On Hilbert Functions of Graded Rings and on the F-rationality of Rees Algebras

by

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Declaration of Authorship

I declare that the thesis titled On Hilbert Functions of Graded Rings and on the F-rationality of Rees Algebras" submitted by me for the degree of Doctor of Philosophy in Mathematics is the record of academic work carried out by me under the guidance of Manoj Kummini and this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

Chennai Mathematical Institute Mitra Koley
Date: 30th October 2017

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I certify that the thesis entitled On Hilbert Functions of Graded Rings and on the F-rationality of Rees Algebras" submitted for the degree of Doctor of Philosophy in Mathematics by Mitra Koley is the record of research work carried out by her under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent her independent work.

Chennai Mathematical Institute　　　　　　　　　　　　Manoj Kummini
Date: 30th October 2017

Abstract

This thesis is divided into two parts, in the first half we study poset embeddings of two hypersurface rings and in second half we study F-rationality of Rees algebras.

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Contents

Dedicated to my parents

Chapter 1

Introduction

This thesis consists of two projects. In Part I we discuss the first project which concerns classifying Hilbert functions of hypersurface toric rings. In part II we discuss my second project which concerns F-rationality of Rees algebras.

Here all the rings we consider are commutative and noetherian with unity.

Part I: In order to talk about first project problem we recall some definition and known results. Let R be a standard graded algebra over a field $\mathbb{K}$, i.e., $R \simeq \bigoplus_{d \geq 0} R_d$ as $\mathbb{K}$-vector-spaces with $R_0 = \mathbb{K}$, $R = \mathbb{K}[R_1]$ and $\dim_{\mathbb{K}} R_1 < \infty$. Let I be a graded ideal, I_n denote n-th graded piece of I. The Hilbert function of I

$$\begin{aligned} H_I : \mathbb{N} &\longrightarrow \mathbb{N}, \\ n &\mapsto \dim_{\mathbb{K}} I_n. \end{aligned}$$

is an important numerical invariant that measures the size of I. When R is a polynomial ring, a theorem of F. Macaulay [Mac27] provides a classification of the Hilbert functions of homogeneous R-ideals; more precisely, a function $H : \mathbb{N} \longrightarrow \mathbb{N}$ is the Hilbert function of some homogeneous ideal if and only if it is the Hilbert function of a LEX-segment ideal (where LEX denotes the graded lexicographic monomial order on the polynomial ring R). Macaulay's theorem was generalized to graded Betti numbers ([Big93],[Hul93],[Par96]): every LEX-segment ideal attains maximal Betti numbers among all the homogeneous ideals which have same Hilbert function as of LEX-segment ideal. So it is natural to ask similar questions for quotients of polynomial rings.

Many researchers including Kruskal, Katona, Mermin, Peeva, Stillman, and others proved the analogue of Macaulay's theorem for various quotient of polynomial rings and related results for graded Betti numbers.

G. Caviglia and M. Kummini [CK13] study Hilbert functions of homogeneous ideals in any standard graded algebra R using an embedding of the poset of Hilbert functions of homogeneous R-ideals into the poset of homogeneous R-ideals.

We classify Hilbert functions of homogeneous ideals in two toric rings $\mathbb{K}[a,b,c,d]/(ad-bc)$ and $\mathbb{K}[a,b,c]/(ac-b^2)$, where $\mathbb{K}$ is a field of arbitrary characteristic and a,b,c,d are indeterminates. We will also prove related results for graded Betti numbers when characteristic of $\mathbb{K}$ is 0. For classification of Hilbert functions we follow the approach of Caviglia and Kummini [CK13]. We prove:

Theorem 1.1. (i) *Let* $R = \mathbb{K}[a,b,c,d]/(ad-bc)$ *and* $S = \mathbb{K}[a,b,c,d]$*, where* $\mathbb{K}$ *is a field and* a,b,c,d *are indeterminates. There exists an embedding of the poset of Hilbert functions of homogeneous ideals of* R *into the poset of homogeneous* R*-ideals i.e. there exists a map as in [CK13]* $\epsilon : \mathcal{H}_R \longrightarrow \mathcal{I}_R$ *as posets.*
For the remaining parts of the theorem we assume characteristic of $\mathbb{K}$ *is* 0.
(ii) *Let* I *be a homogeneous* R*-ideal and* I^ϵ *be the image of* H_I *under* ϵ*. Let* $\tilde{I}$ *and* $\tilde{I}^\epsilon$ *be the preimages of* I *and* I^ϵ *in* S *respectively.* $\beta^S_{i,j}(R/I) \leq \beta^S_{i,j}(R/I^\epsilon)$ *for* $i = 0,1,4$ *and for all* j*. Hence* $\beta^S_{i,j}(\tilde{I}) \leq \beta^S_{i,j}(\tilde{I}^\epsilon)$ *for* $i = 0,3$ *and for all* j*.*
(iii) *Let* I *and* I^ϵ *be as above, then* $\beta^R_{i,i+j}(I) \leq \beta^R_{i,i+j}(I^\epsilon)$ *for all* i,j*.*

Theorem 1.2. (i) *Let* $R = \mathbb{K}[a,b,c]/(ac-b^2)$ *and* $S = \mathbb{K}[a,b,c]$*, where* $\mathbb{K}$ *is a field and* a,b,c *are indeterminates. There exists an embedding of the poset of Hilbert functions of homogeneous* R*-ideals into the poset of homogeneous* R*-ideals i.e. there exists a map as in [CK13]* $\epsilon : \mathcal{H}_R \longrightarrow \mathcal{I}_R$ *as posets.*
(ii) *Assume characteristic of* $\mathbb{K}$ *is* 0*. Let* I *be a homogeneous* R*-ideal and* I^ϵ *be the image of* H_I *under* ϵ*. Let* $\tilde{I}$ *and* $\tilde{I}^\epsilon$ *be the preimages of* I *and* I^ϵ *in* S *respectively.* $\beta^S_{i,j}(R/I) \leq \beta^S_{i,j}(R/I^\epsilon)$ *for all* i *and* j*. Hence* $\beta^S_{i,j}(\tilde{I}) \leq \beta^S_{i,j}(\tilde{I}^\epsilon)$ *for all* i *and* j*.*

The above results have been accepted for publication in the Journal of Commutative Algebra

Part II:

The theory of tight closure of an ideal in a ring or of a submodule of a module was introduced by Hochster and Huneke in [HH90]. It becomes very useful tool in both commutative algebra and algebraic geometry. Tight closure theory gives a simpler proof of theorem of Briançon-Skoda in greater generality. The famous Hochster-Roberts theorem on the Cohen-Macaulayness of rings of invariants has a simple tight closure proof. Also, the existence of big Cohen-Macaulay algebras for rings containing a field was proved using tight closure. Tight closure has been used to study singularities in prime characteristic. F-rational rings were defined by Fedder and Watanabe. F-rationality of a ring is closely related to rational singularity of Spec R. More precisely, in [Smi97] Smith showed that if (R, m) is an excellent F-rational local ring, then it is pseudo-rational. Pseudo-rationality, introduced by Lipman and Teissier in [LT81], is a property of local rings which is an analogue of rational singularities, in situations where desingularization is not known to exist. When the ring is essentially of finite type over a field of characteristic zero, these two notions are the same. In [Smi97] Smith also proved if R is essentially of finite type over a field of characteristic zero and "modulo p reduction" of the ring is F-rational for large enough prime $p > 0$, then it is a rational singularity. Converse of this has been proved in [MS97]and [Har98].

F-rationality of Rees algebra $R[It]$ was studied by Hara, Watanabe and Yoshida in [HWY02]. In [Sin00] A. K. Singh gave an example of a 3-dimensional F-rational hypersurface ring whose Rees algebra is Cohen-Macaulay and normal domain but not F-rational. In [HWY02] they gave a criterion for Rees algebra $R[It]$ to be F-rational in terms of tight integral closure. They also proved that if (R, m) is a two dimensional excellent F-rational local ring and I is an integrally closed $\mathfrak{m}$-primary ideal of R, then $R[It]$ is F-rational. In [Hyr99] Hyry showed that if R is an excellent local ring of characteristic zero, and I is an R-ideal such that $R[It]$ is Cohen-Macaulay, normal and $\operatorname{Proj} R[It]$ has rational singularities, then $R[It]$ has rational singularities. Similar questions about F-rationality were raised and partially answered in [HWY02]. In joint work with Manoj Kummini, we study these problems. We prove:

Theorem 1.3. *Let (R, m) be an excellent normal d-dimensional local ring. Let I be an m-primary ideal. Let $X = \operatorname{Proj} R[It]$ be F-rational and $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$. Then $R[It]^{(n)}$ is F-rational for all $n \gg 0$.*

The following theorem is given as a conjecture in [HWY02](See [HWY02, conjecture 4.1]). We prove:

Theorem 1.4. *Let (R, m) be a d-dimensional F-rational excellent local ring of positive characteristic $p > 0$ and I be an m-primary ideal. If the extended Rees algebra $R[It, t^{-1}]$ is F-rational then so is the Rees algebra $R[It]$.*

We also give an alternative prove of Theorem 4.2 of [HWY02]:

Theorem 1.5. *(see Corollary 6.20) Let (R, m) be an F-rational excellent local ring of positive characteristic $p > 0$ and I be an m-primary ideal. If the Rees algebra $R[It]$ is F-rational so is the extended Rees algebra $R[It, t^{-1}]$.*

We prove Rees algebra over a 2-dimensional excellent F-rational ring of prime characteristic $p > 0$ with respect to an integrally closed m-primary ideal is F-rational by showing extended Rees algebra is F-rational at its homogeneous maximal ideal. This result is also proved in [HWY02, Theorem 3.1].

Theorem 1.6. *Let (R, m) be a 2-dimensional excellent F-rational ring of prime characteristic $p > 0$. Let I be an integrally closed* m*-primary ideal. Then $R[It]$ is F-rational.*

If the Rees algebra is F-rational, then the base ring may or may not be F-rational. In [HWY02], it is shown if a-invariant of the associated graded ring G is less than or equal to 2, then the base ring is F-rational (See Corollary 2.13 of [HWY02]). We extend their result:

Theorem 1.7. *Let (R, m) be a d-dimensional excellent Cohen-Macaulay local ring of prime characteristic $p > 0$ and I be an* m*-primary ideal of R. If $R[It]$ is F-rational and $H^d_{G_+}(G)_{-1} \xrightarrow{F} H^d_{G_+}(G)_{-p}$ is injective, then R is F-rational.*

But the criterion on associated graded ring is not necessary, we also give an example to illustrate this. A manuscript containing the principal results is under preparation.

The organization of the thesis is as follows. In chapter 2 and 3 we discuss first project. In chapter 2 we recall some definitions and results that we need later. We also discuss known results on polynomial rings and some non-polynomial rings. In chapter 3, we discuss new results that we got.

In chapter 4, 5 and 6 we discuss second project. In chapter 4 we recall some definition and results that we need later. In chapter 5 we discuss briefly about tight closure and F-rational rings. In chapter 6 we discuss new results and further questions.

Part I

Poset Embeddings Of Hilbert Functions

Chapter 2

Hilbert functions and Macaulay's theorem

All the rings we consider are noetherian, commutative and with identity.

2.1 Graded rings and modules

Definition 2.1. Let $\mathbb{K}$ be a field. A $\mathbb{K}$-algebra R is called *positively graded* if as $\mathbb{K}$-vector spaces $R \simeq \bigoplus_{d \geq 0} R_d$ with $R_0 = \mathbb{K}$ and $R_i R_j \subseteq R_{i+j}$. R is called *standard graded* if $R = R_0[R_1]$, i.e., as an algebra R is generated by elements of R_1. An element $u \in R$ is called *homogeneous of degree* i if $u \in R_i$, for some i. We write $\deg u$ to denote the degree of a homogeneous element u. An R-ideal is said to be *homogeneous* or *graded* if it is generated by homogeneous elements of R.

If R is a finitely generated $\mathbb{K}$-algebra, then $\dim_{\mathbb{K}} R_1 < \infty$. Note that if I is a graded ideal then I can be written as $\bigoplus_{d=0}^{\infty} I_d$, where I_d is $\mathbb{K}$-subspace of R_d.

Example 2.2. *Let $S = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$. An element of the form $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ is called* monomial*. Let S_i denote the $\mathbb{K}$-vector subspace spanned by the set of monomials $\{x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} : a_1, \dots, a_n \in \mathbb{N}$ and $a_1 + \dots + a_n = i\}$. Then as $\mathbb{K}$-vector spaces we can write $S = \bigoplus_{i=0}^{\infty} S_i$. Hence S is a standard graded algebra over $\mathbb{K}$.*

If R is a finitely generated standard graded $\mathbb{K}$-algebra, then we get a surjective degree zero map from a (standard graded) polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ to R. Let $S := \mathbb{K}[x_1, \dots, x_n]$. We fix a surjective degree zero map $\phi : S \longrightarrow R$. Let $\mathrm{Mon}(S)$ denote the set of monomials in x_i's. It is a $\mathbb{K}$-vector space basis of S. By a *monomial* of R, we mean image of an element of $\mathrm{Mon}(S)$ under ϕ. By a *monomial basis* of R, we mean a subset $\mathbb{B}$ of $\mathrm{Mon}(S)$ whose image under ϕ forms a $\mathbb{K}$-basis for R. For a $\mathbb{K}$-subspace

$V \subseteq R_d$, we say that it is a *monomial space* if it can be spanned by monomials in $\mathbb{B}$ of degree d. An R-ideal I is called *monomial* if it is generated by monomials in $\mathbb{B}$.

Definition 2.3. An R-module M is called *graded* if as $\mathbb{K}$-vector spaces $M \simeq \bigoplus_{i\in\mathbb{Z}} M_i$ such that for each $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, $R_i M_j \subseteq M_{i+j}$.

Definition 2.4. Let M be a finitely generated graded R-module. Write $M = \bigoplus_{n=0}^{\infty} M_n$, where M_n denotes the degree n piece of M. The *Hilbert function* of M is defined as follows:

$$\begin{aligned} H_I : \mathbb{N} \cup \{0\} &\longrightarrow \mathbb{N} \cup \{0\}, \\ n &\mapsto \dim_{\mathbb{K}} M_n. \end{aligned}$$

The Hilbert function of a homogeneous ideal of S is a well-studied and important invariant that measures the size of I. It has applications in many areas, including algebraic geometry, commutative algebra and combinatorics.

Definition 2.5. A *total order* on R is a pair $(\mathbb{B}, \tau)$, where $\mathbb{B}$ is a monomial basis of R and τ is an order on $\mathbb{B}$ such that given two monomials $m, m' \in \mathbb{B}$, exactly one of the following three relations holds:

$$m <_\tau m', \quad m = m', \quad m >_\tau m'.$$

The total order τ is called *graded* if $\deg m < \deg m'$ implies $m <_\tau m'$ for all $m, m' \in \mathbb{B}$.

Definition 2.6. A *monomial order* on R is a graded total order $>$ on $\mathbb{B}$ such that for all $m_1, m_2 \in \mathbb{B}$ with $m_1 > m_2$ and $m' \in \mathbb{B}$ implies $m'm_1 > m'm_2$. If $>$ is a monomial order on R, for $f \in R$, write f as a linear combination of elements of $\mathbb{B}$. We define *initial term of f*, denoted by $\mathrm{in}_>(f)$, to be the greatest term of f with respect to the order $>$. For an ideal I of R, the monomial ideal generated by $\{\mathrm{in}_>(f) : f \in I\}$ is called *initial ideal of* I and is denote by $\mathrm{in}_>(I)$.

Definition 2.7. A linear function $w : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ is called a *weight function* for S. For a weight function we can associate a partial order called *weight order* which we will denote by w: for two monomials $x_1^{a_1} \dots x_n^{a_n}, x_1^{b_1} \dots x_n^{b_n} \in \mathrm{Mon}(S)$,

$$x_1^{a_1} \dots x_n^{a_n} >_w x_1^{b_1} \dots x_n^{b_n} \textit{ if and only if } w(a_1, \dots, a_n) > w(b_1, \dots, b_n).$$

Sometimes we write $w(x_1^{a_1} \dots x_n^{a_n})$ to denote $w(a_1, \dots, a_n)$.

Theorem 2.8 ([Eis95, Theorem 15.3])**.** *Let I be a homogeneous ideal of S. For a monomial order $>$ on S, the set B of all monomials in S not in $\mathrm{in}_>(I)$ forms a $\mathbb{K}$-basis for S/I.*

Theorem 2.9 ([Eis95, Theorem 15.26])**.** *Let $>$ be a monomial order on S and I be a homogeneous ideal of S. Then $H_{S/I} = H_{S/\operatorname{in}_>(I)}$ and hence $H_I = H_{\operatorname{in}_>(I)}$.*

Definition 2.10. Let R be a positively graded $\mathbb{K}$-algebra with unique homogeneous maximal ideal $\mathfrak{m}$. A *graded free resolution* of a graded R-module M is an exact sequence

$$F_\bullet : \quad \cdots \to F_1 \to F_0 \to M \to 0$$

where for each i, F_i is a graded free R-module and the map $F_i \to F_{i-1}$ is degree preserving, i.e., degree n elements of F_i go to degree n elements of F_{i-1}.

Note that if M is finitely generated then F_i can be taken to be finite rank free module.

We say that $F_\bullet$ is a *minimal free resolution* of M if each i, image of F_i is inside $\mathfrak{m}F_{i-1}$. Fix basis for F_i's. $F_\bullet$ is minimal if, for all i, the entries of the matrix associated to the map $F_i \to F_{i-1}$ are contained in the homogeneous maximal ideal $\mathfrak{m}$ of R.

Let M and N be graded R-modules. Let $F_\bullet$ and $G_\bullet$ be graded free resolution of M and N respectively. We define $\operatorname{Tor}_i^R(M,N) := H_i(F_\bullet \otimes_R N) = H_i(M \otimes_R G_\bullet)$. Note that $\operatorname{Tor}_i^R(M,N)$ are also graded modules and independent of choice of resolutions of M and N.

If $F_\bullet$ is a graded free resolution of M, then $\operatorname{rank}_{\mathbb{K}} \operatorname{Tor}_i^R(M,\mathbb{K}) \leq \operatorname{rank}_{\mathbb{K}} F_i \otimes_R \mathbb{K} = \operatorname{rank}_R F_i$. When $F_\bullet$ is minimal, $\operatorname{rank}_{\mathbb{K}} \operatorname{Tor}_i^R(M,\mathbb{K}) = \operatorname{rank}_R F_i$. Minimal graded free resolution of finitely generated R-module M is unique up to isomorphism.

For graded R-module M, we define *graded Betti numbers* of M, denoted by $\beta_{i,j}^R(M)$ as

$$\beta_{i,j}^R(M) := \dim_{\mathbb{K}} \operatorname{Tor}_i^R(M,\mathbb{K})_j.$$

Note that if $F_\bullet$ is a minimal graded free resolution of M, then $F_i = \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{i,j}^R(M)}$.

Definition 2.11. Let M is a finitely generated graded R-module. Suppose M has a minimal graded free R-resolution:

$$\cdots \longrightarrow F_j \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

Let t_j be the maximum of the degrees of a minimal set of homogeneous generators of F_j. The regularity of M, denoted by $\operatorname{reg}^R(M)$, is $\inf\{r \mid t_j - j \leq r \text{ for all } j\}$.

2.2 Hilbert functions and Betti numbers of ideals in polynomial rings

In [Mac27], F. Macaulay gave a classification for Hilbert functions of homogeneous ideals of polynomial rings. For this we introduce some notions.

We recall that $\mathrm{Mon}(S)$ denote the set monomials of S and it is a $\mathbb{K}$-vector space basis of S.

Definition 2.12. We define the *graded lexicographic order* called *lex* on $\mathrm{Mon}(S)$ as follows: given two monomials $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, m' = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, we say $m >_{lex} m'$ if and only if either $\deg m > \deg m'$ or $\deg m = \deg m'$ and there exists an i such that $\alpha_i > \gamma_i$ and $\alpha_j = \gamma_j$ for all $j < i$.

We define the *graded reverse lexicographic order* called *revlex* on $\mathrm{Mon}(S)$ as follows: given two monomials $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, m' = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, we say $m >_{revlex} m'$ if and only if either $\deg m > \deg m'$ or $\deg m = \deg m'$ and there exists i such that $\alpha_i < \gamma_i$ and $\alpha_j = \gamma_j$ for all $j > i$.

Definition 2.13. The *lex-segment* $L_{d,p}$ of length p in degree d is defined as the $\mathbb{K}$-vector space spanned by the lexicographically first (greatest) p monomials in S_d. We say that V is a *lex-segment* in S_d if there exists a p such that $V = L_{d,p}$. The $\mathbb{K}$-vector space generated by a lex-segment in S_d is called *lex-segment subspace* of S_d. For a subspace $V \subseteq S_d$, we say that $L_{d,\dim_{\mathbb{K}} V}$ is its *lexification* in S_d and denote by V^{lex}. A monomial ideal I is said to be *lex* if its each d-th graded piece I_d is a lex-segment subspace of S_d.

Lemma 2.14. *Let I be a monomial ideal in S minimally generated by monomials $m_1, \dots, m_r$. Then I is lex if and only if the following holds: if there exists i, $1 \leq i \leq r$ such that $\deg m = \deg m_i$ and $m >_{lex} m_i$, then $m \in I$.*

Proof. Let I be a lex ideal. By definition, for each d, I_d is the lex-segment subspace of S_d; hence if there exists i, $1 \leq i \leq r$ such that $\deg m = \deg m_i$ and $m >_{lex} m_i$ then $m \in I_{d_i}$

Conversely, let $d_i := \deg m_i$, then by hypothesis a lex-segment in S_{d_i} ending at m_i is inside I. To show I is lex ideal, i.e., to show for each d, I_d is a lex-segment subspace of S_d. Consider a monomial m in I_d, then $m = m' m_i$, for some i. Now all monomials that come before m in the lex order are inside the ideal generated by the lex-segment in S_{d_i} ending with m_i. Hence I_d is a lex-segment subspace of S_d. □

Example 2.15. *Let $S = \mathbb{K}[x_1, x_2, x_3, x_4]$.*
(1) *Let V be the $\mathbb{K}$-vector space spanned by $\{x_2^3, x_2 x_3^2, x_2 x_3 x_4, x_4^3\}$. Then its lexification is the $\mathbb{K}$-vector space spanned by $\{x_1^3, x_1^2 x_2, x_1^2 x_3, x_1^2 x_4\}$.*

(2) *Let* $I = (x_1, x_2, x_3^3, x_3^2x_4)$. *Then it is easy to see that* I *is a lex ideal. Let* $I = (x_1, x_2, x_3^3, x_3x_4^2)$. *Then* I *is not lex ideal, for* $x_3^2x_4 >_{lex} x_3x_4^2$*, but* $x_3^2x_4 \notin I$*, so* I_3 *is not a lex-segment.*

Theorem 2.16. *(Macaulay) [Mac27] For every graded ideal* I *in* S *there exists a lex ideal* L *such that* $H_I = H_L$*.*

The following Proposition is equivalent to the above theorem.

Proposition 2.17. (i)*.* $S_1L_{d,p} = L_{d+1,s}$ *for some* s*.*
(ii)*. Let* V *be an* S_d*-subspace and* V^{lex} *be its lexification in* S_d*. Then* $\dim_{\mathbb{K}} S_1V^{lex} \leq \dim_{\mathbb{K}} S_1V$*.*

We will sketch a proof of the equivalence of the Proposition 2.17 and Macaulay's theorem. Suppose we know that for every graded S-ideal I, there exists a lex ideal L, such that $H_I = H_L$. Now given an S_d-subspace V, we consider the ideal generated by V, say I, then by hypothesis there exists a lex ideal L such that $H_I = H_L$. Now $I_{d+1} = S_1V$ and V^{lex} is the lexification of $I_d = V$. Since $H_I = H_L$, $\dim_{\mathbb{K}} S_1V^{lex} \leq \dim_{\mathbb{K}} S_1V$.

Conversely, suppose that Proposition 2.17 holds. Given a graded S-ideal I we can find a lex ideal L as follows: consider $L = \bigoplus_{i=0}^{\infty} I_d^{lex}$, where I_d^{lex} is the lexification of I_d in S_d. Since $S_1I_d^{lex}$ is again a lex-segment in S_{d+1}, by Proposition 2.17, $S_1I_d^{lex} \subseteq I_{d+1}^{lex}$. Hence L is an S-ideal. By definition it is lex ideal and $H_I = H_L$.

Macaulay's theorem gives a way to check whether a numerical function is the Hilbert function of a homogeneous S-ideal.

Example 2.18. *Let* $H : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ *is a function where* $0 \mapsto 0$*,* $1 \mapsto 3$*,* $2 \mapsto 5$*,* $3 \mapsto 7$*, then* H *is not Hilbert function of a homogeneous* S*-ideal. For if* $H = H_I$ *for some homogeneous* S*-ideal* I*, then by Theorem 2.16* $H = H_L$*, where L is the corresponding lex ideal. Then* $\dim_{\mathbb{K}} L_0 = 0$*,* $\dim_{\mathbb{K}} L_1 = 3$*,* $\dim_{\mathbb{K}} L_2 = 5$*. Since* L_d *is lex-segment subspace in* S_d*, then* L_1 *be the* $\mathbb{K}$*-vector space generated by* $\{x_1, x_2, x_3\}$*. Now* $S_1L_1 \subseteq L_2$*, but* $\dim_{\mathbb{K}} S_1L_1 \geq 6 > 5 = \dim_{\mathbb{K}} L_2$*. So it can not be Hilbert function of a graded ideal.*

Let I be a homogeneous S-ideal. Let L be the corresponding lex ideal in S with $H_I = H_L$. Then $\beta_{0,j}^S(I) \leq \beta_{0,j}^S(L)$ for all j. One can see this as follows, first note that for homogeneous S-ideal I, $\beta_{0,j}^S = \dim_{\mathbb{K}} I_j - \dim_{\mathbb{K}} S_1I_{j-1}$. We have $\dim_{\mathbb{K}} I_j = \dim_{\mathbb{K}} L_j$ and Macaulay's theorem gives $\dim_{\mathbb{K}} S_1I_j \geq \dim_{\mathbb{K}} S_1L_j$, for all j. Hence we have $\beta_{0,j}^S(I) \leq \beta_{0,j}^S(L)$ for all j. This was extended to all graded Betti numbers in [Big93], [Hul93], [Par96] as follows:

Theorem 2.19. *Let* I *be a homogeneous ideal in* S*. If* L *is the lex ideal with the same Hilbert function as* I*, then for all* i, j*;*

$$\beta_{i,j}^S(I) \leq \beta_{i,j}^S(L).$$

Later, Macaulay's theorem and analogous results for Betti number were extended to certain non-polynomial rings.

2.3 Macaulay's theorem for non-polynomial rings

In this section we will see some examples of non-polynomial rings for which analogues of Macaulay's theorem and the related result of graded Betti numbers hold. We also see examples of rings for which an analogue of Macaulay's theorem does not hold.

