Combinatorial and topological aspects of chain and planar polygon spaces

By

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DECLARATION

I declare that the thesis entitled **"Combinatorial and topological aspects of chain and planar polygon spaces"** submitted by me for the degree of **Doctor of Philosophy in Mathematics** is the record of academic work carried out by me under the guidance of Professor "Priyavrat Deshpande" and this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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CERTIFICATE

I certify that the thesis entitled **"Combinatorial and topological aspects of chain and planar polygon spaces"** submitted for the degree of **Doctor of Philosophy in Mathematics** by "Navnath Daundkar" is the record of research work carried out by him under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent his independent work in a very substantial measure.

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Date: June, 2022.

Priyvrat Deshpande
Thesis Supervisor.

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Dedicated to my Mother

Abstract

Let $\ell = (\ell_1, \ldots, \ell_m)$ be an m-tuple of positive real numbers, also called a *length vector*. The *moduli space of planar polygons* (or *planar polygon space*) associated with ℓ, denoted by M_ℓ (respectively $\overline{M}_\ell$), is the collection of all closed piecewise linear paths in the plane upto orientation preserving isometries (respectively isometries) with side lengths $\ell_1, \ell_2, \ldots, \ell_m$. Generically, both M_ℓ and $\overline{M}_\ell$ are closed, smooth manifolds of dimension $m - 3$. We investigate some combinatorial and topological aspects of these moduli spaces in this thesis.

This thesis is divided into three parts:

1. There is a subclass of planar polygonal spaces called *chain spaces* such that each chain space is (topologically) a toric variety. These spaces are the fixed point sets of an involution on a toric manifold known as the *abelian polygon space*, whose elements can be viewed as piece-wise linear paths with side lengths $\ell_1, \ell_2, \ldots, \ell_{m-1}$ terminating on the plane $x = \ell_m$, modulo the rotations about the X-axis. In the first part of the thesis, we show that the moment polytope of the chain space is completely characterized by the combinatorial data, called the *short code* of the length vector. We also classify aspherical chain spaces using the result of Davis, Januszkiewicz, and Scott.

2. In any planar polygon space, the real points of moduli space of genus-zero curves embed as an open dense subset. As a result, polygonal spaces form a compactification of the real moduli space of genus-zero curves. Kapranov showed that the real points of the Deligne-Mumford-Knudson compactification can be obtained from the projective Coxeter complex of type A by blowing up along the minimal building set. In the second part of the thesis, we show that the planar polygon spaces can also be obtained from the projective Coxeter complex of type A by performing an iterative cellular surgery along the subcollection of the minimal building set. Interestingly, this subcollection is generated by the combinatorial data associated with the length vector called the *genetic code*.

3. We obtain the small cover structure on $\overline{\mathrm{M}}$ ’s associated with length vectors having long genetic codes. Using this structure we obtain some numerical invariants. In the end, we study the Borsuk-Ulam theorem for moduli spaces M ’s. Furthermore, we obtain a formula for the Stiefel-Whitney height in terms of genetic code. Finally, we determine for which of these spaces a generalized version of the Borsuk-Ulam theorem hold.

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List of publications/preprints associated with the thesis

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3. Navnath Daundkar and Priyavrat Deshpande. "Building planar polygon spaces from the projective braid arrangement". (2022). arXiv:2204.10278.
4. Navnath Daundkar and Priyavrat Deshpande. "n-dimensional Klein bottle is a real Bott manifold" (2022). arXiv:2201.01063.

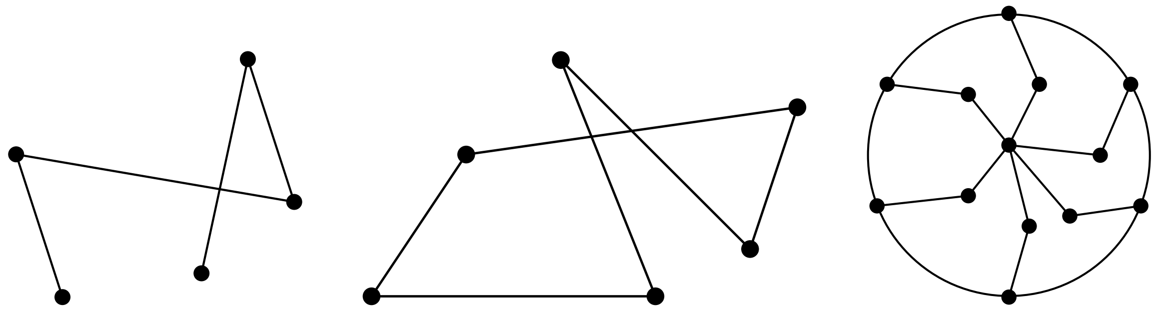
Contents

Chapter 1

Introduction

A *mechanical linkage* is a mechanism in the Euclidean plane, R^2, consisting of rigid bars with fixed side-lengths linked by revolving joints. Assume that a pair of two adjacent joints are fixed on the X-axis and the others are free to move, causing the angles between bars to change but the links to remain connected. The *configuration space* of a mechanical linkage is a space of all its admissible states.

These spaces are useful in a variety of applications; for example, in robotics, they aid in the development of motion planning algorithms, and in molecular biology, they describe molecular shapes. The configuration spaces of various type of mechanical linkages have been extensively studied from a topological standpoint.



Open linkage Polygonal linkage Spidery linkage

FIGURE 1.1: Mechanical linkages

We now formally define the configuration spaces of closed polygonal linkages that we are most interested in. A *length vector* is a tuple of positive real numbers. The *moduli space of planar polygons* associated with a length vector $= (_1,:::, _m)$, denoted by M , is the collection of all closed piecewise linear paths in the plane considered upto orientation preserving isometries. Equivalently, we can define M as

$$M := \left\{ (v_1, v_2, \ldots, v_m) \in (S^1)^m : \sum_{i=1}^{m} {}_i v_i = 0 \right\} \Big/ SO_2,$$

where S^1 is the unit circle and the group of orientation preserving isometries SO_2 acts diagonally. The moduli space of planar polygons (associated with) viewed up to isometries is defined as

$$\overline{M} := \left\{ (v_1, v_2, \ldots, v_m) \in (S^1)^m : \sum_{i=1}^{m} {}_i v_i = 0 \right\} \Big/ O_2.$$

It was shown [23, Theorem 1.3] that, if we choose a length vector such that $\sum_{i=1}^{m} {}_i \neq 0$ then the moduli spaces M and $\overline{M}$ are closed, smooth manifolds of dimension $m - 3$. Such length vectors are called *generic* length vectors. *Unless otherwise stated, the length vectors in the rest of this thesis are assumed to be generic*. One of the tasks in topological robotics is to express topological invariants of these moduli spaces in terms of the length vector.

Many mathematicians contributed to the study of the topological aspects. In 1998, Kamiyama and Tezuka [47] proved that for a length vector $= (1, \ldots, 1, r)$, the integral homology of M is torsion-free and computed the Betti numbers. Later in 2006, Farber and Schutz [22] showed that for an arbitrary length vector, the integral homology groups of M are torsion-free and also described the Betti numbers of M in terms of the combinatorial data associated with the length vector. For the length vector $= (1, 1, \ldots, 1)$, Kamiyama [46] determined the homology groups $H(\overline{M}, \mathbb{Z}_p)$ for odd primes and $H(\overline{M}, \mathbb{Q})$. Hausmann and Knutson computed the $\mathbb{Z}_2$-cohomology ring of $\overline{M}$ in [31, Corollary 9.2]. In his bachelor's thesis [63], K. Walker conjectured that the side lengths of polygonal linkages can be recovered from the intrinsic algebraic properties of cohomology algebra of polygon space. In [25], Farber, Hausmann, and Schutz proved the conjecture in the affirmative for spatial polygon spaces and M in a modified form. Panina [60] constructed a cell structure on M and also studied many combinatorial properties (see [61]).

The spatial version of polygonal linkages were studied by Hausmann-Knutson ([31], [34]), Kapovich-Milson [49], Klyachko [52], Kamiyama [45, 41, 42, 38, 40], Leonor et al. [1], Mandini [54, 53, 55] etc. In this thesis, we only focus on planar polygon spaces.

1.1 Aspherical manifolds

A smooth manifold is said to be *aspherical* if its universal cover is contractible. Whether or not a smooth manifold is aspherical is an interesting question, in general. The question in our case is

Question 1.1. *What is the characterization of length vectors such that the corresponding planar polygon space is aspherical?*

We provide a partial answer to this question. More precisely, we classify aspherical chain spaces, a subclass of planar polygon spaces.

Definition 1.2. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a length vector. The chain space corresponding to α is defined as :

$$\mathrm{Ch}(\alpha) = \left\{ (v_1, v_2, \ldots, v_{m-1}) \in (S^1)^{m-1} : \sum_{i=1}^{m-1} \alpha_i x_i = \alpha_m \right\} \Big/ \mathbb{Z}_2,$$

where $v_i = (x_i, y_i)$ and the group $\mathbb{Z}_2$ acts diagonally.

Note that $\mathrm{Ch}(\alpha) \neq \emptyset$ if $\alpha_m \leq \sum_{i=1}^{m-1} \alpha_{m-1}$. In fact, if α is generic then $\mathrm{Ch}(\alpha)$ is a smooth, closed manifold of dimension $m - 2$. The elements of a chain space can be thought of as a piecewise linear path with side lengths $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ terminating at the line $x = \alpha_m$, considered up to the reflection across the X-axis.

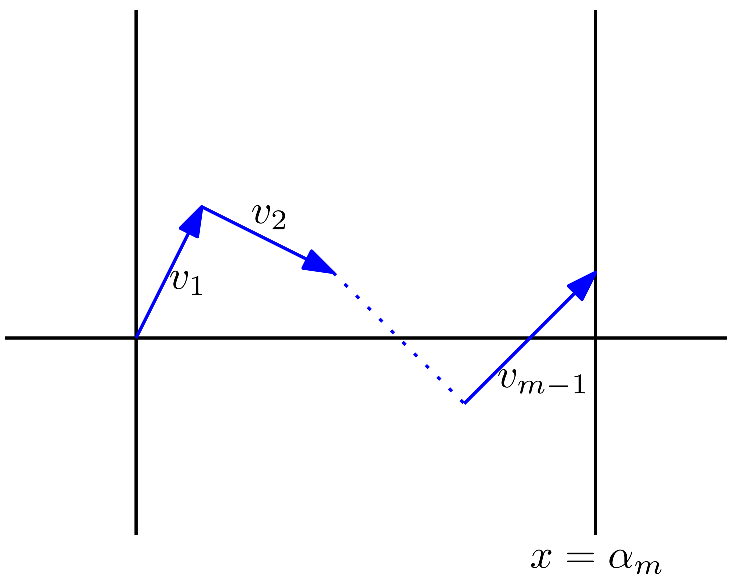


FIGURE 1.2: An $(m - 1)$-sided chain

The spatial version of the chain space (for a generic length vector) was introduced by Hausmann and Knutson; called the *abelian polygon space*. It is a *toric manifold* (see [31, Section 1] for a proof). An element of this spatial version can be viewed as a piece-wise linear path with side lengths $\ell_1, \ell_2, \ldots, \ell_{m-1}$ terminating on the plane $x = \ell_m$, modulo the rotations about the X-axis. It is easy to see that $\mathrm{Ch}(\ell)$ is a fixed point set of an anti-symplectic involution on its spatial version. Thus $\mathrm{Ch}(\ell)$ is a small cover (see [19, Section 1]) a topological analogue of a real toric varieties.

Davis, Januszkiewicz and Scott gave a combinatorial condition to decide whether or not a small cover is aspherical.

Theorem 1.3 ([18, Theorem 2.2.5])**.** *Let M be a small cover and P be the associated quotient polytope. Then the following statements are equivalent:*

1. *M is aspherical.*
2. *The boundary complex of P is dual to a flag complex.*
3. *The dual cubical subdivision of M is nonpositively curved.*

We characterize those ℓ for which $\mathrm{Ch}(\ell)$ is aspherical. To do that we introduce a combinatorial object called the *short code* of a length vector (see, Section 3.2). We first show that the short code of a (generic) length vector completely determines the homeomorphism type of chain space. Moreover, the quotient polytope associated with the chain space is also determined by the short code. Then we establish a combinatorial condition to determine whether or not the chain space is aspherical.

In [10], we prove the following theorem in collaboration with Priyavrat Deshpande.

Theorem 1.4 (Theorem 3.35)**.** *Let ℓ be a generic length vector. Then the corresponding chain space is aspherical if and only if the short code of ℓ is one of the following :*

1. $\langle\{1, 2, \ldots, m-3, m\}\rangle$,
2. $\langle\{1, 2, \ldots, m-2, m\}\rangle$,
3. $\langle\{1, 2, \ldots, m-4, m-2, m\}\rangle$,
4. $\langle\{1, 2, \ldots, m-4, m-1, m\}\rangle$.

1.2 Polygon spaces and the braid arrangement

The moduli space of n-punctured Riemann spheres, M_0^n, is an important object in geometric invariant theory. There is the Deligne-Knudson-Mumford compactification $\overline{M}_0^n$ of this space which has been studied widely. We refer the reader to [50], [51]) for a comprehensive introduction. We are interested in the *real points* $\overline{M}_0^n(\mathbb{R})$ (see Definition 4.10) of this compactification.

The real moduli space of n-punctured Riemann spheres is defined as

$$M_0^n(\mathbb{R}) = \frac{(\mathbb{R}P^1)^n \setminus \Delta}{PGL_2(\mathbb{R})},$$

where $\Delta = \{(x_1, \dots, x_n) \in (\mathbb{R}P^1)^n : \exists\, i, j, x_i = x_j\}$ and the group of projective automorphisms $PGL_2(\mathbb{R})$ acts diagonally.

Since a projective automorphism of $\mathbb{R}P^1$ is uniquely determined by the images of three points, we identify

$$M_0^n(\mathbb{R}) = (\mathbb{R}P^1)^{n-3} \setminus \Delta\ ,$$

where

$$\Delta = \{(x_1, \dots, x_{n-3}) \in (\mathbb{R}P^1)^{n-3} : x_i = x_j,\ x_i = 0, 1, \infty\}.$$

The real part (or real points) of Deligne-Mumford-Knudson compactification, $\overline{M}_0^n(\mathbb{R})$, is obtained by iterated blow-ups of $(\mathbb{R}P^1)^{n-3}$ along non-normal crossings of $\{x_i = x_j, x_i = 0, 1, \infty\}$ in an increasing order of dimension. Kapranov showed that the real points of the Deligne-Mumford-Knudson compactification $M_0^n(\mathbb{R})$, can be obtained from the projective Coxeter complex of type A (equivalently, projective braid arrangement) by iteratively blowing up along the minimal building set.

It is known that for a generic , $\overline{M}$ contains $M_0^n(\mathbb{R})$ as an open dense set. In particular, $\overline{M}$ forms a compactification of $M_0^n(\mathbb{R})$. Therefore, it is natural to ask the following question.

Question 1.5. *Is there a way to obtain* M *(respectively* $\overline{M}$ *) from the Coxeter complex (respectively the projective Coxeter complex) by some iterative topological operation?*

The first step to solve Question 1.5 is to get an appropriate cell structure on both M and $\overline{M}$. We achieve this by constructing a submanifold arrangement on them. Interestingly, this submanifold arrangement locally looks like either a braid arrangement or a product of braid arrangement. Let K and $\overline{K}$ be the induced cell structures on M and $\overline{M}$, respectively. Then we introduce the cellular version of surgery on simple cell complexes. We define a subcollection G of Min(B) and $P(G)$ of Min($P(B)$) which depends only on the length vector.

In [8], we prove the following theorem in collaboration with Priyavrat Deshpande.

Theorem 1.6 (Theorem 4.46)**.** *The iterated cellular surgery on Coxeter complex CA_{m-2} (respectively, on projective Coxeter complex PCA_{m-2}) along the elements of G (respectively $P(G)$) produces a cell complex homotopy equivalent to K (respectively $\overline{K}$).*

1.3 Some numerical aspects

1.3.1 The n-dimensional Klein bottle

Recently Davis [12] initiated a study of an n-dimensional analogue of the Klein bottle, denoted as K_n.

$$K_n := \frac{(S^1)^n}{(z_1, \ldots, z_{n-1}, z_n) \quad (\bar{z}_1, \ldots, \bar{z}_{n-1}, -z_n)}. \tag{1.1}$$

The circle S^1 is considered as the unit circle in $\mathbb{C}$ and $\bar{z}$ is the complex conjugate. It is easy to see that K_2 is the usual Klein bottle. It follows from [32, Proposition 2.1] that K_n is homeomorphic to $\overline{M}$ when the genetic code of is $< 1, 2, \ldots, n-1, n+3 >$. He determined many topological invariants of this space as well as some of its manifold-theoretic properties.

A *real Bott tower* is a sequence of $\mathbb{R}P^1$-bundles:

$$M_n \to M_{n-1} \to \quad \to M_1 \to \{\ \},$$

where each $\mathbb{R}P^1$-bundle $M_i \to M_{i-1}$ is the projectivization of Whitney sum of two real line bundles on M_{i-1}, one of them is the trivial line bundle. For each i, the manifold M_i is called a *real Bott manifold*. The homeomorphism type of real Bott manifolds is completely determined by Stiefel-Whitney classes of the line bundles at each step. Hence, an efficient way to encode the homeomorphism type of these manifolds is using a square matrix, called the Bott matrix, containing 0's and 1's.

We show that this generalized Klein bottle is a real Bott manifold and explicitly determines the associated Bott matrix.

In [9], we prove the following results in collaboration with Priyavrat Deshpande.

Theorem 1.7 (Theorem 5.4)**.** *The n-dimensional Klein bottle K_n is a real Bott manifold corresponding to the Bott matrix*

$$B = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{1.2}$$

As a consequence of the above theorem, we have the following result.

Proposition 1.8 (Theorem 5.8)**.** *Let $\beta_i(K_n, \mathbb{Q})$ be the ith rational Betti number of K_n. Then*

$$\beta_i(K_n, \mathbb{Q}) = \begin{cases} \binom{n-1}{i} & \textit{if } i \textit{ is an even integer} \\ \binom{n-1}{i-1} & \textit{if } i \textit{ is an odd integer.} \end{cases}$$

1.3.2 The Borsuk-Ulam theorem for free $\mathbb{Z}_2$-spaces

The Borsuk-Ulam (BU) theorem has been an object of central attraction in topology for almost a century time. It states that any continuous map from the d-sphere S^d to the Euclidean space $\mathbb{R}^d$ must identify a pair of antipodal points. Recently, the BU theorem has been studied for many different complexes with a free $\mathbb{Z}_2$-action. For instance, Musin [58] considered PL-manifolds, Goncalves et al. [29] considered finite-dimensional CW-complexes with a free cellular involution.

For a topological space X with a fixed-point-free involution τ, we say that (X, τ, d) is a BU triple if for every continuous map $f : X \to \mathbb{R}^d$ there exists $x \in X$ such that $f(x) = f(\tau(x))$. In [29], the authors used index and Stiefel-Whitney height to find BU triples. It is known that the Stiefel-Whitney height gives a lower bound for the index. Moreover, dual to the index notion, there is a notion of coindex of a free $\mathbb{Z}_2$-space which asserts as a lower bound for the Stiefel-Whitney height of the space. More precisely, $\mathrm{coind}(X) \leq \mathrm{ht}(X) \leq \mathrm{ind}(X)$. The complexes for which all these inequalities become equality are called *tidy* and *non-tidy* otherwise. For more discussion about the index, coindex, and tidy spaces, the reader is referred to the book of Matoušek [56] and Csorba's Ph.D. thesis [7].

Observe that M_ℓ admits an involution τ defined by

$$\tau(v_1, v_2, \ldots, v_m) = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m), \tag{1.3}$$

where $\bar{v}_i = (x_i, -y_i)$ and $v_i = (x_i, y_i)$. Geometrically, τ maps a polygon to its reflected image across the X-axis. Since we are dealing with only generic length vectors, τ does not have fixed points. In particular, M_ℓ is a free $\mathbb{Z}_2$-space. It is easy to see that $\overline{M}_\ell = M_\ell/\tau$. Thus M_ℓ is a double cover of $\overline{M}_\ell$.

Panina [60] showed that the orientation preserving moduli spaces admit a CW-structure with free $\mathbb{Z}_2$-action. It is therefore natural to look for BU triples among these spaces and also identify which one of these are tidy.

We compute these parameters for some moduli spaces of polygons. We also determine for which of these spaces a generalized version of the Borsuk-Ulam theorem holds. For a specific class of length vectors, we also obtain a formula for the Stiefel-Whitney height in terms of the genetic code, a combinatorial data associated with side lengths.

In, [11], we proved the following results in collaboration with Priyavrat Deshpande, Shuchita Goyal, and Anurag Singh.

Proposition 1.9 (Theorem 5.49)**.** *Let $\langle\{b, n\}\rangle$ be the genetic code of a length vector ℓ. Then M_ℓ is tidy if and only if b is an odd integer.*

Let

$$S_k = \left\{(b_1, \ldots, b_k) \in \mathbb{Z}^k_{\geq 0} : \sum_{j=1}^{k} b_j = k\right\}.$$

Theorem 1.10 (Theorem 5.53)**.** *Let $\langle\{g_1, \ldots, g_k, n\}, \{b, n\}\rangle$ be the genetic code of ℓ, where $g_1 < g_2 < \cdots < g_k < b$ and let $a_i = g_{k+1-i} - g_{k-i}$ for $1 \leq i \leq k$. Then*

$$R^{n-3} = \sum_{B} \prod_{i=1}^{k} \binom{a_i + b_i - 2}{b_i} + (b - g_k),$$

where $|B| = k$ and $B \in S_k$.

Corollary 1.11 (Corollary 5.65)**.** *Let* $\langle \{g_1, \ldots, g_k, n\}, \{b, n\} \rangle$ *be the genetic code of* α*. If*

$$\sum_{|B|=k,\ B\in S_k} \prod_{i=1}^{k} \binom{a_i + b_i - 2}{b_i} + (b - g_k) \equiv 0 \pmod 2,$$

then M_α *is tidy.*

1.4 Organization of the thesis

We end with the brief description of the chapters mentioned in the thesis.

In Chapter 2, we recall the basics of polygon spaces. We focus on the following three aspects

1. The (co)homology of polygon spaces.
2. The regular cell structure.
3. The geometric description.

These aspects have received a lot of attention and are relevant to the context of this thesis.

In Chapter 3, we introduce the object called the *short code* of a length vector then use the Davis-Januskeiwicz-Scott's techniques to classify aspherical chain spaces. The majority of this chapter's content is drawn from the author's published article [10].

In the Chapter 4, we show that $\overline{M}_\alpha$ is obtained by performing a cellular surgery on the projective Coxeter complex of type A along certain subspaces induced by the braid arrangement. Interestingly, these subspaces are determined by the combinatorial data associated with a length vector called the genetic code. In particular, we answer the Question 1.5. This chapter's content is based on the author's preprint [8], which was a joint work with P. Deshpande.

The Chapter 5 contains four sections. In Section 5.1, we show that the n-dimensional Klein bottle is a real Bott manifold and explicitly determine the associated Bott matrix. In Theorem 5.25, we describe the small cover structure of polygon spaces with long genetic codes and compute their (rational) Betti numbers. In Section 5.1.3, we recall some results on the topological complexity. In Section 5.3, we study the Borsuk-Ulam theorem for polygon spaces. This section's content was adapted from the authors' article [11], which was a joint work with P. Deshpande, S. Goyal, and A. Singh.

Finally, we conclude the thesis in Chapter 6 with a discussion of some potential future directions.

Chapter 2

Moduli spaces of polygons

The purpose of this chapter is to provide an overview of a few polygon space-related results that will be used in subsequent chapters. In the following section, we define a length vector and then recall several definitions of moduli spaces of polygons and chains associated with a length vector. We also investigate some combinatorial invariants related to generic length vectors. Section 2.2 was divided into three sections. In the first section, we discussed Panina's cell structure on planar polygon spaces. The second section is concerned with the homology and cohomology of moduli spaces. In the third section, Hausmann's geometric description of lower-dimensional planar polygon spaces is recalled.

2.1 Definitions

Definition 2.1. A *length vector* $\ell := (\ell_1, \ell_2, \ldots, \ell_m)$ is an m-tuple of positive real numbers.

Definition 2.2. A *polygon* in $\mathbb{R}^2$ with a length vector $\ell = (\ell_1, \ell_2, \ldots, \ell_m)$, is a tuple of vectors $(v_1, \ldots, v_m) \in (S^1)^m$ such that

$$\sum_{i=1}^{m} \ell_i v_i = 0.$$

A polygon can be regarded as a closed piecewise linear path in $\mathbb{R}^2$ which starts at the origin and ends at the origin whose vertex set is

$$V_i = \left\{ \sum_{j=1}^{i} \ell_j v_j : 1 \leqslant i \leqslant m \right\}.$$

Note that $V_m = (0,0)$.

Definition 2.3. A length vector $\ell = (\ell_1, \ell_2, \ldots, \ell_m)$ is said to be *generic* if

$$\sum_{i=1}^{m} \epsilon_i \ell_i \neq 0 \text{, for } \epsilon_i \in \{-1, 1\}. \tag{2.1}$$

Geometrically, the Equation (2.1) says that there is no polygon (or chain) which lies inside a straight line with side lengths $\ell_1, \ell_2, \ldots, \ell_m$.

Example 2.4. The following are few examples and non-examples of polygons.

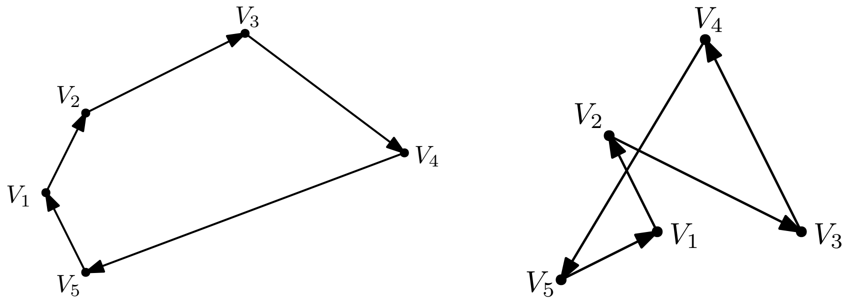


Figure 2.1: Polygons with length vector $(1,1,2,2,3)$.

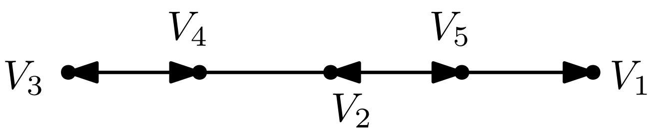


Figure 2.2: Degenerate polygon with length vector $(1,2,2,1,2)$.

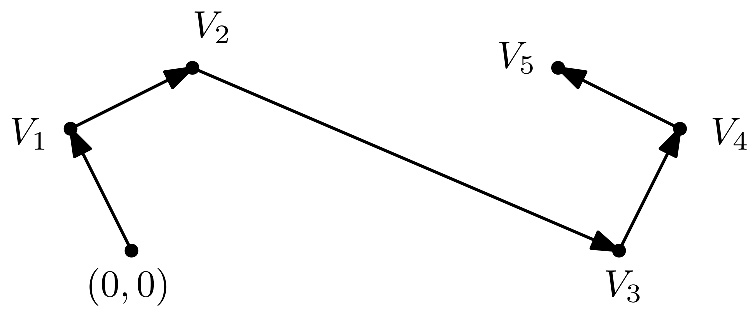


FIGURE 2.3: A non-example of a polygon with a length vector $(1,1,5,1,1)$.

Now we define different kinds of moduli spaces associated with a length vector.

Definition 2.5. The *spatial polygon space* N_ℓ parameterizes all closed piecewise linear paths in $\mathbb{R}^3$, whose side-lengths are prescribed by ℓ, up to rigid motions. Formally we can write N_ℓ as

$$N_\ell = \left\{ (v_1, v_2, \ldots, v_m) \in (S^2)^m : \sum_{i=1}^{m} \ell_i v_i = 0 \right\} \Big/ SO_3,$$

where S^2 is the unit sphere in $\mathbb{R}^3$ and the rotation group SO_3 acts diagonally on a tuple $(v_1, v_2, \ldots, v_m)$.

An element $v = (v_1, v_2, \ldots, v_m)$ of N_ℓ can be regarded as a closed piecewise linear path in $\mathbb{R}^3$ which starts at the origin and ends at the origin whose vertex set is

$$\left\{ V_i = \sum_{j=1}^{i} \ell_j v_j : 1 \leq i \leq m \right\}.$$

Without loss of generality sometimes we may call a closed piecewise linear path as a polygon.

Definition 2.6. Let $\ell = (\ell_1, \ell_2, \ldots, \ell_m)$ be a length vector. The *abelian polygon space* Apol is defined as

$$\text{Apol}(\ell) = \left\{ (v_1, v_2, \ldots, v_{m-1}) \in (S^2)^{m-1} : \pi\Big(\sum_{i=1}^{m-1} \ell_i v_i\Big) = \ell_m \right\} \Big/ SO_2,$$

where π is the projection $\pi(x, y, z) = x$

An element of Apol(ℓ) can be viewed as a piecewise linear path with side lengths $\ell_1, \ell_2, \ldots, \ell_{m-1}$ terminating on the plane $x = \ell_m$, modulo the rotations about the X-axis. We call such path an $(m-1)$-sided *chain*.

The spatial polygon spaces N_ℓ have been studied by Kapovich-Milson [49], Klaychko [52] and Hausmann-Knutson (see, for instance [34], [35].) The space $\mathrm{Apol}(\ell)$ was studied in [31]. It was shown that for a generic length vector ℓ, the corresponding N_ℓ and $\mathrm{Apol}(\ell)$ are both closed, smooth manifolds of dimension $2(m-3)$ and $2(m-2)$, respectively.

Recall the definition of M_ℓ from the Introduction. Observe that each SO_2 orbit of a polygon contains another polygon whose last side is on the X-axis. Therefore, we can rewrite the above definition of M_ℓ as

$$M_\ell := \left\{ (v_1, v_2, \dots, v_{m-1}) \in (S^1)^{m-1} : \sum_{i=1}^{m-1} \ell_i v_i = \ell_m e_1 \right\}. \tag{2.2}$$

Therefore, the moduli space M_ℓ can be thought of as a submanifold of $(S^1)^{m-1}$ as follows.

Remark 2.7. We note the following observation.

1. The involution on S^2 defined by $(x, y, z) \mapsto (x, y, -z)$ induces an involution on N_ℓ. Note that the fixed point set of this involution is $\overline{M}_\ell$.
2. The chain space's spatial version is the same as the abelian polygon space.. Note that the natural involution on S^2 defined by $(x, y, z) \mapsto (x, y, -z)$ induces an involution on $\mathrm{Apol}(\ell)$. It is easy to see that $\mathrm{Ch}(\ell)$ is the fixed point set of this involution.

Recall that a length vector is a tuple of positive real numbers. There are two important combinatorial objects associated with the length vector.

Definition 2.8. Let $\ell = (\ell_1, \ell_2, \dots, \ell_m)$ be a length vector. A subset $I \subset [m]$ is called an ℓ-*short* if

$$\sum_{i \in I} \ell_i < \sum_{j \notin I} \ell_j$$

and *long* otherwise.

We may write short for ℓ-short when the context is clear. The collection of short subsets may be very large. There is another combinatorial object associated with the length vector ℓ which further compactifies the information about the short subsets. Since the diffeomorphism type of a planar polygon spaces does not depend on the ordering of the side lengths of polygons, we assume that the length vector $\ell = (\ell_1, \ell_2, \ldots, \ell_m)$ satisfies $\ell_1 \leqslant \ell_2 \leqslant \cdots \leqslant \ell_m$.

Definition 2.9. For a length vector ℓ, consider the collection of subsets of $[m]$:

$$S_m(\ell) = \{J \subset [m] : m \in J \text{ and } J \text{ is short}\}$$

and a partial order $\leqslant$ on $S_m(\ell)$ by $I \leqslant J$ if $I = \{i_1, \ldots, i_t\}$ and $\{j_i, \ldots, j_t\} \subseteq J$ with $i_s \leqslant j_s$ for $1 \leqslant s \leqslant t$. The *genetic code* of ℓ is the set of maximal elements of $S_m(\ell)$ with respect to this partial order. If $A_1, A_2, \ldots, A_k$ are the maximal elements of $S_m(\ell)$ with respect to $\leqslant$ then the genetic code of ℓ is denoted by $\langle A_1, \ldots, A_k \rangle$.

The maximal elements A_i are called *genes*. A gene without n is called *gee*. Any subset $S \subset [m]$ is called *subgee* if it is dominated by a gee with respect to the partial order defined above.

Example 2.10. The following are examples of genetic codes.

1. The genetic code of $(1, 2, 2, 2, 4)$ is $\langle \{1, 5\} \rangle$.
2. The genetic code of $(1, 1, 3, 3, 3)$ is $\langle \{1, 2, 5\} \rangle$.
3. The genetic code of $(1, 1, 2, 3, 3, 5)$ is $\langle \{126\}, \{36\} \rangle$.

Lemma 2.11 ([35, Lemma 4.1])**.** *Let ℓ be a generic length vector and $S(\ell)$ is the collection of all short subsets of set $[m]$. Then $S(\ell)$ is determined by $S_m(\ell)$.*

Proof. Note that

$$J \in S(\ell) \iff \begin{cases} m \in J \text{ and } J \in S_m(\ell) & \text{or}, \\ m \notin J \text{ and } J^c \notin S_m(\ell). \end{cases} \tag{2.3}$$

This proves the lemma. □

The above lemma was also proved in [31, Proposition 2.5].

As a consequence of Lemma 2.11 the authors in [35] proved that one can reconstruct $S(\ell)$ from the genetic code of ℓ.

Lemma 2.12 ([35, Lemma 4.2]). *Let* $G_\ell = \langle A_1, \ldots, A_k \rangle$ *be the genetic code of* ℓ. *Then* $S(\ell)$ *is determined by* G_ℓ.

Proof. The Equation (2.3) gives

$$J \in S(\ell) \iff \begin{cases} m \in J \text{ and } \exists\, i \in [k] \text{ with } J \not\subseteq A_i \text{ or}, \\ m \notin J \text{ and } J^c \not\subseteq A_i \text{ for all } i \in [k]. \end{cases}$$

□

The following theorem establishes a connection between the genetic code of length vectors and the corresponding polygon spaces.

Theorem 2.13 ([32, 31]). *Suppose* G_ℓ *and* $G_{\ell'}$ *are genetic codes of* ℓ *and* ℓ'. *If* $G_\ell = G_{\ell'}$ *then we have following homemorphisms:*

1. $M_\ell \cong M_{\ell'}$ *and* $\overline{M}_\ell \cong \overline{M}_{\ell'}$.
2. $N_\ell \cong N_{\ell'}$.

As a result, when the genetic code of ℓ is G, we may write $\overline{M}_G$ for $\overline{M}_\ell$ and $S_m(G)$ for $S_m(\ell)$.

2.2 Basic results

2.2.1 The regular cell structure

Definition 2.14. If all the cells of a regular cell complex are combinatorially equivalent to a simple polytope, then it is called a simple cell complex.

An interesting property of simple cell complexes is that it is possible to subdivide each cell into cubes, thus turning it into a cubical complex. Recall that a zonotope is a polytope all of whose faces are centrally symmetric (see [64, Chapter 7] for more details).

Definition 2.15. A regular cell complex is called zonotopal if every cell is combinatorially isomorphic to a zonotope.

Panina [60] described a regular cell structure on M_α. The k-cells of this complex correspond to α-admissible partitions of $[m]$ into $k+3$ blocks. The boundary relations on the cells are described by the partition refinement. Now we briefly describe this cell structure.

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a generic length vector. The most important step in describing this cell structure is to label the elements of M_α by α-admissible partitions.

Labeling polygons

Polygons without parallel sides:
We begin by labelling polygons with no two sides pointing in the same direction. Without loss of generality, we refer to a polygon as an element of M_α or $\overline{M}_\alpha$. Let $P = (v_1, \ldots, v_m)$ be a polygon in M_α such that $v_i \neq v_j$, for all $j \in [m]$. Now we explain how to associate a unique convex polygon P^c to P. Note that $v_i \in S^1$ for all $i \in [m]$. Therefore, we can write $v_i = (\cos\theta_i, \sin\theta_i)$ for $1 \leq i \leq m$. Indeed, it follows from Equation (2.2) that $\theta_m = \pi$. It is worth noting that $\theta_i \neq \theta_j$ for all $1 \leq i < j \leq m$. Arranging all θ_i's in an ascending order yields a unique permutation σ in the symmetric group S_m such that $\theta_{\sigma(1)} < \theta_{\sigma(2)} < \cdots < \theta_{\sigma(m)}$. Observe that $\sigma(m) = m$. Let $P^c := (v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(m)})$ be the polygon with side lengths $\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(m)}$. The following definitions are required to label the polygon P.

The set $\{1, \ldots, m\}$ is denoted by $[m]$. An ordered partition of $[m]$ into k-blocks is a tuple $(J_1, J_2, \ldots, J_k)$ where J_i's are pairwise disjoint subsets of $[m]$ whose union is $[m]$. Under the refinement order relation, the set of all ordered partitions of $[m]$ forms a poset. We now consider a special type of partition known as a cyclically ordered partition, which will allow us to label P.

