On some embedding problems of closed oriented manifolds in the smooth and contact category

Kuldeep Saha

A thesis submitted in partial fulfillment of the requirement for the degree of Doctor of Philosophy in Mathematics

CHENNAI MATHEMATICAL INSTITUTE

On some embedding problems of closed oriented manifolds in the smooth and contact category

by

Kuldeep Saha

Supervisor : Dishant M. Pancholi

Chennai Mathematical Institute

January 2020

To my mother

Smt. Kana Saha

Acknowledgement

Firstly, I am deeply grateful to my advisor Dishant M. Pancholi. I can not thank him enough for his constant encouragement and patience while working towards this thesis.

I am also grateful to Suhas J. Pandit for many helpful discussions and suggestions.

I thank Prof. Francisco Presas Mata for his suggestions and encouragement during a short visit to ICMAT.

I warmly thank Abhijeet Ghanwat for the many stimulating discussions and help with the latex drawing.

I thank Dheeraj Kulkarni and Kashyap Rajeevsarathy for their support and encouragement.

I thank Sayan Chakraborty and Manish Kumar Pandey for their help with latex corrections and typos.

Special thanks to my teachers: Pannalal Chatterjee, Siddharta Shankar Chatterjee, Angshuman Das, Mahan Mj and Subhabrata Das.

Thanks to Tinni and Debarchan, because of whom I could shift to Mathematics.

Thanks to my friends: Sourav Das, Sajal Mukherjee, Arijit Nath, Saheli Pal and Sourav Sen.

I thank my thesis committee and the referees for their efforts.

I gratefully acknowledge supports from the National Board of Higher Mathematics (DAE, Govt. of India) and the Chennai Mathematical Institute.

Abstract

We investigate embedding problems of closed smooth manifolds. First we prove that every open book decomposition of a closed orientable 3-manifold admits a smooth open book embedding in both $S^2 \times S^3$ and $S^2 \tilde{\times} S^3$. Using this, we reprove a well-known result of Hirsch which says that every orientable 3-manifold embeds in the 5-sphere S^5. Next, we discuss contact open book embeddings of contact $(2n+1)$-manifolds in the standard contact $(2N+1)$-sphere (S^{2N+1}, ξ_{std}). We show that there is an infinite family of contact homotopy spheres Σ^{2n+1}, that admit contact open book embedding in (S^{2n+3}, ξ_{std}). We also show that a large class of contact 3-manifolds admit contact open book embedding in (S^5, ξ_{std}). Finally, we consider the contact and isocontact embedding of π-manifolds and prove a contact analogue of a smooth embedding theorem due to R. De Sapio. In particular, we show contact embedding of k-connected $(2n+1)$-dimensional π–manifolds in $(\mathbb{R}^{4n-2k+3}, \xi_{std})$, for $k \leq n-1$.

Contents

CHAPTER 1

Introduction

Embedding of manifolds in the Euclidean spaces has long been a problem of great importance in geometric topology. The first major breakthrough in this field was the Whitney embedding theorem [**Wh**], which says that every n-manifold can be smoothly embedded in $\mathbb{R}^{2n}$. Later, Haefliger and Hirsch ([**HH**]) generalized Whitney's theorem to show that a closed, orientable, k-connected n-manifold can be embedded in $\mathbb{R}^{2n-k-1}$. In lower dimension, a well known result due to Hirsch [**Hi0**] says that every orientable 3-manifold embeds in S^5.

Embedding questions can also be studied with extra geometric structures on the manifolds. For example, John Nash [**Na**] established that every closed Riemannian n-manifold admits a C^∞-isometric embedding in $\frac{n}{2}(3n+11)$-dimensional flat Euclidean space. This initiated the study of embeddings of manifolds preserving a given geometric structure. In the present thesis, we focus on embedding problems preserving open book decompositions of closed orientable manifolds and co-orientable contact manifolds.

An open book, roughly speaking, is a decomposition of a manifold M^n into a co-dimension 2 submanifold B^{n-2} and a mapping torus $\mathcal{MT}(V^{n-1}, \phi)$, such that B is the boundary of V (denoted by ∂V), and ϕ is a diffeomorphism of V that is identity near ∂V. An *equivalent* way to describe an open book is the following.

Definition 1.0.1 (Abstract open book). *Let $M^n = \mathcal{MT}(V^{n-1}, \phi) \cup_{id.} \partial V \times D^2$. Here, one identifies $\partial \mathcal{MT}(V^{n-1}, \phi) = \partial V \times S^1$ with $\partial V \times \partial D^2$ via the identity map. We say, M^n is given by the abstract open book decomposition, $\mathcal{A}ob(V^{n-1}, \phi)$, with page V^{n-1} and monodromy ϕ.*

Following is the corresponding notion of embedding for open book decompositions.

Definition 1.0.2 (Open book embedding). *M^n has an open book embedding in V^N if there is an open book $\mathcal{A}ob(\Sigma_M^{n-1}, \phi_M)$ of M and an open book $\mathcal{A}ob(\Sigma_V^{N-1}, \phi_V)$ of V such that the following conditions hold:*

(1) *there exists a proper embedding $f : (\Sigma_M, \partial\Sigma_M) \to (\Sigma_V, \partial\Sigma_V)$,*
(2) *$\phi_V \circ f = f \circ \phi_M$.*

We also say that M^n open book embeds in V^N with respect to the open book $\mathcal{A}ob(\Sigma_V, \phi_V)$.

Our first investigation is the following natural embedding question for open books.

Question 1.0.3 (**Open book embedding**). *When does a closed, oriented manifold M^{2n+1} open book embed in another closed, oriented manifold V^{2N+1}? In particular, what is the simplest possible 5-manifold into which we can get such embeddings of any orientable 3-manifold?*

In Chapter 3, we study Question 1.0.3 for closed, orientable 3-manifolds. We prove the following.

THEOREM 1.0.4 ([**PPS**]). *(Theorem 3.1.1) Let M be a closed oriented connected 3–dimensional manifold together with an open book decomposition $\mathcal{A}ob(\Sigma, \phi)$. Then, open book $\mathcal{A}ob(\Sigma, \phi)$ admits an open book embedding in any open book decomposition associated to $S^3 \times S^2$ with pages a disk bundle over S^2 of even Euler number and monodromy the identity as well as in any open book of $S^3 \widetilde{\times} S^2$ with pages a disk bundle over S^2 of odd Euler number and monodromy the identity.*

Note that up to isomorphism there exist only two D^4-bundle on S^2 (since $H^2(S^2, \pi_1(SO(4)) = \mathbb{Z}_2$.). One is the trivial bundle $S^2 \times D^4$ and another is the *twisted* bundle $S^2 \widetilde{\times} D^4$. $S^3 \widetilde{\times} S^2$ denotes the boundary of this twisted bundle.

A proper embedding f of a surface $(\Sigma, \partial\Sigma)$ in a 4-manifold $(V^4, \partial V^4)$ is called *flexible* if for every diffeomorphism ϕ of $(\Sigma, \partial\Sigma)$ there is an isotopy Φ_t ($t \in [0,1]$) of $(\Sigma, \partial\Sigma)$ such that $\Phi_0 = id$ and $\Phi_1 \circ f = f \circ \phi$. Our proof of Theorem 3.1.1 relies on finding a flexible embedding of Σ in a D^2-bundle over S^2.

As an application of the methods used to prove Theorem 3.1.1, we reprove the following well known Theorem due to Hirsch.

THEOREM 1.0.5 (Hirsch, [**Hi**]). *Every orientable closed 3-manifold smoothly embeds in S^5.*

We now come to the analogous embedding problem for contact manifolds.

DEFINITION 1.0.6. *A symplectic form on an even dimensional manifold is a non-degenerate, closed 2-form.*

A *contact manifold* is an odd dimensional smooth manifold M^{2n+1}, together with a maximally non-integrable hyperplane distribution $\xi \subset TM$. Let α be a 1-form on M^{2n+1}, representing ξ, i.e., $\xi = Ker\{\alpha\}$ (this means α is non-trivial and $\alpha(v) = 0$ for $v \in \xi$). The contact condition is then equivalent to saying that $\alpha \wedge (d\alpha)^n$ is a volume form on M^{2n+1}. The 1-form α is called a *contact form.* If the line bundle TM/ξ over M is trivial, then the contact structure is said to be co-orientable. We will only consider co-orientable contact structures on closed, orientable manifolds. Let $S^{2n+1} \subset \mathbb{R}^{2n+2}$ be the unit sphere.

Consider the symplectic form $\omega_0 = \Sigma_{i=1}^{n+1} dx_i \wedge dy_i$ on $\mathbb{R}^{2n+2}$. This induces a unique contact structure on S^{2n+1}. We call this the *standard contact structure* on sphere and denote the contact structure by ξ_{std}.

DEFINITION 1.0.7. *A diffeomorphism f between two symplectic manifolds (W_1, ω_1) and (W_2, ω_2) is called a symplectomorphism if $f^*\omega_2 = \omega_1$.*

Due to works of Thurston–Winkelnkemper [**TW**] and Giroux [**Gi**], it is now known that every contact manifold (M^{2n+1}, ξ) can be seen as an open book with an exact symplectic manifold $(V^{2n}, d\alpha)$ as page and a symplectomorphism ϕ as monodromy. In terms of the symplectic page and the monodromy, we denote such an open book by $\mathcal{A}ob(V, d\alpha, \phi) \cong (M, \xi)$.

Definition 1.0.8 (Contact open book decomposition). *$\mathcal{A}ob(V, d\alpha, \phi)$ is called a contact open book with page $(V, d\alpha)$ and binding $(\partial V, \alpha)$. Given a contact manifold (M, α) if one can find an open book $\mathcal{A}ob(V_M, \phi_M)$ of M such that $d\alpha$ restricts to a symplectic form on V_M and α induces positive orientations on M and positive contact orientation on ∂V_M, then one says that $\mathcal{A}ob(V_M, \phi_M)$ is an open book decomposition of M supporting the contact form α.*

If a contact manifold (M, ξ) has a contact form α representing ξ, such that α has a supporting open book, then we say that (M, ξ) has a *supporting open book*. Giroux [**Gi**] has proved that any contact manifold (M, ξ) has a supporting open book.

Definition 1.0.9 (contact open book embedding). *(M_1, ξ_1) contact open book embeds in (M_2, ξ_2) if there exist supporting contact open book of (M_i^{2n+1}, α_i), $\mathcal{A}ob(\Sigma_i, d\alpha_i, \phi_i)$, for $i = 1, 2$, such that the following conditions hold.*

1. *There exists a proper symplectic embedding $g : (\Sigma_1, d\alpha_1) \to (\Sigma_2, d\alpha_2)$, i.e., $g^* d\alpha_2 = d\alpha_1$,*
2. *$g \circ \phi_1 = \phi_2 \circ g$.*

Question 1.0.10 (**Contact open book embedding**). *When does a contact manifold (M^{2n+1}, ξ) contact open book embeds in the standard contact S^{2N+1}?*

In Chapter 4, we consider Question 1.0.10.

An important class of symplectomorphism is the Dehn-Seidel twist. A Dehn-Seidel twist τ, is a proper symplectomorphism of the cotangent bundle T^*S^n (see section 2.4.2). For $n = 2$, these symplectomorphisms form an infinite cyclic subgroup in the group of symplectomorphisms of T^*S^2 [**Se**]. Now observe that an embedding of M^n in V^N induces a proper symplectic embedding of T^*M^n into T^*V^N with the canonical symplectic forms on the cotangent bundles. For details see example 2.4.2. Together with the properties of a Dehn-Seidel twist, this can be used to obtain the following result.

Theorem 1.0.11 ([**S1**], Theorem 4.1.1 in Chapter 4). *For $n \geq 1$ and $k, l \in \mathbb{Z}$, $\mathcal{A}ob(DT^*S^n, \lambda^n_{can}, \tau_k)$ contact open book embeds in $\mathcal{A}ob(DT^*S^{n+1}, \lambda^{n+1}_{can}, \tau_l)$.*

Here, $d\lambda^n_{can}$ denotes the canonical symplectic form on the unit disk cotangent bundle DT^*S^n of S^n, and τ_m denotes the m-fold Dehn-Seidel twist (see section 2.4.2) for $m \in \mathbb{Z}$.

The following result was first established by Casals and Murphy [**CM**].

Corollary 1.0.12 ([**S1**], Corollary 4.1.2 in Chapter 4). *For all $n \geq 1$, there exists an overtwisted contact structure on S^{2n+1} that contact open book embeds in (S^{2n+3}, ξ_{std}).*

We can use Theorem 4.1.1 to produce a large class of contact manifolds that admits co-dimension 2 contact open book embedding in the standard contact sphere. For this we introduce the notion of a *type*-1 contact open book. The operations of boundary connected sum and plumbing will be denoted by $\natural_b$ and $\S$, respectively. For the definitions of these operations see section 2.4.3 and 2.4.4.

Definition 1.0.13. *Consider the canonical symplectic structure $d\lambda_M$ on the cotangent bundle of a manifold M. We call a contact open book $\mathcal{A}ob(V^{2n}, \omega, \phi)$ type-1 if it satisfies the following properties.*

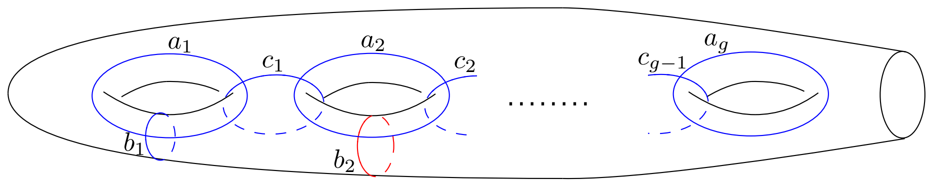

FIGURE 1. Humphreys generators of mapping class groups of Σ_g

(1) (V^{2n}, ω) *is symplectomorphic to*
$(DT^*M_1 \S DT^*M_2 \S ... \S DT^*M_p \#_b DT^*N_1 \#_b DT^*N_2 .... \#_b DT^*N_q, d\lambda_{M_1} \S d\lambda_{M_2} \S ... \S d\lambda_{M_p} \#_b d\lambda_{N_1} \#_b ... \#_b d\lambda_{N_q})$.
Here M_i and N_j are either S^n or a closed n-manifold embedded in S^{n+1}.

(2) ϕ *is generated by Dehn-Seidel twists along the S^ns among M_is and N_js .*

THEOREM 1.0.14 ([**S1**],Theorem 4.1.4 in Chapter 4). *If (M^{2n+1}, ξ) is a contact manifold supported by an open book of type-1, then (M^{2n+1}, ξ) has a contact open book embedding in (S^{2n+3}, ξ_{std}).*

Consider the surface Σ_g as in Figure 1. Note that the mapping class group of $(\Sigma_g, \partial\Sigma_g)$ is generated by Dehn twists along the curves $a_1, c_1, a_2, c_2, ...a_{g-1}, c_{g-1}, a_g, b_1$ and b_2. As an application of Theorem 4.1.4, we get the following result due to Etnyre and Lekili (see Theorem 4.3 in [**EL**]).

COROLLARY 1.0.15 ([**S1**], Corollary 4.1.5). *Let (M^3, ξ) be a contact 3-manifold supported by an open book with page Σ_g and its monodromy is generated by $a_1, c_1, ..., a_{g-1}, c_{g-1}, a_g, b_1$. Then (M^3, ξ) contact open book embeds in (S^5, ξ_{std}).*

Examples of explicit nontrivial contact open book embedding were previously constructed in the works of Casals and Murphy [**CM**] and Etnyre and Lekili [**EL**].

Next we consider the question of isocontact and contact embedding.

DEFINITION 1.0.16 (Isocontact embedding). *(M^{2n+1}, ξ) admits an isocontact embedding in (V^{2N+1}, η), if there is an embedding $\iota : M \hookrightarrow V$ such that for all p in M, $D\iota(T_pM)$ is transverse to $\eta_{\iota(p)}$ and $D\iota(T_pM) \cap \eta_{\iota(p)} = D\iota(\xi_p)$. A manifold M^{2n+1} contact embeds in (V^{2N+1}, η) if there exists a contact structure ξ_0 on M^{2n+1} such that (M, ξ_0) has an isocontact embedding in (V^{2N+1}, η). In case of co-orientable contact structures, it follows from the definition that if α is a contact form representing ξ and β is a contact form representing η, then $\iota^*(\beta) = h \cdot \alpha$ for some positive function h on M. Thus, $D\iota(\xi)$ is a conformal symplectic sub-bundle of $(\eta|_{\iota(M)}, d\beta)$.*

Gromov [**EM**] proved that any contact manifold (M^{2n+1}, ξ) has an isocontact embedding in $(\mathbb{R}^{4n+3}, \xi_{std})$. This result is the contact analog of Whitney's embedding theorem. Haefliger and Hirsch [**HH**] generalized Whitney's theorem to show that a closed, orientable, k-connected n-manifold can be embedded in $\mathbb{R}^{2n-k-1}$. For π-manifolds this result was further refined by Sapio [**Sa**], who proved that every closed, orientable, k-connected π-manifold M^n that bounds a π-manifold, can be embedded in $\mathbb{R}^{2n-2k-1}$ with trivial normal bundle.

QUESTION 1.0.17 (**Contact and isocontact embedding**). *What are the suitable analogues of the Haefliger–Hirsch theorem and Sapio's theorem in the contact category?*

In [**Sa**], Sapio introduced the notion of an almost embedding. A manifold M^n *almost embeds* in a manifold W^N, if there exists a homotopy sphere Σ^n so that $M^n\#\Sigma^n$ smoothly embeds in W^N. We want to define analogous notions for contact and isocontact embeddings. Recall that if (M, ξ_M) and (N^{2n+1}, ξ_N) are two contact manifolds, then by $(M^{2n+1}\#N^{2n+1}, \xi_M\#\xi_N)$ we denote the contact connected sum of them. For details on a contact connected sum we refer to chapter 6 of [**Ge**].

DEFINITION 1.0.18 (Homotopy isocontact embedding). *(M^{2n+1}, ξ) admits a homotopy isocontact embedding in $(\mathbb{R}^{2N+1}, \xi_{std})$, if there exists a contact homotopy sphere (Σ^{2n+1}, η) such that $(M^{2n+1}\#\Sigma^{2n+1}, \xi\#\eta)$ has an isocontact embedding in $(\mathbb{R}^{2N+1}, \xi_{std})$. We say, M^{2n+1} homotopy contact embeds in $(\mathbb{R}^{2N+1}, \xi_{std})$, if there is a homotopy sphere Σ^{2n+1} and a contact structure ξ_0 on $M\#\Sigma^{2n+1}$ such that $(M\#\Sigma^{2n+1}, \xi_0)$ has an isocontact embedding in $(\mathbb{R}^{2N+1}, \xi_{std})$.*

In chapter 5 We prove the following analog of Sapio's Theorem for contact π–manifolds.

THEOREM 1.0.19 ([**S2**], Theorem 5.1.1). *Let M^{2n+1} be a k-connected, π–manifold. Assume that $n \geq 2$ and $k \leq n-1$. Then*

1. *M^{2n+1} homotopy contact embeds in $(\mathbb{R}^{4n-2k+3}, \xi_{std})$.*
2. *If $n \not\equiv 3\ (mod\ 4)$ and for all $i \in \{k+1, \cdots, 2n-k\}$ such that $i \equiv 0, 2, 6, 7\ (mod\ 8)$, $H_{2n-i+1}(M) = 0$, then for any contact structure ξ on M^{2n+1}, (M, ξ) has a homotopy isocontact embedding in $(\mathbb{R}^{4n-2k+3}, \xi_{std})$.*
3. *If $n \not\equiv 3\ (mod\ 4)$ and for all $i \in \{k+1, \cdots, 2n-k\}$ such that $i \equiv 0, 7\ (mod\ 8)$, $H_{2n-i+1}(M) = 0$, then for any SO-contact structure ξ on M^{2n+1}, (M, ξ) has a homotopy isocontact embedding in $(\mathbb{R}^{4n-2k+3}, \xi_{std})$.*
4. *If M^{2n+1} bounds a π–manifold, then we can omit "homotopy" in the above statements.*

In all the statements above, we get contact or isocontact embeddings with a trivial conformal symplectic normal bundle.

The proof of Theorem 5.1.1 is based on Gromov's h-principle for existence of contact structure on open manifold. Roughly speaking, we put a contact structure on a tubular neighborhood of the embedded contact manifold, extend it to an almost contact structure on the ambient manifold using obstruction theory and then apply Gromov's h-principle.

COROLLARY 1.0.20 ([**S2**], Corollary 5.1.2). *Let M^{2n+1} be an $(n-1)$-connected π–manifold that bounds a π–manifold. Then*

1. *M^{2n+1} contact embeds in $(\mathbb{R}^{2n+5}, \xi_{std})$.*
2. *If $n \equiv 4, 5\ (mod\ 8)$, then for any contact structure ξ, (M, ξ) has an isocontact embedding in $(\mathbb{R}^{2n+5}, \xi_{std})$.*

In particular, any contact homotopy sphere Σ^{2n+1} that bounds a parallelizable manifold has an isocontact embedding in $(\mathbb{R}^{2n+5}, \xi_{std})$, for $n \equiv 0, 1, 2\ (mod\ 4)$.

For example, by [**KM**], we get that all 11-dimensional contact homotopy spheres has an isocontact embedding in $(\mathbb{R}^{15}, \xi_{std})$.

Using similar techniques as above and Gromov's h-principles for contact immersion and isocontact embedding we prove the following result for parallelizable manifolds.

THEOREM 1.0.21 ([**S2**], Theorem 5.1.4). *Let M^{2n+1} be a parallelizable manifold.*

(1) *For any contact structure ξ on M^{2n+1}, (M^{2n+1}, ξ) contact immerses in $(\mathbb{R}^{2n+3}, \xi_{std})$.*
(2) *If M^{2n+1} is 5-connected, then for $n \equiv 0, 1 \ (mod\ 4)$ and $n \geq 7$, any contact structure ξ, (M^{2n+1}, ξ) has an isocontact embedding in $(\mathbb{R}^{4n-3}, \xi_{std})$.*

COROLLARY 1.0.22 ([**S2**], Corollary 5.1.5). *Let $M^{2n+1} = N^{2n-1} \times (S^1 \times S^1)$. Where N^{2n-1} is a π–manifold that embeds in $\mathbb{R}^{2N+1}$ with trivial normal bundle. Then M^{2n+1} contact embeds in $(\mathbb{R}^{2N+5}, \xi_{std})$.*

In [**BEM**], S. Borman, Y. Eliashberg and E. Murphy defined the notion of an overtwisted contact ball in all dimensions. Any contact structure that admits a contact embedding of such an overtwisted ball is called an overtwisted contact structure. These contact structures were shown to satisfy the h-principle for homotopy of contact structures. Using this, we prove a uniqueness result for embedding of certain π-manifolds in an overtwisted contact structure η_{ot} on $\mathbb{R}^{2N+1}$, analogous to Theorem 1.25 in [**EF**].

THEOREM 1.0.23 ([**S2**], Theorem 5.1.6). *Let (M^{8k+3}, ξ) be a contact π–manifold such that $H_i(M; \mathbb{Z}) = 0$, for $i \equiv\ 2, 4, 5, 6 \ (mod\ 8)$. Let $\iota_1, \iota_2 : (M^{8k+3}, \xi) \to (\mathbb{R}^{2N+1}, \eta_{ot})$ be two isocontact embeddings with trivial conformal symplectic normal bundle such that both the complements of $\iota_1(M)$ and $\iota_2(M)$ in $(\mathbb{R}^{2N+1}, \eta_{ot})$ are overtwisted. If ι_1 and ι_2 are smoothly isotopic, then there is a contactomorphism $\chi : (\mathbb{R}^{2N+1}, \eta_{ot}) \to (\mathbb{R}^{2N+1}, \eta_{ot})$ such that $\chi \cdot \iota_1 = \iota_2$.*

For example, any two isocontact embeddings of $(S^{8k_1} \times S^{8k_2+3}, \xi_0)$ in $(\mathbb{R}^{8k_1+8k_2+5}, \eta_{ot})$ which satisfy the hypothesis of Theorem 5.1.6, are equivalent.

In recent times much progress has been made on the question of co-dimension 2 contact embedding due to the works of Kasuya [**Ka2**], Etnyre and Furukawa [**EF**], Etnyre and Lekili [**EL**] and Pancholi and Pandit [**PP**]. Recently, the existence and uniqueness questions for co-dimension 2 iso-contact embedding have been completely answered by the works of Casals, Pancholi and Presas [**CPP**], Casals and Etnyre [**CE**] and Honda and Huang [**HoH**].

CHAPTER 2

Preliminaries

We review some basic notions that will be used in later chapters.

2.1. Smooth embedding and isotopy

Let M^n and N^{n+k} denote smooth manifolds of dimension n and $n+k$ respectively. Assume that M^n is compact.

DEFINITION 2.1.1 (Immersion and embedding). *A smooth map $f : M^n \to N^{n+k}$ is called an immersion if the derivative map $Df : TM \to TN$ is injective at each point of M^n. A regular immersion which is an injective map is called an embedding.*

For M^n closed, we shall assume $k \geq 1$. The Whitney embedding theorem [**Wh**] says that every smooth n-manifold embeds in $\mathbb{R}^{2n}$. This fact was later generalized by Haefliger and Hirsch [**HH**] to show that every k-connected n-manifold embeds in $\mathbb{R}^{2n-k}$.

The notion of equivalence for embeddings is called *isotopy*.

DEFINITION 2.1.2 (Isotopy). *Two embeddings $f, g : M^n \to N^{n+k}$ are called isotopic if there exists a family of embeddings $h_t : M^n \to N^{n+k}$ for $t \in [0,1]$, such that $h_0 = f$ and $h_1 = g$.*

A well known theorem of Wu [**Wu**] says that any two embeddings of M^n in $\mathbb{R}^{2n+1}$ are isotopic. This theorem was later generalized by Haefliger and Hirsch [**HH**] for embeddings of k-connected manifolds in $\mathbb{R}^{2n-k+1}$.