Rings of the form $R = S/\mathbf{a}$ where $\mathbf{a} = (x_1^{e_1}, \cdots, x_n^{e_n})$ with $e_1 \le e_2 \le \cdots \le e_n < \infty$ are well studied.

Note that graded lexicographic order on S defined before induces a graded lexicographic order also called lex on R.

Definition 2.20. An R-ideal I is called is *lex* if it is image of a lex ideal in S.

Let $R = S/(x_1^{e_1}, \cdots, x_n^{e_n})$ with $e_1 \le e_2 \le \cdots \le e_n < \infty$. In [CL69] Clements and Lindström proved that every homogeneous R-ideal has the same Hilbert function as the image (in R) of a lex S-ideal. The related result for Betti numbers over S is proved by Mermin and Murai [MM11] and Betti numbers over R is proved by Murai and Peeva in [MP12].

V. Gasharov, N. Horwitz and I. Peeva [GHP08] proved the analogue of Macaulay's theorem for rational normal curves.

In [GMP11] Gasharov, Murai, Peeva proved Macaulay's theorem and results on Betti numbers for Veronese rings.

There are examples of rings for which analogue of Macaulay's theorem does not hold. Let S be a polynomial ring with graded lexicographic order.

Definition 2.21. An graded S-ideal I is *lex-Macaulay* if Hilbert function of graded ideals in the quotient S/I is attained by image of a lex ideal in S/I.

In [Mer10] J. Mermin characterizes the monomial regular sequences which are lex-Macaulay as follows.

Theorem 2.22 ([Mer10, Theorem 4.4])**.** *Let I be a graded S-ideal generated by a regular sequence of monomials. Then I is lex-Macaulay if and only if $I = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r - 1}y)$, with $e_1 \le \cdots \le e_r$ and $y = x_i$ for some $i \ge r$.*

The above theorem shows that there are monomial complete intersection rings where Hilbert function of graded ideals can not be attained by image of a lex ideal.

2.4 Poset embeddings of Hilbert functions

In order to classify Hilbert function of ideals in a standard graded algebra $\mathbb{K}$-algebra R, Caviglia and Kummini (cf.[CK13]) looked at certain embedding of the poset of Hilbert function into the homogeneous R-ideals. We define the terms and discuss their work below.

Definition 2.23. Let $<$ be a graded total order on $\mathbb{B}$. By a $<$-*segment* in R_n, we mean a list of consecutive monomials in the order starting from the first monomial in $\mathbb{B}_n$, where $\mathbb{B}_n$ is the set of monomials of $\mathbb{B}$ of degree n.

Let $\mathcal{I}_R$ be the set $\{I : I$ is a homogeneous R-ideal$\}$, considered as a poset under inclusion and $\mathcal{H}_R$ be the set $\{H_I : I \in \mathcal{I}_R\}$, the poset of Hilbert functions of R-ideals endowed with the partial order: $H \succeq H' \in \mathcal{H}_R$ if, for all $t \in \mathbb{N} \cup \{0\}$, $H(t) \geq H'(t)$. They asked whether there is an (order preserving) embedding $\epsilon : \mathcal{H}_R \longrightarrow \mathcal{I}_R$ as posets, such that $\mathbf{H} \circ \epsilon = id_{\mathcal{H}_R}$, where $\mathbf{H} : \mathcal{I}_R \longrightarrow \mathcal{H}_R$ is the function $I \mapsto H_I$.

If every Hilbert function in $\mathcal{H}_R$ is attained by a image of a lex S-ideal, then $\mathcal{H}_R$ admits an embedding. We define the map by $H_I \mapsto L$, where L is the corresponding lex ideal with $H_I = H_L$.

When such embedding exists, it induce a filtration of R_n as $\mathbb{K}$-subspaces.

Definition 2.24 ([CK13, Definition 2.3])**.** An *embedding filtration* of R is a collection of filtrations $\{0 = V_{n,0} \subsetneq V_{n,1} \subsetneq \cdots \subsetneq V_{n,\dim_{\mathbb{K}}(R_n)} = R_n : n \in \mathbb{N} \cup \{0\}\}$ of R into $\mathbb{K}$-vector spaces that satisfies, for all $n \in \mathbb{N} \cup \{0\}$ and for all $0 \leq r \leq \dim_{\mathbb{K}}(R_n)$,
$(i) R_1 V_{n,r} = V_{n+1,s}$, for some $0 \leq s \leq \dim_{\mathbb{K}}(R_{n+1})$ and
(ii) For all $\mathbb{K}$-subspaces $W \subseteq R_n, \dim_{\mathbb{K}}(R_1 V_{n,\dim_{\mathbb{K}}(W)}) \leq \dim_{\mathbb{K}}(R_1 W)$.

Proposition 2.25 ([CK13, Proposition 2.4])**.** *Let R be a standard graded $\mathbb{K}$-algebra, then $\mathcal{H}_R$ admits an embedding into $\mathcal{I}_R$ if and only if R has an embedding filtration.*

Definition 2.26 ([CK13, Discussion 2.15])**.** Let R be a standard graded algebra with total order τ. Then τ is called an *embedding order* if for all $n \in \mathbb{N} \cup \{0\}$ and for all τ-segment subspace $V \subseteq R_n$,
(1) $R_1 V$ is a τ-segment of R_{n+1} and
(2) $\dim_{\mathbb{K}}(R_1 W) \geq \dim_{\mathbb{K}}(R_1 V)$, for all $\mathbb{K}$-subspaces $W \subseteq R_n$ with $\dim_{\mathbb{K}}(W) = \dim_{\mathbb{K}}(V)$.

An embedding order on R gives an embedding filtration [[CK13, Discussion 2.15]].

2.5 Toric rings

Here we introduce notion of toric rings.
Let $a_1 = (a_{1,1}, \cdots, a_{1,c}), \cdots, a_n = (a_{n,1}, \cdots, a_{n,c})$ be vectors in $\mathbb{N}^c$. Consider the $\mathbb{K}$-algebra homomorphism $\psi : \mathbb{K}[x_1, x_2, \cdots, x_n] \longrightarrow \mathbb{K}[t_1, t_2, \cdots, t_c]$ by $x_i \mapsto t_1^{a_{i,1}} \cdots t_c^{a_{i,c}}$. Since the image of ψ is an integral domain, the kernel of ψ is a prime ideal, called *toric ideal* and the image of ψ is called *toric ring*.

We say that the $\ker(\psi)$ is *projective* (or that $\mathbb{K}[x_1, \cdots, x_n]/\ker(\psi)$ is a *projective toric ring*) if $\ker(\psi)$ is homogeneous in the standard graded ring $\mathbb{K}[x_1, \cdots, x_n]$.

Lemma 2.27 ([Stu96, Lemma 4.1])**.** *Toric ideal is spanned as a $\mathbb{K}$-vector-space by the set of binomials $\{u - v : \psi(u) = \psi(v)\}$, where u, v are monomials of $\mathbb{K}[x_1, \cdots, x_n]$.*

Theorem 2.28 ([GHP08, Theorem 2.5])**.** *Let S be a polynomial ring and $R = S/\mathbf{a}$ be a projective toric ring. Let P be a homogeneous ideal in R. Then there exists a monomial ideal M in R satisfying the following:*
(1) *H_M=H_P.*
(2) *$\beta^R_{i,j}(M) \geq \beta^R_{i,j}(P)$ for all i, j. Furthermore, $\beta^R_{i,j}(P)$ can be obtained from $\beta^R_{i,j}(M)$ by a sequence of consecutive cancellations.*
(3) *Let K and O be the preimages of M and P (respectively) in S. $\beta^S_{i,j}(K) \geq \beta^S_{i,j}(O)$ for all i, j. Furthermore, $\beta^R_{i,j}(O)$ can be obtained from $\beta^S_{i,j}(K)$ by a sequence of consecutive cancellations.*

2.6 Mapping cones and Free resolutions

Basics on mapping cone can be found in [Wei94, Section 1.5].

Let R be a ring that is not necessarily graded.

Definition 2.29. Let $(F_\bullet, d)$ and $(G_\bullet, d')$ be two complexes of R-modules. Let $f : F_\bullet \to G_\bullet$ be a map of complexes. The *mapping cone of f* is the complex $(\mathrm{cone}(f)_\bullet, \delta)$, where $\mathrm{cone}(f)_n = F_{n-1} \oplus G_n$ and $\delta_n : \mathrm{cone}(f)_n \to \mathrm{cone}(f)_{n-1}$ is the map $\delta(b, c) = (-d(b), f(b) + d'(c))$, where $b \in F_{n-1}$ and $c \in G_n$.

One sees that $G_\bullet$ is subcomplex of $\mathrm{cone}(f)_\bullet$ and the quotient is $F[-1]_\bullet$, where $F[-1]_\bullet$ is the complex whose n-th term $F[-1]_n$ is F_{n-1} with differential $-d$. Hence we have a short exact sequence of complexes:

$$0 \to G_\bullet \to \mathrm{cone}(f)_\bullet \to F_\bullet[-1] \to 0.$$

Hence we have the following long exact sequence:

$$\cdots \longrightarrow H_i(G_\bullet) \longrightarrow H_i(\mathrm{cone}(f)_\bullet) \longrightarrow H_i(F_\bullet(-1)) \longrightarrow H_{i-1}(G_\bullet) \cdots .$$

Since $H_i(F_\bullet(-1)) = H_{i-1}(F_\bullet)$, the above exact sequence becomes:

$$\cdots \longrightarrow H_i(G_\bullet) \longrightarrow H_i(\text{cone}(f)_\bullet) \longrightarrow H_{i-1}(F_\bullet) \longrightarrow H_{i-1}(G_\bullet) \cdots . \tag{2.1}$$

One can show that the connecting morphism $H_i(F_\bullet) \longrightarrow H_i(G_\bullet)$ is the map induced by f.

Our reference for the following discussion is [Pee11, Section 27].

Let $f : M \to N$ be morphism of R-modules. Let $F_\bullet$ and $G_\bullet$ be free resolutions of M and N respectively. Then f lifts to a morphism of complexes $\tilde{f} : F_\bullet \to G_\bullet$. Then by long exact sequence (2.1), we have

$$0 \longrightarrow H_1(\text{cone}(\tilde{f})) \longrightarrow M \longrightarrow N \longrightarrow H_0(\text{cone}(\tilde{f})) \longrightarrow 0$$

as $H_0(F_\bullet) = M$ and $H_0(G_\bullet) = N$ and $H_i(\text{cone}(\tilde{f})) = 0$ for all $i \geq 2$. Hence if f is injective, $\text{cone}(\tilde{f})$ is a free resolution of $N/f(M)$. Note that if the ring R is graded and M, N are graded R-modules, f is degree zero morphism and $F_\bullet$ and $G_\bullet$ are graded free resolutions then $\text{cone}(\tilde{f})$ is also a graded free resolution of $N/f(M)$. However even if in addition $F_\bullet$ and $G_\bullet$ are minimal, one can not guarantee $\text{cone}(\tilde{f})$ is minimal.

One can construct examples as follows.

Example 2.30. *Take polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$. Let $\mathrm{m} = (x_1, \ldots, x_n)$. Take a graded ideal I, such that $\operatorname{depth} S/I = 0$, then Auslander-Buchsbaum formula gives $\operatorname{pd}_S S/I = n$, where $\operatorname{pd}_S S/I$ denotes the projective dimension of S/I. Take a homogeneous element $f \in \operatorname{Soc}(S/I)$, then $I : f = \mathrm{m}$. Thus we have exact sequence:*

$$0 \longrightarrow (S/\mathrm{m})(-\deg f) \xrightarrow{m_f} S/I \longrightarrow S/(I, f) \longrightarrow 0,$$

where m_f denotes the map multiplication by f. Koszul complex gives minimal free resolution of $S/\mathrm{m} = \mathbb{K}$ over S. Hence minimal free resolution of S/I and $\mathbb{K}$ have length n. Hence $\text{cone}(m_f)$ has length $n+1$. By Hilbert's syzygy theorem, $\operatorname{pd}_S M \leq n$ for all S-module M. Hence $\text{cone}(m_f)$ is not a minimal free resolution of $S/(I, f)$.

Using mapping cone one can construct free resolution of R/I, where I is an ideal of R. Suppose I is generated by $(f_1, \ldots, f_n)$. We write $J_i = (f_1, \ldots, f_i)$. Then we have an exact sequence of R-modules,

$$0 \longrightarrow (R/J_i : (f_{i+1}))(-\deg f_{i+1}) \xrightarrow{m_{f_{i+1}}} R/J_i \longrightarrow R/J_{i+1} \longrightarrow 0,$$

where $m_{f_{i+1}}$ is the map given by multiplication by f_{i+1}.

Suppose $F_\bullet$ and $G_\bullet$ are graded free resolutions of R/J_i and $R/J_i : (f_{i+1})$ respectively. Then there is a lift $\phi : G_\bullet \to F_\bullet$ of $m_{f_{i+1}}$ and $\mathrm{cone}(\phi)$ gives a graded free resolution of R/J_{i+1}. Iterating this process, we get a graded free resolution of R/I.

Chapter 3

Poset embedding of Hilbert functions for two hypersurface rings

Notation: Here $\mathbb{K}$ will always denote a field.

In this chapter we show poset embedding of Hilbert functions holds for two toric rings $\mathbb{K}[a,b,c,d]/(ad-bc)$ and $\mathbb{K}[a,b,c]/(b^2-ac)$, where $\mathbb{K}$ is a field. In order to show that we follow the approach of Caviglia and Kummini [CK13]. We also prove related result for graded Betti numbers.

3.1 Poset embedding for $\mathbb{K}[a,b,c,d]/(ad-bc)$

In this section R will denote $\mathbb{K}[a,b,c,d]/(ad-bc)$, where a,b,c,d are indeterminates. We write S for the polynomial ring $\mathbb{K}[a,b,c,d]$.

Let $\operatorname{Mon}(R) := \{a^ib^jd^k, a^ic^jd^k : i,j,k \in \mathbb{N} \cup \{0\}\}$. $\operatorname{Mon}(R)$ is a monomial basis for R. Clearly $\operatorname{Mon}(R)$ generates R. Indeed, the initial term of $ad-bc$ with respect to the revlex order in S is bc; so by Theorem 15.3 of [Eis95] $\operatorname{Mon}(R)$ is a monomial basis for R.

Let *lex* be the graded lexicographic order on $\operatorname{Mon}(R)$ with $a \succ_{lex} b \succ_{lex} c \succ_{lex} d$. Note that *lex* is not a monomial order, since $b \succ_{lex} c$ but $b^2 \prec_{lex} ad$, which is the representative for bc in $\operatorname{Mon}(R)$.

Theorem 3.1. *$\succ_{lex}$ is an embedding order for R.*

Definition 3.2. For a $\mathbb{K}$-subspace $V \subseteq R_n$, the subspace generated by first $\dim_{\mathbb{K}} V$ monomials in $\operatorname{Mon}(R)$ of degree n with respect to the lex order is called *lexification* of V and denoted by V^{lex}.

Outline of the proof: In order to prove that for all subspace $V \subseteq R_n$, $\dim_{\mathbb{K}}(R_1 V^{lex}) \leq \dim_{\mathbb{K}}(R_1 V)$, where V^{lex} is the lexification for V in R_n, we define notion of stable vector space and reduce to the case for stable vector space.

Discussion 3.3. Let w be a weight order on S where the weights of a, b, c, d are $(1,0,1,0)$, $(1,0,0,1)$, $(0,1,1,0)$ and $(0,1,0,1)$ respectively. Consider the $\mathbb{K}$-algebra homomorphism $\phi: \ S \longrightarrow \mathbb{K}[x,y,s,t]$ by $a \mapsto xs, b \mapsto xt, c \mapsto ys, d \mapsto yt$. The kernel of this map is generated by binomials i.e. $u - u'$, where u, u' are monomials in S such that $w(u) = w(u')$ (by Lemma 2.27). Since $w(ad) = w(bc)$, then ϕ induces a map $\tilde{\phi}: R \longrightarrow \mathbb{K}[x,y,s,t]$. A simple calculation shows that distinct monomials of R have distinct weights. Hence $\tilde{\phi}$ is injective onto its image. So R is a projective toric ring and the induced weight order on R which we again denote by w is a monomial order. Also note that $\mathrm{in}_w(\tilde{I}) = \widetilde{\mathrm{in}_w(I)}$, where $\tilde{I}$ and $\widetilde{\mathrm{in}_w(I)}$ are the preimages of I and $\mathrm{in}_w(I)$ in S respectively. So for homogeneous R-ideal I, $H_I = H_{\mathrm{in}_w(I)}$. Since w is a monomial order on R, $\mathrm{in}_w(I)$ is a monomial ideal. Hilbert function of a monomial ideal does not depend on the characteristic of the ground field. So for the calculations we can take $\mathrm{in}_w(VR)$ or $\mathrm{in}_w(I)$ instead of VR or I respectively and hence hereafter we assume that $\mathrm{char}(\mathbb{K}) = 0$.

Discussion 3.4. By a *monomial* of $\bigwedge^t R_n$, we mean an element of the form $m_1 \wedge m_2 \wedge \cdots \wedge m_t$, where the m_i's are degree-n monomials of $\mathrm{Mon}(R)$. Any monomial order $>$ on R induces a monomial order on $\bigwedge^t R_n$. Suppose $>$ is a monomial order on R. We say $m_1 \wedge m_2 \cdots \wedge m_t$ is a *normal expression* if m_i's are ordered so that $m_1 > m_2 > \cdots > m_t$. We order the monomials of $\bigwedge^t R_n$ by ordering their normal expression lexicographically i.e. $m_1 \wedge m_2 \wedge \cdots \wedge m_t > {m'}_1 \wedge {m'}_2 \wedge \cdots \wedge {m'}_t$ if and only if $m_i > {m'}_i$ for the smallest i such that $m_i \neq {m'}_i$. Therefore we can define *initial term* of an element $f \in \bigwedge^t R_n$ to be the greatest term with respect to the order.

For $\lambda, \mu \in \mathbb{K}$, we define a $\mathbb{K}$-algebra homomorphism:

$$\begin{aligned} g_{\lambda\mu} : \mathbb{K}[a,b,c,d] &\longrightarrow \mathbb{K}[a,b,c,d], \text{ by} \\ a &\mapsto a \\ b &\mapsto \lambda a + b \\ c &\mapsto \mu a + c \\ d &\mapsto \lambda\mu a + \mu b + \lambda c + d. \end{aligned}$$

Note that $g_{\lambda\mu}$ is an automorphism of $\mathbb{K}[a,b,c,d]$ and the ideal $(ad - bc)$ is fixed under the action of $g_{\lambda\mu}$. Hence it induces an automorphism of R.

Define $\mathfrak{U} = \{g_{\lambda\mu} \mid \lambda, \mu \in \mathbb{K}\}$. Note that $\mathfrak{U}$ forms a group under composition. By a *diagonal* automorphism of R we mean an automorphism of R which sends a to $\lambda_1 a$, b to $\lambda_2 b$, c to $\lambda_3 c$, d to $\lambda_4 d$, where λ_i's are non-zero elements in $\mathbb{K}$. We denote this diagonal automorphism by $\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. A diagonal automorphism of R is of the form $\operatorname{diag}(T_1, T_2, T_3, T_2T_3/T_1)$, where T_i's are non-zero scalars. Let $\mathfrak{B}$ be the group generated by all the diagonal automorphisms of R and $\mathfrak{U}$. Considering $\mathbb{K}^5$ with coordinates $\lambda, \mu, T_1, T_2, T_3$, we see that $\mathfrak{B}$ is isomorphic to $\mathbb{K}^5 \setminus V(T_1T_2T_3)$; hence $\mathfrak{B}$ is dense, open in $\mathbb{K}^5$ and therefore, is irreducible.

Next we will prove theorems analogous to 15.18 and 15.20 of [Eis95].

Theorem 3.5. *Let I be a homogeneous ideal of R. There is a nonempty Zariski open set $U \subset \mathfrak{B}$ and a monomial ideal $J \subset R$ such that for all $g \in U$, $\operatorname{in}_w(gI) = J$, where w is the weight order defined in Discussion 3.3. For each $n \geq 0$, if J_n of J has dimension t, then $\bigwedge^t J_n$ is spanned by the greatest monomial of $\bigwedge^t R_n$ that appears in $\bigwedge^t(gI_n)$ with $g \in \mathfrak{B}$.*

Proof. Let $f_1, f_2, \cdots, f_t$ be a basis for I_n. Consider a matrix g whose entries are indeterminates $\lambda, \mu, T_i's$ such that if we put any value of $\lambda, \mu, T_i's$ from $\mathbb{K}$, $g \in \mathfrak{B}$. Then $g(f_1 \wedge \cdots \wedge f_t) = g(f_1) \wedge \cdots \wedge g(f_t)$ is a linear combination of monomials of $\bigwedge^t R_n$ with coefficients that are rational functions in λ, μ, and T_i's. In that expression let $m = m_1 \wedge \cdots \wedge m_t$ be the first monomial with respect to the induced order on $\bigwedge^t R_n$ with a non-zero function, say $p_n(\lambda, \mu, T_1, T_2, T_3)$. Let U_n be the set of $g \in \mathfrak{B}$ such that $p_n(\lambda, \mu, T_1, T_2, T_3) \neq 0$. Then U_n is a nonempty Zariski open set. The degree-n part of the initial ideal of gI i.e. $\operatorname{in}_w(gI)_n$ will be generated by $m_1, \cdots, m_t$ if and only if $g \in U_n$. Let J_n be the subspace generated by $m_1, \cdots, m_t$.

Write $J = \bigoplus_{n=1}^{\infty} J_n$. To show J is an ideal, it is enough to show for each n, $R_1J_n \subset J_{n+1}$. Since U_n is nonempty Zariski open and $\mathfrak{B}$ is irreducible, U_n is dense; so $U_n \cap U_{n+1} \neq \emptyset$. For $g \in U_n \cap U_{n+1}$, we have $\operatorname{in}_w(gI)_n = J_n$ and $\operatorname{in}_w(gI)_{n+1} = J_{n+1}$. Hence $R_1J_n \subset J_{n+1}$. Note also that by construction J is a monomial ideal. Last statement of the theorem is clear by the definition of J.

Next we will show that $U = \bigcap_{n=1}^{\infty} U_n$ is a Zariski open set. It is enough to show that U is a finite intersection of U_n. For, being finite intersection of open sets, U is open and since each U_n is dense, U is nonempty. Suppose J is generated by forms of degree $\leq e$. We will show that $U = \bigcap_{n=1}^{e} U_n$. Let $g \in \bigcap_{n=1}^{e} U_n$, then $\operatorname{in}_w(gI_n) = J_n$ for all $n \leq e$. Thus $J \subseteq \operatorname{in}_w(gI)$. Since $\dim_{\mathbb{K}} J_n = \dim_{\mathbb{K}} I_n = \dim_{\mathbb{K}}(gI)_n$ for every n, we have $J = \operatorname{in}_w(gI)$. ◻

With I and J as in the above theorem, we write $J := \operatorname{Gin}(I)$.

Definition 3.6. An ideal in R is said to be **$\mathfrak{U}$-stable** if it is fixed under the action of $\mathfrak{U}$.

Theorem 3.7. *Let I be a homogeneous ideal of R. Then $\operatorname{Gin}(I)$ is $\mathfrak{U}$-stable.*

Proof. Let U be as in the previous theorem. Replacing I by gI for some $g \in U$, we may assume by the previous theorem that $\text{in}_w(I) = \text{Gin}(I)$. Therefore we have to show that for $g_{\lambda\mu} \in \mathfrak{U}$, $g_{\lambda\mu}(\text{in}_w(I_n)) = \text{in}_w(I_n)$ for all n.

We choose a basis $f_1, \cdots, f_t$ for I_n with $\text{in}_w(f_1) > \cdots > \text{in}_w(f_t)$. Let $f = f_1 \wedge \cdots \wedge f_t$ be the corresponding generator of the one dimensional subspace $\wedge^t I_n \subset \wedge^t R_n$. We have $\text{in}_w(f) = \text{in}_w(f_1) \wedge \cdots \wedge \text{in}_w(f_t)$.

If $g_{\lambda\mu}(\text{in}_w(I_n)) \neq \text{in}_w(I_n)$, then $g_{\lambda\mu}\,\text{in}_w(f) \neq \text{in}_w(f)$. The terms of $g_{\lambda\mu}\,\text{in}_w(f)$ other than $\text{in}_w(f)$ are all strictly greater than $\text{in}_w(f)$. Let kx be one of these non-zero terms, where k is a non-zero scalar and x is monomial in $\wedge^t R_n$. We will show for a suitable diagonal automorphism T of R, x appears with non-zero coefficient in $g_{\lambda\mu}Tf$ which will contradict the last statement of the previous theorem. Hence $g_{\lambda\mu}(\text{in}_w(I_n)) = \text{in}_w(I_n)$.

For each term $k'm_1 \wedge \cdots \wedge m_t \in \wedge^t R_n$, where $k' \in \mathbb{K}$, we define its weight to be the monomial $v = \prod m_i \in R$. Let $f_v \in \wedge^t R_n$ be the sum of all the terms of f having weight v, so that we have $f = \sum_v f_v$. Let v_0 be the weight of $\text{in}_w(f)$. Here note that different terms of f may have the same weight, but $\text{in}_w(f)$ is the unique term having weight v_0. If $T = \text{diag}(T_1, T_2, T_3, T_2T_3/T_1)$, where T_1, T_2, T_3 are non-zero scalar, is a diagonal automorphism of R, then

$$Tf = \sum_v v(T_1, T_2, T_3, T_2T_3/T_1) f_v.$$

Thus

$$\begin{aligned} g_{\lambda\mu}Tf &= \sum_v g_{\lambda\mu}(v(T_1, T_2, T_3, T_2T_3/T_1) f_v) \\ &= \sum_v v(T_1, T_2, T_3, T_2T_3/T_1) g_{\lambda\mu} f_v \\ &= v_0(T_1, T_2, T_3, T_2T_3/T_1) g_{\lambda\mu} in_w(f) + \sum_{v \neq v_0} v(T_1, T_2, T_3, T_2T_3/T_1) g_{\lambda\mu} f_v. \end{aligned}$$

Thus the coefficient of x in $g_{\lambda\mu}Tf$ has the form

$$h(T_1, T_2, T_3, T_2T_3/T_1) := kv_0(T_1, T_2, T_3, T_2T_3/T_1) + \sum_{v \neq v_0} k_v v(T_1, T_2, T_3, T_2T_3/T_1),$$

where $k_v \in \mathbb{K}$ is the coefficient of x in $g_{\lambda\mu} f_v$. Claim: $v_0(T_1, T_2, T_3, T_2T_3/T_1)$ is a non-zero rational function. Consider the $\mathbb{K}$-algebra map $\mathbb{K}[a, b, c, d] \longrightarrow \mathbb{K}(T_1, T_2, T_3)$ sending $a \mapsto T_1, b \mapsto T_2, c \mapsto T_3, d \mapsto T_2T_3/T_1$. Note that image ring is a domain of dimension 3 as its transcendence degree is 3. So the kernel is a prime of height 1. Hence the kernel is principal. Clearly $ad - bc$ is in the kernel and $ad - bc$ is irreducible, hence prime. Therefore the kernel is precisely the ideal $(ad - bc)$ and R is isomorphic to the image ring. Since v_0 is non-zero in R, $v_0(T_1, T_2, T_3, T_2T_3/T_1)$ is a non-zero rational function. Since the term $kv_0(T_1, T_2, T_3, T_2T_3/T_1)$ is non-zero, we see that h is non-zero rational function. Since $\mathbb{K}$ is infinite, we can find T_1, T_2, T_3 non-zero scalars such that h is non-zero. $\square$

Definition 3.8. A vector space $V \subseteq R_n$ is said to be **$\mathfrak{U}$-stable** if it is fixed under the action of $\mathfrak{U}$.