Definition 2.16. A cyclically ordered partition of $[m]$ is an ordered partition $(J_1, J_2, \ldots, J_k)$ which is equivalent to any ordered partition obtained from it by a cyclic permutation of its blocks, i.e., $(J_1, J_2, \ldots, J_k)$, $(J_2, J_3, \ldots, J_k, J_1), \ldots, (J_k, J_1, \ldots, J_{k-1})$ are all equivalent.

Definition 2.17. A cyclically ordered partition $(J_1, J_2, \ldots, J_k)$ of the set $[m]$ is said to be α-admissible if J_i is α-short for all $1 \leq i \leq k$.

Example 2.18. Let $\alpha = (1, 2, 2, 2, 4)$ be a generic length vector.

1. The ordered partition $(\{1,2\},\{3\},\{4\},\{5\})$ is not same as $(\{3\},\{1,2\},\{4\},\{5\})$, since the orderings of blocks are different.

2. The cyclically ordered partition $(\{1,2\},\{3\},\{4\},\{5\})$ is equal to the partitions $(\{3\},\{4\},\{5\},\{1,2\})$, $(\{4\},\{5\},\{1,2\},\{3\})$ and $(\{5\},\{1,2\},\{3\},\{4\})$.

3. The ordered partition $(\{1\},\{3\},\{4\},\{25\})$ of a set $[5]$ is not ℓ-admissible, since $\{2,5\}$ is not ℓ-short. The following are ℓ-admissible partitions of $[5]$.

$$(\{1,2\},\{3\},\{4\},\{5\}),\ (\{1,2\},\{3,4\},\{5\}),\ (\{1,5\},\{3\},\{4\}),\ (\{1,2,3\},\{4\},\{5\}).$$

The label assigned to a polygon $P \in M_\ell$ is the ℓ-admissible partition consisting of singletons whose order is determined by a unique permutation σ. In other words, the label is

$$\big(\{\sigma(1)\}, \{\sigma(2)\}, \dots, \{\sigma(m-1)\}, \{m\}\big). \tag{2.4}$$

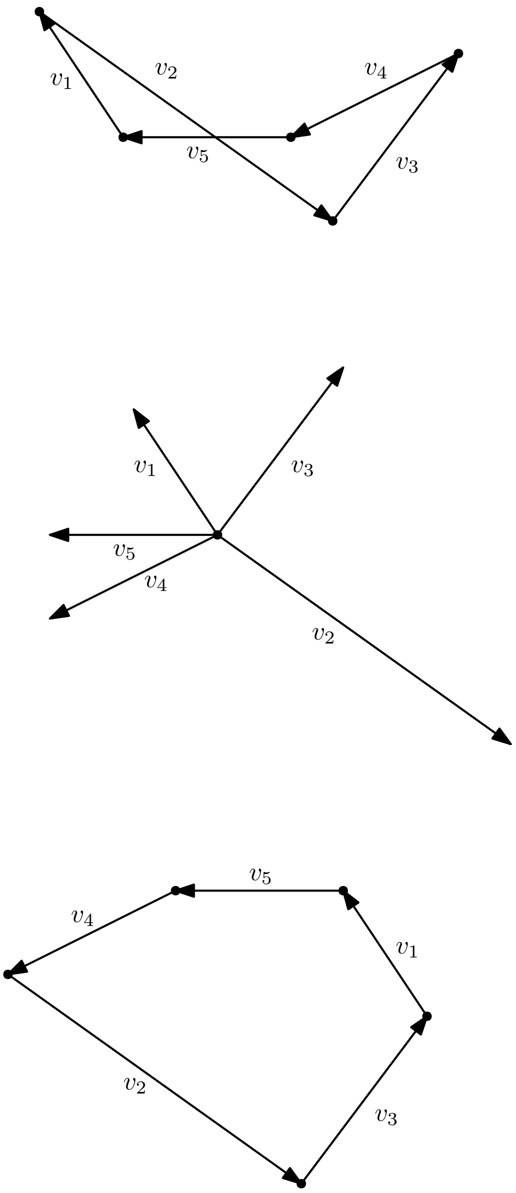
See Figure 2.4.

Remark 2.19. It is important to note that any ordered partition with singleton blocks is an ℓ-admissible partition. As a result, any partition of $[m]$ consisting of singletons is a label. Because applying a cyclic permutation to the blocks in the partition does not change the ordering of the sides, assigning a cyclically ordered partition is justified.

Polygons with parallel sides: Now consider a polygon $P \in M_\ell$ with parallel sides. We can follow the same procedure that we followed for the polygons without parallel sides. First, we arrange the sides in ascending order of increasing angle. Glue all the corresponding sides together and consider them as a single side for each pair of angles $\theta_i = \theta_j$. It is worth noting that there is no unique permutation of $[m]$ that makes P to be convex. In fact, we can order the set of parallel sides in any way we want. Now, we label P by a cyclically ordered partition of $[m]$ by putting all indices of parallel sides in one block. See Figure 2.5.

Lemma 2.20 ([60, Lemma 2.4])**.** *Let ℓ be a generic length vector. Then given an ℓ-admissible partition π of the set $[m]$ into k non-empty blocks, the subset of all polygons labeled by π is an $(k-3)$-dimensional cell.*

Definition 2.21. Let π and π' are partitions of the set $[m]$. Then π is said to be refine (or finer than) π' if , each block of π is contained in a block of π'.



(a) The polygon $P = (v_1, v_2, v_3, v_4, v_5)$ without parallel sides.

(b) The arranged sides of P in an ascending order of angles and the unique permutation $\lambda(1) = 4$, $\lambda(2) = 2$, $\lambda(3) = 3$, $\lambda(4) = 1$ and $\lambda(5) = 5$.

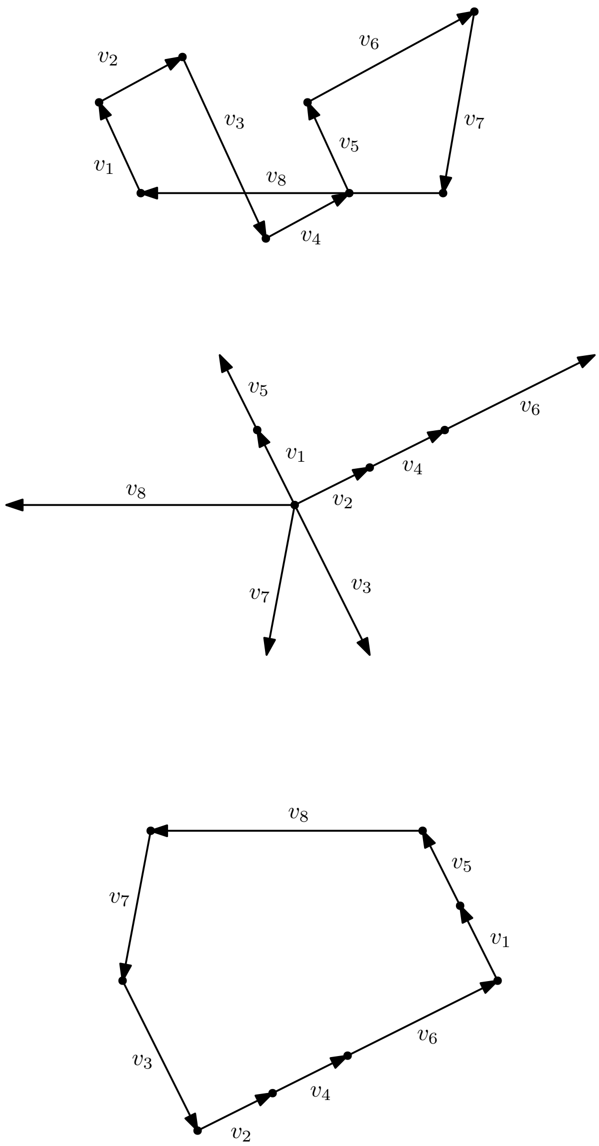
(c) The convex polygon $P^c = (v_4, v_2, v_3, v_1, v_5)$ and the label for P is ({4},{2},{3},{1},{5}).

FIGURE 2.4: Labelling a polygon without parallel sides

Example 2.22. Let $\ = (\{1\}, \{2, 3\}, \{4\}, \{5, 6\})$. The partition $(\{1\}, \{2\}, \{3\}, \{4\}, \{5, 6\})$ refines . However, $(\{1\}, \{4\}, \{2\}, \{3\}, \{5, 6\})$ does not refine .Let $\ = (\{1\}, \{2, 3\}, \{4\}, \{5, 6\})$. The partition $(\{1\}, \{2\}, \{3\}, \{4\}, \{5, 6\})$ refines . However, $(\{1\}, \{4\}, \{2\}, \{3\}, \{5, 6\})$ does not refine .

The following theorem gives a cell structure on M .

Theorem 2.23 ([60, Theorem 2.6])**.** *The space* M *admits a regular cell structure where* k *dimensional cells are labeled by -admissible partitions of* $[m]$ *into* $k + 3$ *blocks. Moreover, a closed-cell* C *is contained in some other closed-cell* C' *if and only if the label of* C' *refines the label of* C.



(a) The polygon $P = (v_1, v_2, v_3, v_4, v_5, v_6.v_7, v_8)$ with parallel sides.

(b) The arranged sides of P in an ascending order of angles, they are $\{7\}$, $\{3\}$, $\{2, 4, 6\}$, $\{1, 5\}$ and $\{8\}$. The ordering of sides within a set doesn't matter.

(c) The convex polygon P^c and the label for P is $(\{7\}, \{3\}, \{2, 4, 6\}, \{1, 3\}, \{8\})$.

FIGURE 2.5: Labelling a polygon with parallel sides

It was shown [60, Proposition 2.12] that the complex K is a simple cell complex. Here, we provide slightly different proof.

Proposition 2.24. *The cell complex* K *is a simple cell complex.*

Proof. It is sufficient to show that the top-dimensional cells are combinatorially equivalent to simple polytopes. Without loss of generality, let $= (1, 2, \ldots, m)$ be a top-dimensional cell and let $v = (I, J, K)$ be a vertex of .

Note that can have at most m facets. We may assume that $I = \{1, 2, \ldots, r\}$, $J = \{r + 1, \ldots, s\}$ and $K = \{s + 1, \ldots, m\}$. Then the following ordered partitions of $[m]$

given by :

$$(1,2,\ldots,\{r,r+1\},\ldots,m),\ (1,2,\ldots,\{s,s+1\},\ldots,m),\ (2,\ldots,\{m,1\})$$

do not refine the ordered partition (I,J,K). Therefore, there are exactly $m-3$ facets which are incident to a vertex v. Hence the top dimensional cell is combinatorially equivalent to a simple polytope. □

Proposition 2.25. *For a generic the dual of* K *is zonotopal.*

Proof. We refer the reader to [60, Section 2] where it is proved that each dual cell is a product of permutohedra (it is a particular type of simple zonotope). It is also proved that the dual cell structure is has the structure of a PL-manifold. □

We will now investigate a method for obtaining a cell structure on $\overline{M}$. Let us begin with some definitions.

Definition 2.26. A map $X \to Y$ of cell complexes is called *cellular* if it takes k-cells to k-cells.

Definition 2.27. A group G acts cellularly on a cell complex X if for each $g \in G$, a map $g : X \to X$ is cellular.

Definition 2.28. A cell complex X is called G-cell complex if G acts on X cellularly and whenever $g \in G$ fixes a cell then it also fixes its points.

We now present the result, which explains how cellular involution induces a cell structure on the quotient space.

Theorem 2.29 ([28, Proposition 3.3.2])**.** *Let X be the G-cell complex and $: X \to X/G$ be the quotient map. Then X/G admits a cell structure whose cells are*

$$\{\ (\ ) : \text{ is a cell of } X\}.$$

Remark 2.30. It is clear that the involution (Equation (1.3)) defined on M is cellular; the cell labeled by $(I_1, I_2, \ldots, I_k)$ mapped to the cell labeled by $(I_k, I_{k-1}, \ldots, I_2, I_1)$. Therefore, $\overline{M}$ has a simple cell structure with cells labeled by bi-cyclically ordered -admissible partitions. We denote this cell structure on $\overline{M}$ by $\overline{K}$. Note that the dual of this cell complex is zonotopal.

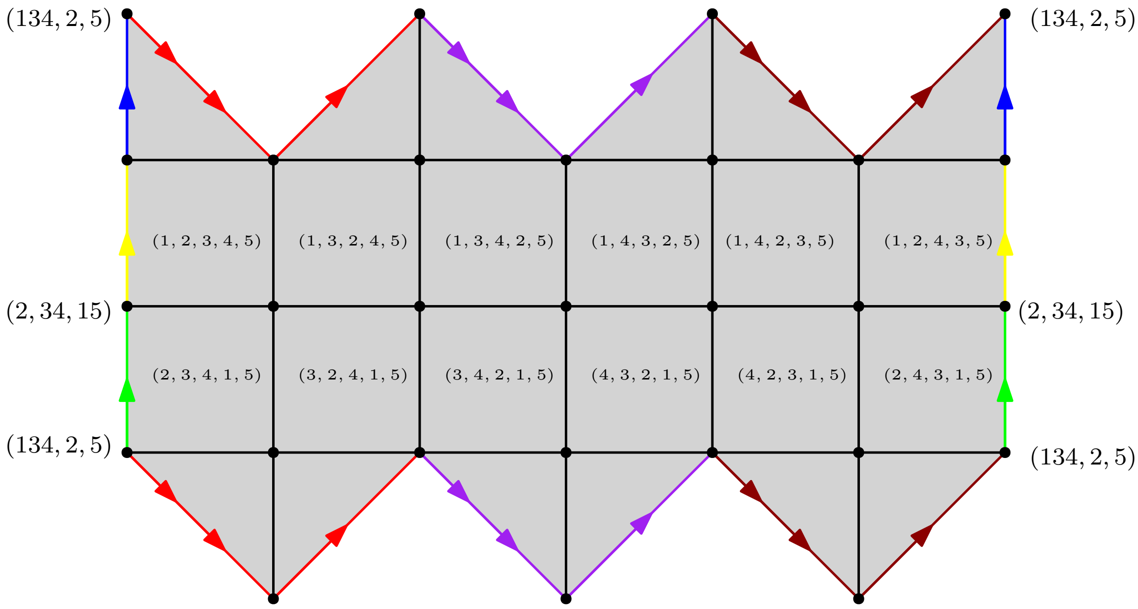


FIGURE 2.6: The cell complex $K_{(1,2,2,2,4)}$

2.2.2 (Co)Homology of polygon spaces

Let $\ell = (\ell_1, \ldots, \ell_m)$ be a generic length vector such that $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_m$. Consider the following collection

$$S_{m,k}(\ell) = \{J \subset [m] : m \in J, |J| = k+1 \text{ and } J \text{ is short}\}.$$

Farber and Schutz [22] proved that the integral homology groups of M_ℓ are torsion-free. They also described the Betti numbers in terms of the combinatorial data associated with the length vector. More precisely, they proved the following result.

Theorem 2.31 ([22, Theorem 1])**.** *Let $a_k = |S_{m,k}|$. The integral homology group $H_k(M_\ell, \mathbb{Z})$ is free abelian with*

$$\operatorname{rank}(H_k(M_\ell, \mathbb{Z})) = a_k + a_{m-3-k},$$

for $0 \leq k \leq m-3$.

The following results are consequences of Theorem 2.31.

Corollary 2.32. *The space M_ℓ (respectively $\overline{M}_\ell$) is homeomorphic to S^{m-3} (respectively $\mathbb{R}P^{m-3}$) if and only if the genetic code of ℓ is $\langle\{m\}\rangle$.*

It is easy to see that, $\mathrm{rank}(H_k(M \,, \mathbb{Z})) = 2$ if and only if the genetic code of is $\langle\{1, 2, \ldots, m-3, m\}\rangle$. In particular we have the following result.

Corollary 2.33 ([22])**.** *The moduli space* M *has two connected components if and only if the genetic code of is* $\langle\{1, 2, \ldots, m-3, m\}\rangle$. *Moreover,*

$$\mathrm{rank}(H_k(M \,, \mathbb{Z})) = 2\binom{m-3}{k}.$$

Kapovich and Milson [48] showed that if M is disconnected then M is diffeomorphic to the disjoint union of two copies of tori T^{m-3}.

The rank of rational homology groups of $\overline{M}$ computed by Kamiyama in [43].

Theorem 2.34 ([43, Theorem A])**.** *Let be a generic length vector. Then*

$$\mathrm{rank}(H_k(\overline{M} \,, \mathbb{Q})) = \begin{cases} a_k & \text{if } k \text{ is an odd integer,} \\ a_{m-3-k} & \text{if } k \text{ is an even integer.} \end{cases}$$

Hausmann and Knutson computed the integral cohomology rings of polygon spaces in [31]. We now list these cohomology rings.

Theorem 2.35 ([31, Theorem 6.4])**.** *The integral cohomology rings of abelian and spatial polygon spaces are shown below.*

1. *The cohomology ring* $H^*(\mathrm{Apol}(\,); \mathbb{Z})$ *is generated by classes* $R, V_1, V_2, \ldots, V_{m-1} \in H^2(\mathrm{Apol}(\,); \mathbb{Z})$ *subject to the following relations:*

   (R1) $RV_i + V_i^2 = 0$, *for* $i \in [m-1]$.

   (R2) $V_S := \prod_{i \in S} V_i = 0$, *unless* S *is a subgee.*

   (R3) For every subgee S *with* $|S| > m - d - 2$,

   $$R \sum_{T \cap S = \emptyset} R^{d-|T|} V_T = 0,$$

   where T *is a subgee.*

2. *The cohomology ring* $H^*(N \,; \mathbb{Z})$ *is generated by classes* $R, V_1, V_2, \ldots, V_{m-1} \in H^2(N \,; \mathbb{Z})$ *subject to the following relations:*

(R1) $RV_i + V_i^2 = 0$, *for* $i \in [m-1]$.

(R2) $V_S = 0$, *unless* S *is a subgee.*

(R3) *For every subgee* S *with* $|S| > m - d - 2$,

$$\sum_{T \cap S = \emptyset} R^{d-|T|} V_T = 0,$$

where T *is a subgee.*

Remember that, the Remark 2.7 states that the fixed point set of an involution on Apol( ) and N , respectively, are Ch( ) and $\overline{M}$. Assume that M represents either N or Apol( ) and M^{invo} represents either $\overline{M}$ or Ch( ). Then Hausmann-Knutson [31, Theorem 9.1] showed the existence of ring isomorphism

$$H^{2}(M, Z_2) \to H(M^{invo}, Z_2).$$

More precisely, the following theorem gives the mod-2 cohomology rings of $\overline{M}$ and Ch( ).

Theorem 2.36 ([31, Corollary 9.2]). *The following are mod-2 cohomology rings* $\overline{M}$ *and* Ch( ).

1. *The cohomology ring* $H(\overline{M}\ ; Z_2)$ *is generated by classes* $R, V_1, V_2, \ldots, V_{m-1} \in H^1(\overline{M}\ ; Z_2)$ *subject to the following relations:*

   (R1) $RV_i + V_i^2 = 0$, *for* $i \in [m-1]$.

   (R2) $V_S := \prod_{i \in S} V_i = 0$, *unless* S *is a subgee.*

   (R3) *For every subgee* S *with* $|S| > m - d - 2$,

   $$\sum_{T \cap S = \emptyset} R^{d-|T|} V_T = 0,$$

   where T *is a subgee.*

2. *The cohomology ring* $H(\mathrm{Ch}(\ ); Z_2)$ *is generated by classes* $R, V_1, V_2, \ldots, V_{m-1} \in H^1(\mathrm{Ch}(\ ); Z_2)$ *subject to the following relations:*

   (R1) $RV_i + V_i^2 = 0$, *for* $i \in [m-1]$.

   (R2) $V_S := \prod_{i \in S} V_i = 0$, *unless* S *is a subgee.*

(R3) *For every subgee* S *with* $|S| > m - d - 2$,

$$R \sum_{T \cap S = \emptyset} R^{d-|T|} V_T = 0,$$

where T *is a subgee.*

2.2.3 Geometric description of polygon spaces

Some of the results from Hausmann's paper [32] are presented in this section. Although his results are more general, we present them in the context of polygonal linkages in $\mathbb{R}^2$ and $\mathbb{R}^3$.

For a generic length vector $\ell = (\ell_1, \ldots, \ell_m)$ and $d \in \{2, 3\}$, consider the following subspace of $(S^{d-1})^{m-1}$

$$C_d^m(\ell) = \left\{ (z_1, \ldots, z_{m-1}) \in (S^{d-1})^{m-1} : \sum_{i=1}^{m-1} \ell_i z_i = \ell_m e_1^d \right\},$$

where $e_1^2 = (1, 0)$ and $e_1^3 = (1, 0, 0)$. An element of C_d^m can be visualized as an $(m-1)$-chain in $\mathbb{R}$ of side lengths $\ell_1, \ldots, \ell_{m-1}$, joining origin and a point $\ell_m e_1^d$. Observe that for $d = 2$, $M_\ell = C_2^m(\ell)$. The collection of rotations in SO_3 who fixes the axis spanned by $(1, 0, 0)$ is isomorphic to SO_2. Note that SO_2 acts naturally on $C_3^m(\ell)$. The following equality is clear.

$$N_\ell = \frac{C_3^m(\ell)}{SO_2}.$$

Let $\langle A_1, \ldots, A_k \rangle$ be the genetic code of $\ell = (\ell_2, \ldots, \ell_m)$. Consider another generic length vector ℓ^+ whose genetic code is $\langle B_1, \ldots, B_k \rangle$ where $B_i = \{a + 1 : a \in A_i\} \cup \{1\}$. Note that ℓ^+ is an m-tuple. With these notations, we state Hausmann's result which justifies the table 2.1 and table 2.3.

Proposition 2.37 ([32, Proposition 2.1]). *There is a* O_{d-1}*-equivariant diffeomorphism*

$$\Psi : C_d^m(\ell^+) \to S^{d-1} \times C_d^m(\ell),$$

where O_{d-1} *acts diagonally on* $S^{d-1} \times C_d^m(\ell)$.

Example 2.38. Let $\ell = (\ell_1, \ell_2, \ell_3)$ be a generic length vector with $\ell_1 \leq \ell_2 \leq \ell_3$. Note that the only possible short subset of $\{1,2,3\}$ containing 3 is $\{3\}$, since singletons are always short. Therefore, $\langle\{3\}\rangle$ is the only possible genetic code for ℓ. It is easy to observe that the genetic code is realized by $\ell = (1,1,1)$. Note that there exist only two triangles in a plane upto orientation preserving isometries, whose side lengths are given by $\ell = (1,1,1)$. Therefore, $M_\ell = S^0$. Since in the quotient of M_ℓ by an involution the two triangles get identified, $\overline{M}_\ell = \{*\}$. Observe that there is only one triangle in $\mathbb{R}^3$ upto the isometries whose side lengths are given by $\ell = (1,1,1)$. Therefore, $N_\ell = \{*\}$.

Genetic code of ℓ	M_ℓ	$\overline{M}_\ell$	N_ℓ
$\langle\{3\}\rangle$	S^0	$\{*\}$	$\{*\}$

TABLE 2.1: Moduli spaces of 3-gons

Proposition 2.39 ([32, Proposition 2.2]). *Let $\ell = (\ell_2, \ldots, \ell_m)$ be a generic length vector. Then N_{ℓ_+} is diffeomorphic to the quotient of $S^2 \times C_3^{m-1}(\ell)$ by the diagonal action of SO_2.*

Example 2.40. Let $\langle\{1,2,\ldots,m-3,m\}\rangle$ be the genetic code of ℓ. Then iteratively applying Proposition 2.37 we get that $M_\ell = T^{n-3} \times S^0 = T^{m-3} \sqcup T^{m-3}$ and $\overline{M}_\ell = T^{m-3}$. Note that $C_4^3(\ell) = S^2$. Therefore, iteratively applying Proposition 2.39 gives us $N_\ell = (S^2)^{m-3}$.

Genetic code of ℓ	M_ℓ	$\overline{M}_\ell$	N_ℓ
$\langle\{1,\ldots,m-3,m\}\rangle$	$T^{m-3} \sqcup T^{m-3}$	T^{m-3}	$(S^2)^{m-3}$

TABLE 2.2

For a generic length vector $\ell = (\ell_1, \ldots, \ell_m)$ and $d \in \{2,3\}$, consider the following manifold

$$V_d(\ell) = \left\{ (z_1, \ldots, z_{m-1}) \in (S^{d-1})^{m-1} : \sum_{i=1}^{m-1} \ell_i z_i = t e_1^d \text{ with } \ell_m \leq t \right\},$$

where $e_1^2 = (1,0)$ and $e_1^3 = (1,0,0)$. Note that the group $O(d-1)$ acts on $V_d(a)$.

Let

$$f : V_d(\ell) \to \mathbb{R}$$

defined by

$$f(z) = -\sum_{i=1}^{m-1} \ell_i z_i .$$

Proposition 2.41 ([32, Proposition 2.5]). *Let $\ell = (\ell_1, \ldots, \ell_m)$ be a generic length vector. Then*

1. $V_d(\ell)$ *is a smooth* $O(d-1)$*-submanifold of* $(S^{d-1})^{m-1}$*, of dimension* $(m-2)(d-1)$*, whose boundary* $C^m{}_d(\ell)$*.*
2. *The function* f *is a* $O(d-1)$*-equivariant Morse function, with one critical point* p_J *for each* $J \in S_m(\ell)$*, where* $p_J = (z_1, \ldots, z_{m-1})$ *such that* $z_i = -e_1^d$ *if* $i \in J$ *and* e_1^d *otherwise (aligned configuration). The index of* p_J *is* $(d-1)(|J|-1)$*.*

Example 2.42. Note that if the genetic code of ℓ is $\langle\{m\}\rangle$ then f has only one critical point of index 0. Therefore, $C_d^m \cong S^{(m-2)(d-1)-1}$ with the $O(d-1)$ action is conjugate to that obtained by the embedding $S^{(m-2)(d-1)-1} \subset (\mathbb{R}^d)^{m-2}$ with the standard diagonal action(see [33, Proposition 4.2]). Therefore, for $d = 2$, $M_\ell \cong S^{m-3}$. Recall that

$$N_\ell = \frac{C_3^m(\ell)}{SO_2}.$$

Therefore, $N_\ell \cong \mathbb{C}P^{m-3}$. In particular for $m = 4$ we have the following table.

Genetic code of ℓ	M_ℓ	$\overline{M}_\ell$	N_ℓ
$\langle\{4\}\rangle$	S^1	$\mathbb{R}P^1$	$\mathbb{C}P^1$
$\langle\{1,4\}\rangle$	$S^1 \sqcup S^1$	S^1	S^2

TABLE 2.3: Moduli spaces of 4-gons

Let ℓ and ℓ' be two length vectors such that $S_m(\ell') = S_m(\ell) \cup J$ for some $J \subset [m]$.

Theorem 2.43 ([32, Proposition 2.9]). *The space* $C_d^m(\ell')$ *is obtained from* $C_d^m(\ell)$ *by an* $O(d-1)$*-equivarient surgery of index* $A = (d-1)(|J|-1)-1$*. i.e.,*

$$C_d^m(\ell') \cong \left(C_d^m(\ell) \setminus S^A \times D^B\right) \bigcup_{S^A \times S^B} D^{A+1} \times S^{B-1},$$

where $B = (m-1-|J|)(d-1)$ *and* $O(d-1)$ *acts antipodally on* $D^{m-1-|J|}$ *and* $D^{|J|-1}$*.*

Proposition 2.44. *Suppose* $|J| = 2$ *in Theorem 2.43. Then*

1. $C_d^m(\ ) = C_d^m(\ )\,\sharp\,(S^{d-1}\ \ S^{(m-3)(d-1)-1})$

2. $N\ \ = N\ \ \sharp\,\overline{CP}^{m-3}$

Example 2.45. Let $\langle\{a, m\}\rangle$ be the genetic code of $\ $. Note that $S_m(\langle\{1, m\}\rangle) = S_m(\langle\{m\}\rangle) \cup \{1, m\}$. Recall that $M_{\langle\{m\}\rangle} = S^{m-3}$. Therefore, using Proposition 2.44 we get $M_{\langle\{1,m\}\rangle} = S^{m-3}\sharp S^1\ \ S^{m-4} = S^1\ \ S^{m-4}$. Now we can apply Proposition 2.44 iteratively to get $M_{\langle\{a,m\}\rangle} = a(S^1\ \ S^{m-4})$, where $a(S^1\ \ S^{m-4})$ is the connected sum of $S^1\ \ S^{m-4}$ with itsef a-times. Similarly, $N\ \ = CP^{m-3}\sharp aCP^{m-3}$. In particular for $m = 5$ we have the following table.

Genetic code of	M	$\overline{\text{M}}$	N
$\langle\{5\}\rangle$	S^2	RP^2	CP^2
$\langle\{1, 5\}\rangle$	$_1$	N_2	$CP^2\sharp\overline{CP^2}$
$\langle\{2, 5\}\rangle$	$_2$	N_3	$CP^2\sharp 2\overline{CP^2}$
$\langle\{3, 5\}\rangle$	$_3$	N_4	$CP^2\sharp 3\overline{CP^2}$
$\langle\{4, 5\}\rangle$	$_4$	N_5	$CP^2\sharp 4\overline{CP^2}$
$\langle\{1, 2, 5\}\rangle$	$T^2 \sqcup T^2$	T^2	$S^2\ \ S^2$

Table 2.4: Moduli spaces of pentagons

Chapter 3

Aspherical chain spaces

In this chapter, we characterize those length vectors for which Ch( ) is aspherical. To do that we introduce a combinatorial object associated with $\overline{K}$ called the *short code*. We first show that the short code of a (generic) length vector completely determines the moment polytope of the associated chain space. Then we establish a combinatorial condition to determine whether or not the chain space is aspherical.

This entire chapter is taken from the author's published article [10], which was co-written with Priyavrat Deshpande.

The results in this chapter are motivated by the techniques developed by Davis, Januszkiewicz, and Scott to conclude that the (real part of) moduli space of certain point configurations is aspherical. In particular, the authors consider the following situation: $M_{\mathbb{C}}$ is an n-dimensional complex manifold and $D_{\mathbb{C}}$ is a smooth divisor. Assume that there is a smooth involution defined on $M_{\mathbb{C}}$ which is locally isomorphic to complex conjugation on $\mathbb{C}^n$. This situation provides a 'real version' of the pair $(M_{\mathbb{C}}, D_{\mathbb{C}})$, which we denote by (M, D), where M is the fixed point set of the involution and $D = D_{\mathbb{C}} \setminus M$. In such a case D is a union of codimension-one smooth submanifolds which is locally isomorphic to an arrangement of hyperplanes. Consequently, the complement of D is a disjoint union of cells, called *chambers*, which are combinatorially equivalent to simple convex polytopes. The cell structure induced by D has a cubical subdivision (i.e., one can subdivide the polytopal cells to obtain a tiling by cubes). For (smooth) manifolds equipped with such a cell structure, the authors established a combinatorial condition to check whether or not the manifold is aspherical.

The motivation for the results in this chapter is stated first in the following section. We also explain why it is difficult to investigate whether or not general planar polygon space is aspherical using the techniques of Davis-Januszkiewicz-Scott. In Section 3.2, we introduce the notion of *short codes*. We then show that a chain space is diffeomorphic to some planar polygon space. In Section 3.3, we describe the moment polytope of a chain space and explicitly describe it in some special cases. Next, we determine its number of facets in terms of the short subset information. We then introduce the poset of *admissible subsets* corresponding to a generic length vector and prove that it is isomorphic to the face poset of the corresponding moment polytope. We also determine the characteristic functions on these moment polytopes such that corresponding small covers are homeomorphic to chain spaces. In Section 3.4, we prove our main result Theorem 3.35.

3.1 Motivation

In this section, we formally state the results of Davis-Januszkiewicz-Scott, that are useful to us and also show that the planar polygonal spaces possess a similar structure.

Definition 3.1. An n-dimensional convex polytope is called *simple* if each vertex is an intersection of n codimension-1 faces (also called *facets*).

Example 3.2. The following are some examples and non-examples of simple polytopes.

1. An n-dimensional simplex is a simple polytope. Furthermore, any vertex truncation of a simple polytope results in another simple polytope.

2. The octahedron is not a simple polytope.

Definition 3.3. A simple polytope is called a *flagtope* if every collection of its pairwise intersecting facets has a nonempty intersection.

Example 3.4. The following are some examples and non-examples of falgtopes.

1. P_m I^n where P_m is m-gon.

2. Product of two flagtopes is a flagtope.

3. An n-dimensional simplex and vertex truncations of simple polytopes are not flagtopes.

Flagtopes have many interesting combinatorial properties. For example, an n-dimensional flagtope has at least $2n$ facets. In fact, the only flagtope with the fewest possible facets is an n-cube.

Recall that a $2n$-dimensional symplectic manifold is called a toric manifold if it admits a Hamiltonian action of an n-dimensional torus. It turns out that for some choices of the manifold N is a toric manifold. The half-dimensional torus action is given by bending flows [49]. Hausmann and Roudrigue [35, Proposition 6.8] provided a (combinatorial) sufficient condition for N to be a toric manifold.

A smooth $2n$-dimensional (respectively n-dimensional) manifold M is called a quasi-toric (respectively small cover) if it has a locally standard action of n-dimensional torus (respectively $\mathbb{Z}_2^n$) such that the orbit space can be identified with a simple n-polytope. These manifolds were introduced by Davis and Januszkeiwicz in [19]. They showed that many topological invariants of these spaces are encoded in the combinatorics of the associated quotient polytope. In the same paper, they prove that the small cover can be realized as a fixed point set of an involution on a quasi-toric manifold. Note that in this article we will refer to the quotient polytope as the moment polytope even in the context of small covers.

Since the Hamiltonion action of a torus is locally standard (see [19, Section 7.3] for proof) and the image of the moment map is a simple convex polytope, hence toric manifolds are quasi-toric. Moreover, the fixed point set of an anti-symplectic involution on a toric manifold is a small cover. This is true since the image under the restriction of the moment map to the fixed point set is again the same moment polytope. It can be seen that $\overline{M}$ is the fixed point set of an anti-symplectic involution on N. Therefore it has the structure of a small cover whenever N is a toric manifold. We refer the interested reader to the paper of Hausmann and Knutson [34] for the terminologies related to symplectic structure that are not defined here.

We use Theorem 1.3 to classify aspherical chain spaces.

We can now state an important consequence of Gromov's lemma that provides a combinatorial condition to check whether the given cubical complex is non-positively curved or not. The details of the following result can be found in [18, Section 1.6]

Theorem 3.5. *Suppose that* K *is a simple cell complex structure on an* n*-dimensional smooth manifold such that its dual cell complex is zonotopal. If each* n*-cell* P *of* K *is combinatorially isomorphic to a flagtope then* K *is aspherical.*

We now know that for a generic   the complex $\overline{K}$ satisfies the premise of Theorem 3.5. So we find out whether or not the top-cells of $\overline{K}$ are flagtopes. We do this analysis by considering the number of sides. If $m = 3$ then there is only one possibility, $\overline{K}$ is always a point. If $m = 4$ then again $\overline{K}$ $= S^1$ for any generic . If $m = 5$ then $\overline{K}$ is either a torus or the nonorientable surfaces of genus $1, 2, 3, 4$ and 5. It is not difficult to verify that in the case of genus 5 all of the 12 top-cells are pentagonal, hence they are flagtopes. However, in all other cases, the cell structure contains at least one triangular top-cell, see [60, Section 2] for details. So we can't appeal to Theorem 3.5 in this case. In fact, for the same reason, we can't use this theorem for the larger values of m.

Proposition 3.6. *Let* $m > 6$ *and*   *be a generic length vector with* m *components. Then, the cell structure* $\overline{K}$ *contains no top-cell isomorphic to a flagtope.*

Proof. It is straightforward to see that any top-dimensional (i.e. $(m - 3)$-dimensional) cell of $\overline{K}$ has at most m facets. Therefore, the only possible values of m for which the number of facets are greater than or equal to $2(m - 3)$ are $m = 3, 4, 5$ and 6. Therefore, if $m > 7$ then none of the top-dimensional cells of $\overline{K}$ can be combinatorially isomorphic to a flagtope. The cases $m = 3, 4, 5$ are dealt with above, so here we deal with the case $m = 6$, where the top-dimensional cells of $\overline{K}$ may have 6 facets.

From [2, Theorem 3] we know that there is only one 3-dimensional flagtope with 6 facets and that is a 3-cube. We now show that it is impossible for the corresponding $\overline{K}$ to have each top-dimensional cell of $\overline{K}$ isomorphic to a 3-cube. To see this assume the contrary that a cell denoted by $= (1, 2, 3, 4, 5, 6)$ is combinatorially isomorphic to a 3-cube. We write fi, jg as ij for short. Then   have following 6 facets,

$$(12, 3, 4, 5, 6),\ (1, 23, 4, 5, 6),\ (1, 2, 34, 5, 6),\ (1, 2, 3, 45, 6),\ (1, 2, 3, 4, 56),\ (2, 3, 4, 5, 16).$$

Each of the facet is isomorphic to the 2-dimensional cube. Consider the following 2-faces and their edges.