Let $(V, \partial V)$ and $(W, \partial W)$ denote manifolds with nonempty boundaries. Assume that the dimension of W is strictly greater than the dimension of V.

DEFINITION 2.1.3 (Proper embedding). *A proper embedding $f_0 : (V, \partial V) \to (W, \partial W)$ is an embedding of V in W that maps the interior of V to the interior of W and the boundary ∂V to the boundary ∂W.*

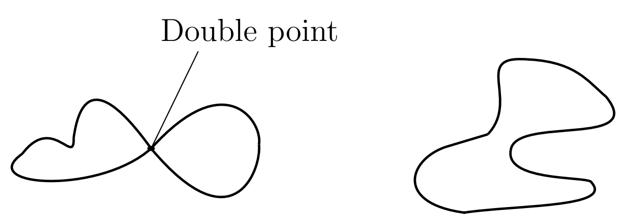

FIGURE 1. On the left is an immersion of S^1 in $\mathbb{R}^2$ and on the right is an embedding.

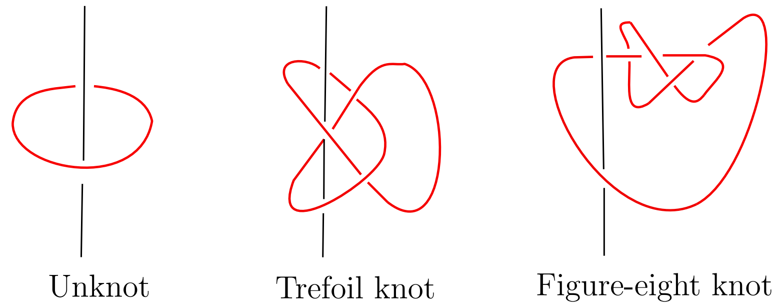

FIGURE 2. Non-isotopic embeddings of S^1 in $\mathbb{R}^3$.

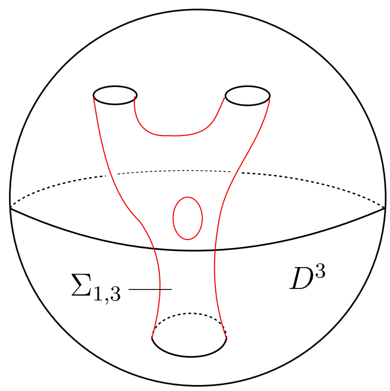

FIGURE 3. Proper embedding of $\Sigma_{1,3}$, a surface with genus 1 and 3 boundary components, into D^3.

DEFINITION 2.1.4 (Relative isotopy). *A* relative isotopy h_t^0 *between two proper embeddings* $f_0, g_0 : (V, \partial V) \to (W, \partial W)$ *such that* $f_0|_{\partial V} = g_0|_{\partial V}$*, is a family of proper embeddings* h_t^0 *such that* $h_0^0 = f_0$ *and* $h_1^0 = g_0$ *and* $h_t^0 = f_0|_{\partial V}$ *for* $t \in [0, 1]$.

For example, any oriented surface with boundary admits a proper embedding in the unit disk in $\mathbb{R}^3$. A relative version of the Whitney theorem says that every n-manifold with boundary has proper embedding in the unit disk $D^{2n} \subset \mathbb{R}^{2n}$. Similarly, a relative version of Wu's theorem says that any two proper embeddings of an n-manifold in $(D^{2n+1}, \partial D^{2n+1})$ are relatively isotopic.

2.2. Open books

Roughly speaking, an open book is a decomposition of a manifold into a co-dimension 2 submanifold and a fibration over S^1.

DEFINITION 2.2.1 (Open book decomposition). *An open book decomposition of a closed oriented manifold* M *consists of a co-dimension* 2 *oriented submanifold* B *with a trivial normal bundle in* M *and a locally trivial fibration* $\pi : M \setminus B \to S^1$ *such that* $\pi^{-1}(\theta)$ *is the interior of a co-dimension* 1 *submanifold* N_θ *and* $\partial N_\theta = B$*, for all* $\theta \in S^1$*. The submanifold* B *is called the binding and* N_θ *is called a page of the open book. We denote the open book decomposition of* M *by* $(M, \mathcal{O}b(B, \pi))$ *or sometimes simply by* $\mathcal{O}b(B, \pi)$.

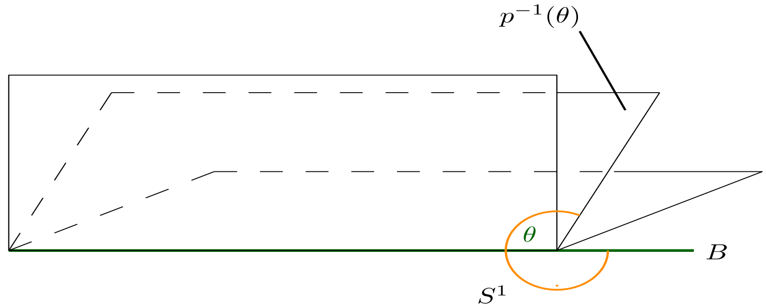

FIGURE 4. Fibration of an open book over S^1, in the complement of the binding B, with page $p^{-1}(\theta)$,

EXAMPLE 2.2.1. *If one removes the co-dimension* 2 *submanifold* $\mathbb{R}^n \times \{0\}$ *from* $\mathbb{R}^{n+2} \cong \mathbb{R}^n \times \mathbb{R}^2$*, then the complement trivially fibers over* S^1 *with fiber* $\mathbb{R}^{n+1}$*. This open book on* $\mathbb{R}^{n+2}$ *induces an open book on* S^{n+2} *via the one point compactification, with binding* S^n*.*

Another way to look at the above example is taking the natural embedding $S^{n+2} \subset \mathbb{R}^{n+3}$ *as unit sphere and then restricting the open book decomposition of* $\mathbb{R}^{n+3}$ *with binding* $\mathbb{R}^{n+1}$ *on* S^{n+2} *to obtain an open book decomposition of* S^{n+2} *with binding* $S^n \subset \mathbb{R}^{n+1}$*.*

There is another notion of an *abstract open book* that is equivalent to the above definition and in many cases is easier to work with. Let us first recall the following.

DEFINITION 2.2.2 (Mapping torus). *Let* Σ *be a manifold with non-empty boundary* $\partial\Sigma$*. Let* ϕ *be an element of the mapping class group of* $(\Sigma, \partial\Sigma)$*. By the mapping torus* $\mathcal{MT}(\Sigma, \phi)$*, we mean the quotient manifold*

$$\Sigma \times [0, 1]/\sim$$

obtained by identifying $(x, 0)$ *with* $(\phi(x), 1)$*.*

DEFINITION 2.2.3 (Abstract open book). *Let* Σ *and* ϕ *be as in the previous definition. An* abstract open book *decomposition of* M *is a pair* (Σ, ϕ) *such that* M *is diffeomorphic to* $\mathcal{MT}(\Sigma, \phi) \cup_{id} \partial\Sigma \times D^2$*, where one identifies* $\partial\mathcal{MT}(V^{n-1}, \phi) = \partial V \times S^1$ *with* $\partial V \times \partial D^2$ *via the identity map.*

Note that the isotopy class of ϕ uniquely determines M, up to diffeomorphisms. The map ϕ is called the *monodromy* of the open book.

Two abstract open books $\mathcal{A}ob(\Sigma_1, \phi_1)$ and $\mathcal{A}ob(\Sigma_2, \phi_2)$ are *equivalent* if there is a diffeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $h \circ \phi_1 = \phi_2 \circ h$. This holds because of a symmetry property of open books. If $(\Sigma, \partial\Sigma)$ is as above and ϕ_1 and ϕ_2 are two monodromies, then $\mathcal{A}ob(V, \phi_1 \circ \phi_2) \cong \mathcal{A}ob(V, \phi_2 \circ \phi_1)$.

It is not hard to see that an abstract open book decomposition of M, up to equivalence, gives an open book decomposition of M up to diffeomorphism and vice versa. The boundary of the page, $\partial\Sigma$, gives the binding in the first definition and a fiber of the fibration $\pi : M \setminus B \to S^1$ gives the page in the second definition. Hence, we *will not distinguish between open books and abstract open books.* For more on open books, see [**Et**] and [**Ge**].

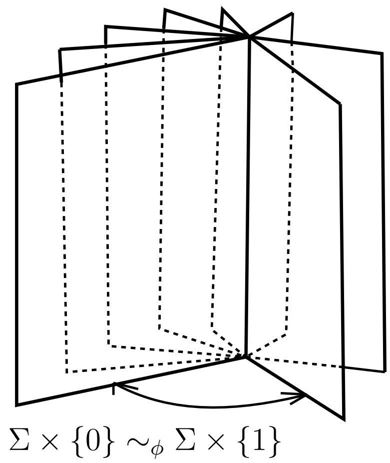

FIGURE 5. Mapping torus of an abstract open book

EXAMPLE 2.2.2. *S^n admits an open book decomposition with pages D^{n-1} and monodromy the identity map of D^{n-1}. We call this open book the trivial open book of S^n.*

EXAMPLE 2.2.3. *$S^3 \times S^2$ admits an open book decomposition with pages the unit disk bundle DT^*S^2 and monodromy the identity. We call this open book the standard open book decomposition of $S^3 \times S^2$. Recall that DT^*S^2 is homeomorphic to a 2-disk bundle over S^2 with Euler number -2. We will later see that this manifold can also be obtained by attaching a 4-dimensional 2-handle to D^4 along an unknot in its boundary $\partial D^4 = S^3$ with framing -2. This implies that $\partial DT^*S^2 = \mathbb{R}P^3$.*

In [**Al**], J. Alexander proved that every closed oriented 3–manifold admits an open book. Open book decompositions of closed oriented simply connected manifolds were studied by H. Winkelnkemper in [**Wi**], where he proved the existence of such decompositions in dimension $n \geq 6$, provided n is not divisible by 4. He also established that if the dimension $n > 6$ of a closed simply connected manifold is divisible by 4, then it admits an open book decomposition if and only if its signature is zero. Winkelnkemper's results were then extended to the non-simply connected case by J. Lawson [**La**], F.Quinn [**Qu**] and I. Tamura [**Ta**]. Due to these works, the conditions under which a manifold admits an open book decomposition is now well known. In particular, every closed orientable odd dimensional manifold admits an open book decomposition. For more on open book decompositions we refer to [**Et**] and [**Ko**].

Now we define the notion of an open book embedding. Let M^n and V^N be manifolds admitting open book decompositions. Assume $N \geq n+1$.

DEFINITION 2.2.4 (Open book embedding). *M^n has an open book embedding in V^N if there is an open book $\mathcal{A}ob(\Sigma_M^{n-1}, \phi_M)$ of M and an open book $\mathcal{A}ob(\Sigma_V^{N-1}, \phi_V)$ of V such that the following conditions hold:*

(1) *there exists a proper embedding $f : (\Sigma_M, \partial\Sigma_M) \to (\Sigma_V, \partial\Sigma_V)$,*
(2) *$\phi_V \circ f = f \circ \phi_M$.*

We also say that M^n open book embeds in V^N with respect to the open book $\mathcal{A}ob(\Sigma_V, \phi_V)$.

EXAMPLE 2.2.4. *$S^n \cong \mathcal{A}ob(D^{n-1}, id)$ canonically open book embeds in $S^N \cong \mathcal{A}ob(D^{N-1}, id)$ for $N - n \geq 1$.*

EXAMPLE 2.2.5. *The Whitney embedding theorem says that every m-dimensional manifold embeds in $\mathbb{R}^{2m+1}$. By* [**Wu**]*, any two embeddings of a closed manifold M^m in $\mathbb{R}^{2m+1}$ are isotopic. These results also hold for proper embedding of manifolds with boundary. Let $M^m = \mathcal{A}ob(V^{m-1}, \phi)$. Let $f : (V, \partial V) \to (D^{2m-1}, S^{2m-2})$ be a proper embedding. By* [**Wu**]*, f and $f \circ \phi$ are relative isotopic. By the isotopy extension theorem one can extend this isotopy to an ambient isotopy Φ_t of (D^{2m-1}, S^{2m-2}) such that $\Phi_0 = id$ and $\Phi_1 \circ f = f \circ \phi$. Hence, every manifold M^m, admitting open book decomposition, open book embeds in S^{2m} with respect to the trivial open book $\mathcal{A}ob(D^{2n-1}, id)$. This result can be generalized to show open book embedding of k-connected closed oriented n-manifolds in $\mathcal{A}ob(D^{2n-k-1}, id) = S^{2n-k}$* [**NS**].

2.3. Contact and isocontact embedding

2.3.1. Contact structures. We start with the notion of a symplectic structure.

DEFINITION 2.3.1. *A* symplectic form *ω on an even dimensional real vector space L is a non-degenerate, skew-symmetric 2-form.*

EXAMPLE 2.3.1. *The 2-form $\omega_0 = \Sigma_{i=1}^n dx_i \wedge dy_i$ is called the* standard symplectic form *or symplectic structure on $\mathbb{R}^{2n}$.*

An *almost symplectic structure* on an even dimensional manifold is a distribution of non-degenerate, skew-symmetric 2-forms ω on its tangent bundle. It is called a symplectic form or symplectic structure on that manifold if $d\omega = 0$.

EXAMPLE 2.3.2. *Given a smooth manifold M^n, there exists a natural symplectic form on its cotangent bundle T^*M^n. We denote this canonical symplectic form by $d\lambda_M$. Here, λ_M is the canonical 1-form which, in terms of local co-ordinates $(q_1, ...q_n, p_1, ..., p_n)$ on T^*M^n, is given by the 1-form $\Sigma_{i=1}^n p_i dq_i$.*

We recall the notion of a *symplectic vector bundle.* A symplectic vector bundle over a manifold M is a pair (E, ω) consisting of a real vector bundle $\pi : E \to M$ and a family of symplectic bilinear forms ω_q on the fibers $E_q = \pi^{-1}(q)$ of the vector bundle that vary smoothly with $q \in M$.

DEFINITION 2.3.2. *A* contact manifold *is an odd dimensional smooth manifold M^{2n+1}, together with a maximally non-integrable co-dimension 1 distribution $\xi \subset TM$. A* contact form *α representing ξ is a local 1-form on M such that $\xi = Ker\{\alpha\}$. The contact condition then says that $\alpha \wedge (d\alpha)^n$ is a volume form.*

If the line bundle TM/ξ over M is trivial, then the contact structure is said to be *co-orientable.* In this thesis, we will only consider co-orientable contact structures on closed, orientable manifolds.

EXAMPLE 2.3.3. *The contact structure on $\mathbb{R}^{2n+1}$, given by the contact 1-form $\alpha_0 = -dz + \Sigma_{i=1}^n x_i dy_i$, is called the standard contact structure and is denoted by $(\mathbb{R}^{2n+1}, \xi_{std})$. See Figure 6 for pictures of $(\mathbb{R}^3, \xi_{std})$.*

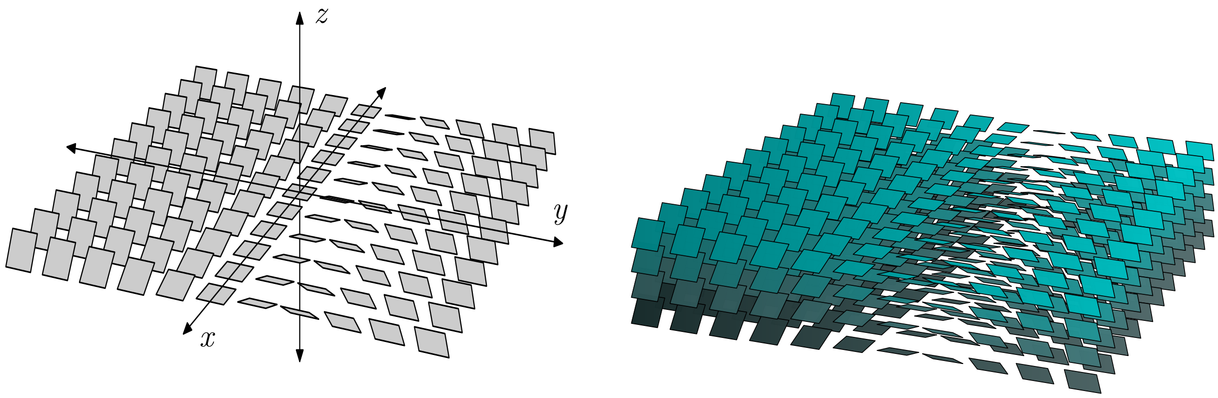

FIGURE 6. The standard contact structure on $\mathbb{R}^3$ given by the contact form $-dz + ydx$. (Courtesy: Momotaro [Public domain], WIKIMEDIA COMMONS)

EXAMPLE 2.3.4. *Let $S^{2n+1} \subset \mathbb{R}^{2n+2}$ be the unit sphere. Consider the symplectic form $\omega = \Sigma_{i=1}^{n+1} dx_i \wedge dy_i$ on $\mathbb{R}^{2n+2}$. The 1-form $\lambda = \Sigma_{i=1}^{n+1}(x_i dy_i - y_i dx_i)$, which is the ω-dual of the radial vector field $R = \Sigma_{i=1}^{n+1}(x_i \partial_{x_i} + y_i \partial_{y_i})$, defines a contact structure on S^{2n+1}. The restriction of this contact structure on $S^{2n+1} \setminus \{pt.\}$ is known to be contactomorphic to $(\mathbb{R}^{2n+1}, \xi_{std})$. We call this contact sphere as the standard contact $(2n+1)$-sphere and denote it by (S^{2n+1}, ξ_{std}).*

If α is a contact form on M^{2n+1}, then the 2-form $d\alpha$ induces a conformal symplectic bundle structure on ξ. We will denote a manifold M together with a contact structure ξ by (M, ξ).

Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are equivalent if there is a diffeomorphism h between them such that $Dh(\xi_1) = \xi_2$. We say, the two contact manifolds are *contactomorphic* to each other. For more details on contact manifolds see [**Ge**].

Two fundamental theorems in contact geometry are the Darboux theorem and the Gray Stability theorem. These theorems illustrate the fact that contact manifolds do not have local invariants. For proofs of these theorems see [**Ge**].

THEOREM 2.3.3 (Gray stability, [**Ge**]). *Let ξ_t be a smooth family of contact structures on a closed manifold M for $t \in [0, 1]$. Then there is an isotopy χ_t of M such that $D\chi_t(\xi_0) = \xi_t$ for each $t \in [0, 1]$.*

THEOREM 2.3.4 (Darboux's Theorem). *Let α be a contact 1-form on M^{2n+1} and p be a point in M^{2n+1}. There exist coordinates $\{x_1, ..., x_n, y_1, ..., y_n, z\}$ on a neighborhood $U \subset M$ of p, such that $p = (0, ..., 0)$ and $\alpha|_U = -dz + \Sigma_{i=1}^{n} x_i dy_i$.*

2.3.2. Isocontact embedding and isocontact immersion.

DEFINITION 2.3.5 (Isocontact and contact embedding). *(M^{2n+1}, ξ) admits an isocontact embedding in (V^{2N+1}, η) if there is an embedding $\iota : M \hookrightarrow V$ such that for all p in M, $D\iota(T_pM)$ is transverse to $\eta_{\iota(p)}$ and $D\iota(T_pM) \cap \eta_{\iota(p)} = D\iota(\xi_p)$. A manifold M^{2n+1} contact embeds in (V^{2N+1}, η) if there exists a contact structure ξ_0 on M^{2n+1} such that (M, ξ_0) has an isocontact embedding in (V^{2N+1}, η).*

For an isocontact embedding ι, it follows from the definition that if α is a contact form representing ξ and β is a contact form representing η, then $\iota^*(\beta) = h \cdot \alpha$ for some positive function h on M. In other words, $D\iota(\xi)$ is a conformal symplectic sub-bundle of $(\eta|_{\iota(M)}, d\beta)$.

DEFINITION 2.3.6 (Isocontact immersion). *An isocontact immersion of (M, ξ) in $(\mathbb{R}^{2N+1}, \eta)$ is an immersion $j : (M, \xi) \looparrowright (\mathbb{R}^{2N+1}, \eta)$ such that $Dj(TM)$ is transverse η and $Dj(TM) \cap \eta = Dj(\xi)$.*

EXAMPLE 2.3.5. *Let c be a positive number. Consider an embedding $g : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+3}$ given by $g(x_1, x_2, ..., x_n, y_1, y_2, ..., y_n, z) = (a \cdot x_1, a \cdot x_2, ...a \cdot x_n, 0, b \cdot y_1, b \cdot y_2, ..., b \cdot y_n, 0, ab \cdot z)$, for $a, b > 0$. Then, $g^*(-dz + \Sigma_{i=1}^n x_i dy_i) = ab(-dz + \Sigma_{i=1}^n x_i dy_i)$. Thus, g is an isocontact embedding of $(\mathbb{R}^{2n+1}, \xi_{std})$ in $(\mathbb{R}^{2n+3}, \xi_{std})$*

EXAMPLE 2.3.6. *Let $S^{2n+1} \subset \mathbb{R}^{2n+2}$ be the unit sphere. Let J_0 be the standard complex structure on $\mathbb{R}^{2n+2}$. Then $TS^{2n+1} \cap J_0 TS^{2n+1}$ defines a contact structure on S^{2n+1}, which is contactomorphic to the standard contact structure ξ_{std} on S^{2n+1}. The standard inclusion of $\mathbb{R}^{2n+2}$ in $\mathbb{R}^{2n+2k+2}$ induces an isocontact embedding of (S^{2n+1}, ξ_{std}) in $(S^{2n+2k+1}, \xi_{std})$.*

2.4. Contact open book and contact open book embedding

2.4.1. Contact manifolds as open books. We start with a contact analogue of the abstract open book decomposition. This construction of an *abstract contact open book* is due to Thurston and Winkelnkemper [**TW**]. The discussion here is based on the lecture notes by Otto Van Koert [**Ko**].

Let $(V, \partial V, d\alpha)$ be an exact symplectic manifold, with a collar neighborhood $N(\partial V)$ symplectomorphic to $((-1, 0] \times \partial V, d(e^s \cdot \alpha))$, for $s \in (-1, 0]$. The Liouville vector field Y for $d\alpha$ is defined by $i_Y d\alpha = \alpha$. So, near boundary it looks like $\frac{\partial}{\partial s}$ and is transverse to ∂V, pointing outwards. The 1-form $e^s \cdot \alpha$ induces a contact structure on ∂V. Let ϕ be a symplectomorphism of $(V, d\alpha)$ such that ϕ is identity in a neighborhood of the boundary. The following lemma, due to Giroux, shows that we can assume $\phi^*\alpha - \alpha$ to be exact.

LEMMA 2.4.1 (Giroux). *The symplectomorphism ϕ of $(V, d\alpha)$ is isotopic, via symplectomorphisms which are identity near ∂V, to a symplectomorphism ϕ_1 such that $\phi_1^*\alpha - \alpha$ is exact.*

For a proof of the above lemma see [**Ko**].

Let $\phi^*\alpha - \alpha = dh$. Here, $h : V \to \mathbb{R}$ is a function well defined up to addition by constants. Note that $dt + \alpha$ is a contact form on $\mathbb{R} \times V$, where the t coordinate is along $\mathbb{R}$. Consider the mapping torus $\mathcal{MT}(V, \phi)$ defined by the following map.

$$\begin{aligned} \Delta : (\mathbb{R} \times V, dt + \alpha) &\longrightarrow (\mathbb{R} \times V, dt + \alpha) \\ (t, x) &\longmapsto \quad (t - h, \phi(x)) \end{aligned}$$

The contact form $dt + \alpha$ then descends to a contact form λ on $\mathcal{MT}(V, \phi)$. Since ϕ is identity near ∂V, a contact neighborhood of the boundary of $(\mathcal{MT}(V, \phi), \lambda)$ looks like $((-\frac{1}{2}, 0) \times \partial V \times S^1, e^r \cdot \alpha|_{\partial V} + dt)$. Let $A(r, R) = \{z \in \mathbb{C} \mid r < |z| < R\}$. Define Φ as follows.

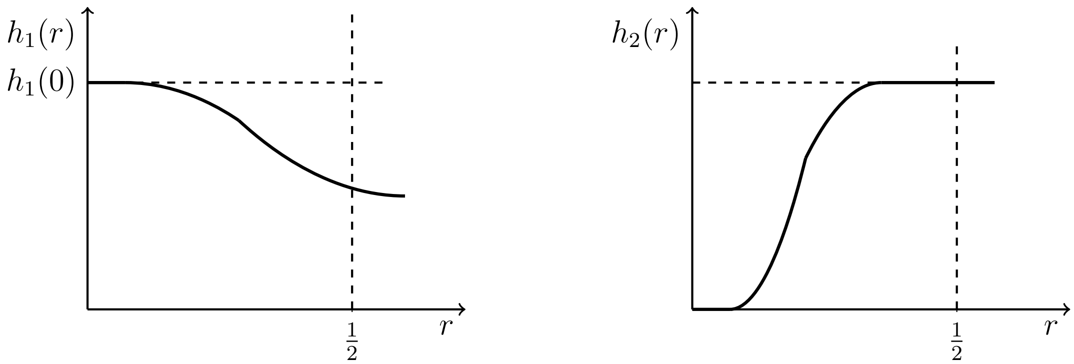

FIGURE 7. Functions for the contact form near binding

$$\begin{aligned} \Phi : \partial V \times A(\frac{1}{2}, 1) &\longrightarrow (-\frac{1}{2}, 0) \times \partial V \times S^1 \\ (v, re^{it}) &\longmapsto \qquad (\frac{1}{2} - r, v, t) \end{aligned}$$

Using Φ, we can glue $\mathcal{MT}(V, \phi)$ and $\partial V \times D^2$ along a neighborhood of their boundary, such that under Φ, the 1-form λ pulls back to $(e^{\frac{1}{2}-r} \cdot \alpha|_{\partial V} + dt)$ on $\partial V \times A(\frac{1}{2}, 1)$. We want to extend this 1-form to a form $\beta = h_1(r) \cdot \alpha|_{\partial V} + h_2(r) \cdot dt$, such that β is contact in the interior of $\partial V \times D^2$. We can choose the functions h_1 and h_2 (see Figure 7 2.4.1) so that β becomes a globally defined contact form on $W^{2n+1} = \mathcal{MT}(V, \phi) \cup_{id} \partial V \times D^2$, and it coincides with λ on $\mathcal{MT}(V, \phi)$ and with $\alpha + r^2 dt$ on $\partial V \times D^2$. We will denote the resulting contact manifold (W^{2n+1}, β) as $\mathcal{A}ob(V, d\alpha; \phi)$.

