R}^{(\mathrm{D})\,k}_{ikj} &= \Gamma^{(\mathrm{D})\,k}_{\sigma k} \Gamma^{(\mathrm{D})\,\sigma}_{ij} - \Gamma^{(\mathrm{D})\,k}_{\sigma j} \Gamma^{(\mathrm{D})\,\sigma}_{ik} \ , \\ \mathcal{R}^{(\mathrm{D})\,\rho}_{\mu\rho i} &= 0 \ , \qquad \mathcal{R}^{(\mathrm{D})\,k}_{\mu k i} = 0 \ . \end{aligned} \tag{C.34}$$

Using $g^{(D)}_{ij} = \delta_{ij} g^{(D)}_{xx}$, we compute the components of the Ricci tensor as

$$\begin{aligned} \mathcal{R}^{(D)}_{\mu\nu} &= \mathcal{R}^{(\mathrm{D})\,\rho}_{\mu\rho\nu} + \mathcal{R}^{(\mathrm{D})\,k}_{\mu k\nu} \\ &= \mathcal{R}^{(2)}_{\mu\nu} - \frac{(D-2)}{2} \nabla_{(2)\,\nu} (g^{(D)\,xx} \partial_\mu g^{(D)}_{xx}) - \frac{(D-2)}{4} (g^{(D)\,xx})^2 \partial_\mu g^{(D)}_{xx} \partial_\nu g^{(D)}_{xx} \ , \\ \mathcal{R}^{(D)}_{ij} &= \mathcal{R}^{(\mathrm{D})\,\rho}_{i\rho j} + \mathcal{R}^{(\mathrm{D})\,k}_{ikj} \\ &= \delta_{ij} \Big[-\frac{1}{2} \nabla_{(2)\,\mu} \big(g^{(2)\,\mu\nu} \partial_\nu g^{(D)}_{xx} \big) - \frac{(D-4)}{4} g^{(D)\,xx} g^{(2)\,\mu\nu} \partial_\mu g^{(D)}_{xx} \partial_\nu g^{(D)}_{xx} \Big] \ , \\ \mathcal{R}^{(D)}_{\mu j} &= \mathcal{R}^{(\mathrm{D})\,\rho}_{\mu\rho j} + \mathcal{R}^{(\mathrm{D})\,k}_{\mu kj} = 0 \ , \end{aligned} \tag{C.35}$$

where $\nabla_{(2)\,\mu}$ is a covariant derivative with respect to the 2-dimensional metric $g^{(2)}_{\mu\nu}$.

Substituting $g^{(D)}_{xx} = \Phi^{\frac{4}{D-2}}$ and simplifying further, the components of Ricci tensor become

$$\mathcal{R}^{(D)}_{\mu\nu} = \mathcal{R}^{(2)}_{\mu\nu} - \frac{\nabla_{(2)\,\mu}\nabla_{(2)\,\nu}\Phi^2}{\Phi^2} + \frac{(D-3)}{(D-2)}\frac{\partial_\mu\Phi^2\partial_\nu\Phi^2}{\Phi^4} \;, \qquad \mathcal{R}^{(D)}_{ij} = \frac{\delta_{ij}\Phi^{\frac{4}{D-2}}}{(D-2)}\Big(-\frac{\nabla^2_{(2)}\Phi^2}{\Phi^2}\Big) \;. \tag{C.36}$$

Then the Ricci scalar is

$$\begin{aligned}\mathcal{R}^{(D)} = g^{(2)\,\mu\nu}\mathcal{R}^{(D)}_{\mu\nu} + g^{(D)\,ij}\mathcal{R}^{(D)}_{ij} &= \mathcal{R}^{(2)} - 2\frac{\nabla^2_{(2)}\Phi^2}{\Phi^2} + \frac{(D-3)}{(D-2)}\frac{(\nabla_{(2)}\Phi^2)^2}{\Phi^4} \\ &= \mathcal{R}^{(2)} - 4\frac{\nabla^2_{(2)}\Phi}{\Phi} - \frac{4}{(D-2)}\frac{(\nabla_{(2)}\Phi)^2}{\Phi^2} \;.\end{aligned} \tag{C.37}$$