1. The possible faces of $(2,3,4,5,16)$:

$$(23,4,5,16),\ (2,34,5,16),\ (2,3,45,16),\ (2,3,4,156) \text{ or } (3,4,5,126).$$

2. The possible faces of $(1,2,34,5,6)$:

$$(12,34,5,6),\ (1,2,34,56),\ (2,34,5,16),\ (1,234,5,6) \text{ or } (1,2,345,6).$$

3. The possible faces of $(1,2,3,45,6)$:

$$(12,3,45,6),\ (1,23,45,6),\ (2,3,45,16),\ (1,2,345,6) \text{ or } (1,2,3,456).$$

Note that the set 156 cannot be short. Otherwise, both $(1,2,34,5,6)$ and $(1,2,3,45,6)$ will be isomorphic to 2-simplex, because then 234, 345 and 456 will be long subsets. Since we assumed $(2,3,4,5,16)$ is a 2-cube, 126 must be short. But then a 2-face $(1,2,3,45,6)$ will be isomorphic to a 2-simplex. Which is a contradiction. Therefore, if $m = 6$, it is impossible to have each top-dimensional cell of $\overline{K}_\ell$ isomorphic to a 3-cube. $\square$

Although the natural cell structure of $\overline{K}_\ell$ is simple we cannot apply the Davis-Januszkiewicz-Scott schema as the top-cells are not flagtopes. However, some of these polygon spaces are real toric varieties and the Davis-Januszkiewicz-Scott theorem can be applied to this situation since they have a natural cell structure tiled by the moment polytope. If the moment polytope is a flagtope then the corresponding real toric variety is aspherical (see [18, Theorem 2.2.5]).

Hausmann and Roudrigue [35, Proposition 6.8] establish a sufficient condition for spatial polygon space N_ℓ to be a complex toric variety, which we now state.

Theorem 3.7. *Let $\ell = (\ell_1, \ell_2, \ldots, \ell_m)$ be a generic length vector with $\ell_m > \sum_{i=1}^{m-5} \ell_i$ then N_ℓ is diffeomorphic to a toric variety.*

Recall that $\overline{M}_\ell$ is a fixed point set of an anti-syplectic involution on N_ℓ. Hence when ℓ satisfies the condition of Theorem 3.7 corresponding polygon space is a small cover.

Remark 3.8. For a generic length vector $\ell = (\ell_1, \ell_2, \ldots, \ell_m)$, the half-dimensional torus action on $N(\ell)$ is given by bending flows (see [34, Section 5], [35, Section 6]).

The moment polytope is given by the following triangle inequalities

$$x_i + {}_i > x_i,\ x_i + {}_i > x_{i-1}, x_{i-1} + x_i > {}_i \text{ for } 1 \leq i \leq m-5,$$

$$x_{m-4} + {}_{m-4} > {}_{m-3},\ x_{m-4} + {}_{m-3} > {}_{m-4},\ {}_{m-3} + {}_{m-4} > x_{m-4},$$

$$x_{m-3} + {}_{m-2} > {}_{m-1},\ x_{m-3} + {}_{m-1} > {}_{m-2},\ {}_{m-1} + {}_{m-2} > x_{m-3},$$

$$x_{m-5} + x_{m-4} > x_{m-3},\ x_{m-5} + x_{m-3} > x_{m-4},\ x_{m-4} + x_{m-3} > x_{m-5}.$$

It is possible to visualize these polytopes in dimensions 2 and 3. But in general, it is hard to characterize the face poset of the moment polytope using the above equations. For example, we cannot determine the number of facets of the moment polytope, one of the important information to determine whether a simple polytope is flagtope or not. So, whenever N is a toric variety, it is not straightforward to use Theorem 1.3 to conclude whether its real part $\overline{M}$ is aspherical or not.

In view of the discussion in this Section, we now focus on a particular sub-class of planar polygon spaces, called chain spaces (or abelian polygon spaces), because they are toric varieties and it is possible to explicitly describe their moment polytope.

3.2 The short code

As stated above we now focus solely on chain spaces (i.e., real abelian polygon spaces). In this section, we introduce the notion of a combinatorial object associated with generic length vector, called the short code. The short code of a generic length vector is closely related to the *genetic code* defined by Hausmann in [32, Section 1.5] in the context of polygon spaces. We also show that chain spaces are planar polygon spaces, up to diffeomorphism.

Since the diffeomorphism type of a chain space does not depend on the ordering of its side lengths, we assume that our (generic) length vector $=$ $({}_1, {}_2, \ldots, {}_{m-1}, {}_m)$ satisfies ${}_1 \leq {}_2 \leq \cdots \leq {}_{m-1}$. Note that the only restriction on ${}_m$ is that it is less than the sum $\sum_{i=1}^{m-1} {}_i$.

Definition 3.9. For a generic length vector , we define the following collection of subsets of $[m]$:

$$S_m( ) = J \subset [m] : m \in J \text{ and } J \text{ is } \text{-short} .$$

A partial order $\leq$ is defined on $S_m(\alpha)$ by declaring $I \leq J$ if $I = \{i_1, \ldots, i_t\}$ and $\{j_i, \ldots, j_t\} \subseteq J$ with $i_s \leq j_s$ for $1 \leq s \leq t$. The *short code* of α is the set of maximal elements of $S_m(\alpha)$ with respect to this partial order. If $A_1, A_2, \ldots, A_k$ are the maximal elements of $S_m(\alpha)$, we denote the short code as $\langle A_1, \ldots, A_k \rangle$.

Example 3.10. The following are examples of short codes.

1. The short codes of length vector $(1, 1, 3, 3, 3)$ is $\langle\{1, 2, 5\}\rangle$.

2. The short codes of length vector $(1, 2, 2, 5, 3)$ is $\langle\{1, 3, 5\}\rangle$,

Note that $\langle\{1, 3, 5\}\rangle$ never be a genetic code of a length vector.

Given a generic length vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{m-1}, \alpha_m)$ of a chain space and a positive real number $\lambda > \sum_{i=1}^{m-1} \alpha_i$, define a new generic length vector (for a polygon space) as

$$\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_{m-1}, \lambda, \alpha_m + \lambda). \tag{3.1}$$

It was shown in [31] that the spatial polygon space $N_{\alpha'}$ is a toric manifold. In particular, the authors proved that the moment polytope of the abelian polygon space corresponding to α and that of $N_{\alpha'}$ are isomorphic [31, Proposition 1.3]. Hence the two spaces are diffeomorphic. Here we prove the real version of their result by providing an explicit diffeomorphism.

Proposition 3.11 ([10, Proposition 3.3]). *Let α be a generic length vector and α' be the vector defined in Equation (3.1). Then the corresponding chain space* $\mathrm{Ch}(\alpha)$ *is diffeomorphic to the planar polygon space* $\overline{M}_{\alpha'}$.

Proof. Given a chain with side lengths $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ we can associate unique polygon with side lengths $\alpha_1, \alpha_2 \ldots \alpha_{m-1}, \lambda, \alpha_m + \lambda$ in the following way:

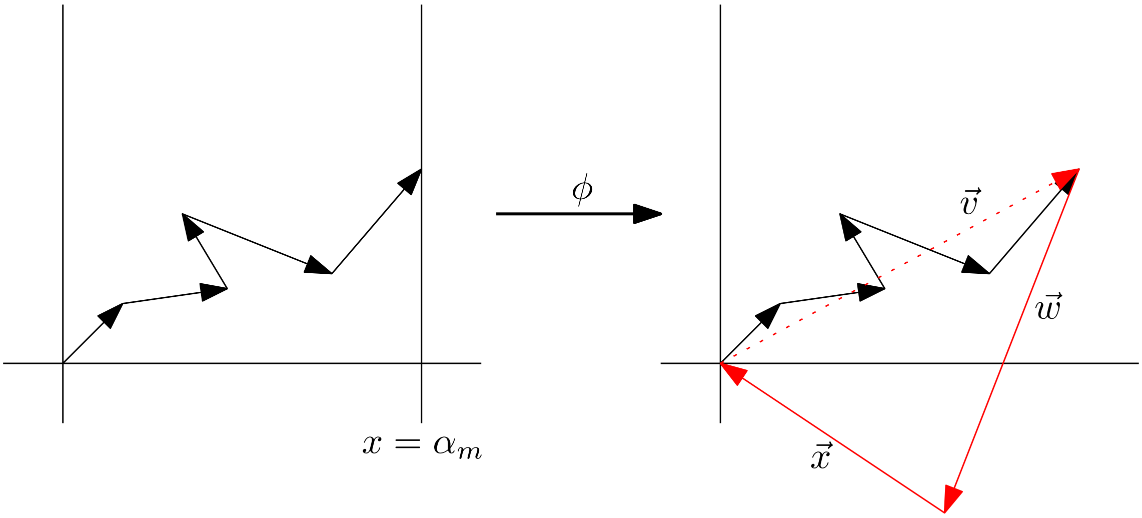


FIGURE 3.1

Let $(v_1, v_2, \ldots, v_{m-1}) \in \mathrm{Ch}(\ )$ and let $\vec{v} = \sum_{i=1}^{m-1} v_i$. We have following inequalities :

$$|\vec{v}| + (\ + \ _m) > \ , \quad + (\ + \ _m) > |\vec{v}|, \ |\vec{v}| + \ > \ + \ _m$$

as $\vec{v} = (\ _m, y)$ for some $y \in \mathbb{R}$. Therefore, the sides lengths $|\vec{v}|$, and $+ \ _m$ satisfy the triangle inequalities. In fact there exist unique triangle (see Section 3.2) with direction vectors $\vec{v}$, $\vec{w}$ and $\vec{x}$ up to isometries. Consequently, we have an $m+1$-gon $(v_1, v_2, \ldots, v_{m-1}, \vec{w}, \vec{x})$ with side length $\ _1, \ _2, \ldots, \ _{m-1}, \ , \ _m + \ $. Its not hard to see that the map

$$: \mathrm{Ch}(\ ) \to \overline{\mathrm{M}}_{\ '}$$

defined by

$$((v_1, v_2, \ldots, v_{m-1})) = (v_1, v_2, \ldots, v_{m-1}, \vec{w}, \vec{x})$$

is a diffeomorphism. □

Remark 3.12. If the short code of a generic length vector $= (\ _1, \ _2, \ldots, \ _{m-1}, \ _m)$ is $\langle A_1 \cup \{m\}, A_2 \cup \{m\}, \ldots, A_k \cup \{m\} \rangle$ where $A_i \ \ [m-1]$ for all $1 \leq i \leq k$ then $\langle A_1 \cup \{m+1\}, A_2 \cup \{m+1\}, \ldots, A_k \cup \{m+1\} \rangle$ is the genetic code of the length vector $\ '$ defined in Equation 3.1.

Now we show that the short code of a length vector determines the diffeomorphism type of a chain space.

Proposition 3.13 ([10, Proposition 3.5]). *Let and be two generic length vectors with the same short code. Then the corresponding chain spaces* $\mathrm{Ch}(\ )$ *and* $\mathrm{Ch}(\ )$ *are diffeomorphic.*

Proof. Consider $'$ and $'$ be two generic length vectors defined as in Equation (3.1). Note that the genetic codes of $'$ and $'$ coincide, since the short code of and are same. Consequently [35, Lemma 4.2] and [32, Lemma 1.2] give that the the planar polygon spaces $\overline{M}_{'}$ and $\overline{M}_{'}$ corresponding to $'$ and $'$ are diffeomorphic. It follows from the Theorem 3.11 that Ch($'$) and Ch($'$) are diffeomorphic. □

Corollary 3.14 ([10, Corollary 3.6]). *Let* $\langle\{1, 2, \ldots, m-2, m\}\rangle$ *be the short code of a length vector . Then the corresponding chain space* Ch( ) *is diffeomorphic to the* $(m-2)$-*dimensional torus* T^{m-2}.

Proof. Let $'$ be the new length vector defined as in Equation 3.1. Using Theorem 3.12 we get the genetic code of $'$; which is $\langle\{1, 2, \ldots, m-2, m+1\}\rangle$. By [48, Theorem 1] we infer that $\overline{M}_{'}$ is diffeomorphic to T^{m-2}. Now the corollary follows from Proposition 3.11. □

3.3 The face poset of the moment polytope

Recall that the chain space Ch( ) is the real part of a toric manifold. Our aim is to understand the topology of this space via the combinatorics of the associated moment polytpoe. In this section we describe the face poset of the moment polytope. We show that it is completely determined by the short code of the corresponding length vector. The moment polytopes corresponding to two different length vectors are strongly isomorphic if they have the same short code. In particular, the results in this section imply that if the short codes are same then the equations defining the corresponding moment polytopes differ by a constant (equivalently their facets are parallel).

Given a generic length vector , the moment polytope of the corresponding chain space was first described in [31] as an intersection of a parallelepiped with a hyperplane. We begin with the description of the moment map:

$$: \mathrm{Ch}(\ ) \to \mathbb{R}^{m-1}$$

such that

$$(v_1, \ldots, v_{m-1}) = (\ _1 x_1, \ldots, \ _{m-1} x_{m-1})$$

where $v_i = (x_i, y_i)$. We have

$$P(\ell) := \mu(\mathrm{Ch}(\ell)) = \left\{ (w_1, w_2, \dots, w_{m-1}) \in \prod_{i=1}^{m-1} [-\ell_i, \ell_i] : \sum_{i=1}^{m-1} w_i = \ell_m \right\}.$$

Let

$$C^{m-1}(\ell) = \prod_{i=1}^{m-1} [-\ell_i, \ell_i]$$

and

$$H(\ell) = \left\{ (w_1, w_2, \dots, w_{m-1}) \in \mathbb{R}^{m-1} : \sum_{i=1}^{m-1} w_i = \ell_m \right\}.$$

Hence the moment polytope $P(\ell) = C^{m-1}(\ell) \cap H(\ell)$.

It is clear that the facets of $P(\ell)$ are given by intersections of the facets of $C^{m-1}(\ell)$ with the hyperplane $H(\ell)$. Note that the facets of $C^{m-1}(\ell)$ are described by the equations $x_j = \pm\ell_j$. We call the facets described by equations $x_j = \ell_j$ and $x_j = -\ell_j$ as *positive facets* and *negative facets* respectively.

Corollary Theorem 3.14 says that, if the short code of a length vector is $\langle \{1, 2, \dots, m-2, m\} \rangle$ then the corresponding chain space is diffeomorphic to $(m-2)$-dimensional torus. So it is clear that the corresponding moment polytope must be a $(m-2)$-cube. Here we explicitly describe it in terms of intersections of half-spaces.

Lemma 3.15. *Let* $\langle \{1, 2, \dots, m-2, m\} \rangle$ *be the short code of a length vector* ℓ. *Then the moment polytope* $P(\ell) = I^{m-2}$, *the* $(m-2)$*-dimensional cube.*

Proof. Let F_{m-1} and $\bar{F}_{m-1}$ be the two opposite facets of $C^{m-1}(\ell)$ represented by the equations $x_{m-1} = \ell_{m-1}$ and $x_{m-1} = -\ell_{m-1}$ respectively. Note that the collection

$$(\pm\ell_1, \pm\ell_2, \dots, \pm\ell_{m-2}, \ell_{m-1})$$

forms the vertices of F_{m-1} and the collection

$$(\pm\ell_1, \pm\ell_2, \dots, \pm\ell_{m-2}, -\ell_{m-1})$$

forms the vertices of $\bar{F}_{m-1}$. Since $\{m-1\}$ is a long subset we have the following inequality,

$$-\ell_{m-1} + \sum_{i=1}^{m-2} \ell_i < \ell_m < \ell_{m-1} + \sum_{i=1}^{m-2} \ell_i. \tag{3.2}$$

The rightmost inequality above implies that, the hyperplane H(α) does not intersect the facet $\bar{F}_{m-1}$. Similarly the leftmost inequality says that H(α) does not intersect the facet F_{m-1}. Therefore, the hyperplane H(α) passes through $C^{m-1}(\alpha)$ dividing it into two isomorphic polytopes which are combinatorially isomorphic to $(m-1)$-cubes. We conclude that $C^{m-1}(\alpha) \cap H(\alpha) = I^{m-2}$, being a facet of both the divided parts. □

Since our aim is to classify the aspherical chain spaces, henceforth we discard length vectors whose short code is $\langle\{1, 2, \dots, m-2, m\}\rangle$, i.e., we discard the length vector where $\alpha_{m-1} > \sum_{i=1}^{m-2} \alpha_i + \alpha_m$. In particular we assume that $\alpha_i < \sum_{j\neq i} \alpha_j$.

Proposition 3.16. *Let* $\alpha = (\alpha_1, \dots, \alpha_m)$ *be a length vector with* $\alpha_i < \sum_{j\neq i} \alpha_j$. *Then the hyperplane* H(α) *intersects all the positive facet of* $C^{m-1}(\alpha)$.

Proof. We argue by contradiction. Suppose that the hyperplane H(α) does not intersect all the positive facets. In particular, let it miss the facet given by the equation $x_{j_0} = \alpha_{j_0}$. The coordinate sum of the elements of H(α) (which is α_m) is then strictly less than the coordinate sum of points on the missed facet, i.e.,

$$\alpha_m < \alpha_{j_0} - \sum_{i\neq j_0} \alpha_i$$

equivalently

$$\alpha_m + \sum_{i\neq j_0} \alpha_i < \alpha_{j_0}.$$

This is impossible since α is generic. □

The Theorem 3.16 describes $m-1$ many facets of the moment polytope. We now characterize length vectors α when P(α) is a simplex.

Lemma 3.17. *Let* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ *be a generic length vector. The hyperplane* H(α) *intersects exactly* $m-1$ *facets of* $C^{m-1}(\alpha)$ *given by an equations* $x_j = \alpha_j$ *for* $1 \leq j \leq m-1$ *if and only if* $\{m, j\}$ *is long subset for all* $1 \leq j \leq m-1$. *In particular* $P(\alpha) = \Delta^{m-2}$, *an* $(m-2)$*-simplex.*

Proof. Consider a vertex $v = (\alpha_1, \dots, \alpha_{m-1})$ of $C^{m-1}(\alpha)$ and

$$v(j) = (\alpha_1, \dots, -\alpha_j, \dots, \alpha_{m-1}) : 1 \leq j \leq m-1$$

be its neighboring vertices. We can observe that the hyperplane $H(\ )$ cannot intersects v since is generic. Note that $H(\ )$ intersects exactly $m-1$ positive facets of $C^{m-1}(\ )$ if and only if the coordinate sum (which is ${}_m$) of an element of $H(\ )$ is greater than or equal the coordinate sum of adjacent vertices of $v(j)$. Let ${}_j = \sum_{i \neq j} {}_i - {}_j$ be the coordinate sum of $v(j)$ for $1 \leq j \leq m-1$. The hyperplane $H(\ )$ intersects exactly $m-1$ positive facets of $C^{m-1}(\ )$ if and only if

$$ {}_m \geq {}_j = \sum_{i \neq j} {}_i - {}_j, $$

for all $1 \leq j \leq m-1$. Which gives

$$ {}_m + {}_j \geq \sum_{i \neq j} {}_i. $$

This proves the lemma. $\square$

The next result determines the remaining facets of the moment polytope. It shows that these facets can be determined using the short subset information.

Lemma 3.18. *Let* $= ({}_1, {}_2, \ldots, {}_m)$ *be a generic length vector. The hyperplane* $H(\ )$ *intersects a facet of* $C^{m-1}(\ )$ *given by an equation* $x_j = - {}_j$ *for some* $1 \leq j \leq m-1$ *if and only if* $\{m, j\}$ *is an -short subset.*

Proof. Let $v(j) = ({}_1, \ldots, - {}_j, \ldots, {}_{m-1})$ and $w = (- {}_1, \ldots, + {}_j, \ldots, - {}_{m-1})$ be the two (extreme) vertices of the facet supported by $x_j = - {}_j$. The coordinate sum of $v(j)$ is

$$ \sum_{i \neq j} {}_i - {}_j. $$

The hyperplane $H(\ )$ intersects the facet supported by $x_j = - {}_j$ if and only if the sum of the coordinates of an element of $H(\ )$ is between the coordinate sums of points $v(j)$ and w, i.e., $- \sum_{i}^{m-1} {}_i \leq {}_m \leq \sum_{i \neq j} {}_i - {}_j$, equivalently

$$ {}_m + {}_j \leq \sum_{i \neq j} {}_i. $$

This proves the lemma. $\square$

Both the Theorem 3.17 and Theorem 3.18 give the exact count of the facets in terms of the short subset information.

Lemma 3.19. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a generic length vector and k be the number of two element α-short subsets containing m. Then $P(\alpha)$ has $m - 1 + k$ many facets.*

Proof. Follows from the Theorem 3.17 and Theorem 3.18. □

We now describe the face poset of the moment polytope $P(\alpha)$ in terms of certain subsets of $[m]$. We further show that this face poset is completely determined by the short code.

Notations:

1. Define $[m-1]^+ := \{1, 2, \ldots, m-1\}$ and $[m-1]^- := \{\bar{1}, \bar{2}, \ldots, \overline{m-1}\}$ where $\bar{i} = -i$.
2. Denote by $[m-1]^{\pm}$ the union $[m-1]^+ \cup [m-1]^-$.
3. Let $J_1 \subseteq [m-1]^+$ and $J_2 \subseteq [m-1]^-$. We write α_{J_1} for $\sum_{j_i \in J_1} \alpha_{j_i}$ and α_{J_2} for $\sum_{j_s \in -J_2} \alpha_{j_s}$.

Definition 3.20. Let $J \subseteq [m-1]^{\pm}$, such that $J = J_1 \cup J_2$ where $J_1 \subseteq [m-1]^+$, $J_2 \subseteq [m-1]^-$ and $J_1 \cap -J_2 = \emptyset$. Then J is called an *admissible subset* of $[m-1]^{\pm}$ if the shorter length vector

$$\alpha(J) := (\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_k}, |\alpha_m + \alpha_{J_1} - \alpha_{J_2}|)$$

is generic, where $(J_1 \cup -J_2)^c = \{j_1, \ldots, j_k\}$.

We denote the set of all admissible subsets of $[m-1]^{\pm}$ by $Ad(\alpha)$. We have a natural partial order on $Ad(\alpha)$, $J \leq S$ if and only if $S \subseteq J$. With this partial order $Ad(\alpha)$ becomes a poset.

Theorem 3.21. *The poset $Ad(\alpha)$ is isomorphic to the face poset of $P(\alpha)$.*

Proof. We start by associating a face of the moment polytope to an admissible subset of $[m-1]^{\pm}$. Let J be an admissible subset of $[m-1]^{\pm}$. Define

$$F_J := \{(x_1, x_2, \ldots, x_{m-1}) \in P(\alpha) : x_{j_i} = \alpha_{j_i}, j_i \in J_1, x_{j_s} = -\alpha_{j_s}, j_s \in -J_2\}$$

Let F_i and $\bar{F}_i$ be the facets of $C^{m-1}(\alpha)$ given by an equations $x_i = \alpha_i$ and $x_i = -\alpha_i$. Note that

$$F_J = \left(\bigcap_{j_i \in J_1} F_{j_i}\right) \cap \left(\bigcap_{j_s \in -J_2} \bar{F}_{j_s}\right) \cap H(\alpha).$$

In fact it is easy to see that any face of $P(\ )$ looks like F_J for some admissible subset of $[m-1]$. The map

$$: \mathrm{Ad}(\ ) \to P(\ )$$

defined by

$$(J) = F_J,$$

is a poset isomorphism. □

Remark 3.22. We observe straightforwardly that for an admissible subset J, $\dim(F_J) = m-2-|J|$. Moreover, let and be two generic length vectors with an isomorphism between $\mathrm{Ad}(\ )$ and $\mathrm{Ad}(\ )$. Any such isomorphism restricts to an isomorphism between $S_m(\ )$ and $S_m(\ )$. Therefore, the short code of determines the poset $\mathrm{Ad}(\ )$.

Proposition 3.23. *Let $\langle\{k-1,m\}\rangle$ and $\langle\{k,m\}\rangle$ be short codes of generic length vectors and respectively. Then the moment polytope $P(\ )$ is obtained by truncating a vertex of $P(\ )$.*

Proof. Let

$$Z = \ \bar{1},\bar{2},\dots,\overline{k-1},\overline{k+1},\dots,\overline{m-1}$$

be an admissible subset corresponding to . Note that Z represents a vertex of $P(\ )$. Let

$$Z_i = \ Z\setminus\{\bar{i}\} \bigcup \ \{k\} \text{ for all } i \in \{1,2,\dots,k-1,k+1,\dots,m-1\}.$$

Each Z_i is an admissible subset corresponding to and represents a vertex of $P(\ )$. Let $\mathrm{Ad}(\ )$ and $\mathrm{Ad}(\ )$ be the posets of an admissible subsets corresponding to the length vector , respectively. Then we have,

$$\mathrm{Ad}(\ ) = \ \mathrm{Ad}(\ )\setminus Z \ \bigsqcup \ \ \{k\}, J : J \ \ Z_i \ \ , \tag{3.3}$$

where $i \in \{1,2,\dots,k-1,k+1,\dots,m-1\}$. Note that

$$\{k\},\ J : J \ \ Z_i$$

represents an $(m-3)$-simplex in $P(\ )$. The Equation (3.3) implies the vertex Z of $P(\ )$ is replaced by an $(m-3)$-simplex represented by the admissible subset $\{k\}$ thus proving the proposition. □

Example 3.24. If the short code of a generic length vector is $\langle\{5\}\rangle$ then the corresponding moment polytope is a 3-simplex. On the other hand, if the short code is $\langle\{1,5\}\rangle$ then the corresponding moment polytope is obtained by truncating a vertex of the 3-simplex.

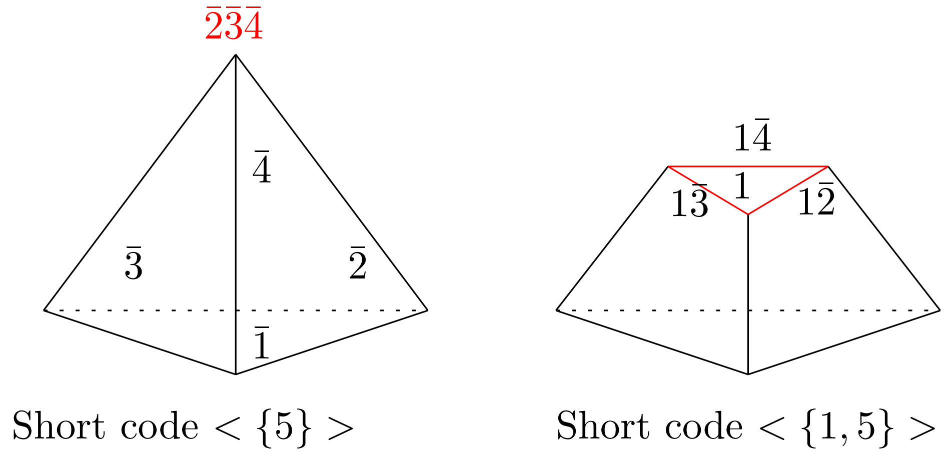


FIGURE 3.2: Vertex truncation

Now we show that the poset of admissible subsets $Ad(\alpha)$ is determined by the short code of the corresponding length vector.

Theorem 3.25. *Let* $\alpha = (\alpha_1, \ldots, \alpha_m)$ *be a generic length vector. Let* $J = J_1 \cup J_2$ *where* $J_1 \subseteq [m-1]^+$ *and* $J_2 \subseteq [m-1]^-$ *with* $J_1 \cap J_2 = \emptyset$. *Then* J *is an admissible subset of* $[m-1]$ *if and only if* $J_1 \cup \{m\}$ *and* $-J_2$ *are* α*-short subsets of* $[m]$.

Proof. Suppose $J = J_1 \cup J_2$ is an admissible subset of $[m-1]_\alpha$. i.e.

$$\alpha_m + \sum_{i\in J_1} \alpha_i - \sum_{j\in -J_2} \alpha_j < \sum_{k\in (J_1\cup -J_2)^c} \alpha_k. \tag{3.4}$$

The above equation gives us two inequalities, the first of which is

$$\alpha_m + \sum_{i\in J_1} \alpha_i - \sum_{j\in J_2} \alpha_j < \sum_{k\in (J_1\cup -J_2)^c} \alpha_k \tag{3.5}$$

and the second one is

$$-\sum_{k\in (J_1\cup -J_2)^c} \alpha_k < \alpha_m + \sum_{i\in J_1} \alpha_i - \sum_{j\in J_2} \alpha_j. \tag{3.6}$$

Equation (3.5) gives

$$\ell_m + \sum_{i \in J_1} \ell_i < \sum_{k \in (J_1 \cup -J_2)^c} \ell_k + \sum_{j \in J_2} \ell_j,$$

i.e., $J_1 \cup \{m\}$ is a short subset. Equation (3.6) gives

$$\sum_{j \in J_2} \ell_j < \ell_m + \sum_{i \in J_1} \ell_i + \sum_{k \in (J_1 \cup -J_2)^c} \ell_k,$$

i.e., $-J_2$ is a short subset.

Conversely, we can obtain the Equation (3.4) using the inequalities in Equation (3.5) and Equation (3.6). It proves the converse since the length vector $\ell(J)$ is generic. As the short code determines the collection of short subsets, it automatically determines the poset of admissible subsets. □

Recall the definition of small cover from the Motivation. Note that these manifolds are a topological generalization of real toric varieties as proved by Davis and Januszkeiwicz in [19]. One of their important results specifies how to build a small cover from the quotient polytope (see [19, Section 1.5] for details). It says that there is a regular cell structure on the manifold consisting of 2^n copies of the quotient polytope as the top-dimensional cells. We describe their construction briefly.

Consider an n-dimensional simple polytope P with the facet set $\mathcal{F} = \{F_1, \ldots, F_m\}$. A function $\lambda : \mathcal{F} \to \mathbb{Z}_2^n$ is called characteristic for P if for each vertex $v = F_{i_1} \cap \cdots \cap F_{i_n}$, the $n \times n$ matrix whose columns are $\lambda(F_{i_1}), \ldots, \lambda(F_{i_n})$ is unimodular. Given the pair (P, λ) the corresponding small cover $X(P, \lambda)$ is constructed as follows:

$$X(P, \lambda) := \frac{(\mathbb{Z}_2)^n \times P}{\{(t, p) \sim (u, q)\}} \quad \text{if } p = q \text{ and } t^{-1}u \in \operatorname{stab}(F_q)$$

where F_q is the unique face of P containing q in its relative interior.

In the remaining section we explicitly determine the entries of the characteristic matrix for chain spaces. The moment polytope $P(\ell)$ is $(m-2)$-dimensional but described as a subset of $\mathbb{R}^{m-1}$, so project it onto an affinely isomorphic polytope $Q(\ell)$. This new polytope is embeded in $\mathbb{R}^{m-2}$ and we determine its outward normals, which determine the characteristic function.

Given a generic length vector $\ell = (\ell_1, \ldots, \ell_{m-1}, \ell_m)$, consider the following hyperplanes in $\mathbb{R}^{m-2}$:

$$\overline{H}_{m-1}(\ell) = \left\{(x_1, x_2, \ldots, x_{m-2}) \in \mathbb{R}^{m-2} : \sum_{i=1}^{m-2} x_i = \ell_m - \ell_{m-1}\right\}$$

and

$$H_{m-1}(\ell) = \left\{(x_1, x_2, \ldots, x_{m-2}) \in \mathbb{R}^{m-2} : \sum_{i=1}^{m-2} x_i = \ell_m + \ell_{m-1}\right\}.$$

Let $\overline{H}^{\geq 0}_{m-1}$ and $H^{\leq 0}_{m-1}$ be a positive and negative part of $\overline{H}_{m-1}$ and H_{m-1} respectively.

Theorem 3.26. *Define an* $(m-2)$*-dimensional polytope as follows:*

$$Q(\ell) = \begin{cases} \prod_{i=1}^{m-2} [-\ell_i, \ell_i] \cap \overline{H}^{\geq 0}_{m-1}, & \text{if } \{m-1, m\} \text{ is long subset,} \\ \prod_{i=1}^{m-2} [-\ell_i, \ell_i] \cap \overline{H}^{\geq 0}_{m-1} \cap H^{\leq 0}_{m-1}, & \text{if } \{m-1, m\} \text{ is short subset.} \end{cases}$$

Then $P(\ell)$ *is affinely isomorphic to* $Q(\ell)$*.*

Proof. Let

$$\pi : \mathbb{R}^{m-1} \to \mathbb{R}^{m-2}$$

be the projection defined by

$$\pi(x_1, x_2, \ldots, x_{m-1}) = (x_1, x_2, \ldots, x_{m-2}).$$

Note that π restricts to an isomorphism on $H(\ell)$. Since $P(\ell) \subset H(\ell)$, π gives a bijection between $P(\ell)$ and $\pi(P(\ell))$. We show that $Q(\ell) = \pi(P(\ell))$. Suppose $\{m-1, m\}$ is a long subset. Then $H(\ell)$ does not intersects a facet of $C^{m-1}(\ell)$ given by $x_{m-1} = -\ell_{m-1}$. Also H_{m-1} does not intersect $C^{m-2}(\ell)$. Let $(x_1, x_2, \ldots, x_{m-1}) \in P(\ell)$. Then the following inequality is clear:

$$\sum_{i=1}^{m-2} x_i = \ell_m - x_{m-1} \geq \ell_m - \ell_{m-1}.$$

This gives us $\pi(P(\ell)) \subset Q(\ell)$. Now we show the other inclusion. Let $(y_1, y_2, \ldots, y_{m-2}) \in Q(\ell)$. Then it follows that $\sum_{i=1}^{m-2} y_i \geq \ell_m - \ell_{m-1}$. Let $a =$

$\alpha_m - \sum_{i=1}^{m-2} y_i$ and $\bar{y} = (y_1, y_2, \dots, y_{m-2}, a)$. Note that $\bar{y} \in P(\alpha)$ as $\sum_{i=1}^{m-2} y_i + a = \alpha_m$ and $|a| \leq \alpha_{m-1}$. Since $\pi(\bar{y}) = (y_1, y_2, \dots, y_{m-2})$, $(y_1, y_2, \dots, y_{m-2}) \in \pi(P(\alpha))$. Which gives $Q(\alpha) \subseteq \pi(P(\alpha))$. We conclude that $Q(\alpha) = \pi(P(\alpha))$. Similar arguments works when $\{m-1, m\}$ is a short subset. $\square$

Note that the facets of $Q(\alpha)$ are given by following equations.

- $F_i : x_i = \alpha_1$ for $1 \leq i \leq m-2$.
- $\overline{F}_{m-1} : \sum_{i=1}^{m-2} x_i = \alpha_m - \alpha_{m-1}$ when $\{m-1, m\}$ is long subset.
- $F_i : x_i = -\alpha_i$ when $\{i, m\}$ is short subset.
- $F_{m-1} : \sum_{i=1}^{m-2} x_i = \alpha_m + \alpha_{m-1}$ when $\{m-1, m\}$ is short subset.

Let $\mathcal{F}(Q(\alpha))$ be the collection of facets of $Q(\alpha)$. We define a map

$$\lambda : \mathcal{F}(Q(\alpha)) \to \mathbb{Z}_2^{m-2}$$

as follows:

$$\lambda(\overline{F}_i) = \begin{cases} -e_i, & 1 \leq i \leq m-2, \\ \sum_{i=1}^{m-2} e_i, & \text{if } \{m-1, m\} \text{ is long}, \end{cases}$$

and

$$\lambda(F_i) = \begin{cases} e_i, & \text{if } \{i, m\} \text{ is short}, \\ -\sum_{i=1}^{m-2} e_i, & \text{if } \{m-1, m\} \text{ is short}. \end{cases}$$

The following result is clear.

Lemma 3.27. *The function λ is characteristic for $Q(\alpha)$.*

We can now state the main equivalence.

Theorem 3.28. *With the notation as above the chain space $\mathrm{Ch}(\alpha)$ is the small cover corresponding to the pair $(Q(\alpha), \lambda)$.*

Proof. The natural action of $\mathbb{Z}_2^{m-1}$ on $(S^1)^{m-1}$ descends to Ch( ). However, this action is not effective. The element $(r, \ldots, r)$ of $\mathbb{Z}_2^{m-1}$ (where r generates a copy of $\mathbb{Z}_2$) xes every element of Ch( ). So we get the effective action by dividing by the diagonal subgroup. The remaining details are easy to verify hence omitted here. □

Remark 3.29. Note that the above charcterisitc vectors are normal to the corresponding facets. Hence, if all the $_i$'s are rational then the moment polytope is a lattice polytope and the corresponding small cover is a non-singular, real toric variety. We refer the reader to [19, Section 7] for more on the characteristic functions of non-singular toric varieties.

Example 3.30. Let $= (1, 2, 2, 2)$ $= (1, 1, 2, 1)$ and $= (2, 2, 2, 1)$ be generic length vectors. The shaded regions denotes the corresponding moment polytopes. The corresponding characterstic functions are given by matrices

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \text{ and } = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Figure 3.3: Moment polytopes

3.4 Main theorem

In this section we prove the main theorem of this chapter. We begin by de ning a special class of polytopes.