The contact manifold $\mathcal{A}ob(V, d\alpha, \phi)$ depends on the symplectic isotopy class of the monodromy ϕ. If ϕ_1 and ϕ_2 are symplectomorphisms of $(V, d\alpha)$, then $\mathcal{A}ob(V, d\alpha, \phi_1 \circ \phi_2)$ is contactomorphic to $\mathcal{A}ob(V, d\alpha, \phi_2 \circ \phi_1)$.

DEFINITION 2.4.2 (Contact open book). *$\mathcal{A}ob(V, d\alpha, \phi)$ is called a* contact open book *with* page $(V, d\alpha)$ *and* monodromy ϕ. *The contact manifold $(\partial V, \alpha)$ is called the* binding.

Given a contact manifold (M, α) with a contact 1-form α, if one can find an open book $\mathcal{A}ob(V_M, \phi_M)$ of M, such that $d\alpha$ restricts to a symplectic form on V_M and α induces positive orientation on M and positive contact orientation on ∂V_M, then one says that $\mathcal{A}ob(V_M, \phi_M)$ is an open book decomposition of M *supporting* the contact form α.

If a contact manifold (M, ξ) has a contact form α representing ξ, such that α has a supporting open book, then we say that (M, ξ) has a *supporting open book*. Sometimes, we may write $(M, \xi) = \mathcal{A}ob(V_M, d\alpha; \phi_M)$ to say that (M, ξ) is *supported* by the open book with page $(V_M, d\alpha)$ and monodromy ϕ_M. Giroux [**Gi**] then says that any contact manifold (M, ξ) has a supporting open book.

EXAMPLE 2.4.1. *(S^{2n+1}, ξ_{std}) has a contact open book decomposition with page $(D^{2n}, \Sigma_{i=1}^n r_i dr_i \wedge d\theta_i)$ and monodromy identity.*

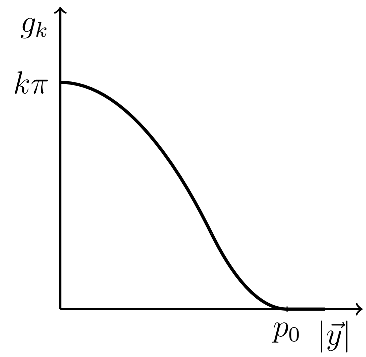

FIGURE 8.

DEFINITION 2.4.3 (contact open book embedding). (M_1, ξ_1) contact open book embeds *into* (M_2, ξ_2)*, if there exist supporting contact open books* $\mathcal{A}ob(\Sigma_1, d\alpha_1, \phi_1)$ *and* $\mathcal{A}ob(\Sigma_2, d\alpha_2, \phi_2)$*, for* (M_1, ξ_1) *and* (M_2, ξ_2) *respectively, such that the following conditions hold.*

(1) *There exists a symplectic proper embedding* $g : (\Sigma_1, d\alpha_1) \to (\Sigma_2, d\alpha_2)$*, i.e.,* $g^* d\alpha_2 = d\alpha_1$ *,*
(2) $g \circ \phi_1 = \phi_2 \circ g$*.*

The above definition implies that the mapping torus $\mathcal{MT}(\Sigma_1, \phi_1)$ iso-contact embeds into the mapping torus $\mathcal{MT}(\Sigma_2, \phi_2)$. Since $g|_{\partial\Sigma_1}$ pulls back the contact form α_2 to $h \cdot \alpha_1$ for some positive function h on $\partial\Sigma_1$, we can extend this embedding to an iso-contact embedding $\mathcal{I}$ of $\mathcal{A}ob(\Sigma_1, d\alpha_1, \phi_1)$ into $\mathcal{A}ob(\Sigma_2, d\alpha_2, \phi_2)$, such that the restriction of $\mathcal{A}ob(\Sigma_2, d\alpha_2, \phi_2)$ on the image of $\mathcal{I}$ gives the supporting contact open book $\mathcal{A}ob(\Sigma_1, d\alpha_1, \phi_1)$ of (M_1, ξ_1).

2.4.2. Dehn-Seidel twist. Consider the symplectic structure on the cotangent bundle $(T^*S^n, d\lambda_{can})$. Here, λ_{can} is the canonical 1-form on T^*S^n. We regard T^*S^n as a submanifold of $\mathbb{R}^{2n+2} \cong \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. A point $(\vec{x}, \vec{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, represents a point in T^*S^n if and only if it satisfies the relations: $|\vec{x}| = 1$ and $\vec{x} \cdot \vec{y} = 0$. Here, $\vec{y} \equiv (y_1, .., y_{n+1})$ and $\vec{x} \equiv (x_1, .., x_{n+1})$. In these coordinates, λ_{can} is given by the form $\sum y_i dx_i$.

Let $\sigma_t : T^*S^n \to T^*S^n$ be a map (for $|\vec{y}| > 0$) defined as follows.

$$\sigma_t(\vec{x}, \vec{y}) = \begin{pmatrix} \cos t & |\vec{y}|^{-1} \sin t \\ -|\vec{y}| \sin t & \cos t \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$$

For $k \in \mathbb{Z}_{>0}$, let $g_k : [0, \infty) \to \mathbb{R}$ be a smooth function that satisfies the following properties.

(1) $g_k(0) = k\pi$ and $g_k'(0) < 0$.
(2) Fix $p_0 > 0$. The function $g_k(|\vec{y}|)$ decreases to 0 at p_0 and then remains 0 for all $\vec{y}$ with $|\vec{y}| > p_0$. See Figure 8(2.4.2).

Now we can define the *positive k-fold Dehn-Seidel twist* as follows.

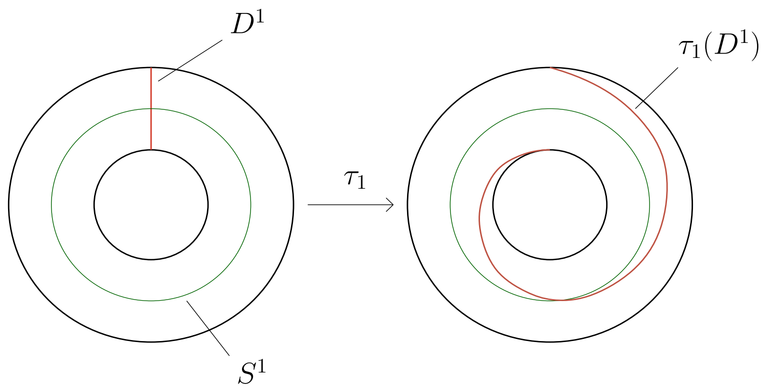

FIGURE 9. The 1-dimensional positive Dehn twist on T^*S^1 which is the usual Dehn twist on an annulus

$$\tau_k(\vec{x},\vec{y}) = \begin{cases} \sigma_{g_k(|\vec{y}|)}(\vec{x},\vec{y}) & for \quad \vec{y} \neq \vec{0} \\ -Id \quad for \quad \vec{y} = \vec{0} \end{cases}$$

The Dehn-Seidel twist is a proper symplectomorphism of T^*S^n. From Figure 8(2.4.2), we see that τ_k has compact support. Therefore, choosing p_0 properly, τ_k can be defined on the unit disk bundle $(DT^*S^n, d\lambda_{can})$, such that it is identity near boundary. In fact, we can choose the support as small as we wish without affecting the symplectic isotopy class of the resulting τ_k. More precisely, let g_k^1 and g_k^2 be two functions similar to g_k as above. Say, g_k^1 has support p_1 and g_k^2 has support p_2. Then $\tau_k^t = \sigma_{tg_k^1(|\cdot|)+(1-t)g_k^2(|\cdot|)}$ gives a symplectic isotopy between $\tau_k^1 = \sigma_{g_k^1(|\cdot|)}$ and $\tau_k^0 = \sigma_{g_k^2(|\cdot|)}$.

Similarly, for $k < 0$, we can define the *negative k-fold Dehn-Seidel twist.* For $k = 0$, τ_0 is defined to be the identity map of DT^*S^n. Sometimes we may say just *Dehn twist* instead of *Dehn-Seidel twist.*

EXAMPLE 2.4.2 (Contact open book embedding and Dehn-Seidel twist). *An important open book decomposition of (S^{2n+1}, ξ_{std}) is given with page $(DT^*S^n, d\lambda^n_{can})$ and monodromy a positive Dehn-Seidel twist. In terms of the coordinates discussed above, the standard inclusion of S^n in S^{n+k} is given by $(x_1, x_2, ..., x_{n+1}) \mapsto (x_1, x_2, ..., x_{n+1}, 0, 0, ..., 0)$. This induces a proper symplectic embedding of $(DT^*S^n, d\lambda^n_{can})$ in $(DT^*S^{n+k}, d\lambda^{n+k}_{can})$. In general, any embedding between manifolds induces a proper symplectic embedding between the corresponing cotangent bundles. The map can be described in the following way.*

Let $W_1 = DT^*M_1$ and $W_2 = DT^*M_2$. A diffeomorphism $f : M_1 \to M_2$, induces the diffeomorphism $f_\# : W_1 \to W_2$, given by $f_\#(x_1, \rho_1) = (f(x_1), \rho_2)$, such that $\rho_1 = df^*_{x_1}\rho_2$. Here, $f_\#$ pulls back the canonical 1-form on W_2 to the canonical 1-form on W_1. Now, consider an embedding $\iota : M^m \to N^{m+k}$. Note that $\iota^*(DT^*N) = DT^*\iota(M) \bigoplus \nu^*(\iota)$. Here, $\nu^*(\iota) = \{(\iota(x), \eta(\iota(x))) \in DT^*N \mid \iota^*(\iota(x), \eta(\iota(x))) = (x, 0) \in DT^*M\}$. Thus, every 1-form $(\iota(x), \rho(\iota(x))) \in DT^*N$, over a point $\iota(x) \in N$, can be uniquely decomposed into sum of two forms $(\iota(x), \rho_M(x)) \in DT^*\iota(M)$ and $(\iota(x), \rho_\nu(x)) \in \nu^*(\iota)$. Now, we can define the induced map $\iota_\# : DT^*M \to DT^*N$ by $(x, \rho_0(x)) \mapsto (\iota(x), \rho_M(\iota(x)))$, such that $\iota^*_x(\rho_M(\iota(x))) = \rho_0(x)$.

One can then show that $\iota_{\#}$ *pulls back the canonical* 1*-form on* DT^*N *to the canonical* 1*-form on* DT^*M*.*

However, note that in case of $S^n \subset S^{n+k}$*, we can take advantage of the simple coordinate system on* $T^*S^m \subset \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$*, to write the induced symplectic embedding on the cotangent bundles. In section 4.2, we will take this approach to write down the symplectic embedding explicitly in coordinates.*

Now, observe that an $(n+k)$*-dimensional Dehn-Seidel twist on* DT^*S^{n+k} *induces an* n*-dimensional Dehn-Seidel twist on the embedded* DT^*S^n*. Therefore,* $(S^{2n+1}, \xi_{std}) = \mathcal{A}ob(DT^*S^n, d\lambda^n_{can}, \tau_1)$ *contact open book embeds in* $(S^{2n+2k+1}, \xi_{std}) = \mathcal{A}ob(DT^*S^{n+k}, d\lambda^{n+k}_{can}, \tau_1)$*. Refer to section 2.4.2 for the notation* τ_1*.*

2.4.3. Stabilization of contact open books and overtwisted contact structure. Let $\pi_i : E_i \to B^n_i$ be an n-disk bundle over B^n_i, for $i = 1, 2$. Choose a point x_i and a disk neighborhood D^n_i of x_i in B_i, such that $\pi_i^{-1}(D^n_i)$ is diffeomorphic to $D^n_i \times D^n$, for $i = 1, 2$.

The *plumbing* of E_1 and E_2, at $(x_1, x_2) \in E_1 \times E_2$, is obtained by identifying $D^n_1 \times D^n$ with $D^n_2 \times D^n$ by the following map and then smoothing the corners.

$$\begin{aligned} D^n_1 \times D^n &\xrightarrow{\chi} D^n_2 \times D^n \\ (q, p) &\longmapsto (-q, p) \end{aligned}$$

Here $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n)$ and $-q$ denotes $(-q_1, -q_2, \ldots, -q_n)$. We denote the plumbing of E_1 and E_2 by $E_1 \S E_2$. For details of plumbing operation see [**Ge**].

Now, consider two copies of the unit disk cotangent bundles of sphere, $(DT^*S^n_1, dp_1 \wedge dq_1)$ and $(DT^*S^n_2, dp_2 \wedge dq_2)$, with the canonical symplectic structures on them. Since locally $\chi^*(dp_2 \wedge dq_2) = dp_1 \wedge dq_1$, we get an induced symplectic structure on $DT^*S^n_1 \S DT^*S^n_2$, denoted by $dp_1 \wedge dq_1 \S dp_2 \wedge dq_2$. See Figure 10(2.4.3.)

Definition 2.4.4. *Consider an open book decomposition given by* $\mathcal{A}ob(DT^*M^n, d\lambda_M, \phi_M)$*. We call the modified open book,* $\mathcal{A}ob(DT^*M^n \S DT^*S^n, \phi_M \circ \tau_1)$*, a positive stabilization of* $\mathcal{A}ob(DT^*M^n, \phi_M)$*. If* τ_1 *is replaced by* τ_{-1}*, we call the modified open book, a negative stabilization of* $\mathcal{A}ob(DT^*M^n, \phi_M)$*.*

We should mention that the above definition is not the most general definition of stabilization used in the literature. However, it suffices for the purpose of the present article. In general, consider a contact open book $\mathcal{A}ob(W^{2n}, \omega = d\lambda, \phi)$. Topologically one can attach a k-handle to W along some attaching $(k-1)$-sphere in ∂W. The symplectic analogue of such topological handles are called *Weinstein handles*. For details on Weinstein handle theory we refer to [**W**] and [**Ko**]. In the contact–symplectic category, one demands that the boundary is convex and the attaching sphere S^{k-1} is *isotropic* (i.e. $\lambda|_{S^{k-1}} = 0$) with a trivialization of its symplectic normal bundle in $(\partial W, \lambda)$. When $k = n$, we call the attaching sphere *Legendrian*. Let $S^{n-1}_L \subset (\partial W, \lambda)$ be a Legendrian sphere, i.e., $\lambda|_{S^{n-1}_L} = 0$. Attach a Weinstein n-handle H_{2n} to W, along S_L. Say, $D^n_L \subset W$ is a Lagrangian disk (i.e. $d\lambda|_{D^n_L} = 0$) with boundary S_L. Let $\mathcal{L}$ be the Lagrangian n-sphere in $W \cup H_{2n}$, formed by D^n_L and the Lagrangian core disk of H_{2n}. By the Weinstein neighborhood theorem, a symplectic neighborhood of a Lagrangian sphere S^n is symplectomorphic to T^*S^n. Let $\tilde{\omega}$ denote the resulting

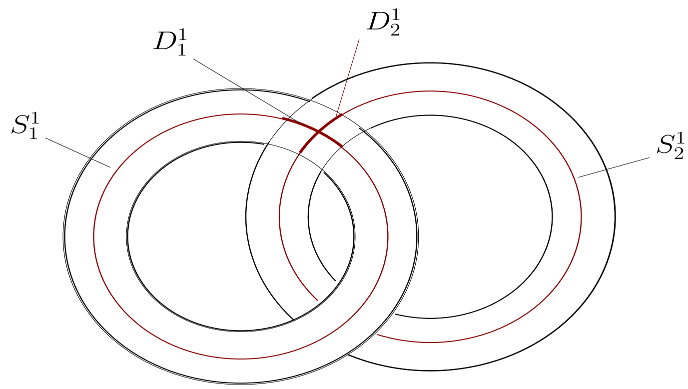

FIGURE 10. A stabilization of the open book $\mathcal{A}ob(DT^*S_1^1, \tau_1)$ of (S^3, ξ_{std}) is obtained by plumbing $DT^*S_1^1$ with $DT^*S_2^1$ and composing the monodromy with a positive Dehn twist along S_2^1. Composing with a negative Dehn twist along S_2^1 induces an overtwisted contact structure on S^3.

symplectic structure on $W^{2n} \cup H_{2n}$ and let $\tau_{\mathcal{L}}$ denote a positive Dehn-Seidel twist along $\mathcal{L}$. Then, $\mathcal{A}ob(W^{2n} \cup H_{2n}, \tilde{\omega}, \phi \circ \tau_{\mathcal{L}})$ is called a positive stabilization of $\mathcal{A}ob(W^{2n}, \omega, \phi)$. It is well known that a positive stabilization does not change the contactomorphism type of the total manifold. For a proof of this fact and more details on Weinstein handles, see [**W**] and [**Ko**].

In [**CMP**], Casals, Murphy and Presas gave a characterization of an *overtwisted* contact structure in terms of open books. They showed that every overtwisted contact structure is a negative stabilization of some open book decomposition. In particular, $\mathcal{A}ob(DT^*S^n, d\lambda_{can}, \tau_{-1})$ gives an overtwisted contact structure on S^{2n+1}. We will denote this overtwisted contact structure by ξ_{ot}.

EXAMPLE 2.4.3. *Let S_1^n and S_2^n be two copies of S^n and let $(\tau_1)_i$ denote the positive Dehn twist along S_i^n. Then $\mathcal{A}ob(DT^*S_1^n \S DT^*S_2^n, d\lambda_{can} \S d\lambda_{can}, (\tau_1)_1 \circ (\tau_{-1})_2) = (S^{2n+1}, \xi_{ot})$.*

2.4.4. Boundary connected sum. Consider two disjoint connected symplectic manifolds $(W_1, \partial W_1, \omega_1)$ and $(W_2, \partial W_2, \omega_2)$ of dimension $2n$ with convex boundaries. If we attach a Weinstein 1-handle along two points $w_1 \in \partial W_1$ and $w_2 \in \partial W_2$, then we get the symplectic boundary connected sum of (W_1, ω_1) and (W_2, ω_2), denoted by $(W_1 \#_b W_2, \omega_1 \#_b \omega_2)$. It follows from the theory of Weinstein handlebody that the operation of symplectic boundary connected sum is well defined upto symplectomorphism.

Consider two contact open books $\mathcal{A}ob(\Sigma_1, d\alpha_1, \phi_1)$ and $\mathcal{A}ob(\Sigma_2, d\alpha_2, \phi_2)$. It is known that

$$\mathcal{A}ob(\Sigma_1 \#_b \Sigma_2, d\alpha_1 \#_b d\alpha_2, \phi_1 \circ \phi_2) \cong \mathcal{A}ob(\Sigma_1, d\alpha_1, \phi_1) \# \mathcal{A}ob(\Sigma_2, d\alpha_2, \phi_2)$$

(see section 2.4 in [**DGK**]). Here the connected sum denoted by #, means the contact connected sum.

EXAMPLE 2.4.4. *Let ω_0 denote the standard symplectic form on D^{2n}. Then $(S^{2n+1}, \xi_{std}) \cong (S^{2n+1} \# S^{2n+1}, \xi_{std} \# \xi_{std}) \cong \mathcal{A}ob(DT^*S^n, \omega_0) \# \mathcal{A}ob(DT^*S^n, \omega_0) \cong \mathcal{A}ob(DT^*S^n \#_b DT^*S^n, \omega_0 \#_b \omega_0)$*

2.5. Almost contact structure

We recall the notion of an almost contact structure.

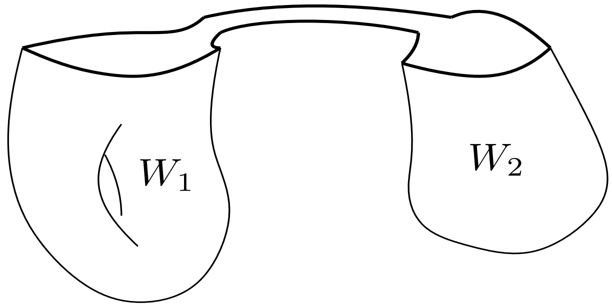

FIGURE 11. Boundary connected sum of W_1 and W_2.

DEFINITION 2.5.1. *Consider a real vector bundle $p : E \to B$ of rank $2m$, with fiber $F \cong \mathbb{R}^{2m}$. An almost complex structure on E is a smooth assignment of linear automorphisms $J_p : F_p \to F_p$ for all point p in B such that $J_p^2 = -Id$.*

Take the standard metric and orientation on $\mathbb{R}^{2m}$. The set of all complex structures on $\mathbb{R}^{2m}$ is then homeomorphic to $\Gamma_m = SO(2m)/U(m)$ ([**Ge**], Lemma 8.1.7).

Let us briefly recall the notion of an associated fiber bundle.

DEFINITION 2.5.2 (Associated bundle). *Let $\Pi : P \to B$ be a principal G-bundle and let $\rho : G \to Homeo(Fb)$ be a continuous left action of G on a space Fb. Consider the right G-action on $P \times Fb$ given by $(p, f) \cdot g = (p \cdot g, \rho(g^{-1}) \cdot f)$. The quotient by this action is a fiber bundle with fiber Fb that we call the associated Fb-bundle of P. The transition functions of this bundle are given by the images of the transition functions of P under ρ. It is well known that every smooth rank n vector bundle E can be identified with a unique principal $SO(n)$-bundle. By an associated Fb-bundle of E, we understand the associated Fb-bundle of this principal $SO(n)$-bundle.*

For related terminologies and notions from fiber bundle and obstruction theory and we refer to [**St**].

Thus, in terms of principle bundles, the existence of an almost complex structure on E is equivalent to the existence of a section of the associated Γ_m-bundle.

DEFINITION 2.5.3. *An almost contact structure on an odd dimensional manifold N^{2n+1} is an almost complex structure on its stable tangent bundle $TN \oplus \varepsilon_N^1$.*

Thus, an almost contact structure on N is an almost complex structure on $N \times \mathbb{R}$. Note that every almost contact structure on N is given by a section of the associated Γ_{n+1}–bundle of $T(N \times \mathbb{R})$.

DEFINITION 2.5.4. *Two almost contact structures are said to be in the same homotopy class, if their corresponding sections to the associated Γ_{n+1}–bundle are homotopic.*

The existence of an almost contact structure on N is a necessary condition for the existence of a contact structure. For open manifolds, Gromov ([**Gr**]) proved the following h-principle showing that this condition is also sufficient.

THEOREM 2.5.5 (Gromov,[**Gr**]). *Let K be a sub-complex of an open manifold V. Let $\bar{\xi}$ be an almost contact structure on V which restricts to a contact structure in a neighborhood $Op(K)$ of K. Then one can homotope $\bar{\xi}$, relative to $Op(K)$, to a contact structure ξ on V.*

For closed manifolds, the corresponding h-principle follows from the work of Borman, Eliashberg and Murphy [**BEM**]. In particular, they showed that in every homotopy class of an almost contact structure there is at least one contact structure called *overtwisted* (see [**BEM**]). Two such overtwisted contact structures are isotopic if and only if they are homotopic as almost contact structures. [**BEM**] gives a parametric version of Theorem 2.5.5 that holds for both open and closed contact manifolds.

THEOREM 2.5.6 (Borman, Eliashberg and Murphy, [**BEM**]). *Let $K \subset M^{2n+1}$ be a closed subset. Let ξ_1 and ξ_2 be two overtwisted contact structures on M that agree on some $Op(K)$. If ξ_1 and ξ_2 are homotopic as almost contact structures over $M \setminus K$ relative to $Op(K)$, then ξ_1 and and ξ_2 are homotopic as contact structures relative to $Op(K)$.*

So, by Gray's stability, ξ_1 and ξ_2 are isotopic contact structures.

The obstructions to the existence of an almost contact structure on N^{2n+1} lie in the groups $H^i(N; \pi_{i-1}(\Gamma_{n+1}))$, for $1 \leq i \leq 2n+1$. The homotopy obstructions between two almost contact structures on N lie in the groups $H^i(N; \pi_i(\Gamma_{n+1}))$, for $1 \leq i \leq 2n+1$. The stable homotopy groups of Γ_n were computed by Bott ([**B**]).

THEOREM 2.5.7 (Bott, [**B**]). *For $q \leq 2n-2$*

$$\pi_q(\Gamma_n) = \pi_{q+1}(SO) = \begin{cases} 0 & \text{for } \ q \equiv 1, 3, 4, 5 \ (mod\ 8) \\ \mathbb{Z} & \text{for } \ q \equiv 2, 6 \ (mod\ 8) \\ \mathbb{Z}_2 & \text{for } \ q \equiv 0, 7 \ (mod\ 8) \end{cases}$$

EXAMPLE 2.5.1. *Note that $\pi_i(\Gamma_3) = Z$ for $i = 2$ and 0 otherwise. Thus, for a contact 5-manifold M^5, the only obstruction to the existence of an almost contact structure arises in $H^3(M^5; \mathbb{Z})$. The obstruction here is known to be the integral Stiefel-Whitney class $W_3(M)$ of M.*

EXAMPLE 2.5.2. *Since, $H^i(S^5, \pi_i \Gamma_3) = 0$ for all i, any two almost contact structures on S^5 are homotopic. Hence, there is a unique overtwisted contact structure on S^5.*

2.6. Obstructions to isocontact immersion and isocontact embedding

We first recall some facts from the Smale-Hirsch immersion theory that will be useful later.