Define $\succ_{stb}$ be the graded partial order on R with $a \succ_{stb} b \succ_{stb} d$, $a \succ_{stb} c \succ_{stb} d$; b and c are not comparable such that $a^i b^j d^k \succ_{stb} a^l b^m d^n$ if and only if either $i+j+k > l+m+n$ or $i+j+k = l+m+n$ and $(i,j,k) > (l,m,n)$, similarly $a^i c^j d^k \succ_{stb} a^l c^m d^n$ if and only if either $i+j+k > l+m+n$ or $i+j+k = l+m+n$ and $(i,j,k) > (l,m,n)$.

Definition 3.9. A vector space $V \subseteq R_n$ is said to be **stable** if it is monomial and a monomial $u \in V$, all the monomials of degree n that come before u in $\succ_{stb}$ are also in V.

Example 3.10. *Let $V \subseteq R_4$ generated by $\{a^4, a^3b, a^2b^2, ab^3\}$, is a stable vector space. Vector space generated by $\{a^4, a^3b, a^3d, a^2b^2, ab^3\}$ is not stable, because $a^3d \in V$ and $a^3c \succ_{stb} a^3d$ but $a^3c \notin V$.*

Lemma 3.11. *Let $V \subseteq R_n$ be a monomial vector space. V is $\mathfrak{U}$-stable if and only if V is stable.*

Proof. Suppose that V is $\mathfrak{U}$-stable. Let $u \in V$ be a monomial. Since V is $\mathfrak{U}$-stable, $g_{\lambda\mu}u \in V$, for all $g_{\lambda\mu} \in \mathfrak{U}$. Note that for some general λ and μ by definition of $g_{\lambda\mu}$, all monomials that appear with non-zero coefficients in the expression for $g_{\lambda\mu}u$ are those that come before u in the partial order $\succ_{stb}$ and u itself (here we have used characteristic of $\mathbb{K}$ is 0). As V is monomial vector space all these monomials also belong to V. Hence V is stable.

Conversely, let V be stable. By definition, V is monomial. Therefore it remains to show that if a monomial $u \in V$, then $g_{\lambda\mu}u \in V$, for all $g_{\lambda\mu} \in \mathfrak{U}$. Let u be a monomial in V. Note that each term that appears with a non-zero coefficient in $g_{\lambda\mu}u$ is of the form kv, where $k \in \mathbb{K}$ and v is either u or a monomial that comes before u in the partial order $\succ_{stb}$. Since V is stable, each term of $g_{\lambda\mu}u$ is in V. Hence $g_{\lambda\mu}u \in V$ and V is $\mathfrak{U}$-stable. $\square$

Definition 3.12. Let I be a homogeneous ideal of R, I is said to be **stable** if I is monomial and for each $n \geq 0$, I_n is a stable vector space.

Proposition 3.13. *Let I be an R-ideal. Then I is monomial and $\mathfrak{U}$-stable if and only if I is stable.*

Proof. Since $\mathfrak{U}$ is consists of degree zero automorphisms of R, the proposition follows from Lemma 3.11. $\square$

For an arbitrary vector space $V \subseteq R_n$, we consider the ideal VR, the ideal generated by V, then by Theorem 3.7 and Proposition 3.13, there exists a stable ideal $\operatorname{Gin}(VR)$ with same Hilbert function as of VR. So $\dim_{\mathbb{K}}(R_1 V) \geq \dim_{\mathbb{K}}(R_1 \operatorname{Gin}(VR)_n)$. So we can take $\operatorname{Gin}(VR)_n$ instead of V. Therefore it is enough to consider only stable vector spaces.

Let V be a stable vector-space in R_n.

Notation 3.14. *Write* $V = \bigoplus_{i=0}^{k} B_i(V)b^i \oplus \bigoplus_{j=1}^{l} C_j(V)c^j$, *where* $B_i(V) \subseteq \mathbb{K}[a,d]$ *and* $C_j(V) \subseteq \mathbb{K}[a,d]$ *are* $\mathbb{K}$*-subspaces.*
Let V *be generated by* $\{a^4, a^3b, a^3c, a^3d, a^2b^2, a^2bd\}$. *Here* $B_0(V) =< a^4, a^3d >, B_1(V) =< a^3, a^2d >, B_2(V) =< a^2 >$ *and* $B_i(V) = 0$ *for all* $i \geq 3$. $C_1(V) =< a^3 >$ *and* $C_j(V) = 0$ *for all* $j \geq 2$.

Let $\Gamma_i^b(V)$ *and* $\Gamma_i^c(V)$ *denote the dimension of* $B_i(V)$ *and* $C_i(V)$ *respectively.*

Define $\nu_b(V) = max\ \{i : B_i(V) \neq 0\}$. *Similarly we define* $\nu_c(V)$.

Define $\delta(V) = |\{i \geq 0 : B_i(V) \neq 0\}| + |\{i \geq 1 : C_i(V) \neq 0\}|$, *where* $|.|$ *denotes cardinality of the set.*

Lemma 3.15. $B_j(V)$ *and* $C_j(V)$ *are monomial subspaces of* $\mathbb{K}[a,d]$ *and have monomial basis consist of a lex-segment in variable* a, d *with respect to the graded lexicographic order with* $a \succ d$.

Proof. It is clear that $B_j(V)$ and $C_j(V)$ are monomial subspaces of $\mathbb{K}[a,d]$, as V is so. If $B_j(V) = \mathbb{K}$, there is nothing to prove.
Otherwise, if $B_j(V) \neq 0$, consider the last monomial, say $a^i d^k$ such that $i + k = n - j$, in the monomial basis of $B_j(V)$ with respect to the *lex* order in $\mathbb{K}[a,d]$. Hence $a^i b^j d^k \in V$. Since V is stable, for $i + 1 \leq i_0 \leq n$, $a^{i_0} b^j d^{n-i_0-j} \in V$. Therefore, for $i + 1 \leq i_0 \leq n$, $a^{i_0} d^{n-i_0-j} \in B_j(V)$. Similarly one can show that $C_j(V)$ has monomial basis consists of a *lex*-segment in variable a, d. $\square$

Observation: $\nu_b(R_1V) = \nu_b(V) + 1$ and $\nu_c(R_1V) = \nu_c(V) + 1$

Lemma 3.16. $\Gamma_i^b(R_1V) = \Gamma_i^b(V) + 1$, *for all* $i \leq \nu_b(R_1V)$ *and* $\Gamma_i^c(R_1V) = \Gamma_i^c(V) + 1$, *for all* $i \leq \nu_c(R_1V)$.

Proof. First observe that $B_0(R_1V) = (a,d)B_0(V) + adC_1(V) + adB_1(V)$ and for $i > 0$, $B_i(R_1V) = (a,d)B_i(V) + B_{i-1}(V) + adB_{i+1}(V)$.

Since $C_1(V)c \subseteq V$ and V is stable, $aC_1(V) \subseteq B_0(V) \subseteq V$, hence $adC_1(V) \subseteq (a,d)B_0(V)$. Since $B_{i+1}(V)b^{i+1} \subseteq V$ and V stable, we have $B_{i+1}(V)ab^i \subseteq V$ i.e. $B_{i+1}(V)a \subseteq B_i(V)$. Hence $B_{i+1}(V)ad \subseteq dB_i(V)$.

Suppose that $\Gamma^b_{i-1}(V) = 1$. Then by Lemma 3.15, $B_{i-1}(V)$ has monomial basis $\{a^{n-i+1}\}$. If $B_i(V) = 0$ i.e. $\Gamma^b_i(V) = 0$, we have the desired equality. If $B_i(V) \neq 0$, then $a^{n-i} \in B_i(V)$. So $(a,d)B_i(V) \supseteq B_{i-1}(V)$. Since $B_i(V)$ is *lex*-segment subspace in a, d we have the equality.

If $\Gamma^b_{i-1}(V) \geq 2$, then $B_{i-1}(V)$ is a subspace generated by, say, $\{a^{n-i+1}, a^{n-i}d, \cdots, a^{n-i-k}d^{k+1}\}$, for some $k \geq 0$. Hence $B_i(V)$ must contain a^{n-i}, $a^{n-i-1}d, \cdots, a^{n-i-k}d^k$. So $(a,d)B_i(V) \supseteq B_{i-1}(V)$. Hence in this case $B_i(R_1V) = (a,d)B_i(V)$. Since $B_i(V)$ is a *lex*-segment subspace in a, d we have first part of the lemma.

There is also similar expression for $C_i(R_1V)$. Similar calculations also hold for $C_i(R_1V)$. □

Proposition 3.17. *(i).* $\dim R_1V - \dim V = \delta(V) + 2$.
(ii). $\delta(R_1V) = \delta(V) + 2$.

Proof. (i). Immediate from Lemma 3.16. (ii). Follows from the above observation. □

By above proposition in order to show that $\dim_{\mathbb{K}}(R_1V) \geq \dim_{\mathbb{K}}(R_1V^{lex})$, it is enough to show that $\delta(V) \geq \delta(V^{lex})$ which will be shown in the following two propositions.

Given a stable vector space V, with its ordered monomial basis $\mathscr{B}$.
We define $\theta_1(V), \theta_2(V), \theta_3(V)$ as follows:
$\theta_1(V) :=$ maximal *lex*-segment of V.
$\theta_2(V) :=$ the segment starting from the monomial that comes just after $\theta_1(V)$ in the *lex* order(not in $\mathscr{B}$) to the monomial which comes after $\theta_1(V)$ in $\mathscr{B}$.
$\theta_3(V) := \mathscr{B} \setminus \theta_1(V)$.
Note that $\theta_1(V) \neq \emptyset$.

Example 3.18. *: Let V be the subspace of R_5 generated by $\{a^5, a^4b, a^4c, a^4d, a^3b^2, a^3bd, a^2b^3, a^2b^2d, ab^4\}$. Here $\theta_1(V) = \{a^5, a^4b, a^4c, a^4d, a^3b^2, a^3bd\}$, $\theta_2(V) = \{a^3c^2, a^3cd, a^3d^2\}$, $\theta_3(V) = \{a^2b^3, a^2b^2d, ab^4\}$.*

Proposition 3.19. *Let V be a stable subspace of R_n. If V is not a lex-segment subspace, there exists a stable vector space $V' \subseteq R_n$ such that $\dim V = \dim V'$ and $\delta(V') \leq \delta(V)$ and $|\theta_1(V')| > |\theta_1(V)|$.*

Proof. Observe that first element in the $\theta_3(V)$ is either a^ib^{n-i} or a^ic^{n-i}. For if not, it must be either $a^ib^jd^k$ or $a^ic^jd^k$ for some $i,\ j,\ k$ such that $i+j+k = n$ and $k > 0$.

If it is $a^ib^jd^k$, $a^ib^{j+1}d^{k-1} \in \theta_2(V)$ i.e. not in V. Since V is stable, $a^ib^{j+1}d^{k-1} \in V$, contradiction. By similar reasoning it can not be $a^ic^jd^k$.

Suppose that first element of $\theta_3(V)$ is a^ib^{n-i}. Now $a^ib^{n-i} \in V$ and V stable, hence $a^{i+1}b^{n-i-1} \in \theta_1(V)$. Hence the *lex*-segment ending with $a^{i+1}b^{n-i-1}$ is in V. Now we will explore the possibilities of the element that comes first in $\theta_2(V)$ in *lex* order. Note that it can be any element between $a^{i+1}b^{n-i-1}$ to a^ib^{n-i} in *lex* order.

Case1: If it is $a^{i+1}b^{n-i-1-k}d^k$ with $k \geq 1$, we replace the last element of $\mathscr{B}$ with $a^{i+1}b^{n-i-1-k}d^k$ and get a new stable vector space V' such that $\dim V = \dim V'$. Note that $\nu_c(V') \leq \nu_c(V)$. Since V is stable, $a^{i+1+k}b^{n-i-1-k} \in V$, we have $\nu_b(V') \leq \nu_b(V)$. Hence $\delta(V') \leq \delta(V)$. By construction $|\theta_1(V')| > |\theta_1(V)|$.

Case2: If it is $a^{i+1}d^{n-i-1}$, then similarly replacing the last element of $\mathscr{B}$ by $a^{i+1}d^{n-i-1}$ we get V' such that $\dim V = \dim V'$ and $|\theta_1(V')| > |\theta_1(V)|$. Also we have $\nu_c(V') \leq \nu_c(V)$. Since $\Gamma_0^b(V) \neq 0$, $\nu_b(V') \leq \nu_b(V)$. Therefore $\delta(V') \leq \delta(V)$.

Case3: If it is $a^{i+1}c^{n-i-1-k}d^k$ with $k \geq 1$, by replacing last element of $\mathscr{B}$ by $a^{i+1}c^{n-i-1-k}d^k$ we get stable vector space V' with $|\theta_1(V')| > |\theta_1(V)|$ and $\dim V = \dim V'$. Here we note that $\nu_b(V') \leq \nu_b(V)$. Since $a^{i+1}b^{n-i-1} \in \theta_1(V)$ and $k \geq 1$ we have $a^{i+1+k}c^{n-i-1-k} \in \theta_1(V) \subseteq V$. Hence $\nu_c(V') \leq \nu_c(V)$. Therefore $\delta(V') \leq \delta(V)$.

Case4: If it is $a^{i+1}c^{n-i-1}$, then the $\theta_2(V)$ is $\{a^{i+1}c^{n-i-1}, \ \cdots, \ a^{i+1}d^{n-i-1}\}$. Note that V does not contain monomials of the form $a^{i_1}c^{j_1}d^{k_1}$ with $i_1 \leq i$ and $i_1 + j_1 + k_1 = n$. Note also $a^ibd^{n-i-1} \notin V$, because if it belongs to V then that would imply that $a^{i+1}d^{n-i-1} \in V$. We denote the segment $\subseteq \{a^pb^q, \ \cdots, \ a^pbd^{q-1}\}$ that is in V by α_p, here by segment we mean list of consecutive monomials. Therefore by above observation $\alpha_i \subseteq \{a^ib^{n-i}, \ \cdots, \ a^ib^2d^{n-i-2}\}$, $\alpha_{i-1} \subseteq \{a^{i-1}b^{n-i+1}, \ \cdots, \ a^{i-1}b^3d^{n-i-2}\}$, $\cdots$, $\alpha_0 \subseteq \{b^n, \ ,\cdots \ b^{i+2}d^{n-i-2}\}$. Let i_0 be the smallest such that $\alpha_{i_0} \neq 0$. We replace α_{i_0} by the initial segment of $\{a^{i+1}c^{n-i-1}, \ \cdots, \ a^{i+1}d^{n-i-1}\}$ of equal size as α_{i_0}. Call the new monomial vector space V' which is stable and $\dim V = \dim V'$. By construction, $\delta(V') = \delta(V)$ and $|\theta_1(V')| > |\theta_1(V)|$.

Suppose that first element in $\theta_3(V)$ is a^ic^{n-i}, then $a^ic^{n-i} \in V$. Since V is stable, $a^{i+1}c^{n-i-1} \in \theta_1(V)$, hence *lex*-segment ending with $a^{i+1}c^{n-i-1}$ is in V. Now we see the possibilities of the element that comes first in $\theta_2(V)$ in *lex* order. Note that the element can be any element between $a^{i+1}c^{n-i-1}$ to a^ic^{n-i} in *lex* order. Again we do case by case analysis as before. When that first monomial in $\theta_2(V)$ is either $a^{i+1}c^{n-i-1-k}d^k$ with $k \geq 1$ or $a^{i+1}d^{n-i-1}$ or $a^ib^{n-i-k}d^k$ with $k \geq 1$ the arguments are similar as case 1,2,3 with necessary changes.

We will do the case analogous to the Case 4 i.e. when the first monomial in $\theta_2(V)$ is a^ib^{n-i}. In that case the $\theta_2(V)$ is $\{a^ib^{n-i}, \ \cdots, \ a^ibd^{n-i-1}\}$. Note that V does not contain monomials of the form $a^{i_1}b^{j_1}d^{k_1}$ with $i_1 \leq i$ and $i_1+j_1+k_1 = n$. Note also $a^i d^{n-i} \notin V$, because if not that would imply $a^i b^{n-i} \in V$. We denote the segment $\subseteq \{a^p c^q, \ \cdots, \ a^p d^q\}$ that is in V by σ_p. Therefore by above observation $\sigma_i \subseteq \{a^i c^{n-i}, \ \cdots, a^i c d^{n-i-1}\}$, $\sigma_{i-1} \subseteq \{a^{i-1}c^{n-i+1}, \ \cdots, a^{i-1}c^2 d^{n-i-1}\}, \ \cdots$. Let i_0 denote the smallest such that $\sigma_{i_0} \neq 0$. We replace σ_{i_0} by the initial segment of $\{a^i b^{n-i}, \ \cdots, \ a^i b d^{n-i-1}\}$ of equal size as σ_{i_0}. Call the new monomial vector space V' which is stable and $\dim V = \dim V'$. By construction, $\delta(V') = \delta(V)$ and $|\theta_1(V')| > |\theta_1(V)|$. □

Proposition 3.20. *For a stable vector space V, $\delta(V) \geq \delta(V^{lex})$.*

Proof. We will use induction on $t = |\theta_1(V^{lex})| - |\theta_1(V)|$. When $t = 0$, $V = V^{lex}$, the proposition follows. Now for $t > 0$, by Proposition 3.19 there exists a stable vector space V' of same dimension as V such that $\delta(V') \leq \delta(V)$ and $|\theta_1(V')| > |\theta_1(V)|$. Since V and V' have same dimension, $V'^{lex} = V^{lex}$. Again since $|\theta_1(V')| > |\theta_1(V)|$, then $|\theta_1(V^{lex})| - |\theta_1(V')| < t$. Therefore by induction $\delta(V') \geq \delta(V^{lex})$. By proposition 3.19, we know that $\delta(V) \geq \delta(V')$, hence the proposition follows. □

Proof of Theorem 3.1. (1). We will show if $V \subset R_n$ is a *lex*-segment subspace then $R_1 V$ is also a *lex*-segment subspace of R_{n+1} i.e. if a monomial $u \in V$, then we need to show that all monomials that come before ud in the *lex* order in R_{n+1} also belong to $R_1 V$.

Case 1: If $u = a^i b^j d^k \in V$, then the monomials that come before ud are of the form $a^{i'} b^{j'} d^{k'}$ with $i'+j'+k' = i+j+k+1$ and $(i', j', k') > (i, j, k+1)$ or $a^{i'} c^{j'} d^{k'}$ with $i' > i$ and $j' > 0$.

If $i' = i+1$, since V is *lex*-segment, $a^{i+1} c^{j'-1} d^{k'} \in V$. Hence $a^{i+1} c^{j'} d^{k'} \in R_1 V$. If $j' \neq 0$, then $a^{i+1} b^{j'-1} d^{k'} \in V$, giving $a^{i'} b^{j'} d^{k'} \in R_1 V$. If $k' \neq 0$, this case is similar to above. If $(j', k') = (0, 0)$, then $(j, k) = (0, 0)$. Hence $a^{i+1} \in R_1 V$.

If $i' > i+1$, then $a^{i'-1} b^{j'} d^{k'}, a^{i'-1} c^{j'} d^{k'} \in V$. Hence $a^{i'} b^{j'} d^{k'}, a^{i'} c^{j'} d^{k'} \in R_1 V$.

Case 2: $u = a^i c^j d^k$ case can be done in similar way as of Case 1.

(2) Now Proposition 3.17(i), Proposition 3.19 and Proposition 3.20 together give that for arbitrary stable monomial space V, $\dim R_1 V^{lex} \leq \dim R_1 V$. By Theorem 3.7 we know that for arbitrary subspace $V \subseteq R_n$, there exists a stable monomial space $\tilde{V}$ such that $\dim V = \dim \tilde{V}$ and $\dim R_1 \tilde{V} \leq \dim R_1 V$. Since $\dim V = \dim \tilde{V}$, we have $V^{lex} = \tilde{V}^{lex}$. Hence $\dim R_1 V^{lex} \leq \dim R_1 \tilde{V} \leq \dim R_1 V$. □

3.2 Graded Betti numbers over $\mathbb{K}[a, b, c, d]$

Theorem 3.21. *Assume characteristic of $\mathbb{K}$ is 0. Let $\epsilon : \mathcal{H}_R \longrightarrow \mathcal{I}_R$ be the poset embedding for R induced by the embedding order $\succ_{lex}$. Let I be a homogeneous R-ideal* *and I^ϵ be the image of H_I under ϵ. Let $\tilde{I}$ and $\tilde{I^\epsilon}$ be the preimages of I and I^ϵ in S respectively. $\beta^S_{i,j}(R/I) \leq \beta^S_{i,j}(R/I^\epsilon)$ for $i = 0, 1, 4$ and for all j. Hence*

$$\beta^S_{i,j}(\tilde{I}) \leq \beta^S_{i,j}(\tilde{I^\epsilon}) \text{ for } i = 0, 1, 4 \text{ and for all } j.$$

Discussion 3.22. Let $\epsilon : \mathcal{H}_R \longrightarrow \mathcal{I}_R$ be the embedding induced by lex and I be a homogeneous R-ideal. Then $I^\epsilon_n = {I_n}^{lex}$ for all n. Then $\beta^R_{1,j}(R/I) \leq \beta^R_{1,j}(R/I^\epsilon)$ [CK13, Remark 2.5]. For arbitrary R whether $\beta^R_{i,j}(R/I) \leq \beta^R_{i,j}(R/I^\epsilon)$ for all i and j is not known. In general, there are examples with $\beta^R_{i,j}(R/I) > \beta^R_{i,j}(R/I^\epsilon)$ (See[MP12]).

For homogeneous ideal I of R, by Theorem 3.7 we have $\operatorname{Gin}(I) = \operatorname{in}_w(gI)$, for all $g \in U$, is a stable ideal. Since g is an automorphism, graded Betti numbers of R/I and R/gI over S are equal. Let $\widetilde{gI}$ be the preimage of gI in S. Note that for all homogeneous R-ideal I, $\operatorname{in}_w(\tilde{I}) = \widetilde{\operatorname{in}_w(I)}$, where $\widetilde{\operatorname{in}_w(I)}$ denotes the preimage of $\operatorname{in}_w(I)$ in S. Hence $S/\operatorname{in}_w \widetilde{gI}$ and $R/\operatorname{in}_w(gI)$ are isomorphic as S-modules. Also R/gI and $S/\widetilde{gI}$ are isomorphic as S-modules. Therefore $\operatorname{Tor}^S_i(R/gI, \mathbb{K}) \simeq \operatorname{Tor}^S_i(S/\widetilde{gI}, \mathbb{K})$. Again by Theorem 2.28(3), we have graded Betti numbers of $S/\widetilde{gI}$ over S are smaller than or equal to those of $S/\operatorname{in}_w(\widetilde{gI})$ (Here P=gI and $M = \operatorname{in}_w(gI)$). Since $\operatorname{Gin}(I)^\epsilon = I^\epsilon$ in order to show that, for all homogeneous ideal I, graded Betti numbers of R/I over S are smaller than or equal to those of R/I^ϵ it is enough to consider only stable ideal.

Since a, b, c, d form a regular sequence for S, Koszul complex gives a graded S-free resolution of $\mathbb{K}$. So $\beta^S_i(R/I) = 0$, for all $i > 4$ and $\operatorname{Tor}^S_4(R/I, \mathbb{K}) = \operatorname{Soc}(R/I)(-4)$, considering S is standard graded with $\deg a = \deg b = \deg c = \deg d = 1$.

Proposition 3.23. *For all stable ideal I, $\beta^S_{1,j}(R/I) \leq \beta^S_{1,j}(R/I^\epsilon)$ for all j.*

Proof. Let $\tilde{I}$ and $\tilde{I^\epsilon}$ be the preimages of I and I^ϵ respectively in S. Since $R/I \simeq S/\tilde{I}$ as S-modules, $\operatorname{Tor}^S_i(R/I, \mathbb{K}) \simeq \operatorname{Tor}^S_i(S/\tilde{I}, \mathbb{K})$. We have an exact sequence of S-modules

$$0 \longrightarrow \tilde{I} \longrightarrow S \longrightarrow S/\tilde{I} \longrightarrow 0.$$

Tensoring with $\mathbb{K}$, we get corresponding long exact sequence in homology

$$\cdots \to \operatorname{Tor}^S_1(S, \mathbb{K}) \to \operatorname{Tor}^S_1(S/\tilde{I}, \mathbb{K}) \to \tilde{I} \otimes \mathbb{K} \to S \otimes \mathbb{K} \to S/\tilde{I} \otimes \mathbb{K} \to 0.$$

Since S is free over S, $\operatorname{Tor}^S_1(S, \mathbb{K}) = 0$. Hence $\operatorname{Tor}^S_1(S/\tilde{I}, \mathbb{K}) = \ker(\tilde{I} \otimes \mathbb{K} \longrightarrow S \otimes \mathbb{K}) = \tilde{I}/(a, b, c, d)\tilde{I}$. Therefore we have to show that $\dim_{\mathbb{K}}(\tilde{I}/(a, b, c, d)\tilde{I})_j \leq \dim_{\mathbb{K}}(\tilde{I^\epsilon}/(a, b, c, d)\tilde{I^\epsilon})_j$ for all j. We have an exact sequence

$$0 \longrightarrow S(-2) \overset{ad-bc}{\longrightarrow} \tilde{I} \longrightarrow I \longrightarrow 0.$$

Tensoring with $\mathbb{K}$, we get the long exact sequence in homology

$$\cdots \to \mathrm{Tor}_1^S(I, \mathbb{K}) \to \mathbb{K}(-2) \to \tilde{I}/(a,b,c,d)\tilde{I} \to I/(a,b,c,d)I \to 0.$$

Similarly, get long exact sequence for I^ϵ

$$\cdots \to \mathrm{Tor}_1^S(I^\epsilon, \mathbb{K}) \to \mathbb{K}(-2) \to \tilde{I}^\epsilon/(a,b,c,d)\tilde{I}^\epsilon \to I^\epsilon/(a,b,c,d)I^\epsilon \to 0.$$

Hence for each j, we get exact sequence for $\mathbb{K}$-vector spaces.

$$\to (\mathrm{Tor}_1^S(I, \mathbb{K}))_j \to (\mathbb{K}(-2))_j \to (\tilde{I}/(a,b,c,d)\tilde{I})_j \to (I/(a,b,c,d)I)_j \to 0.$$

and similar exact sequence for I^ϵ.