De nition 3.31. A simple polytope is *triangle-free* if it does not contain a triangular face of dimension 2.

The following result can nd in [3].

Lemma 3.32. Flagtopes are triangle-free.

Proof. Let P be the n-dimensional flagtope. On contrary, suppose P is not triangle-free. Then there exist a 2-face Δ which is 2-simplex. Since P is simple, $\Delta = \cap_{j=1}^{n-2} F_{i_j}$. Let v_1, v_2, v_3 be vertices and e_1, e_2, e_3 are edges of Δ such that e_i is opposite to v_i. Note that for each i there exist a facet F_{e_i} which is different from F_{i_j}'s such that $e_i = F_{e_i} \cap \Delta$. i.e. $e_i = F_{e_i} \cap (\cap_{j=1}^{n-2} F_{i_j})$. For each $1 \leq i \leq 3$ consider,

$$S_i = \{F_{e_i}\} \cup \{F_{i_j} : 1 \leq j \leq n-2\}.$$

Since $e_i \cap e_j \neq \emptyset$ for $1 \leq i < j \leq 3$, we have three families of subsets of facets which are pairwise disjoint. But P is flagtope therefore,

$$\cap_{i=1}^{3} F_{e_i} \bigcap \left(\bigcap_{j=1}^{n-2} F_{i_j} \right) \neq \emptyset.$$

Clearly, this is a contradiction to the fact that P is a simple polytope, since we have nonempty intersection of $n+1$ facets. Therefore, P must be triangle-free. □

The following theorem gives a partial converse of previous lemma.

Theorem 3.33 ([2, Theorem 3]). If P is a triangle-free convex polytope of dimension n then $f_i(P) \geq f_i(I^n)$ for $i = 0, \dots, n$. In particular, such a polytope has at least $2n$ facets. Furthermore, if P is simple then

1. $f_{n-1}(P) = 2n$ implies that $P = I^n$;
2. $f_{n-1}(P) = 2n+1$ implies that $P = P_5 \times I^{n-2}$ where P_5 is a pentagon;
3. $f_{n-1}(P) = 2n+2$ implies that $P = P_6 \times I^{n-2}$ or $P = Q \times I^{n-3}$ or $P = P_5 \times P_5 \times I^{n-4}$ where P_6 is a hexagon and Q is the 3-polytope obtained from a pentagonal prism by truncating one of the edges forming a pentagonal facet.

The Theorem 3.33 helps us to classify length vectors ℓ's for which the moment polytope $P(\ell)$ is a flagtope.

Theorem 3.34. Let ℓ be a generic length vector then the moment polytope $P(\ell)$ is a flagtope if and only if the short code of ℓ is one of the following:

1. $\langle\{1,2,\ldots,m-3,m\}\rangle$
2. $\langle\{1,2,\ldots,m-4,m-2,m\}\rangle$
3. $\langle\{1,2,\ldots,m-4,m-1,m\}\rangle$

Proof. First we describe the idea of the proof, which is the same in each case. It follows from Theorem 3.19 that, the number of facets of $P(\ell)$ corresponding to each short code mentioned above, in that order, are $2(m-2)$, $2(m-2)+1$ and $2(m-2)+2$, respectively. Therefore, to show that the polytope $P(\ell)$ is triangle-free, we need to make sure that each 2-dimensional face is not a 2-simplex. Note that the 2-dimensional faces of $P(\ell)$ correspond to admissible subsets of cardinality $m-4$. Therefore, in each case we determine these subsets and show that the corresponding 2-dimensional faces have at least 4 edges. Then we use Theorem 3.33 to conclude $P(\ell)$ is a flagtope. We now analyze each short code.

Case 1. The short code of ℓ is $\langle\{1,2,\ldots,m-3,m\}\rangle$.

Let $J = J_1 \cup J_2$ be an admissible subset of $[m-1]$ where $J_1 \subseteq [m-1]^+$, $J_2 \subseteq [m-1]^-$ with $J_1 \cap -J_2 = \emptyset$ and $|J| = m-4$. By Theorem 3.25 we have that $J_1 \cup \{m\}$ and $-J_2$ are ℓ-short subsets. Moreover $J_1 \subseteq [m-3]$ as short code of ℓ is $\langle\{1,2,\ldots,m-3,m\}\rangle$. Note that $\{m-2,m-1\}$ cannot be a subset of $-J_2$ since it is long. There are following three possibilities for $-J_2$.

1. We have $-J_2 \subseteq [m-3]$: Note that the facets of F_J correspond to admissible subsets of cardinality $m-3$ containing J. We have, $\{i,m-2,m-1\} \subseteq (J_1 \cup -J_2)^c$ for $i \in [m-3]$. Clearly, the subsets $\{i\} \cup J_1 \cup \{m\}$ and $\{i\} \cup -J_2$ are ℓ-short. Moreover, the subsets $\{m-2\} \cup -J_2$ and $\{m-1\} \cup -J_2$ are ℓ-short, since their complements contains long subsets, $\{m-1,m\}$ and $\{m-2,m\}$ respectively. Therefore, the subsets

$$\{i\} \cup J,\ \{-i\} \cup J,\ \{-(m-2)\} \cup J,\ \{-(m-1)\} \cup J$$

are admissible. Note that these admissible subsets represents the facets of F_J. Clearly $F_J = I^2$.

2. We have $m-2 \in -J_2$ but $m-1 \notin -J_2$: In this case we have $\{i,j,m-1\} \subseteq (J_1 \cup -J_2)^c$ where $\{i,j\} \subseteq [m-3]$. Note that the subsets,

$$\{i\} \cup J_1 \cup \{m\},\ \{j\} \cup J_1 \cup \{m\},\ \{i\} \cup -J_2,\ \{j\} \cup -J_2$$

are α-short. Therefore, the subsets

$$\{i\}\cup J,\ \{j\}\cup J,\ \{-i\}\cup J,\ \{-j\}\cup J$$

are admissible. Therefore, 2-dimensional faces F_J have at least four facets.

3. We have $m-1 \in -J_2$ but $m-2 \notin -J_2$: This is exactly similar to the earlier case.

As a consequence of Theorem 3.19 we have, $f_{m-3}(P(\alpha)) = 2(m-2)$. Therefore, using Theorem 3.33 we get $P(\alpha) = I^{m-2}$, the $(m-2)$-cube.

Case 2. The short code of α is $\langle\{1,2,\ldots,m-4,m-2,m\}\rangle$.

Note that $f_{m-3}(P(\alpha)) = 2(m-2)+1$. The short code information gives following possibilities for J_1 and J_2.

- Possibilities for J_1:
  1. $J_1 \subseteq [m-3]$
  2. $m-2 \in J_1$ and $J_1 \setminus \{m-2\} \subseteq [m-4]$
- Possibilities for J_2:
  1. $-J_2 \subseteq [m-2]$
  2. $-J_2 = S \cup \{m-1\}$ where $S \subseteq [m-4]$ as $\{m-3, m-1\}$ and $\{m-2, m-1\}$ are long subsets.

We now consider all four possible combinations of J_1 and J_2.

1. $J_1 \subseteq [m-3]$ and $-J_2 \subseteq [m-2]$:

   Note that $\{i, j, m-1\} \subseteq (J_1 \cup -J_2)^c$ where $\{i, j\} \subseteq [m-2]$. Consider the following possibilities for $\{i, j\}$:

   (a) $\{i, j\} \subseteq [m-3]$:

   Observe that $S \cup \{m\}$ is α-short if $S \subseteq [m-3]$. Therefore, $\{i\}\cup J_1 \cup \{m\}$ and $\{j\}\cup J_1\cup\{m\}$ are α-short subsets. Since $\{m-1, m\}$ is long, any subset of

$[m-2]$ is -short. Consequently, $\{i\} \cup -J_2$ and $\{j\} \cup -J_2$ are -short subsets. Hence

$$\{i\} \cup J,\ \{j\} \cup J,\ \{-i\} \cup J,\ \{-j\} \cup J$$

are admissible subsets. Therefore, the 2-dimensional faces F_J have at least four edges.

(b) $i \in [m-3]$ and $j = m-2$:

In this case we have $\{i, m-2, m-1\} \subseteq (J_1 \cup -J_2)^c$. Consider the following three sub-cases :

i. If $i \in [m-4]$ and $m-3 \in J_1$:

Note that $\{i\} \cup J_1 \cup \{m\}$, $\{i\} \cup -J_2$ and $\{m-2\} \cup -J_2$ are short subsets. Since $m-3 \in J_1$, $m-3 \notin -J_2$. Therefore, $-J_2 \subseteq [m-4]$. Hence, a subset $\{m-1\} \cup -J_2$ is -short. Therefore,

$$\{i\} \cup J,\ \{-i\} \cup J,\ \{-(m-2)\} \cup J,\ \{-(m-1)\} \cup J$$

are admissible subsets.

ii. If $i \in [m-4]$ and $m-3 \in -J_2$: This case is exactly same as above case.

iii. If $i = m-3$:

Note that $\{m-3, m-2, m-1\} \subseteq (J_1 \cup -J_2)^c$. It is easy to see that the following subsets

$$\{m-3\} \cup J,\ \{m-2\} \cup J,\ \{-(m-3)\} \cup J,\ \{-(m-2)\} \cup J,\ \{-(m-1)\} \cup J$$

are admissible containing J and represents the facets of F_J. Therefore, 2-dimensional faces F_J have exactly five edges.

2. $J_1 \subseteq [m-3]$ and $-J_2 = S \cup \{m-1\}$ where $S \subseteq [m-4]$:

In this case we have $\{i, j, m-2\} \subseteq (J_1 \cup -J_2)^c$ where $\{i, j\} \subseteq [m-3]$. Consider the following possibilities for $\{i, j\}$:

(a) Suppose $\{i, j\} \subseteq [m-4]$:

It is easy to see that the following subsets

$$\{i\} \cup J_1 \cup \{m\},\ \{j\} \cup J_1,\ \{j\} \cup -J_2,\ \{j\} \cup -J_2$$

are -short. Hence, the subsets

$$\{i\} \cup J,\ \{j\} \cup J,\ \{-i\} \cup J,\ \{-j\} \cup J$$

are admissible.

(b) Suppose $i \in [m-4]$ and $j = m-3$:

In this case we have $\{i, m-3, m-2\} \subseteq (J_1 \cup -J_2)^c$. Since $m-3 \notin J_1$, $J_1 \subseteq [m-4]$. Hence the subsets, $\{m-3\} \cup J_1 \cup \{m\}$ and $\{m-2\} \cup J_1 \cup \{m\}$ are -short. Note that, $\{i\} \cup J_1$ and $\{i\} \cup -J_2$ are also -short subsets. Consequently,

$$\{m-3\} \cup J,\ \{m-2\} \cup J,\ \{i\} \cup J,\ \{-i\} \cup J$$

are admissible subsets.

3. $m-2 \in J_1$, $J_1 \setminus \{m-2\} \subseteq [m-4]$ and $-J_2 \subseteq [m-2]$:

Observe that, $\{i, j, m-1\} \subseteq (J_1 \cup -J_2)^c$ where $\{i.j\} \subseteq [m-3]$. Consider the following possibilities for $\{i.j\}$:

(a) If $\{i, j\} \subseteq [m-4]$:

Clearly the subsets,

$$\{i\} \cup J,\ \{j\} \cup J,\ \{-i\} \cup J,\ \{-j\} \cup J$$

are admissible.

(b) If $i \in [m-4]$ and $j = m-3$:

In this case we have $\{i, m-3, m-1\} \subseteq (J_1 \cup -J_2)^c$. Since $m-2 \in J_1$, $m-2 \notin -J_2$. On the other hand, since $m-3 \notin -J_2$, $-J_2 \subseteq [m-4]$. Note that, $S \cup \{m-1\}$ is -short for $S \subseteq [m-4]$. Therefore, $\{m-1\} \cup -J_2$ is -short subset. Consequently $\{-(m-1)\} \cup J$ is -admissible subset. Hence,

$$\{i\} \cup J,\ \{-i\} \cup J,\ \{-(m-3)\} \cup J,\ \{-(m-1)\} \cup J$$

are admissible subsets.

4. $m-2 \in J_1$, $J_1 \setminus \{m-2\} \subseteq [m-4]$ and $J_2 = S \cup [m-1]$, $S \subseteq [m-4]$:

Here, we have $\{i, j, m-3\} \subseteq (J_1 \cup -J_2)^c$ where $\{i, j\} \subseteq [m-4]$. Clearly the subsets

$$\{i\} \cup J, \ \{-i\} \cup J, \ \{j\} \cup J, \ \{-j\} \cup J$$

are admissible subsets.

Finally we conclude that $P(\ell)$ is triangle-free if the short code of ℓ is $\langle\{1, 2, \ldots, m-4, m-1, m\}\rangle$. Moreover using Theorem Theorem 3.33 and Lemma Theorem 3.19, we get $P(\ell) = P_5 \times I^{m-4}$, where P_5 is a pentagon.

Case 3. The short code of ℓ is $\langle\{1, 2, \ldots, m-4, m-1, m\}\rangle$.

Note that the short code of ℓ gives us the following possibilities of J_1 and J_2 :

- Possibilities for J_1:
  1. $J_1 \subseteq [m-3]$
  2. $J_1 \subseteq [m-4]$ and $m-2 \in J_1$
  3. $J_1 \subseteq [m-4]$ and $m-1 \in J_1$
- Possibilities for J_2
  1. $-J_2 \subseteq [m-3]$
  2. $-J_2 \subseteq [m-4]$ and $m-2 \in -J_2$
  3. $-J_2 \subseteq [m-4]$ and $m-1 \in -J_2$

We consider the all nine possible combinations of J_1 and J_2. Let's begin with the first combination:

1. $J_1 \subseteq [m-3]$ and $-J_2 \subseteq [m-3]$:

   Observe that, $\{i, m-2, m-1\} \subseteq (J_1 \cup -J_2)^c$ where $i \in [m-3]$. Since $\{m-2, m-1\}$ is a long, the subsets $\{i\} \cup J_1 \cup \{m\}$ and $i \cup -J_2$ are ℓ-short. Consider the following possibilities :

(a) $m-3 \in J_1$:

Since $J_1 \cap -J_2 = \emptyset$, $-(m-3) \notin J_2$. Hence $-J_2 \subseteq [m-4]$. Therefore, $\{m-2\} \cup -J_2$ and $\{m-1\} \cup -J_2$ are ℓ-short subsets. Consequently, the subsets

$$\{i\} \cup J,\ \{-i\} \cup J,\ \{-(m-2)\} \cup J,\ \{-(m-1)\} \cup J$$

are admissible.

(b) $m-3 \notin J_1$:

Therefore $J_1 \subseteq [m-4]$. Hence, the subsets $\{m-2\} \cup J_1 \cup \{m\}$ and $\{m-1\} \cup J_1 \cup \{m\}$ are ℓ-short. Consequently, the subsets

$$\{i\} \cup J,\ \{-i\} \cup J,\ \{m-2\} \cup J,\ \{m-1\} \cup J$$

are admissible.

2. $J_1 \subseteq [m-3]$ and $-J_2 \subseteq [m-4]$, $m-2 \in -J_2$:

Since $\{m-3, m-2\}$ is long, $m-3 \notin -J_2$. In this case we have $\{i, j, m-1\} \subseteq (J_1 \cup -J_2)^c$ where $\{i, j\} \subseteq [m-3]$. Therefore, the subsets $\{i\} \cup J_1 \cup \{m\}$ and $\{j\} \cup J_1 \cup \{m\}$ are ℓ-short. Consider the following possibilities:

(a) $m-3 \in J_1$:

Hence $\{i, j\} \subseteq [m-4]$. Therefore, $\{i\} \cup -J_2$ and $\{j\} \cup -J_2$ are ℓ-short subsets. Hence, the subsets

$$\{i\} \cup J,\ \{j\} \cup J,\ \{-i\} \cup J,\ \{-j\} \cup J$$

are admissible.

(b) $m-3 \notin J_1$:

Hence $J_1 \subseteq [m-4]$. Therefore, $\{m-1\} \cup J_1 \cup \{m\}$ is a ℓ-short subset. Since $m-3 \notin -J_2$, $\{i, m-3, m-1\} \subseteq (J_1 \cup -J_2)^c$ where $i \in [m-4]$. So $\{i\} \cup -J_2$ is ℓ-short subset. Therefore, the subsets,

$$\{i\} \cup J,\ \{-i\} \cup J,\ \{m-3\} \cup J,\ \{m-1\} \cup J$$

are admissible.

3. $J_1 \subseteq [m-3]$ and $-J_2 \subseteq [m-4], m-1 \in -J_2$: This case is exactly similar to the above case.

4. $J_1 \subseteq [m-4], m-2 \in J_1$ and $-J_2 \subseteq [m-3]$:

   In this case we have $\{i, j, m-1\} \subseteq (J_1 \cup -J_2)^c$ where $\{i, j\} \subseteq [m-3]$. Consider the following possibilities :

   (a) $m-3 \in -J_2$:

   Hence $\{i, j\} \subseteq [m-4]$. Therefore, the subsets,

   $$\{i\} \cup J_1 \cup \{m\}, \{j\} \cup J_1 \cup \{m\}, \{i\} \cup -J_2, \{j\} \cup -J_2$$

   are α-short. Hence the subsets,

   $$\{i\} \cup J, \{j\} \cup J, \{-i\} \cup J, \{-j\} \cup J$$

   are admissible.

   (b) $m-3 \notin -J_2$:

   In this case we have $\{i, m-3, m-1\} \subseteq (J_1 \cup -J_2)^c$ where $i \in [m-4]$. Therefore,

   $$\{i\} \cup J_1 \cup \{m\}, \{i\} \cup -J_2, \{m-3\} \cup -J_2, \{m-1\} \cup -J_2$$

   are α-short subsets. Consequently, the subsets

   $$\{i\} \cup J, \{-i\} \cup J, \{-(m-3)\} \cup J, \{-(m-1)\} \cup J$$

   are admissible.

5. $J_1 \subseteq [m-4], m-2 \in J_1$ and $-J_2 \subseteq [m-4]$ and $m-2 \in -J_2$:

   This case doesn't not arise as $J_1 \cap -J_2 \neq \emptyset$.

6. $J_1 \subseteq [m-4], m-2 \in J_1$ and $-J_2 \subseteq [m-4], m-1 \in -J_2$:

   In this case we have $\{i, j, m-3\} \subseteq (J_1 \cup -J_2)^c$ where $\{i.j\} \subseteq [m-4]$. It is easy to see that the subsets,

   $$\{i\} \cup J_1 \cup \{m\}, \{j\} \cup J_1 \cup \{m\}, \{i\} \cup -J_2, \{j\} \cup -J_2$$

are -short subsets. Hence the subsets

$$\{i\} \cup J, \{j\} \cup J, \{-i\} \cup J, \{-j\} \cup J$$

are admissible.

7. $J_1 \quad [m-4], m-1 \in J_1$ and $-J_2 \quad [m-3]$: This case is exactly similar to the case where, $J_1 \quad [m-4], m-2 \in J_1$ and $-J_2 \quad [m-3]$.

8. $J_1 \quad [m-4], m-1 \in J_1$ and $-J_2 \quad [m-4], m-2 \in -J_2$: This case is exactly similar to the case where, $J_1 \quad [m-4], m-2 \in J_1$ and $-J_2 \quad [m-4], m-1 \in -J_2$:

9. $J_1 \quad [m-4]$ and $m-1 \in J_1$ and $-J_2 \quad [m-4]$ and $m-1 \in -J_2$ This case doesn't arise as $J_1 \cap -J_2 \neq \emptyset$.

Moreover, using Theorem 3.33 and Theorem 3.19, we get $P(\ ) = P_6 \quad I^{m-4}$, where P_6 is a hexagon.

Now we prove the converse. Recall that, the Theorem 3.22 says that the poset of an admissible subsets determine the short code of a length vector. In particular, for generic length vectors and , if $P(\ ) = P(\ )$ then the short codes of and coincides. Suppose, $P(\ )$ is a agtope for a generic length vector . Note that Theorem 3.19 give,

$$2(m-2) \leq f_{m-3}(P(\ )) \leq 2(m-2) + 2.$$

Therefore, there are only three possibilities for the number of facets of $P(\ )$. Suppose that $f_{m-3}(P(\ )) = 2(m-2)$. Theorem 3.33 shows that $P(\ ) = I^{m-2}$, a $m-2$-cube. But we have shown that I^{m-2} is the moment polytope for short code $\langle\{1, 2, \ldots, m-3, m\}\rangle$. Therefore, the short code of is $\langle\{1, 2, \ldots, m-3, m\}\rangle$ whenever $P(\ )$ is agtope and $f_{m-3}(P(\ )) = 2(m-2)$. Similar arguments works when $f_{m-3}(P(\ )) = 2(m-2)+1$ and $f_{m-3}(P(\ )) = 2(m-2)+2$. □

Now we state the main theorem of this section.

Theorem 3.35. Let be a generic length vector. Then the corresponding chain space is aspherical if and only if the short code of is one of the following:

1. $\langle\{1, 2, \ldots, m-3, m\}\rangle$,

2. $\langle\{1, 2, \ldots, m-2, m\}\rangle$,

3. $\langle\{1, 2, \ldots, m-4, m-2, m\}\rangle$,

4. $\langle\{1, 2, \ldots, m-4, m-1, m\}\rangle$.

Proof. The proof now follows from Theorem 3.34 and Theorem 1.3. □

Example 3.36. Below are some examples.

1. If $= (1, 1, 3, 3, 3)$ then the short code of is $\langle\{1, 2, 5\}\rangle$ and $P( ) = I^3$.

2. If $= (1, 2, 2, 5, 3)$ then the short code of is $\langle\{1, 3, 5\}\rangle$ and $P( ) = P_5 \quad I$.

3. If $= (1, 3, 3, 3, 1)$ then the short code of is $\langle\{1, 4, 5\}\rangle$ and $P( ) = P_6 \quad I$.

We now describe the homeomorphism type of these aspherical chain spaces. However, before that we mention an important and relevant result about polygon spaces. Let $\langle A_1, \dots, A_k \rangle$ be the genetic code of $\ell = (\ell_2, \dots, \ell_m)$. Consider another generic length vector ℓ^+ whose genetic code is $\langle B_1, \dots, B_k \rangle$ where $B_i = \{a+1 : a \in A_i\} \cup \{1\}$. Note that ℓ^+ is an m-tuple. Hausmann [32, Proposition 2.1] described a relationship between the polygon spaces $\overline{M}_{\ell^+}$ and $\overline{M}_{\ell}$.

Proposition 3.37. The polygon space $\overline{M}_{\ell^+}$ is diffeomorphic to the fibered product $S^1 \times_{O(1)} M_{\ell}$, where $O(1)$ acts diagonally.

Recall that

$$q : M_{\ell} \to \overline{M}_{\ell}$$

is a double cover. This double cover helps us to define a natural map

$$\phi : \overline{M}_{\ell^+} \to \overline{M}_{\ell}.$$

It is easy to see that ϕ is an S^1-fibration.

Now we return to aspherical chain spaces. Recall Theorem 3.11; it says that given a generic length vector $\ell = (\ell_1, \dots, \ell_m)$ we have an isomorphism between

$$\mathrm{Ch}(\ell) \to \overline{M}_{\ell'},$$

where

$$\ell' = (\ell_1, \ell_2, \dots, \ell_{m-1}, \lambda, \ell_m + \lambda)$$

for some positive real number $\lambda > \sum_{i=1}^{m-1} \ell_i$.

Suppose the short code of is $<1,2,\ldots,m-3,m>$. Then the genetic code of ^{0} is $<1,2,\ldots,m-3,m+1>$. Let be a generic length vector whose genetic code is $<1,2,\ldots,m-4,m>$. Clearly, $^{+} = {}^{0}$. Let $(1) = (_2,\ldots, _m + _1)$ be a generic length vector. Note that the short code of (1) is $<1,2,\ldots,m-4,m-1>$ with $(1)^0 =$. We have an isomorphism $Ch((1)) \to \overline{M}_{(1)^0}$. Using Hausmann's S^1- bration described in the proof of Proposition Theorem 3.37, we have the following maps.

$$Ch( ) \xrightarrow{=} \overline{M}_{0} \xrightarrow{1} \overline{M} \xrightarrow{=} Ch((1)).$$

Clealry, the above composition of maps gives a S^1- bration $\tilde{}$ from

$$Ch( ) \xrightarrow{\tilde{}1} Ch((1)).$$

Let $I_j = \{1,2,\ldots,j\}$. By induction we get the chain of S^1- brations

$$Ch( ) \xrightarrow{\tilde{}1} Ch((I_1)) \xrightarrow{\tilde{}2} Ch((I_2)) \xrightarrow{\tilde{}3} \cdots \xrightarrow{\tilde{}m-3} Ch((I_{m-3})) \xrightarrow{\tilde{}m-2} \{?\} \tag{3.7}$$

where $(I_j) = (_{j+1},\ldots, _{m-1}, _m + \sum_{i=1}^{j} {}_i)$.

Recall that the moment polytope is the $(m-2)$-cube. Hence,the above chain of brations is a real Bott tower. One can easily check that the characterstic matrix of $Ch( )$ is

$$\left[\, I_{m-2} \;\middle|\; I^0_{m-3} \;\middle|\; 1 \,\right]$$

where I_{m-2} is the block of $(m-2)\ (m-2)$ identity matrix, I^0_{m-3} is the $(m-2)$ $(m-3)$ block containing size $(m-3)$ identity matrix with the last row of zeros, the last block 1 is the column of 1's. The corresponding Bott matrix (that encodes the Stiefel-Whitney class of the bration at each stage) is

$$\begin{bmatrix} 0 & 1 & & 1 \\ 0 & 0 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 0 \end{bmatrix}.$$

Similarly, the remaining three aspherical chain spaces are towers of S^1- brations starting from a non-orientable surface.

Chapter 4

Building planar polygon spaces from the Projective braid arrangement

Observe that the collection of polygons in M (respectively $\overline{M}$) with exactly two parallel sides is a codimension-1 submanifold of M (respectively $\overline{M}$). It turns out that the collection A (respectively $\overline{A}$) of such codimesnion-1 submanifolds of M (respectively $\overline{M}$) forms a submanifold arrangement. Consequently, there is a cell structure on both M and $\overline{M}$ induced by A and $\overline{A}$, respectively. These cell structures on M and $\overline{M}$, denoted by K and $\overline{K}$, respectively.

In this chapter, we answer the Question 1.5 affirmatively. In order to achieve this, we introduce the notion of the (projective) cellular surgery on certain cell complexes (cells are combinatorially equivalent to simple polytopes (see Definition 4.42).) Then we show that M and $\overline{M}$ can be obtained from the iterated (projective) cellular surgery on the (projective) Coxeter complex along certain subspaces in the intersection lattice of the (projective) braid arrangement.

The results presented in this chapter can be found in [8], which is a joint work with P. Deshpande.

4.1 The braid arrangement and the Coxeter complex

In this section, we set up some notations and prove some results related to the Coxeter complex.

We begin with some basic definitions.

Definition 4.1. A finite collection of codimension-1 subspaces in the Euclidean space is called an arrangement of hyperplanes(or a hyperplane arrangement).

Definition 4.2. The braid arrangement is the collection

$$B_m = \{H_{ij} : 1 \leqslant i < j \leqslant m\}$$

where

$$H_{ij} = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_i - x_j = 0\}$$

An arrangement of hyperplanes is said to be an *essential* if the intersection of all hyperplanes is the origin. We can see that the arrangement B_m is not an essential, since

$$\bigcap H_{ij} = \{(t, \ldots, t) \in \mathbb{R}^m : t \in \mathbb{R}\} \neq \{0\}$$

Nevertheless, there is a way to make B_m essential by taking the quotient of $\mathbb{R}^m$ by $\bigcap H_{ij}$, i.e.,

$$V := \{(x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} x_i = 0\}$$

Now it is easy to see that the collection

$$B_V = \{H_{ij} \cap V : 1 \leqslant i < j \leqslant m\}$$

is an essential arrangement in V. The arrangement B_V is called an *essentialization* of B_m. Let SV be the unit sphere in V.

Definition 4.3. The intersection of hyperplanes in B_V gives a cellular decomposition of SV into $m!$ simplices of dimension $m-2$. This cellular decomposition of SV is called the Coxeter complex of type A_{m-1} and it is denoted by CA_{m-1}. The projective Coxeter complex PCA_{m-1} of type A_{m-1} is the quotient of Coxeter complex CA_{m-1} by the antipodal action.

It is clear that PCA_{m-1} is tiled by $\frac{1}{2}m!$ simplices of dimension $m-2$.

Example 4.4. The Coxeter complex CA_3 is a 2-sphere cellulated by 24 triangles and PCA_3 is the projective plane cellulated by 12 triangles (see Figure 4.1).

Figure 4.1: The Coxeter complex CA_3 and the projective Coxeter complex PCA_3

The collection of all possible intersections of hyperplanes of the hyperplane arrangement A forms a lattice under reverse inclusion as the partial order. We denote this lattice by $I(A)$, which is known as the intersection lattice Let $I(B_m)$ be the intersection lattice of B_m. It is clear that the lattices $I(B_m)$ and $I(B_V)$ are isomorphic. Moreover, it is known that $I(B)$ is isomorphic to the lattice of partitions of the set $[m]$, denoted by $\ _m$. In fact, if $\ = (J_1, \ldots, J_k)$ be a partition of $[m]$ then one can associate to

$$X \ = \{(x_1, \ldots, x_m) \in V : x_i = x_j \text{ whenever } i \text{ and } j \text{ in some } J_s \text{ for } 1 \leq s \leq k\}$$

an element of $I(B_V)$. The map

$$: \ _m \to \ I(B_V)$$

de ned by

$$(\ ) = X$$

is an isomorphism.

De Concini and Procesi [20] identi ed a special collection of elements of the intersection lattice of an arrangement such that the blow-ups along these subspaces commute for a given dimension and the resulting arrangement has normal crossings.

De nition 4.5. Let $X \in I(A)$. Consider the collection

$$A_X = \{H \in A : X \quad H\}$$

Then X is said to be reducible if there exist Y and Z in $I(A)$ such that $A_X = A_Y \sqcup A_Z$, otherwise X is irreducible.

Definition 4.6. The minimal building set $\mathrm{Min}(A)$ of A is the collection of all irreducible elements of $I(A)$.

Example 4.7. The following are some examples of minimal building sets corresponding to the braid arrangement.

1. $\mathrm{Min}(B_2) = \Pi_3 \setminus \{1-2-3\}$
2. In general, $\mathrm{Min}(B_m)$ consist of all partitions of $[m]$ which has at most one block of size greater or equal 2.

Now we prove that, for an element $X \in I(B_V)$, the induced cell decomposition on the unit sphere in X is a lower dimensional Coxeter complex.

Lemma 4.8. Let $X \in I(B_V)$ and $S_X = X \cap CA_{m-1}$. Then S_X is isomorphic to the Coxeter complex $CA_{\dim(X)-1}$.

Proof. Recall that $X = X_\pi$ for some partition $\pi = (J_1, \ldots, J_k)$ of $[m]$. Moreover, $\dim(X_\pi) = k-2$. Note that S_X is a sphere in X. We can think of the k blocks of X as the elements $\{1, 2, \ldots, k\}$ Then the induced cell structure on S_X is equivalent to the cell structure on the unit sphere in $\mathbb{R}^k$ induced by the braid arrangement. Therefore, $S_X = CA_{k-1}$. This proves the lemma. □

4.2 Motivation

The moduli space of m-punctured Riemann spheres M_0^m is an important object in geometric invariant theory. The Deligne-Knudson-Mumford compactification $\overline{M}_0^m$ of this space has been studied widely. We refer the reader to [50], [51]) for comprehensive introduction.

In [37], Hu introduced the notion of "stable polygons" (see [37, Definition 4.13]). Roughly speaking, a stable polygon is obtained from the following procedure: Let $P = (v_1, \ldots, v_m)$ be a polygon and $J \subseteq [m]$ such that $v_i = v_j$ for $i, j \in J$. That is, sides of P indexed by J are parallel. Now introduce a new polygon without parallel edges, whose all sides except the longest one are indexed by J. The longest side is $\sum_{j \in J} \ell_j - \epsilon$, where ϵ is a carefully chosen small positive real number. Denote this new polygon by P_J. Follow the same procedure for all sets of parallel sides and obtain such polygons without parallel edges. The stable polygon is a tuple of all such newly constructed polygons without parallel sides whose first coordinate is P.

Let Y be the collection of subvarities of N_ℓ defined in [37, Section 6]. The following theorem gives a relation between the moduli space of stable polygons $M_{\ell,\epsilon}$, the Deligne-Knudson-Mumford compactification $\overline{M}_0^m$ and the spatial polygon space N_ℓ.

Theorem 4.9 ([37, Theorem 7.3, Theorem 6.5]). With the above notations,

1. The moduli space $M_{\ell,\epsilon}$ is a complex manifold biholomorphic to $\overline{M}_0^m$.
2. The moduli space $M_{\ell,\epsilon}$ is obtained from N_ℓ, by iteratively blowing up along the elements of Y.

Note that the red and the black arrow in the Figure 4.2 denotes the iterated blow-up and the dotted line shows no relationship between the objects.

Figure 4.2: Hu's theorem

The ordered configuration space of m distinct points on a manifold M is

$$C_m(M) := M^m \setminus \Delta,$$

where $\Delta = \{(x_1, \ldots, x_m) \in M^m : \exists i, j, x_i = x_j\}$

Definition 4.10. The real moduli space of m-punctured Riemann spheres is

$$M_0^m(\mathbb{R}) := \frac{C_m(\mathbb{R}P^1)}{PGL_2(\mathbb{R})}.$$

Let $P(B_{m-1})$ be the projective braid arrangement in PV and $M(P(B_{m-1}))$ be the complement of $P(B_{m-1})$ in PV. Let $(PCA_{m-1})_{\#}$ denote the space obtained from PCA_{m-1} by iterated blow-ups along the minimal building set of $P(B_{m-1})$. Let $\overline{M}_0^m(R)$ be the real points of the Deligne-Mumford-Knudson compacti cation $\overline{M}_0^m$. Kapranov proved the following relation between $\overline{M}_0^m(R)$ and $(PCA_{m-1})_{\#}$.

Theorem 4.11 ([51]). With the above notations,

1. There are homeomorphisms $M(P(B)) = M_0^{m+1}(R)$ and $\overline{M}_0^{m+1}(R) = (PCA_{m-1})_{\#}$.
2. The space $\overline{M}_0^{m+1}(R)$ and its double cover $\tilde{M}_0^{m+1}(R)$, both are tiled by the convex polytopes combinatorially equivalent to associahedron.

The red arrow in the Fig. 4.3 denotes the iterated blow-up and the dotted lines shows no relation between the objects.

Figure 4 .3: Kapranov's theorem

It is known that for a generic , $\overline{M}$ contains $M_0^n(R)$ as an open dense set. In particular, $\overline{M}$ form a compacti cation of $M_0^n(R)$ (see [52], [48] and [60] for more details). Therefore, it is natural to ask whether the conclusions of Theorem 4.9 and Theorem 4.11 hold for planar polygon spaces.

4.3 The genetic order

Recall the de nition of genetic code from the Section 2.1. It can be observed that the partial order de ned in De nition 2.9 doesn't depend on the length vector. In particular, this partial order remains a partial order on the set of all subsets of $[m]$ containing m. This fact will help us to introduce the partial order on the collection of genetic codes.

Definition 4.12. Let $\langle A_1, \ldots, A_k \rangle$ and $\langle B_1, \ldots, B_l \rangle$ be two genetic codes. We say that

$$\langle A_1, \ldots, A_k \rangle \quad \langle B_1, \ldots, B_l \rangle$$

if for each $1 \leq i \leq k$ there exist $1 \leq j \leq l$ such that $A_i \leq B_j$. We call this partial order the genetic order

Remark 4.13. Note that in the Definition 4.12 we may have $l \leq k$. For example, $\langle 126, 36 \rangle \quad \langle 136 \rangle$.

Recall the collection $S_m(\ )$ from Theorem 2.9; it is obtained by adding one element in $S_m(\ )$. Then the genetic code of covers the genetic code of in the genetic order.

Proposition 4.14. The genetic code of covers the genetic code of in the genetic order if and only if $S_m(\ ) = S_m(\ ) \cup \{J\}$ for some $J \quad [m]$.

Proof. Observe that $S_m(\ ) = S_m(\ ) \cup \{J\}$ for some $J \quad [m]$ if and only if $G \ = \langle G \ , J \rangle$. Now the proposition is straightforward. □

Now we show that, for given genetic code G with a single gene, how to construct a saturated chain of genetic codes which starts with $\langle \{m\} \rangle$ and end with G.