2.6.1. h-principle for immersion. Let $f, g : M^{2n+1} \looparrowright V^{2N+1}$ be two immersions. We say that f is *regularly homotopic* to g, if there is a family $h_t : M \looparrowright V$ of immersions joining f and g. Being an immersion, f induces $Df : TM \to TV$ such that for all $p \in M$, Df restricts to the monomorphism Df_p from T_pM to $T_{f(p)}V$.

Definition 2.6.1 (Formal immersion). *A formal immersion of M in V is a bundle map $F : TM \to TV$ that restricts to a monomorphism F_p on each tangent space T_pM, for $p \in M$. It can be represented by the following diagram, where f_0 is any smooth map making it commutative.*

$$\begin{array}{ccc} TM & \xrightarrow{F} & TV \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M & \xrightarrow{f_0} & V \end{array}$$

We say that F is a formal immersion covering f_0. The existence of a formal immersion from TM in TV is a necessary condition for the existence of an immersion of M in V.

Definition 2.6.2 (Homotopy between formal immersions). *Two formal immersions, F and G, are called formally homotopic (or just homotopic) if there is a homotopy $H_t : TM \to TV$ of formal immersions such that $H_0 = F$ and $H_1 = G$.*

Two immersions f and g are called formally homotopic if Df and Dg are homotopic as formal immersions. Assume that $dim(V) \geq dim(M) + 1$. Let $Imm(M, V)$ denote the set of all immersions of M in V and let $Mono(TM, TV)$ denote the set of all formal immersions. Let $I : Imm(M, V) \to Mono(TM, TV)$ be the inclusion map given by the tangent bundle monomorphism induced by an immersion.

Theorem 2.6.3 (The Smale-Hirsch h-principle for immersion). *([**Hi**]) The map I is a homotopy equivalence.*

So, I induces an isomorphism from $\pi_0(Imm(M, V))$ to $\pi_0(Mon(TM, TV))$. This implies that the existence of a formal immersion is also sufficient for the existence of an immersion. Moreover, the isomorphism implies that if f_0 and f_1 are two immersions which are formally homotopic, then they are regularly homotopic.

We now discuss the obstruction theoretic problem for the existence of a formal immersion and the classification of formal immersions. For more details on immersion theory we refer to [**Hi**].

Obstructions to formal immersion and homotopy: Given a manifold N, let T_kN denote the bundle of k-frames associated to TN. A formal immersion of M^n into $\mathbb{R}^N$ defines an $SO(n)$–equivariant map from T_nM to $V_{N,n}$. Here, $V_{N,n}$ denotes the real Stiefel manifold consisting of all oriented n-frames in $\mathbb{R}^N$. According to [**Hi**], the homotopy classes of the formal immersions are in one-one correspondence with the homotopy classes of such $SO(n)$–equivariant maps to $V_{N,n}$, i.e., with homotopy classes of cross sections of the associated bundle of T_nM with fiber $V_{N,n}$. Thus, the problem is reduced to looking at the obstructions to the existence of a section s of this associated bundle. Such obstructions lie in the groups $H^i(M^n; \pi_{i-1}(V_{N,n}))$, for $1 \leq i \leq n$.

Moreover, two formal immersions F and G are homotopic if and only if their corresponding sections s_F and s_G to the associated $V_{N,n}$–bundle are homotopic. Thus, the homotopy obstructions between two formal immersions F and G lie in $H^i(M^n; \pi_i(V_{N,n}))$ for $1 \leq i \leq n$.

2.6.2. Obstructions to contact immersion and embedding. Similar to the notion of formal immersion one can define a *formal contact immersion* or a *contact monomorphism.*

DEFINITION 2.6.4. *A formal immersion $F : TM \to TV$ of (M, ξ) in (V, η) is called a contact monomorphism if F is conformal symplectic (i.e. $F^*\eta$ is positive multiple of ξ) and $F(\xi) = F(TM) \cap \eta$.*

Gromov ([**Gr**]) proved the following h-principle for contact immersions.

THEOREM 2.6.5 (Gromov). *Let (M, ξ) and (V, η) be contact manifolds of dimensions $2n + 1$ and $2N + 1$ respectively. Assume that $n \leq N - 1$. A contact monomorphism $F_0 : TM \to TV$, covering an immersion $f_0 : M \to V$, is formally homotopic to $F_1 = df_1$ for some contact immersion $f_1 : M \to V$.*

For contact embedding, Gromov ([**Gr**]) proved the following theorem. The statement here is taken from [**EM**].

THEOREM 2.6.6 (Gromov). *Let (M, ξ) and (V, η) be contact manifolds of dimension $2n + 1$ and $2N + 1$ respectively. Suppose that the differential $F_0 = Df_0$ of an embedding $f_0 : (M, \xi) \to (V, \eta)$ is homotopic via a homotopy of monomorphisms $F_t : TM \to TV$ covering f_0 to a contact monomorphism $F_1 : TM \to TV$.*

(1) *Open case: If $n \leq N - 1$ and the manifold M is open, then there exists an isotopy $f_t : M \to V$ such that the embedding $f_1 : M \to W$ is contact and the differential Df_1 is homotopic to F_1 through contact monomorphisms.*
(2) *Closed case: If $n \leq N - 2$, then the above isotopy f_t exists even if M is closed.*

Thus, if an embedding is regularly homotopic to a contact immersion, then it is isotopic to a contact embedding. We now discuss the analogous obstruction problem for the existence of a contact monomorphism.

Obstructions to contact monomorphism : Consider a symplectic vector space (X, ω). Let J be an ω-compatible almost complex structure (i.e., $\omega(Ju, Jv) = \omega(u, v)$ and $\omega(u, Ju) > 0$ for all $u, v \in X \setminus \{0\}$). If Y is a J-subspace of (X, J), then Y is a symplectic subspace of (X, ω). An almost complex structure J_ξ on the contact hyperplane bundle $\xi = Ker\{\alpha\}$ is called ξ–compatible, if it is compatible with the conformal symplectic structure on ξ induced by $d\alpha$. A contact monomorphism takes ξ to a symplectic sub-bundle of η. There exists an η–compatible almost complex structure J_η such that the contact monomorphism takes ξ to a J_η-sub-bundle of (η, J_η). So, finding a contact monomorphism from (TM, ξ) to $(T\mathbb{R}^{2N+1}, \eta)$ is equivalent to finding a $U(n)$–equivariant map from the complex n-frame bundle associated to ξ to $V^{\mathbb{C}}_{N,n}$. In other words, finding a contact monomorphism is equivalent to the existence of a section of the associated $V^{\mathbb{C}}_{N,n}$–bundle of TM (see 2.2 in [**Ka**]). Here, $V^{\mathbb{C}}_{N,n}$ denotes the complex Stiefel manifold. Thus, (M, ξ) has a contact monomorphism in (V, η) if and only if all the obstructions classes in $H^i(M; \pi_{i-1}(V^{\mathbb{C}}_{N,n}))$ vanish for $1 \leq i \leq 2n + 1$.

From contact immersion to contact embedding : In the previous sections, we saw that any formal immersion of M^{2n+1} in $\mathbb{R}^{2N+1}$ is given by a section s_F of the associated

$V_{2N+1,2n+1}$-bundle of TM. For a contact monomorphism $F_{\mathbb{C}}$, let $s_{F_{\mathbb{C}}}$ denote the corresponding section to the associated $V_n^{\mathbb{C}}(\eta)$–bundle. $s_{F_{\mathbb{C}}}$ also induces a section map $s_{\mathbb{C}}$ to the associated $V_{2N+1,2n+1}$-bundle via the inclusion $V_{N,n}^{\mathbb{C}} \subset V_{2N,2n} \subset V_{2N+1,2n+1}$. Let $\iota : M \hookrightarrow V$ be an embedding and let s_ι denote the corresponding section to the associated $V_{2N+1,2n+1}$-bundle. The homotopy obstructions between s_ι and $s_{\mathbb{C}}$ lie in the groups $H^i(M;\pi_i(V_{2N+1,2n+1}))$, for $1 \leq i \leq 2n+1$. If all of these obstructions vanish, then by Theorem 2.6.6, ι can be isotoped to a contact embedding.

EXAMPLE 2.6.1. *Let $(V,\eta) = (\mathbb{R}^{4n+3}, \xi_{std})$. Note that $V_{4n+3,2n+1}$ is $(2n+1)$-connected and $V_{2n+1,n}^{\mathbb{C}}$ is $(2n+2)$-connected. So, all of the groups $H^i(M;\pi_{i-1}(W_{2n+1,n})$ and $H^i(M;\pi_i(V_{2N+1,2n+1})$ vanish, for $1 \leq i \leq 2n+1$. By the Whitney embedding theorem, any smooth $(2n+1)$-manifold embeds into $\mathbb{R}^{4n+3}$. Thus, we get a result of Gromov* [**Gr**]*, saying that every contact manifold (M^{2n+1},ξ) has an isocontact embedding in $(\mathbb{R}^{4n+3},\xi_{std})$.*

REMARK 2.6.7. *When the embedding co-dimension is $\leq dim(M) - 1$, there is a natural topological obstruction to contact embedding. It has the following description. If $\iota : (M^{2n+1},\xi) \hookrightarrow (\mathbb{R}^{2N+1},\eta)$ is a contact embedding, then the normal bundle $\nu(\iota) = \iota^*(\eta)/\xi$ has an induced complex structure on it. So we have the following relation of total Chern classes.*

$$c(\xi \oplus \nu(\iota)) = \iota^*(\eta) = 1$$

. Let $\bar{c}_j(\xi)$ denote the j^{th} order cohomology class in $(1 + c_1(\xi) + c_2(\xi) + ... + c_n(\xi))^{-1}$. Since the Euler class of the normal bundle of an embedding in $\mathbb{R}^{2N+1}$ is zero,

$$c_{N-n}(\nu(\iota)) = 0 \Leftrightarrow \bar{c}_{N-n}(\xi) = 0$$

This gives a condition on the Chern classes of ξ. Thus, for isocontact embedding of co-dimension $\leq dim(M) - 1$, one has to restrict the problem on the contact structures whose Chern classes satisfy this condition. For isocontact embedding with trivial symplectic normal bundle, the following holds.

$$\xi \oplus \nu(\iota) \cong \xi \oplus \varepsilon_M^{N-n}(\mathbb{C}) = \eta|_{\iota(M)} \cong \varepsilon_M^N(\mathbb{C})$$

Thus, $c_i(\xi \oplus \varepsilon_M^{N-n}(\mathbb{C})) = 0 \Leftrightarrow c_i(\xi) = 0$, for $1 \leq i \leq n$.

CHAPTER 3

Open book embedding of closed, orientable manifolds

In this chapter, we produce open book embeddings of closed oriented 3-manifolds in open books of $S^3 \times S^2$ and $S^2 \widetilde{\times} S^3$ with pages any disk bundle over S^2 and monodromy the identity. We also show that every k-connected, closed, orientable n-manifold M^n admitting open book decomposition, has an open book embedding in the trivial open book of S^{2n-k}.

3.1. Open book embedding of closed 3-manifolds

We prove the following theorem.

THEOREM 3.1.1. *Let M be a closed oriented connected 3–dimensional manifold together with an open book decomposition $\mathcal{A}ob(\Sigma, \phi)$. Then, $\mathcal{A}ob(\Sigma, \phi)$ admits an open book embedding in any open book decomposition associated to $S^3 \times S^2$ with pages a disk bundle over S^2 of even Euler number and monodromy the identity as well as in any open book of $S^3 \widetilde{\times} S^2$ with pages a disk bundle over S^2 of odd Euler number and monodromy the identity.*

Using the methods of the proof of Theorem 3.1.1, we establish the following:

THEOREM 3.1.2. *Every closed orientable 3-manifold admits a smooth embedding in S^5.*

This theorem was first discovered by M. Hirsch in [**Hi0**]. There are other proofs of Theorem 3.1.2. See, for example, the article [**HLM**] for a proof using what is now known as braided embeddings and also the article [**Kp**] for embeddings of closed orientable 3–manifolds in S^5 using surgery description of 3-manifolds and Kirby calculus. We refer to [**Wa**] and [**Ro**] for embeddings of non-orientable 3–manifolds in $\mathbb{R}^5$.

We begin by reviewing quickly some well known results about embedded Hopf band in S^3. We can view S^3 as the unit sphere in $\mathbb{C}^2$. The Hopf links $H^{\pm}$ are the pre-images of 0 under the maps $(z_1, z_2) \to z_1 z_2$ and $(z_1, z_2) \to z_1 \bar{z}_2$, respectively, restricted to the unit sphere S^3 of $\mathbb{C}^2$. A Hopf annulus is a Seifert surface for a Hopf link and a positive/negative Hopf band in S^3 is an embedded annulus with the boundary $H^{\pm}$. See, for example, [**Et**] for more details.

3.1.1. Hopf band in S^3 and the mapping class group of an annulus.

To begin with, we go through the proofs of the following well known results. These are also proved in [**HY**].

LEMMA 3.1.3. *Let A be an annulus and let $[\phi]$ be an element of the mapping class group $\mathcal{MCG}(A)$ of A. Then, there exists an embedding f of A in S^3 that satisfies the following:*

(1) *$f(A)$ is a Hopf annulus in S^3.*

(2) *There exists a diffeomorphism* Ψ_1 *of* S^3*, isotopic to the identity via an isotopy* Ψ_t *such that* $f^{-1} \circ \Psi_1 \circ f = \phi$.
(3) *The isotopy* Ψ_t *fixes the boundary of* A *point-wise for all* t.

Proof. We know that S^3 admits an open book decomposition with pages a Hopf annulus and the monodromy a Dehn twist around its center circle. This implies that there exists a flow Φ_t on S^3 whose time 1 map Φ_1 maps a Hopf annulus page – say $\mathcal{A}$ – to itself and Φ_1 restricted to $\mathcal{A}$ is a Dehn twist along the center circle on $\mathcal{A}$. We consider an embedding f of A in S^3 such that $f(A) = \mathcal{A}$. The lemma is now a straight forward consequence of the fact that every element of the mapping class group of an annulus is just an isotopy class of a power of the Dehn twist along its center circle. $\square$

Note that $S^3 \times [0,1]$ can be regarded as a collar of ∂D^4 in D^4 with $\partial D^4 = S^3 \times \{1\}$. Since Ψ, as constructed in the Lemma 3.1.3, is isotopic to the identity, we have the following:

Corollary 3.1.4. *There exists a proper embedding* f *of an annulus* A *in* $(D^4, \partial D^4)$ *which satisfies the property that for every element* $\phi \in \mathcal{MCG}(A)$*, there exists a diffeomorphism* Γ_1 *of* $(D^4, \partial D^4)$ *isotopic to the identity such that* $\phi = f^{-1} \circ \Gamma_1 \circ f$.

Proof. First, we consider a proper embedding of A in $S^3 \times [0,1]$ as follows: we smoothly push a Hopf annulus, say $\mathcal{A}$ from $\partial D^4 = S^3 \times \{1\}$ to the level $S^3 \times \{0\}$ keeping the boundary of the Hopf annulus fixed such that $S^3 \times \{t\} \cap \mathcal{A}$ is a Hopf link for each $t \in (0,1]$. We consider the proper embedding f of A such that image of f is the pushed Hopf annulus $\mathcal{A}$. Now, let Ψ_t be the isotopy of S^3 such that Ψ_1 realizes the given element of $\mathcal{MCG}(A)$. Using the isotopy Ψ_t, we construct a diffeomorphism Γ_1 of $S^3 \times [-1,1]$ that satisfies the following:

(1) Γ_1 is isotopic to the identity via a family of diffeomorphisms Γ_t.
(2) Γ_1 restricted to $S^3 \times \{0\}$ is Ψ_1.

This diffeomorphism is defined as follows:

$$\Gamma_1(x,t) = \begin{cases} \Psi_{1-t}(x) & \text{if } t \geq 0 \\ \Psi_{t+1}(x) & \text{if } t \leq 0 \end{cases}$$

Since $S^3 \times [-1,1]$ can be regarded as a collar of ∂D^4 in $(D^4, \partial D^4)$, we are through as Γ_1 clearly can be extended smoothly to a diffeomorphism of $(D^4, \partial D^4)$ by the identity in the complement of the collar. $\square$

Now, we have an easy consequence using the Corollary 3.1.4.

Proposition 3.1.5. *Any closed oriented* 3*–manifold with an open book decomposition having pages an annulus* A *and the monodromy any mapping class* ϕ *of the annulus admits an open book embedding in the trivial open book of* S^5.

Proof. Corollary 3.1.4 implies that the abstract open book $\mathcal{A}ob\,(A, \phi)$ associated to M abstract open book embeds in the abstract open book $\mathcal{A}ob\,(D^4, Id)$ associated to S^5. Hence, the result follows. $\square$

3.1.2. The proof of Theorem 3.1.1. Recall that we need to show that every closed oriented 3-dimensional manifold with a given open book decomposition, open book embeds in any open book decomposition of $S^3 \times S^2$ and $S^2 \widetilde{\times} S^3$ having pages a disk bundle over S^2 and monodromy the identity map.

We refer to [**GS**] for the notions of surgery and handle decomposition in the theory of three and four manifolds. We know that when we add a 2–handle to a 4–ball D^4 along an unknot on the boundary with framing $m, m \in \mathbb{Z}$ we produce a 2-disk bundle over S^2 with Euler number m. Let us denote this disk bundle by $\mathcal{D}E(m)$.

Next, we establish a lemma. The techniques used in the proof of this lemma is adopted from the techniques developed by Hirose and Yasuhara [**HY**] to establish *flexible* embeddings of closed surfaces in certain 4–manifolds. Hirose and Yasuhara called an embedding f of a surface Σ in a 4–manifold M *flexible*, provided for every $\phi \in \mathcal{MCG}(\Sigma)$ there exists a diffeomorphism Ψ of M, isotopic to the identity, which maps $f(\Sigma)$ to itself and $f^{-1} \circ \Psi|_{f(\Sigma)} \circ f = \phi$.

Lemma 3.1.6. *Let $(\Sigma, \partial\Sigma)$ be an oriented surface with non-empty boundary. There exists an embedding f of Σ in a disk bundle $\mathcal{D}E(m)$, for any $m \in \mathbb{Z}$, which satisfies the following:*

(1) *The embedding is proper.*
(2) *Given any diffeomorphism ϕ of $(\Sigma, \partial\Sigma)$, there exists a family Ψ_t of diffeomorphisms of $\mathcal{D}E(m)$ with $\Psi_0 = id$ such that Ψ_1 maps Σ to itself and satisfies the property that $f^{-1} \circ \Psi_1 \circ f$ is isotopic to the given diffeomorphism ϕ of $(\Sigma, \partial\Sigma)$.*

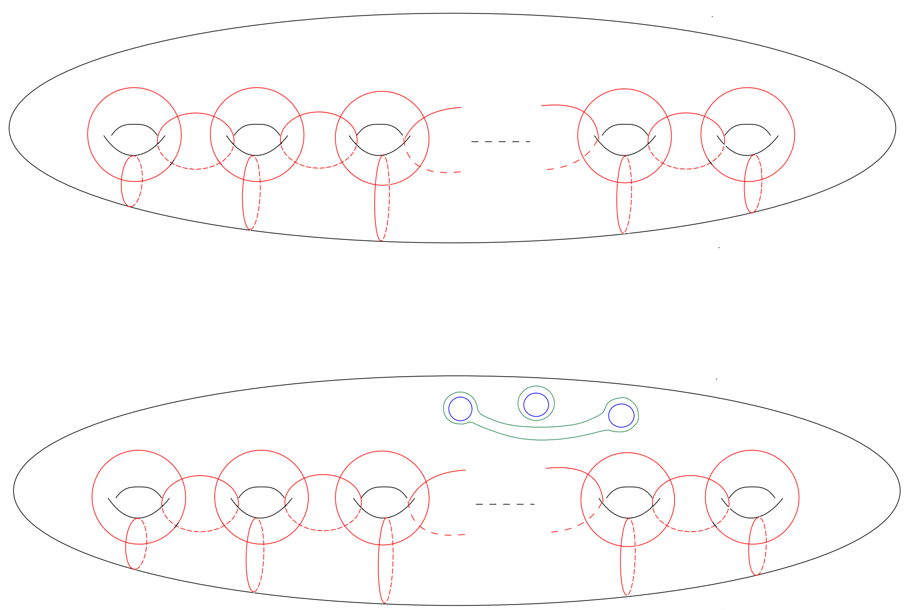

Figure 1. Figure depicts genus g compact orientable surface Σ with one boundary component. The embedded curves on the surface represents the standard Lickorish generators corresponding to the presentation of the mapping class group of Σ as given in [**J**].

Proof. We know that $\mathcal{D}E(m)$ is obtained by attaching a 2-handle to B^4 along an unknot with its framing m. This implies that we can regard it as a union of B^4 with $D^2 \times D^2$. We first describe an embedding of $(\Sigma, \partial\Sigma)$ in $S^3 = \partial B^4$ that we will need in order to establish the Lemma. Let us assume that $\partial\Sigma$ has $n \in \mathbb{N}$ boundary components. Let us denote by Σ_0 the closed surface obtained from $(\Sigma, \partial\Sigma)$ after attaching disks to each boundary component of $\partial\Sigma$. First, embed Σ_0 in S^3 such that it bounds the standard unknotted handle-body as shown in the Figure 1.

Observe that by removing the disks $D_1, D_2, \cdots, D_n$ as shown in Figure 2, we get an embedding of $(\Sigma, \partial\Sigma)$ in S^3 such that each boundary component of $(\Sigma, \partial\Sigma)$ is the boundary of D_i for some i.

Next, we attach a band with one full-twist around a properly embedded arc in the disk D_1 to the surface Σ as shown in Figure 2. This produces an embedded surface S with $(n+1)$ boundary components in S^3. Notice that out of these $n+1$ boundary components, $n-1$ boundary components correspond to boundaries of the disks $D_i, i = 2, \cdots, n$. The remaining two boundary components form a Hopf link as depicted in Figure 2. We denote these boundary components by H_1 and H_2. We use this embedding of the surface S with $n+1$ boundary components to properly embed the surface $(\Sigma, \partial\Sigma)$ in $\mathcal{D}E(m)$ in the following way:

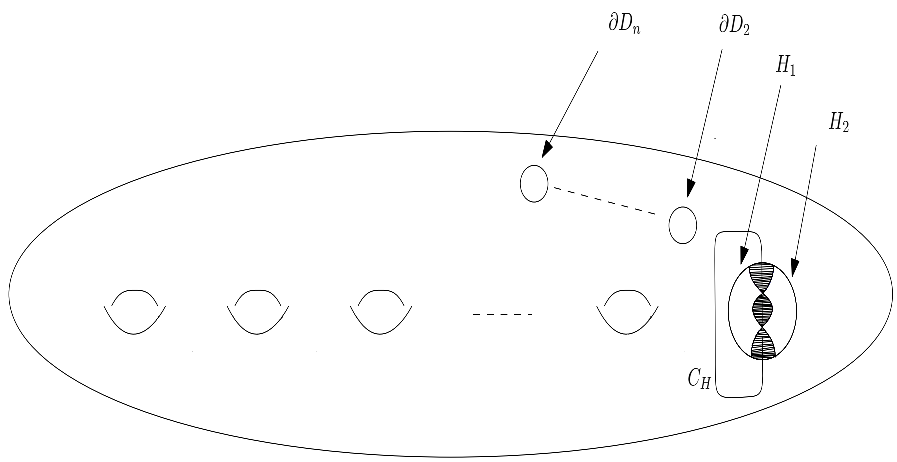

Figure 2. Embedding of the surface S with $n+1$ boundary components which contains a Hopf band H as a subsurface

Observe that by construction, S admits an embedding of a Hopf band H with the boundary components H_1 and H_2 as shown in Figure 2. Now, consider S^3 being embedded as $S^3 \times \{\frac{1}{2}\}$ in $S^3 \times [0,1]$, where we regard $S^3 \times [0,1]$ as a collar of ∂B^4. We now observe that we can attach a 2-handle along one of the boundary components of the Hopf band in such a way that we obtain $\mathcal{D}E(m)$ from B^4. More precisely, consider one of the boundary components – say H_1 – of the Hopf band and consider the cylinder $H_1 \times [\frac{1}{2}, 1]$ and assume that $H_1 \times \{1\}$ is the unknot along which the 2–handle with framing m is attached. In Figure 3, the boundary component H_1 is denoted by a dashed circle. Thus, H_1 bounds a disk D in

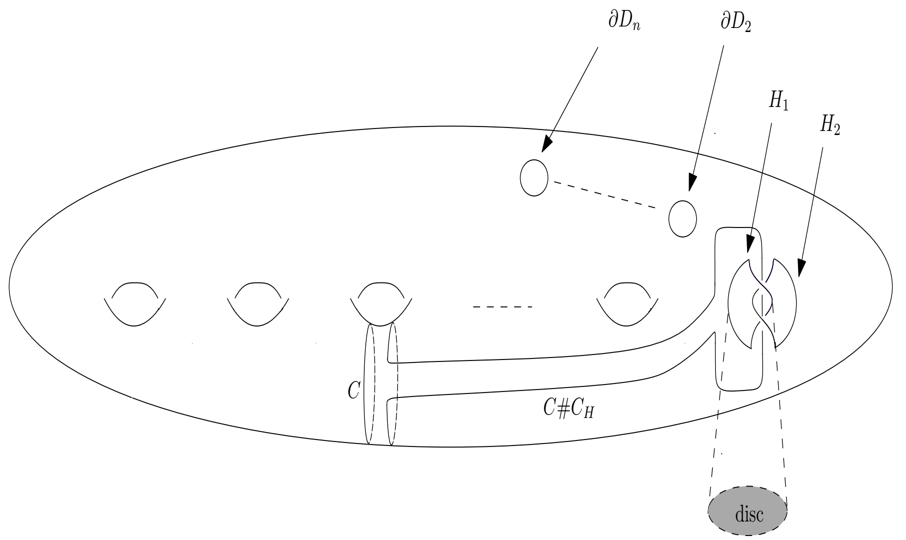

FIGURE 3. Embedding of the surface S with $n+1$ boundary components. The boundary component with dashed line bounds a properly embedded disk in $\mathcal{D}E(m)$

$\mathcal{D}E(m)$. We attach this disk to the surface S to get a new embedding – say $\widetilde{f}$ – of $(\Sigma, \partial\Sigma)$ in $\mathcal{D}E(m)$ with its n boundary components. Let us denote these boundary components by $\partial D_1, \partial D_2, \cdots, \partial D_n$ as shown in Figure 3. Here ∂D_1 actually represents H_2.