Weyl transformation, $g_{\mu\nu} = \Phi^n g^{(2)}_{\mu\nu}$:

Under the Weyl transformation $g_{\mu\nu} = \Phi^n g^{(2)}_{\mu\nu}$ for arbitrary n, the various quantities transform as

$$\begin{aligned}&\nabla^2_{(2)}\Phi = \Phi^n\nabla^2\Phi \;, \qquad (\nabla_{(2)}\Phi)^2 = \Phi^n(\nabla\Phi)^2 \;, \\ &\Gamma^{(2)\,\rho}_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \frac{n}{2\Phi}(\delta^\rho_\nu\partial_\mu\Phi + \delta^\rho_\mu\partial_\nu\Phi - g^{\rho\sigma}g_{\mu\nu}\partial_\sigma\Phi) \;, \\ &\mathcal{R}^{(2)}_{\mu\nu} = \mathcal{R}_{\mu\nu} + \frac{n\,g_{\mu\nu}\,\nabla^2\Phi}{2\,\Phi} - \frac{n\,g_{\mu\nu}\,(\nabla\Phi)^2}{2\,\Phi^2} \;, \\ &\mathcal{R}^{(2)} = \Phi^n\Big[\mathcal{R} + n\Big(\frac{\nabla^2\Phi}{\Phi} - \frac{(\nabla\Phi)^2}{\Phi^2}\Big)\Big] = \Phi^n\Big[\mathcal{R} + \frac{n}{2}\Big(\frac{\nabla^2\Phi^2}{\Phi^2} - \frac{(\nabla\Phi^2)^2}{\Phi^4}\Big)\Big].\end{aligned} \tag{C.38}$$

Substituting this expression for $\mathcal{R}^{(2)}$ in (C.37), we get

$$\mathcal{R}^{(D)} = \Phi^n\Big[\mathcal{R} + \frac{(n-4)}{2}\frac{\nabla^2\Phi^2}{\Phi^2} + 2\Big(n - 2\frac{(D-3)}{(D-2)}\Big)\frac{(\nabla\Phi)^2}{\Phi^2}\Big] \;. \tag{C.39}$$

Now for $n = \frac{2(D-3)}{(D-2)}$ *i.e.* $g_{\mu\nu} = \Phi^{\frac{2(D-3)}{(D-2)}} g^{(D)}_{\mu\nu}$, we get

$$\mathcal{R}^{(D)} = \Phi^{\frac{2(D-3)}{(D-2)}}\Big[\mathcal{R} - \frac{(D-1)}{(D-2)}\frac{\nabla^2\Phi^2}{\Phi^2}\Big] \;. \tag{C.40}$$

To summarize, upon dimensional reduction on T^{D-2} and a further Weyl transformation in 2-dimensions, we get

$$\begin{aligned}\sqrt{-g^{(D)}}\,\mathcal{R}^{(D)} &= \sqrt{-g^{(2)}}\,\Phi^2\Big[\mathcal{R}^{(2)} - 4\frac{\nabla^2_{(2)}\Phi}{\Phi} - \frac{4}{(D-2)}\frac{(\nabla_{(2)}\Phi)^2}{\Phi^2}\Big] \\ &= \sqrt{-g^{(2)}}\,\Phi^2\Big[\mathcal{R}^{(2)} - 2\frac{\nabla^2_{(2)}\Phi^2}{\Phi^2} + \frac{(D-3)}{(D-2)}\frac{(\nabla_{(2)}\Phi^2)^2}{\Phi^4}\Big]\end{aligned} \tag{C.41}$$

and

$$\sqrt{-g^{(D)}}\,\mathcal{R}^{(D)} = \sqrt{-g}\Big[\Phi^2\mathcal{R} - \frac{(D-1)}{(D-2)}\nabla^2\Phi^2\Big] \;. \tag{C.42}$$