Existence of a saturated chain of genetic codes :

Let $G = \langle \{g_1, \ldots, g_r, m\} \rangle$ be the genetic code. We construct a saturated chain of genetic codes starting with the genetic code $\langle m \rangle$. For each $1 \leq i \leq g_1 - 1$ consider the genetic code $G_i(g_1) = \langle \{g_1 - i, g_2, \ldots, g_r, m\} \rangle$. Clearly, $G_i(g_1)$ covers $G_{i+1}(g_1)$ since

$$S_m(G_i(g_1)) = S_m(G_{i+1}(g_1)) \cup \{g_1 - i, \ldots, g_r, m\}$$

Note that we have the following saturated chain

$$G_{g_1-1}(g_1) \quad G_{g_1-2}(g_1) \qquad G_1(g_1) \quad G. \tag{4.1}$$

Remember that our aim is to reach $\langle m \rangle$. To achieve that we need to find a suitable genetic code covered by $G_{g_1-1}(g_1)$. For $1 \leq i \leq g_2 - 2$, consider the following genetic code

$$G_i(1, g_2) = \langle \{1, g_2 - i, g_3, \ldots, g_r, m\}\{g_2, \ldots, g_r, m\} \rangle. \tag{4.2}$$

Clearly, $G_i(1,g_2)$ covers $G_{i+1}(1,g_2)$ since

$$S_m(G_i(1,g_2)) = S_m(G_{i+1}(1,g_2)) \cup \{1, g_2-i, g_3, \dots, g_r, m\}.$$

Since $G_{g_1-1}(g_1) = \langle\{1, g_2, \dots, g_r, m\}\rangle$, it is easy to see that

$$S_m(G_{g_1-1}(g_1)) = S_m(G_1(1,g_2)) \cup \{1, g_2, \dots, g_r, m\}.$$

Therefore, we got the genetic code $G_1(1,g_2)$ covered by $G_{g_1-1}(g_1)$. Now we can further extend the chain 4.1 as follows:

$$G_{g_2-2}(1,g_2) < \cdots < G_1(1,g_2) < G_{g_1-1}(g_1) < G_{g_1-2}(g_1) < \cdots < G_1(g_1) < G. \tag{4.3}$$

Now it is easy to see that the genetic code $\langle\{g_2, \dots, g_r, m\}\rangle$ is covered by $G_{g_2-2}(1,g_2)$. Now we can repeat the same procedure for $\langle\{g_2, \dots, g_r, m\}\rangle$ that we did for the genetic code $\langle\{g_1, \dots, g_r, m\}\rangle$ and arrive at the stage where we get the genetic code $\langle\{1, g_3, \dots, g_r, m\}\rangle$. Then we construct the genetic code $G_i(1,g_3)$ similar to Equation (4.2) and arrive at the stage where we get the genetic code $\langle\{g_3, \dots, g_r, m\}\rangle$. Continuing this way we can reduce the size of genes and after finite steps, get the genetic code $\langle\{m\}\rangle$.

Example 4.15. Let $G = \langle\{169\}\rangle$. The following is a saturated chain.

$$\langle\{9\}\rangle < \langle\{19\}\rangle < \cdots < \langle\{69\}\rangle < \langle\{129\},\{69\}\rangle < \cdots < \langle\{159\},\{69\}\rangle < \langle\{169\}\rangle.$$

For a generic length vector α, consider the collection

$$S(\alpha) = \{J \subseteq [m] : J \text{ is } \alpha\text{-short}\}.$$

Remark 4.16. Suppose $S_m(\alpha) = S_m(\beta) \cup \{J\}$ for some $J \subseteq [m]$. Let $J' < J$ with $m \in J'$. Note that $J' \in S_m(\beta)$. Since $S_m(\alpha) = S_m(\beta) \cup \{J\}$, $J' \in S_m(\beta)$. Consequently, $S_m(\beta)$ generates all α-short subsets except J. Therefore, $S(\alpha) = (S(\beta) \setminus \{J^c\}) \cup \{J\}$.

4.4 The submanifold arrangement

Corresponding to every 2-element short subset there is a codimension-1 submanifold embedded in any planar polygon space. In fact, the collection of all such submanifolds forms a submanifold arrangement. In this section, we study some combinatorial properties of this arrangement. Furthermore, we study the cell structure on M_α and $\overline{M}_\alpha$ induced by the submanifold arrangement.

Definition 4.17. Let X be a finite dimensional smooth, closed manifold. The submanifold arrangement is a finite collection $\mathcal{A} = \{N_1, \ldots, N_r\}$ of codimension-1 submanifolds such that,

1. each element of $\mathcal{A}$ is smoothly embedded as a closed subset;
2. for every point $x \in \bigcup_{i=1}^{r} N_i$ has a co-ordinate neighbourhood V_x such that the collection $\{N_1 \cap V_x, \ldots, N_r \cap V_x\}$ is a hyperplane arrangement in V_x with x as the origin;
3. the intersections of members of $\mathcal{A}$ induces a regular cell structure on X and each cell is combinatorially equivalent to simple convex polytope of an appropriate dimension.

There is an important combinatorial object associated with the submanifold arrangement.

Definition 4.18. The intersection poset $L(\mathcal{A})$ is the set of connected components of all possible intersections of N_i's ordered by reverse inclusion.

Now we describe the collection of submanifolds of planar polygon spaces which form a submanifold arrangement. Corresponding to every 2-element short subset $\{i, j\}$ we have a configuration with i-th and j-th sides in the same direction. Collection of such polygonal configurations forms a codimension-1 submanifold of M_α. In particular we write

$$N_{i,j} = \left\{(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_m) \in M_\alpha : v_i = v_j\right\}.$$

Let

$$\alpha(i,j) = (\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_i + \alpha_j, \ldots, \alpha_m)$$

be the $(m-1)$-tuple such that α_i and α_j are replaced by $\alpha_i + \alpha_j$ in $\alpha(i,j)$. Observe that $\alpha(i,j)$ is a generic length vector. It is easy to see that $N_{i,j} = M_{\alpha(i,j)}$. Similarly, we define

$$\overline{N}_{i,j} = \left\{(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_m) \in \overline{M}_\alpha : v_i = v_j\right\}.$$

We have $\overline{N}_{i,j} = \overline{M}_{(i,j)}$. For a generic length vector , we define the finite collections of submanifolds of M and $\overline{M}$ as follows

$$A = \{ N_{i,j} : \{i,j\} \text{ is an } \text{ - short} \},$$

$$\overline{A} = \{ \overline{N}_{i,j} : \{i,j\} \text{ is an } \text{ - short} \}.$$

Let

$$_m( ) = \{ \in {}_m : \text{blocks of are -short} \}$$

and

$$\overline{}_m( ) = \{ \overline{} : \in {}_m \text{ and } M_{( )} \text{ is disconnected} \}.$$

Let $L = {}_m( ) \sqcup \overline{}_m( )$ be the poset under the reverse refinement as a partial order.

Lemma 4.19. The intersection posets $I(A)$ and $I(\overline{A})$ are isomorphic to posets L and $_m( )$, respectively.

Proof. Consider the following intersection

$$X = N_{i_1 j_1} \cap N_{i_2 j_2} \cap \cdots \cap N_{i_r j_r}.$$

Then by clubbing together pairwise intersecting 2-element short subsets

$$\{i_l, j_l\} : 1 \leq l \leq r\}$$

we can write

$$X = N_{I_1} \cap N_{I_2} \cap \cdots \cap N_{I_s},$$

where $N_{I_t} = \bigcap_{\{i,j\} \subseteq I_t} N_{ij}$ For $1 \leq t \leq s$. Note that $I_1 - I_2 - \cdots - I_s$ is a partition of $\{i_1, j_1, \dots, i_r, j_r\}$. By putting together remaining singletons we get the partition of $[m]$. Let's denote this partition by . Recall that if X is disconnected then it is the disjoint union of tori. We label one of the connected components of and the other one by $\overline{}$. Otherwise, label X by . Conversely, we define an element of $I(A)$ corresponding to a partition $= J_1 - \cdots - J_k$ of $[m]$ with all J_i's are short. Consider the following intersection.

$$X = (\cap_{\{i_1,j_1\} \subseteq J_1} N_{i_1 j_1}) \cap \cdots \cap (\cap_{\{i_k,j_k\} \subseteq J_k} N_{i_k j_k}).$$

As done above if X is disconnected we label one of the connected components by and the other one by $\overline{}$.

Note that if the intersection corresponding to 2-element short subsets $\{i_l, j_l : 1 \le l \le r\}$

$$\overline{X} = \overline{N}_{i_1 j_1} \cap \overline{N}_{i_2 j_2} \cap \cdots \cap \overline{N}_{i_r j_r}$$

is nonempty then $\overline{X}$ is connected. Now the isomorphism between $I(\overline{\mathcal{A}}_\alpha)$ and $\overline{\mathcal{P}}_m(\alpha)$ is clear. □

Remark 4.20. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a generic length vector and $\pi = J_1 - \cdots - J_k$ be a partition of $[m]$ with all J_i's are α-short. Consider the shorter length vector $\alpha(\pi) = (\alpha_{J_1}, \ldots, \alpha_{J_k})$ where $\alpha_{J_l} = \sum_{i \in J_l} \alpha_i$ for $1 \le l \le k$. Let

$$X = (\cap_{\{i_1, j_1\} \subseteq J_1} N_{i_1 j_1}) \cap \cdots \cap (\cap_{\{i_k, j_k\} \subseteq J_k} N_{i_k j_k})$$

and

$$\overline{X} = (\cap_{\{i_1, j_1\} \subseteq J_1} \overline{N}_{i_1 j_1}) \cap \cdots \cap (\cap_{\{i_k, j_k\} \subseteq J_k} \overline{N}_{i_k j_k}).$$

Then it is easy to see that $X = M_{\alpha(\pi)}$ and $\overline{X} = \overline{M}_{\alpha(\pi)}$.

Corollary 4.21. Both the collections $\mathcal{A}_\alpha$ and $\overline{\mathcal{A}}_\alpha$ are locally isomorphic to either braid arrangement or a product of braid arrangements.

Proof. Let $X \in I(\mathcal{A}_\alpha)$ be a connected submanifold. Then without loss of generality, assume that $X = J_1 - J_2 - \cdots - J_k$, where J_i's are α-short. Consider the collection

$$I(\mathcal{A})_X = \{Y \in I(\mathcal{A}_\alpha) : X \subseteq Y\}.$$

Note that any element of $\mathcal{A}_X$ has the labelled by the refined partition of X. Therefore, the poset $I(\mathcal{A})_X$ is isomorphic to the poset of all refinements of partition $J_1 - J_2 - \cdots - J_k$. This concludes

$$I(\mathcal{A})_X \cong \prod_{i=1}^{k} I(\mathcal{B}_{|J_i|}).$$

Similar arguments prove the case of $\overline{\mathcal{A}}_\alpha$. □

The following result is an immediate consequence of the above corollary.

Corollary 4.22. The collections $\mathcal{A}_\alpha$ and $\overline{\mathcal{A}}_\alpha$ induces the regular cell structure on M_α and $\overline{M}_\alpha$, where all cells are combinatorially equivalent to simple polytopes.

The following proposition is now clear.

Proposition 4.23. The collections $\mathcal{A}$ and $\overline{\mathcal{A}}$ are the submanifold arrangements in M and $\overline{M}$, respectively.

We denote the cell structures induced from the submanifold arrangements $\mathcal{A}$ and $\overline{\mathcal{A}}$ on M and $\overline{M}$ by K and $\overline{K}$, respectively.

Recall the Definition 2.15 of zonotopal complexes.

Proposition 4.24. The dual of K and $\overline{K}$ is zonotopal. In particular, the dual cells are either permutohedrons or the product of permutohedrons.

Proof. This follows from the fact that $\mathcal{A}$ and $\overline{\mathcal{A}}$ are locally isomorphic to either braid arrangement or a product of braid arrangements. $\square$

Example 4.25. Let $\langle m \rangle$ be the genetic code of . Then we have

$$\overline{\mathcal{A}} = \left\{ \overline{N}_{ij} : \{i, j\} \subset [m-1] \right\}.$$

Note that in this case any subset of $[m-1]$ is -short. Therefore, corresponding to any partition of $[m-1]$, we have a nonempty intersection of $\overline{N}_{i,j}$'s. Therefore, it is easy to see that

$$I(\overline{\mathcal{A}}_{\langle m \rangle}) = {}_{m-1} \setminus \{\hat{1}\}$$

where ${}_{m-1}$ is the lattice of partitions of $[m-1]$. Therefore, $\overline{\mathcal{A}}$ is the projective braid arrangement PB_{m-1}.

Note that $M = S^{m-3}$ and the arrangement

$$\mathcal{A} = \left\{ N_{ij} : \{i, j\} \subset [m-1] \right\}$$

is the braid arrangement B_{m-1} intersected with S^{m-3}. Note that $I(\mathcal{A}) = {}_{m-1} \sqcup \overline{}_{m-1}$.

Figure 4.4: $\overline{K}_{\langle 5\rangle} = PCA_3$ and $I(\overline{A}) = {}_4 \setminus \{\hat{1}\}$

Figure 4.5: $K_{\langle 5\rangle} = CA_3$ with $I(A)$

Proposition 4.26. The cell complex $K_{\langle m\rangle}$ (respectively $\overline{K}_{\langle m\rangle}$) is isomorphic to the Coxeter complex (respectively projective Coxeter complex) of type A_{m-2}.

Proof. Recall that $M_{\langle m\rangle} = S^{m-3}$ and $\overline{M}_{\langle m\rangle} = \mathbb{RP}^{m-3}$. Moreover, the submanifold arrangement $A_{\langle m\rangle}$ is isomorphic to the braid arrangement B_{m-1}; see Example 4.25. Therefore, it is evident that $K_{\langle m\rangle} = CA_{m-2}$ and $\overline{K}_{\langle m\rangle} = PCA_{m-2}$. □

4.5 Hausmann's Theorem

Let α and β be two length vectors such that $S_m(\beta) = S_m(\alpha) \cup J$ for some $J \subseteq [m]$. Hausmann [32] used techniques from Morse theory to obtain the relation between corresponding planar polygon spaces M_α and M_β. He proved the following theorem.

Theorem 4.27 ([32]). The space M_β is obtained from M_α by an $O(1)$-equivariant surgery of index $|J|-2$. i.e.,

$$M_\beta = \left(M_\alpha \setminus S^{|J|-2} \times D^{m-1-|J|}\right) \bigcup_{S^{|J|-2} \times S^{m-2-|J|}} \left(D^{|J|-1} \times S^{m-2-|J|}\right),$$

where $O(1)$ acts antipodally on $D^{m-1-|J|}$ and $D^{|J|-1}$.

Note that using Theorem 4.14, we can say that if the genetic code G covers G' then M_G is obtained from $M_{G'}$ by an $O(1)$-equivariant surgery. In fact, one can iterate this process to any saturated chain of genetic codes. Note that $M_{\langle m\rangle} = S^{m-3}$ and $\overline{M}_{\langle m\rangle} = \mathbb{R}P^{m-3}$. The iterated version of Theorem 4.27 is given by the following proposition.

Proposition 4.28. Let $\langle m\rangle = G_1 \subset G_2 \subset \cdots \subset G_r = G$ be the saturated chain of genetic codes. Then the space M_G is obtained from S^{m-3} by an iterated $O(1)$-equivariant surgery.

Proof. Note that $S_m(G_{i+1}) = S_m(G_i) \cup J_i$ for $1 \leqslant i \leqslant r-1$. Therefore, $M_{G_{i+1}}$ is obtained from M_{G_i} by an $O(1)$-equivariant surgery along $S^{|J_i|-2}$. Observe that $S_m(G_r) = \{m\} \cup \bigcup_{i=1}^{r-1} J_i$. Now the propositions follows from iteratively applying Theorem 4.27. □

Remark 4.29. Observe that the above proposition doesn't say anything about the cell structure on M_G.

Now we define the projective version of surgery that is applicable in the context of taking quotient under a free $\mathbb{Z}_2$-action. Let M be a smooth manifold of dimension n with a free $\mathbb{Z}_2$-action. Suppose the k-dimensional sphere S^k and its trivial tubular neighbourhood $S^k \times D^{n-k}$, embed $\mathbb{Z}_2$-equivariantly in M. Let $\overline{M}$ denotes the quotient of M by a free $\mathbb{Z}_2$-action. Note that $\mathbb{R}P^k$ and the quotient $\frac{S^k \times D^{n-k}}{(x,y)\sim(-x,-y)}$ embed in $\overline{M}$. With this information, we introduce the following notations.

$$\overline{DP}(k) := \frac{S^k \times D^{n-k}}{(x,y) \sim (-x,-y)},$$

$$\underline{DP}(k) := \frac{D^k \times S^{n-k}}{(x,y) \sim (-x,-y)},$$

$$\partial(\overline{DP}(k)) = \frac{S^k \times S^{n-k-1}}{(x,y) \sim (-x,-y)} = \partial(\underline{DP}(k+1)).$$

Remark 4.30. The space $\partial(\overline{DP}(k))$ is the total space of the sphere bundle of the $(n-k)$-direct sum of canonical line bundles over RP^k and $\overline{DP}(k)$ is the total space of disc bundle of the $(n-k)$-direct sum of canonical line bundles over RP^k.

With the above notations, we now define projective cellular surgery.

Definition 4.31. An index k-projective surgery on a manifold $\overline{M}$ along RP^k, produces a manifold $PS_k(\overline{M})$ defned as follows

$$PS_k(\overline{M}) := \left(\overline{M} \setminus \overline{DP}(k)\right) \bigcup_{\partial(\overline{DP}(k))} \left(\underline{DP}(k+1)\right).$$

We denote the usual index-k surgery on M by $S_k(M)$.

Proposition 4.32. With the above definition we have the following results.

1. The index-0 surgery on a manifold M along S^0, produces a manifold homeomorphic to $M \# (S^1 \times S^{n-1})$.
2. The index-0 projective surgery on a manifold $\overline{M}$ along RP^0, produces a manifold homeomorphic to $\overline{M} \# RP^n$.

Proof of (1). Firstly, we assume that $M = S^n$. Let D^+ and D^- be two antipodal discs containing the north pole and the south pole, respectively. Then the surgery on S^n along S^0 tells us that, remove D^+ and D^- from S^n and attach $D^1 \times S^{n-1}$ to $S^n \setminus (D^+ \sqcup D^-)$. This clearly gives $S_0(S^n) = S^1 \times S^{n-1}$. Observe that $S^n \# (S^1 \times S^{n-1}) = S^1 \times S^{n-1}$. Without loss of generality, we can assume that there is a bigger disc D such that $D^+ \sqcup D^- \subset D$ Now observe that the index-0 surgery on S^n is an equivalent operation to removing D from S^n and attaching $S^1 \times S^{n-1} \setminus D'$ to $S^n \setminus D$, for some disc D' in $S^1 \times S^{n-1}$. This is same as the connected sum of S^n and $S^1 \times S^{n-1}$. The same idea works for general M.

Proof of (2). We make the following observations:

1.
$$\overline{DP}(0) = \frac{S^0 \times D^n}{(x,y) \sim (-x,-y)} = D^n,$$

2.
$$\underline{DP}(1) = \frac{D^1 \times S^{n-1}}{(x,y) \sim (-x,-y)} = \frac{S^n \setminus (D^n_+ \sqcup D^n_-)}{x \sim -x} = RP^n \setminus D^n,$$

3.
$$\partial(\overline{DP}(0)) = \frac{S^0 \times S^{n-1}}{(x,y) \sim (-x,-y)} = S^{n-1} = \partial(\underline{DP}(1)).$$

Therefore,
$$PS_0(\overline{M}) := \left(\overline{M} \setminus D^n\right) \bigcup_{S^{n-1}} \left(RP^n \setminus D^n\right) = \overline{M} \sharp RP^n.$$

This proves the result. □

Theorem 4.33. If the genetic code G covers G' i.e., $S_m(G) = S_m(G') \cup J$ for some $J \subseteq [m]$ then M_G is homeomorphic to $PS_{|J|-2}(M_{G'})$.

One can iterate the projective surgery to any chain $G_1 \subset G_2 \subset \cdots \subset G_r = G$ such that for each $1 \leq i \leq r-1$, G_i is covered by G_{i+1}. We denote the space after iterated projective surgery as $PS_{(j_1,\ldots,j_r)}(M_{G_1})$ where $j_i = |J_i| - 2$ such that $S_m(G_{i+1}) = S_m(G_i) \cup J_i$. In fact, we have $S_m(G_r) = S_m(G_1) \cup \bigcup_{i=1}^{r-1} J_i$. With this, we have the following version of Theorem 4.27.

Proposition 4.34. The planar polygon space $\overline{M}_G$ is homeomorphic to $PS_{(j_1,\ldots,j_r)}(RP^{m-3})$.

4.6 Combinatorial surgery on a meet semi-lattice

The notion of combinatorial blow-up was introduced by Feichtner and Kozlov in [26]. Here, we introduce the similar notion in the contexts of surgery.

Definition 4.35. Let L be a meet semilattice. For an element $x \in L$, we define a poset $CS_x(L)$, the combinatorial surgery on L along x, as follows:

- elements of $CS_x(L)$:
  1. $y \in L$, $y \neq x$ and $y \not< x$
  2. $[x,y]$, $y < x$

- order relations in $CS_x(L)$:

  1. $y > z$ in $CS_x(L)$ if $y > z$ in L
  2. $[x, y] > [x, z]$ in $CS_x(L)$ if $y > z$ in L
  3. $[x, y] > z$ in $CS_x(L)$ if $y > z$ in L.
  4. $y < [x, \hat{0}]$ if $y \vee x \in L$.

Remark 4.36. The element $[x, \hat{0}]$ can be thought of as a result of combinatorial surgery along x.

Theorem 4.37. The poset $CS_x(L)$ is a meet semilattice. Moreover, for $x \in L$, the posets L and $CS_x(L)$ are of equal rank and if k be the rank of L, then

$$rk([x, y]) = k - rk(x) + rk(y) + 1.$$

Example 4.38. Let $G = \langle\{2, 6\}\rangle$ be the genetic code and $I(A_G)$ be the corresponding meet semilattice. Let $(1, 2, 345, 6) \in I(A_G)$. We denote this partition by 345. Then

$$\begin{aligned} CS_{345}(I(A_G)) &= \left(I(A_G) \setminus I(A_G)_{\geq 345}\right) \bigsqcup \{[345, y] : y < 345\} \\ &= \left(I(A_G) \setminus I(A_G)_{\geq 345}\right) \bigsqcup \{(126, \ ) : \ < (1, , 2, 345, 6)\} \\ &= I(A_{\langle\{1,2,6\}\rangle}), \end{aligned}$$

where $(126, \ )$ denotes an unordered partition of $[6]$. Observe that the genetic code $\langle\{1, 2, 6\}\rangle$ covers $\langle\{2, 6\}\rangle$ with respect to the genetic order.

Figure 4.6: Combinatorial surgery along 345

Let G and G' be two genetic codes of m-length vectors such that G' covers G. It follows from Proposition 4.14 that there exists a subset $J \subseteq [m]$ with $S_m(G') = S_m(G) \cup J$. With this, now the following result is straightforward.

Proposition 4.39. $CS_{\mathcal{J}^c}(I(A_G)) = I(A_{G'})$.

4.7 Cellular surgery on a simple cell complex

Let K be the simple cell complex of dimension n such that S^k embeds in K as a subcomplex for some k. Let's denote this subcomplex by KS^k. Moreover, assume that for any k-simplices $\sigma, \sigma' \in KS^k$, $Lk(\sigma, K) = Lk(\sigma', K) = S^{n-k-1}$. With this assumption we have the following definition.

Definition 4.40. The index k cellular surgery on K along KS^k is defined in two steps:

Step 1: Truncate all cells whose closure intersects KS^k.

Step 2: Let $D(KS^k)$ be the cellular disc with the boundary KS^k. Note that the boundary complex of the truncated part around KS^k is $KS^k \times Lk(\sigma, K)$ for $\sigma \in KS^k$. Now attach another simple cell complex $D(KS^k) \times \partial Lk(\sigma, K)$ to K along $KS^k \times \partial Lk(\sigma, K)$.

In particular, if $\tilde{K}$ denotes the cell complex obtained by the cellular surgery on K then

$$\tilde{K} := \left(K \setminus KS^k \times D(Lk(\sigma, K))\right) \bigcup_{KS^k \times Lk(\sigma,K)} \left(D(KS^k) \times Lk(\sigma, K)\right).$$

Recall that a simple cell complex is one in which all of the cells are combinatorially equivalent to simple polytopes. Let K be a simple cell complex with free Z_2-action such that S^k embeds in K as a Z_2-equivariant subcomplex. Assume that, for any k-simplices $\sigma, \sigma' \in KS^k$ we have $Lk(\sigma, K) = Lk(\sigma', K) = S^{n-k-1}$ such that the quotient of $Lk(\sigma, K)$ by Z_2-action is again a cell complex. We are now ready to define the projective version of a cellular surgery on the quotient of K by the Z_2-action based on these assumptions.

Definition 4.41. Let PKS^k and $\overline{K}$ be the quotients of KS^k and K by Z_2-action, respectively. The index k projective cellular surgery on $\overline{K}$ along PKS^k is a cell complex $\tilde{\overline{K}}$ defined as

$$\tilde{\overline{K}} := \left(\overline{K} \setminus KS^k \times_{Z_2} D(Lk(\sigma, K))\right) \bigcup_{KS^k \times_{Z_2} Lk(\sigma,K)} \left(D(KS^k) \times_{Z_2} Lk(\sigma, K)\right),$$

where $KS^k \times_{Z_2} D(Lk$ denotes the quotient of $KS^k \times D(Lk$ by diagonal Z_2-action. Similarly, $KS^k \times_{Z_2} Lk(\sigma, K)$ and $D(KS^k) \times_{Z_2} Lk(\sigma, K)$ are defined.

Let CA_{m-1} be the Coxeter complex correspond to the braid arrangement B_m. We introduce the notion of cellular surgery on CA_{m-1} along an element of the minimal building set $\mathrm{Min}(B_m)$ of the braid arrangement B_m. Let $X \in \mathrm{Min}(B_m)$. Recall that X can be represented by the partition of $[m]$ with at most one block of size greater equal 2. Let $X = J - i_1 - i_2 - \cdots - i_k$. Let

$$B_X = \{H \in B_m : X \subseteq H\}$$

be the subcollection of B_m. It is easy to see that,

$$B_X = \{H_{ij} \in B_m : \{i,j\} \subseteq J\} = B_{|J|}.$$

Let $\sigma \in S_X$ be a cell such that $\dim(\sigma) = \dim(S_X)$. From the above discussion, it is clear that $Lk(\sigma, CA_{m-1}) = CA_{|J|-1}$.

Definition 4.42. Let $X \in \mathrm{Min}(B_m)$. Cellular surgery on CA_{m-1} along S_X is defined in two steps.

1. Truncate all cells which are adjacent to S_X.
2. Note that the boundary complex of the truncated part around S_X is $S_X \times CA_{|J|-1}$. Let $D(S_X)$ be the cellular disc whose boundary is S_X. Now attach the another complex $D(S_X) \times CA_{|J|-1}$ along the boundary $S_X \times CA_{|J|-1}$.

Similarly, we can define a cellular surgery on the projective Coxeter complex by replacing S_X and CA_{m-1} by PS_X and PCA_{m-1} respectively in the Theorem 4.42. Note that after truncating cells adjacent PS_X, the boundary of the truncated part will be $S_X \times_{O(1)} CA_{|J|-1}$. Accordingly, attach the $D(S_X) \times_{O(1)} CA_{|J|-1}$ to the truncated complex.

Remark 4.43. 1. It is easy to see that truncation of all cells adjacent to S_X in CA_{m-1} is an equivalent operation to removing $S^{m-|J|-1} \times D^{|J|-1}$ (tubular neighbourhood of S_X), since $S_X = S^{m-|J|-1}$ and $D^{|J|-1}$ is the $(|J|-1)$-dimensional disc. In step 2 of the above definition, we attach $D^{m-|J|} \times S^{|J|-2}$ since, $CA_{|J|-1} = S^{|J|-2}$. Therefore, the Theorem 4.42 is a cellular analogue of the original definition of surgery on manifold.

2. If $\dim(X) = 0$ then the cellular surgery on CA_{m-1} along S_X gives the cell complex homeomorphic to $S^1 \times S^{m-3}$. On the other hand, the cellular surgery

on PCA_{m-1} along PS_X gives a cell complex which is homeomorphic to $\mathbb{R}P^{m-2} \# \mathbb{R}P^{m-2}$.

Example 4.44. Let $X = 123$ be an element of $\mathrm{Min}(B_4)$. Note that $S_{123} = S^0 = \{123, \overline{123}\}$ a 0-dimensional sphere. Observe that $B_X = B_3$. Therefore, if we truncate all the cells of CA_3 adjacent to 123 then the boundary of the truncated part is CA_2 (see the red hexagonal circle in Figure 4.7). Similarly, truncating cells adjacent to $\overline{123}$ we get the another hexagonal circle. Therefore, truncating cells adjacent to S_{123} gives the disjoint union of two hexagonal circles as the boundary of the truncated part. Note that this boundary is isomorphic to $S_{123} \times CA_2$. Now in the next step of cellular surgery along 123, we have to attach $D(S_{123}) \times CA_2$, a hexagonal cylinder to the complex $S_{123} \times CA_2$, the boundary complex of the truncated part in the previous step . The resulting complex is the torus cellulated by 18 squares and 12 triangles.

Figure 4.7: Cellular surgery on CA_3 along S_{123}.

Example 4.45. Let $X \in \mathrm{Min}(B_4)$ such that it is represented by an unordered partition $123-4$. Without loss of generality, we can omit the singletons and write $X = 123$. Consider the 0-dimensional projective coxeter complex PS_{123} in PX. Similarly, as in the previous example we have $PB_{PX} = B_3$. Now truncating cells of PCA_3 adjacent to PS_{123} gives boundary of the truncated part to be $S_{123} \times_{O(1)} CA_2$, a hexagonal circle. Note that the boundary $\partial(D(S_{123}) \times_{O(1)} CA_2) = S_{123} \times_{O(1)} CA_2$. Now in the next step, we attach $D(S_{123}) \times_{O(1)} CA_2$ to $S_{123} \times_{O(1)} CA_2$. Note that $D(S_{123}) \times_{O(1)} CA_2$ is the cellular Mobius band. Note that the resulting complex is cellulated by 6 triangles and 9 squares.

Figure 4.8: Cellular surgery on PCA_3 along PS_{123}.

Let $\langle m \rangle = G_1 \lessdot G_2 \lessdot \cdots \lessdot G_r = G$ be the saturated chain of genetic codes such that G_{i+1} covers G_i for $1 \leq i \leq r-1$. Note that $S_m(G) = \{m\} \cup \bigcup_{i=1}^{r-1} J_i$. Note that $m \notin J_i^c$. Therefore, J_i^c's are short subsets with respect to the genetic code $\langle m \rangle$. Note that each J_i^c represents the partition $J_i^c - j_1 - j_2 - \cdots - j_k$ of $[m]$. Now it follows from the Theorem 4.7 that $\{J_1^c, \ldots, J_{r-1}^c\} \subseteq \mathrm{Min}(B_{m-1})$. Consider the collections

$$\mathcal{G}_G = \left\{ S_{J_1^c}, S_{J_2^c}, \ldots, S_{J_{r-1}^c} \right\}$$

and

$$P\mathcal{G}_G = \left\{ PS_{J_1^c}, PS_{J_2^c}, \ldots, PS_{J_{r-1}^c} \right\}.$$

Theorem 4.46. Let G be the genetic code of a length vector. Then the iterated cellular surgery on CA_{m-2} (respectively on PCA_{m-2}) along the elements of $\mathcal{G}_G$ (respectively $P\mathcal{G}_G$) produces the cell complex $\tilde{K}$ (respectively $\tilde{\tilde{K}}$) homotopy equivalent to K (respectively $\overline{K}$).

Proof. Following the inductive argument, it is enough to prove the theorem for a saturated chain of length 1. Let $G \lessdot G'$ be a saturated chain of length 1. It follows from the Theorem 4.14 that, $S_m(G') = S_m(G) \cup J$ for some $J \subseteq [m]$. Since J^c is the maximal short subset (i.e., adding an extra element in J^c makes it into long), the subcomplex S_{J^c} of K_G is isomorphic to the Coxeter complex $CA_{|J|-1}$ of dimension $|J|-2$. Note that J is short subset with respect to the genetic code G'. We also have

$G^0 = \langle G, J \rangle$. Since J is maximal short subset the subcomplex S_J of K_{G^0} represents the Coxeter complex $CA_{m-2-|J|}$. Now we see that the $Lk(\ , K_G)$ is isomorphic to the Coxeter complex for $\ 2\ S_{J^c}$ with $\dim(\ ) = |J| - 1$. Recall that M_G is a PL-manifold. Therefore, $Lk(\ , K_G) = S^{n-|J|-2}$ if $\dim(\ ) = |J| - 1$. The cell structure on $S^{n-|J|-2}$ is induced by the collection

$$\{N_{i,j} : \{i, j\} \quad J^c\}$$

Note that the above collection is isomorphic to the braid arrangement $B_{m-|J|}$. Therefore, $Lk(\ , K_G) = CA_{m-2-|J|}$. Let $\tilde{K}_G$ be the complex obtained by the index $|J| - 1$ cellular surgery on K_G along S_{J^c}. Then

$$\tilde{K}_G = \quad K_G \setminus S_{J^c} \quad D(CA_{m-2-|J|}) \bigcup_{S_{J^c} \quad CA_{m-2-|J|}} D(S_{J^c}) \quad CA_{m-2-|J|} \quad .$$

Now if we collapse $D(S_{J^c}) \quad CA_{m-2-|J|}$ onto $CA_{m-2-|J|}$, $\tilde{K}_G$ becomes homotopy equivalent the complex $(K_G \setminus S_{J^c}) \cup S_J$. It follows from Section 4.5 that $\tilde{K}_G = M_{G^0}$. Note that collapsing $D(S_{J^c}) \quad CA_{m-2-|J|}$ onto $CA_{m-2-|J|}$ doesn't change the homeomorphism type of $\tilde{K}_G$. Therefore, $(K_G \setminus S_{J^c}) \cup S_J = M_{G^0}$. Now it follows from Theorem 4.16 that the cell complex $(K_G \setminus S_{J^c}) \cup S_J$ is induced from the submanifold arrangement A . Therefore, $(K_G \setminus S_{J^c}) \cup S_J = K_{G^0}$.

Let PS_{J^c} be the projective Coxeter complex $PCA_{|J|-1}$ in $\overline{K}_G$ represented by a partition J^c of $[m]$ and let PS_J be the subcomplex of $\tilde{\overline{K}}_{G^0}$ isomorphic to the projective Coxeter complex $PCA_{m-2-|J|}$. The index $|J| - 1$ projective cellular surgery on $\overline{K}_G$ along PS_{J^c} gives

$$\tilde{\overline{K}}_G = \quad \overline{K}_G \setminus S_{J^c} \quad {}_{O(1)} D(CA_{m-2-|J|}) \bigcup_{S_{J^c} \quad {}_{O(1)} CA_{m-2-|J|}} D(S_{J^c}) \quad {}_{O(1)} CA_{m-2-|J|} \quad .$$

Note that $S_{J^c} \quad {}_{O(1)} D(CA_{m-2-|J|})$ and $D(S_{J^c}) \quad {}_{O(1)} CA_{m-2-|J|}$ are the total spaces of disc bundles over PS_{J^c} and PS_J, respectively. Therefore, $J^c \quad {}_{O(1)} D(CA_{m-2-|J|})$ and $D(J^c) \quad {}_{O(1)} CA_{m-2-|J|}$ are homotopy equivalent to PS_{J^c} and $PCA_{m-2-|J|}$, respectively. Therefore, $\tilde{\overline{K}}_G$ is homotopy equivalent to the complex $(\overline{K}_G \setminus PS_{J^c}) \cup PS_J$. Now the theorem follows from similar arguments as did for the cellular surgery.

□

Example 4.47. Consider the chain of genetic codes $\langle 5 \rangle \quad \langle 15 \rangle \quad \langle 25 \rangle \quad \langle 125 \rangle$. Recall that $M_{\langle 125 \rangle} = T^2 \sqcup T^2$. Note that $G_{\langle 125 \rangle} = \{S_{234}, S_{134}, S_{34}\}$ Now we explain how to obtain the cell complex $K_{\langle 125 \rangle}$ (resp. $\overline{K}_{\langle 125 \rangle}$) by performing the cellular surgery on

CA_3 (resp. PCA_3) along $G_{\langle 125\rangle}$ (resp. $PG_{\langle 125\rangle}$). We begin by doing surgery on CA_3 along S_{234}. Then we get the complex $\tilde{K}_{15}$ isomorphic to the torus. Note that, if we collapse the hexagonal cylinder onto one of its boundary components we get the complex again isomorphic to the torus. It is easy to see that this complex is isomorphic to the complex $K_{\langle 15\rangle}$. Later we follow the same process for S_{134} and get the complex $K_{\langle 25\rangle}$. Now we need to do the surgery along S_{34}. Note that S_{34} represents the hexagonal circle in $K_{\langle 25\rangle}$. In this case, the first step is to truncate all the cells adjacent to S_{34}. After truncating adjacent cells we get the two disjoint complexes, each of them is isomorphic to the complex obtained from the torus removing the hexagonal disc. In the second step, we attach the two disjoint unions of the hexagonal disc to the hexagonal boundary of each complex obtained in the previous step. Then we get the complex isomorphic to the disjoint union of two torus. Note that, if we collapse the attached hexagonal discs to corresponding points, then again the resulting complex is isomorphic to the disjoint union of the torus which is exactly the complex K_{125}. (see Figure 4.9)

At every step of the iterated cellular surgery on CA_3, we can take the quotients by antipodal action and get the cellular surgery on PCA_3. In particular, at the last step, we get the complex isomorphic to $\overline{K}_{125}$, the torus.