Consider n cylinders $\partial D_i \times [\frac{1}{2}, 1]$ for $i = 1$ to n. Using these cylinders, we now modify the embedding $\widetilde{f}$ to get a proper embedding f of $(\Sigma, \partial\Sigma)$ in $\mathcal{D}E(m)$. This we do by considering the union $\widetilde{f}(\Sigma) \cup \partial D_1 \times [\frac{1}{2}, 1] \cup \cdots \cup \partial D_n \times [\frac{1}{2}, 1]$. This completes the proof of part (1).

The embedding described above then clearly gives a proper embedding of $(\Sigma, \partial\Sigma)$ in $\mathcal{D}E(m)$. We perturb this embedding – if necessary – to make it into a smooth and proper embedding. By slight abuse of notation, let us again denote this embedding of $(\Sigma, \partial\Sigma)$ by f. Note that such an embedding was not possible to obtain in dimension below 4.

We now observe that the embedding f satisfies the property that any simple closed curve C and its ambient band connected sum with the center curve C_H of the Hopf band H, $C\#_b C_H$ (depicted by the longer simple closed curve passing through the Hopf band in Figure 3), are ambiently isotopic. This holds because C_H is isotopic to the boundary component of H_1 which bounds the disk D. Hence, C_H can be shrunk to a point in the interior of $f(\Sigma)$. This implies that we can isotope C to $C\#_b C_H$ using the disk D in $\mathcal{D}E(m)$.

Note that a regular neighborhood of the curve $C\#_b C_H$ is a Hopf annulus. We claim that there is an isotopy – say Φ_t – of $\mathcal{D}E(m)$ which is fixed near the boundary of $\mathcal{D}E(m)$ and which induces a Dehn twist along $C\#_b C_H$. In fact, the isotopy can be assumed to be the identity when restricted to the 2–handle as well. This can be done as follows:

To begin with, recall that the whole surface Σ, except the 2-disk D coming from the attached 2-handle, is still embedded in B^4. In fact, all of $f(\Sigma)$, except the cylinders $\partial D_i \times [\frac{1}{2}, 1]$ and the disk D, is still embedded in the level $S^3 \times \{\frac{1}{2}\}$ of the collar $S^3 \times [0, 1]$ of ∂B^4. In particular, a fixed neighborhood $\mathcal{N}(C\#_b C_H)$ of $C\#_b C_H$ is contained in $S^3 \times \{\frac{1}{2}\}$.

In order to get the isotopy Ψ_t as claimed we first describe how to produce an isotopy Φ_t of $\mathcal{D}E(m)$ which induces the Dehn twist along $C\#_b C_H$ on Σ. Push the neighborhood $\mathcal{N}(C\#_b C_H)$ slightly towards $S^3 \times \{0\}$ in the collar of ∂B^4, such that at a fixed level between 0 and $\frac{1}{2}$, the intersection of this pushed neighborhood is a Hopf annulus and this Hopf annulus contains the pushed curve $C\#_b C_H$ as its center curve. Let us denote this level by $S^3 \times \{s_0\}$. We now perform an isotopy to induce a Dehn twist along the pushed $C\#_b C_H$ in such a way that this isotopy is supported in a small neighborhood of $S^3 \times \{s_0\}$ not intersecting $S^3 \times \{\frac{1}{2}\}$. After performing this isotopy, we further isotope the pushed neighborhood $\mathcal{N}(C\#_b C_H)$ back to its original place in $S^3 \times \{\frac{1}{2}\}$. Clearly, the effect of successive compositions of these isotopies is an isotopy Φ_t of $\mathcal{D}E(m)$, which induces the Dehn twist along $C\#_b C_H$ on Σ.

We now recall that the mapping class group of $(\Sigma, \partial\Sigma)$ is generated by Dehn twists along the Lickorish curves as described in the Figure 1 (see [**FM**, p. 133]). Since on each Lickorish curve it is possible to perform a Dehn twist via an ambient isotopy of $\mathcal{D}E(m)$, we get the isotopy Ψ_t with the required properties.

$\square$

Remark 3.1.7. *Note that Lemma 3.1.6 shows that in $\mathcal{D}E(m)$, there exists a proper flexible embedding of the surface Σ.*

Proof of Theorem 3.1.1. Consider the abstract open book $\mathcal{A}ob\,(\mathcal{D}E(m), Id)$. Recall from [**Ko**] that if m is even, then $\mathcal{A}ob\,(\mathcal{D}E(m), Id)$ represents the manifold $S^3 \times S^2$ and if m is odd then it represents $S^2 \widetilde{\times} S^3$. Then by Lemma 3.1.6, $M^3 = \mathcal{A}ob(\Sigma, \phi))$ admits an open book embedding in any open book decomposition associated to $S^3 \times S^2$ and $S^2 \widetilde{\times} S^3$ with page a disk bundle over S^2 and monodromy the identity.

$\square$

3.2. Embeddings of 3–manifolds in S^5

In this section, we use the techniques developed to establish Lemma 3.1.6 to prove that every closed orientable 3–manifold embeds in S^5.

Proof of Theorem 3.1.2. In order to produce an embedding of a closed orientable 3–manifold M in S^5, we first notice that it is sufficient to embed M in $S^3 \times \mathbb{R}^2$ as S^3 is embedded in S^5 with a trivial normal bundle. In fact, S^3 sits in S^5 as the binding of the trivial open book decomposition of S^5.

Given a closed connected orientable manifold M, we first consider an abstract open book $\mathcal{A}ob(\Sigma, \phi)$ of M. As every closed connected orientable manifold admits an open book decomposition with connected binding (see, [**My**]), we can assume that Σ has connected boundary.

We will first produce a proper flexible embedding of Σ in the disk bundle $\mathcal{D}E(1)$ over S^2 such that Σ does not intersect the zero section of $\mathcal{D}E(1)$. For such an embedding, the mapping torus $\mathcal{M}T(\Sigma, \phi)$ associated to the abstract open book $M = \mathcal{A}ob(\Sigma, \phi)$ is properly embedded in a manifold diffeomorphic to $S^1 \times \mathcal{D}E(1)$. In fact, as Σ is disjoint from the zero section of $\mathcal{D}E(1)$, the mapping torus $\mathcal{M}T(\Sigma, \phi)$ is properly embedded in a manifold diffeomorphic to $S^1 \times S^3 \times (0, 1]$. This follows from the fact that the boundary of $DE(1)$ is S^3 and the complement of the zero section in $\mathcal{D}E(1)$ is $S^3 \times (0, 1]$, for more details we refer to [**GS**, p. 119]. Next, consider the disjoint union of $S^1 \times S^3 \times (0,1]$ and $S^3 \times D^2$. The quotient manifold, obtained by identifying the boundary $S^1 \times S^3 \times \{1\}$ of $S^1 \times S^3 \times (0,1]$ with the boundary $S^3 \times S^1$ of $S^3 \times D^2$ by the identity map, is diffeomorphic to $S^3 \times D^2$. Thus, one can see that M embedded in $S^2 \widetilde{\times} S^3$ obtained via the open book embedding of M in $S^2 \widetilde{\times} S^3 = \mathcal{A}ob(\mathcal{D}E(1), Id)$ is actually contained in a manifold diffeomorphic to $S^3 \times \mathbb{R}^2$ as required.

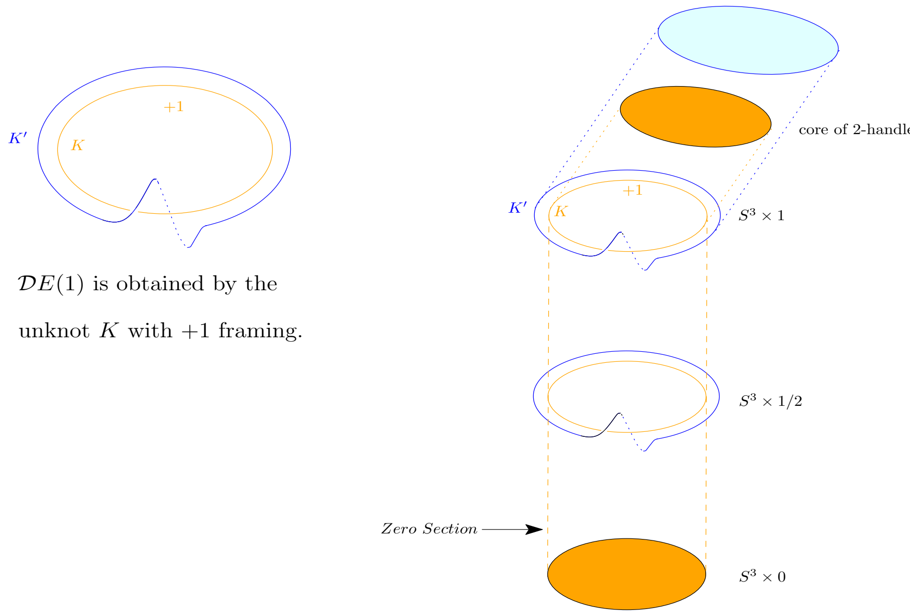

FIGURE 4. The left of the figure shows the Kirby diagram for $\mathcal{D}E(1)$ and the knot K' which links K once. The unknots K and K' bound disks in the 2–handle $D^2 \times D^2$ attached to B^4 in $\mathcal{D}E(1)$. The right of the figure depicts the zero section $\mathcal{S}$ in $\mathcal{D}E(1)$ which is union of the core of the 2–handle, $K \times [0,1]$ and the disk bounded by $k \times 0$ in $S^3 \times 0$.

As discussed above, we will now give a precise construction of a proper flexible embedding of Σ in $\mathcal{D}E(1)$ such that it is disjoint from the zero section of $\mathcal{D}E(1)$.

Recall that $\mathcal{D}E(1)$ is obtained from a 4–ball B^4 by attaching a 2–handle $D^2 \times D^2$ along an unknot with framing $+1$ contained in the boundary ∂B^4 of B^4. Now, we describe the zero section of $DE(1)$. We parameterize a collar of ∂B^4 by $S^3 \times [0,1]$ such that $\partial B^4 = S^3 \times \{1\}$. The unknot K which is the attaching circle of the 2–handle is then contained in $S^3 \times \{1\}$. This is depicted in the left of the Figure 4 by orange circle with framing $+1$. Let us denote the zero section of the bundle $\mathcal{D}E(1)$ by $\mathcal{S}$. We can regard $\mathcal{S}$ as the sphere obtained by considering the union of attaching disk of the 2–handle with $K \times [0,1]$ and the obvious disk $K \times \{0\}$ bounds in $S^3 \times \{0\}$, see right of Figure 4. Next, we denote by $\mathcal{N}(K)$ a tubular neighborhood of K which is the attaching region of the 2–handle $H_2 = D^2 \times D^2$. If p is a point on the boundary of D^2, then the disk $D^2 \times \{p\}$ embedded in the 2–handle H_2 intersects the boundary $S^3 \times \{1\}$ in a curve K' which links K once. This is depicted by the blue curve in the Figure 4. Notice that K' lies on the boundary of the attaching region $\mathcal{N}(K)$.

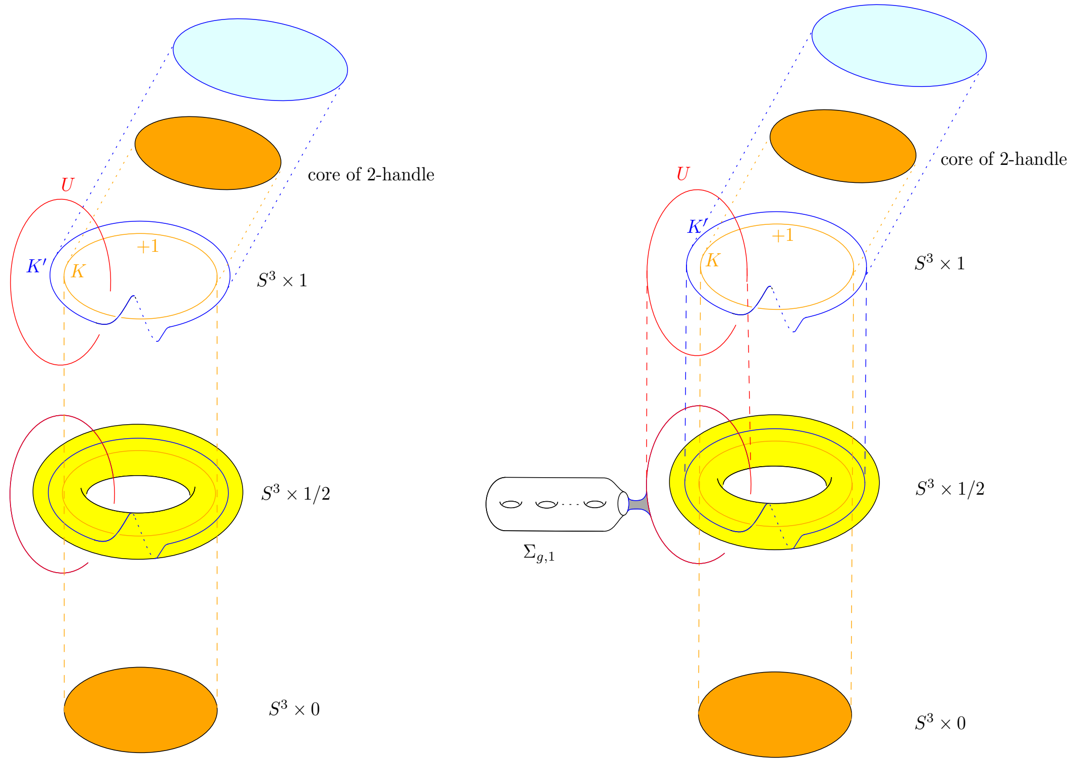

FIGURE 5. The left of the figure depicts the zero section of $\mathcal{D}E(1)$ together with the unknot U which is the core of the complement of $\mathcal{N}(K)$. The right of the figure depicts an embedding of Σ away from the zero section of $\mathcal{D}E(1)$.

In order to produce the required embedding Σ in $\mathcal{D}E(1)$, we proceed as follow: Consider an unknot U which links once the attaching region $\mathcal{N}(K)$ as depicted in red colour in the left of Figure 5. Consider the two circles $U \times \{\frac{1}{2}\}$ and $K' \times \{\frac{1}{2}\}$ in the sphere $S^3 \times \{\frac{1}{2}\}$. Notice that the complement of $\mathcal{N}(K) \times \{\frac{1}{2}\}$ in $S^3 \times \{\frac{1}{2}\}$ is a solid torus $\tau = S^1 \times D^2$. The circle $U \times \{\frac{1}{2}\}$ is the center circle $S^1 \times \{0\}$ of this solid torus while $K' \times \{\frac{1}{2}\}$ is a curve going once around the longitude and once around the meridian of the solid torus. This implies circles $U \times \{\frac{1}{2}\}$ and $K' \times \{\frac{1}{2}\}$ bound a Hopf annulus in $S^3 \times \{\frac{1}{2}\}$ which is disjoint from $K \times \{\frac{1}{2}\}$ as it lies inside the solid torus τ. Let us call this Hopf annulus $\mathcal{A}$. Now, observe that the boundary component $K' \times \{\frac{1}{2}\}$ of the annulus $\mathcal{A}$ bounds a disk – say $\mathcal{D}$ – in $\mathcal{D}E(1)$ by construction. The disk $\mathcal{D}$ is the union of the disk bounded by $K' \times \{1\}$ in the 2–handle and the annulus $K' \times [\frac{1}{2}, 1]$. By attaching the annulus $U \times [\frac{1}{2}, 1]$ to $\mathcal{A} \cup \mathcal{D}$ along its boundary $U \times \{\frac{1}{2}\}$, we produce a properly embedded disk – say $\mathcal{D}'$– in $\mathcal{D}E(1)$.

Next, let $\widetilde{\Sigma}$ be the boundary of a standard unknottedly embedded handle-body contained in the solid torus τ which is disjoint from the Hopf annulus $\mathcal{A}$. We can perform an ambient connected sum of $\widetilde{\Sigma}$ with $\mathcal{A}$ in $S^3 \times \{\frac{1}{2}\}$ such that the surface obtained after the ambient connected sum is still contained in the complement of $K \times \{\frac{1}{2}\}$ in $S^3 \times \{\frac{1}{2}\}$. Notice that since $\mathcal{A}$ is a Hopf annulus in a properly embedded disk $\mathcal{D}'$ described in the previous paragraph, this connected sum operation is actually a connected sum of $\widetilde{\Sigma}$ with the disk $\mathcal{D}'$ and this produces a properly embedded surface with one boundary component, refer right of Figure 5. Let us denote this surface by Σ.

Observe that an argument similar to the one used in the proof of the Lemma 3.1.6 implies that Σ is a properly embedded flexible surface in $\mathcal{D}E(1)$. This is because, the embedded surface Σ admits an embedding of a Hopf annulus such that one of the boundary component of this Hopf annulus bounds a disk in the surface. Hence, we can isotope every generator of the mapping class group of Σ in such a way that it admits a neighborhood which is a Hopf annulus embedded in $S^3 \times \{\frac{1}{2}\}$. Furthermore, notice that Σ does not intersect the zero section $\mathcal{S}$ of the bundle $\mathcal{D}E(1)$ as it does not intersect the core disk of the 2–handle as well as the annulus $K \times [\frac{1}{2}, 1]$. This completes our argument.

$\square$

CHAPTER 4

Contact open book embedding

In this chapter we prove some contact open book embedding results.

4.1. Statements of the main results

We start with the following theorem.

THEOREM 4.1.1. *For $n \geq 1$ and $k, l \in \mathbb{Z}$, $\mathcal{A}ob(DT^*S^n, d\lambda^n_{can}, \tau_k)$ contact open book embeds in $\mathcal{A}ob(DT^*S^{n+1}, d\lambda^{n+1}_{can}, \tau_l)$.*

Here, $d\lambda^n_{can}$ denotes the canonical symplectic form on the unit disk cotangent bundle DT^*S^n of S^n, and τ_m denotes the m-fold Dehn-Seidel twist (section 2.4.2) for $m \in \mathbb{Z}$.

Theorem 4.1.1 can also be obtained from the construction of contact open book embedding of $\mathcal{A}ob(DT^*S^n, d\lambda^n_{can}, \tau_k)$ in $\mathcal{A}ob(D^{2n+2}, d\lambda_0, id)$, due to Casals and Murphy [**CM**]. For that we have to stabilize the target page $(D^{2n+2}, d\lambda_0)$, away from the image of DT^*S^n under the embedding. The proof given here has a different approach, and will come in handy while proving Theorem 4.1.4 below.

In [**CMP**], Casals, Murphy and Presas gave a characterization of overtwisted contact structures in terms of open books. They showed that every overtwisted contact structure is a negative stabilization of some open book decomposition. In particular, $\mathcal{A}ob(DT^*S^n, d\lambda_{can}, \tau_{-1})$ gives an overtwisted contact structure on S^{2n+1}. Thus, an immediate corollary of Theorem 4.1.1 is the following fact, which was first shown by Casals and Murphy [**CM**].

COROLLARY 4.1.2. *For all $n \geq 1$, there exists an overtwisted contact structure on S^{2n+1} that contact open book embeds in (S^{2n+3}, ξ_{std}).*

Using Theorem 4.1.1, we can find a large class of contact manifolds that admit co-dimension 2 contact open book embeddings in the standard contact sphere. For definition of plumbing and boundary connected sum, see section 2.4.3 and section 2.4.4. The boundary connected sum and plumbing will be denoted by $\#_b$ and § respectively.

DEFINITION 4.1.3. *Consider the canonical symplectic structure $d\lambda_M$ on the cotangent bundle of a manifold M. We call a contact open book $\mathcal{A}ob(V^{2n}, \omega, \phi)$ of type-1, if it satisfies the following properties.*

1. *(V^{2n}, ω) is symplectomorphic to$(DT^*M_1 \S DT^*M_2 \S .. \S DT^*M_p \#_b DT^*N_1 \#_b DT^*N_2 .. \#_b DT^*N_q,$ $d\lambda_{M_1} \S d\lambda_{M_2} \S .. \S d\lambda_{M_p} \#_b d\lambda_{N_1} \#_b d\lambda_{N_2} \#_b .. \#_b d\lambda_{N_q})$.*

   Here, M_is and N_js will always be either the equator sphere $S^n \subset S^{n+1}$, or a closed n-dimensional submanifold of S^{n+1}.
2. *The monodromy ϕ is generated by Dehn-Seidel twists along the S^ns among M_is and N_js .*

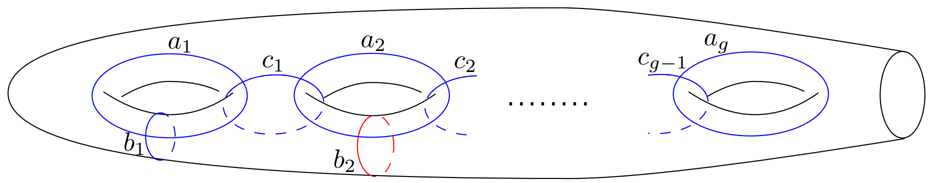

FIGURE 1. The Humphreys generators of mapping class groups of Σ_g consists of the $(2g+1)$ curves given by $a_1, c_1, a_2, c_2, ...a_{g-1}, c_{g-1}, a_g, b_1, b_2$.

THEOREM 4.1.4. *If (M^{2n+1}, ξ) is a contact manifold supported by an open book of type-1, then (M^{2n+1}, ξ) has a contact open book embedding in (S^{2n+3}, ξ_{std}).*

The next corollary gives an application of Theorem 4.1.4 to contact open book embedding of 3-manifolds in (S^5, ξ_{std}). It was previously proved by Etnyre and Lekili (Theorem 4.3, [**EL**]).

COROLLARY 4.1.5. *Let (M^3, ξ) be a contact 3-manifold supported by an open book with page Σ_g as in Figure 4.1. Say, the monodromy of the supporting open book is generated by Dehn twists along the blue curves: $b_1, a_1, c_1, a_2, c_2, ..., a_{g-1}, c_{g-1}, a_g$. Then (M^3, ξ) contact open book embeds in (S^5, ξ_{std}).*

By Giroux [**Gi**], if two contact structures on M^3 are supported by the same open book, then the contact structures are contactomorphic. Moreover, any two open books supporting a contact structure are related by stabilizations [**Gi**]. Given any supporting open book of a contact 3-manifold, we can stabilize it finitely many times until the resulting open book has a page homeomorphic to Σ_g. Stabilization does not change the contactomorphism type of the manifold.

Note that the relative mapping class group $\mathcal{MCG}(\Sigma_g, \partial\Sigma_g)$ is generated by Dehn twists along the curves $a_1, c_1, ..., a_{g-1}, c_{g-1}, a_g, b_1, b_2$ and a curve d, parallel to the boundary $\partial\Sigma_g$. But, $\tau_d = (\tau_{b_1} \circ \tau_{a_1} \circ \tau_{c_1} \circ ... \circ \tau_{a_{g-1}} \circ \tau_{c_{g-1}} \circ \tau_{a_g})^{4g+2}$ (see Proposition 4.12 in [**FM**]). Thus, $\mathcal{MCG}(\Sigma_g, \partial\Sigma_g)$ can be generated by Dehn twists along the curves $a_1, c_1, ..., a_{g-1}, c_{g-1}, a_g, b_1, b_2$. Every contact open book embedding also gives an iso-contact embedding. This shows that the only class of 3-dimensional contact open books with page Σ_g, which may not admit iso-contact embedding in (S^5, ξ_{std}), are the ones with monodromy involving Dehn twists along the red curve b_2 in Figure 4.1.

We also note that a contact manifold (M^3, ξ) may have a supporting open book with page Σ_g and monodromy involving Dehn twists along b_2, and still can iso-contact embed in (S^5, ξ_{std}). Examples of such open books are discussed in section 4.3.5.