C.4.1 Reduction of the Gibbons-Hawking boundary term

The Gibbons-Hawking action on the $(D-1)$ dimensional boundary is

$$S_{GH}^{(D)} = \frac{1}{8\pi G_D}\int d^{D-1}x\sqrt{-\gamma^{(D-1)}}\,K^{(D-1)}\ , \tag{C.43}$$

where $\gamma_{MN}^{(D-1)}$ is the induced metric on the boundary and $K^{(D-1)}$ is the extrinsic curvature. Upon dimensional reduction with the KK metric (C.32), we get

$$K^{(D-1)} = K^{(1)} + \frac{\tilde{n}_\mu \nabla^\mu_{(2)}\Phi^2}{\Phi^2}\ , \qquad \sqrt{-\gamma^{(D-1)}} = \sqrt{-\gamma^{(1)}}\,\Phi^2\ , \tag{C.44}$$

where $\tilde{n}_\mu$ is the outward pointing normal and the Gibbons Hawking term reduces to

$$S_{GH}^{(D)} = \frac{1}{8\pi G_2}\int dt\,\sqrt{-\gamma^{(1)}}\left[\Phi^2\,K^{(1)} + \tilde{n}_\mu\nabla^\mu_{(2)}\Phi^2\right]\ . \tag{C.45}$$

The Weyl transformation $g_{\mu\nu} = \Phi^{\frac{2(D-3)}{(D-2)}}g^{(2)}_{\mu\nu}$ induces a Weyl transformation on the boundary metric $\gamma_{tt} = \Phi^{\frac{2(D-3)}{(D-2)}}\gamma^{(1)}_{tt}$, $\sqrt{-\gamma} = \sqrt{-\gamma^{(1)}}\Phi^{\frac{(D-3)}{(D-2)}}$ and $n^\mu\partial_\mu\Phi^2 = g^{\mu\nu}n_\mu\partial_\nu\Phi^2 = \Phi^{-\frac{D-3}{D-2}}\tilde{n}^\mu\partial_\mu\Phi^2$. The extrinsic curvature becomes

$$K^{(1)} = \gamma^{(1)\,tt}K^{(1)}_{tt} = \Phi^{\frac{D-3}{D-2}}K - \frac{(D-3)}{2(D-2)}\Phi^{\frac{D-3}{D-2}-2}n^\mu\partial_\mu\Phi^2\ . \tag{C.46}$$

The Gibbons Hawking term becomes

$$S_{GH}^{(D)} = \frac{1}{8\pi G_2}\int dt\,\sqrt{-\gamma}\left[\Phi^2\,K + \frac{(D-1)}{2(D-2)}n_\mu\nabla^\mu\Phi^2\right]\ . \tag{C.47}$$

From (C.41), (C.42) (C.45) and (C.47), we get

$$\begin{aligned}
&\frac{1}{16\pi G_D}\Big(\int d^Dx\,\sqrt{-g^{(D)}}\,\mathcal{R}^{(D)} + 2\int d^{D-1}x\,\sqrt{-\gamma^{(D-1)}}\,K^{(D-1)}\Big)\\
&= \frac{1}{16\pi G_2}\Big(\int d^2x\,\sqrt{-g^{(2)}}\Big[\Phi^2\mathcal{R}^{(2)} + \frac{(D-3)}{(D-2)}\frac{(\nabla_{(2)}\Phi^2)^2}{\Phi^2}\Big] + 2\int dt\,\sqrt{-\gamma^{(1)}}\,\Phi^2\,K^{(1)}\Big)\\
&= \frac{1}{16\pi G_2}\Big(\int d^2x\,\sqrt{-g}\,\Phi^2\mathcal{R}^{(2)} + 2\int dt\,\sqrt{-\gamma}\,\Phi^2\,K\Big)\ .
\end{aligned} \tag{C.48}$$

The Gibbons-Hawking action terms for the 1-dimensional boundary before and after Weyl transformation are

$$S_{GH}^{(1)} = \frac{1}{8\pi G_2}\int dt\,\sqrt{-\gamma^{(1)}}\,\Phi^2\,K^{(1)}\ , \qquad S_{GH} = \frac{1}{8\pi G_2}\int dt\,\sqrt{-\gamma}\,\Phi^2\,K\ . \tag{C.49}$$