The following arrows summarize the above process.

$$CA_3 \xrightarrow{234} \tilde{K}_{15} \xrightarrow{h.e.} K_{\langle 15\rangle} \xrightarrow{134} \tilde{K}_{\langle 25\rangle} \xrightarrow{h.e.} K_{\langle 25\rangle} \xrightarrow{34} \tilde{K}_{\langle 125\rangle} \xrightarrow{h.e.} K_{\langle 125\rangle}.$$

$$PCA_3 \xrightarrow{234} \overline{\tilde{K}}_{15} \xrightarrow{h.e.} \overline{K}_{\langle 15\rangle} \xrightarrow{134} \overline{\tilde{K}}_{\langle 25\rangle} \xrightarrow{h.e.} \overline{K}_{\langle 25\rangle} \xrightarrow{34} \overline{\tilde{K}}_{\langle 125\rangle} \xrightarrow{h.e.} \overline{K}_{\langle 125\rangle}.$$

Figure 4 .9: Iterated cellular surgery on CA_3 along G_{h125i}

Chapter 5

Numerical aspects of planar polygon spaces

In this chapter, we compute some numerical invariants associated with the planar polygon spaces. In the next section, we study the n-dimensional Klein bottle introduced by Davis in [12]. In Section 5.2, we obtain the small cover structure on polygon spaces associated with the long genetic codes. We also compute their Betti numbers. In Section 5.3, we study the Borsuk-Ulam theorem for oriented polygon spaces. The results presented in the Section 5.3 of this chapter are from the author's two preprints [9] and [11]. These are joint work with Deshpande, Goyal, and Singh.

5.1 The n-dimensional Klein bottle is a real Bott manifold

The generalized version of an n-dimensional Klein bottle K_n is introduced by Davis in [12] as follows:

$$K_n = \frac{(S^1)^n}{(z_1, \ldots, z_{n-1}, z_n) \sim (\bar{z}_1, \ldots, \bar{z}_{n-1}, -z_n)}. \tag{5.1}$$

It is easy to see that K_2 is the usual Klein bottle.

Davis computed the fundamental group, integral cohomology algebra, and the stable homotopy type of K_n. Moreover, he obtained an explicit immersion of K_n in $\mathbb{R}^{n+1}$ and an embedding in $\mathbb{R}^{n+2}$. The following result follows from [32, Proposition 2.1] and justifies a connection between polygon spaces and n-Klein bottle.

Theorem 5.1. Let be a length vector with genetic code $<1, 2, \ldots, n-1, n+3>$. Then

$$\overline{M} = K_n.$$

Recall that the real Bott manifolds of dimension n are the special examples of small covers where the quotient polytope is an n-dimensional cube.

In this section, we show that the n-dimensional Klein bottle is a real Bott manifold. Moreover, we determine the corresponding Bott matrix. Consequently, we get the characteristic function corresponding to this Bott matrix and the small cover over n-dimensional cube. Then we compute the rational Betti numbers of K_n using the Suciu-Trevesan formula. There are n-dimensional closed manifolds defined in the same spirit of n-dimensional Klein bottle by replacing the last two copies of S^1 in K_n by an orientable surface of genus 3 and 4. The authors [10, Theorem 4.14 and Theorem 5.2] show that these manifolds are small covers over $P_5 \quad I^{n-2}$ and $P_6 \quad I^{n-2}$ and also compute their characteristic functions. Here we compute their rational Betti numbers.

5.1.1 Small cover structure

The n-dimensional cube is given by

$$I^n = [-1, 1]^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : -1 \leq x_i \leq 1 \text{ for } 1 \leq i \leq n\}.$$

Consider the following labeling of the facets of I^n. For each $1 \leq i \leq n$, we set

$$F_i = I \quad \{-1\} \quad I$$

and

$$F_{n+i} = I \quad \{1\} \quad I,$$

where $\{-1\}$ and $\{1\}$ is at the ith place. Let $F(I^n)$ be the collection of facets of I^n. Define a function

$$: F(I^n) \to \mathbb{Z}_2^n$$

as

$$\lambda(F) = \begin{cases} e_i & \text{if } F = F_i \text{ or } F = F_{n+i},\ 2 \leq i \leq n\ , \\ e_1 & \text{if } F = F_1, \\ \sum_{i=1}^{n} e_i & \text{if } F = F_{n+1}. \end{cases}$$

It is clear that $n \times 2n$-matrix of λ is

$$\lambda = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 1 \end{bmatrix}. \tag{5.2}$$

Now we prove that λ is indeed a characteristic function on the facets of I^n.

Lemma 5.2. *The function λ is a characteristic for I^n.*

Proof. Let v be the vertex of I^n. Consider the following subcollection of facets of $F(I^n)$

$$F(v) = \{F \in F(I^n) : v \in F\}.$$

Then it is easy to see that

$$\lambda(F(v)) = \begin{cases} \{e_2, \ldots, e_n, \sum_{i=1}^{n} e_i\} & \text{if } v \in F_{n+1}, \\ \{e_1, \ldots, e_n\} & \text{otherwise.} \end{cases}$$

Clearly, in both the cases $\lambda(F(v))$ forms a basis for $\mathbb{Z}_2^n$. Therefore, λ is the characterstic function on $F(I^n)$. □

We follow [4] for the basics of real Bott manifold. Corresponding to a strictly upper triangular binary matrix, a real Bott manifold can be described as the quotient of the n-dimensional torus by a free action of $\mathbb{Z}_2^n$.

Definition 5.3. A binary square matrix A is said to be a Bott matrix if there exist a permutation matrix P and a strictly upper triangular binary matrix B such that $A = PBP^{-1}$

Let $z \in S^1$ and $a \in \{0, 1\}$ Define the notation

$$z(a) := \begin{cases} z & \text{if } a = 0, \\ \bar{z} & \text{if } a = 1. \end{cases}$$

Let A^i_j be the (i, j) entry of a Bott matrix A. For $1 \leq i \leq n$ define the involution a_i on $(S^1)^n$ as follows:

$$a_i((z_1, \ldots, z_n)) = (z_1(A^i_1), \ldots, z_{i-1}(A^i_{i-1}), -z_i, z_{i+1}(A^i_{i+1}), \ldots, z_n(A^i_n)).$$

Note that these involutions commute with each other and generate a multiplicative group $\mathbb{Z}_2^n$ denoted as $G(A)$. Moreover, it was shown in [4, Lemma 2.1] the action of this group on $(S^1)^n$ is free. The real Bott manifold associated with the Bott matrix is defined as the quotient

$$\frac{(S^1)^n}{G(A)}.$$

Recall that the n-dimensional real Bott manifolds are small covers over an n-dimensional cube, for which the characteristic function is determined by the Bott matrix. Let $B = [b_{i,j}]$ be the Bott matrix and $F_1, \ldots, F_n, F_{n+1}, \ldots, F_{2n}$ are the facets of I^n. Then the corresponding characteristic function is given as follows:

$$\lambda(F) = \begin{cases} e_i & \text{if } F = F_i \text{ for } 1 \leq i \leq n, \\ e_i + \sum_{k=i+1}^{n} b_{i,k} e_k & \text{if } F = F_{n+i} \text{ for } 1 \leq i \leq n-1, \\ e_n & \text{if } F = F_{2n}. \end{cases} \tag{5.3}$$

It is easy to see that the matrix of this characteristic function is given by

$$\left[I_n \,\middle|\, I_n + B^T \right]$$

where I_n is the block of $n \times n$ identity matrix.

Now we prove that K_n is indeed a real Bott manifold.

Theorem 5.4. The n-dimensional Klein bottle K_n is a real Bott manifold corresponding to the Bott matrix

$$B = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{5.4}$$

In particular, K_n is homeomorphic to the small cover $X(I^n, \lambda)$, where λ is defined by the Equation 5.2.

Proof. By the qoutient construction of real Bott manifold we have

$$M(B) = \frac{(S^1)^n}{G(B)},$$

where $G(B) = < a_1, \ldots, a_n >$ and

$$a_1((z_1, z_2, \ldots, z_n)) = (-z_1, \bar{z}_2, \ldots, \bar{z}_n)$$

$$a_i((z_1, \ldots, z_n)) = (z_1, \ldots, -z_i, \ldots, z_n),$$

for $2 \leqslant i \leqslant n$. It is clear that

$$M(B) = \frac{S^1 \times S^{n-1}}{< a_1 > \times < a_2, \ldots, a_n >}.$$

This gives

$$M(B) = S^1 \times_{Z_2} (\mathbb{R}P^1)^{n-1},$$

where the action of Z_2 is given by $a_1((z_1, [z_2, \ldots, z_n])) = (-z_1, [\bar{z}_2, \ldots, \bar{z}_n])$. Now it is clear that $M(B)$ is homeomorphic to K_n given by the Equation 5.1.

Recall that real Bott manifolds are small covers where the characteristic function is generated by Bott matrix (see Equation 5.3). Note that the characteristic matrix given by Equation 5.2 coincides with the characteristic matrix generated by the Bott matrix B. Thus, K_n is the small cover $X(I^n, \lambda)$. □

Now we describe the Suciu-Trevesan formula which gives the formula of the rational Betti numbers of a small cover. Let P be an n-dimensional, simple polytope with m facets and let K be the simplicial complex dual of ∂P. Let λ be an $n \times m$

characterstic matrix of P with entries from $\mathbb{Z}_2$. For a subset $T \subseteq [n]$, define

$$\lambda_T := \sum_{i \in T} \lambda_i,$$

where λ_i is the ith row of Λ. Let $K_{\lambda,T}$ be the induced subcomplex of K on the vertex set

$$\mathrm{supp}(\lambda_T) := \{i \in [m] : i\text{th entry of } \lambda_T \text{ is nonzero}\}.$$

Theorem 5.5 ([62]). Let $\beta_i(X(P,\lambda),\mathbb{Q})$ be the ith rational Betti number of a small cover $X(P,\lambda)$. Then

$$\beta_i = \sum_{T \subseteq [n]} \tilde{\beta}_{i-1}(K_{\lambda,T},\mathbb{Q}),$$

where $\tilde{\beta}_{i-1}(K_{\lambda,T},\mathbb{Q})$ is the $(i-1)$th reduced rational Betti number of $K_{\lambda,T}$.

5.1.2 Betti numbers

Lemma 5.6. Let λ be the characteristic function of and $T \subseteq [n]$. Then

$$|\mathrm{supp}(\lambda_T)| = \begin{cases} 2|T| & \text{if } |T| \text{ is an even integer and } 1 \notin T \\ 2|T| - 1 & \text{if } |T| \text{ is an even integer and } 1 \in T \\ 2|T| & \text{if } |T| \text{ is an odd integer and } 1 \in T \\ 2|T| + 1 & \text{if } |T| \text{ is an odd integer and } 1 \notin T. \end{cases}$$

Proof. Let λ_i be the ith row of the characteristic matrix of λ. Note that for $2 \leq i \leq n$, λ_i contains contains exactly three 1's and λ_1 contains exactly two 1's. Moreover, the ith and $(n+i)$th colomn are same for $2 \leq i \leq n$. It is easy to see that for a subset $T \subseteq [n]$, $i \in T \setminus \{1\}$ the 1 occurs as the ith and $(n+i)$th coordinate of vector λ_T.

Suppose $|T|$ is an odd integer and $1 \notin T$. Then it is clear that 1 occurs in λ_T at the $(n+1)$th position. Note that $T \subseteq [n] \setminus \{1\}$. Therefore, for $i \in T$, 1 occurs at ith, $n+i$th and $(n+1)$th position of λ_T. In particular 1 occurs $2|T|+1$ many times in λ_T.

Now assume that $|T|$ is an odd integer and $1 \in T$. Note that for $i \in T \setminus \{1\}$ 1 already occurred at the ith and $(n+i)$th position of λ_T. So λ_T contains $2(|T|-1)$ such

1's. Since $1 \in T$ and $|T|$ is an odd integer, two more 1's gets added in ${}_T$ at its 1st and $(n+1)$th position. In particular, 1 occurs $2(|T|-1)+2 = 2|T|$ many times in ${}_T$.

Suppose $|T|$ is an even integer with $1 \notin T$. Then observe that for each $i \in T$, 1 will occur at ith and $(n+i)$th position of ${}_T$ but won't occur at the $(n+1)$th position. It is now clear that in this case 1 occurs in ${}_T$ exactly $2|T|$ times.

We now assume that $|T|$ is an even integer and $1 \in T$. Then again as observed above we have, for each $i \in T \setminus \{1\}$ 1 occurs at the ith and $(n+i)$th position but wont occurs at the $(n+1)$th position of ${}_T$. So there are $2(|T|-1)$ such 1's in ${}_T$. Since $1 \in T$, one more extra 1 gets added in ${}_T$. Therefore, there are $2(|T|-1)+1 = 2|T|-1$ many 1 occurs in ${}_T$. Finally this proves the lemma. $\square$

Now we determine the homotopy types of the subcomplexes $K_{,T}$ for any subset $T \subseteq [n]$.

Lemma 5.7. Let $K_{,T}$ be the subcomplex of K

$$K_{,T} = \begin{cases} S^{|T|-1} & \text{if } |\mathrm{supp}({}_T)| \text{ is an even integer,} \\ \{\emptyset\} & \text{if } |\mathrm{supp}({}_T)| \text{ is an odd integer.} \end{cases}$$

Proof. Suppose $|\mathrm{supp}({}_T)|$ is an even integer. Then it follows from Lemma 5.6 that, either $|T|$ is an even integer and $1 \notin T$ or $|T|$ is an odd integer and $1 \in T$.

Consider the first possibility that $|T|$ is an even integer and $1 \notin T$. Let K be the boundary of the cross polytope of dimension n. Observe that for each $1 \leq i \leq n$ the vertex i of K is antipodal to another vertex $n+i$. Note that $T \subseteq [n] \setminus \{1\}$ Therefore for each $i \in T$, 1 occurs at the ith and $(n+i)$th position of vector ${}_T$. Consequently, $K_{,T}$ can be obtained from K by removing the star of the antipodal vertices which does not belong to $\mathrm{supp}({}_T)$. Therefore, the subcomplex $K_{,T}$ is the boundary of $|T|$-dimensional cross polytope. This gives us $K_{,T} = S^{|T|-1}$.

Now consider another possibility that $|T|$ is an odd integer and $1 \in T$. Clearly, 1 occurs at the 1st and $(n+1)$th position of ${}_T$. Recall that the vertices in $\mathrm{supp}({}_T) \setminus \{n\}$ are antipodal. Therefore, for each $i \in T$, 1 occurs at the ith and $(n+i)$th position of vector ${}_T$. Then it is clear that $K_{,T}$ is obtained from K by removing the star of the antipodal vertices which does not belong to $\mathrm{supp}({}_T)$. Therefore, again $K_{,T}$ is the boundary of $|T|$-dimensional cross polytope. This gives $K_{\sigma,T} \simeq S^{|T|-1}$. This proves the lemma in the first case.

Now assume that $|supp(\sigma_T)|$ is an odd integer. Then by Lemma 5.6, either $|T|$ is an even integer and $1 \in T$ or $|T|$ is an odd integer and $1 \notin T$. Consider the first possibility that $|T|$ is an even integer and $1 \in T$. Therefore, 1 occurs at the 1st position but not at the $(n+1)$th position of σ_T. Since the vertices in $supp(\sigma_T) \setminus \{1\}$ are antipodal, it can be easily checked that

$$K_{\sigma,T} = S^{|T|-1} \setminus star(\{n+1\}).$$

Clearly, $K_{\sigma,T} \simeq \{\star\}$. Similarly in the second possibility, we get that

$$K_{\sigma,T} = S^{|T|-1} \setminus star(\{1\}).$$

Therefore, $K_{\sigma,T} \simeq \{\star\}$. This proves the lemma in the second case. □

Since the Lemma 5.7 already determined the homotopy types of subcomplexes $K_{\sigma,T}$, it is easy to compute the rational Betti numbers of K_n using the Suciu-Trevesan formula.

Theorem 5.8. *Let β_i be the ith rational Betti number of K_n. Then*

$$\beta_i = \begin{cases} \binom{n-1}{i} & \text{if } i \text{ is an even integer} \\ \binom{n-1}{i-1} & \text{if } i \text{ is an odd integer} \end{cases}$$

Proof. It follows from the Lemma 5.6 and Lemma 5.7 that the reduced rational homology of $K_{\sigma,T}$ is

$$\tilde{H}_{i-1}(K_{\sigma,T}, \mathbb{Q}) = \mathbb{Q}$$

if and only if

1. $|T| = i$ is an even integer and $1 \notin T$.

2. $|T| = i$ is an odd integer and $1 \in T$.

Now we can use the Suciu-Trevesan formula to compute the Betti numbers of K_n. If i is an even integer then the corresponding Betti number is number of i-element subsets $[n]$ not containing 1 and if i is an odd integer then the corresponding Betti number is the number of i-element subsets $[n]$ containing 1. This proves the theorem. □

Remark 5.9. Observe that, if n is an odd integer then $\epsilon_{[n]} = (1, 1, \ldots, 1)$. Therefore, $K_{\epsilon,[n]} = K$. In particular, $\beta_{2k+1}(K_{2k+1}) = 1$ for all k. Consequently, for each k, K_{2k+1} is orientable.

Example 5.10. The following table contains first five Betti numbers of K_n upto the dimension 5.

$n \backslash i$	0	1	2	3	4	5
1	1	1	0	0	0	0
2	1	1	0	0	0	0
3	1	1	1	1	0	0
4	1	1	3	3	0	0
5	1	1	6	6	1	1

Table 5.1: $\beta_i(K_n)$

Now we prove some properties of K_n which easily follow from its real Bott structure. Recall that a closed manifold M of dimension $2n$ is cohomologically symplectic if there exists a cohomology class $\omega \in H^2(M)$ such that $\omega^n \neq 0$.

Proposition 5.11. Let K_n be the n-dimensional Klein bottle. Then

1. K_n is orientable if and only if n is an odd integer,
2. for no value of $n > 1$ the manifold K_n is cohomologically symplectic.

Proof. It was shown in the first part of [4, Lemma 2.2] that the real Bott manifold $M(A)$ corresponding to a Bott matrix $A = [A^i_j]$ is orientable if and only if all row sums of A are zero in $\mathbb{Z}_2$. Recall that the Bott matrix B associated with K_n is given by Equation (5.4). It is easy to see that all row sums of B are zero if and only if n is an odd integer. This proves the first of the lemma.

The second part of [4, Lemma 2.2] says, $M(A)$ admits a symplectic form if and only if $|\{k : A_k = A_j\}|$ is even for every $1 \leq j \leq n$. Let B_i is the ith column of B. For each $j \in [n]$, consider the collection

$$B(j) = \{k \in [n] : B_k = B_j\}.$$

Note that $|B(1)| = 1$. Therefore, K_n never admits a symplectic form. □

Remark 5.12. The first part of the above lemma also follows from [12, Proposition 3.1].

Let $M(A)$ be the real Bott manifold corresponding to a Bott matrix A. The rational cohomology ring $H^*(M(A), \mathbb{Q})$ of $M(A)$ was computed by Choi and Park in [5]. Moreover, they showed that $H^*(M(A), \mathbb{Q})$ completely depends on the binary matroid associated with a Bott matrix A. We refer the reader to [5, Section 4] for more details.

Let A be a Bott matrix and $E = \{A_j : 1 \leq j \leq n\}$ be the set of its columns. A subset $C \subseteq E$ is said to be *minimally dependent* if every proper subset of C is linearly independent. Consider the collection

$$\mathcal{C} = \{C : C \subseteq E \text{ is minimally dependent}\}.$$

The matroid $T(A) = (E, \mathcal{C})$ is called a *binary matroid* associated with A and the elements $C \in \mathcal{C}$ are called *circuits*.

Theorem 5.13 ([5, Proposition 4.3])**.** *Let x_C be the formal symbol for the cohomology class corresponding to a circuit C. Then*

$$H^*(M(A), \mathbb{Q}) = \frac{\mathbb{Q}\langle x_C : C \in \mathcal{C}\rangle}{\ },$$

where the relations are given as follows:

$$x_C x_{C'} = \begin{cases} (-1)^{|C||C'|} x_C x_{C'} & \text{if } C \cap C' = \emptyset; \\ 0 & \text{if } C \cap C' \neq \emptyset, \end{cases}$$

with $\deg(x_C) = |C|$.

Now recall that K_n is a real Bott manifold corresponding to the Bott matrix given by Equation (5.4). In this case the corresponding matroid is

$$\mathcal{C} = \{\{1\}, \{i, j\} : 2 \leq i < j \leq n\}.$$

Let Y be the formal symbol of degree-1 cohomology class corresponding to the singleton set $\{1\}$ and for each $\{i, j\} \in \mathcal{C}$, let X_{ij} be the formal symbol of degree-2 cohomology class. Then we have

$$H^*(K_n, \mathbb{Q}) = \frac{\mathbb{Q}[Y, X_{ij} : 2 \leq i < j \leq n]}{},$$

where the following relations hold for $2 \leq i < j \leq n$ and $2 \leq k < l \leq n$.

1. $Y^2 = X_{ij}^2 = 0$,
2. $YX_{ij} = X_{ij}Y$,
3. $X_{ij}X_{kl} = X_{kl}X_{ij}$ if $\{i,j\} \cap \{k,l\} = \emptyset$,
4. $X_{ij}X_{kl} = 0$ if $\{i,j\} \cap \{k,l\} \neq \emptyset$.

5.1.3 Topological complexity of K_n

For a topological space X, Farber introduced the notion of a topological complexity $TC(X)$ in [23]. It is an important homotopy invariant of a topological space X. Let X be a path connected space and PX be the space of all paths in X. Let $f : [0,1] \to X$ be any path in X. There is a fibration $\pi : PX \to X \times X$ defined by $\pi(f) = (f(0), f(1))$. The topological complexity is the smallest k such that $X \times X$ admits an open cover $V_1, \ldots, V_k$ such that there exist continuous sections of π on V_i for $1 \leq i \leq k$. The topological complexity of X is denoted by $TC(X)$. This invariant is closely related to the Lusternik-Schnirelmann category (LS category) of a space X, denoted as $\mathrm{cat}(X)$. The $\mathrm{cat}(X)$ is the smallest integer r such that X can be covered by r open subsets $V_1, \ldots, V_r$ with each inclusion $V_i \hookrightarrow X$ is null-homotopic.

The product inequality for the Lusternik-Schnirelmann category was proved in [27].

Theorem 5.14 ([27, Theorem 9]). If X and Y are the path connected spaces. Then

$$\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y) - 1.$$

The similar product inequality for topological complexity is proved by Farber in [23].

Theorem 5.15 ([23, Theorem 11]). If X and Y are path connected spaces. Then

$$TC(X \times Y) \leq TC(X) + TC(Y) - 1.$$

The topological complexity of fibrations has been studied by many mathematicians. We recall some results here. Let G be a group acting on E. The strong equivariant topological complexity $TC^*_G(E)$ was introduced by Dranishnikov in [21, Section 3].

Theorem 5.16 ([21]). *Let E, B be locally compact metric ANR-spaces. and $p : E \to B$ be a fibre bundle with fibre F with structure group properly acting on F. Then*

$$TC(E) \leq TC(B) + TC^*_G(F).$$

The upper bound for fiber spaces is given by Mark Grant in [24].

Theorem 5.17 ([24, Lemma 7]). *Let $p : E \to B$ be a Hurewicz fibration with fibre F. Then*

$$TC(E) \leq TC(F)\,\mathrm{cat}\,(B \times B).$$

Mark Grant improved the upper bound in the above theorem (see [30, Theorem 3.1]).

In [59], Sarkar and Naskar extend the product inequality formulas to some classes of fibre bundles. In particular, they gave an upper bound for the topological complexity of the total space of a fibre bundle.

Theorem 5.18 ([59, Corollary 3.5]). *Let $p : E \to B$ be a fibre bundle with fibre F. Then*

$$TC(E) \leq TC(F) + \mathrm{cat}(B \times B).$$

Theorem 5.19 ([59, Theorem 3.4]). *Let F, E and B be path-connected spaces and $F \to E \xrightarrow{p} B$ be a fiber bundle with fibre F and $V_1, \ldots, V_m$ be an open cover of $B \times B$ with homotopy sections $\sigma_j : V_j \to PR_j \subseteq PB$ of $\pi : PB \to B \times B$ such that over R_j the bundle $E \xrightarrow{p} B$ is trivial for $j = 1, \ldots, m$. Let*

$$h_j : V_j \times (F \times F) \to (p \times p)^{-1}(V_j)$$

be a local trivialization for the bundle $E \times E \xrightarrow{p \times p} B \times B$ with fibre $F \times F$ for $j = 1, \ldots, m$. Then

$$TC(E) \leq TC(F) + m - 1.$$

Recall the definition of n-dimensional Klein bottle. It follows from [32, Proposition 2.1] that the K_n is homeomorphic to $\overline{M}(\ell)$, where $\ell = (\ell_1, \ldots, \ell_{n-1}, 1, 1, 1, 2)$ with

$\sum_{i=1}^{n-1} \ell_i < 1$. Davis [12, Proposition 5.2] showed that the $TC(K_n)$ is bounded below by $n+3$.

Theorem 5.20. There exist a fibre bundle $p : K_n \to S^1$ with fibre $(S^1)^{n-1}$.

Proof. Let

$$p : K_n \to \mathbb{R}P^1,$$

defined by

$$p([z_1, z_2, \dots, z_n]) = [z_1].$$

Let $[a] \in \mathbb{R}P^1$. Then note that

$$p^{-1}([a]) = \frac{\{a, -a\} \times (S^1)^{n-1}}{(a, z_2, \dots, z_n) \sim (-a, \bar{z}_2, \dots, \bar{z}_n)} \cong (S^1)^{n-1}.$$

Let U be a neighbourhood of $[a]$ in $\mathbb{R}P^1$ with U^+ and U^- being the neighbourhoods of a and $-a$ in S^1, respectively. Clearly,

$$p^{-1}([U]) = \frac{\{U^+, U^-\} \times (S^1)^{n-1}}{(x, z_2, \dots, z_n) \sim (-x, \bar{z}_2, \dots, \bar{z}_n)} \cong U \times (S^1)^{n-1},$$

where $x \in U^+ \cup U^-$. Now it is also clear that $p : K_n \to \mathbb{R}P^1$ is a fibre bundle with fibre $(S^1)^{n-1}$. □

Remark 5.21. After proving Theorem 5.20, we were planning to apply Theorem 5.18 to reduce the upper bound on $TC(K_n)$. But Prof. Donald Davis brought to our notice that Theorem 5.18 is not true in general. Prof. Mark Grant also informed us that the Lense spaces give a counterexample to Theorem 5.18. We later tried to use Theorem 5.19, but we end up getting a counterexample. So in the end we do not succeed in improving the upper bound on $TC(K_n)$. This is a work in progress. We are trying to prove a version of Theorem 5.19 in a special case where the base space of the fibre bundle is S^1 and the fibre space is the product of circles.

Counterexample to Theorem 5.18 suggested by Prof. Mark Grant

Let $\mathbb{Z}_m$ be the cyclic group considered as a multiplicative group $\{1, w, \dots, w^{m-1}\} \subset \mathbb{C}$ of m-th roots of unity. Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$. The lens space is defined as the quotient

$$L_m^{2n+1} = \frac{S^{2n+1}}{\mathbb{Z}_m},$$

where $\mathbb{Z}_m$ acts freely on S^{2n+1} via pointwise multiplication. The lens space L_m^{2n+1} can be considered as the total space of a fibre bundle over $\mathbb{C}P^n$ whose fibre is S^1: Let

$U(1)$ be the unitary group of dimension 1. Since CP^n is the quotient of S^{2n+1} by $U(1)$ and $\mathbb{Z}_m \subset U(1)$, there is a natural projection $L_m^{2n+1} \to CP^n$. It is easy to see that the fibre of this projection is S^1.

Farber and Grant [24, Theorem 12, 13, 14] shown that for some specific values of m and n the $TC(L_m^{2n+1}) = 4n+2$. On the other hand Theorem 5.18 gives $TC(L_m^{2n+1}) \leq 2n+4$. Therefore, many different lens spaces provide a counterexample to Theorem 5.18.

Counterexample to Theorem 5.19

Let K be the Klein bottle. Note that K can be considered as the total space of fibre bundle over S^1 with fibre S^1. Now we construct two open covers of $S^1 \times S^1$ with three open sets satisfying the hypothesis of Theorem 5.19. Let $a, b, c \in S^1$ be three distinct points on circle. We write a instead $\{a\}$ for short. Similarly for b and c. Consider $R_1 = S^1 \setminus a$, $R_2 = S^1 \setminus b$, $R_3 = S^1 \setminus c$ and $V_1 = R_1 \times R_1$, $V_2 = R_2 \times R_2$, $V_3 = R_3 \times R_3$. Note that $S^1 = \cup_{i=1}^{3} R_i$ and $S^1 \times S^1 = \cup_{i=1}^{3} V_i$. Observe that given $(x,y) \in V_i$, there is a unique (counterclockwise) path $\gamma_{(x,y)}$ from x to y which lies inside R_i. Therefore, we can define sections $\sigma_i : V_i \to PS^1$ of $\pi : PS^1 \to S^1 \times S^1$ by $\sigma_i(x,y) = \gamma_{(x,y)}$. Note that $\sigma_i(V_i) \subset PR_i$. Since each V_j is contractible, we have local trivialization

$$h_j : V_j \times (S^1 \times S^1) \to (p \times p)^{-1}(V_j)$$

for the bundle $K \times K \xrightarrow{p \times p} S^1 \times S^1$ with fibre $S^1 \times S^1$ for $j = 1, 2, 3$.

The above discussion shows that the hyothesis of Theorem 5.19 is satisfied. Consequently, we get $TC(K) \leq 4$. Which is not possible, since $TC(K) = 5$ (see [6]).

5.2 Polygon spaces with long genetic codes

In this section we define the characteristic functions on the facets of $P_5 \times I^{n-2}$ and $P_6 \times I^{n-2}$ where P_5 is the pentagon and P_6 is the hexagon. We also show that the corresponding small covers $X(P_5 \times I^{n-2})$ and $X(P_6 \times I^{n-2})$ are homeomorphic to the planar polygon spaces associated with the genetic codes $\langle\{1, 2, \ldots, n-5, n-3, n\}\rangle$ and $\langle\{1, 2, \ldots, n-5, n-2, n\}\rangle$.

5.2.1 Betti numbers of $X(P_5 \times I^{n-2}, \Lambda)$

We refer reader to [36] for the following definition and remark.

Definition 5.22. Let P and P' are two convex polytopes of dimension d and d', both containing the origin. Then their direct sum is the $(d+d')$ dimensional polytope

$$P \oplus P' = \mathrm{conv}(\{(p,0) \in \mathbb{R}^{d+d'} : p \in P\} \cup \{(0,p') \in \mathbb{R}^{d+d'} : p' \in P'\}).$$

Remark 5.23. Let $P^{\triangle}$ and $P'^{\triangle}$ be the dual polytopes of P and P' containing the origin. Then their direct sum and product is related by the following equation :

$$P \oplus P' = (P^{\triangle} \times P'^{\triangle})^{\triangle}.$$

In particular, if P_m is the m-gon then

$$(P_m \times I^{n-2})^{\triangle} = P_m \oplus (I^{n-2})^{\triangle}.$$

To construct the characteristic function over $P_5 \times I^{n-2}$, we give a specific labeling for the facets of $P_5 \times I^{n-2}$ as follows :

- For each $1 \leq i \leq n-2$,

$$F_i = P_5 \times I \times \cdots \times \{-1\} \times \cdots \times I,$$

  where $\{-1\}$ is at the ith place.

- For each $1 \leq i \leq n-2$,

$$F_{n+i} = P_5 \times I \times \cdots \times \{1\} \times \cdots \times I,$$

  where $\{1\}$ is at the ith place.

- For $1 \leq i \leq 5$, let E_i is the ith side of P_5. We set

$$F_{n-1} = E_1 \times I^{n-2}, F_n = E_2 \times I^{n-2}, F_{2n-1} = E_3 \times I^{n-2},$$

$$F_{2n} = E_4 \times I^{n-2}, F_{2n+1} = E_5 \times I^{n-2}.$$

Let $F(P_5 \times I^{n-2})$ be the collection of facets of $P_5 \times I^{n-2}$. We define a function

$$\lambda : F(P_5 \times I^{n-2}) \to \mathbb{Z}_2^n$$

by

$$\lambda(F) = \begin{cases} e_i & \text{if } F = F_i \text{ and } F = F_{n+i},\ 1 \leqslant i \leqslant n \\ \sum_{i=1}^{n} e_i & \text{if } F = E_5 \times I^{n-2}. \end{cases} \tag{5.5}$$

Now we prove that λ is a characteristic function.

Lemma 5.24. The function λ is a characteristic function for $P_5 \times I^{n-2}$.

Proof. Observe that

$$\lambda(F(v)) = \begin{cases} \{e_1, \ldots, e_{n-1}, \sum_{i=1}^{n} e_i\} & \text{if } v \in F_{2n+1}, \\ \{e_1, \ldots, e_n\} & \text{otherwise.} \end{cases}$$

Therefore, for any vertex, $\lambda(F(v))$ forms a basis of $\mathbb{Z}_2^n$. Consequently, λ is the characteristic function. $\square$

It is clear that the $n \times (2n+1)$-matrix of λ is

$$\Lambda = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.$$

Theorem 5.25. There is a following homeomorphism

$$X(P_5 \times I^{n-2}, \lambda) = \frac{(S^1)^{n-2} \times \Sigma_3}{(z_1, \ldots, z_{n-2}, z) \sim (\bar{z}_1, \ldots, \bar{z}_{n-2}, -z)},$$

where Σ_3 is the orientable surface of genus 3.

Proof. The authors [10, Theorem 4.14, Theorem 5.2] shows that the chain space corresponding to the genetic code $< 1, 2, \ldots, n-1, n+1, n+3 >$ is a small cover with the quotient polytope as $P_5 \times I^{n-2}$. Moreover, they compute the corresponding characteristic function as well which coincides with the Equation 5.5. Now the theorem follows from the [32, Proposition 2.1]. □

Lemma 5.26.

$$|\mathrm{supp}(\ _T)| = \begin{cases} 2|T| & \text{if } |T| \text{ is an even integer} \\ 2|T|+1 & \text{if } |T| \text{ is an odd integer} \end{cases}$$

Proof. Observe that, each row of the characteristic matrix contains three 1's and for each $1 \leq i \leq n$, the ith and $(n+i)$th column coincides.

It is easy to see that, for each $i \in T$ with $1 \leq i \leq n$, 1 occurs at the ith and $(n+i)$th position of vector $\ _T$. Moreover, if $|T|$ is an odd integer then 1 occurs in $\ _T$ at the $(2n+1)$th position as well. In particular, 1 occurs $2|T|+1$ many times in $\ _S$.

Suppose $|T|$ is an even integer. Then 1 will always occur at ith and $(n+i)$th position of $\ _T$ but wont occur at the $(2n+1)$th position. Therefore, in this case 1 occurs in $\ _T$ exactly $2|T|$ times.

□

Lemma 5.27.

$$K_{,T} = \begin{cases} S^{|T|-1} & \text{if } \{n-1, n\} \quad T \text{ and } |T| \text{ is an odd integer} \\ \{\emptyset\} & \text{if } \{n-1, n\} \quad T \text{ and } |T| \text{ is an even integer.} \end{cases}$$

Proof. Suppose $\{n-1, n\} \quad T$ and $|T|$ is an odd integer. Therefore,

$$\{n-1, n, 2n-1, 2n, 2n+1\} \quad \mathrm{supp}(\ _T).$$

Clearly, $P_5 \quad K_{,T}$. Therefore, the antipodal vertices which does not belongs to

$$([n-2] \cup \{n+i : i \in [n-2]\}) \cap \mathrm{supp}(\ _T)$$

gets removed from the $K_{,T}$. Since we have $K = \partial P_5 \quad (I^{n-2})^4)$, $K_{,T} = \partial P_5 \quad (I^{|T|-2})^4)$. Consequently, $K_{,T} = S^{|T|-1}$. If $|T|$ is an even integer then clealry $2n+1 \notin \mathrm{supp}(\ _T)$. Therefore, $K_{,T} = S^{|T|-1} \setminus \mathrm{star}(\{2n+1\})$. Clearly, $K_{,T} = \{\emptyset\}$ □

Lemma 5.28.

$$K_{,T} = \begin{cases} S^{|T|-1} & \text{if } \{n-1,n\} \subseteq T \text{ and } |T| \text{ is an even integer} \\ \{\emptyset\} & \text{if } \{n-1,n\} \subseteq T \text{ and } |T| \text{ is an odd integer.} \end{cases}$$

Proof. Suppose $\{n-1,n\} \subseteq T$ and $|T|$ is an even integer. Therefore,

$$\{n-1, n, 2n-1, 2n, 2n+1\} \subseteq \mathrm{supp}(\ _T).$$

In particular $P_5 \subseteq K_{,T}$. It follows from the Remark 5.23 $K = \partial(P_5 \quad (I^{n-2})^4)$. Therefore, it is easy to see that $K_{,T} = \partial((I^{|T|})^4)$. Now suppose that $\{n-1,n\} \subseteq T$ and $|T|$ is an odd integer. This gives

$$\{n-1, n, 2n-1, 2n\} \subseteq \mathrm{supp}(\ _T)$$

and $2n+1 \in \mathrm{supp}(\ _T)$. Note that the vertex $2n+1$ in K is adjacent to all the vertices in

$$[n-2] \cup \{n+i : i \in [n-2]\}.$$

Therefore, in $K_{,T}$ the vertex $2n+1$ is adjacent to

$$([n-2] \cup \{n+i : i \in [n-2]\}) \cap \mathrm{supp}(\ _T).$$

This gives $K_{,T}$ is isomorphic to the cone over $S^{|T|-1}$ with the apex vertex $2n+1$ since

$$K_{,T} \setminus \{2n+1\} \cong S^{|T|-1}.$$

This proves the lemma. □

Lemma 5.29. If one of the following condition satisfies

1. Suppose $n-1 \notin T$ and $n \in T$.

2. Suppose $n-1 \in T$ and $n \notin T$.

Then $K_{,T} = S^{|T|-1}$.