4.2. Proof of the theorems

The standard inclusion of $S^n \subset \mathbb{R}^{n+1}$ in $S^{n+1} \subset \mathbb{R}^{n+2}$ induces a canonical proper symplectic embedding $j_0 : (DT^*S^n, \lambda^n_{can}) \hookrightarrow (DT^*S^{n+1}, \lambda^{n+1}_{can})$. As discussed in section 2.4.2, we consider $DT^*S^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and $DT^*S^{n+1} \subset \mathbb{R}^{n+2} \times \mathbb{R}^{n+2}$. So, in terms of the coordinates, $j_0(\vec{x}, \vec{y}) = (\vec{x}, 0, \vec{y}, 0)$. Recall that $(\vec{x}, \vec{y}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ satisfies $|\vec{x}| = 1$ and $\vec{x} \cdot \vec{y} = 0$. Let $j_1 : (DT^*S^n, \lambda^n_{can}) \hookrightarrow (DT^*S^{n+1}, \lambda^{n+1}_{can})$ be another proper symplectic embedding given

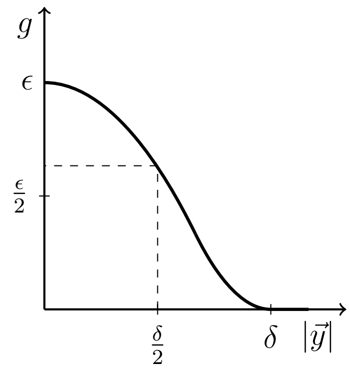

FIGURE 2.

by : $j_1(\vec{x},\vec{y}) = (\vec{x},0,\vec{y},g(|\vec{y}|))$. Here, g is a smooth cut-off function as described in Figure 2. Note that g has support $[0,\delta]$ and $g(0)=\epsilon$, $g(\frac{\delta}{2}) > \frac{\epsilon}{2}$.

The next two lemmas are the main ingredients to prove our theorems.

LEMMA 4.2.1. *j_0 is symplectic isotopic to j_1.*

PROOF. Define $j_t(\vec{x},\vec{y}) = (\vec{x},0,\vec{y},t\cdot g(|\vec{y}|))$. Now, $j_t^*(d\lambda_{can}^{n+1}) = j_t^*(\sum_{i=1}^{n+2} dx_i \wedge dy_i) = \sum_{i=1}^{n+1} dx_i \wedge dy_i + j_t^*(dx_{n+2}\wedge dy_{n+2})$. Since $j_t^*(dx_{n+2}) = 0$, we have $j_t^*(d\lambda_{can}^{n+1}) = d\lambda_{can}^n$. □

In general, let M^n be an oriented closed hyper surface in S^{n+1}. Then the normal bundle is $M\times\mathbb{R}$. Thus we get an induced symplectic embedding of $(DT^*M, d\lambda_M)$ in $(DT^*S^{n+1}, d\lambda_{can}^{n+1})$ so that a symplectic tubular neighborhood of this embedding is isomorphic to $(DT^*M\times T^*\mathbb{R}^1, d\lambda_M \oplus dx\wedge dy)$. We can now define a similar symplectic isotopy sending $(p,v,0,0)\mapsto (p,v,0,g(|v|))$ in $T^*M\times T^*\mathbb{R}^1$.

The next lemma is an adaptation of Proposition 4 in [**Au**] to our setting. The proof is the same as in [**Au**] with slight modifications.

LEMMA 4.2.2. *Let $(V,\partial V, d\lambda_V)$ and $(W,\partial W, d\lambda_W)$ be two exact symplectic manifolds with convex boundaries of dimension $2m$ and $2m+2s$ respectively. Let $\psi_t : (V,\partial V)\to(W,\partial W)$ be a family of proper symplectic embeddings such that near ∂V $\psi_t=\psi_0$ for all $t\in[0,1]$. Then there is a symplectic isotopy Ψ_t of $(W,\partial W,d\lambda_W)$ such that $\Psi_0 = Id$ and $\Psi_1\circ\psi_0(V) = \psi_1(V)$.*

PROOF. Let V_t denote $\psi_t(V)$. $\psi_t\cdot\psi_0^{-1}$ gives a family of symplectomorphisms from $(V_0, d\lambda_W|_{V_0})$ to $(V_t, d\lambda_W|_{V_t})$. Since the symplectic normal bundles to all V_t are isomorphic, using Weinstein symplectic neighborhood theorem we can extend $\psi_t\cdot\psi_0^{-1}$ to a family of symplectomorphisms $L_t : U_0 \to U_t$, where U_t is a small symplectic tubular neighborhood of V_t in W. Let $\rho_t : (W,\partial W)\to(W,\partial W)$ be any family of diffeomorphism extending ψ_t. We can assume that ρ_t is identity near ∂W. Let $\omega_t = \rho_t^*(d\lambda_W)$ and $\Omega_t = -\frac{d\omega_t}{dt}$. We want to find vector fields Y_t on W such that $d\iota_{Y_t}\omega_t = \Omega_t$ and Y_t is tangent to V_0. Let $\omega = d\lambda_W$. For such a Y_t if χ_t denotes the corresponding flow, then we have

$$\frac{d}{dt}(\chi_t^*\rho_t^*\omega) = \chi_t^*(\frac{d}{dt}(\rho_t^*\omega) + L_{Y_t}(\rho_t^*\omega)) = \chi_t^*(-\Omega_t + d\iota_{Y_t}\omega_t) = 0$$

. Thus, $\rho_t \circ \psi_t$ is a family of symplectomorphisms of W. Let $\alpha_t = \iota_{Y_t}\omega_t$. Then equivalently we have to find a 1-form α_t on W such that $d\alpha_t = \Omega_t$ and at every point of $v \in V_0$, the ω_W-symplectic orthogonal N_vV_0 to T_vV_0 lies in the kernel of α_t. Now,

$$\Omega_t = -\frac{d\omega_t}{dt} = -\frac{d}{dt}(\rho_t^* d\lambda_W) = -d(\frac{d}{dt}\rho_t^*\lambda_W)$$

So, defining $\beta_t = -\frac{d}{dt}\rho_t^*\lambda_W$ gives $d\beta_t = \Omega_t$. Note that $d\beta_t = 0$ over U_0 and since ρ_t is identity near ∂W, $\beta_t = 0$ near $\partial U_0 \cap \partial W$. Thus, $\beta_t \in H^1(\bar{U_0}, \partial\bar{U_0} \cap \partial W; \mathbb{R})$ and $\beta_t|_{V_0} \in H^1(V_0, \partial V_0; \mathbb{R})$. Let $\pi : U_0 \to V_0$ be the projection map of the symplectic normal bundle and $i_0 : V_0 \hookrightarrow U_0$ be the zero section. Let $\gamma_t = \pi^*\beta_t|_{V_0}$. By construction, γ_t is closed over U_0 and for any $x \in V_0$ the normal fiber N_xV_0 lies in the kernel of γ_t. Moreover, the composition $\pi^* \circ i_0^*$ induces the identity map over $H^1(\bar{U_0}, \partial\bar{U_0} \cap \partial W; \mathbb{R})$. Thus, $[\gamma_t] = [\beta_t|U_0]$ in $H^1(\bar{U_0}, \partial\bar{U_0} \cap \partial W; \mathbb{R})$. Therefore there is a smooth real valued function f_t over U_0 such that $\gamma_t = \beta_t + df_t$ over U_0. Now, we can extend f_t to some smooth real function g_t over W and define $\alpha_t = \beta_t + dg_t$. The 1-forms α_t satisfy $d\alpha_t = \Omega_t$ and since $\alpha_t|U_0 = \gamma_t$, N_xV_0 lies in the kernel of α_t for all $x \in V_0$. □

Proof of Theorem 4.1.1. (S^{2n+3}, η_{st}) has an open book decomposition with page DT^*S^{n+1} and monodromy τ_1^{n+1}. First we embed $(DT^*S^n, \lambda_{can}^n)$ in $(DT^*S^{n+1}, \lambda_{can}^{n+1})$ via j_0 and then apply a k-Dehn twist τ_k^{n+1} on DT^*S^{n+1}. This will induce the monodromy τ_k^n on DT^*S^n. Next, using Lemma 4.2.1 we can isotope $\tau_k^n(DT^*S^n)$ in DT^*S^{n+1} to $j_1 \circ \tau_k^n(DT^*S^n)$. Next we apply an $(l-k)$-fold Dehn twist $\tau_{(l-k)}^{n+1}$ on DT^*S^{n+1} with support $\frac{m_0}{2}$, where the number m_0 is constructed as follows.

As discussed in section 2.4.2, while defining the Dehn twist, we can make the support of the function g_k smaller, without changing the symplectic isotopy class of the resulting Dehn twist map. Now, let S_0^{n+1} denote the zero section of DT^*S^{n+1}. Let $\mathcal{D} = d(j_1(DT^*S^n), S_0^{n+1})$ be the distance between $j_1(DT^*S^n)$ and S_0^{n+1} in $\mathbb{R}^{n+2} \times \mathbb{R}^{n+2}$, under the standard euclidean metric. For the discussion below, we will be referring to Figure 2. Let $A_0 = (\vec{w}, 0) \in S_0^{n+1}$ and $A_1(\vec{y}) = (\vec{x}, 0, \vec{y}, g(|\vec{y}|)) \in j_1(DT^*S^n)$. Then the square of distance between A_0 and $A_1(\vec{y})$ is given by

$$d(A_0, A_1(\vec{y}))^2 = |w - (\vec{x}, 0)|^2 + |\vec{y}|^2 + |g(|\vec{y}|)|^2$$

. Note that for $|\vec{y}| < \frac{\delta}{2}$, $g(|\vec{y}|) > \frac{\epsilon}{2}$ and therefore, $d(A_0, A_1(\vec{y}))^2 > \frac{\epsilon^2}{4}$. For $|\vec{y}| > \frac{\delta}{2}$, $d(A_0, A_1(\vec{y}))^2 > \frac{\delta^2}{4}$. Thus, $\mathcal{D} > \frac{1}{2}min\{\epsilon, \delta\}$. Define, $m_0 = \frac{1}{2}min\{\epsilon, \delta\}$.

The above choice of m_0 will ensure that $j_1 \circ \tau_k^n(DT^*S^n)$ is not affected by $\tau_{(l-k)}^{n+1}$. Figure 3 describes the situation for $n = 0$. Lastly, we isotope $j_1 \circ \tau_k^n(DT^*S^n)$ back to $\tau_k^n(DT^*S^n)$ and finish gluing the mapping torus. Using Lemma 4.2.2, we can extend j_t to a symplectic isotopy J_t of DT^*S^{n+1}, such that $J_0 = id$. The resultant monodromy on DT^*S^{n+1} then becomes $J_1^{-1} \circ \tau_{(l-k)}^{n+1} \circ J_1 \circ \tau_k^{n+1}$, which is equivalent to getting a mapping torus with monodromy τ_l^{n+1}. When restricted to DT^*S^n, it induces the monodromy τ_k^n.

□

Proof of Corollary 4.1.2. By Theorem 4.1.1, $\mathcal{A}ob(DT^*S^n, d\lambda_{can}^n, \tau_{-1})$ contact open book embeds in (S^{2n+3}, η_{st}) with open book $\mathcal{A}ob(DT^*S^{n+1}, d\lambda_{can}^{n+1})$. According to [**CMP**], $\mathcal{A}ob(DT^*S^n, d\lambda_{can}$ $, \tau_{-1})$ gives an overtwisted contact structure on S^{2n+1}. □

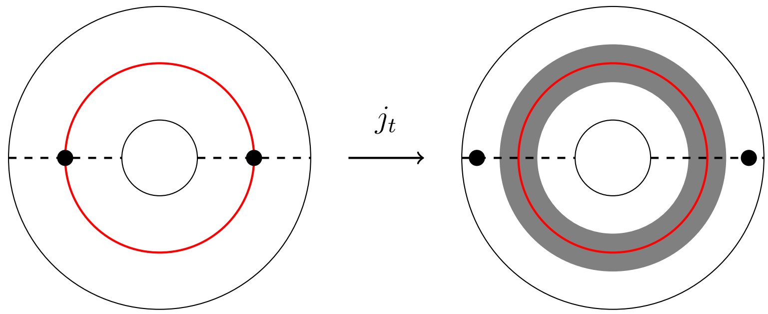

FIGURE 3. Using the isotopy j_t, we push DT^*S^0 away from the zero section S^1_0 (denoted by red circles) of DT^*S^1. Then we apply Dehn twist along S^1_0, with support in the shaded region.

Proof of Theorem 4.1.4. Theorem 4.1.1 implies that $\mathcal{A}ob(DT^*S^n, d\lambda^n_{can}, \tau_k)$ contact open book embeds in $\mathcal{A}ob(DT^*S^{n+1}, d\lambda^{n+1}_{can}, \tau_1)$. Following the discussion after Lemma 4.2.1 and the proof of Theorem 4.1.1, one can also see that for $V^n \subset S^{n+1}$, $\mathcal{A}ob(DT{*}V^n, id)$ contact open book embeds in $\mathcal{A}ob(DT^*S^{n+1}, \tau_1)$. Moreover, there is an ambient symplectic isotopy of the identity map of $(DT^*S^{n+1}, d\lambda^{n+1}_{can})$, relative to the boundary, that pushes a symplectic neighborhood of the zero section in DT^*V^n away from the zero section of DT^*S^{n+1}. Recall that a page of a type-1 open book is constructed by taking plumbing and boundary connected sum of such DT^*V^ns and DT^*S^ns.

(1) *Boundary connected sum :*
Using the boundary connected sum operation described in section 2.4.4, we get a symplectic embedding of $(DT^*M^n_1 \#_b DT^*M^n_2$ in $DT^*S^{n+1}_1 \#_b DT^*S^{n+1}_2$. Assume that the monodromy map of $(DT^*M^n_1 \#_b DT^*M^n_2$ is identity. Here, S^{n+1}_1 and S^{n+1}_2 are used to denote two copies of S^{n+1}. Let Φ^t_i be the ambient symplectic isotopy of $DT^*S^{n+1}_i$ that pushes a symplectic neighborhood of the zero section in $DT^*M^n_i$ away from the zero section of $DT^*S^{n+1}_i$, for $i = 1, 2$. Since Φ^t_1 and Φ^t_2 are identity near boundary for all $t \in [0, 1]$, we can extend the isotopy to $\tilde{\phi}^t$ on $DT^*S^{n+1}_1 \#_b DT^*S^{n+1}_2$ by defining it identity on the attached 1-handle of boundary connected sum. Now, following the proof of Theorem 4.1.1, apply $\tilde{\phi}^t$ to push away a neighborhood of M^n_1 and M^n_2 and then apply positive Dehn-twists in a small enough neighborhood of S^{n+1}_1 and S^{n+1}_2. Bring back the neighborhoods of M^n_1 and M^n_2 by $\tilde{\phi}^{1-t}$ and complete the mapping torus. The effective monodromy on $(DT^*M^n_1 \#_b DT^*M^n_2$ is identity and on $DT^*S^{n+1}_1 \#_b DT^*S^{n+1}_2$ it is composition of Dehn twists along S^{n+1}_1 and S^{n+1}_2. Thus, by section 2.4.4 the result follows.

(2) *Plumbing :*
For plumbing, we do the following. Consider $DT^*S^n_1 \S DT^*S^n_2 \hookrightarrow DT^*S^{n+1}_1 \S DT^*S^{n+1}_2$. Say, ϕ_0 is a symplectomorphism of $DT^*S^n_1 \S DT^*S^n_2$ generated by Dehn twists along S^n_1 and S^n_2, denoted by τ^1_1 and τ^2_1, respectively. For the moment, assume that $\phi_0 = \tau^1_l \circ \tau^2_k$. Let $t \in (0, 1)$ denote the S^1 direction in the mapping torus of $\mathcal{A}ob(DT^*S^n_1 \S DT^*S^n_2, \phi_0)$. In the time interval $[\frac{1}{4}, \frac{1}{3}]$, we apply an l-fold Dehn twist τ^1_l along S^{n+1}_1. Next, we isotope the $DT^*S^n_1$ part inside $DT^*S^{n+1}_1 \subset DT^*S^{n+1}_1 \S DT^*S^{n+1}_2$

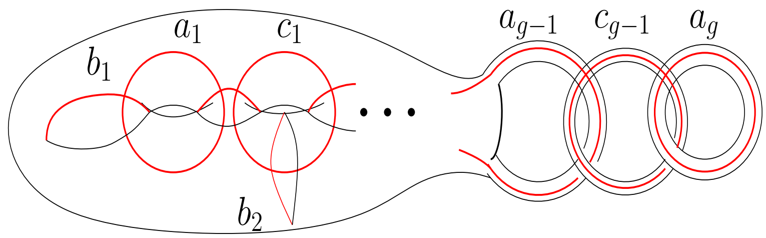

FIGURE 4.

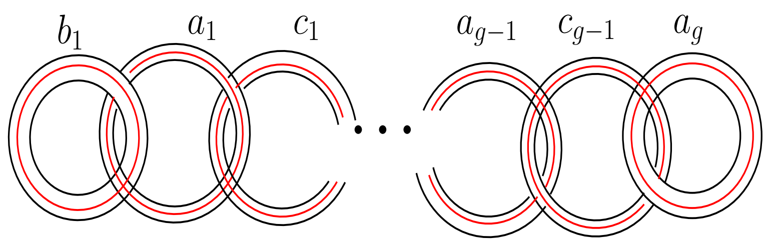

FIGURE 5.

away from S_1^{n+1} and apply $(-l+1)$-fold Dehn twist along S_1^{n+1}. Finally, we isotope $DT^*S_1^n$ back to its original place. The procedure is similar to that in the proof of Theorem 4.1.1. Only here we extend the isotopy in the complement of $DT^*S_1^n$ in $DT^*S_1^n \S DT^*S_2^n$ by the identity map. Next, we apply the same procedure in the interval $[\frac{1}{2}, \frac{3}{4}]$ starting with a k-fold Dehn twist τ_k^2 along S_2^{n+1}, before completing the mapping torus. This produces an open book embedding of $\mathcal{A}ob(DT^*S_1^n \S DT^*S_2^n, \phi_0)$ in (S^{2n+3}, η_{st}). For the general case, we can factor the monodromy into Dehn twists along various S^ns and divide the S^1-interval of the mapping torus accordingly to apply the same argument finitely many times.

(3) *General case :*

For the cases $\mathcal{A}ob(DT^*M\#_b DT^*S^n, id\#_b\tau_k)$ and $\mathcal{A}ob(DT^*M\S DT^*S^n, id\S\tau_l)$ we can easily combine the above two methods and thus the general case follows.

□

Proof of Corollary 4.1.5. By Giroux [**Gi**], if two contact structures on M^3 are supported by the same open book, then the contact structures are contactomorphic. If the supporting pages of (M^3, ξ) are plumbed copies of DT^*S^1s, and the monodromy is generated by Dehn twists along each S^1's, then by Theorem 4.1.4, (M^3, ξ) has a contact open book embedding in (S^5, η_{st}). Now, the surface described in Fig 1 is can be deformation retracted onto a diffeomorphic image as in Figure 4. We can further deformation retract it to Figure 5, which is just the plumbing of the cotangent bundles of the circles, described by the curves $\{b_1, a_1, c_1, a_2, c_2, a_3, ..., c_{g-1}, a_g\}$ on Figure 1. The result now follows from Theorem 4.1.4.

□

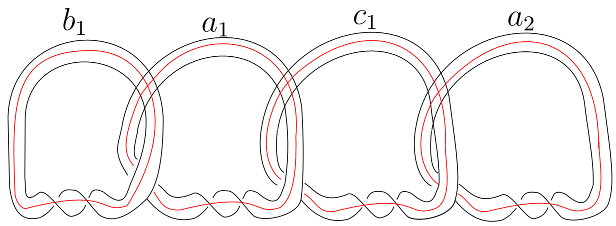

FIGURE 6. embedding of Σ as plumbing of Hopf bands when genus of Σ is 2.

4.2.1. A remark on smooth open book embedding of 3-manifolds in $\mathcal{A}ob(D^4, id.)$. As we saw in the proof of Theorem 4.1.5, the surface Σ is nothing but plumbing of the cotangent bundles of the circles, described by the curves $\{b_1, a_1, c_1, a_2, c_2, a_3, ..., c_{g-1}, a_g\}$ on Figure 1. In particular, we can embed this plumbing of annuli in S^3 as plumbing of Hopf bands. Then similar to Lemma 3.1.6, one can show the following.

LEMMA 4.2.3. *Let $(\Sigma, \partial\Sigma)$ be a surface with non-empty boundary. There exists an embedding f of Σ in D^4, which satisfies the following:*

(1) *The embedding is proper.*
(2) *Given any diffeomorphism ϕ of $(\Sigma, \partial\Sigma)$, which is generated by Dehn twists along the curves $\{b_1, a_1, c_1, a_2, c_2, a_3, ..., c_{g-1}, a_g\}$, there exists a family Ψ_t of diffeomorphisms of D^4 with $\Psi_0 = id$ such that Ψ_1 maps Σ to itself and satisfies the property that $f^{-1} \circ \Psi_1 \circ f$ is isotopic to the given diffeomorphism ϕ of $(\Sigma, \partial\Sigma)$.*

The proof of Lemma 4.2.3 is similar to that of Lemma 3.1.6. Recall that in the proof of Lemma 3.1.6, the main idea was to convert a standard annulus neighborhood of a simple closed curve on the surface $\Sigma \subset S^3$ to a Hopf annulus neighborhood, by attaching a Hopf band along one of the boundary components and then killing one of the two new boundary components by attaching a 2-handle along. In the case of Lemma 4.2.3, we take the embedding of $(\Sigma, \partial\Sigma)$ in S^3 as plumbing of Hopf bands with core curves $\{b_1, a_1, c_1, a_2, c_2, a_3, ..., c_{g-1}, a_g\}$. Given this embedding, each of the considered curves already have a Hopf annulus neighborhood in S^3. Thus, we can push a neighborhood of such a curve in a collar of $\partial D^4 \subset D^4$ and use the isotopy of the S^3 to induce Dehn twist along the pushed neighborhood and then bring back the neighborhood to their original place as in the proof of Lemma 3.1.6.

As a corollary of Lemma 4.2.3 we get the following.

THEOREM 4.2.4. *Let M^3 be a closed 3-manifold supported by an open book with page Σ_g as in Figure 1 (4.1). Say, the monodromy of the supporting open book is generated by Dehn twists along the blue curves: $b_1, a_1, c_1, a_2, c_2, ..., a_{g-1}, c_{g-1}, a_g$. Then M^3 open book embeds in $\mathcal{A}ob(D^4, id.) = S^5$.*

Abhijeet Ghanwat pointed out to the author that the tube trick (see [**HY**]) implies that the square of a Dehn twist along the core circle of an annulus admits flexible proper embedding in D^4. Thus we get the following.

COROLLARY 4.2.5. *Consider the Humphreys generators in Figure 1 (4.1). Let G_0 be the subgroup of $\mathcal{MCG}(\Sigma, \partial\Sigma)$ generated by $\{\tau_{b_2}^2, \tau_{b_1}, \tau_{a_1}, \tau_{c_1}, \tau_{a_{g-1}}, \tau_{c_{g-1}}, \tau_{a_g}\}$. If $\phi \in G_0$, then $\mathcal{A}ob(\Sigma, \phi)$ open book embeds in $\mathcal{A}ob(D^4, id.)$.*

4.3. Some comments and remarks

4.3.1. Contact open book embedding of overtwisted contact 3-manifolds. Let $\mathcal{A}ob(\Sigma,\phi)$ be a contact open book of (M^3,ξ), such that $\mathcal{A}ob(\Sigma,\phi)$ contact open book embeds in (N^5,η), for some open book $\mathcal{A}ob(W^4,\psi)$ supporting (N^5,η). Assume that M^3 has no 2-torsion in $H_1(M;\mathbb{Z})$. If the first Chern class $c_1(\xi)=0$, then it is known that the overtwisted contact structures on M^3 are in one to one correspondence with the overtwisted contact structures on S^3 (see section 2.2 and 2.5 of [**EF**]). The overtwisted contact structures on S^3 corresponds to elements in the group $\mathbb{Z}$. They are denoted by ξ_n for $n\in\mathbb{Z}$. For $n\geq 1$, ξ_n is supported by the open book $\mathcal{A}ob(DT^*S^1\S ...n\ \ copies...\S DT^*S^1,\tau_{-1}\circ ...n\ \ times...\circ\tau_{-1})$. Thus, by Theorem 4.1.1, (S^3,ξ_n) contact open book embeds in $\mathcal{A}ob(V_n^2,\chi_n)=\mathcal{A}ob(DT^*S^2\S ...n\ \ copies...\S DT^*S^2,\tau_1\circ ...n\ \ times...\circ\tau_1)=(S^5,\xi_{std})$, for $n\geq 1$. Thus, if $c_1(\xi)=0$, then at least half of the overtwisted contact structures on M^3, admit contact open book embedding in $\mathcal{A}ob(W^4\#_b V_n^2,\psi\circ\chi_n)=(N^5,\eta)$. If we know W^4 explicitely, then we get explicit open book embeddings of those overtwisted contact structures on M^3.

For example, consider the real projective space RP^3 with the unique tight contact structure ξ^2 on it. Let ξ_n^2 denote the overtwisted contact structures on RP^3, obtained by taking connected sum with (S^3,ξ_n), for $n\geq 1$. By [**CM**], $(RP^3,\xi^2)=\mathcal{A}ob(DT^*S^1,\tau_2)$ contact open book embeds in $\mathcal{A}ob(D^4,id)=(S^5,\xi_{std})$. Therefore, (RP^3,ξ_n^2) contact open book embeds in $\mathcal{A}ob(V_n^2,\chi_n)=(S^5,\xi_{std})$.

4.3.2. Contact open book embedding of exotic contact spheres. Koert and Niederkrüger [**KN**] has proved that for $m\geq 2$, all the Ustilovsky spheres of dimension $4m+1$ admit open book decompositions with page DT^*S^{2m} and monodromy a k-fold Dehn twist, for some odd k. Since, plumbing of pages of the supporting open book gives contact connected sum of the corresponding contact manifolds (see Proposition 2.6 in [**CM**]), every contact exotic sphere Σ^{4m+1} which is a connected sum of the Ustilovsky spheres, has a contact open book embedding in (S^{4m+3},ξ_{std}). In fact, Theorem 4.1.1 shows that an Ustilovsky sphere of dimension $4m+1$ contact open book embeds in all the contact homotopy $(4m+3)$-spheres given by $\mathcal{A}ob(DT^*S^{2m+1},\tau_k)$ for $k\in\mathbb{Z}$.