Note that since *lex* is an embedding order, $\dim_{\mathbb{K}}(R_1 I_j) \geq \dim_{\mathbb{K}}(R_1 I_j^\epsilon)$, for all j. Now for all j, $\dim_{\mathbb{K}}(I/(a,b,c,d)I)_j = \dim_{\mathbb{K}} I_j - \dim_{\mathbb{K}}(R_1 I_{j-1})$. Hence $\dim_{\mathbb{K}}(I/(a,b,c,d)I)_j \leq \dim_{\mathbb{K}}(I^\epsilon/(a,b,c,d)I^\epsilon)_j$, for all j.

Note also that the map $\mathbb{K}(-2) \xrightarrow{ad-bc} (\tilde{I}/(a,b,c,d)\tilde{I})$ is either zero or injective. Similarly for $\tilde{I}^\epsilon$ also. So for all j, $\dim_{\mathbb{K}}(\tilde{I}/(a,b,c,d)\tilde{I})_j \geq \dim_{\mathbb{K}}(I/(a,b,c,d)I)_j$ and $\dim_{\mathbb{K}}(\tilde{I}^\epsilon/(a,b,c,d)\tilde{I}^\epsilon)_j \geq \dim_{\mathbb{K}}(I^\epsilon/(a,b,c,d)I^\epsilon)_j$. If the above map is zero, $(\tilde{I}/(a,b,c,d)\tilde{I})_j \simeq (I/(a,b,c,d)I)_j$, for all j and the proposition follows. Now $\mathbb{K}(-2) \xrightarrow{ad-bc} (\tilde{I}/(a,b,c,d)\tilde{I})$ is injective if and only if $\mathbb{K}(-2)_2 \xrightarrow{ad-bc} (\tilde{I}/(a,b,c,d)\tilde{I})_2$ is injective because $\mathbb{K}(-2)_j = 0$ for $j \neq 2$. Then $ad - bc \notin (a,b,c,d)\tilde{I}_1$ and $\dim_{\mathbb{K}}(\tilde{I}/(a,b,c,d)\tilde{I})_2 = 1 + \dim_{\mathbb{K}}(I/(a,b,c,d)I)_2$. We will show that $\mathbb{K}(-2)_2 \longrightarrow (\tilde{I}^\epsilon/(a,b,c,d)\tilde{I}^\epsilon)_2$ is injective. If not, then that $ad - bc \in (a,b,c,d)\tilde{I}_1^\epsilon$ implies $a, b \in I_1^\epsilon$. Hence $\dim_{\mathbb{K}} I_1 \geq 2$. Since I is stable, either a, b or a, c are in I. Hence $ad - bc \in (a,b,c,d)\tilde{I}_1$, a contradiction. Therefore $\mathbb{K}(-2)_2 \longrightarrow (\tilde{I}^\epsilon/(a,b,c,d)\tilde{I}^\epsilon)_2$ is injective. Hence $\dim_{\mathbb{K}}(\tilde{I}^\epsilon/(a,b,c,d)\tilde{I}^\epsilon)_2 = 1 + \dim_{\mathbb{K}}(I^\epsilon/(a,b,c,d)I^\epsilon)_2$. Therefore $\dim_{\mathbb{K}}(\tilde{I}/(a,b,c,d)\tilde{I})_2 \leq \dim_{\mathbb{K}}(\tilde{I}^\epsilon/(a,b,c,d)\tilde{I}^\epsilon)_2$. □

Lemma 3.24. *For a monomial ideal I, $\mathrm{Soc}(R/I)$ is monomial.*

Proof. Let I be generated by the monomials $u_1, \cdots, u_n$. Let $f \in \mathrm{Soc}(R/I)_j$. Write $f = x_1 + \cdots + x_t$ as a sum of monomials. Now $af \in I_{j+1}$ implies $ax_1 + \cdots + ax_t = \sum f_i u_i$, where f_i's are homogeneous elements in R. Linear independence of monomials implies $ax_i \in (u_k) \subset I$, for some $k \in \{1, \cdots, n\}$. Similarly bx_i, cx_i, dx_i are in I. Therefore $x_i \in \mathrm{Soc}(R/I)$. □

Proposition 3.25. *For a stable ideal I,*

$$\dim_{\mathbb{K}}(\mathrm{Soc}(R/I)_i) \leq \dim_{\mathbb{K}}(\mathrm{Soc}(R/I^\epsilon)_i), \text{ for all } i.$$

Proof. For $i \in \mathbb{N} \cup \{0\}$ and a subset V of R_i, define

$$k_1(V) := |\{a^l b^m, a^l c^m \in V \mid l, m \in \mathbb{N} \cup \{0\}|,$$

$$k_2(V) := |\{a^l b^m d^n, a^l c^m d^n \in V \mid l, m \in \mathbb{N} \cup \{0\} \text{ and } n \geq 1\}|.$$

Note that if V is a stable subspace of R_i, then $k_1(V) = \delta(V)$ and $\dim_{\mathbb{K}}(V) = k_1(V) + k_2(V)$.

We now argue that $\dim_{\mathbb{K}} \operatorname{Soc}(R/I)_i = k_2(I_{i+1} \setminus R_1 I_i)$. First note that for any non-zero monomial x in $\operatorname{Soc}(R/I)_i$, $xd \in I_{i+1} \setminus R_1 I_i$, giving an injective map from $\operatorname{Soc}(R/I)_i$ to the set $\{a^l b^m d^n, a^l c^m d^n \in R_{i+1} \setminus R_1 I_i \mid l, m \in \mathbb{N} \cup \{0\} \text{ and } n \geq 1\}|$. In the other direction, suppose that $a^l b^m d^n \in I_{i+1} \setminus (R_1 I_i)$ with $n \geq 1$; since I is stable, we see that

$$a^{l+1} b^m d^{n-1}, a^l b^{m+1} d^{n-1}, a^{l+1} b^{m-1} d^n \in I_{i+1}.$$

Hence $a^l b^m d^{n-1} \in \operatorname{Soc}(R/I)_i$. A similar argument applies to $a^l c^m d^n \in I_{i+1} \setminus (R_1 I_i)$ with $n \geq 1$. Hence $\dim_{\mathbb{K}} \operatorname{Soc}(R/I)_i = k_2(I_{i+1} \setminus R_1 I_i)$. Similarly, since I^ϵ is stable, $\dim_{\mathbb{K}} \operatorname{Soc}(R/I^\epsilon)_i = k_2(I^\epsilon_{i+1} \setminus R_1 I^\epsilon_i)$.

We need to show that

$$k_2(I_{i+1} \setminus R_1 I_i) \leq k_2(I^\epsilon_{i+1} \setminus R_1 I^\epsilon_i).$$

Note that

$$k_2(I_{i+1}) = k_2(R_1 I_i) + k_2(I_{i+1} \setminus R_1 I_i) \text{ and}$$

$$k_2(I^\epsilon_{i+1}) = k_2(R_1 I^\epsilon_i) + k_2(I^\epsilon_{i+1} \setminus R_1 I^\epsilon_i).$$

Now $k_1(I_{i+1}) = \delta(I_{i+1}) \geq \delta(I^\epsilon_{i+1}) = k_1 I^\epsilon_{i+1}$, where second inequality follows from Proposition 3.20. Hence $k_2(I_{i+1}) \leq k_2 I^\epsilon_{i+1}$. Note that since I is stable, $R_1 I_i$ is also a stable vector space; hence by Proposition 3.17(ii), $\delta(R_1 I_i) = \delta(I_i) + 2$. Similarly, since I^ϵ is stable, $\delta(R_1 I^\epsilon_i) = \delta(I^\epsilon_i) + 2$. Hence by Proposition 3.17(i), we have

$$\begin{aligned}\dim(R_1 I_i) - \dim(R_1 I^\epsilon_i) &= \delta(R_1 I_i) - \delta(R_1 I^\epsilon_i) \\ &= k_1(R_1 I_i) - k_1(R_1 I^\epsilon_i).\end{aligned}$$

Therefore $k_2(R_1 I_i) = k_2(R_1 I^\epsilon_i)$. Hence the proposition.

$\square$

Proof of Theorem 3.21. Since $\mathrm{Tor}_0^S(R/I, \mathbb{K}) = \mathrm{Tor}_0^S(R/I^\epsilon, \mathbb{K}) = \mathbb{K}$, then $\beta_{0,j}^S(R/I) \leq \beta_{0,j}^S(R/I^\epsilon)$ for all j. The $i = 1$ and $i = 4$ cases follow from Discussion 3.22, Theorem 3.23 and Proposition 3.25. □

3.3 Graded Betti numbers over $\mathbb{K}[a, b, c, d]/(ad - bc)$

Theorem 3.26. *Assume characteristic of $\mathbb{K}$ is 0. Let $\epsilon : \mathcal{H}_R \longrightarrow \mathcal{I}_R$ be the poset embedding for R induced by the embedding order $\succ_{lex}$. Let I be a homogeneous R-ideal and I^ϵ be the image of H_I under ϵ.*

$$\beta_{i,j}^R(I) \leq \beta_{i,j}^R(I^\epsilon), \text{ for all } i, j.$$

Discussion 3.27. Using a similar argument as in Discussion 3.22 and in Theorem 2.28 (2), we see that the graded Betti numbers of I over R are smaller than or equal to those of $\mathrm{in}_w(gI)$, where $g \in U$ and U is as in the Theorem 3.5. So for proving Theorem 3.26 we again reduce to the case of stable ideals. Next we define the notion of linear resolution analogous for polynomial ring [cf.[HH99]].

Definition 3.28. Let I be a graded R-ideal. We say that I has a **linear resolution** if there exists an integer n such that $\beta_{i,i+j}^R(I) = 0$, for all i and j with $j \neq n$.

Note that if I has a linear resolution, then I is generated by homogeneous elements in R of the same degree.

Notation 3.29. *Let I be a stable R-ideal and* $\mathrm{Mon}(I)$ *be its minimal monomial generating set. Order the monomials in* $\mathrm{Mon}(I)$ *with respect to the lex order. Let f be the last monomial in* $\mathrm{Mon}(I)$ *with respect to the lex order. Let J denote the ideal generated by* $\mathrm{Mon}(I) \setminus \{f\}$*. Then we can write $I = J + (f)$.*

Lemma 3.30. *Let I be a stable R-ideal. We write $I = J + (f)$, as in the above Notation 3.29. Then $J : (f)$ is a monomial ideal generated by linear forms.*

Proof. Similar to Lemma 3.24, one can show that $J : (f)$ is a monomial ideal. If $I = (f)$, then there is nothing to prove. Hence we assume $J \neq (0)$. Therefore $f = a^i b^j d^k$ with $(j, k) \neq (0, 0)$ or $a^i c^j d^k$ with $(j, k) \neq (0, 0)$.
Case 1: If $f = a^i b^j d^k$ with $k \geq 1$, then $af = a^{i+1} b^j d^k = a^{i+1} b^j d^{k-1} d$. Since $a^{i+1} b^j d^{k-1} \succ_{lex} a^i b^j d^k$ and I is stable, $a^{i+1} b^j d^{k-1} \in J$; hence $a^{i+1} b^j d^{k-1} d \in J$. Similarly $bf, cf \in J$. Hence $(a, b, c) \subseteq J : (f)$.

We will show next that $J : (f) = (a, b, c)$. If possible, let $a^i b^j d^{k+l} \in J$ with $l > 0$. We choose l minimum such that $a^i b^j d^{k+l} \in J$. If $a^i b^j d^{k+l} = a^{i_1} b^{j_1} d^{k_1} a^{i_2} b^{j_2} d^{k_2}$, with $a^{i_1} b^{j_1} d^{k_1}$ is in the minimal generating set J, then $i_1 \leq i, j_1 \leq j, k_1 \leq k + l$. Note that $(i_1, j_1, k_1) < (i, j, k + l)$, as $a^i b^j d^{k+l}$ is not part of minimal monomial generating set of J. If $k_1 < k + l$, then $a^i b^j d^{k+l-1} \in J$, which contradicts minimality of l. If $k_1 = k + l$, then $(i_1, j_1) < (i, j)$. Since J is stable, we have $a^i b^j d^{k+l-1} \in J$ which again contradicts minimality of l. If $a^i b^j d^{k+l} = a^{i_1} b^{j_1} d^{k_1} a^{i_2} c^{j_2} d^{k_2}$ where $a^{i_2} c^{j_2} d^{k_2}$ is in the minimal generating set of J with $j_2 \neq 0$, then $j_1 = j + j_2$ and $i = i_1 + i_2 + j_2$. Since J is stable, $a^{i_2+j_2} d^{k_2} \in J$. Hence $a^i b^j d^{k_2} \in J$. But $k_2 < k + l$, which gives a contradiction. So $J : (f) = (a, b, c)$.

Case 2: If $f = a^i b^j$, with $j > 0$, then $af = a^{i+1} b^{j-1} b \in J$. Since I is stable, $a^{i+1} b^{j-1} \in J$. Again $cf = a^{i+1} b^{j-1} d \in J$. So $(a, c) \subseteq J : (f)$. Similar calculation as above shows that $J : (f) = (a, c)$.

Case 3: If $f = a^i c^j$, with $j > 0$, then it is easy to see that $(a, b) \subseteq J : (f)$. Similar calculation as in case 1 shows that $J : (f) = (a, b)$.

Case 4: If $f = a^i c^j d^k$, with $k > 0$, then similar calculation shows that $J : (f) = (a, b, c)$.

□

Proposition 3.31. *Let I be a stable R-ideal. Let t be the maximum degree of an element in its minimal monomial generating set.*
(i). Then $\operatorname{reg}^R(R/I) = t - 1$.
(ii). Write $I = J + (f)$, as in Notation 3.29. Then, $\beta^R_{i,i+j}(I) = \beta^R_{i,i+j}(J) + \beta^R_{i,i+j-t}(R/J : (f))$.

That is, for $j \neq t$, $\beta^R_{i,i+j}(I) = \beta^R_{i,i+j}(J)$ and $\beta^R_{i,i+t}(I) = \beta^R_{i,i+t}(J) + \beta^R_i(R/J : (f))$.

Proof. (i): We will prove if I is a stable R-ideal and t is the maximal degree of the minimal monomial generating set of I, $\operatorname{reg}^R(R/I) \leq t - 1$, hence $\operatorname{reg}^R(R/I) = t - 1$. We first check that the assertion holds for stable ideals generated by linear forms i.e., when $I = (a), (a, b), (a, c), (a, b, c)$ or (a, b, c, d). The minimal free R-resolution of $R/(a)$ is

$$0 \longrightarrow R(-1) \xrightarrow{a} R \longrightarrow 0.$$

Hence $R/(a)$ has regularity 0. The minimal free R-resolution of $R/(a, b)$ is periodic of periodicity 2:

$$\cdots \longrightarrow R^2(-3) \xrightarrow{\begin{bmatrix} c & d \\ -a & -b \end{bmatrix}} R^2(-2) \xrightarrow{\begin{bmatrix} b & d \\ -a & -c \end{bmatrix}} R^2(-1) \xrightarrow{\begin{bmatrix} a & b \end{bmatrix}} R \cdot$$

Hence $R/(a, b)$ has regularity 0. Similarly $R/(a, c)$ has regularity 0. For minimal free resolution of $R/(a, b, c)$: we consider the following complex of R-modules:

$$R^4(-3) \xrightarrow{\begin{bmatrix} a & -a & b & 0 \\ -b & 0 & 0 & d \\ c & c & d & 0 \\ 0 & -a & -b & -c \end{bmatrix}} R^4(-2) \xrightarrow{\begin{bmatrix} 0 & -c & -b & -d \\ -c & 0 & a & 0 \\ b & a & 0 & b \end{bmatrix}} R^3(-1) \xrightarrow{\begin{bmatrix} a & b & c \end{bmatrix}} R\cdot$$

It is easy to show that the above complex is exact. Depth of $R/(a,b,c) = 1$, as an R-module. Hence depth of the image of the map $R^4(-2) \longrightarrow R^3(-1)$ is 3. So the image of the map $R^4(-2) \longrightarrow R^3(-1)$ is a maximal Cohen-Macaulay module over Cohen-Macaulay ring R, hence it has a periodic minimal free resolution with periodicity 2 [Yos90, Chapter 7]. As f is quadratic, entries of the matrices in matrix factorization of f are linear. Hence $R/(a,b,c)$ has regularity 0.
By [Fro99] it is known that R is a Koszul ring and hence regularity of $R/(a,b,c,d)$ is 0.

For arbitrary stable ideals we use induction on the number of minimal monomial generators of I. When I is generated by single monomial i.e., $I = (a^t)$, the assertion is true. Write $I = J + (f)$, as in Notation 3.29. Then $t = \deg(f)$. Then we have an exact sequence of R-modules:

$$0 \longrightarrow (R/J:(f))(-t) \xrightarrow{f} R/J \longrightarrow R/I \longrightarrow 0.$$

By Lemma 3.30, $J:(f)$ is generated by linear monomials. By induction R/J has regularity $\leq t-1$, hence using long exact sequence of Tor modules one can show that $\operatorname{reg}^R(R/I) \leq t-1$.

(ii): Consider the exact sequence of R modules:

$$0 \longrightarrow (R/J:(f))(-t) \xrightarrow{f} R/J \longrightarrow R/I \longrightarrow 0.$$

Let $F_\bullet$ and $G_\bullet$ be a minimal graded free resolution of R/J and $R/J:(f)(-t)$ respectively. Since $R/J:(f)(-t)$ has t-linear resolution, for each i, $G_i \simeq R(-i-t)^{\beta_i^R(R/J:(f)(-t))}$. Since by (i) $\operatorname{reg}^R(R/J) \leq t-1$, F_i involves $R(-j)$ for only $j \leq i+t-1$. Hence the comparison map $G_i \xrightarrow{\phi_{f_i}} F_i$ is minimal. So the mapping cone of $\phi_f : G_\bullet \to F_\bullet$ gives a minimal free resolution of R/I. Therefore $\beta^R_{i,i+j}(I) = \beta^R_{i,i+j}(J) + \beta^R_{i,i+j-t}(R/J:(f))$.

□

Proof of Theorem 3.26. Let I be a stable ideal. Let I be generated minimally by the ordered monomials $\{f_1, \cdots, f_n\}$ with respect to the lex order and $f_l, f_{l+1}, \cdots, f_m$ be all the monomials of degree j in the minimal monomial generating set. Write $J_k = (f_1, \cdots, f_{k-1})$, for all k. Then by Proposition 3.31 (ii) we have

$$\beta^R_{i,i+j}(I) = \sum_{k=l}^{m} \beta_i^R(R/J_k : f_k)(-j).$$

First note that the number of monomials of degree j in the generating set is smaller than or equal to that of I^ϵ. We saw in the proof of Lemma 3.30 that, $R/J_k : f_k = (a,b),(a,c)$ *or* (a,b,c) and that depends on the pair (J_k, f_k). $\beta_i^R(R/(a,b)) = \beta_i^R(R/(a,c))$ and $\beta_i^R(R/(a,b)) \leq \beta_i^R(R/(a,b,c))$, for all i. So in order to show $\beta_{i,i+j}^R(I) \leq \beta_{i,i+j}^R(I^\epsilon)$ it is enough to show that number of monomials of the form $a^l b^m d^n$ with $n > 0$ or $a^l c^m d^n$ with $n > 0$ of degree j in the minimal monomial generating set of I^ϵ is more than or equal to that of I. Now let us look at how do we choose minimal monomial generating set for a monomial ideal I. Since I_1 is a monomial subspace, we take all its monomial basis, then we take all the monomials of $I_2 \setminus R_1 I_1$ and so on. So minimal monomial generators of I of degree j are the monomials in $I_j \setminus R_1 I_{j-1}$. In the proof of Proposition 3.25 we argue that the number of monomials of the form $a^l b^m d^n$ with $n > 0$ or $a^l c^m d^n$ with $n > 0$ in $I_j \setminus R_1 I_{j-1}$ is less than or equal to that of $I_j^\epsilon \setminus R_1 I_{j-1}^\epsilon$, for all j. Hence $\beta_{i,i+j}^R(I) \leq \beta_{i,i+j}^R(I^\epsilon)$ for all i and j. $\square$

3.4 Poset embedding for $\mathbb{K}[a,b,c]/(ac-b^2)$

Consider $R = \mathbb{K}[a,b,c]/(ac-b^2)$, where $\mathbb{K}$ is a field of arbitrary characteristic and a,b,c are indeterminates. In R we choose ac over b^2 i.e. all monomials of R are of the form $a^i b^j c^k$ with $j = 0,1$. Monomials of the form $a^i b^j c^k$, where $j = 0,1$ form a monomial basis for R, this can be seen using revlex order with $a \succ b \succ c$ and Theorem 15.3 of [Eis95]. Let $S = \mathbb{K}[a,b,c]$.

Theorem 3.32. *Let lex be the graded lexicographic order on monomials of R with $a \succ_{lex} b \succ_{lex} c$. Then lex is an embedding order for R.*

Discussion 3.33. Let w be a weight order on S where the weights of a,b,c are $(2,0),(1,1)$ and $(0,2)$ respectively. Consider the $\mathbb{K}$-algebra homomorphism $\phi : S \to \mathbb{K}[s,t]$ where $a \mapsto s^2, b \mapsto st, c \mapsto t^2$. The kernel of this map is generated by binomials i.e. $u - u'$, where u,u' are monomials in S with $w(u) = w(u')$ (by Lemma 2.27). Since $w(ac) = w(b^2)$, then ϕ induces a map $\tilde{\phi} : R \longrightarrow \mathbb{K}[s,t]$. A simple calculation shows that distinct monomials of R have distinct weights. Hence $\tilde{\phi}$ is injective onto its image. So R is a projective toric ring and induced weight order w on R is a monomial order. Also note that $\mathrm{in}_w(\tilde{I}) = \widetilde{\mathrm{in}_w(I)}$, where $\tilde{I}$ and $\widetilde{\mathrm{in}_w(I)}$ are the preimages of I and $\mathrm{in}_w(I)$ in S respectively. So for all homogeneous R-ideal I, we have $H_I = H_{\mathrm{in}_w(I)}$. Since w is a monomial order on R, $\mathrm{in}_w(I)$ is a monomial ideal. Now for an arbitrary $\mathbb{K}$-subspace V of R_n, We have $H_{VR} = H_{\mathrm{in}_w(VR)}$, where VR is the ideal generated by V. Since $\dim_{\mathbb{K}}(R_1 V) \geq \dim_{\mathbb{K}}(R_1(\mathrm{in}_w(VR))_n)$, we can take $(\mathrm{in}_w(VR))_n$ instead of taking V. Therefore without loss of generality we can assume V is a monomial subspace of R_n.

Lemma 3.34. *For all monomial vector space $V \subseteq R_n$, $\dim_{\mathbb{K}}(R_1V) \geq \dim_{\mathbb{K}}(R_1V^{lex})$, where V^{lex} is the lex-segment subspace of R_n of same dimension as of V.*

Proof. Let V be a monomial subspace of R_n. Let B denote its monomial basis ordered by $\succ_{lex}$. We want to calculate the monomial basis of R_1V. Note that $\{af, bf, cf : f \in B\}$ is the monomial basis of R_1V. Let $f = a^ic^k$, with $k \geq 1$ be a monomial in B, then R_1f is a monomial subspace of R_{n+1} with basis $a^{i+1}c^k, a^ibc^k, a^ic^{k+1}$. If $a^{i+1}c^{k-1} \in B$, then it comes before f in B and $a^{i+1}c^k = ca^{i+1}c^{k-1}$. In that case it has already been counted in the basis of R_1V. Again if $a^ibc^{k-1} \in B$, then it comes before f in B and $a^ibc^k = ca^ibc^{k-1}$. Similarly then it has already been counted in the basis of R_1V. But note that a^ic^{k+1} always contributes to the monomial basis of R_1V. For, if $a^ic^{k+1} \in R_1f'$, where f' is a monomial, then $f' = f(= a^ic^k)$ or $a^{i-1}c^{k+1}$ or $a^{i-1}bc^k$. But later two come after f in *lex* order. Similar calculation holds for $a^ibc^k \in B$. Therefore, if f is the first vector in B, then it always contributes 3 basis vectors for R_1V. Otherwise it contributes at most 3 and at least 1 basis vector for R_1V. Note that if B is *lex*-segment and f is not first vector in B, then f contributes exactly one basis vector for R_1V. So $\dim_{\mathbb{K}}(R^1V^{lex}) = 3 + 1 + 1 + \cdots + 1$, where number of $1's = \dim_{\mathbb{K}}(V) - 1$. Hence the lemma. □

Proof of Theorem 3.32. Similarly, as in the proof of Theorem 3.1 one can show that if $V \subset R_n$ is a *lex*-segment subspace then R_1V is also a *lex*-segment subspace of R_{n+1}. Hence condition (1) for embedding order follows. Condition (2) for embedding order follows from Lemma 3.34. □

3.5 Graded Betti numbers over $\mathbb{K}[a, b, c]$

Hereafter we assume that characteristic of $\mathbb{K}$ is 0.

Theorem 3.35. *Let $\epsilon : \mathcal{H}_R \longrightarrow \mathcal{I}_R$ be the poset embedding for R induced by the embedding order lex. Let I be a homogeneous R-ideal and I^ϵ be the image of I under ϵ. Let $\tilde{I}$ and $\tilde{I^\epsilon}$ be the preimages of I and I^ϵ in S respectively. Then $\beta^S_{i,j}(R/I) \leq \beta^S_{i,j}(R/I^\epsilon)$, for all i and j. Hence*

$$\beta^S_{i,j}(\tilde{I}) \leq \beta^S_{i,j}(\tilde{I^\epsilon}), \text{ for all } i \text{ and } j.$$

For $\lambda \in \mathbb{K}$, we define $\mathbb{K}$-algebra homomorphism:

$$\begin{aligned} g_\lambda : \mathbb{K}[a, b, c] &\longrightarrow \mathbb{K}[a, b, c], \text{ by} \\ a &\mapsto a \\ b &\mapsto \lambda a + b \\ c &\mapsto \lambda^2 a + 2\lambda b + c. \end{aligned}$$

Note that g_λ is an automorphism of $\mathbb{K}[a,b,c]$ and the ideal $(ac-b^2)$ is fixed under the action of g_λ. Hence it induces an automorphism of R.

Define $\mathfrak{U} = \{g_\lambda \mid \lambda \in \mathbb{K}\}$. Note that $\mathfrak{U}$ forms a group under composition. One can define diagonal automorphism of R similarly as we defined in section 1. A diagonal automorphism of R is of the form $\mathrm{diag}(T_1, T_2, T_2^2/T_1)$, where T_i's are non-zero scalars. Let $\mathfrak{B}$ be the group generated by diagonal automorphisms of R and $\mathfrak{U}$.