Proof. Suppose that $n-1 \notin T$ and $n \in T$ with $|T|$ is an even integer. This gives

$$\{n-1, 2n-1, 2n+1\} \subseteq \mathrm{supp}(\ _T) \text{ and } \{n, 2n\} \quad \mathrm{supp}(\ _T).$$

Therefore, $\mathrm{supp}(\sigma_T)$ contains two antipodal vertices from P_5 and $2(|T|-1)$ vertices from $(I^{n-2})^4$. It is easy to see that $K_{\sigma,T} = \partial(I \times I^{(|T|-1)})^4)$. Clearly, $K_{\sigma,T} = S^{|T|-1}$. Now suppose $|T|$ is an odd integer. Then we clearly have

$$\{n-1, 2n-1\} \not\subseteq \mathrm{supp}(\sigma_T) \text{ and } \{n, 2n, 2n+1\} \subseteq \mathrm{supp}(\sigma_T).$$

Since $\{2n, 2n+1\}$ are adjacent vertices and n is antipodal to $2n$, we can collapse an edge $\{2n, 2n+1\}$ to $2n$. In particular, we have $P_5 \setminus K_{\sigma,T} = S^0$. Therefore, again we have $K_{\sigma,T} = \partial(I \times I^{(|T|-1)})^4)$. This proves the lemma in the context of first case. Similar arguments can be used to prove the lemma in second case. $\square$

Theorem 5.30. Let β_i be the ith rational Betti number of $X(P_5 \times I^{n-2}, \lambda)$. Then

$$\beta_i = \begin{cases} 2\binom{n-2}{i-1} + \binom{n-2}{i} & \text{if } i \text{ is an even} \\ 2\binom{n-2}{i-1} + \binom{n-2}{i-2} & \text{if } i \text{ is an odd integer} \end{cases}$$

Proof. Using Lemma 5.27, Lemma 5.28 and Lemma 5.29 we have $\tilde{H}_{i-1}(K_{\sigma,T}, \mathbb{Q}) = \mathbb{Q}$ if the following conditions holds :

1. If $|T| = i$ is an odd integer with $\{n-1, n\} \subseteq T$.

2. If $|T| = i$ is an even integer with $\{n-1, n\} \not\subseteq T$.

3. If $|T| = i$ with $n-1 \notin T$ and $n \in T$.

4. If $|T| = i$ with $n-1 \in T$ and $n \notin T$.

Now we can use the Suciu-Trevesan formula to compute the Betti numbers of $X(P_5 \times I^{n-2}, \lambda)$. If i is an even integer then the corresponding rational Betti number is the sum of the cardinalities of i-element subsets of $[n]$ of type (2), (3) and (4). Similarly, if i is an odd integer then the corresponding Betti number is the sum of the cardinalities of i-element subsets of $[n]$ of type (1), (3) and (4). $\square$

Example 5.31. The following table contains first five Betti numbers of $X(P_5 \times I^{n-2}, \lambda))$ up to the dimension 5.

$n \backslash i$	0	1	2	3	4	5
2	1	2	0	0	0	0
3	1	2	2	1	0	0
4	1	2	5	4	0	0
5	1	2	9	9	2	1

Table 5.2: $\beta_i(X(P_5 \times I^{n-2}, \lambda))$.

5.2.2 Betti numbers of $X(P_6 \times I^{n-2}, \lambda)$

To construct the characteristic function over $P_6 \times I^{n-2}$, we give a specific labeling for its facets:

- For each $1 \leq i \leq n-2$, we set

  $$F_i = P_6 \times I \times \cdots \times \{-1\} \times \cdots \times I$$

  where $\{-1\}$ is at the ith place.

- For each $1 \leq i \leq n-2$, we set

  $$F_{n+1+i} = P_6 \times I \times \cdots \times \{1\} \times \cdots \times I$$

  where $\{1\}$ is at the ith place.

- For $1 \leq i \leq 6$, let E_i is the ith side of P_6. Then we set

  $$F_{n-1} = E_1 \times I^{n-2},\ F_n = E_2 \times I^{n-2},\ F_{2n-1} = E_3 \times I^{n-2},$$

  $$F_{2n} = E_4 \times I^{n-2},\ F_{2n+1} = E_5 \times I^{n-2},\ F_{2n+1} = E_6 \times I^{n-2}.$$

Let $F(P_6 \times I^{n-2})$ be the collection of facets of $P_6 \times I^{n-2}$. We define a function

$$\lambda : F(P_6 \times I^{n-2}) \to \mathbb{Z}_2^n$$

as

$$\lambda(F) = \begin{cases} e_i & \text{if } F = F_i \text{ and } F = F_{n+1+i},\ 1 \leq i \leq n \\ \sum_{i=1}^{n} e_i & \text{if } F = F_{n+1} \text{ and } F = F_{2n+2}. \end{cases}$$

Now we prove the function λ is characteristic for $P_6 \times I^{n-2}$.

Lemma 5.32. The function λ is a characteristic function for $P_6 \times I^{n-2}$.

Proof. Note that

$$\lambda(F(v)) = \begin{cases} \{e_1, \ldots, e_{n-1}, \sum_{i=1}^{n} e_i\} & \text{if either } v \in F_{n+1} \text{ or } v \in F_{2n+2}, \\ \{e_1, \ldots, e_n\} & \text{otherwise.} \end{cases}$$

□

It is clear that the $(n \times 2n)$-matrix of λ is

$$\Lambda = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.$$

Theorem 5.33. There is a following homeomorphism

$$X(P_6 \times I^{n-2}, \lambda) = \frac{(S^1)^{n-2} \times N_4}{(z_1, \ldots, z_{n-2}, z) \sim (\bar{z}_1, \ldots, \bar{z}_{n-2}, -z)},$$

where N_4 is the orientable surface of genus 4.

Proof. The proof of this theorem is the same as that of Theorem 5.25. We just have to replace the genetic code by $< 1, 2, \ldots, n-1, n+2, n+3 >$. □

Lemma 5.34.

$$|\mathrm{supp}(\lambda_T)| = \begin{cases} 2|T| & \text{if } |T| \text{ is an even integer} \\ 2|T| + 2 & \text{if } |T| \text{ is an odd integer} \end{cases}$$

Proof. Observe that, each row of the characteristic matrix contains four 1's and for each $1 \leq i \leq n+1$, the ith and $(n+1+i)$th column coincides.

It is easy to see that, for each $i \in T$ with $1 \leq i \leq n$, 1 occurs at the ith and $(n+1+i)$th position of vector ${}_T$. Moreover, if $|T|$ is an odd integer then 1 occurs in ${}_T$ at the $(n+1)$th and $(2n+2)$th position. In particular, 1 occurs $2|T|+2$ many times in ${}_S$.

Suppose $|T|$ is an even integer. Then 1 will always occur at ith and $(n+1+i)$th position of ${}_T$ but wont occur at the $(n+1)$th and $(2n+2)$th position. Therefore, in this case 1 occurs in ${}_T$ exactly $2|T|$ times. □

Lemma 5.35.

$$K_{,T} = \begin{cases} S^{|T|-1} & \text{if } \{n-1, n\} \quad T \text{ and } |T| \text{ is an odd integer} \\ S^{|T|-2} & \text{if } \{n-1, n\} \quad T \text{ and } |T| \text{ is an even integer.} \end{cases}$$

Proof. Suppose $\{n-1, n\} \quad T$ and $|T|$ is an odd integer. Note that for each $i \in T$ with $1 \leq i \leq n$, 1 occurs at the ith and $(n+1+i)$th position of vector ${}_T$. Since $|T|$ is an odd integer, 1 occurs at $(n+1)$th and $(2n+2)$th position of ${}_T$ as well. Therefore,

$$\{n-1, n, n+1, 2n, 2n+1, 2n+2\} \quad \mathrm{supp}({}_T).$$

Since the above set forms a vertex set of P_6, we have $P_6 \quad K_{,T}$. The remaining vertices of $K_{,T}$ are given by

$$\{i : i \in T\} \cup \{n+1+i : i \in T\}$$

Note that $K = P_6 \quad (I^{n-2})^4$. Observe that the above vertices are from the $(I^{n-2})^4$ factor of K. Therefore, $K_{,T} = \partial(P_6 \quad {}_{i \in T \cap [n-2]} I_i)$, where $I_i = I$. Now it is clear that

$$K_{,T} = \partial(P_6 \quad (I^{|T|-2})^4) = S^{|T|-1}.$$

Now assume that $\{n-1, n\} \quad T$ and $|T|$ is an even integer. Therefore, 1 does not occur at the $(n+1)$th and $(2n+2)$th position of vector ${}_T$. This gives

$$\{n+1, 2n+2\} \not\subseteq \mathrm{supp}({}_T), \text{ and } \{n-1, n, 2n, 2n+1\} \quad \mathrm{supp}({}_T)$$

since $\{n-1, n\} \subseteq T$. Clearly, we have $P_6 \cap K_{,T} = S^0$. Now it is easy to see that $K_{,T} = @(\ast_{i=1}^{|T|-2} I_i)$, where $I_i = I$ for all i. Therefore, $K_{,T} = @((I^{|T|-1})^4) = S^{|T|-2}$. This proves the lemma. □

Lemma 5.36.

$$K_{,T} = \begin{cases} S^{|T|} & \text{if } \{n-1, n\} \not\subseteq T \text{ and } |T| \text{ is an odd integer,} \\ S^{|T|-1} & \text{if } \{n-1, n\} \not\subseteq T \text{ and } |T| \text{ is an even integer.} \end{cases}$$

Proof. Suppose $\{n-1, n\} \not\subseteq T$ and $|T|$ is an even integer. Therefore, 1 does not occur at the $(n+1)$th and $(2n+2)$th position of vector $_T$. This gives

$$\{n+1, 2n+2\} \not\subseteq \operatorname{supp}(_T) \text{ and } \{n-1, n, 2n, 2n+1\} \not\subseteq \operatorname{supp}(_T).$$

Therefore, $P_6 \not\subseteq K_{,T}$. Since $T \subseteq [n-2]$, $K_{,T} = @(\ast_{i=1}^{|T|} I_i)$, where $I_i = I$ for all i. Therefore, $K_{,T} = @((I^{|T|})^4) = S^{|T|-1}$.

Now assume that $\{n-1, n\} \not\subseteq T$ and $|T|$ is an odd integer. Therefore, 1 occurs at the $(n+1)$th and $(2n+2)$th position of vector $_T$. Therefore,

$$\{n+1, 2n+2\} \subseteq \operatorname{supp}(_T) \text{ and } \{n-1, n, 2n, 2n+1\} \not\subseteq \operatorname{supp}(_T).$$

Since $T \subseteq [n-2]$, $K_{,T} = @(\ast_{i=1}^{|T|} I_i)$, where $I_i = I$ for all i. Note that the first factor in the previous direct sum is corresponding to $\{n+1, 2n+2\}$. Therefore,

$$K_{,T} = @((I^{|T|+1})^4) = S^{|T|}.$$

This proves the lemma. □

Lemma 5.37. If one of the following condition satisfies

1. Suppose $n-1 \notin T$ and $n \in T$.

2. Suppose $n-1 \in T$ and $n \notin T$.

Then $K_{,T} = S^{|T|-1}$.

Proof. Suppose $n-1 \notin T$ and $n \in T$ with $|T|$ is an odd integer. Therefore, 1 occurs at the nth, $(2n+1)$th, $(n+1)$th, $(2n+2)$th position of vector $_T$ but doesn't occur at the

$(n-1)$th and $(2n)$th position. This clearly gives

$$\{n, n+1, 2n+1, 2n+2\} \quad supp(\ {}_T) \text{ and } \{n-1, 2n\}^* \quad supp(\ {}_T).$$

Since $T \setminus \{n\} \quad [n-2]$, $K_{\ ,T} = @I^{|T|-1}_{i=1} I_i)$, where $I_i = I$ for all i. Note that the rst factor in the above direct sum is corresponding to $\{n, 2n+1\}$. Therefore,

$$K_{\ ,T} = @(I^{|T|})^4\ ) = S^{|T|-1}.$$

Now suppose $n-1 \notin T$ and $n \in T$ with $|T|$ is an even integer. Therefore, 1 does not occurs at the $n-1$th, $(2n)$th, $(n+1)$th, $(2n+2)$th position of vector $\ {}_T$ but occurs at the nth and $(2n+1)$th position. In particular, we have

$$\{n-1, 2n, n+1, 2n+2\}^* \quad supp(\ {}_T) \text{ and } \{n, 2n+1\} \quad supp(\ {}_T).$$

Since $T \setminus \{n\} \quad [n-2]$, $K_{\ ,T} = @I^{|T|-1}_{i=1} I_i)$, where $I_i = I$ for all i. Note that the rst factor in the above direct sum is corresponding to $\{n, 2n+1\}$. Therefore,

$$K_{\ ,T} = @(I^{|T|})^4\ ) = S^{|T|-1}.$$

This proves the lemma in the rst case. Similar steps can be followed to prove the second case. □

Theorem 5.38. Let $\ _i$ be the ith rational Betti number of $X(P_6 \quad I^{n-2}, \ )$. Then

$$\ _i = \begin{cases} 3\binom{n-2}{i-1} + \binom{n-2}{i} & \text{if } i \text{ is an even integer,} \\ 3\binom{n-2}{i-1} + \binom{n-2}{i-2} & \text{if } i \text{ is an odd integer} \end{cases}$$

Proof. Let i be an odd integer. Then from Lemma 5.35, Lemma 5.36 and Lemma 5.37 we have $\tilde{H}_{i-1}(K_{\ ,T}, Q) = Q$ if the following conditions holds :

1. If $|T| = i$ with $\{n-1, n\} \quad T$.
2. If $|T| = i+1$ with $\{n-1, n\} \quad T$.
3. If $|T| = i$ with $n-1 \notin T$ and $n \in T$.
4. If $|T| = i$ with $n-1 \in T$ and $n \notin T$.

Note that the cardinality of type (1) sets is $\binom{n-2}{i-2}$ and the cardinalities of type (2), type (3) and type (4) sets are same and it is equal to $\binom{n-2}{i-1}$ in each case. Now theorem follows by adding these cardinalities.

Now suppose i is an even integer. Then again we can use Lemma 5.35, Lemma 5.36 and Lemma 5.37 to get the $(i-1)$th reduced rational homology of $K_{\tau,T}$. We have $\tilde{H}_{i-1}(K_{\tau,T}, \mathbb{Q}) = \mathbb{Q}$ if the following conditions holds :

1. If $|T| = i$ with $\{n-1, n\} \ast T$.
2. If $|T| = i-1$ with $\{n-1, n\} \ast T$.
3. If $|T| = i$ with $n-1 \notin T$ and $n \in T$.
4. If $|T| = i$ with $n-1 \in T$ and $n \notin T$.

Note that the cardinality of type (1) sets is $\binom{n-2}{i}$ and the cardinalities of type (2), type (3), type (4) sets are same and it is equal to $\binom{n-2}{i-1}$ in each case. This proves the theorem. □

Example 5.39. The following table contains first five Betti numbers of $X(P_6 \times I^{n-2}, \tau))$ upto the dimension 5.

n \ i	0	1	2	3	4	5
2	1	3	0	0	0	0
3	1	3	3	1	0	0
4	1	3	7	5	0	0
5	1	3	12	12	3	1

Table 5.3: $\beta_i(X(P_6 \times I^{n-2}, \tau))$.

5.3 The Borsuk-Ulam theorem for planar polygon spaces

In this chapter, We investigate the tidyness and existence of BU triples among the class of moduli spaces of planar polygons with a free cellular involution.

In Section 5.3.1 we collect relevant results about $\mathbb{Z}_2$-spaces and polygonal spaces. Section 5.3.2 contains results about those moduli spaces for which the genetic code of the length vector contains exactly one gene. Then in Section 5.3.3 we consider the two genes case and mainly focus on deriving an expression for the Stiefel-Whitney height. The formula we obtain generalizes a result of Don Davis. Finally, in Section 5.3.4 we tackle the class of quasi-equilateral polygons.

5.3.1 $\mathbb{Z}_2$-spaces

Let X be a topological space with a free $\mathbb{Z}_2$ action and consider the n-sphere, S^n_a, with the antipodal action. Then we have the following numerical data associated with $\mathbb{Z}_2$-spaces. Note that these numbers are not homotopy invariant.

Definition 5.40. The *coindex* of X is

$$\operatorname{coind}(X) := \max\{n \geq 0 : \exists\, \mathbb{Z}_2\text{- map } S^n_a \to X\}$$

The *index* of X is

$$\operatorname{ind}(X) := \min\{n \geq 0 \mid \exists\, \mathbb{Z}_2 \text{ map } X \to S^n_a\}$$

The *Stiefel-Whitney height* of X is

$$\operatorname{ht}(X) := \sup\{n \geq 0 \mid (w_1(X))^n \neq 0\}$$

where $w_1(X)$ is the first Stiefel-Whitney class of the double cover $X \to X/\mathbb{Z}_2$.

For a free $\mathbb{Z}_2$ space X, the following inequality relates these three parameters:

$$0 \leq \operatorname{coind}(X) \leq \operatorname{ht}(X) \leq \operatorname{ind}(X) \leq \dim(X). \tag{5.6}$$

Remark 5.41. If the genetic code for $\ell = (\ell_1, \ldots, \ell_b, \ldots, \ell_n)$ is $\langle\{b, n\}\rangle$, then the genetic code for the $(n-1)$-length vector $\ell' = (\ell_1, \ldots, \ell_{b-1}, \ell_{b+1} \ldots, \ell_n + \ell_b)$, is $\langle\{n-1\}\rangle$. Therefore, $M_{\ell'} = S^{n-4}$. Since $M_{\ell'} \subset M_\ell$, it inherits the free $\mathbb{Z}_2$-action from M_ℓ and hence $\operatorname{coind}(M_\ell) \geq n-4$.

Below are some equivalent formulations of BU-triples in terms of above defined numerical parameters. The proof of their equivalence can be found in [29]. In fact, the authors prove 10 equivalent conditions, we omit some those here since they are not relevant and will need us introduce additional terms that are not used in the present article.

Proposition 5.42 ([29, Proposition 2.2]). Let $(X, \ )$ be a free $\mathbb{Z}_2$-space. Then the following are equivalent.

1. The triple $(X, \ , n)$ is a BU-triple.
2. One has $\mathrm{ind}(X) > n$.
3. For every $\mathbb{Z}_2$-equivariant map $f : X \to \mathbb{R}^n$, $0 \in \mathrm{Im}(f)$.
4. There is no $\mathbb{Z}_2$-equivariant map $f : X \to S^{n-1}$.

Proposition 5.43 ([29, Theorem 3.4]). Let X be an m-manifold with a free $\mathbb{Z}_2$ action . Then $(X, \ , m)$ is a BU-triple if and only if $\mathrm{ind}(X) = \mathrm{ht}(X) = m$.

Corollary 5.44. Let X be an m-manifold with a free $\mathbb{Z}_2$ action . If $\mathrm{ht}(X) = m - 1$ then $\mathrm{ind}(X)$ is also $m - 1$ and $(X, \ , m - 1)$ is a BU-triple.

5.3.2 Length vectors with monogenetic codes

A genetic code with only one gene is called *monogenetic*. This section deals with computations of coindex, index, and height of certain planar polygon spaces having monogenetic code.

Proposition 5.45. Let $\langle n \rangle$ be the genetic code of . Then M is tidy.

Proof. For a length vector with genetic code $\langle\{n\}\rangle$, It is well known that M is homeomorphic to S^{n-3}. The identity morphism on S^{n-3} implies that the coindex of M is at least $n - 3$. By Borsuk-Ulam theorem, there does not exist a $\mathbb{Z}_2$-equivariant map from $S^{n-2}_a \to S^{n-3}$. Therefore, $\mathrm{coind}(M\ ) = n - 3 \leq \mathrm{ind}(M\ ) \leq \dim(M\ ) = n - 3$. Hence M is tidy. □

Lemma 5.46. If size of the smallest gee in the genetic code corresponding to an n-length vector is k, then $n - 3 - k \leq \mathrm{coind}(M\ )$.

Proof. Let $= (\ _1, \ldots, \ _n)$ and $S \ [n]$ be the smallest gee of the genetic code of . Consider the reduced length vector

$$(S) = (\ _{i_1}, \ldots, \ _{i_k}, \ _n + \sum_{j \in S} \ _j),$$

where $S = \{i_1, \ldots, i_k\}$. Since $S \cup \{n\}$ is a gene of the genetic code of α, the set $S \cup \{n\} \cup \{i_s\}$ is long for each $1 \leq s \leq k$. Therefore, $M_{\alpha(S)} = S^{n-l-3}$ as $\alpha(S)$ is the $(n-l-3)$-length vector. Note that any polygon with side lengths given by length vector $\alpha(S)$ can be considered as a polygon with side lengths given by α whose sides indexed by S are parallel. This gives a $\mathbb{Z}_2$-equivariant embedding of S^{n-l-3} in M_α. This proves the lemma. □

The above result will be used throughout this section. In particular, this helps in characterizing monogenetic codes of length vector α with a gene of size 2 such that the space M_α is tidy. First we collect two results we will need.

Lemma 5.47. Let X be a $\mathbb{Z}_2$-space such that $\mathrm{ind}(X) = \mathrm{coind}(X) = n$ for $n \geq 1$. Then the homotopy group $\pi_n(X)$ is torsion free. In particular $\pi_n(X)$ has an infinite cyclic quotient.

Proof. Since the index and the coindex of X are the same we have

$$S^n \to X \to S^n$$

such that both the maps are $\mathbb{Z}_2$ equivariant. The composition gives us a self-map which is antipodal-preserving. Recall that such a map has odd degree. Therefore the induced map on $\pi_n(S^n)$ is a non-trivial homomorphism with infinite cyclic image that factors through $\pi_n(X)$. □

Lemma 5.48. Let $\langle\{b, n\}\rangle$ be the genetic code of a length vector α and $n \geq 6$. Then the universal cover of $\widetilde{M}_\alpha$ has the homotopy type of wedge of countably many spheres of dimension $n-4$.

Proof. A proof can be found in [39, Theorem B]. However, we sketch the steps here for the benefit of the reader. It follows from [32] that the moduli space, in this case, is homeomorphic to $\#_b(x^1 \times x^{m-1})$. The universal cover of a connected sum can be constructed by taking the universal cover of each summand minus an open ball and then gluing as dictated by the fundamental group. In the present case each summand is $S^1 \times S^{n-4}$ and the universal cover of this space minus an open ball is precisely $\mathbb{R} \times S^{n-4}$ with countably many open discs removed. The next step is take b copies of this punctured space and identify the boundary components according to the deck transformations. The fact that the resulting space is the universal cover and it has the homotopy type of wedge of countable copies of S^{n-4} is a bit long and technical to reproduce here. For the interested reader we provide one more reference [57]. □

Theorem 5.49. Let $\langle\{b, n\}\rangle$ be the genetic code of a length vector ℓ. Then M_ℓ is tidy if and only if b is an odd integer.

Proof. Since the genetic code is $\langle\{b, n\}\rangle$, the collection of gees is $\{\{b\}\}$ and hence

$$\{\emptyset, \{1\}, \{2\}, \ldots, \{b\}\}$$

is the set of all the subgees. For $i \neq j$, the relation (R2) of Theorem 2.36 implies that $V_iV_j = 0$ if $i \neq j$. Let $m = n - 3$ and $d \geq m$. Consider the subgee $S = \{1\}$ then subgees disjoint from S are $\emptyset, \{2\}, \{3\}, \ldots, \{b\}$. Using the relation (R3) of Theorem 2.36, we get $R^d + \sum_{i=2}^{b} R^{d-1}V_i = 0$. For the subgee $S = \{j\}$, $1 \leq j \leq b$, above equation gives:

$$R^d + \sum_{j\neq i=1}^{b} R^{d-1}V_i = 0. \tag{5.7}$$

Comparing these relations with each other, we get that

$$RV_1 = RV_2 = RV_3 = \cdots = RV_b.$$

Note that Equation (5.7) now implies $R^d = \sum_{i=2}^{b} R^{d-1}V_1 = (b-1)R^{d-1}V_1$.

For $d = n-3$, it implies

$$R^{n-3} = (b-1)R^{n-4}V_1. \tag{5.8}$$

By Remark 5.41, we see that $\mathrm{coind}(M_\ell) \geq n-4$. Also use of Equation (5.6) implies that

$$n-4 \leq \mathrm{coind}(M_\ell) \leq \mathrm{ht}(M_\ell) \leq \mathrm{ind}(M_\ell) \leq \dim(M_\ell) = n-3.$$

We now analyse Equation (5.8) further. Since $R^{m-1} \neq 0 \neq V_1$, $R^m = 0$ if and only if $b-1$ is 0, that is, $(b-1) \equiv 0 \pmod 2$, that is, b is odd. Thus for odd values of b, $R^m = 0$, implying that the Stiefel-Whitney height is not full. By [29, Proposition 2.2], we get that $\mathrm{ind}(M_{\ell}) = \mathrm{ht}(M_{\ell}) = m - 1 = \mathrm{coind}(M_{\ell})$. Hence M_{ℓ} is tidy whenever b is an odd integer.

Now let b be even. Then using [29, Proposition 2.2], we get that $\mathrm{ind}(M_{\ell}) = \mathrm{ht}(M_{\ell}) = m$. Therefore to find whether M_{ℓ} is tidy of not, it suffices to find if $\mathrm{coind}(M_{\ell})$ is $m - 1$ or m. We first observe that the space $M_{\ell} = \#_b(S^1 \times S^{m-1})$. If $m = 2$, then $M_{\ell} = \#_b(S^1 \times S^1)$ and hence is not tidy ((see [56, Section 5.3, page 100]).).

For $m > 4$, the universal cover $\tilde{M}_{\ell}$ of M_{ℓ} is homotopy equivalent to $\bigvee_{\mathbb{Z}} S^{m-1}$. Recall that for a topological space X, if $\mathrm{coind}(X) = \mathrm{ind}(X) = k$, then $\pi_k(X)$ has a $\mathbb{Z}$-summand. Since $m - 1 \leq \mathrm{coind}(M_{\ell}) \leq \mathrm{ind}(M_{\ell}) = m$, assume that $\mathrm{coind}(M_{\ell}) = m$. So $\pi_m(M_{\ell}) = \pi_m(\tilde{M}_{\ell})$ has a $\mathbb{Z}$-summand. Since the homotopy group of a wedge sum of spheres is a colimit of the homotopy groups in that dimension taken over all finite subsets, we have

$$
\begin{aligned}
\pi_m(\textstyle\bigvee_{\mathbb{Z}} S^{m-1}) &= \mathrm{colim}_{n \in F \subset \mathbb{Z}}\, \pi_m(\textstyle\bigvee_F S^{m-1}) \\
&= \mathrm{colim}_{n \in F \subset \mathbb{Z}} \textstyle\bigoplus_F(\pi_m(S^{m-1})) \\
&= \mathrm{colim}_{n \in F \subset \mathbb{Z}} \textstyle\bigoplus_F(\mathbb{Z}_2).
\end{aligned}
$$

Since $\pi_m(\tilde{M}_{\ell})$ does not have a $\mathbb{Z}$-summand, so does $\pi_m(M_{\ell})$ which contradicts the assumption that $\mathrm{coind}(M_{\ell}) = m$. Therefore, $m - 1 = \mathrm{coind}(M_{\ell}) < \mathrm{ind}(M_{\ell}) = m$. Hence M_{ℓ} is non-tidy for even values of b. $\square$

Remark 5.50. Let Σ_g be the orientable surface of genus g. Recall that, for $1 \leq g \leq 4$ if the genetic code of a length vector $\ell(g)$ is $\langle g, 5\rangle$ then $M_{\ell(g)} = \Sigma_g$. Therefore, $M_{\ell(g)}$ is tidy if and only if $g = 1, 3$.

Consider the following genetic codes:

1. $G_1 = \langle\{1, \ldots, n-4, n\}\rangle$
2. $G_2 = \langle\{1, \ldots, n-5, n-3, n\}\rangle$
3. $G_3 = \langle\{1, \ldots, n-5, n-2, n\}\rangle$
4. $G_4 = \langle\{1, \ldots, n-5, n-1, n\}\rangle$

Let M_{G_i} be the planar polygon corresponding to G_i. It follows from [32, Proposition 2.1] that $M_{G_i} = (S^1)^{n-2} \times M_{(i)}$ where the genetic code of (i) is $\langle i, 5\rangle$ for $1 \leq i \leq 4$. The free $\mathbb{Z}_2$-action on M_{G_i} is given by an Equation (1.3).

With the above notations, we have the following proposition.

Proposition 5.51. For $1 \leq i \leq 4$ the space M_{G_i} is tidy if and only if $i = 1, 3$.

Proof. Note that for each $1 \leq i \leq 4$, the projection $M_{G_i} \to M_{(i)}$ is a $\mathbb{Z}_2$-map. Therefore, $\mathrm{ind}(M_{G_i}) \leq \mathrm{ind}(M_{(i)})$. Since $M_{(i)} = {}_i$, $\mathrm{ind}(M_{G_i}) \leq 2$ for $1 \leq i \leq 4$. It is easy to see that

$$\mathrm{ind}(\ _i) = \begin{cases} 1 & \text{if } i \text{ is an odd integer,} \\ 2 & \text{if } i \text{ is an even integer.} \end{cases}$$

Therefore,

$$\mathrm{ind}(M_{G_i}) \leq \begin{cases} 1 & \text{if } i = 1, 3, \\ 2 & \text{if } i = 2, 4. \end{cases}$$

Let (G_i) be the length vector corresponding to the genetic code G_i. Note that $J = \{1, \ldots, n-4\}$ is a short subset with respect to each genetic code G_i. Therefore, we can reduce (G_i) to the length vector

$$(G_i, J) = (\ _{m-3},\ _{m-2},\ _{m-1},\ _n + \sum_{i=1}^{n-4}\ _i).$$

Observe that, $M_{(G_i, J)} = S^1$. Therefore, we have $\mathbb{Z}_2$-equivarient embedding $M_{(G_i, J)}$ of in M_{G_i}. This gives

$$1 \leq \mathrm{coind}(M_{G_i}), \text{ for } 1 \leq i \leq 4.$$

Now it is clear that M_{G_i} is tidy if $i = 1, 3$.

For $i = 2, 4$ we now prove that $\mathrm{coind}(M_{G_i}) = 1$ and $\mathrm{ind}(M_{G_i}) = 2$. It follows from [13, Theorem 2.3] that the $2 > \mathrm{ht}(M_{G_i}))$ for $i = 2, 4$. Therefore,

$$\mathrm{ind}(M_{G_i}) = 2 \text{ for } i = 2, 4.$$

Now it follows from Theorem 5.49 that

$$\mathrm{coind}(M_{G_i})) = 1 \text{ for } i = 2, 4.$$

Therefore, M_{G_i} is not tidy for $i = 2, 4$. This proves the proposition. □

5.3.3 Formula for the Stiefel-Whitney height

In this section, we deal with the n-length vectors that correspond to the genetic code having two genes with one gene of size 2. We begin with obtaining a formula for R^{n-3} when the genetic code is $\langle\{g_1, \ldots, g_k, n\}\{b, n\}\rangle$, where $g_1 < g_2 < \cdots < g_k < b$. This formula is the generalization of the Davis's formula [14, Theorem 1.4].

Let us fix the following notations for the sake of simplicity of writing.

Notations:

1. Let $\mathbb{Z}_{\geq 0}$ be the set of non-negative integers.

$$S_k = \left\{(b_1, \ldots, b_k) \in \mathbb{Z}^k_{\geq 0} : \sum_{j=0}^{i-1} b_{k-j} \leq i \text{ for } 1 \leq i \leq k\right\}.$$

2. Let $B = (b_1, \ldots, b_k) \in \mathbb{Z}^k_{\geq 0}$. Denote $|B| = \sum_{i=1}^{k} b_i$.

3. Let $J = \{j_1, \ldots, j_r\}$ be the set of distinct positive integers such that $j_i \leq h_k$ for $1 \leq i \leq r$. Let $\mu(J) = (\mu_1, \ldots, \mu_k)$ where,

$$\mu_i = |\{j \in J : g_{i-1} < j \leq g_i\}|$$

Throughout this section, $|A|$ denotes the cardinality of the set A, and we take $g_0 = 0$.

It is easy to see that J is a subgee dominated by $\{g_1, \ldots, g_k\}$ if and only if $\mu(J) \in S_k$. With the above notations, Davis proved the following result.

Theorem 5.52 ([14, Theorem 1.4]). Let $\langle\{g_1, \ldots, g_k, n\}\rangle$ be the genetic code of and $a_i = g_{k+1-i} - g_{k-i}$ for $1 \leq i \leq k$. Let $\phi : H^{n-3}(\overline{M}; \mathbb{Z}_2) \to \mathbb{Z}_2$ be the Poincare-duality isomorphism and J be a subgee of cardinality r. Then

$$\phi(R^{n-3-r} V_J) = \sum_{B} \prod_{i=1}^{k} \binom{a_i + b_i - 2}{b_i},$$

where

$$B \in \{(b_1,\ldots,b_k) \in \mathbb{Z}^k_{\geq 0} : \sum_{i=1}^{k} b_i = k - r,\ (b_1,\ldots,b_k) + (J) \in S_k\}$$

We extend the above result for genetic codes $\langle\{g_1,\ldots,g_k,n\}\{b,n\}\rangle$. In particular, we prove the following theorem.

Theorem 5.53. Let $\langle\{g_1,\ldots,g_k,n\}\{b,n\}\rangle$ be the genetic code of and $a_i = g_{k+1-i} - g_{k-i}$ for $1 \leq i \leq k$. Then

$$R^{n-3} = \sum_{B} \prod_{i=1}^{k} \binom{a_i + b_i - 2}{b_i} + (b - g_k),$$

where $|B| = k$ and $B \in S_k$.

Now we prove some results to prove Theorem 5.52. We need the following notations.

1. For a subgee $S = \{i_1,\ldots,i_k\}$, $V_S = V_{i_1} \cdots V_{i_k}$.
2. The collection $S_i = \{P \subseteq [n-1] : P \text{ is a subgee and } |P| = i\}$.
3. For a subgee S,
$$\mathrm{supp}(S) := \{s \in [b] : s \notin S \text{ and } S \cup \{s\} \text{ a subgee}\}.$$
4. For a subgee S and $s \in \mathrm{supp}(S)$,
$$S_i(S,s) = \{P \in S_i : S \subseteq P,\ s \notin P\}.$$
5. For $1 \leq i \leq k$, the set $(g_{i-1}, g_i] := \{j \in \mathbb{Z}_{>0} : g_{i-1} < j \leq g_i\}$ and $(g_k, b]$ defined similarly. We call these sets by blocks.

Proposition 5.54. Let P be a subgee and $\{s, s'\} \subseteq \mathrm{supp}(P)$. If s and s' are from same block then for each $t \in [k]$,

$$|S_t(P,s)| = |S_t(P,s')|.$$

Proof. Define $f : S(P,s) \to S(P,s')$ by

$$f(S) = \begin{cases} S & \text{if } s' \notin S, \\ (S \setminus \{s'\}) \cup \{s\} & \text{if } s' \in S. \end{cases}$$

It is easy to see that f is a bijection. □

The next result shows that the conclusion of Theorem 5.54 holds even if s, s' supp(P) are not in the same block, provided $|P| = k - 1$.

Proposition 5.55. If $P \in S_{k-1}$ and $\{s, s'\}$ supp(P) then

$$|S_k(P, s)| = |S_k(P, s')|.$$

Proof. We observe that both $P \cup \{s'\}$ and $P \cup \{s\}$ are subgees of cardinality k, and thus Now the proposition follows from the observation

$$S_k(P, s) \setminus (P \cup \{s'\}) = S_k(P, s') \setminus (P \cup \{s\}).$$

This proves the result. □

In [14, Corollary 1.6], the author proved that for the monogenetic code $\langle\{g_1, g_2, \ldots, g_k, n\}\rangle$ the image of $R^{m-|J|}V_J$ under (the Poincare duality isomorphism) is 1 whenever the subgee J is of cardinality k. Observe that the same proof works for any genetic code whenever J is of maximum cardinality. Here we reproduce the proof in our case for completeness.