4.3.3. A class of contact $(2n+1)$-manifolds that contact open book embed in (S^{4n-1},ξ_{std}). Every oriented closed manifold V^n embeds in S^{2n-1} with a non-zero normal vector field. This implies that $(DT^*V^n,d\lambda_{can})$ admits proper symplectic embedding in $(DT^*S^{2n-1},d\lambda_0)$. Thus we can follow the proof of Theorem 4.1.1 to show that the contact manifold $(N^{2n+1},\eta)=\mathcal{A}ob(DT^*V^n,d\lambda_{can}$
$,id)$ contact open book embeds in $(S^{4n-1},\xi_{std})=\mathcal{A}ob(DT^*S^{2n-1},\tau_1)$. Note that for $n\geq 4$, $\pi_1(N^{2n+1})=\pi_1(V^n)$ and given any finitely presented group $\mathcal{G}$, we can find V^n with $\pi_1(V^n)=\mathcal{G}$. Thus, given any finitely presented group $\mathcal{G}$, we can find a contact manifold (N^{2n+1},ξ) such that $\pi_1(N)=\mathcal{G}$ and (N^{2n+1},ξ) contact open book embeds in $(S^{4n-1},\xi_{std})=\mathcal{A}ob(DT^*S^{2n-1},\tau_1)$. So, in dimension greater than 9, the fundamental group of a contact manifold (M^{2n+1},ξ) does not pose any obstruction to contact open book embedding in (S^{4n-1},ξ_{std}). Kasuya has shown that every 2-connected contact $(2n+1)$-manifold admits an iso-contact embedding in (S^{4n+1},ξ_{std}), for $n\geq 3$ (see Theorem 1.5 in [**Ka**]). Moreover,

Torres [**Tor**] has proved that every $(2n+1)$-dimensional contact manifold admits a contact open book embedding in (S^{4n+3}, ξ_{std}).

4.3.4. Contact manifolds with first Chern class zero. As discussed in [**Ka2**], a necessary condition for iso-contact embedding of (M^{2n+1}, ξ) in (S^{2n+3}, ξ_{std}) is $c_1(\xi) = 0$. Thus, Theorem 4.1.4 provides a class of contact manifolds with vanishing first Chern class.

4.3.5. Contact open books with page Σ_g and monodromy involving τ_{b_2}. Consider the surface $(\Sigma_g, \partial\Sigma_g)$ of Figure 4.1. Let $\{m_1, m_2, ..., m_{2k+1}\}$ be an odd chain of curves on $(\Sigma_g, \partial\Sigma_g)$, in minimal position. Consider a closed regular neighborhood $N(m_1, ..., m_{2k+1})$ of their union. The boundary of $N(m_1, ..., m_{2k+1})$ then consists of two simple closed curves, d_1 and d_2. It is known that $(\tau_{m_1} \circ \tau_{m_2} \circ ... \circ \tau_{m_{2k+1}})^{2k+2} = \tau_{d_1} \circ \tau_{d_2}$ (see section 4.4 of [**FM**]). The map $\tau_{d_1} \circ \tau_{d_2}^{-1}$ is called the *chain map* corresponding to $\{m_1, m_2, ..., m_{2k+1}\}$. The *length* of a chain is the number of simple closed curves in it. A chain is called *odd* or *even*, depending on the parity of its length. In [**J**], Johnson proved the following remarkable fact.

THEOREM 4.3.1 (Johnson,[**J**]). *For $g \geq 3$, the odd subchain maps of the two chains* $\{b_1, a_1, c_1, ...$
$, a_{g-1}, c_{g-1}, a_g\}$ *and* $\{\tau_{b_2}(a_2), c_2, a_3, c_3, .., a_{g-1}, c_{g-1}, a_g\}$ *generate the Torelli subgroup of* $\mathcal{MCG}(\Sigma_g, \partial\Sigma_g)$.

Recall that the Torelli subgroup consists of elements in $\mathcal{MCG}(\Sigma_g, \partial\Sigma_g)$, which has trivial action on $H_1(\Sigma_g, \partial\Sigma_g; \mathbb{Z})$. Now, consider the odd subchain $\{\tau_{b_2}(a_2), c_2, a_3\}$ on $(\Sigma_g, \partial\Sigma_g)$. Let e_1, e_2 be the boundary curves of $N(\tau_{b_2}(a_2), c_2, a_3)$. The discussion above then implies the following.

$$\tau_{e_1} \circ \tau_{e_2}^{-1} = (\tau_{\tau_{b_2}(a_2)} \circ \tau_{c_2} \circ \tau_{a_3})^4 \circ \tau_{e_2}^{-2} = (\tau_{b_2} \circ \tau_{a_2} \circ \tau_{b_2}^{-1} \circ \tau_{c_2} \circ \tau_{a_3})^4 \circ \tau_{e_2}^{-2}$$

. So, $\tau_{e_1} \circ \tau_{e_2}^{-1}$ has a factorization involving τ_{b_2}. Since $\tau_{e_1} \circ \tau_{e_2}^{-1}$ is an element of the Torelli subgroup, $\mathcal{A}ob(\Sigma_g, \tau_{e_1} \circ \tau_{e_2}^{-1})$ is a homology sphere. Pancholi and Pandit (Theorem 3 in [**PP**]) have proved that if a closed 3-manifold has no 2-torsion in the first integer homology group, then a contact structure η on that manifold admits iso-contact embedding in (S^5, ξ_{std}), if and only if $c_1(\eta) = 0$. Therefore, the contact open book $\mathcal{A}ob(\Sigma_g, \tau_{e_1} \circ \tau_{e_2}^{-1})$ admits an iso-contact embedding in (S^5, ξ_{std}). Many such examples can be formed by finding elements in the Torelli subgroup that involves τ_{b_2} in its factorization.

CHAPTER 5

Contact and isocontact embedding of π–manifolds

In this chapter we prove contact analogs of some smooth embedding theorems for closed π–manifolds. We show that a closed, k-connected, π–manifold of dimension $(2n+1)$ that bounds a π–manifold, contact embeds in the $(4n-2k+3)$-dimensional Euclidean space with the standard contact structure. We also prove some isocontact embedding results for π–manifolds and parallelizable manifolds.

The following generalization of the Whitney embedding theorem is due to Haefliger and Hirsch ([**HH**]).

THEOREM 5.0.1 (Haefliger-Hirsch). *If M^n is a closed orientable k-connected n-manifold $(0 \leq k \leq \frac{1}{2}(n-4))$, then M^n embeds in $\mathbb{R}^{2n-k-1}$.*

Recall that a manifold M is called a π-manifold, provided the direct sum of its tangent bundle with the trivial real line bundle is trivial. In [**Sa**], Sapio introduced the notion of an almost embedding. A manifold M^n admits an *almost embedding* in N^{n+l} if there exists a homotopy n-sphere Σ^n such that $M^n \# \Sigma^n$ embeds in N^{n+l}. For M^n a π–manifold, we have the following result due to Sapio ([**Sa**]).

THEOREM 5.0.2 (Sapio). *Let M^n be a k-connected, n-dimensional π–manifold $(n \geq 5$, and $k \leq [n/2])$. Assume that $n \not\equiv 6$ (mod 8). Then*

1. *M^n almost embeds in $\mathbb{R}^{2n-2k-1}$ with a trivial normal bundle.*
2. *If M^n bounds a π–manifold, then M^n embeds in $\mathbb{R}^{2n-2k-1}$ with a trivial normal bundle.*

Sapio, in a sense improved the Haefliger-Hirsch embedding theorem for k-connected π-manifolds to produce embeddings with trivial normal bundle in $\mathbb{R}^{2n-2k-1}$.

Gromov [**Gr**] reduced the existence of an isocontact embedding of a contact manifold (M^{2n+1}, ξ) in a contact manifold (V^{2N+1}, η), for $N \geq n+2$, to a problem in obstruction theory. Gromov [**EM**] proved that any contact manifold (M^{2n+1}, ξ) has an isocontact embedding in $(\mathbb{R}^{4n+3}, \xi_{std})$. This result, which is essentially the contact analog of Whitney's embedding theorem, was reproved later by A. Mori [**Mo**] for $n = 1$ and by D. M. Torres [**Tor**] for all n using different techniques. For isocontact embeddings of a contact manifold (M, ξ) of co-dimension $\leq dim(M)-1$, there is a condition on the Chern classes of ξ. This condition comes from the normal bundle of the embedding. See the Remark 2.6.7 for a precise statement. This implies, one has to restrict the isocontact embedding question to contact structures which satisfy that condition. Given this, a theorem of N. Kasuya ([**Ka**], Theorem 1.5) says that for 2-connected $(2n+1)$-contact manifolds, the Haefliger-Hirsch theorem has a contact analog giving isocontact embedding in $(\mathbb{R}^{4n+1}, \xi_{std})$. Before stating our results, we introduce some terminologies.

First we define a notion analogous to the almost embedding for contact manifolds. Recall that if (M^{2n+1}, ξ_M) and (N^{2n+1}, ξ_N) are two contact manifolds, then by $(M^{2n+1} \# N^{2n+1}, \xi_M \# \xi_N)$ we denote the contact connected sum of them. For details on a contact connected sum we refer to chapter 6 of [**Ge**].

Definition 5.0.3 (Homotopy isocontact embedding). *(M^{2n+1}, ξ) admits a homotopy isocontact embedding in $(\mathbb{R}^{2N+1}, \xi_{std})$, if there exists a contact homotopy sphere (Σ^{2n+1}, η) such that $(M^{2n+1} \# \Sigma^{2n+1}, \xi \# \eta)$ has an isocontact embedding in $(\mathbb{R}^{2N+1}, \xi_{std})$. We say, M^{2n+1} homotopy contact embeds in $(\mathbb{R}^{2N+1}, \xi_{std})$, if there is a homotopy sphere Σ^{2n+1} and a contact structure ξ_0 on $M \# \Sigma^{2n+1}$ such that $(M \# \Sigma^{2n+1}, \xi_0)$ has an isocontact embedding in $(\mathbb{R}^{2N+1}, \xi_{std})$.*

Before stating our results, we describe the contact structures we will be considering. For an isocontact embedding of (M^{2n+1}, ξ) of co-dimension $2(N-n)$ with trivial symplectic normal bundle, we need that the Chern classes $c_i(\xi)$ vanish for $1 \leq i \leq n$. For details see the Remark 2.6.7. Note that by a theorem of Peterson (Theorem 2.1, [**Ke**]), if (M^{2n+1}, ξ) is torsion free, then this condition is true if and only if ξ is trivial as a complex vector bundle over the $2n$-skeleton of M. Consider the fibration map $SO(2n+2) \to \Gamma_{n+1}$ with fiber $U(n+1)$, where Γ_{n+1} denotes the space of almost complex structures on $\mathbb{R}^{2n+2}$. Since $TM \oplus \varepsilon^1_M$ is trivial for a π–manifold, one can postcompose a trivialization map to $SO(2n+2)$ with the above fibration map to get an almost contact structure on M. For notions of almost complex and almost contact structures see section 2.5.

Definition 5.0.4. *A contact structure on a π–manifold M, in the homotopy class of an almost contact structure that factors through a map from M to $SO(2n+2)$ as mentioned above, will be called an SO-*contact structure.

5.1. Statements of the theorems

We now state the analog of Sapio's Theorem for contact π–manifolds.

Theorem 5.1.1. *Let M^{2n+1} be a k-connected, π–manifold. Assume that $n \geq 2$ and $k \leq n-1$. Then*

1. *M^{2n+1} homotopy contact embeds in $(\mathbb{R}^{4n-2k+3}, \xi_{std})$.*
2. *Assume that $n \not\equiv 3$ (mod 4). If for all $i \in \{k+1, \cdots, 2n-k\}$ such that $i \equiv 0, 2, 6, 7$ (mod 8), $H_{2n-i+1}(M) = 0$, then for any contact structure ξ on M^{2n+1}, (M, ξ) has a homotopy isocontact embedding in $(\mathbb{R}^{4n-2k+3}, \xi_{std})$.*
3. *Assume that $n \not\equiv 3$ (mod 4). If for all $i \in \{k+1, \cdots, 2n-k\}$ such that $i \equiv 0, 7$ (mod 8), $H_{2n-i+1}(M) = 0$, then for any SO-contact structure ξ on M^{2n+1}, (M, ξ) has a homotopy isocontact embedding in $(\mathbb{R}^{4n-2k+3}, \xi_{std})$.*
4. *If M^{2n+1} bounds a π–manifold, then we can omit "homotopy" in the above statements.*

We remark that in all the statements above, we get contact or isocontact embeddings with a trivial conformal symplectic normal bundle.

Note that Theorem 5.1.1 provides criteria to find examples of isocontact embeddings of π-manifolds in the standard contact euclidean space. For example, a straightforward application of statement 2 and 4 in Theorem 5.1.1 shows that every contact structure on $S^4 \times S^5$ has an isocontact embedding in $(\mathbb{R}^{15}, \xi_{std})$. Here, we have $k = 2$ and $n = 4$. Similarly, one can check that all contact structures on $S^4 \times S^9$ ($k = 3$) and $S^{11} \times S^{12}$ ($k = 10$) admit isocontact embeddings in $(\mathbb{R}^{21}, \xi_{std})$ and $(\mathbb{R}^{27}, \xi_{std})$, respectively.

The proof of Theorem 5.1.1 is based on Gromov's h-principle for existence of contact structure on open manifold (see Theorem 2.5.5). Roughly speaking, we put a contact structure on a tubular neighborhood of the embedded contact manifold, extend it to an almost contact structure on the ambient manifold using obstruction theory and then apply Gromov's h-principle.

COROLLARY 5.1.2. *Let M^{2n+1} be an $(n-1)$-connected π–manifold that bounds a π–manifold. Then*

(1) *M^{2n+1} contact embeds in $(\mathbb{R}^{2n+5}, \xi_{std})$.*
(2) *If $n \equiv 4, 5$ (mod 8), then for any contact structure ξ, (M, ξ) has an isocontact embedding in $(\mathbb{R}^{2n+5}, \xi_{std})$.*

In particular, any contact homotopy sphere Σ^{2n+1} that bounds a parallelizable manifold has an isocontact embedding in $(\mathbb{R}^{2n+5}, \xi_{std})$, for $n \equiv 0, 1, 2$ (mod 4).

For example, by [**KM**], we get that all 11-dimensional contact homotopy spheres has an isocontact embedding in $(\mathbb{R}^{15}, \xi_{std})$.

REMARK 5.1.3. *(*On optimal dimension of embedding*) In section 4 of* [**Sa**]*, Sapio constructs a family of $(r - \rho(r) - 1)$-connected $(2r - \rho(r) - 1)$-dimensional manifolds, denoted by $M(r)$. Here, $r = (2a + 1)2^{b+4c}, 0 \leq b \leq 3, a, b, c \in \mathbb{Z}_{\geq 0}$ and $\rho(r) = 2^b + 8c$. Note that $[2(2r - \rho(r) - 1) - 2(r - \rho(r) - 1) - 1] = 2r - 1$. Sapio shows that $M(r)$ bounds a π-manifold. So, by Theorem 5.0.2, $M(r)$ embeds in $\mathbb{R}^{2r-1}$. But $M(r)$ does not embed in $\mathbb{R}^{2r-2}$. It follows from the discussion in section 5.2.1 that if we assume $n - k \geq 2$ in Theorem 5.1.1, then we can improve the dimension of embedding to $4n - 2k + 1 = 2(2n + 1) - 2k - 1$. Therefore, the family of examples given by the manifolds $M(r)$, for $\rho(r) \geq 4$, actually show that for $n - k \geq 2$, $(4n - 2k + 1)$ is the optimal dimension of contact embedding.*

Using similar techniques as above and Gromov's h-principles for contact immersion and isocontact embedding (see 2.6.5 and 2.6.6) we prove the following result for parallelizable manifolds.

THEOREM 5.1.4. *Let M^{2n+1} be a parallelizable manifold.*

(1) *For any contact structure ξ on M^{2n+1}, (M^{2n+1}, ξ) contact immerses in $(\mathbb{R}^{2n+3}, \xi_{std})$.*
(2) *If M^{2n+1} is 5-connected, then for $n \equiv 0, 1$ (mod 4) and $n \geq 7$, and for any contact structure ξ, (M^{2n+1}, ξ) has an isocontact embedding in $(\mathbb{R}^{4n-3}, \xi_{std})$.*

COROLLARY 5.1.5. *Let $M^{2n+1} = N^{2n-1} \times (S^1 \times S^1)$. Where N^{2n-1} is a π–manifold that embeds in $\mathbb{R}^{2N+1}$ with trivial normal bundle. Then M^{2n+1} contact embeds in $(\mathbb{R}^{2N+5}, \xi_{std})$.*

In [**BEM**], S. Borman, Y. Eliashberg and E. Murphy defined the notion of an overtwisted contact ball in all dimensions. Any contact structure that admits a contact embedding of such an overtwisted ball is called an overtwisted contact structure. These contact structures were shown to satisfy the h-principle for homotopy of contact structures. For details see Theorem 1.1 of 2.5.6. Using this, we prove a uniqueness result for embedding of certain π-manifolds in an overtwisted contact structure ξ_{ot} on $\mathbb{R}^{2N+1}$, analogous to Theorem 1.25 in [**EF**].

THEOREM 5.1.6. *Let (M^{8k+3}, ξ) be a contact π–manifold such that $H_i(M;\mathbb{Z}) = 0$, for $i \equiv\ 2, 4, 5, 6$ $(mod\ 8)$. Let $\iota_1, \iota_2 : (M^{8k+3}, \xi) \to (\mathbb{R}^{2N+1}, \xi_{ot})$ be two isocontact embeddings with trivial conformal symplectic normal bundle such that both the complements of $\iota_1(M)$ and $\iota_2(M)$ in $(\mathbb{R}^{2N+1}, \xi_{ot})$ are overtwisted. If ι_1 and ι_2 are smoothly isotopic, then there is a contactomorphism $\chi : (\mathbb{R}^{2N+1}, \xi_{ot}) \to (\mathbb{R}^{2N+1}, \xi_{ot})$ such that $\chi \cdot \iota_1 = \iota_2$.*

For example, any two isocontact embeddings of $(S^{8k_1} \times S^{8k_2+3}, \xi_0)$ in $(\mathbb{R}^{8k_1+8k_2+5}, \xi_{ot})$ which satisfy the hypothesis of Theorem 5.1.6, are equivalent. In particular, take $k_1 = 1$ and $k_2 = 1$. Then Theorem 5.1.6 says that any two isocontact embeddings of $S^8 \times S^{11}$ in $(\mathbb{R}^{21}, \xi_{ot})$, which are smoothly isotopic and does not intersect an overtwisted disk in $(\mathbb{R}^{21}, \xi_{ot})$, are equivalent as isocontact embeddings.

5.2. Proofs of the theorems

Recall that $TN^n \bigoplus \epsilon_N^{k+1} \cong \epsilon_N^{n+k+1} \Leftrightarrow TN^n \bigoplus \epsilon_N^1 = \epsilon_N^{n+1}$ (see corollary 1.4, p-70 of [**Kos**]). Therefore, M^{2n+1} is a π-manifold if and only if M^{2n+1} embeds in the Euclidean space $\mathbb{R}^d$ with a trivial normal bundle, for some $d \geq 2n + 2$. The following lemma is the main ingredient to prove Theorem 5.1.1 .

LEMMA 5.2.1. *If an almost contact manifold M^{2n+1} embeds in $\mathbb{R}^{2N+1}$ with a trivial normal bundle, then there exists a contact structure ξ_0 such that (M, ξ_0) isocontact embeds into $(\mathbb{R}^{2N+3}, \xi_{std})$ $(N - n \geq 1)$.*

PROOF. By assumption, there is an embedding $\iota : M \hookrightarrow \mathbb{R}^{2N+1}$ with normal bundle of embedding $\nu(\iota)$ trivial. Since M is a π–manifold, $TM \oplus \epsilon_M^1 \cong \epsilon_M^{2n+2}$. So, any section to the associated Γ_{n+1}-bundle of $TM \oplus \epsilon_M^1$ is given by a homotopy class of map $s : M \to \Gamma_{n+1}$. By [**BEM**], in every homotopy class of an almost contact structure there is a genuine contact structure. Fix a homotopy class of an almost contact structure on M^{2n+1}. Let ξ be a contact structure representing it. Let $E(\nu)$ denote the total space of $\nu(\iota)$. Since ξ is co-orientable,

$$TE(\nu) \oplus \varepsilon^1 \cong TM \oplus \nu(\iota) \oplus \varepsilon^1 \cong \xi \oplus \varepsilon^2 \oplus \varepsilon^{2(N-n)}$$

. We now define a contact structure on the tubular neighborhood $E(\nu)$ of M, such that its restriction to M is contact. Let α be a contact form representing ξ. Let $(r_1, \theta_1, r_2, \theta_2, ..., r_{N-n}, \theta_{N-n})$ be a cylindrical co-ordinate system on $\mathbb{D}^{2(N-n)}$. The 1-form $\tilde{\alpha} = \alpha + \Sigma_{i=1}^{N-n}(r_i^2 d\theta_i)$ defines a contact structure on $E(\nu) \cong M \times \mathbb{D}^{2N-2n}$ that restricts to the contact structure ξ on M. Let J_ξ be an almost complex structure on the stable tangent bundle of TM that induces the contact structure ξ on M. Put the standard complex structure $J_0(N-n)$ on the normal bundle $\nu(\iota)$ and define an almost complex structure on $TE(\nu(\iota)) \oplus \epsilon_E^1$ given by $J_\xi \oplus J_0(N-n)$. Note that over each fiber of $E(\nu)$, $d\tilde{\alpha}$ restricts to the standard symplectic structure on $\mathbb{D}^{2N-2n}$ compatible with $J_0(N-n)$. So, the almost contact structure associated to $\tilde{\alpha}$ is the same as the almost contact structure induced by $J_\xi \oplus J_0(N-n)$. If we can extend this almost contact structure on $E(\nu)$ to all of $\mathbb{R}^{2N+1}$, then by Theorem 2.5.5, we will get a contact embedding of (M,ξ) into $(\mathbb{R}^{2N+1},\eta_0)$ for some contact structure η_0.

Now we show how to extend the section $s_\xi : M \to \Gamma_{N+1}$, given by $J_\xi \oplus J_0(N-n)$, to all of $\mathbb{R}^{2N+1}$. The obstructions to such an extension lie in $H^{i+1}(\mathbb{R}^{2N+1}, M; \pi_i(\Gamma_{N+1})) \cong H^i(M;\pi_i(\Gamma_{N+1}))$, for $1 \le i \le 2n+1$. Consider the section $s_\eta : M \to \Gamma_{N+1}$ induced by $\eta|_{\iota(M)}$, for some contact structure η on $\mathbb{R}^{2N+1}$. If s_ξ is homotopic to s_η, then the obstructions vanish and s_ξ extends to all of $\mathbb{R}^{2N+1}$.

Let $In : \Gamma_{n+1} \hookrightarrow \Gamma_{N+1}$ be the inclusion map given by $J(n+1) \longmapsto J(n+1) \oplus J_0(N-n)$. Consider the fibration $\Gamma_m \xrightarrow{j_m} \Gamma_{m+1} \to S^{2m}$ [**Ha**]. Here, j_m denotes the inclusion map that sends $J(m)$ to $J(m) \oplus J_0(2)$. From this we get the following long exact sequence.

$$\ldots \longrightarrow \pi_{i+1}(S^{2m}) \longrightarrow \pi_i(\Gamma_m) \longrightarrow \pi_i(\Gamma_{m+1}) \longrightarrow \pi_i(S^{2m}) \longrightarrow \ldots$$

It follows that j_m induces isomorphism on π_i for $i \le 2m-2$ and onto homomorphism for $i = 2m-1$. Note that the composition map $\Gamma_{n+1} \xrightarrow{j_N \circ j_{N-1} \circ \ldots \circ j_{n+1}} \Gamma_{N+1}$ is the same as the one defined by In. Here, j_{n+1} induces isomorphism on π_i for $i \le 2n$ and onto homomorphism for $i = 2n+1$. For $l \ge n+2$, j_l's induce isomorphisms on π_i's, for all $i \le 2n+1$. Therefore, In induces isomorphism on the i^{th}-homotopy group, for $1 \le i \le 2n$ and onto homomorphism for $i = 2n+1$. So, we can choose the homotopy class of ξ so that the homotopy class of the image of the corresponding section $s_\xi : M \to \Gamma_{n+1}$ under In is the same as the homotopy class of the map $s_\eta : M \to \Gamma_{N+1}$. Thus, there is a contact structure ξ_0 on M such that the corresponding section $s_{\xi_0} : M \to \Gamma_{N+1}$ extends to all of $\mathbb{R}^{2N+1}$. Therefore, (M,ξ_0) contact embeds in $(\mathbb{R}^{2N+1},\eta_0)$ for some contact structure η_0 on $\mathbb{R}^{2N+1}$. By Theorem 2.6.6, $(\mathbb{R}^{2N+1},\eta_0)$ contact embeds in $(\mathbb{R}^{2N+3},\xi_{std})$. Hence, (M,ξ_0) isocontact embeds in $(\mathbb{R}^{2N+3},\xi_{std})$.

$\square$

REMARK 5.2.2. *For $n \ge 4$, the groups $\pi_{2n+1}(\Gamma_{n+1})$ have the following values* [**Ha**]*:*

$$\pi_{2n+1}(\Gamma_{n+1}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n \equiv 3 \ (mod\ 4) \\ \mathbb{Z}_{(n-1)!} & \text{for } n \equiv 0 \ (mod\ 4) \\ \mathbb{Z} & \text{for } n \equiv 1 \ (mod\ 4) \\ \mathbb{Z}_{\frac{(n-1)!}{2}} & \text{for } n \equiv 2 \ (mod\ 4) \end{cases} \tag{1}$$

Whereas, $\pi_{2n+1}(\Gamma_{N+1})$ is either 0 or $\mathbb{Z}_2$. So, for $n \ge 4$, the onto map induced by In on π_{2n+1} has a non-trivial kernel. Thus, we can actually choose a homotopy class of almost contact structures on M which is not null-homotopic and isocontact embeds in $(\mathbb{R}^{2N+3},\xi_{std})$.