Similar to Discussion 3.4, we have a notion of monomial of $\bigwedge^t R_n$ and given a monomial order on R, we have an induced order on $\bigwedge^t R_n$. Also one can define *initial term* of an element $f \in \bigwedge^t R_n$ similarly as in Discussion 3.4.

The following two theorems are analogous to Theorem 3.5 and Theorem 3.7 with correspondingly analogous proofs.

Theorem 3.36. *Let I be a homogeneous ideal of R. There is a nonempty Zariski open set $U \subset \mathfrak{B}$ and a monomial ideal $J \subset R$ such that for all $g \in U$, $\mathrm{in}_w(gI) = J$, where w is the weight order defined in Discussion 3.33. For each $n \geq 0$, if J_n of J has dimension t, then $\bigwedge^t J_n$ is spanned by the greatest monomial of $\bigwedge^t R_n$ that appears in $\bigwedge^t(gI_n)$ with $g \in \mathfrak{B}$.*

Proof. Let $f_1, f_2, \cdots, f_t$ be a basis for I_n. Consider a matrix g whose entries are indeterminates $\lambda, T_i's$ such that if we put any value of $\lambda, T_i's$ from $\mathbb{K}$, $g \in \mathfrak{B}$. Then $g(f_1 \wedge \cdots \wedge f_t) = g(f_1) \wedge \cdots \wedge g(f_t)$ is a linear combination of monomials of $\bigwedge^t R_n$ with coefficients that are rational functions in λ, and T_i's. In that expression let $m = m_1 \wedge \cdots \wedge m_t$ be the first monomial with respect to the induced order on $\bigwedge^t R_n$ with a non-zero function, say $p_n(\lambda, T_1, T_2)$. Let U_n be the set of $g \in \mathfrak{B}$ such that $p_n(\lambda, T_1, T_2) \neq 0$. Then U_n is a nonempty Zariski open set. The degree-n part of the initial ideal of gI i.e. $\mathrm{in}_w(gI)_n$ will be generated by $m_1, \cdots, m_t$ if and only if $g \in U_n$. Let J_n be the subspace generated by $m_1, \cdots, m_t$.

Write $J = \bigoplus_{n=1}^{\infty} J_n$. To show J is an ideal, it is enough to show for each n, $R_1 J_n \subset J_{n+1}$. Since U_n is nonempty Zariski open and $\mathfrak{B}$ is irreducible, U_n is dense; so $U_n \cap U_{n+1} \neq \emptyset$. For $g \in U_n \cap U_{n+1}$, we have $\mathrm{in}_w(gI)_n = J_n$ and $\mathrm{in}_w(gI)_{n+1} = J_{n+1}$. Hence $R_1 J_n \subset J_{n+1}$. Note also that by construction J is a monomial ideal. Last statement of the theorem is clear by the definition of J.

Next we will show that $U = \bigcap_{n=1}^{\infty} U_n$ is a Zariski open set. It is enough to show that U is a finite intersection of U_n. For, being finite intersection of open sets, U is open and since each U_n is dense, U is nonempty. Suppose J is generated by forms of degree $\leq e$. We will show that $U = \bigcap_{n=1}^{e} U_n$. Let $g \in \bigcap_{n=1}^{e} U_n$, then $\mathrm{in}_w(gI_n) = J_n$ for all $n \leq e$. Thus $J \subseteq \mathrm{in}_w(gI)$. Since $\dim_{\mathbb{K}} J_n = \dim_{\mathbb{K}} I_n = \dim_{\mathbb{K}}(gI)_n$ for every n, we have $J = \mathrm{in}_w(gI)$. $\square$

With I and J as in the above theorem, we write $J := \mathrm{Gin}(I)$.

Definition 3.37. An ideal in R is said to be **$\mathfrak{U}$-stable** if it is fixed under the action of $\mathfrak{U}$.

Theorem 3.38. *Let I be a homogeneous ideal of R. Then $\mathrm{Gin}(I)$ is $\mathfrak{U}$-stable.*

Proof. Let U be as in the previous theorem. Replacing I by gI for some $g \in U$, we may assume by the previous theorem that $\mathrm{in}_w(I) = \mathrm{Gin}(I)$. Therefore we have to show that for all $g_\lambda \in \mathfrak{U}$, $g_\lambda(\mathrm{in}_w(I_n)) = \mathrm{in}_w(I_n)$ for all n.

We choose a basis $f_1, \cdots, f_t$ for I_n with $\mathrm{in}_w(f_1) > \cdots > \mathrm{in}_w(f_t)$. Let $f = f_1 \wedge \cdots \wedge f_t$ be the corresponding generator of the one dimensional subspace $\wedge^t I_n \subset \wedge^t R_n$. We have $\mathrm{in}_w(f) = \mathrm{in}_w(f_1) \wedge \cdots \wedge \mathrm{in}_w(f_t)$.

If $g_\lambda(\mathrm{in}_w(I_n)) \neq \mathrm{in}_w(I_n)$, then $g_\lambda \mathrm{in}_w(f) \neq \mathrm{in}_w(f)$. The terms of $g_\lambda \mathrm{in}_w(f)$ other than $\mathrm{in}_w(f)$ are all strictly greater than $\mathrm{in}_w(f)$. Let kx be one of these non-zero terms, where k is a non-zero scalar and x is monomial in $\wedge^t R_n$. We will show for a suitable diagonal automorphism T of R, x appears with non-zero coefficient in $g_\lambda T f$ which will contradict the last statement of the previous theorem. Hence $g_\lambda(\mathrm{in}_w(I_n)) = \mathrm{in}_w(I_n)$.

For each term $k' m_1 \wedge \cdots \wedge m_t \in \wedge^t R_n$, where $k' \in \mathbb{K}$, we define its weight to be the monomial $v = \prod m_i \in R$. Let $f_v \in \wedge^t R_n$ be the sum of all the terms of f having weight v, so that we have $f = \sum_v f_v$. Let v_0 be the weight of $\mathrm{in}_w(f)$. Here note that different terms of f may have the same weight, but $\mathrm{in}_w(f)$ is the unique term having weight v_0. If $T = \mathrm{diag}(T_1, T_2, T_2^2/T_1)$, where T_1, T_2 are non-zero scalar, is a diagonal automorphism of R, then

$$Tf = \sum_v v(T_1, T_2, T_2^2/T_1) f_v.$$

Thus

$$\begin{aligned} g_\lambda T f &= \sum_v g_\lambda(v(T_1, T_2, T_2^2/T_1) f_v) \\ &= \sum_v v(T_1, T_2, T_2^2/T_1) g_\lambda f_v \\ &= v_0(T_1, T_2, T_2^2/T_1) g_\lambda \in_w (f) + \sum_{v \neq v_0} v(T_1, T_2, T_2^2/T_1) g_\lambda f_v. \end{aligned}$$

Thus the coefficient of x in $g_\lambda T f$ has the form

$$h(T_1, T_2, T_2^2/T_1) := k v_0(T_1, T_2, T_2^2/T_1) + \sum_{v \neq v_0} k_v v(T_1, T_2, T_2^2/T_1),$$

where $k_v \in \mathbb{K}$ is the coefficient of x in $g_\lambda f_v$. Claim: $v_0(T_1, T_2, T_2^2/T_1)$ is a non-zero rational function. Consider the $\mathbb{K}$-algebra map $\mathbb{K}[a, b, c] \longrightarrow \mathbb{K}(T_1, T_2)$ sending

$a \mapsto T_1, b \mapsto T_2, c \mapsto T_2^2/T_1$. Note that image ring is a domain of dimension 2 as its transcendence degree is 2. So the kernel is a prime of height 1. Hence the kernel is principal. Clearly $ac - b^2$ is in the kernel and $ac - b^2$ is irreducible, hence prime. Therefore the kernel is precisely the ideal $(ac - b^2)$ and R is isomorphic to the image ring. Since v_0 is non-zero in R, $v_0(T_1, T_2, T_2^2/T_1)$ is a non-zero rational function. Since the term $kv_0(T_1, T_2, T_2^2/T_1)$ is non-zero, we see that h is non-zero rational function. Since $\mathbb{K}$ is infinite, we can find T_1, T_2 non-zero scalars such that h is non-zero. $\square$

Let $\epsilon : \mathcal{H}_R \longrightarrow \mathcal{I}_R$ be the poset embedding for R induced by the embedding order *lex*. Let I be a homogeneous R-ideal and I^ϵ be the image of I under ϵ.

Proposition 3.39. $\operatorname{Gin}(I) = I^\epsilon$.

Proof. Note that by definition $I_n^\epsilon = I_n^{lex}$.

Case 1: Let $a^i c^k \in \operatorname{Gin}(I)$. Then by above theorem $g_\lambda(a^i c^k) \in \operatorname{Gin}(I)$ for all $g_\lambda \in \mathfrak{U}$. Now $g_\lambda(a^i c^k) = a^i(\lambda^2 a + 2\lambda b + c)^k$. Note that for some general $\lambda \in \mathbb{K}$, all monomials that appear with non-zero coefficients in the expression of $g_\lambda(c^k)$ are those that come before c^k with respect to the *lex* order and c^k itself (Here we have used $\operatorname{char} \mathbb{K} = 0$). Hence the monomials that appear with non-zero coefficients in the expression of $g_\lambda(a^i c^k)$ are those that come before $a^i c^k$ with respect to the lex order and $a^i c^k$ itself. Since $\operatorname{Gin}(I)$ is a monomial ideal, those monomials that appear with non-zero constants in the expression for $g_\lambda(a^i c^k)$ belong to $\operatorname{Gin}(I)$.

Case 2: Let $a^i b c^k \in \operatorname{Gin}(I)$. Then $g_\lambda(a^i b c^k) \in \operatorname{Gin}(I)$. Again, for some general $\lambda \in \mathbb{K}$, all monomials that appear with non-zero coefficients in the expression of $g_\lambda(a^i b c^k)$ are those that come before $a^i b c^k$ with respect to the *lex* order and $a^i b c^k$ itself ($\operatorname{char} \mathbb{K} = 0$ is again used here). Since $\operatorname{Gin}(I)$ is monomial, those monomials that appear with non-zero coefficients in the expression of $g_\lambda(a^i b c^k)$ belong to $\operatorname{Gin}(I)$. Hence the proposition.

$\square$

Proof of Theorem 3.35. Let I be a homogeneous R-ideal. Let $\tilde{I}$ and $\widetilde{gI}$ denote the preimages of I and gI in S respectively, where $g \in U$ and U as in the Theorem 4.3. Since g is an isomorphism, $\operatorname{Tor}_i^S(R/I, \mathbb{K}) \simeq \operatorname{Tor}_i^S(R/gI, \mathbb{K})$ for all i. Now for all homogeneous R-ideal I, $R/I \simeq S/\tilde{I}$ as S-module. So $\operatorname{Tor}_i^S(\underbrace{S/\tilde{I}, \mathbb{K}}) \simeq \operatorname{Tor}_i^S(\underbrace{S/\widetilde{gI}}, \mathbb{K})$ for all i. Also note that for all homogeneous R-ideal, $\operatorname{in}_w(\tilde{I}) = \widetilde{\operatorname{in}_w(I)}$, where $\widetilde{\operatorname{in}_w(I)}$ denotes the preimage of $\operatorname{in}_w(I)$ in S. Hence for all i and j, $\beta_{i,j}^S(R/I) = \beta_{i,j}^S(S/\tilde{I}) \le \beta_{i,j}^S(S/\widetilde{\operatorname{Gin}(I)}) = \beta_{i,j}^S(R/\operatorname{Gin}(I)) = \beta_{i,j}^S(R/I^\epsilon)$, second inequality follows from Theorem 2.28 (3) and the last equality follow from the previous proposition. Hence we have the theorem. $\square$

Part II

F-rationality of Rees algebras

Chapter 4

Preliminaries

All rings are commutative with identity and noetherian unless otherwise specified.

4.1 Excellent rings

Excellent rings form a subclass of noetherian rings with many good properties that finitely generated algebras over fields and their localizations have. Typically, noetherian rings arising in algebraic geometry, number theory and several complex variables are excellent. Before going to its definition we introduce some notions.

Definition 4.1. A ring R is called *catenary* if for all prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ of R, all saturated chains of prime ideals joining $\mathfrak{p}$ and $\mathfrak{q}$ have the same length. A ring R is called *universally catenary* if every finitely generated R-algebra is catenary.

Example 4.2 ([Eis95, Corollary 13.6])**.** *Every finitely generated algebra over a field is universally catenary.*

It is easy to see that if R is universally catenary, then every localization of R, every homomorphic image of R and every finitely generated R-algebra is also universally catenary.

Definition 4.3. Let K be a field. A noetherian K-algebra R is called *geometrically regular* over K if for every finite algebraic extension L of K, $L \otimes_K R$ is regular.

This condition is equivalent to the condition that every finite purely inseparable field extension L of K, $L \otimes_K R$ is regular. Note that if R is geometrically regular, then R is regular (take $L = K$).

For a ring homomorphism $\psi : R \longrightarrow S$, we get a map $\psi^* : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ by $\mathfrak{q} \mapsto \mathfrak{q} \cap R$. For any prime $\mathfrak{p} \in \operatorname{Spec} R$, ${\psi^*}^{-1}(\mathfrak{p})$, called the fibre over $\mathfrak{p}$, is $(\operatorname{Spec} S) \otimes_R \kappa(\mathfrak{p})$, where $\kappa(\mathfrak{p})$ denotes the field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Via the natural map $\kappa(\mathfrak{p}) \longrightarrow \kappa(\mathfrak{p}) \otimes_R S$, $\kappa(\mathfrak{p}) \otimes_R S$ becomes a $\kappa(\mathfrak{p})$-algebra.

Definition 4.4. A homomorphism $R \longrightarrow S$ of noetherian rings is *geometrically regular* if it is flat and for each prime $\mathfrak{p}$ in R, the fibre ring $\kappa(\mathfrak{p}) \otimes_R S$ is geometrically regular over $\kappa(\mathfrak{p})$, where the $\kappa(\mathfrak{p})$-algebra structure comes from the natural map $\kappa(\mathfrak{p}) \longrightarrow \kappa(\mathfrak{p}) \otimes_R S$.

Definition 4.5. A noetherian ring R is called *excellent* if it is universally catenary, for every local ring $R_\mathfrak{p}$ of R, the map $R_\mathfrak{p} \longrightarrow \hat{R}_\mathfrak{p}$ is geometrically regular and for every finitely generated R-algebra S, the regular locus $\{\mathfrak{p} \in \operatorname{Spec} S : S_\mathfrak{p} \text{ is regular}\}$ is Zariski-open.

Theorem 4.6 ([Mat80, (28.P), Theorem 68, Theorem 74])**.** *All complete noetherian local rings are excellent.*

In view of the above theorem, we have the following examples.

Example 4.7. *All fields are excellent.*

Example 4.8. *The rings of convergent power series in a finite number of variables over $\mathbb{R}$ or $\mathbb{C}$ are excellent.*

Proofs of the following results can be found in [Mat80, Chapter 13].

Theorem 4.9. *Let R be an excellent ring. Then every localization of R, every homomorphic image of R and every finitely generated R-algebra is excellent. Hence every algebra essentially of finite type over R is excellent.*

Since fields are excellent, then by above theorem finitely generated algebras over a field are excellent.

Theorem 4.10. *Let R be an excellent ring.*
(1) *If R is local and reduced, then its completion $\hat{R}$ is reduced.*
(2) *If R is reduced, then the normalization of R is module-finite over R.*
(3) *If R is local and normal, then $\hat{R}$ is normal.*
(4) *If R is local and equidimensional, then $\hat{R}$ is equidimensional.*

Note that completion of an excellent local domain need not be a domain. For example consider $R = \mathbb{C}[x, y]/(y^2 - x^2 - x^3)$. This is a domain because $x^2 + x^3$ is not a square in $\mathbb{C}[x, y]$. Let $\mathrm{m} = (x, y)$, then R_m is also a local domain. In $\hat{R}_\mathrm{m} \simeq \mathbb{C}[[x, y]]/(y^2 - x^2 - x^3)$, $x^2 + x^3$ becomes a square, as $(1 + x)^{1/2}$ exists. Hence $\hat{R}_\mathrm{m}$ is not a domain.

4.2 Local cohomology

Our references for local cohomology are [Gro65],[BS13], [ILL$^+$07].
Let R be a noetherian ring and I be an R-ideal.

Let M be an R-module. Define

$$\Gamma_I(M) = \{m \in M : I^t m = 0 \text{ for some } t \in \mathbb{N}\}.$$

It is easy to see that $\Gamma_I(M)$ is a submodule of M. An R-module M is called *I-torsion* if for $m \in M$, there exists a positive integer t such that $I^t m = 0$. If M is I-torsion then $\Gamma_I(M) = M$. Given an R-module map $\psi : M \longrightarrow M'$, we have an R-module map $\Gamma_I(\psi) : \Gamma_I(M) \longrightarrow \Gamma_I(M')$. It is easy to see that if ϕ is an R-module map from $M' \to M''$ then $\Gamma_I(\psi \circ \phi) = \Gamma_I(\psi) \circ \Gamma_I(\phi)$ and $\Gamma_I(id_M) = id_{\Gamma_I(M)}$. Hence $\Gamma_I(-)$ is a functor called *I-torsion functor*. One can show that the functor $\Gamma_I()$ is a left exact functor. Its i-th right derived functor is called i-th *local cohomology* functor denoted by $H_I^i()$. We recall that $H_I^i(M) := H^i(\Gamma_I(\mathcal{I}^\bullet))$, where $\mathcal{I}^\bullet$ is an injective resolution of M. Note that $H_I^i(M)$ has an induced R-module structure from $\mathcal{I}^\bullet$ and $H_I^i(M)$ is called i-th *local cohomology* module of M with support in I.

Now we describe other equivalent definitions of local cohomology.
Local cohomology can be defined as $\varinjlim_t \operatorname{Ext}_R^i(R/I^t, M)$ ([ILL$^+$07, Theorem 7.8]). Any sequence of ideals cofinal with the powers of I may be used instead of $\{I^t \ : \ t \geq 1\}$. Let $I = (x_1, \cdots, x_n)$, then collection of ideals $\{(x_1^{p^e}, \cdots, x_n^{p^e}) \ : \ e \geq 1\}$ cofinal with the $\{I^t \ : \ t \geq 1\}$.

Alternatively, local cohomology can be computed via the Čech complex ([ILL$^+$07, Construction 7.12]). For $f \in R$, we define the Čech complex $\check{C}^\bullet(f; R)$, to be the complex $0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow 0$, where $C^0 = R$, $C^1 = R_f$ and the map between them is the canonical map $R \longrightarrow R_f$ sending $r \mapsto \frac{r}{1}$, $r \in R$. If $\underline{f}$ is a sequence of elements $f_1, \ldots, f_n$ in R, we define $\check{C}^\bullet(\underline{f}; R)$ to be the tensor product of the n complexes $\check{C}^\bullet(f_i; R)$, i.e.,

$$\check{C}^\bullet(\underline{f}; R) := \check{C}^\bullet(f_1; R) \otimes_R \check{C}^\bullet(f_2; R) \otimes_R \cdots \otimes_R \check{C}^\bullet(f_n; R).$$

We define $\check{C}^\bullet(\underline{f}; M) := \check{C}^\bullet(\underline{f}; R) \otimes_R M$. Note that $\check{C}^\bullet(\underline{f}, M)$ is of the form:

$$0 \longrightarrow M \longrightarrow \bigoplus_i M_{f_i} \longrightarrow \bigoplus_{i<j} M_{f_i f_j} \longrightarrow \cdots \longrightarrow M_{f_1 \cdots f_n} \longrightarrow 0.$$

The cohomology of this complex turns out to be $H_I^\bullet(M)$, where $I = (f_1, \ldots, f_n)$, [ILL$^+$07, Theorem 7.13].

The following are few basic properties of local cohomology:

Proposition 4.11. *Let M be an R-module, I, J be ideals of R and $i \in \mathbb{N} \cup \{0\}$*
(1) *One has $H_I^0(M) = \Gamma_I(M)$ and $H_I^i(M)$ is I-torsion for all i.*
(2) *If $\operatorname{rad} I = \operatorname{rad} J$, then $H_I^i(M) = H_J^i(M)$ for all i.*
(3) *An exact sequence of R-modules*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

induces an exact sequence in local cohomology

$$\cdots \longrightarrow H_I^i(M') \longrightarrow H_I^i(M) \longrightarrow H_I^i(M'') \longrightarrow H_I^{i+1}(M') \longrightarrow \cdots .$$

(4) *If S is a multiplicative set of R, then*

$$H_I^i(S^{-1}M) \simeq S^{-1}H_I^i(M).$$

(5) *If $R \longrightarrow S$ is a ring homomorphism and N is an S-module, then*

$$H_I^i(N) = H_{IS}^i(N).$$

(6) *If $R \longrightarrow S$ is flat, then there is a natural isomorphism of S-modules*

$$S \otimes_R H_I^i(M) \simeq H_{IS}^i(S \otimes_R M).$$

(7) *If M is finitely generated, then*

$$\operatorname{depth}_R(I, M) = \inf\{i : H_I^i(M) \neq 0\}.$$

(8)*(Grothendieck) If (R, m) is local and M is finitely generated R-module, then*

$$\dim_R(M) = \sup\{i : H_{\mathrm{m}}^i(M) \neq 0\}.$$

Note that if (R, m) is a d-dimensional local ring and M is finitely generated R-module, then M is Cohen-Macaulay if and only if

$$\begin{aligned} H_{\mathrm{m}}^i(M) &\neq 0 \quad \text{if } i = d \\ &= 0 \quad \text{if } i \neq d. \end{aligned}$$

The following proposition is given as an exercise in [ILL$^+$07] and also in [BS13], we include its proof for the sake of completeness:

Proposition 4.12. *Let I be an R-ideal and x be an element in R. There is an exact sequence*

$$\cdots \longrightarrow H^i_{I+Rx}(R) \longrightarrow H^i_I(R) \longrightarrow H^i_{I_x}(R_x) \longrightarrow H^{i+1}_{I+Rx}(R) \longrightarrow \cdots .$$

Proof. Let I be generated by $f_1, \ldots, f_n$. We write $\underline{f}$ for $f_1, \ldots, f_n$. Then by definition

$$\check{C}^\bullet(\underline{f}, x; R) = \check{C}^\bullet(\underline{f}; R) \otimes_R \check{C}^\bullet(x; R).$$

So for each i,

$$(\check{C}^i(\underline{f}, x; R)) = (\check{C}^i(\underline{f}; R)) \otimes_R R \oplus (\check{C}^{i-1}(\underline{f}; R)) \otimes_R R_x.$$

We also know that $\check{C}^\bullet(\underline{f}; R) \otimes_R R_x = \check{C}^\bullet(\underline{f}; R_x)$. It is easy to see that the following diagram commutes and the rows are split-exact.

$$\begin{tikzcd}
0 \arrow[r] & \check{C}^{i-1}(\underline{f}; R_x) \arrow[r] \arrow[d] & \check{C}^i(\underline{f}, x; R) \arrow[r] \arrow[d] & \check{C}^i(\underline{f}; R) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \check{C}^i(\underline{f}; R_x) \arrow[r] & \check{C}^{i+1}(\underline{f}, x; R) \arrow[r] & (\check{C}^{i+1}(\underline{f}; R) \arrow[r] & 0
\end{tikzcd}$$

Hence we have a short exact sequence of complexes:

$$0 \longrightarrow \check{C}^\bullet(\underline{f}; R_x)[-1] \longrightarrow \check{C}^\bullet(\underline{f}, x; R) \longrightarrow \check{C}^\bullet(\underline{f}; R) \longrightarrow 0.$$

Therefore we get the desired long exact sequence:

$$\cdots \longrightarrow H^i_{I+Rx}(R) \longrightarrow H^i_I(R) \longrightarrow H^i_{I_x}(R_x) \longrightarrow H^{i+1}_{I+Rx}(R) \longrightarrow \cdots .$$

$\square$

Discussion 4.13. Note that the natural map $R \to R_x$ induces a map on Čech complexes

$$\check{C}^i(\underline{f}; R) \to \check{C}^{i-1}(\underline{f}; R_x).$$

One can see that in the proof of above proposition $\check{C}^\bullet(\underline{f}, x; R)$ is the mapping cone of $\check{C}^i(\underline{f}; R) \to \check{C}^{i-1}(\underline{f}; R_x)$. Hence the connecting morphism $H^i_I(R) \to H^i_{I_x}(R_x)$ is the induced map from $R \to R_x$.

Definition 4.14. Let (R, m) be a d-dimensional positively graded algebra over a local ring with unique homogeneous maximal ideal m. Then

$$a_i(R) := \max\{k \mid [H^i_{\mathrm{m}}(R)]_k \neq 0\}.$$

Notation 4.15. *We write $a(R)$ for $a_d(R)$.*

Note that since for each i, $H^i_{\mathfrak{m}}(R)$ is Artinian, $a_i(R)$ exists. Note also that if $R = K[x_1, \cdots, x_d]/(f_1, \cdots, f_m)$, where K is field, and $\{f_1, \cdots, f_m\}$ is a regular sequence; then $a(R) = \sum \deg(f_i) - \sum \deg(X_i)$.

Theorem 4.16 ([Har77, Theorem 5.2])**.** *Let X be a projective scheme over a noetherian ring R, and $\mathcal{O}_X(1)$ be a very ample sheaf on X over* $\operatorname{Spec} R$*. Let $\mathcal{F}$ be coherent sheaf on X. Then*
(i) for each $i \geq 0$, $H^i(X, \mathcal{F})$ is a finitely generated R-module.
(ii) there is an integer n_0 depending on $\mathcal{F}$ such that for all $i > 0$ and each $n \geq n_0$, $H^i(X, \mathcal{F}(n)) = 0$.