Appendix D

Appendix to chapter 5

D.1 Orthonormal basis for N-level irreducible ghost-spin

We give the transformations which transform the defining basis for N-level irreducible ghost-spin to an orthonormal basis. For even N-level ghost-spin, the transformations to orthonormal basis are

$$
\begin{aligned}
|+\rangle &= \frac{\sum_{i=1}^{N} |e_i\rangle}{\sqrt{N^2 - N}} \ , \qquad |2\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle - |e_2\rangle) \ , \\
&\vdots \\
\left|\frac{N}{2} + 1\right\rangle &= \frac{1}{\sqrt{2}}(|e_{N-1}\rangle - |e_N\rangle) \ , \\
\left|\frac{N}{2} + 2\right\rangle &= \frac{1}{2}(|e_1\rangle + |e_2\rangle - |e_3\rangle - |e_4\rangle) \ , \\
\left|\frac{N}{2} + 3\right\rangle &= \frac{1}{\sqrt{12}}\Big(\sum_{i=1}^{4} |e_i\rangle - 2(|e_5\rangle + |e_6\rangle)\Big) \ , \\
&\vdots \\
|N-1\rangle &= \frac{1}{\sqrt{\frac{(N-1)(N-3)}{2}}} \left(\sum_{i=1}^{N-4} |e_i\rangle - \left(\frac{N-1}{2} - 1\right)(|e_{N-3}\rangle + |e_{N-2}\rangle)\right) \ , \\
|N\rangle &= \frac{1}{\sqrt{\frac{N(N-2)}{2}}} \left(\sum_{i=1}^{N-2} |e_i\rangle - \left(\frac{N}{2} - 1\right)(|e_{N-1}\rangle + |e_N\rangle)\right) \ . \qquad \text{(D.1)}
\end{aligned}
$$

For odd $N+1$-level ghost-spin (where N is even), the transformations to orthonormal basis are

$$\begin{aligned}
|+\rangle &= \frac{\sum_{i=1}^{N+1}|e_i\rangle}{\sqrt{N(N+1)}} \ , \qquad |2\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle - |e_2\rangle) \ , \\
&\vdots \\
\left|\frac{N}{2}+1\right\rangle &= \frac{1}{\sqrt{2}}(|e_{N-1}\rangle - |e_N\rangle) \ , \\
\left|\frac{N}{2}+2\right\rangle &= \frac{1}{2}(|e_1\rangle + |e_2\rangle - |e_3\rangle - |e_4\rangle) \ , \\
\left|\frac{N}{2}+3\right\rangle &= \frac{1}{\sqrt{12}}\Big(\sum_{i=1}^{4}|e_i\rangle - 2(|e_5\rangle + |e_6\rangle)\Big) \ , \\
&\vdots \\
|N-1\rangle &= \frac{1}{\sqrt{\frac{(N-1)(N-3)}{2}}}\left(\sum_{i=1}^{N-4}|e_i\rangle - \left(\frac{N-1}{2}-1\right)(|e_{N-3}\rangle + |e_{N-2}\rangle)\right) \ , \\
|N\rangle &= \frac{1}{\sqrt{\frac{N(N-2)}{2}}}\left(\sum_{i=1}^{N-2}|e_i\rangle - \left(\frac{N}{2}-1\right)(|e_{N-1}\rangle + |e_N\rangle)\right) \ , \\
|N+1\rangle &= \frac{1}{\sqrt{(N+1)^2-(N+1)}}\left(\sum_{i=1}^{N}|e_i\rangle - N|e_{N+1}\rangle\right) \ .
\end{aligned} \tag{D.2}$$

To illustrate these transformations, we write them explicitly for $N=3$ and $N=4$. For a 3-level ghost-spin, the orthonormal basis are

$$|+\rangle = \frac{1}{\sqrt{6}}(|e_1\rangle+|e_2\rangle+|e_3\rangle) \ , \quad |2\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle-|e_2\rangle) \ , \quad |3\rangle = \frac{1}{\sqrt{6}}(|e_1\rangle+|e_2\rangle-2|e_3\rangle) \tag{D.3}$$

and for $N=4$, the orthonormal basis are

$$\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{12}}(|e_1\rangle + |e_2\rangle + |e_3\rangle + |e_1\rangle) \ , \qquad |2\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle - |e_2\rangle) \ , \\
|3\rangle &= \frac{1}{\sqrt{2}}(|e_3\rangle - |e_4\rangle) \ , \qquad |4\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle - |e_3\rangle - |e_4\rangle) \ .
\end{aligned} \tag{D.4}$$

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