Proof of Theorem 5.1.1. Since every homotopy sphere Σ^{2n+1} admits an almost contact structure, $M^{2n+1}\sharp\Sigma^{2n+1}$ also admits an almost contact structure. By statement 2 of the Theorem 5.0.2, if M^{2n+1} bounds a π–manifold then it satisfies the hypothesis of Lemma 5.2.1, for $N = 2n-k$. By statement 1 of the Theorem 5.0.2, whenever M^{2n+1} does not bound a π–manifold, we can take connected sum with a suitable homotopy sphere and embed the resulting π–manifold in $\mathbb{R}^{4n-2k+1}$ with a trivial normal bundle. This hypothesis on the normal bundle of embedding is the only thing that we need to prove the present theorem. Therefore, it is enough to prove statements (1) to (3), for manifolds that bound π–manifolds.

(1) The result follows from Lemma 3.1.

(2) As discussed in the proof of Lemma 3.1, given any contact structure ξ on M^{2n+1} and an embedding $\iota : M \to \mathbb{R}^{4n-2k+1}$ with a trivial normal bundle ν, the obstructions to extend the almost contact structure on $E(\nu)$ to all of $\mathbb{R}^{4n-2k+1}$, lie in the groups $H^i(M; \pi_i(\Gamma_{2n-k+1}))$, for $1 \leq i \leq 2n+1$. Since M is k-connected, there are no obstructions in the dimensions 1 to k. Being k-connected also implies that $M \setminus \{pt.\}$ deformation retracts onto the $(2n-k)$-skeleton of M. So, the obstructions in the dimensions $(2n-k+1)$ to $2n$ also vanish. Thus, we now only consider values of i in $\{k+1, k+2, ..., 2n-k+1\}$. Since $k \leq n-1$, by the theorem of Bott (2.5.7), we get the following for $1 \leq i \leq 2n+1$.

$$\pi_i(\Gamma_{2n-k+1}) = \begin{cases} 0 & for \;\; i \equiv 1,3,4,5 \ (mod\ 8) \\ \mathbb{Z}_2 & for \;\; i \equiv 0,7 \ (mod\ 8) \\ \mathbb{Z} & for \;\; i \equiv 2,6 \ (mod\ 8) \end{cases}$$

So, for $i \equiv 1,3,4,5$ $(mod\ 8)$, there are no obstructions. For $i \equiv 0,2,6,7$ $(mod\ 8)$, $H^i(M^{2n+1}, \tilde{G}) \cong H_{2n-i+1}(M^{2n+1}, \tilde{G}) = 0$ by hypothesis. Here, $\tilde{G}$ is either $\mathbb{Z}$ or $\mathbb{Z}_2$. Moreover, for $n \not\equiv 3$ $(mod\ 4)$, $\pi_{2n+1}(\Gamma_{2n-k+1}) = 0$. Hence, there is no obstructions in the top dimension. Thus, we can extend the almost contact structure for any xi and the result follows.

(3) Note that every assumption in statement (2) holds for statement (3), except that now we have $H^i(M^{2n+1}, \tilde{G}) \cong H_{2n-i+1}(M^{2n+1}, \tilde{G}) = 0$ for $i \equiv 0,7$ $(mod\ 8)$. Thus, we are left to show that the obstructions vanish for $i \equiv 2,6$ $(mod\ 8)$. We now claim that in the proof of Lemma 3.1, both s_ξ and s_η factors through the map $\tilde{j}$ in the fibration

$$U(N+1) \to SO(2N+2) \xrightarrow{\tilde{j}_{N+1}} \Gamma_{N+1}$$

. Since η was a contact structure on $\mathbb{R}^{2N+1}$, the assertion is clear for s_η. The reason for s_ξ is the following. Since ξ is an SO-contact structure, the almost contact structure associated to ξ, $M \to \Gamma_{n+1}$, factors through the map $SO(2n+2) \xrightarrow{\tilde{j}_{n+1}} \Gamma_{n+1}$. Let $\hat{i} : SO(2n+2) \to SO(2N+2)$ denote the inclusion map given by $A \mapsto A \oplus I_{2(N-n)}$. Recall that $\tilde{j}$ takes a matrix $A \in SO(2m+2)$ to $A^{-1}J_0(m+1)A \in \Gamma_{m+1}$. The assertion then follows from the commutative diagram below.

$$\begin{array}{ccc} SO(2n+2) & \xrightarrow{\hat{i}} & SO(2N+2) \\ \downarrow{\scriptstyle \tilde{j}_{n+1}} & & \downarrow{\scriptstyle \tilde{j}_{N+1}} \\ \Gamma_{n+1} & \xrightarrow{In} & \Gamma_{N+1} \end{array}$$

Thus, the homotopy obstructions come from the groups $H^i(M; \pi_i(SO(2N+2)))$. Since $\pi_i(SO) = 0$ for $i \equiv 2,6$ $(mod\ 8)$, the obstructions vanish. Therefore, by following the proof of Lemma 3.1, we get an isocontact embedding of (M, ξ) into $(\mathbb{R}^{4n-2k+3}, \xi_{std})$.

□

Proof of Corollary 5.1.2. **(1)** Follows from Theorem 5.1.1 by putting $k = n-1$.

(2) Following the proof of Theorem 5.1.1, we can see that the only obstructions to extending the almost contact structure on the normal bundle to all of $\mathbb{R}^{2n+3}$ lie in the groups $H^n(M;\pi_n(SO))$ and $H^{n+1}(M;\pi_{n+1}(SO))$. Since both $\pi_n(SO)$ and $\pi_{n+1}(SO)$ vanish for $n \equiv 4, 5$ $(mod\ 8)$, the result follows.

□

Proof of Theorem 5.1.4. **(1)** The existence of a contact monomorphism from (TM^{2n+1}, ξ) to $(T\mathbb{R}^{2N+1}, \eta_{st})$ is equivalent to the existence of a section $s : M \to V^{\mathbb{C}}_{N,n}$ of the associated bundle of TM. Since TM is trivial, such a section always exists. Since any parallelizable manifold M^{2n+1} immerses in $\mathbb{R}^{2n+3}$ and has a contact monomorphism in $(T\mathbb{R}^{2n+3}, \xi_{std})$, by Theorem 2.6.5, (M, ξ) isocontact immerses in $(\mathbb{R}^{2n+3}, \xi_{std})$.

(2) Any section corresponding to a contact monomorphism also induces a section s_0 to the associated $V_{2N+1,2n+1}$–bundle of TM. Assume that M is $(2k-1)$-connected. By Theorem 5.0.1, there exists an embedding $f : M^{2n+1} \to \mathbb{R}^{4n-2k+3}$. Let s_f be the corresponding section to $V_{4n-2k+3,2n+1}$. The homotopy obstructions between s_0 and s_f lie in the groups $H^i(M^{2n+1}, \pi_i(V_{4n-2k+3,2n+1}))$, for $1 \leq i \leq 2n+1$. Since $M^{2n+1} \setminus \mathbb{D}^{2n+1}$ deformation retracts onto the $(2n-2k+1)$-skeleton of M and $V_{4n-2k+3,2n+1}$ is $(2n-2k+1)$-connected, there are no obstructions till dimension $2n$. Therefore, the only homotopy obstruction lies in $H^{2n+1}(M^{2n+1}, \pi_{2n+1}(V_{4n-2k+3,2n+1}))$. By [**HM**], for $k = 3$ and $n \equiv 0, 1$ $(mod\ 4)$, $\pi_{2n+1}(V_{4n-3,2n+1}) = 0$. Thus, by Theorem 2.6.6, (M^{2n+1}, ξ) has an isocontact embedding in $(\mathbb{R}^{4n-3}, \xi_{std})$.

□

Note that the proof of statement (2) in Theorem 5.1.4 does not necessarily require a parallelizable manifold. In general, the following can be said.

PROPOSITION 5.2.3. *A* 5*-connected contact manifold* (M^{2n+1}, ξ) *admits an isocontact embedding in* $(\mathbb{R}^{4n-3}, \xi_{std})$ *for* $n \geq 3$ *and* $n \equiv 0, 1$ $(mod\ 4)$*, if it admits an isocontact immersion in* $(\mathbb{R}^{4n-3}, \xi_{std})$.

Proof of Corollary 5.1.5. Consider $\mathbb{R}^N$ as $\mathbb{R}^{N-2} \times \mathbb{R}^2$. It is well known that $\mathbb{R}^N \setminus (\mathbb{R}^{N-2} \times \{0\})$ can be decomposed as $\mathbb{R}^{N-1} \times S^1$. This is the so called standard open book decomposition of $\mathbb{R}^N$ (see [**Ge**] or [**E**]). Say, M is embedded in $\mathbb{R}^{N-1}$. Then using this open book description we can see that $M \times S^1$ naturally embeds in $\mathbb{R}^{N-1} \times S^1 \subset \mathbb{R}^N$. Starting with an embedding of N^{2n-1} in $\mathbb{R}^{2N+1}$ with trivial normal bundle, we can then apply this procedure twice to get an embedding of $N^{2n-1} \times (S^1 \times S^1)$ in $\mathbb{R}^{2N+3}$ with trivial normal bundle. The result then follows from Lemma 3.1.

□

5.2.1. Contact embedding of co-dimension ≥ 4**.** For embeddings of co-dimension ≥ 4, we can actually get isocontact embedding in the standard contact structure. Let ι be an isocontact embedding of (M, ξ) in $(\mathbb{R}^{2N+1}, \eta)$. Any two contact structures on $\mathbb{R}^{2N+1}$ are homotopic as almost contact structures. Let $H_t : T\mathbb{R}^{2N+1} \to T\mathbb{R}^{2N+1}$ be a formal homotopy covering the identity map of $\mathbb{R}^{2N+1}$ such that $H_0 = Id$, $H_1(\eta) = \xi_{std}$ and $H_t(\eta)$ is an almost contact structure on $\mathbb{R}^{2N+1}$ for all $t \in (0, 1)$. Then $H_t \cdot D\iota$ gives a formal homotopy covering ι and $H_1 \cdot D\iota$ is a contact monomorphism of (TM, ξ) into $(T\mathbb{R}^{2N+1}, \xi_{std})$. If we assume that $N - n \geq 2$, then by Theorem 2.6.6, (M^{2n+1}, ξ) isocontact embeds in $(\mathbb{R}^{2N+1}, \xi_{std})$. Thus, for embedding of co-dimension ≥ 4, we also get contact embedding in the standard contact structure. Using this fact with Lemma 3.1, one can find interesting examples of contact embedding of non-simply connected manifolds. For example, by [**MR**], the 7-dimensional real projective space RP^7 embeds in $\mathbb{R}^{13}$ with trivial normal bundle. Thus, RP^7 contact embeds in $(\mathbb{R}^{13}, \xi_{std})$. Let us look at another simple class of such examples. It is a well known theorem of Hirsch that every oriented, closed 3-manifold M^3 embeds in $\mathbb{R}^5$. By [**CS**], every closed, oriented 4-manifold V^4, whose second Stiefel-Whitney class and signature vanish, embeds in $\mathbb{R}^6$. Since, the Euler class of the normal bundle of an embedding vanishes, each of these embeddings has trivial normal bundle. Thus, $W^7 = M^3 \times V^4$ embeds in $\mathbb{R}^{11}$ with trivial normal bundle. Hence, W^7 contact embeds in $(\mathbb{R}^{11}, \xi_{std})$.

We now prove Theorem 5.1.6. The idea of the proof is essentially contained in [**EF**] (proof of Theorem 1.25).

Proof of Theorem 5.1.6. .

Since ι_1 and ι_2 are contact embeddings with trivial normal bundle, there exist a contact form α representing ξ and suitable neighborhoods $\mathcal{N}_j$ of $\iota_j(M)$ in $(\mathbb{R}^{2N+1}, \eta_{ot})$, for $j = 1, 2$, which are contactomorphic to $(M \times \mathbb{D}^{2(N-n)}, \alpha + \Sigma_{i=1}^{N-n} r_i^2 d\theta_i)$. Now, ι_1 and ι_2 are isotopic and have isomorphic symplectic normal bundle. Thus, one can use the contact tubular neighborhood theorem for contact submanifold (Theorem 2.5.15, [**Ge**]) to get an ambient isotopy $\Phi_t : \mathbb{R}^{2N+1} \longrightarrow \mathbb{R}^{2N+1}$ such that Φ_1 restricts to a contactomorphism from $(\mathcal{N}_1, \eta_{ot}|_{\mathcal{N}_1})$ to $(\mathcal{N}_2, \eta_{ot}|_{\mathcal{N}_2})$ and $\Phi_1(\iota_1(M)) = \iota_2(M)$. Moreover, since we can assume that the isotopy between ι_1 and ι_2 lies in the complement of an overtwisted contact ball, Φ_t can be chosen so that it restricts to the identity map on that overtwisted contact ball in the complement of $\mathcal{N}_1$ in $(\mathbb{R}^{2N+1}, \eta_{ot})$. Thus, the distribution $(\Phi_1)_*\eta_{ot}$ induces a contact structure on $\mathbb{R}^{2N+1}$ that is overtwisted in the complement of $\mathcal{N}_2$. Now, we look at the homotopy obstructions between $(\Phi_1)_*\eta_{ot}$ and η_{ot} relative to $\mathcal{N}_2$. All such obstructions lie in $H^i(\mathbb{R}^{2N+1}, \mathcal{N}_2; \pi_i(\Gamma_{N+1}))$, for $1 \leq i \leq 2n+1$. By Theorem 2.5.7, these groups vanish for $i \equiv 1, 3, 4, 5 \ (mod\ 8)$. Note that $H^i(\mathbb{R}^{2N+1}, \mathcal{N}_2; \pi_i(\Gamma_{N+1})) \cong H^{i-1}(M; \pi_i(\Gamma_{N+1})) \cong H_{2n+2-i}(M; \pi_i(\Gamma_{N+1}))$. The assumptions that $2n+1$ is of the form $8k+3$ and that the homology groups of M vanish for $i \equiv 2, 4, 5, 6 \ (mod\ 8)$ then ensures that the rest of the obstructions also vanish. Theorem 2.5.6 then implies that there is a contact isotopy Ψ_t of $\mathbb{R}^{2N+1}$ relative to $\mathcal{N}_2$ such that $(\Psi_1)_* \cdot (\Phi_1)_*\eta_{ot} = \eta_{ot}$. So, $\Psi_1 \cdot \Phi_1$ gives a contactomorphism of $(\mathbb{R}^{2N+1}, \eta_{ot})$ that takes $\iota_1(M)$ to $\iota_2(M)$.

$\square$

Bibliography

[Al] J. Alexander, A lemma on systems of knotted curves, *Proc. Nat. Acad. Sci.*, vol. 9, (1923), 93–95.

[Au] D. Auroux, Asymptotically homomorphic families of symplectic submanifolds. *Geom. Funct. Anal.* 7 (1997), 971-995.

[B] R. Bott, The Stable Homotopy of the Classical Groups, *Annals of Mathematics*, Vol. 70, NO. 2(Sep., 1959), 313-337.

[BEM] J. Borman, Y. Eliashberg, E. Murphy, Existence and classification of overtwisted contact structures in all dimensions, *Acta Math.*, vol. 215, No. 2 (2015), 281-361.

[CE] R. Casals, J.B. Etnyre, Non-simplicity of isocontact embeddings in all higher dimensions, *arXiv:1811.05455.*

[CMP] R. Casals, E. Murphy and F. Presas, Geometric criteria for overtwistedness, *Journal of American Mathematical Society, 32 (2019), 563-604.*

[CM] R. Casals, E. Murphy, Contact topology from the loose viewpoint *Proceedings of* 22^{nd} *Gökova Geometry-Topology Conference, 81-115*

[CPP] R. Casals, D. Pancholi, F. Presas, The Legendrian Whitney Trick, *arXiv:1908.04828.*

[CS] S. E. Cappell, J. L. Shaneson, Embedding and immersion of four-dimensional manifolds in $\mathbb{R}^6$, Proceedings of the 1977 Georgia Topology Conference, Academic Press 1979, 301-305.

[DGK] F. Ding, H. Geiges, O. Van Koert, Diagrams for contact 5-manifolds, *Journal of the London Mathematical Society, Volume 86, Issue 3, December 2012, 657682.*

[DK] S. Durst , M. Klukas, Nested Open books and the binding sum, *arxiv:1610.07356v2[math.GT]*, (2017).

[E] J. B. Etnyre, Lectures on open book decomposition and contact structures, *arXiv:math/0409402.*

[EF] J. B. Etnyre, R. Furukawa, Braided embeddings of contact 3-manifolds in the standard contact 5-sphere, *Journal of Topology, Volume 10, Issue 2, June 2017, 412-446..*

[EL] J. Etnyre and Y. Lekili, Embedding all contact 3–manifolds in a fixed contact 5–manifold, *Journal of the London Mathematical Society, Volume 99, Issue 1, February 2019, 52-68*(*arXiv:1712.09642v1 [math.GT]*).

[EM] Y. Eliashberg, N. Mishachev, Introduction to the h-principle, *Graduate Studies in Mathematics* 48 (2002), AMS.

[EO] J. B. Etnyre, B. Ozbagci, Invariants of contact structures from open books, *arXiv:math/0605441v1*, 2006.

[Et] J. Etnyre, Lectures on open book decompositions and contact structures, *Floer homology, gauge theory, and low-dimensional topology* 5, 103–141.

[FM] B. Farb and D. Margalit, A primer on mapping class groups, *Princeton Mathematical series*, vol. 49, Princeton University Press, (2012).

[Ga] D. Gabai, The Murasugi sum is a natural geometric operation. *Low-dimensional topology (San Francisco, Calif., 1981)*, Contemp. Math., 20, Amer. Math. Soc., Providence, RI, 1983. 57M25 (57N10), 131–143.

[Ge] H. Geiges, An Introduction to Contact Topology, *Cambridge Studies in Advanced Mathematics* 109.

[Gi] E. Giroux, Géométrie de contact: de la dimension trois vers les dimensions supérieures, *Proceedings of the ICM, Beijing* 2002, vol. 2, 405–414.

[Gr] M. Gromov, Partial Differential Relations, *Ergeb. Math. Grenzgeb.* 9 (1986), Springer–Verlag.

[GS] R. Gompf and A. Stipsticz, 4–manifolds and Kirby Calculus. *Graduate studies in Mathematics*, vol. 20, AMS.

[Ha] B. Harris, Some calculations of homotopy groups of symmetric spaces, *Transactions of the American Mathematical Society* Vol. 106, No. 1 (Jan., 1963), pp. 174-184.

[Hae] A. Haefliger, Plongements differentiables de varietes dans varietes, *Comment. Math. Helv.* 36 (1961), 47-82.

[HH] A. Haefliger, M. Hirsch, On the existence and classification of differentiable embeddings, *Topology*, vol. 2, (1963), 129–135.

[Hi] M. W. Hirsch, Immersions of Manifolds, *Transactions of the American Mathematical Society* Vol. 93, No. 2(Nov.,1959), 242-276.

[Hi0] M. Hirsch, The imbedding of bounding manifolds in euclidean space. *Ann. of Math.* vol. 74 (3), (1961), 494–497.

[Hi2] M. Hirsch, On imbedding differentiable manifolds in Euclidean space. *Ann. of Math.* vol 73 (3), (1961) , 566-571.

[HLM] H. M. Hilden, M.T. Lozano and J.M. Montesinos, All three-manifolds are pullbacks of a branched covering S^3 to S^3, *Trans. Amer. Math. Soc.*, vol. 279 (2), (1983), 729–735.

[HM] C. S. Hoo, M. E. Mahowald, Some homotopy groups of Stiefel manifolds, *Bulletin of AMS* 71 (1965), no. 4, 661-667.

[HoH] K. Honda, Y. Huang, Convex hypersurface theory in contact topology, *arXiv:1907.06025*.

[HY] S. Hirose and A. Yasuhara, Surfaces in 4-manifolds and their mapping class groups, *Topology*, vol. 47 (1), 2008, 41–50.

[J] D. Johnson, The structure of the Torelli group I: A finite set of generators for $\mathcal{I}$, *Annals of Math.*, vol. 118 (2), (1983), 423–442.

[Ka] N. Kasuya, On contact embeddings of contact manifolds in the odd dimensional Euclidan spaces, *International Journal of Mathematics* vol. 26, No. 7 (2015) 1550045.

[Ka2] N. Kasuya, An obstruction for co-dimension two contact embeddings in the odd dimensional Euclidean spaces, *J. Math. Soc. Japan* Volume 68, Number 2 (2016), 737-743.

[Ke] M. A. Kervaire, A note on obstructions and characteristic classes, *American Journal of Mathematics* Vol. 81, No. 3 (Jul.,1959), 773-784.

[KM] M. A. Kervaire, J. W. Milnor, Groups of Homotopy Spheres: One, *Annals of Mathematics* Vol. 77, No. 3. (May, 1963), pp. 504-537.

[KN] O. Van Koert, Klaus Niederkrüger, Open book decompositions for contact structures on Brieskorn manifolds, *Pro. of the American Mathematical Society*, Vol. 133, No. 12, 3679 - 3686.

[Ko] O. van Koert, Lecture notes on stabilization of contact open books, *Mnster J. Math., 10 (2017), 2, 425-455.arXiv:1012.4359v1* (2010).

[Kos] A. A. Kosinski, Differential Manifolds, *Academic Press,Inc.*

[Kp] S. Kaplan, Constructing framed 4-manifolds with given almost framed boundaries, *Trans. Amer. Math. Soc.*, vol. 254, (1979), 237–263.

[La] T. Lawson, Open book decomposition for odd dimensional manifolds, *Topology*, vol 17, (1979), 189–192.

[Li] W. Lickorish, A representation of orientable combinatorial 3–manifolds, *Ann. of Math.*, vol. 76, (1962), 531–540,

[Ma] D. Martińez-Torres, Contact embeddings in standard contact spheres via approximately holomorphic geometry, *J. Math. Sci. Univ. Tokyo*, vol. 18 (2), (2011), 139–154.

[Mo] A. Mori, Global models of contact forms. *J. Math. Sci. Univ. Tokyo*, vol. 11(4), 447454, (2004).

[MR] R. J. Milgram, E. Rees, On the normal bundle to an embedding, *Topology* Vol. 10, pp. 299-308, 1971.

[Mr] A. Mori, Global models of contact forms. *J. Math. Sci. Univ. Tokyo*, vol. 11 (4), 447–454, (2004).

[My] R. Myers, Open book decompositions of 3–manifolds, *Proceedings of the AMS*, vol.72 (2), (1978), 397–402.

[Na] J. Nash, The Imbedding Problem for Riemannian Manifolds, *Annals of Mathematics*, Vol. 63 (1), 2063, (1956).

[NS] A. Nath, K. Saha, Some open book embeddings of higher dimensional smooth and contact manifolds, arXiv:2005.10772v1 [math.GT].

[PP] D. M. Pancholi, S. Pandit, Iso-contact embeddings of manifolds in co-dimension 2, *arXiv:1808.04059* (2018).

[PPS] D. M. Pancholi, S. Pandit, K. Saha, Embedding of 3-manifolds via open books, arXiv:1806.09784v2, 2018.

[Qu] F. Quinn, Open book decompositions and the bordism of automorphisms, *Topology*, vol. 18 (1), (1979), 55–73.

[Ro] V. Rohlin, The embedding of non-orientable three manifolds into five-dimensional Euclidean space (Russian). *Dokl. Akad. Nauk. SSSR*, vol. 160 (1965), 153-156.

[S1] K. Saha, On open book embedding of contact manifolds in the standard contact sphere, *Canadian Mathematical Bulletin*, 1-16 (2019), arXiv:1811.07333v3 [math.SG]

[S2] K. Saha, Contact and isocontact embedding of π-manifolds, to appear in *Math. Proc. of Indian Academy of Science*, arXiv:2005.10135v1 [math.SG].

[Sa] R. De Sapio, Embedding π-manifolds, *Annals of Mathematics* vol. 82. No. 2(Sep.,1965). 213-224.

[Se] P. Seidel, Lectures on four-dimensional Dehn twist, arXiv:math/0309012v3 [math.SG].

[St] N. Steenrod, The Topology of Fibre BUndles. *PMS(14)*, Princeton University Press.

[Ta] I. Tamura, Spinnable structures on differentiable manifolds, *Proc. Japan Acad.* vol. 48, (1972), 293–296.

[Tor] D. Martinez-Torres, Contact embeddings in standard contact spheres via approximately holomorphic geometry, *J. Math. Sci. Univ. Tokyo*, vol. 18 (2), (2011), 139154.

[TW] W. Thurston, H Winkelnkemper, On the existence of contact forms, *Proceedings of AMS* vol. 52,

[W] A. Weinstein, Contact surgery and symplectic handlebodies, *Hokkaido Mathematical Journal*, 2, 241-251, (1991).

[Wa] C. Wall, All 3–manifolds imbed in 5–space, *Bull. A.M.S (N.S)*, vol. 71, (1965), 564–567.

[Wh] H. Whitney, The self-intersections of a smooth n–manifold in $2n$–space, *Ann. of Math,* vol. 45(2), (1944), 200–246.

[Wi] H. Winkelnkemper, Manifolds as open books, *Bull. Amer. Math. Soc.*, vol. 7, (1973), 45–51.

[Wu] W. T. Wu, On the isotopy of C^r-manifolds of dimension n in Euclidean $(2n + 1)$-space, Sci. Record (N.S.) 2 271-275 (1958).