4.3 Local cohomology and the Frobenius endomorphism

Let R be a ring of prime characteristic $p > 0$. Define $F_R : R \longrightarrow R$ by $r \mapsto r^p$ is a ring homomorphism as $(r_1 + r_2)^p = r_1^p + r_2^p$, called *Frobenius endomorphism*. For $g \in R$, Frobenius homomorphism of R induces an endomorphism of R_g denoted by F_{R_g} also called *Frobenius endomorphism* on R_g such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r] \arrow[d, "F_R"] & R_g \arrow[d, "F_{R_g}"] \\ R \arrow[r] & R_g. \end{tikzcd}$$

Sometimes we ignore the subscript R in F_R when the ring R in the context is clear. Hence for $f_1, \ldots, f_n \in R$, we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d, "F"] & \bigoplus_i R_{f_i} \arrow[r] \arrow[d, "F"] & \cdots \arrow[r] & R_{f_1 \cdots f_n} \arrow[r] \arrow[d, "F"] & 0 \\ 0 \arrow[r] & R \arrow[r] & \bigoplus_i R_{f_i} \arrow[r] & \cdots \arrow[r] & R_{f_1 \cdots f_n} \arrow[r] & 0. \end{tikzcd}$$

In other words, we have a map of complexes of groups $\check{C}^\bullet(\underline{f}; R) \xrightarrow{F} \check{C}^\bullet(\underline{f}; R)$, where $\underline{f}$ denotes the sequence of elements $f_1, \ldots, f_n$. Hence it induces a homomorphism $F : H^i_I(R) \to H^i_I(R)$ and also called the *Frobenius* map, where $I = (f_1, \ldots, f_n)$. Let $\eta = [(\cdots, \frac{r_{j_1 \cdots j_i}}{(f_{j_1} \cdots f_{j_i})^k}, \cdots)] \in H^i_I(R)$, where $\{j_i, \ldots, j_i\} \subseteq \{1, \ldots, n\}$ and $(\cdots, \frac{r_{j_1 \cdots j_i}}{(f_{j_1} \cdots f_{j_i})^k}, \cdots)$ is a cycle in $C^i(\underline{f}; R)$ and $[(\cdots, \frac{r_{j_1 \cdots j_i}}{(f_{j_1} \cdots f_{j_i})^k}, \cdots)]$ denotes its image in $H^i_I(R)$. Then $F(\eta) =$

$[(\cdots, \frac{r^p_{j_1 \ldots j_i}}{(f_{j_1} \cdots f_{j_i})^{pk}}, \cdots)] \in H^i_I(R)$. The map F on $H^i_I(R)$ is independent of choice of generators of I.

Discussion 4.17. Let R, x and I be as in the Proposition 4.12. One can see that the following diagram commutes where F are for the respective Čech complexes.

$$\begin{tikzcd} 0 \arrow[r] & \check{C}^\bullet(\underline{f}; R_x)[-1] \arrow[r] \arrow[d, "F"] & \check{C}^\bullet(\underline{f}, x; R) \arrow[r] \arrow[d, "F"] & \check{C}^\bullet(\underline{f}; R) \arrow[r] \arrow[d, "F"] & 0 \\ 0 \arrow[r] & \check{C}^\bullet(\underline{f}; R_x)[-1] \arrow[r] & \check{C}^\bullet(\underline{f}, x; R) \arrow[r] & \check{C}^\bullet(\underline{f}; R) \arrow[r] & 0. \end{tikzcd}$$

Hence it induces a commutative diagram in homology:

$$\begin{tikzcd} \cdots \arrow[r] & H^i_{I+Rx}(R) \arrow[r] \arrow[d, "F"] & H^i_I(R) \arrow[r] \arrow[d, "F"] & H^i_{I_x}(R_x) \arrow[r] \arrow[d, "F"] & H^{i+1}_{I+Rx}(R) \arrow[r] \arrow[d, "F"] & \cdots \\ \cdots \arrow[r] & H^i_{I+Rx}(R) \arrow[r] & H^i_I(R) \arrow[r] & H^i_{I_x}(R_x) \arrow[r] & H^{i+1}_{I+Rx}(R) \arrow[r] & \cdots . \end{tikzcd}$$

4.4 Rees algebras and blow-up

Basics on Rees algebra can be found in [HS06, Chapter 5].

Convention: For an ideal I of a ring R, $I^n = R$ for $n \leq 0$.

Notation: $\operatorname{Min} R$ denotes the set of minimal primes of R.

Definition 4.18. Let R be a ring, I be an ideal of R and t be an indeterminate over R. The *Rees algebra of I* is graded subring of $R[t]$, denoted by $R[It]$ and defined by $\{\sum_{i=0}^{n} r_i t^i \mid r_i \in I^i, n \in \mathbb{N} \cup \{0\}\} = \bigoplus_{n \geq 0} I^n t^n$.

The *extended Rees algebra of I* is a graded subring of $R[t, t^{-1}]$, denoted by $R[It, t^{-1}]$ and defined as $\{\sum_{i=-n'}^{n} r_i t^i \mid r_i \in I^i; n, n' \in \mathbb{N} \cup \{0\}\} = \bigoplus_{n \in \mathbb{Z}} I^n t^n$.

Theorem 4.19 ([HS06, Theorem 5.1.4])**.** (1) *Minimal primes of $R[It]$ and $R[It, t^{-1}]$ are the contracted minimal prime ideals of $R[t]$ and $R[t, t^{-1}]$ respectively. More precisely,*

$$\operatorname{Min} R[It] = \{\mathfrak{p}R[t] \cap R[It] \mid \mathfrak{p} \in \operatorname{Min} R\}$$

and

$$\operatorname{Min} R[It, t^{-1}] = \{\mathfrak{p}R[t, t^{-1}] \cap R[It, t^{-1}] \mid \mathfrak{p} \in \operatorname{Min} R\}.$$

(2) *If* $\dim R$ *is finite, then*

$$\begin{aligned}\dim R[It] &= \dim R + 1 \quad \textit{if } I \not\subseteq \mathfrak{p} \textit{ for some prime } \mathfrak{p} \textit{ with } \dim(R/\mathfrak{p}) = \dim R,\\ &= \dim R \qquad\quad \textit{otherwise.}\end{aligned}$$

$\dim R[It, t^{-1}] = \dim R + 1.$

Definition 4.20. The *associate graded ring* of I is denoted by $\mathrm{gr}_I(R)$ and defined as $\bigoplus_{n\geq 0}(I^n/I^{n+1})$.

Note that $\mathrm{gr}_I(R) = R[It]/IR[It] = R[It, t^{-1}]/t^{-1}R[It, t^{-1}]$.

Theorem 4.21 ([HS06, Theorem 5.1.6])**.** *If* (R, m) *is local and* $I \subseteq \mathrm{m}$*, then* $\dim \mathrm{gr}_I(R) = \dim R$*.*

Theorem 4.22 ([GS82, Theorem 1.1 and Remark 3.10 and equations (*) and(**) on page 203])**.** *Suppose* (R, m) *is a Cohen-Macaulay local ring. Let* I *be an* m*-primary ideal of* R*. Then the following are equivalent:*
1. *The Rees algebra* $R[It]$ *is Cohen-Macaulay.*
2. *The associated graded ring* $\mathrm{gr}_I(R)$ *is Cohen-Macaulay and* $a(\mathrm{gr}_I(R)) < 0$*.*

Definition 4.23. Let R be a ring and I be an R-ideal, then the *blow-up* of $\operatorname{Spec} R$ along the sheaf of ideals I is $\operatorname{Proj} R[It]$.

Notation 4.24. *Let* R *and* I *be as above, and* $\pi : \operatorname{Proj} R[It] \to \operatorname{Spec} R$ *be the natural map, write* $\mathcal{U} = \operatorname{Spec} R \setminus \operatorname{Spec}(R/I)$*,* $X = \operatorname{Proj} R[It]$*,* $\mathscr{R} := R[It]$*,* $\mathcal{O}_X(n) = \widetilde{\mathscr{R}(n)}$*, and* $I\mathcal{O}_X := \mathrm{image}(I \otimes_R \mathcal{O}_X \to \mathcal{O}_X)$*.*

Theorem 4.25 ([Har77, Chapter II, Proposition 7.3])**.** (1) $I\mathcal{O}_X$ *is invertible.*
(2) $\pi : \pi^{-1}(\mathcal{U}) \to \mathcal{U}$ *is an isomorphism.*

The closed subscheme defined by $I\mathcal{O}_X$ is $\operatorname{Proj} \mathrm{gr}_I(R)$ and is denoted by E.

The following lemma is well-known. One can look at [ILL+07, Theorem 13.21], where it is proved when base ring is a field, but one can see same proof will work when base ring is not a field.

Lemma 4.26. *With notation as in Notation 4.24, there is an exact sequence of graded* $\mathscr{R}$*-modules:*

$$0 \longrightarrow H^0_{\mathscr{R}_+}(\mathscr{R}) \longrightarrow \mathscr{R} \longrightarrow \bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{O}_X(n)) \longrightarrow H^1_{\mathscr{R}_+}(\mathscr{R}) \longrightarrow 0.$$

More over for all $i \geq 1$ *one has:*

$$\bigoplus_{n\in\mathbb{Z}} H^i(X, \mathcal{O}_X(n)) \simeq H^{i+1}_{\mathscr{R}_+}(\mathscr{R}).$$

Observation 4.27. *Since $\mathcal{O}_X(1)$ is very ample on X over $\operatorname{Spec} R$, in view of Lemma 4.26, and Theorem 4.16 we have for all $i \geq 2$ there exists an integer N, such that $[H^i_{\mathscr{R}_+}(\mathscr{R})]_n = 0$ for all $n \geq N$.*

4.5 Integral closure of ideals and Reductions

Definition 4.28. Let R be a ring and I be an ideal of R. An element $r \in R$ is said to be *integral over I* if r satisfies an equation of the form $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$, where for all j, $a_j \in I^j$ and $n \in \mathbb{N}$.

The set of elements that are integral over I is called *integral closure of I* and denoted by $\overline{I}$. An ideal I is called *integrally closed* if $I = \overline{I}$.

Example 4.29. *Let R be a ring and $r_1, r_2 \in R$, then $r_1r_2 \in \overline{(r_1^2, r_2^2)}$ since $(r_1r_2)^2 - r_1^2r_2^2 = 0$ and $r_1^2r_2^2 \in (r_1^2, r_2^2)^2$.*

Proposition 4.30 ([HS06, Corollary 1.3.1])**.** *Let R be a ring and I be an ideal of R. Then $\overline{I}$ is an R-ideal and $I \subseteq \overline{I}$.*

Theorem 4.31 ([HS06, Theorem 5.2.4])**.** *Let R be a ring and $\overline{R}$ denote the integral closure of R in its total ring of fractions. Then integral closure of $R[It]$ in its total ring of fractions is*

$$\overline{R} \oplus \overline{I\overline{R}}t \oplus \overline{I^2\overline{R}}t^2 \oplus \cdots,$$

and the integral closure of $R[It, t^{-1}]$ in its total ring of fractions is

$$\cdots \overline{R}t^{-2} \oplus \overline{R}t^{-1} \oplus \overline{R} \oplus \overline{I\overline{R}}t \oplus \overline{I^2\overline{R}}t^2 \oplus \cdots.$$

Definition 4.32. Let R be a ring and I be an R-ideal. $J \subseteq I$ is called *reduction* of I if $I^n = JI^{n-1}$, for some $n \in \mathbb{N}$.

Definition 4.33. A reduction J of I is called *minimal* if $K \subseteq J$ is any other reduction for I, then $J = K$.

For ideals in arbitrary noetherian ring minimal reduction may not exists, however for ideal in noetherian local ring, minimal reductions exist.

Theorem 4.34 ([HS06, Theorem 8.3.6])**.** *Let (R, m) be a noetherian local ring and I be an R-ideal. If $J \subseteq I$ is a reduction for I, then there exists at least one ideal K in J such that K is minimal reduction for I.*

Theorem 4.35 ([HS06, Corollary 1.2.5])**.** *Let $J \subseteq I$ be R-ideals. Assume I is finitely generated. Then J is a reduction of I if and only if $I \subset \overline{J}$.*

Observation 4.36. *Observe that if I is an integrally closed ideal in a local ring (R, m) and $J \subseteq I$ is its reduction, then $I \subseteq \overline{J} \subseteq \overline{I} = I$; $\overline{J} = I$. If I is* m*-primary, R/m is infinite and J be its minimal reduction then minimal generators for J is a system of parameters for R.*

Chapter 5

Tight Closure

All rings are excellent and of prime characteristic $p > 0$ unless otherwise specified.

5.1 Tight closure

We will denote R^0 to be the complement of the minimal primes of R. Note that R^0 is a multiplicative set of R. If R is a domain, then $R^0 = R \setminus \{0\}$.

Let R be a ring of prime characteristic $p > 0$. Recall the Frobenius endomorphism $F : R \longrightarrow R$ is given by $r \mapsto r^p$. By F^e, we denote the e-th iteration of F. We write q for powers of p.

Let R be a reduced ring of prime characteristic p. Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ be the minimal primes of R. Then we have $R \hookrightarrow \prod_{i=1}^{n} R/\mathfrak{p}_i \hookrightarrow \prod_{i=1}^{n} \overline{Q(R/\mathfrak{p}_i)}$, where $\overline{Q(R/\mathfrak{p}_i)}$ is an fixed algebraic closure of $Q(R/\mathfrak{p}_i)$, the quotient field of $R/\mathfrak{p}_i$. Define $R^{1/q} := \{x \in \prod_{i=1}^{n} \overline{Q(R/\mathfrak{p}_i)} : x^q \in R\}$. Let I be an ideal of R. We write $IR^{1/q}$ for the $R^{1/q}$-ideal generated by the elements of I. Note that if R is reduced, then $F : R \longrightarrow R$ can be viewed as the inclusion $R \subseteq R^{1/q}$.

Definition 5.1. Let I be an ideal of R. The q-th *Frobenius power* of I is the R-ideal generated by $\{x^q \mid x \in I\}$ and is denoted by $I^{[q]}$.

Note that if the ideal I is generated by $x_1, \ldots, x_n$, then $I^{[q]}$ is generated by $x_1^q, \ldots, x_n^q$.

Definition 5.2. Let I be an R-ideal. Define

$$I^* = \{x \in R \mid \text{ there exists } c \in R^0 \text{ such that } cx^q \in I^{[q]}, \text{ for all } q \gg 0\}.$$

I^* is called *tight closure* of I. If $I^* = I$, then we say that I is *tightly closed*.

The choice of c can depends on I and x. Note that if R is reduced, then $cx^q \in I^{[q]}$ if and only if $c^{1/q}x \in IR^{1/q}$, where $c^{1/q} \in R^{1/q}$ is the unique q-th root of c.

Example 5.3. *Let $R = \mathbb{F}_p[x^2, x^3]$. Then $x^3 \notin (x^2)$. But we have $x^{3q} = x^q x^{2q} \in (x^{2q})$ for each q. Hence $x^3 \in (x^2)^*$.*

Example 5.4. *Let $R = \mathbb{F}_p[x, y, z]/(x^3 + y^3 + z^3)$ and $I = (x, y)$.*

Note that $z^2 \in R^0$. Then $z^2 z^{2q} = z^{2q+2}$. Write $2q + 2 = 3k + i$, where $k \geq 0$ and $0 \leq i \leq 2$. Then $z^{2q+2} = z^{3k+i} = z^i(x^3 + y^3)^k \in (x^{\lfloor 3k/2 \rfloor}, y^{\lfloor 3k/2 \rfloor})$. A simple calculation shows that $\lfloor 3k/2 \rfloor \geq q$. So $z^2 z^{2q} \in I^{[q]}$ for all q. Hence $z \in I^*$.

The following are some basic properties of tight closure. Proofs can be found in [HH90, Proposition 4.1, Theorem 4.4].

Proposition 5.5. *Let R be a noetherian ring of characteristic p and I, J be ideals of R.*
(1) I^* is an ideal of R and $I \subseteq I^*$.
(2) If $I \subseteq J$, then $I^* \subseteq J^*$. The intersection of an arbitrary family of tightly closed ideals is tightly closed.
(3) Let $x \in R$. Then $x \in I^*$ if and only if $\overline{x} \in (I(R/\mathfrak{p}))^*$, for all minimal prime $\mathfrak{p}$ of R, where $\overline{x}$ denotes the image of x in $R/\mathfrak{p}$.
(4) If I has positive height or if R is reduced, then $x \in I^*$ if and only if there exists $c \in R^0$ such that $cx^q \in I^{[q]}$ for all $q = p^e$.
(5) $I^* = I^{**}$.
(6) $(I \cap J)^* \subseteq I^* \cap J^*$.
(7) $(I + J)^* = (I^* + J^*)^*$.
(8) $(IJ)^* = (I^* J^*)^*$.
(9) $(0)^* = \mathrm{rad}(0)$. In particular, I^* contains the nilradical of R for all ideal I.
(10) If I is tightly closed, then $I : J$ is tightly closed for all ideal J.
(11) (Colon-capturing) Let (R, m, K) be reduced, excellent, equidimensional local ring and $x_1, \ldots, x_n$ be part of a system of parameters for R, then $(x_1, \ldots, x_{n-1}) :_R x_n \subseteq (x_1, \ldots, x_{n-1})^*$.
(12) If R is regular, then every ideal of R is tightly closed.

The above Proposition (3) tells us that the study of tight closure can be reduce to the case of domains and (5) shows that $*$ is actually a closure operation.

In general tight closure does not commute localization([BM10]). But for some special cases tight closure commutes with localization:

Theorem 5.6 ([HH90, Proposition 4.14])**.** *Let R be a noetherian ring of prime characteristic $p \geq 0$. Let I be an ideal of R primary to a maximal ideal m of R, then $I^* R_{\mathrm{m}} = (IR_{\mathrm{m}})^*$.*

There is also a notion of tight closure of submodule of a module. We will describe it now.

Discussion 5.7. For an R-module M, the assignment $M \mapsto {}^eR \otimes_R M$, where ${}^eR$ is R as a group, considered right R-module via the e-th power of the Frobenius endomorphism and left R-module by usual multiplication in R, is a functor from R-modules to R-modules called *Peskine-Szpiro* functor and is denoted by $F^e(M)$. There is a natural map $M \to F^e(M)$ sending $x \mapsto 1 \otimes x$. The image of x in $F^e(M)$ is often denoted by x^q, where $q = p^e$. For $N \subseteq M$, we have map $F^e(N) \to F^e(M)$, the image of $F^e(N)$ in $F^e(M)$ is denoted by $N_M^{[q]}$. In other words, $N^{[q]}$ is the R-submodule generated by the set $\{x^q \in F^e(M) : x \in N\}$. Note that if $M = R$, $F^e(R) \simeq R$ as R-modules. If $N = I$ an ideal of R, then $I_R^{[q]}$ is the ideal generated by $x^q : x \in I$ matches with the definition of $I^{[q]}$ defined earlier in 5.2.

Definition 5.8. Let $N \subseteq M$, the *tight closure* of N in M, denoted by N_M^*, is the set

$$\{z \in M : \text{ there exists } c \in R^0 \textit{ such that } cz^q \in N_M^{[q]} \text{ for all sufficiently large } q\}.$$

It is easy to see that N_M^* is a submodule of M. N is called *tightly closed* if $N_M^* = N$.

Definition 5.9. Let R be a ring of prime characteristic $p > 0$. Then R is said to be *weakly F-regular* if every ideal of R is tightly closed. R is said to be *F-regular* if $S^{-1}R$ is weakly F-regular for every multiplicative set S of R.

Example 5.10. *By* (12) *of Proposition 5.5, regular rings are weakly F-regular. Since localization of regular ring is regular, regular rings are F-regular.*

Definition 5.11. Let R be a ring of prime characteristic $p > 0$. Then R is said to be *F-pure* if for any R-module M, the map $F \otimes id_M : R \otimes M \to R \otimes M$ is injective.

Theorem 5.12 ([Fed83, Theorem 1.12])**.** *(Fedder's criterion) Let (S, m) be a regular local ring of prime characteristic $p > 0$. Let $R = S/I$, R is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathrm{m}^{[p]}$.*

Definition 5.13. A sequence of elements $x_1, \ldots, x_n$ in R are called *parameters* if they can be extended to a system of parameters in every local ring $R_{\mathfrak{p}}$ of R for all prime ideal $\mathfrak{p}$ of R that contains them.

An ideal of R is said to be *parameter ideal* if it can be generated by parameters.

Note that $x_1, \ldots, x_n$ are parameters if and only if $\mathrm{ht}(x_1, \ldots, x_i) = i$, for each $1 \leq i \leq n$. Note also that if R is a local ring which is both equidimensional and catenary, then elements $x_1, \ldots, x_i$ are parameters if and only if they form part of a system of parameters for R.

Definition 5.14. A ring of prime characteristic is *F-rational* if every parameter ideal is tightly closed.

Example 5.15. *Regular rings are F-rational.*

Example 5.16. *Weakly F-regular rings are F-rational.*

Definition 5.17. Let X be an excellent scheme. We say that X is *F-rational* if local ring at every point of X is F-rational.

Next we summarize some of the main properties of F-rational rings. Proofs can be found in [HH94].

Proposition 5.18. *Let R be a ring of prime characteristic $p > 0$. Then the following hold:*
(a) An F-rational ring is normal.
(b) An F-rational ring which is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay.
(c) A local ring (R, m) which is a homomorphic image of a Cohen-Macaulay ring is F-rational if and only if it is equidimensional and the ideal generated by one system of parameter is tightly closed.
(d) A homomorphic image of a Cohen-Macaulay ring is F-rational if and only if its localization at every maximal ideal is F-rational.
(e) A Gorenstein ring is weakly F-regular if and only if it is F-rational.
(f) If (R, m) is local ring which is a homomorphic image of a Cohen-Macaulay ring and $x \in \mathrm{m}$ is a nonzerodivisor such that R/xR is F-rational, then R is F-rational.
(g) Localization of F-rational ring is F-rational.

The following theorem is well known we give a proof for the sake of completeness.

Theorem 5.19. *If $R \to S$ is faithfully flat map. If S is F-rational, then R is so.*

Proof. Since $R \to S$ is faithfully flat, then parameters of R go to parameters of S. Let $I \subset R$ be a parameter ideal. Then $(IS)^* = IS$, as S is F-rational. Now $I^*S \subseteq (IS)^* = IS$, hence $I^* = I^*S \cap R \subseteq IS \cap R = I$, first and the third equality follows because $R \to S$ is faithfully flat. Therefore $I^* = I$. $\square$

Theorem 5.20 ([Smi97, Lemma 1.4])**.** *If (R, m) is an excellent local ring, then R is F-rational if and only if $\hat{R}$ is F-rational.*

Theorem 5.21 ([HH94, Proposition 6.27])**.** *An excellent F-rational local ring is Cohen-Macaulay.*

Proof. Let $(R, \mathfrak{m})$ be an excellent F-rational local ring. By Theorem 5.20, $\hat{R}$ is F-rational. Hence by (b) of Proposition 5.18, $\hat{R}$ is Cohen-Macaulay; therefore R is Cohen-Macaulay. □

Definition 5.22. An element $c \in R^0$ is called *test element* if for every ideal I of R and for all $u \in R$, $u \in I^*$ if and only if $cu^q \in I^{[q]}$ for all $q \geq 1$. If this is true only for ideals generated by parameters, c is called *parameter test element*.

The element c is called a *locally (respectively, completely) stable test element* if its image in (respectively, in the completion of) every local ring of R is a test element.

Theorem 5.23 ([HH94, Theorem 6.1])**.** *Let R be a reduced algebra of finite type over an excellent local ring. Let c be an element of R^0 such that R_c is regular. Then c has a power which is a completely stable test element for R.*

Theorem 5.24 ([VÝ95, Theorem 3.9])**.** *Let R be a reduced finitely generated algebra over an excellent local ring. If c is an element of R^0 such that R_c is F-rational, then there is a power of c, which is a test element for parameter ideals of R.*

Definition 5.25. The *parameter test ideal* of R is the ideal

$$\{c \in R \ : \ cI^* \subseteq I \text{ for all parameter ideals } I \text{ of } R\}.$$

Note that if an element $c \in R^0$ is in the parameter test ideal then c is a parameter test element.

Discussion 5.26. [Smi97, Section 2] Let $(R, \mathfrak{m})$ be a d-dimensional local ring. Let $x_1, \ldots, x_d$ be a system of parameters, then

$$H^d_{\mathfrak{m}}(R) \simeq \varinjlim_t R/(x_1^t, \ldots, x_d^t),$$

where the direct system is

$$\cdots R/(x_1^t, \ldots, x_d^t) \to R/(x_1^{t+1}, \ldots, x_d^{t+1}) \cdots ,$$

where the maps are multiplication by $x_1 \ldots x_d$. An elements of $\varinjlim_t R/(x_1^t, \ldots, x_d^t)$ is of the form $[z + (x_1^t, \ldots, x_d^t)]$, where $z + (x_1^t, \ldots, x_d^t) \in R/(x_1^t, \ldots, x_d^t)$ and $[.]$ denotes the image in $H^d_{\mathfrak{m}}(R)$. The natural isomorphism $\varinjlim_t R/(x_1^t, \ldots, x_d^t) \to H^d_{\mathfrak{m}}(R)$ is given by $[z + (x_1^t, \ldots, x_d^t)] \mapsto [z/x^t]$, where $x^t = x_1^t \cdots x_d^t$. Under this isomorphism the Frobenius endomorphism on $H^d_{\mathfrak{m}}(R)$ is given by $F([z + (x_1^t, \ldots, x_d^t)]) = [z^p + (x_1^{pt}, \ldots, x_d^{pt})]$.

Discussion 5.27. The discussion below is taken from [Smi94, Proposition 3.3 (i)] and [Smi97, Proposition 2.5]. If R is Cohen-Macaulay, the maps in the direct system are injective. Assume R is Cohen-Macaulay, let $z \in (x_1, \ldots, x_d)^*$, then there exists $c \in R^0$ such that $cz^q \in (x_1^q, \ldots, x_d^q)$ for all $q \gg 0$. Now $[z + (x_1, \ldots, x_d)]$ is an element in $H^d_{\mathfrak{m}}(R)$ and $c[z^q + (x_1^q, \ldots, x_d^q)] = 0$ for all $q \gg 0$. Hence $[z + (x_1, \ldots, x_d)] \in 0^*_{H^d_{\mathfrak{m}}(R)}$.

If $[z + (x_1^t, \ldots, x_d^t)] \in 0^*_{H^d_{\mathfrak{m}}(R)}$, there exists $c \in R^0$ such that $c[z^q + (x_1^{qt}, \ldots, x_d^{qt})] = 0$ for all $q \gg 0$. Since R is Cohen-Macaulay, the maps in the direct system defining $H^d_{\mathfrak{m}}(R) = 0$ are injective, hence $cz^q \in (x_1^{qt}, \ldots, x_d^{qt})$; $z \in (x_1^t, \ldots, x_d^t)^*$. Also note that since R is Cohen-Macaulay, $z \in (x_1, \ldots, x_d)^* \setminus (x_1, \ldots, x_d)$ if and only if $[z + (x_1, \ldots, x_d)]$ is a non-zero element in $0^*_{H^d_{\mathfrak{m}}(R)}$. Hence we have:

Theorem 5.28. *Let $(R, \mathfrak{m})$ be a d-dimensional excellent Cohen-Macaulay local ring of prime characteristic $p > 0$. Then R is F-rational if and only if $0^*_{H^d_{\mathfrak{m}}(R)} = 0$.*

It is easy to see that $0^*_{H^d_{\mathfrak{m}}(R)}$ is an F-stable submodule of $H^d_{\mathfrak{m}}(R)$. In fact, when R is domain, it is the largest F-stable submodule $H^d_{\mathfrak{m}}(R)$ ([Smi97, Proposition 2.5]). Smith shows the connection between parameter ideal and tight closure of zero in [Smi95]:

Proposition 5.29 ([Smi95, Proposition 4.4])**.** *Let $(R, \mathfrak{m})$ be an excellent equidimensional local ring of dimension d and J be its parameter test ideal.*
(i) $J = \{c \in R \mid cI^* \subseteq I,$ where I is a full system of parameters for $R\}$.
(ii) When R is Cohen-Macaulay, $J = \mathrm{Ann}_R(0^*_{H^d_{\mathfrak{m}}(R)})$.
(iii) When R is Cohen-Macaulay, and $x_1, \cdots, x_d$ is a fixed system of parameters for R, $J = \{c \in R \mid c(x_1^t, \cdots, x_d^t)^* \subseteq (x_1^t, \cdots, x_d^t),$ for all $t \in \mathbb{N}\}$.

The following theorem of K. Smith (cf.[Smi97]) gives another useful characterization of F-rational rings.

Theorem 5.30 ([Smi97, Theorem 2.6])**.** *Let $(R, \mathfrak{m})$ be an excellent local Cohen-Macaulay ring of dimension d and prime characteristic $p > 0$. The ring R is F-rational if and only if $H^d_{\mathfrak{m}}(R)$ has no proper non-trivial F-stable submodule.*

Definition 5.31. A *desingularization* of an integral scheme X is a pair (W, f) where W is a non-singular scheme and $W \xrightarrow{f} X$ is a proper birational map.

A scheme X is a *rational singularity* if there exists a desingularization (W, f) such that the natural map $\mathcal{O}_X \to Rf_*\mathcal{O}_W$ is a quasi-isomorphism. That is $\mathcal{O}_X = f_*\mathcal{O}_W$ and for all $i > 0$, $R^i f_*\mathcal{O}_W = 0$.

Rational singularity is a local property. When X is affine, $R^i f_*\mathcal{O}_W$ is the sheaf determined by the module $H^i(W, \mathcal{O}_W)$, where $H^i(W, \mathcal{O}_W)$ is the usual sheaf cohomology on W.

In [LT81], Lipman and Teissier defined notion of pseudo-rational rings. Pseudo-rationality is a property of local rings which is an analog of rational singularities for more general schemes, e.g. rings which may not have a desingularization. When the ring is essentially of finite type over a field of characteristic zero these two notions are the same. We now recall definition of pseudo-rationality (cf. [LT81]):

Definition 5.32. Let (R, m) be a d-dimensional local ring. Then R is *pseudo-rational* if it is normal, Cohen-Macaulay, and its completion $\hat{R}$ is reduced and if for every proper, birational map $\pi : W \longrightarrow X = \operatorname{Spec} R$ with W normal and closed fiber $E = \pi^{-1}(\mathrm{m})$, the canonical map (an edge-homomorphism in the Leray spectral sequence for cohomology with support)

$$H^d_{\mathrm{m}}(\pi_* \mathcal{O}_W) = H^d_{\mathrm{m}}(R) \longrightarrow H^d_E(\mathcal{O}_W)$$

is injective.

Theorem 5.33 ([LT81, Corollary 5.4])**.** *Let R be two dimensional pseudo-rational local ring and I be an ideal. Then for every integer $\lambda > 1$ we have,*

$$\overline{I^{\lambda+1}} = I\overline{I^{\lambda}} = I^{\lambda}\overline{I}.$$

The following theorem of Smith (cf.[Smi97]) shows the connection between F-rationality and pseudo-rationality.

Theorem 5.34 ([Smi97, Theorem 3.1])**.** *Let (R, m) be an excellent local ring of prime characteristic $p > 0$. If R is F-rational, then it is pseudo-rational.*

Definition 5.35. Let (R, m) be local ring or positively graded algebra over a local ring with unique homogeneous maximal ideal m of prime characteristic $p > 0$. R is called *F-injective* if

$$F : H^i_{\mathrm{m}}(R) \longrightarrow H^i_{\mathrm{m}}(R) \text{ is an injective map for all } i.$$

Since $\ker F$ is an F-stable submodule of $H^d_{\mathrm{m}}(R)$, from Theorem 5.30 we have the following proposition:

Proposition 5.36. *An excellent F-rational local ring is F-injective.*

Discussion 5.37. Let (R, m) is a positively graded algebra over a local ring of dimension n with unique homogeneous maximal ideal m. If R is F-injective, then $a_i(R) \leq 0$. But the converse is not true:

Example 5.38. *Let $R = K[x, y, z]/(x^2 + y^3 + z^5)$, where K is a field of characteristic 2. Note that R is a 2-dimensional Cohen-Macaulay ring. Let $\deg x = 15, \deg y = 10, \deg z = 6$. $\deg(x^2 + y^3 + z^5) = 30$ and $a(R) = -1$. We will show that R is not F-injective. Note that y, z is a system of parameters for R, then its local cohomology can be computed from the Čech complex $\check{C}^{\bullet}(y, z; R)$. Consider $[x/yz] \in [H^2_{\mathrm{m}}(R)]_{-1}$, since $x \notin (y, z)$, $[x/yz] \neq 0$. Note that $F([x/yz]) = [x^2/y^2z^2] = 0$ in $H^2_{\mathrm{m}}(R)$.*

Theorem 5.39 ([V95́, Proposition 3.2])**.** *Let R be an excellent F-rational ring. Then any polynomial ring extension of R, is F-rational.*

Converse of the theorem also holds i.e.

Theorem 5.40. *Let R be an excellent ring. If $R[t]$ is F-rational, then R is so.*

Proof. Since the natural map $R \to R[t]$ is faithfully flat, the proof follows from Theorem 5.19. □

5.2 F-rational rings

In this section we discuss the characterization of F-rationality of excellent rings in terms of F-injectivity and F-unstability as in [FW89]. Definition of F-unstability is given below. We also extend their result [FW89, Theorem 2.8] for local rings. Here onwards all rings are assumed to be excellent.

Definition 5.41. Let (R, m) be a local ring or positively graded ring with R_0 a local ring of dimension d. Let S_i denote the socle of $H^i_{\mathrm{m}}(R)$. We say that $H^i_{\mathrm{m}}(R)$ is *F-unstable* if there exists $N > 0$ such that $S_i \cap F^e(S_i) = 0$ for every $e > N$. We say that R is *F-unstable* if for each $0 \leq i \leq d$, $H^i_{\mathrm{m}}(R)$ is F-unstable.

Lemma 5.42 ([FW89, Lemma 2.3])**.** *Let (R, m) be as in the above definition. Assume R is an F-injective ring of dimension d which is not F-unstable. Denote the socle of $H^i_{\mathrm{m}}(R)$ by S_i. Then, for each S_i which does not satisfy the F-unstable property (i.e. for which $S_i \cap F^e(S_i) \neq 0$ holds for infinitely many choices of $e > 0$), there exists $0 \neq \eta \in S_i$ such that $F^e(\eta) \in S_i$ for every $e \geq 0$.*

Lemma 5.43 ([FW89, Remark 1.17])**.** *Let (R, m) is a positively graded algebra over a local ring with unique homogeneous maximal ideal m. If $a_i(R) < 0$, for all i, then R is F-unstable.*

Lemma 5.44 ([FW89, Remark 1.17])**.** *Let (R, m) be a positively graded algebra over a field. If R is F-injective, then R is F-unstable if and only if for all i, $a_i(R) < 0$.*

Proposition 5.45 ([FW89, Proposition 2.4])**.** *Let (R, m) be as in the Definition 5.41. If R is an F-rational ring, then R is both F-injective and F-unstable.*

Proof. F-injectivity of R follows from Proposition 5.36.

Suppose R is not F-unstable, then by Lemma 5.42 there exists a nonzero $\eta \in \mathrm{Soc}(H^d_{\mathrm{m}}(R))$ such that $F^e(\eta) \in \mathrm{Soc}(H^d_{\mathrm{m}}(R))$ for every $e \geq 0$. Now R-submodule generated by set $\{F^e(\eta) : e \geq 0\}$ forms a nonzero F-stable submodule, say M whose annihilator is m. Since annihilator of $H^d_{\mathrm{m}}(R) = 0$, M is proper which contradicts F-rationality of R. So R is F-unstable. □

Example 5.46. *Let* $R = K[x, y, z]/(x^2 + y^3 + z^7)$*, where* K *is a field of prime characteristic* $p > 0$*. Let* $\deg x = 21, \deg y = 14, \deg z = 6$*, then* $a(R) = 42 - 41 = 1$*. Hence* R *is not* F*-injective; by Proposition 5.36,* R *is not* F*-rational.*

Example 5.47. *Let* $R = K[x, y, z, w]/(x^4 + y^4 + z^4 + w^4)$ *where* K *is a field of prime characteristic* $p > 0$*. Then* $a(R) = 0$*; hence* R *is not* F*-rational.*

In [FW89], Fedder and Watanabe characterizes F-rationality in terms of F-injectivity and F-unstability. They prove

Theorem 5.48 ([FW89, Theorem 2.8])**.** *Let* (R, m) *be a local ring or positively graded ring with* R_0 *being field. Assume* R *is* F*-finite ring of dimension* d *with isolated singularity. If* R *is an equidimensional quotient of a Cohen-Macaulay ring and* $H^i_{\mathrm{m}}(R)$ *has finite length (possibly* 0*) for every* $i < d$*, then:*
R is F-rational if and only if R is F-injective and F-unstable.

The above Theorem still holds if we change the assumption punctured spectrum being regular to F-rational. The proof will be same as their proof using Theorem 5.24 and replace every occurance of 'test element' by 'parameter test element'.

Since rings where we want to apply the theorem are reduced and Cohen-Macaulay we write an alternative proof for reduced and Cohen-Macaulay rings.

Theorem 5.49. *Let* (R, m) *be a reduced Cohen-Macaulay local ring of dimension* d *such that its punctured spectrum* $\operatorname{Spec} R \setminus \{\mathrm{m}\}$ *is* F*-rational. If* R *is* F*-injective and* F*-unstable then* R *is* F*-rational.*

Proof. Since R is Cohen-Macaulay it is enough to show that $0^*_{H^d_{\mathrm{m}}(R)} = 0$. By Proposition 5.29 $\operatorname{Ann}_R(0^*_{H^d_{\mathrm{m}}(R)})$ is the parameter test ideal J. Since the punctured spectrum is F-rational, if J is proper it is m-primary. Now we will show that J is radical ideal. It is enough to show that if $c^2 \in J$, then $c \in J$. Let $c^2 \in J$, $c^p \in J$. Let $\xi \in 0^*_{H^d_{\mathrm{m}}(R)}$, since $0^*_{H^d_{\mathrm{m}}(R)}$ is F-stable, $\xi^p \in 0^*_{H^d_{\mathrm{m}}(R)}$. So $(c\xi)^p = c^p\xi^p = 0$. Since R is F-injective, $c\xi = 0$. Hence $c \in J$. Hence J is radical ideal; $J = \mathrm{m}$. Since $0^*_{H^d_{\mathrm{m}}(R)}$ is Artinian, it has non-zero socle elements, let $\eta \in 0^*_{H^d_{\mathrm{m}(R)}}$ be a non-zero socle element. Since $J = \mathrm{m}$ and R is F-injective, for all $e \geq 1$, η^{p^e} is non-zero socle element of $H^d_{\mathrm{m}}(R)$, which contradicts R is F-unstable. $\square$

Example 5.50. *Let* $R = K[[x, y, z]]/(x^2 + y^3 + z^5)$*, where* K *is a field of characteristic* 7 *and* $\mathrm{m} = (x, y, z)$*. By Jacobian criterion* R *is regular on* $\operatorname{Spec} R \setminus \{\mathrm{m}\}$*. Let* $\deg x = 15, \deg y = 10, \deg z = 6$*,* $\deg(x^2 + y^3 + z^5) = 30$*.* $a(R) = -1$*. Next we will show that* R *is* F*-injective. Now* $(x^2 + y^3 + z^5)^6$ *has term* $(x^2)^3(y^3)^2z^5$ *with non-zero coefficient and* $(x^2)^3(y^3)^2z^5 \notin \mathrm{m}^{[7]}$*; Hence by Fedder's criterion (Theorem 5.12)* R *is* F*-pure, in particular* F*-injective. Hence* R_{m} *is* F*-injective. Therefore by above Theorem* R_{m} *is* F*-rational.*

Chapter 6

F-rationality of Rees algebra

Here all the rings are excellent of prime characteristic $p > 0$, unless otherwise stated.

6.1 F-rationality of extended Rees algebras

Let (R, m) be an excellent ring and I be an m-primary ideal. We want to study F-rationality of the Rees algebra $R[It]$. In [Sin00], Singh gave an example of an 3-dimensional hypersurface F-rational ring, such that its Rees algebra with respect to its maximal ideal is Cohen-Macaulay and normal but not F-rational by showing that its Proj not F-rational. In [HWY02] Hara, Watanabe,Yoshida gave criterion for F-rationality of Rees algebra in terms of tight integral closure. In [Hyr99], Hyry proved that if (R, m) is excellent local ring of characteristic 0 and I be an m-primary ideal such that $R[It]$ is Cohen-Macaulay and normal, then $R[It]$ is rational singularity if and only if $\operatorname{Proj} R[It]$ is rational singularity. We prove partial analogue of that in prime characteristic $p > 0$.

Notation 6.1. *Let (R, m) be a d-dimensional excellent local ring with R/m infinite, I be an* m*-primary ideal. Let $J = (f_1, \cdots, f_d)$ be a minimal reduction for I. Then $\{f_1, \cdots, f_d\}$ is a system of parameter for R. We write $\mathscr{R}$ for $R[It]$ and $\mathfrak{M}$ for the unique homogeneous maximal ideal of $R[It]$. Let $X := \operatorname{Proj} \mathscr{R}$, $\mathcal{O}_X(n) = \widetilde{\mathscr{R}(n)}$, for $n \in \mathbb{Z}$. Let $\pi : X \to \operatorname{Spec} R$ be the natural map. Let E denote the exceptional divisor defined by $I\mathcal{O}_X$.*

We write G for $\mathrm{gr}_I(R)$, G_+ to denote the $\mathrm{gr}_I(R)$-ideal $I/I^2 \oplus I^2/I^3 \oplus \cdots$. We write $\mathscr{R}'$ for $R[It, t^{-1}]$, $\mathfrak{M}'$ for the homogeneous maximal ideal of $\mathscr{R}'$ and $\mathscr{R}'_+$ for the homogeneous $\mathscr{R}'$-ideal generated by It.

Theorem 6.2. *Let (R, m) be an excellent normal d-dimensional local ring. Let I be an* m*-primary ideal. Let $X = \operatorname{Proj} \mathscr{R}$ be F-rational and $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$. Then $\mathscr{R}^{(n)}$ is F-rational for all $n \gg 0$.*

Notation: For any graded module M, we write $[M]_n$ to denote its degree-n piece.

Definition 6.3. Let S be a graded ring i.e. $S \simeq \bigoplus_{i\in\mathbb{Z}} S_i$. For $n > 0$, the *n-th Veronese subring* is denoted by $S^{(n)}$ and defined by $S^{(n)} := \bigoplus_{in\in\mathbb{Z}} S_{in}$. For a graded S-module M, we define $M^{(n)} := \bigoplus_{in\in\mathbb{Z}} [M]_{in}$.

Lemma 6.4. *If X is F-rational, then the punctured spectrum* $\operatorname{Spec} R \setminus \{\mathrm{m}\}$ *is also F-rational.*

Proof. Since I is m-primary, $\operatorname{Spec}(R/I) = \{\mathrm{m}\}$. Now the proof follows from the fact $\operatorname{Spec} R \setminus \{\mathrm{m}\}$ is isomorphic to $\operatorname{Proj} \mathscr{R} \setminus E$. □

Lemma 6.5. *For $ft \in It$, $\mathscr{R}_{ft} = \mathscr{R}_{(ft)}[z, z^{-1}]$.*

Proof. We define a homomorphism $\rho : \mathscr{R}_{(ft)}[z, z^{-1}] \longrightarrow \mathscr{R}_{ft}$ by sending z to ft, z^{-1} to $(ft)^{-1}$ and elements of $\mathscr{R}_{(ft)}$ to itself. It is easy to see that ρ is an isomorphism. □

Lemma 6.6. *Let R and I be same as above, if X is F-rational, then* $\operatorname{Spec} \mathscr{R} \setminus \{\mathfrak{M}\}$ *is F-rational.*

Proof. Let $P \in \operatorname{Spec} \mathscr{R} \setminus \{\mathfrak{M}\}$. If $\mathscr{R}_+ \subseteq P$, then contraction of P in R is not m, as $P \neq \mathfrak{M}$. Since I is m-primary, $(R \backslash P \cap R) \cap I \neq \phi$. Hence $\mathscr{R}_P$ is a localization of $R_{P\cap R}[t]$. As $P \cap R \neq \mathrm{m}$, $R_{P\cap R}$ is F-rational by Lemma 6.4, so is $R_{P\cap R}[t]$. Hence any localization of $R_{P\cap R}[t]$ is also F-rational. If $\mathscr{R}_+ \not\subseteq P$, then $P \in \operatorname{Spec} \mathscr{R} \setminus V(\mathscr{R}_+)$. Now $\operatorname{Spec} \mathscr{R} \setminus V(\mathscr{R}_+)$ is covered by the open sets $\operatorname{Spec} \mathscr{R}_{f_i t}$ for $i = 1, \cdots, n$. We also know that $\operatorname{Proj} \mathscr{R}$ is covered by the open sets $\operatorname{Spec} \mathscr{R}_{(f_i t)}$ for $i = 1, \cdots, n$. By hypothesis $\mathscr{R}_{(f_i t)}$ is F-rational, hence $\mathscr{R}_{f_i t} = \mathscr{R}_{(f_i t)}[z, z^{-1}]$ is also F-rational. □

Proposition 6.7 ([GN94, Part II, Theorem 3.3])**.** *With notation as in 6.1 and 4.15, $a(\mathscr{R}) = -1$.*

Proof of Theorem 6.2. Since R is normal, $H^0(X, \mathcal{O}_X) = R$ and by hypothesis $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$, so by Theorem 4.1 of [Lip94] for all sufficiently large n', $\mathscr{R}^{(n')}$ is Cohen-Macaulay. By Lemma 6.6 $\operatorname{Spec} \mathscr{R} \setminus \{\mathfrak{M}\}$ is F-rational. Hence if parameter test ideal of $\mathscr{R}$ is proper then by Theorem 5.24 it is $\mathfrak{M}$-primary. Hence $\mathfrak{M}^l 0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})} = 0$, for some $l > 0$. So by the following Lemma 6.8 $0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})}$ is of finite length. Then there exists an integer $k > 0$ such that $[0^*_{H^{d+1}_{\mathscr{M}}(\mathscr{R})}]_{-k'} = 0$ for all $k' > k$. Since $a(\mathscr{R}) = -1$ and for all $n \geq 0$, $(H^{d+1}_{\mathfrak{M}}(\mathscr{R}))^{(n)} \simeq H^{d+1}_{\mathfrak{M}^{(n)}}(\mathscr{R}^{(n)})$; for all sufficiently large n,

$$0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})} \cap H^{d+1}_{\mathfrak{M}^{(n)}}(\mathscr{R}^{(n)}) = 0.$$

Let $\xi \in 0^*_{H^{d+1}_{\mathfrak{M}^{(n)}}(\mathscr{R}^{(n)})}$ be a homogeneous element. By definition, $\xi \in 0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})}$. Hence

$$\xi \in 0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})} \cap H^{d+1}_{\mathfrak{M}^{(n)}}(\mathscr{R}^{(n)}) = 0.$$

So by Theorem 5.28, for all $n \gg 0$, $\mathscr{R}^{(n)}$ is F-rational. □

Lemma 6.8. *Let (A, m, K) be a local ring. If N be an Artinian A-module such that $m^l N = 0$ for some positive integer l. Then N has finite length.*

Proof. Since N is Artinian N can be embedded in E^a where E is the injective hull of K and $a = \dim_K \operatorname{Soc}(N)$. Since $m^l N = 0$, $N \subseteq (0 :_{E^a} m^l)$. Now $0 :_E m^l \simeq \operatorname{Hom}_A(A/m^l, E)$ is finite length A-module, as $\operatorname{Hom}_A(A/m^l, E) = E_{A/m^l}(K)$ ([ILL+07, Theorem A.25]) is finite length ([Mat86, Theorem 18.6]). Hence the lemma. □

In [HWY02], they study the connection between F-rationality of Rees algebras and extended Rees algebras. They prove:

Theorem 6.9 ([HWY02, Theorem 4.2])**.** *Let (R, m) be an F-rational excellent local ring of positive characteristic $p > 0$ and I be an m-primary ideal. If the Rees algebra $R[It]$ is F-rational so is the extended Rees algebra $R[It, t^{-1}]$.*

The converse of the theorem is given as a conjecture in (See [HWY02, conjecture 4.1]). We prove:

Theorem 6.10. *Let (R, m) be a d-dimensional F-rational excellent local ring of positive characteristic $p > 0$ and I be an m-primary ideal. If the extended Rees algebra $R[It, t^{-1}]$ is F-rational then so is the Rees algebra $R[It]$.*

Discussion 6.11. [HWY02, Corollary 1.10] Since R is excellent domain, then there exists a non-zero element c such that R_c is regular. Take e any non-zero element in I, then $ce \in I$ and R_{ce} is also F-rational; hence $\mathscr{R}_{ce} = R_{ce}[t]$ and $\mathscr{R}'_{ce} = R_{ce}[t, t^{-1}]$ are also F-rational. Hence we can take a common power of ce so that it is a test element for parameters for $R, \mathscr{R}, \mathscr{R}'$. We write c for the common test element for parameters for $R, \mathscr{R}, \mathscr{R}'$.

Observation 6.12. *If $P \in \operatorname{Proj} \mathscr{R}$, then $\mathscr{R}_P$ is a localization of $\mathscr{R}_{(P)}[z, z^{-1}]$. Hence $\mathscr{R}_{(P)} \to \mathscr{R}_P$ is faithfully flat map.*

Lemma 6.13. *If $\mathscr{R}_{\mathfrak{M}}$ is F-rational, the $\mathscr{R}_P$ is F-rational for all prime $P \in \operatorname{Spec} \mathscr{R}$.*

Proof. Since $\mathscr{R}_{\mathfrak{M}}$ is F-rational, by Prop 5.18(g) $\mathscr{R}_P$ is F-rational for all $P \in \operatorname{Proj} \mathscr{R}$. By Observation 6.12, $\mathscr{R}_{(P)} \to \mathscr{R}_P$ is faithfully flat hence by Theorem 5.19 $\mathscr{R}_{(P)}$ is F-rational. Hence $\operatorname{Proj} \mathscr{R}$ is F-rational, hence by Lemma 6.6 $\operatorname{Spec} \mathscr{R} \setminus \{\mathfrak{M}\}$ is F-rational. Hence the lemma. □

It is easy to see that for $ft \in It$, $\mathscr{R}_{ft} = \mathscr{R}'_{ft}$. Hence for all $i \geq 2$,

$$H^i_{\mathscr{R}'_+}(\mathscr{R}') = H^i_{\mathscr{R}_+}(\mathscr{R}). \tag{6.1}$$

Lemma 6.14. $H^{d+1}_{\mathscr{R}'_+}(\mathscr{R}') = 0$.

Proof. Since I is m-primary, then $\mathscr{R}'_+$ is generated by d elements up to radical; hence $H^{d+1}_{\mathscr{R}'_+}(\mathscr{R}') = 0$. □

Discussion 6.15. Hypothesis on $R, \mathscr{R}, \mathscr{R}'$ are as in the Theorem 6.10. Recall $J \subseteq I$ is a minimal reduction and $J = (f_1, \cdots, f_d)$. We write f for the product $f_1 \cdots f_d$. By Observation 4.36, $Jt + (t^{-1})$ is $\mathfrak{M}'$ primary and $\mathrm{rad}(Jt) = \mathscr{R}'_+$. Hence for all i,

$$H^i_{(Jt+(t^{-1}))}(\mathscr{R}') = H^i_{\mathfrak{M}'}(\mathscr{R}') \text{ and}$$

$$H^i_{(Jt)}(\mathscr{R}') = H^i_{\mathscr{R}'_+}(\mathscr{R}').$$

So by Proposition 4.12, we have a long exact sequence:

$$\cdots \to H^d_{\mathfrak{M}'}(\mathscr{R}') \to H^d_{\mathscr{R}'_+}(\mathscr{R}') \to H^d_{\mathscr{R}'_+}(\mathscr{R}'_{t^{-1}}) \to H^{d+1}_{\mathfrak{M}'}(\mathscr{R}') \to H^{d+1}_{\mathscr{R}'_+}(\mathscr{R}') \to \cdots.$$

Since $\mathscr{R}'$ is Cohen-Macaulay, $H^d_{\mathfrak{M}'}(\mathscr{R}') = 0$. Also $H^{d+1}_{\mathscr{R}'_+}(\mathscr{R}') = 0$. Now $\mathscr{R}'_{t^{-1}} = R[t, t^{-1}]$ and $\mathrm{rad}(Jt\mathscr{R}'_{t^{-1}}) = \mathrm{m}[t, t^{-1}]$. Hence $H^d_{\mathscr{R}'_+}(\mathscr{R}'_{t^{-1}}) = H^d_{\mathrm{m}}(R)[t, t^{-1}]$. Hence the above long exact sequence becomes:

$$0 \longrightarrow H^d_{\mathscr{R}'_+}(\mathscr{R}') \longrightarrow H^d_{\mathrm{m}}(R)[t, t^{-1}] \longrightarrow H^{d+1}_{\mathfrak{M}'}(\mathscr{R}') \longrightarrow 0.$$

Now $H^d_{\mathscr{R}'_+}(\mathscr{R}')$ can be computed via Čech cohomology with respect to the elements $f_1t, \cdots, f_dt$. So a homogeneous element of degree n in $H^d_{\mathscr{R}'_+}(\mathscr{R}')$ is of the form $[\frac{a}{f^l}t^n]$, where $[-]$ denotes the image in $H^d_{\mathscr{R}'_+}(\mathscr{R}')$, $l \geq 0$ and $a \in I^{dl+n}$. Since $\{f_1, \cdots, f_d\}$ is a system of parameter for R then $H^d_{\mathrm{m}}(R)$ can be computed from the Čech complex $\check{C}^\bullet(f_1, \cdots, f_d; R)$. From Discussion 4.13, we see that the map in the above exact sequence $H^d_{\mathscr{R}'_+}(\mathscr{R}') \longrightarrow H^d_{\mathrm{m}}(R)[t, t^{-1}]$, is given by $[\frac{a}{f^l}t^n]$ goes to $[\frac{a}{f^l}]t^n$.

Lemma 6.16. *With hypothesis as in the above theorem, $R[It]$ is Cohen-Macaulay.*

Proof. By above discussion we have the following exact sequence:

$$0 \longrightarrow H^d_{\mathscr{R}'_+}(\mathscr{R}') \longrightarrow H^d_{\mathrm{m}}(R)[t, t^{-1}] \longrightarrow H^{d+1}_{\mathfrak{M}'}(\mathscr{R}') \longrightarrow 0.$$

By Discussion 4.17 the above exact sequence is compatible with Frobenius map, i.e. the following diagram commutes, where F denote the respective Frobenius map on cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^d_{\mathscr{R}'_+}(\mathscr{R}') \arrow[r] \arrow[d, "F"] & H^d_{\mathfrak{m}}(R)[t,t^{-1}] \arrow[r] \arrow[d, "F"] & H^{d+1}_{\mathfrak{M}'}(\mathscr{R}') \arrow[r] \arrow[d, "F"] & 0 \\ 0 \arrow[r] & H^d_{\mathscr{R}'_+}(\mathscr{R}') \arrow[r] & H^d_{\mathfrak{m}}(R)[t,t^{-1}] \arrow[r] & H^{d+1}_{\mathfrak{M}'}(\mathscr{R}') \arrow[r] & 0. \end{tikzcd}$$

From Observation 4.27 and (6.1), we know for all large n, $[H^d_{\mathscr{R}'_+}(\mathscr{R}')]_n = 0$, choose n such that $[H^d_{\mathscr{R}'_+}(\mathscr{R}')]_n \neq 0$ and $[H^d_{\mathscr{R}'_+}(\mathscr{R}')]_i = 0$ for all $i > n$. If $n \geq 0$, then the R-submodule $[H^d_{\mathscr{R}'_+}(\mathscr{R}')]_n$ is F-stable. Since the above exact sequence is compatible with Frobenius map, $[H^d_{\mathscr{R}'_+}(\mathscr{R}')]_n$ is an F-stable R-submodule of $H^d_{\mathfrak{m}}(R)$, which gives a contradiction to the fact R is F-rational (Theorem 5.30). Hence $\max\{i \mid H^d_{\mathscr{R}'_+}(\mathscr{R}')_i \neq 0\} < 0$. Next we will show that $a(G) < 0$. Since $\mathscr{R}'$ is Cohen-Macaulay and t^{-1} is a non-zero divisor of $\mathscr{R}'$, $G = \mathscr{R}'/(t^{-1})$ is also Cohen-Macaulay. We have an exact sequence of $\mathscr{R}'$-modules:

$$0 \longrightarrow \mathscr{R}'(1) \xrightarrow{t^{-1}} \mathscr{R}' \longrightarrow G \longrightarrow 0.$$

Hence we get a long exact sequence:

$$\cdots \longrightarrow H^i_{\mathscr{R}'_+}(\mathscr{R}')(1) \longrightarrow H^i_{\mathscr{R}'_+}(\mathscr{R}') \longrightarrow H^i_{\mathscr{R}'_+}(G) \cdots .$$

Note that $H^i_{\mathscr{R}'_+}(G) = H^i_{G_+}(G)$ for all i. Since $H^{d+1}_{\mathscr{R}'_+}(\mathscr{R}') = 0$ and $\operatorname{rad}(G_+)$ is the maximal homogeneous ideal of G, the above exact sequence becomes:

$$0 \longrightarrow H^d_{\mathscr{R}'_+}(\mathscr{R}')(1) \longrightarrow H^d_{\mathscr{R}'_+}(\mathscr{R}') \longrightarrow H^d_{\mathscr{R}'_+}(G) \longrightarrow 0.$$

Since $\max\{i \mid [H^d_{\mathscr{R}'_+}(\mathscr{R}')]_i \neq 0\} < 0$ and $H^i_{\mathscr{R}'_+}(G) = H^i_{G_+}(G)$ we have $a(G) < 0$. Hence by Theorem 4.22, $R[It]$ is Cohen-Macaulay. $\square$

Discussion 6.17. Let c be a common test element for parameters of $R, \mathscr{R}, \mathscr{R}'$ (Discussion 6.11). Since $\mathscr{R}$ is Cohen-Macaulay, by the commutative diagram 2.10.2 of [HWY02] we have

$$\begin{tikzcd} 0 \arrow[r] & H^d_{\mathscr{R}_+}(\mathscr{R}) \arrow[r, "\phi"] \arrow[d, "cF^e"] & \bigoplus_{n<0} H^d_{\mathfrak{m}}(R)t^n \arrow[r, "\psi"] \arrow[d, "cF^e"] & H^{d+1}_{\mathfrak{M}}(\mathscr{R}) \arrow[r] \arrow[d, "cF^e"] & 0 \\ 0 \arrow[r] & H^d_{\mathscr{R}_+}(\mathscr{R}) \arrow[r, "\phi"] & \bigoplus_{n<0} H^d_{\mathfrak{m}}(R)t^n \arrow[r, "\psi"] & H^{d+1}_{\mathfrak{M}}(\mathscr{R}) \arrow[r] & 0, \end{tikzcd} \tag{6.2}$$

where $\phi([(a/f^l)t^n]) = [a/f^l]t^n$ (See Remark after Lemma 2.7 in [HWY02]). One can see that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \check{C}^\bullet(\underline{ft}; R_{t^{-1}})[-1] \arrow[r] \arrow[d, "cF^e"] & \check{C}^\bullet(\underline{ft}, t^{-1}; R) \arrow[r] \arrow[d, "cF^e"] & \check{C}^\bullet(\underline{ft}; R) \arrow[r] \arrow[d, "cF^e"] & 0 \\
0 \arrow[r] & \check{C}^\bullet(\underline{ft}; R_{t^{-1}})[-1] \arrow[r] & \check{C}^\bullet(\underline{ft}, t^{-1}; R) \arrow[r] & \check{C}^\bullet(\underline{ft}; R) \arrow[r] & 0.
\end{tikzcd}$$

Hence we have the following commutative diagram in local cohomology:

$$\begin{tikzcd}
0 \arrow[r] & H^d_{\mathscr{R}'_+}(\mathscr{R}') \arrow[r, "\phi'"] \arrow[d, "cF^e"] & \bigoplus_{n\in\mathbb{Z}} H^d_{\mathfrak{m}}(R)t^n \arrow[r, "\psi'"] \arrow[d, "cF^e"] & H^{d+1}_{\mathfrak{M}'}(\mathscr{R}') \arrow[r] \arrow[d, "cF^e"] & 0 \\
0 \arrow[r] & H^d_{\mathscr{R}'_+}(\mathscr{R}') \arrow[r, "\phi'"] & \bigoplus_{n\in\mathbb{Z}} H^d_{\mathfrak{m}}(R)t^n \arrow[r, "\psi'"] & H^{d+1}_{\mathfrak{M}'}(\mathscr{R}') \arrow[r] & 0.
\end{tikzcd} \tag{6.3}$$

By discussion 6.15 we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^d_{\mathscr{R}_+}(\mathscr{R}) \arrow[r, "\phi"] \arrow[d, "="] & \bigoplus_{n<0} H^d_{\mathfrak{m}}(R)t^n \arrow[r, "\psi"] \arrow[d] & H^{d+1}_{\mathfrak{M}}(\mathscr{R}) \arrow[r] & 0 \\
0 \arrow[r] & H^d_{\mathscr{R}'_+}(\mathscr{R}') \arrow[r, "\phi'"] & \bigoplus_{n\in\mathbb{Z}} H^d_{\mathfrak{m}}(R)t^n \arrow[r, "\psi'"] & H^{d+1}_{\mathfrak{M}'}(\mathscr{R}') \arrow[r] & 0.
\end{tikzcd}$$

Hence we get an $\mathscr{R}$-module map $\theta : H^{d+1}_{\mathfrak{M}}(\mathscr{R}) \to H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')$ such that the above diagram commutes. Thus we have:

$$\begin{tikzcd}
0 \arrow[r] & H^d_{\mathscr{R}_+}(\mathscr{R}) \arrow[r, "\phi"] \arrow[d, "="] & \bigoplus_{n<0} H^d_{\mathfrak{m}}(R)t^n \arrow[r, "\psi"] \arrow[d] & H^{d+1}_{\mathfrak{M}}(\mathscr{R}) \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & H^d_{\mathscr{R}'_+}(\mathscr{R}') \arrow[r, "\phi'"] & \bigoplus_{n\in\mathbb{Z}} H^d_{\mathfrak{m}}(R)t^n \arrow[r, "\psi'"] & H^{d+1}_{\mathfrak{M}'}(\mathscr{R}') \arrow[r] & 0.
\end{tikzcd} \tag{6.4}$$

Take any $\xi \in H^{d+1}_{\mathfrak{M}}(\mathscr{R})$ homogeneous, choose $\eta \in H^d_{\mathfrak{m}}(R)$ such that $\psi(\eta) = \xi$. Define $\theta(\xi) = \psi'(\eta)$. Applying Snake lemma to diagram (6.4) we see that θ is injective. Next we will show that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & H^{d+1}_{\mathfrak{M}}(\mathscr{R}) \arrow[r, "\theta"] \arrow[d, "cF^e"] & H^{d+1}_{\mathfrak{M}'}(\mathscr{R}') \arrow[d, "cF^e"] \\ 0 \arrow[r] & H^{d+1}_{\mathfrak{M}}(\mathscr{R}) \arrow[r, "\theta"] & H^{d+1}_{\mathfrak{M}'}(\mathscr{R}'). \end{tikzcd} \tag{6.5}$$

Let $\xi \in H^{d+1}_{\mathfrak{M}}(\mathscr{R})$ be homogeneous element. Choose $\eta \in H^d_{\mathfrak{m}}(R)$ such that $\psi(\eta) = \xi$. Then $\theta(\xi) = \psi'(\eta)$. We need to show that

$$\theta(cF^e(\xi)) = cF^e(\theta(\xi)).$$

From the commutative diagram (6.2) we have $\psi cF^e(\eta) = cF^e\psi(\eta)$. Hence $\psi cF^e(\eta) = cF^e(\xi)$. Then $\theta(cF^e(\xi)) = \psi'(cF^e(\eta))$. From the commutative diagram (6.3) we have $\psi'(cF^e(\eta)) = cF^e(\psi'(\eta))$. Hence $\theta(cF^e(\xi)) = \psi'(cF^e(\eta)) = cF^e(\psi'(\eta)) = cF^e(\theta(\xi))$.

Lemma 6.18. *Let $\mathscr{R}$ be Cohen-Macaulay. Then*

$$\theta(0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})}) \subseteq 0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')}.$$

Proof. Proof follows from the commutative diagram (6.5). □

Proof of Theorem 6.10. Since $\mathscr{R}'$ is F-rational, then $\operatorname{Proj}\mathscr{R}$ is F-rational. Hence by Lemma 6.6, $\operatorname{Spec}\mathscr{R} \setminus \{\mathfrak{M}\}$ is also F-rational. By Lemma 6.16 $\mathscr{R}$ is Cohen-Macaulay. Since $\mathscr{R}'$ is F-rational, $0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')} = 0$. By above Lemma 6.18, $\theta(0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})}) = 0$. Since θ is injective, $0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})} = 0$. Hence $\mathscr{R}_{\mathfrak{M}}$ is F-rational. □

Theorem 6.19. *Let $\mathscr{R}$ be F-rational. R is F-rational if and only if $0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')} = 0$.*

Proof. Since $\mathscr{R}$ is F-rational and excellent, it is Cohen-Macaulay. Hence G is Cohen-Macaulay. As t^{-1} is a non-zero divisor on $\mathscr{R}'$, $\mathscr{R}'$ is also Cohen-Macaulay. As $\mathscr{R}'_{t^{-1}} = R[t, t^{-1}]$; R is also Cohen-Macaulay. Assume $0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')} = 0$. To see R is F-rational, its enough to show that $0^*_{H^d_{\mathfrak{m}}(R)} = 0$. Applying Snake lemma to the diagram (6.4) we see that for all $n \geq 0$,

$$\begin{tikzcd} 0 \arrow[r] & H^d_{\mathfrak{m}}(R)t^n \arrow[r] \arrow[d, "cF^e"] & {[H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')]_n} \arrow[r] \arrow[d, "cF^e"] & 0 \\ 0 \arrow[r] & H^d_{\mathfrak{m}}(R)t^{qn} \arrow[r] & {[H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')]_{qn}} \arrow[r] & 0 \end{tikzcd} \tag{6.6}$$

Hence we have $0^*_{H^d_{\mathfrak{m}}(R)} = 0$. Conversely, assume R is F-rational; hence $0^*_{H^d_{\mathfrak{m}}(R)} = 0$. Again by Snake lemma applied to the diagram (6.4) we see that for all $n < 0$, $[0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')}]_n =$

$[\theta(0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})})]_n \simeq [0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})}]_n = 0$. Also from diagram (6.6) we see that for all $n \geq 0$, $[0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')}]_n = 0$. □

Corollary 6.20. *Let (R, m) be an excellent d-dimensional F-rational ring and I be an m-primary ideal. If $\mathscr{R}$ is F-rational, $\mathscr{R}'$ is so.*

Proof. First note that $xt \in It$, the inclusion $\mathscr{R} \hookrightarrow \mathscr{R}'$, induces an equality $\mathscr{R}_{xt} = \mathscr{R}'_{xt}$. Since R is F-rational, $\mathscr{R}'_{t^{-1}} = R[t, t^{-1}]$ is also F-rational. Hence $\operatorname{Spec} \mathscr{R}' \setminus \{\mathfrak{M}'\}$ is F-rational. By Theorem 6.19 $0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')} = 0$; since $\mathscr{R}'$ is Cohen-Macaulay, $\mathscr{R}'_{\mathfrak{M}}$ is F-rational. □

The proof of the following proposition is word by word translation of Proposition 2.13 in [FW89] with necessary changes.

Proposition 6.21. *Let (A, m) be an n-dimensional excellent Cohen-Macaulay reduced ring. Let f be a regular element of A such that*
(i) $A/(f)$ is F-injective.
(ii) A_f is F-rational.
Then A is F-rational.

Proof. Since $f \in A$ is regular, f can be extended to a system of parameter of A, say $f = f_1, f_2, \ldots, f_n$. Let $I = (f_1, \ldots, f_n)$. Since A is Cohen-Macaulay it is enough to show that $I^* = I$. Since A is a reduced excellent local ring such that A_f is F-rational, then by Theorem 5.24 there exists an positive integer k such that f^k is a parameter test element. Let $x \in I^*$, then $f^k x^q \in I^{[q]}$ for all $q \geq 1$. Since A is Cohen-Macaulay, $x^q \in (f_1^{q-k}, f_2^q, \ldots, f_n^q)$. Reducing modulo (f), we get $\overline{x^q} = (\overline{f_2^q}, \ldots, \overline{f_n^q})$ in $A/(f)$, where $\bar{\cdot}$ denotes the image of elements of A in $A/(f)$. Since $A/(f)$ is F-injective and $(f_2, \ldots, f_n)$ is a system of parameters of $A/(f)$, we have $\overline{x} \in (\overline{f_2} \ldots \overline{f_n})$. Hence $x \in I$. □

Discussion 6.22. By above Proposition we can say the following thing. If (R, m) is an excellent F-rational ring and I is an m-primary ideal such that G is F-injective and $\mathscr{R}$ is Cohen-Macaulay, then $\mathscr{R}$ is F-rational. This can be seen in the following way. First note that since $\mathscr{R}$ is Cohen-Macaulay, G is Cohen-Macaulay. Now $G = \mathscr{R}'/(t^{-1})$ and t^{-1} is a non-zero divisor in $\mathscr{R}'$; hence $\mathscr{R}'$ is also Cohen-Macaulay domain. By above proposition we see that $Jt + (t^{-1})$ is tightly closed in $\mathscr{R}'_{\mathfrak{M}'}$. Hence Lemma 6.18, $0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})} = 0$. Hence by Lemma 6.13 $\mathscr{R}$ is F-rational. This result is useful. Let $R = K[x_1, x_2, \cdots, x_n]/(f)$, where f is a homogeneous element in $K[x_1, \cdots, x_n]$, be F-rational ring. Let G denote the associated graded ring with respect to its homogeneous maximal ideal $(x_1, \cdots, x_n)$. We know that $G \simeq R$; hence the Rees algebra $R[\mathrm{m}t]$ is F-rational.

6.2 F-rationality of Rees algebras over two dimensional F-rational rings

In this section we study F-rationality of Rees algebra over two dimensional excellent F-rational local ring. We prove:

Theorem 6.23. *Let (R, m) be a 2-dimensional excellent F-rational ring of prime characteristic $p > 0$. Let I be an integrally closed m-primary ideal. $\mathscr{R}$ is also F-rational.*

The above theorem is proved in [HWY02, Theorem 3.1].
Outline of the proof: We first prove that that $\mathscr{R}'_{\mathfrak{M}'}$ is F-rational, then will show $\mathscr{R}$ is F-rational.

Let $J = (x, y)$ be a minimal reduction for I. Since R is F-rational, it is pseudo-rational; hence by Theorem 5.33 $I^2 = JI$. Hence $I^q = J^{q-1}I$, for all $q \geq 2$.

Lemma 6.24. $(xt, yt, t^{-1})^* = (xt, yt, t^{-1})$.

Proof. Let $\alpha \in (xt, yt, t^{-1})^*$ be a homogeneous element. We write $\alpha = at^k$, $a \in I^k$.
Case 1: If $k < 0$, then $\alpha \in (t^{-1}) \subset (xt, yt, t^{-1})$.
Case 2: If $k = 0$, then for all $q \gg 0$, write

$$c\alpha^q = ca^q = a_1 t^{-q} x^q t^q + a_2 t^{-q} y^q t^q + a_3 t^q t^{-q}, \text{ where } a_1, a_2 \in R \text{ and } a_3 \in I^q.$$

Hence $ca^q \in I^q$ for all $q \gg 0$. Since I is integrally closed, $a \in I$. Hence $a \in Itt^{-1} \subset (xt, yt, t^{-1})$.
Case 3: If $k = 1$, then for all $q \gg 0$, write

$$c\alpha^q = ca^q t^q = a_1 x^q t^q + a_2 y^q t^q + a_3 t^{2q} t^{-q}, \text{ where } a_1, a_2 \in R \text{ and } a_3 \in I^{2q}.$$

Since $I^2 = JI$, $I^{2q} = J^{2q-1}I$. Since $\dim R = 2$, $J^{2q-1} \subseteq J^{[q]}$. Hence $ca^q \in J^{[q]}$ for all $q \gg 0$. Since R is F-rational, $a \in J$; $at \in Jt$.
Case 4: If $k \geq 2$, $a \in I^k = JI^{k-1}$. Hence $at^k \in JtIt^{k-1}$. □

Proof of Theorem 6.23. Since R is excellent F-rational, it is Cohen-Macaulay. So $\mathscr{R}'$ is homomorphic image of a Cohen-Macaulay ring. Since (xt, yt, t^{-1}) is a homogeneous system of parameter of $\mathscr{R}'$ and $((xt, yt, t^{-1})^* = (xt, yt, t^{-1}))$, Hence $((xt, yt, t^{-1})\mathscr{R}'_{\mathfrak{M}'})^* = (xt, yt, t^{-1})^*\mathscr{R}'_{\mathfrak{M}'} = (xt, yt, t^{-1})\mathscr{R}'_{\mathfrak{M}'}$. Hence $\mathscr{R}'_{\mathfrak{M}'}$ is F-rational. Hence by Theorem 6.16 $\mathscr{R}$ is Cohen-Macaulay and by Lemma 6.18, $\mathscr{R}_{\mathfrak{M}}$ is F rational. By Lemma 6.13, $\mathscr{R}$ is F-rational. □

6.3 F-rationality of base ring

In this section we study the following question: Let (R, m) be an excellent Cohen-Macaulay ring. If I be an m-primary ideal such that $\mathscr{R}$ is F-rational, is R F-rational?

In general the answer of this question is no. Later we will see examples given in [HWY02] where $\mathscr{R}$ is F-rational, but R is not. If $\mathscr{R}$ is Gorenstein and F-rational, then R is weakly F-regular, in particular F-rational. This can be seen easily. Since $\mathscr{R}$ is Gorenstein it is F-regular. Note that R is normal, as $\mathscr{R}$ is so. Hence $R[It]$ is domain. Let $S = \mathscr{R}_{\mathfrak{M}}$. Let J be an ideal in R. Now $J^*S \subseteq (JS)^* = (JS)$, as S is weakly F-regular. Hence $J^* = J^*S \cap R \subseteq JS \cap R = J$, first and the third equality follows since $R \overset{\oplus}{\hookrightarrow} S$.

In [HWY02], they prove:

Corollary 6.25 ([HWY02, Corollary 2.13])**.** *Let (R, m) be an excellent Cohen-Macaulay ring with $\dim R = d \geq 2$ and I be an m-primary ideal of R. If $\mathscr{R}$ is F-rational and $a(G) \neq -1$, then R is F-rational.*

Note that since $\mathscr{R}$ is F-rational, then $a(G) \leq -1$. If $\mathscr{R}$ is F-rational with $a(G) = -1$, then R might be F-rational or might not be.

Example 6.26. *Let $R = K[x_1, x_2, x_3]/(x_1^2 + x_2^2 + x_3^2)$, where K is a field of prime characteristic 5. Let $\mathrm{m} = (x_1, x_2, x_3)$ and $\mathscr{R} = R[\mathrm{m}t]$. Then $G = \mathscr{R}/\mathrm{m}\mathscr{R} \simeq R$ and $a(G) = -1$. By Discussion 6.22 $\mathscr{R}$ is F-rational. By Theorem 5.49 R is F-rational.*

Example 6.27. *Let $R = K[x, y, z]/(z^2 + x^2y + xy^2)$ where $\operatorname{char} R = 2$. Let $(x, y, z) = \mathrm{m}$ and $\mathscr{R} = R[\mathrm{m}t]$. Then $G = \mathscr{R}/\mathrm{m}\mathscr{R} = K[x, y, z]/z^2$. Hence $a(G) = -1$. By Example 3.9 of [HWY02], $\mathscr{R}$ is F-rational. But R is not, as $z \in (x, y)^*$.*

Theorem 6.28. *Let (R, m) be a d-dimensional excellent Cohen-Macaulay local ring of prime characteristic $p > 0$ and I be an m-primary ideal of R. If $\mathscr{R}$ is F-rational and $H^d_{G_+}(G)_{-1} \xrightarrow{F} H^d_{G_+}(G)_{-p}$ is injective, then R is F-rational.*

Proof. First we will show that $0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')} = 0$. Note that we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{R}'(1) \arrow[r, "t^{-1}"] \arrow[d, "t^{1-p}F"] & \mathscr{R}' \arrow[r] \arrow[d, "F"] & G \arrow[r] \arrow[d, "F"] & 0 \\ 0 \arrow[r] & \mathscr{R}'(1) \arrow[r, "t^{-1}"] & \mathscr{R}' \arrow[r] & G \arrow[r] & 0. \end{tikzcd} \tag{6.7}$$

Since $\mathscr{R}$ is Cohen-Macaulay, G is Cohen-Macaulay; hence $\mathscr{R}'$ is so. Hence by above commutative diagram we have the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & H^d_{\mathfrak{M}'}(G) \arrow[r] \arrow[d, "F"] & H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')(1) \arrow[r, "t^{-1}"] \arrow[d, "t^{1-p}F"] & H^{d+1}_{\mathfrak{M}'}(\mathscr{R}') \arrow[r] \arrow[d, "F"] & 0 \\ 0 \arrow[r] & H^d_{\mathfrak{M}'}(G) \arrow[r] & H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')(1) \arrow[r, "t^{-1}"] & H^{d+1}_{\mathfrak{M}'}(\mathscr{R}') \arrow[r] & 0 \end{tikzcd} \tag{6.8}$$

By diagram (6.4) for all $n < 0$, $[H^{d+1}_{\mathfrak{M}'}\mathscr{R}']_n \simeq [H^{d+1}_{\mathfrak{M}}\mathscr{R}]_n$. Let $\xi \in 0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')}$ be a homogeneous element of degree n. If $n < 0$, then $\theta^{-1}(\xi) \in 0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})}$. Hence $\theta^{-1}(\xi) = 0$, as $\mathscr{R}$ is F-rational. Hence $[0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')}]_n = 0$ for all $n < 0$. Let $k+1 = \min\{n \geq 0 \mid [0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')}]_n \neq 0\}$. Note that minimum exists since $[0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')}]_n = 0$ for $n < 0$. Let $\xi \in [0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')}]_{k+1}$ be a non-zero homogeneous element. Then $\xi t^{-1} = 0$; hence there exists non-zero $\eta \in [H^d_{\mathfrak{M}'}(G)]_k$ such that $\eta \mapsto \xi$ under the map $H^d_{\mathfrak{M}'}(G) \to H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')(1)$. Since $a(G) \leq -1$, $k = -1$. By hypothesis $F(\eta) \neq 0$. Since $0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')}$ is F-stable submodule of $H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')$, $F(\xi) \in [0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')}]_0$, $t^{1-p}F(\xi) = 0$, which contradicts the commutativity of left square of the diagram (6.8). Hence $0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')} = 0$. By Theorem 6.19, R is F-rational. $\square$

The above proof also gives $\mathscr{R}'$ is F-rational, because in the proof we show $0^*_{H^{d+1}_{\mathfrak{M}'}(\mathscr{R}')} = 0$ and $\operatorname{Spec}\mathscr{R}' \setminus \{\mathfrak{M}'\}$ is F-rational, as $\mathscr{R}$ and R is F-rational.

As a corollary we get result of Hara, Watanabe and Yoshida [HWY02, Corollary 2.13]:

Corollary 6.29. *Let R and $\mathscr{R}$ be as above in the theorem. If $a(G) < -1$, R is F-rational*

Proof. Since $a(G) < -1$, the criterion on G vacuously holds; hence the corollary. $\square$

While the above condition is sufficient for F-rationality of R but not necessary.

Example 6.30. *Let $R = K[[x, y, z]]/(x^2 + y^3 + z^5)$, where K is a field of characteristic 7 and $\mathrm{m} = (x, y, z)$. By Example 5.50, R is a 2-dimensional F-rational ring. $m = \overline{m}$. By Theorem 5.33, for each i, $m^i = \overline{m^i}$. Then by Theorem 6.23, $R[mt]$ is F-rational. Now $G = K[x, y, z]/(x^2)$, $H^2_{G_+}(G)_{-1} \to H^2_{G_+}(G)_{-7}$ is not injective because $H^2_{G_+}(G)$ can be computed from the Čech complex $\check{C}^\bullet(y, z; G)$. Since $x \notin (y, z)$, $[x/yz]$ is a non-zero element of $\in H^2_{G_+}(G)_{-1}$ and $F([x/yz]) = x^7/y^7z^7 = 0$.*

Further questions:

Q1. Let (R, m) be an F-rational excellent ring and I be its ideal such that $\operatorname{Proj} R[It]$ is Cohen-Macaulay and normal. If $\operatorname{Proj} R[It]$ is F-rational, is $R[It]$ F-rational?

Q2. Find a necessary condition such that $R[It]$ F-rational will imply R F-rational.

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