Theorem 5.56 ([14, Corollary 1.6]). Let $\langle\{g_1, \ldots, g_k, n\}\{b, n\}\rangle$ be the genetic code of a length vector and J be a subgee of cardinality k. Then $(R^{m-|J|}V_J) = 1$.

Proof. Observe that the relation (R2) of Theorem 2.36 gives, $V_J \neq 0$. By Poincare duality, there must be an $X = \sum_i X_i \in H^{m-|J|}(\overline{M}\ ; Z_2)$ such that $X\ V_J = 1$ and thus $(X\ V_J) = 1$. Since J is maximal, if X_i contains a factor V_t such that $t \notin J$, then $X_i\ V_J = 0$. Now using the relation (R1) of Theorem 2.36, we can replace V_t^2 by RV_t, whenever $t \in J$. Therefore if $X_i\ V_J \neq 0$, then each X_i can be replaced by $R^{m-|J|}$. Since $(X\ V_J) = 1$, number of such X_i's must be odd. This proves the theorem. □

For a generic length vector $= (\ _1, \ldots, \ _n)$, $m = n - 3$ is the dimension of M . The following result is an important tool to compute R^m.

Lemma 5.57. Let P be a nonempty subgee and $s \in$ supp(P). Then

$$R^{m-|P|} = \sum_{|P|+1 \leq t \leq k} \left(\sum_{S \in S_t,\, P \subseteq S \text{ and } s \notin S} R^{m-|S|} V_S \right).$$

Before we prove Lemma 5.57, we make a few observations which play a critical role in its proof. For a nonempty subgee P, let E_P denotes the equation obtained from the relation (R3) of Theorem 2.36, i.e.,

$$E_P : \sum_{T \cap P = \emptyset} R^{m-|T|} V_T = 0. \tag{5.9}$$

Remark 5.58. If $P \neq \emptyset$ and $s \in \mathrm{supp}(P)$, then adding E_P and $E_{P \cup \{s\}}$, we get the following equation

$$E_P + E_{P \cup \{s\}} : \sum_{T \cap P = \emptyset,\ s \in T} R^{m-|T|} V_T = 0. \tag{5.10}$$

Remark 5.59. Let $|P| > 2$, $s' \in P$ and $s \in \mathrm{supp}(P)$. Then adding $E_P + E_{P \cup \{s\}}$ and $E_{P \setminus \{s'\}} + E_{(P \setminus \{s'\}) \cup \{s\}}$, we get the following

$$R^{m-2} V_{\{s,s'\}} = \sum_{\substack{T \cap (P \setminus \{s'\}) = \emptyset \\ \{s,s'\} \subseteq T}} R^{m-|T|} V_T. \tag{5.11}$$

Proof of Lemma 5.57. We prove this using a binary tree representation. Given a subgee $P = \{i_1, i_2, \ldots, i_k\}$ and $s \in \mathrm{supp}(P)$, we construct a binary tree B_k of height k as follows. For $r \in \{0, 1, \ldots k-1\}$ all the vertices of B_k at depth r are names as i_r. To label a vertex at depth k, we proceed as follows: Since B_k is a binary tree, each non-leaf vertex has two children, say a left child and a right child. For any leaf vertex, there is a unique path that connects the root to this leaf vertex. Consider this path, take an empty set. We start moving from the root vertex on this path. If on the path, we move to the left child from the vertex, then we include this vertex in the set; if we move to the right child instead then we do not include this vertex (see Figure 5.1 for the illustration when $k = 3$.).

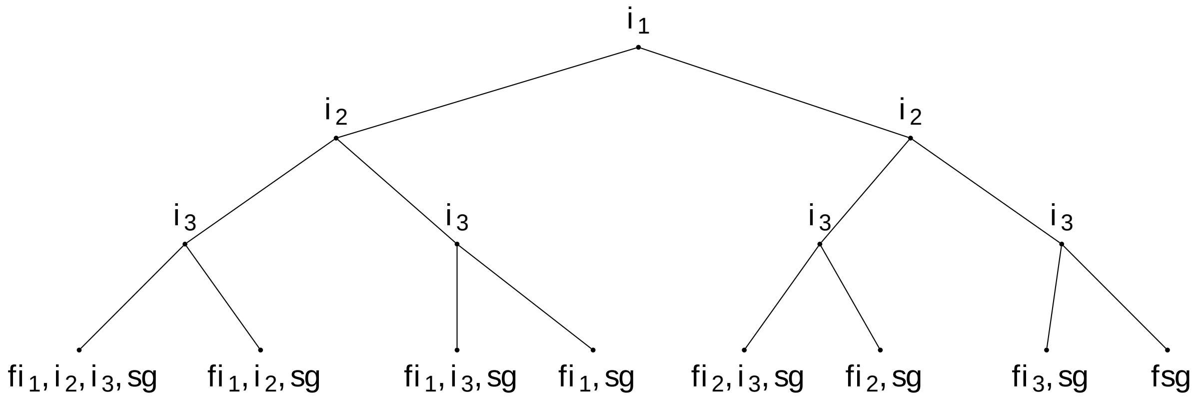


Figure 5.1: Tree B_k, for $k = 3$ before relabeling

We note that $s \in \mathrm{supp}(P)$ implies that $P \cup \{s\}$ is a subgee, and hence every subset of it is also a subgee. We now associate an equation to every vertex of B_k starting from the leaf vertices. If a leaf vertex corresponds to a subset, say A of $P \cup \{s\}$ then the equation associated with the leaf A is E_A (see Equation (5.9)). We now associate equations to the parents of leaves by adding the equation associated with their children. We do this recursively in the decreasing order of the depth of the vertices by using Remark 5.58 and Remark 5.59. Finally, we see that the equation corresponding to the root vertex is

$$R^{m-|P|} = \sum_{|P|+1 \leq t \leq k} \left(\sum_{S \in S_t,\, P \subseteq S \text{ and } s \notin S} R^{m-|S|} V_S \right).$$

□

Corollary 5.60. If $P \in S_{k-1}$ and $s, s' \in \mathrm{supp}(P)$ then $R^{m-k} V_{P \cup \{s\}} = R^{m-k} V_{P \cup \{s'\}}$.

Proof. The proof follows from the Theorem 5.57 and the observation

$$S_k(P, s) \setminus (P \cup \{s'\}) = S_k(P, s') \setminus (P \cup \{s\}).$$

□

Proposition 5.61. If $S, S' \in S_k$ then $R^{m-k} V_S = R^{m-k} V_{S'}$.

Proof. Let $T = \{g_1, \ldots, g_k\}$ be the gee of size k. We will show that for any $S \in S_k$, $R^{m-k}V_S = R^{m-k}V_T$. Let $S = \{i_1, \ldots, i_k\}$ where $i_1 < \cdots < i_k$. It follows from the Corollary 5.3.3 that

$$R^{m-k}V_T = R^{m-k}V_{\{g_1,\ldots,g_{k-1},i_k\}}.$$

Similarly

$$R^{m-k}V_{\{g_1,\ldots,g_{k-1},i_k\}} = R^{m-k}V_{\{g_1,\ldots,g_{k-2},i_{k-1},i_k\}}.$$

By continuing this way we get

$$R^{m-k}V_{\{g_1,i_2,\ldots,i_k\}} = R^{m-k}V_S.$$

Therefore, for any two subgees of size k, the above equation holds thereby proving the result. $\square$

Lemma 5.62. Let P be a subgee with $|P| = r \leq k-1$ and $s, s' \in \mathrm{supp}(P)$. Then

$$R^{m-r-1}V_{P \cup \{s\}} = R^{m-r-1}V_{P \cup \{s'\}},$$

whenever s and s' are in the same block.

Proof. The proof is by induction. Note that the Corollary 5.3.3 implies that the result is true for $k > t$. We now prove the lemma for subgees of size t. Let $|P| = t$ and $s, s' \in \mathrm{supp}(P)$. Usng Theorem 5.57, we get

$$\sum_{t+1 \leq i \leq k} \left(\sum_{S \in S_i, P \subseteq S \text{ and } s \notin S} R^{m-i}V_S \right) = \sum_{t+1 \leq j \leq k} \left(\sum_{S \in S_j, P \subseteq S \text{ and } s' \notin S} R^{m-j}V_S \right).$$

This gives

$$\sum_{t+1 \leq i \leq k} \left(\sum_{S \in S_i(P,s)} R^{m-i}V_S \right) = \sum_{t+1 \leq j \leq k} \left(\sum_{S \in S_j(P,s')} R^{m-j}V_S \right).$$

Using induction and Theorem 5.54 we get

$$\sum_{S \in S_{t+1}(P,s)} R^{m-t-1}V_S = \sum_{S \in S_{t+1}(P,s')} R^{m-t-1}V_S.$$

The result now follows Proposition 5.55. $\square$

Proposition 5.63. Let $G = \langle \{g_1, \dots, g_k\} \mid \{b, n\} \rangle$ and $G' = \langle \{g_1, \dots, g_k\} \rangle$ be the genetic codes of and , respectively. Let P be a nonempty subgee such that $\mathrm{supp}(P) \neq \emptyset$. Then the value of $R^{m-|P|}V_P$ under the corresponding Poincare duality isomorphisms from $H^m(\overline{M}, Z_2)$ and $H^m(\overline{M}, Z_2)$ remains same.

Proof. Observe that if $|P| = k$, then the result follows from Theorem 5.56. Now we assume that $|P| < k$. In this case, P is dominated by $\{g_1, \dots, g_k\}$. Note that the collections of subgees of cardinality greater than 1 with respect to both G and G' coincides. Moreover, the equations in Lemma 5.57 also remain unchanged for both the genetic codes. Therefore, Theorem 5.56 implies the result. $\square$

The following is now an immediate consequence of the above result and Theorem 5.52.

Corollary 5.64. Let $\langle \{g_1, \dots, g_k, n\} \mid \{b, n\} \rangle$ be the genetic code of and $a_i = g_{k+1-i} - g_{k-i}$ for $1 \leq i \leq k$. Let P be a subgee of cardinality $r \in \{1, \dots, k-1\}$ dominated by $\{g_1, \dots, g_k\}$. Then

$$(R^{m-r}V_P) = \sum_{B} \prod_{i=1}^{k} \binom{a_i + b_i - 2}{b_i},$$

where

$$B \in \{(b_1, \dots, b_k) \in Z^k_{\geq 0} : \sum_{i=1}^{k} b_i = k - r, \text{ and } (b_1, \dots, b_k) + (P) \in S_k\}$$

We now have all the necessary machinery for Theorem 5.53. So we next see its proof.

Proof of the Theorem 5.53. Without loss of generality, we omit the powers of R just to simplify the notations. Comparing $E_{\{g_k\}}$ with $E_{\{b\}}$ we get.

$$V_b = V_{g_k} + \sum_{\substack{|S| \geq 2 \\ g_k \in S}} V_S. \tag{5.12}$$

Let $P = \{g_k\}$ and $s = g_{k-1}$. Then Lemma 5.57 gives

$$V_{g_k} = \sum_{\substack{|S| \geq 2 \\ g_k \in S,\ g_{k-1} \notin S}} V_S.$$

Comparing it with Equation (5.12), we get

$$V_b = V_{\{g_k,g_{k-1}\}} + \sum_{\substack{|S|\geqslant 3 \\ \{g_k,g_{k-1}\}\subseteq S}} V_S. \tag{5.13}$$

Recall that $a_j = g_{k+1-j} - g_{k-j}$ for $1 \leqslant j \leqslant k$, therefore Equation (5.13) can be rewritten as

$$V_b = V_{\{g_k,g_{k-1}\}} + \sum_{i=1}^{k-2}\left(\sum_{\substack{C+(1,1,0,\ldots,0)\in S_k \\ C_+=i}}\left(\prod_{j=3}^{k}\binom{a_j}{c_j} V_{g_k}V_{g_{k-1}}V_{g_{k-j+1}}V_{g_{k-j+1}-1}\cdots\right)\right).$$

Allowing $C_+ = 0$ enables us to rewrite the above equation as

$$V_b = \sum_{i=0}^{k-2}\left(\sum_{\substack{C+(1,1,0,\ldots,0)\in S_k \\ C_+=i}}\left(\prod_{j=3}^{k}\binom{a_j}{c_j} V_{g_k}V_{g_{k-1}}V_{g_{k-j+1}}V_{g_{k-j+1}-1}\cdots\right)\right). \tag{5.14}$$

From Corollary 5.64 we get,

$$\begin{aligned} V_b &= \sum_{i=0}^{k-2}\left(\sum_{\substack{C+(1,1,0,\ldots,0)\in S_k \\ C_+=i}}\left(\prod_{j=3}^{k}\binom{a_j}{c_j} \sum_{\substack{B+C+(1,1,0,\ldots,0)\in S_k \\ B_+=k-2-i}} \prod_{q=1}^{k}\binom{a_q+b_q-2}{b_q}\right)\right) \\ &= \sum_{C+(1,1,0,\ldots,0)\in S_k}\left(\prod_{j=3}^{k}\binom{a_j}{c_j} \sum_{\substack{B+C+(1,1,0,\ldots,0)\in S_k \\ B_+=k-2-C_+}} \prod_{q=1}^{k}\binom{a_q+b_q-2}{b_q}\right). \end{aligned}$$

For $P = \{1\}$ the relation (R3) of Theorem 2.36 gives

$$R^m = \sum_{\substack{|S|\geqslant 1 \\ 1\notin S}} V_S = \left(\sum_{\substack{|S|\geqslant 1 \\ \max(S)\leqslant g_k,\ 1\notin S}} V_S\right) + (b-g_k)V_b. \tag{5.15}$$

Note that $C+(1,1,0,\ldots,0)\in S_k$ implies $C=(0,0,c_3,\ldots,c_k)$ and $\binom{a_1}{0}\binom{a_2}{0}=1$. Therefore, we can rewrite V_b as

$$V_b = \sum_{C+(1,1,0,\ldots,0)\in S_k}\left(\prod_{j=1}^{k}\binom{a_j}{c_j}\sum_{\substack{B+C+(1,1,0,\ldots,0)\in S_k\\ B_+=k-2-C_+}}\prod_{q=1}^{k}\binom{a_q+b_q-2}{b_q}\right).$$

Let $T=B+C$. Then it is easy to see that

$$V_b = \sum_{\substack{T+(1,1,0,\ldots,0)\in S_k\\ |T|=k-2}}\left(\sum_{B\leq T}\prod_{j=1}^{k}\binom{a_j}{t_j-b_j}\binom{a_j+b_j-2}{b_j}\right).$$

Let $\tilde{S}_{k-2}=\{0\}\times\{0\}\times S_{k-2}$. Now using binomial identity $\binom{a_j+b_j-2}{b_j}\equiv\binom{1-a_j}{b_j}$ we get,

$$V_b = \sum_{\substack{T\in\tilde{S}_{k-2}\\ |T|=k-2}}\left(\prod_{j=1}^{k}\sum_{b_j}\binom{a_j}{t_j-b_j}\binom{1-a_j}{b_j}\right).$$

Now the Vandermonde identity $\binom{m+n}{r}=\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$ gives

$$V_b = \sum_{\substack{T\in\tilde{S}_{k-2}\\ |T|=k-2}}\prod_{j=1}^{k}\binom{1}{t_j}.$$

Note that $(0,0,1,\ldots,1)$ is the only possibile choice for T in the above equation. Therefore, $V_b=1$.

Hence, using Corollary 5.64 and Equation (5.15) we get that

$$R^m = \sum_{\substack{B\in S_k\\ B_+=k}}\prod_{i=1}^{k}\binom{a_i+b_i-2}{b_i}+(b-g_k).$$

This completes the proof of Theorem 5.53.

□

The following is now a straightforward corollary of the above result.

Corollary 5.65. Assuming the notations introduced in the proof of the previous result, and let $\langle\{g_1, \ldots, g_k, n\}\{b, n\}\rangle$ be the genetic code of ℓ. If

$$\sum_{|B|=k,\ B\in S_k} \prod_{i=1}^{k} \binom{a_i+b_i-2}{b_i} + (b-g_k) \equiv 0 \pmod 2,$$

then M_ℓ is tidy.

Proof. By Theorem 5.53, we have that

$$R^m = \sum_{|B|=k,\ B\in S_k} \prod_{i=1}^{k} \binom{a_i+b_i-2}{b_i} + (b-g_k) = 0.$$

Since the size of the smallest gee in $\langle\{g_1, \ldots, g_k, n\}\{b, n\}\rangle$ is 1, Theorem 5.46 implies that

$$n-4 \leq \mathrm{coind}(M_\ell) \leq \mathrm{ht}(M_\ell) \leq \mathrm{ind}(M_\ell) \leq n-3.$$

Using [29, Proposition 2.2], the height of the Stiefel-Whitney class is full if and only if the index is full. Here $R^m = 0$, thus $\mathrm{ind}(M_\ell)$ is not full. In particular, $\mathrm{ind}(M_\ell) \leq n-4$. This proves the result. □

Example 5.66. Suppose the genetic code of a length vector is $\langle\{2, 4, n\}\{b, n\}\rangle$ and $b \geq 5$. Note that the collection of subgees is

$$\{\emptyset, \{1\}, \{2\}, \ldots, \{b-1\}, \{b\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}\}. \tag{5.16}$$

For a subgee $\{1,2\}$ the relation (R3) of Theorem 2.36 gives

$$R^d = \sum_{i=3}^{b} R^{d-1} V_i.$$

Now the Proposition 5.61 and Equation (5.16) together gives

$$R^m = (b-2) R^{m-1} V_b. \tag{5.17}$$

Now we compute the value of R^m using Theorem 5.53. Note that only possible values of $B \in S_2$ are $(1,1)$ and $(2,0)$. Therefore, Theorem 5.53 gives

$$R^{n-3} = \binom{1}{1}\binom{1}{1} + \binom{0}{0}\binom{2}{2} + b - 4 = b - 2,$$

as desired.

5.3.4 The case of quasi-equilateral polygon spaces

The planar polygon space M_ℓ associated with the length vector $\ell = (1,\ldots,1,r)$ is called a quasi-equilateral planar polygon space. Suppose r is a natural number. Then it is easy to see that ℓ is generic if both r and n have the same parity. Moreover, if $r > n-1$ then $M_\ell = \emptyset$. We make the following observations.

1. If $r = n-2$ then the genetic code of ℓ is $\langle\{n\}\rangle$. It follows from [32, Example 2.6] $M_\ell = S^{n-3}$ and $\overline{M}_\ell = \mathbb{R}P^{n-3}$.
2. If $r = n-4$ then the genetic code of ℓ is $\langle\{n-1,n\}\rangle$. The [32, Example 2.12] gives $M_\ell = (n-1)\sharp(S^1 \times S^{n-4})$ and $\overline{M}_\ell = n\sharp\mathbb{R}P^{n-3}$. Here $\sharp_n X$ represents the connected sum of n copies of the space X.

From Theorem 2.36 the cohomology ring of $H^*(\overline{M}_\ell;\mathbb{Z}_2)$ for $\ell = (1,\ldots,1,r)$ is generated by classes $R, V_1, V_2, \ldots, V_{n-1} \in H^1(\overline{M}_\ell;\mathbb{Z}_2)$ subject to the following relations:

(R1) $RV_i + V_i^2 = 0$, for $i \in [n-1]$,

(R2) $V_S = 0$ if $|S| > \frac{n-r}{2}$,

(R3) For $L \subseteq [n-1]$ such that $|L| > \frac{n+r}{2}$,

$$\sum_{S \subset L} R^{|L-S|-1} V_S = 0.$$

Recall that the class R coincides with the first Stiefel-Whitney class of the double cover $M_\ell \to \overline{M}_\ell$. Kamiyama [44] computed the height of R in terms of the values of n and r. Before stating this result, we recall the following notations from [44].

Notations:

1. $D(n) = n-2$, $e(n,r) = \frac{n-r}{2} - 1$. Note that $e(n,r)$ is the largest size of the gee (Here it is smallest as well since the genetic code has a single gene).
2. $k(n,r) = \max\left\{ i : 0 \leqslant i \leqslant e(n,r)-1,\ \binom{D(n)-e(n,r)+i}{i} \equiv 1 \pmod 2 \right\}$.

3.

$$(n,r) = \begin{cases} n-3, & \text{if } \frac{D(n)}{e(n,r)} \equiv 1 \pmod 2, \\ \frac{n+r}{2} + k(n,r) - 2, & \text{if } \frac{D(n)}{e(n,r)} \equiv 0 \pmod 2. \end{cases}$$

Theorem 5.67 ([44, Theorem A]). Let $h(n,r)$ be the height of a class R. Then for all $n > 4$ and $r \in \mathbb{N}$ such that $r \leq n-2$ with same parity as n, $h(n,r) = (n,r)$.

We use Theorem 5.67 to decide for which values of n and r, the corresponding M is tidy. Note that the cases $r = n-2$ and $r = n-4$ have already been discussed. Thus, here we consider the cases $r = n-6$ and $r = n-8$.

Case 1: Let $r = n-6$.

For $r = n-6$, $D(n) = n-2$, $e(n, n-6) = 2$, and

$$k(n, n-6) = \max \left\{ i : 0 \leq i \leq 1, \binom{n-4+i}{i} \equiv 1 \pmod 2 \right\}.$$

Note that if n is an odd integer, then $n-4+1$ is even. Moreover, if n is an even integer, then $n-4+1$ is odd. In particular,

$$k(n, n-6) = \begin{cases} 0, & \text{if } n \text{ is an odd integer}, \\ 1, & \text{if } n \text{ is an even integer}. \end{cases} \tag{5.18}$$

Therefore,

$$(n, n-6) = \begin{cases} n-3, & \text{if } \frac{n-2}{2} \equiv 1 \pmod 2, \\ n-5+k(n,n-6), & \text{if } \frac{n-2}{2} \equiv 0 \pmod 2. \end{cases} \tag{5.19}$$

Remark 5.68. Since $e(n, n-6) = 2$, it follows from Lemma 5.46 that $n-5 \leq \operatorname{coind}(M)$. Equation 5.18 and Equation (5.19) gives, if $\frac{n-2}{2}$ is even and n is an odd integer then $\operatorname{coind}(M) = \operatorname{ht}(M) = n-5$.

Since $n-5 \leq \operatorname{coind}(M)$, the $\operatorname{ht}(M)$ may take values $n-5$, $n-4$ and $n-3$. We classify the values of n for which $\operatorname{ht}(M) = n-5$. Observe that for odd values of n, $\operatorname{ht}(M) = n-5$ if and only if $\frac{n-2}{2} \equiv 0 \pmod 2$. Therefore, we need to find values of n for which $\frac{(n-2)(n-3)}{2} \equiv 0 \pmod 2$. Since n is odd, $n-2$ is odd and $n-3$ is even. We have, $\frac{(n-2)(n-3)}{2} = 2k$ for some $k \in \mathbb{Z}$. Let $n-3 = 2l$ for some $l \in \mathbb{Z}$. This gives

$(n-2)l = 2k$. Since $n-2$ is odd, l has to be even. Therefore, $n-3 = 4r'$ for some r'. The above discussion proves that $\mathrm{coind}(M_\alpha) = \mathrm{ht}(M_\alpha) = n-5$ if and only if $n = 4r+3$.

Now we classify the values of n for which $\mathrm{ht}(M_\alpha) = n-4$. Observe that $\mathrm{ht}(M_\alpha) = n-4$ if and only if n is even and $\frac{n-2}{2} \equiv 0 \pmod 2$. Similar calculations as above tell us that for an even n, $\frac{n-2}{2} \equiv 0 \pmod 2$ if and only if $n = 4r'+2$.

We now classify the values of n for which $\mathrm{ht}(M_\alpha) = n-3$. Note that $\mathrm{ht}(M_\alpha) = n-3$ if and only if $\frac{(n-2)(n-3)}{2} \equiv 1 \pmod 2$. Therefore, $\frac{(n-2)(n-3)}{2} = 2k+1$ for some k. We first consider the case when n is even, i.e., $n-3$ is odd and $n-2$ is even. In this case, $(n-3)\frac{n-2}{2} = 2k+1$ implies that $\frac{n-2}{2}$ is odd. Suppose $n = 2k'$ for some k', then $(n-3)(k'-1) = 2k+1$. Therefore k' is even implying that $n = 4r'$ for some r'. Now if n is odd, then similar arguments give that $n = 4r'+1$ for some r. Consequently, $\mathrm{ht}(M_\alpha) = n-3$ if and only if either $n = 4r'$ or $n = 4r'+1$.

The above discussion is summarized in the following.

Proposition 5.69. Let $\alpha = (1, \ldots, 1, n-6)$ be an n-length vector. Then,

$$\mathrm{ht}(M_\alpha) = \begin{cases} n-3, & \text{if } n = 4r' \text{ or } 4r'+1, \\ n-4, & \text{if } n = 4r'+2, \\ n-5 & \text{if } n = 4r'+3. \end{cases}$$

Case 2: Let $r = n-8$.

Since $e(n, n-8) = 3$, $n-6 \leq \mathrm{coind}(M_\alpha)$, and

$$k(n, n-8) = \max\left\{ i : 0 \leq i \leq 2, \binom{n-5+i}{i} \equiv 1 \pmod 2 \right\}.$$

Note that $k(n, n-8)$ may take values 0, 1 or 2. We begin with classifying values of n for which $k(n, n-8) = 2$. Observe that $k(n, n-8) = 2$ if and only if $\binom{n-3}{2} \equiv 1 \pmod 2$. Therefore, we find values of n such that $\frac{(n-4)(n-3)}{2} \equiv 1 \pmod 2$. Firstly, consider $n = 2k$. Then $n-3$ is odd and $\frac{(2k-4)(n-3)}{2} = 2r'+1$ for some r'. This implies that $(k-2)(n-3) = 2r'+1$ and $k-2 = 2k'+1$ for some k'. Hence, $n = 4t+2$ for some t. Secondly, consider the case $n = 2k+1$. Then $n-3 = 2l$ for some l. Since $\frac{(n-4)(n-3)}{2} = 2r'+1$ for some r', we get $(n-4)l = 2r'+1$. Note that l must be odd as

$n-4$ is odd. Therefore, $n = 4t+1$. Consequently, $k(n,n-8) = 2$ if and only if either $n = 4t+1$ or $n = t+2$.

Now observe that $k(n,n-8) < 2$ if and only if $\binom{n-3}{2} \equiv 0 \pmod 2$. Therefore, we need to find the values of n for which $\frac{(n-4)(n-3)}{2} = 2r'$, for some r'. Fisrt consider the case when n is even, i.e., $n = 2k$. Then $n-3$ is odd and $(k-2)(n-3) = 2r'$, and hence k is even. This implies that $n = 4t$ for some t. Now let $n = 2k+1$. Then $n-3$ is even and $(n-4)(k-1) = 2r'$. Since $n-4$ is odd, $k-1$ is even. Therefore, k is odd implying that $n = 4t+3$ for some t. We conclude that $k(n,n-8) < 2$ if and only if either $n = 4t$ or $n = 4t+3$. Moreover, it is easy to see that $\binom{n-4}{1} \equiv 0 \pmod 2$ if and only if n is odd. Therefore, we get that

$$k(n,n-8) = \begin{cases} 0, & \text{if } n = 4t+3, \\ 1, & \text{if } n = 4t, \\ 2, & \text{if } n = 4t+1 \text{ or } n = 4t+2. \end{cases} \tag{5.20}$$

Thus,

$$(n,n-8) = \begin{cases} n-3, & \text{if } \binom{n-2}{3} \equiv 1 \pmod 2, \\ n-6+k(n,n-6), & \text{if } \binom{n-2}{3} \equiv 0 \pmod 2. \end{cases} \tag{5.21}$$

Since $n-6 \leq \text{coind}(M)$, the $\text{ht}(M)$ may take values $n-6$, $n-5$, $n-4$ or $n-3$. From Equation (5.20) and Equation (5.21), for odd values of n, $\text{ht}(M) \in \{n-4, n-6\}$ if and only if $\binom{n-2}{3} \equiv 0 \pmod 2$. Note that $\frac{(n-4)(n-3)(n-2)}{6} = 2k$ if and only if either 3 divides $n-2$ or 3 divides $n-4$ and 2 divides $n-3$. This implies that $\frac{(n-4)(n-3)(n-2)}{6} = 2k$ if and only if either $n = 3r_1+2$ or $3r_2+4$ and $n = 2r_3+3$ for some r_1, r_2, r_3. Since n is odd, both r_1 and r_2 must be odd. Therefore, if $n = 6k+1$ or $6k+5$ then

$$\text{ht}(M) = \begin{cases} n-6, & \text{if } k \text{ is odd}, \\ n-4, & \text{if } k \text{ is even}. \end{cases}$$

It is easy to see that $\text{ht}(M) = n-3$ if $n = 6k \pm 3$.

Similarly, for even values of n, we obtain that $\frac{(n-4)(n-3)(n-2)}{6} = 2k$ if and only if 3 divides $n-3$. Since n is even, $n = 6k$ for some k. Therefore,

$$\operatorname{ht}(M_\alpha) = \begin{cases} n-5, & \text{if } k \text{ is even}, \\ n-4, & \text{if } k \text{ is odd}. \end{cases}$$

Note that if $n = 6k+2$ or $6k+4$ then $\operatorname{ht}(M_\alpha) = n-3$.

The above discussion of the quasi-equilateral case for $\alpha = (1,\ldots,1,n-8)$ has been summarized in the following.

Proposition 5.70. Let $\alpha = (1,\ldots,1,n-8)$ be an n-length vector. Then,

$$\operatorname{ht}(M_\alpha) = \begin{cases} n-3, & n = 6k \pm 3 \text{ or } n = 6k \pm 2, \\ n-4, & n = 6k \text{ and } k \text{ odd, or } n = 6k \pm 1 \text{ and } k \text{ even}, \\ n-5 & n = 6k,\ k \text{ is even}, \\ n-6 & n = 6k \pm 1,\ k \text{ is odd} \end{cases}$$

We now consider quasi-equilateral polygon spaces for $r = 1$ and 2.

1. Let $r = 1$. Observe that for α to be generic, n must be odd.

   (a) Consider $n = 2^{s+1} - 1$.

   In this case, the genetic code of α is $\langle\{2^s+1,\ldots,2^{s+1}-1\}\rangle$ and the size of the gee is $2^s - 2$. Therefore, using Lemma 5.46 and [44, Proposition C] we get the following inequality

   $$2^s - 2 \leq \operatorname{coind}(M_\alpha) \leq \operatorname{ht}(M_\alpha) = 2^s - 2.$$

   Therefore, $\operatorname{coind}(M_\alpha) = \operatorname{ht}(M_\alpha) = 2^s - 2$.

   (b) Consider $n = 2^s + 1$.

   From [44, Proposition C], we have $\operatorname{ht}(M_\alpha) = 2^s - 2$. Here, the genetic code of α is $\langle\{2^{s-2}+2,\ldots,2^s+1\}\rangle$ and the size of gee is $2^{s-1} - 1$. Therefore,

   $$2^{s-1} - 1 \leq \operatorname{coind}(M_\alpha) \leq \operatorname{ht}(M_\alpha) = 2^s - 2.$$

2. Let $r = 2$. Observe that for to be generic, n must be even.

   (a) Consider $n = 2^{s+1} - 2$.

   From [44, Proposition C], we have $\mathrm{ht}(M) = 2^s - 2$. The genetic code of in this case is $\langle\{2^s + 1, \ldots, 2^{s+1} - 2\}\rangle$ and the size of gee is $2^s - 3$. Therefore,

$$2^s - 2 \leq \mathrm{coind}(M) \leq \mathrm{ht}(M) = 2^s - 2.$$

   Therefore, $\mathrm{coind}(M) = \mathrm{ht}(M) = 2^s - 2$.

   (b) Consider $n = 2^s$.

   From [44, Proposition C], we know that $\mathrm{ht}(M) = 2^s - 3$. Therefore, $\mathrm{ind}(M) = 2^s - 3$. Here, the genetic code of is $\langle\{2^{s-1} + 2, \ldots, 2^s\}\rangle$. Since the size of gee is $2^{s-1} - 2$, we have the following inequality

$$2^{s-1} - 1 \leq \mathrm{coind}(M) \leq \mathrm{ht}(M) = 2^s - 3.$$

Genetic code	Coindex	Height	Index	T/NT	BUT
$\langle n\rangle$	$n - 3$	$n - 3$	$n - 3$	T	Y
$\langle\{b, n\}\rangle$ b is odd b is even	 $n - 4$ $n - 4$	 $n - 4$ $n - 3$	 $n - 4$ $n - 3$	 T NT	 Y Y
$\langle\{2^s + 1, \ldots, 2^{s+1} - 1\}\rangle$	$2^s - 2$	$2^s - 2$	$\leq 2^{s+1} - 4$	?	Y
$\langle\{2^{s-1} + 2, \ldots, 2^s + 1\}\rangle$	$2^{s-1} - 1 \leq$	$2^s - 2$	$2^s - 2$	?	Y
$\langle\{2^s + 1, \ldots, 2^{s+1} - 2\}\rangle$	$2^s - 2$	$2^s - 2$	$\leq 2^{s+1} - 6$	?	Y
$\langle\{2^{s-1} + 2, \ldots, 2^s\}\rangle$	$2^{s-1} - 1 \leq$	$2^s - 3$	$2^s - 3$	?	Y
$\langle\{1, 2, \ldots, n - 4, n\}\rangle$	1	1	1	T	Y
$\langle\{1, 2, \ldots, n - 5, n - 2, n\}\rangle$	1	1	1	T	Y
$\langle\{1, 2, \ldots, n - 5, n - 3, n\}\rangle$	1	2	2	NT	Y
$\langle\{1, 2, \ldots, n - 5, n - 1, n\}\rangle$	1	2	2	NT	Y

Table 5.4: Tidyness of Polygon spaces M

Value of r	Condition on n	Coindex	Height	Index	T/NT	BUT
$n - 2$	-	$n - 3$	$n - 3$	$n - 3$	T	Y
$n - 4$	n even	$n - 4$	$n - 4$	$n - 4$	T	Y
	n odd	$n - 4$	$n - 3$	$n - 3$	NT	Y
$n - 6$	$n = 4k + 3$	$n - 5$	$n - 5$	$6\ n - 4$	?	Y
	$n = 4k + 2$	$n - 5\ 6$	$n - 4$	$n - 4$	?	Y
	$n = 4k$ or $4k + 1$	$n - 5\ 6$	$n - 3$	$n - 3$	?	Y
$n - 8$	$n = 6k\quad 1$, k odd	$n - 6$	$n - 6$	$6\ n - 4$	?	Y
	$n = 6k$, k even	$n - 6\ 6$	$n - 5$	$6\ n - 4$	?	Y
	$n = 6k$, k odd or $n = 6k\quad 1$, k even	$n - 6\ 6$	$n - 4$	$n - 4$	?	Y
	$n = 6k\quad 2$ or $6k\quad 3$	$n - 6\ 6$	$n - 3$	$n - 3$	?	Y

Table 5 .5: Tidyness of Polygon spaces $M_{(1,1,:::,1,r)}$

Concluding remarks :

In most of the results of this paper, we have found the exact value of the index for various planar polygon spaces. However, in some cases, we could only nd a lower bound for the index (for instance, Proposition 5.69 and Proposition 5.70). It would be interesting to see if in those cases the height is equal to the index.

In Lemma 5.46, we showed that $\mathrm{coind}(M\ ) > n - k - 3$ for any generic n-length vector with the smallest gee of size k. Based on our computations, we believe that this bound is tight. Therefore, we conjecture the following.

Conjecture 5.71. Let be a generic n-length vector. If the size of the smallest gee is k, then

$$\mathrm{coind}(M\ ) = n - k - 3.$$

In the proof of Lemma 5.49, we showed that the Conjecture 5.71 is true for $= fb, ngif\ n \ 6\ 6.$

Chapter 6

Future directions

6.1 Classifying all aspherical planar polygon spaces

Recall that the chain spaces form a subclass of planar polygon spaces. We classified length vectors whose corresponding chain space is aspherical in Chapter 3. We would like to respond to Question 1.1 by characterizing genetic codes in such a way that the corresponding planar polygon space is aspherical. This is an ongoing project.

6.2 Multi-branched chains

Recall that a chain is a peicewise linear path that terminates on a line parallel to the Y-axis. Here, we introduce a generalized version of a chain called a multi-branched chain (see Figure 6.1).

Notations:

1. For each $0 \leq u \leq k$ and $1 \leq t \leq i_u$, we set

$$\vec{v}_t^u = (v_1^u, v_2^u, \ldots, v_t^u) \in (S^1)^{i_u},$$

   where $i_0 = r$ and $\Lambda = [\lambda_t^u]$ is a matrix of positive real numbers such that for each $1 \leq u \leq k$, the vector $(\lambda_1^0, \ldots, \lambda_r^0, \lambda_1^u, \ldots, \lambda_{i_u}^u)$ is a generic length vector.

2. Let $\vec{w}_t^u = \sum_{t=1}^{i_u} \lambda_t^u v_t^u$.

3. Let a_i's are positive real numbers such that $a_1 < \cdots < a_k$ are real numbers.

With the above notations we introduce the generalized version of the chain space.

Definition 6.1. Let $1 \leq u \leq k$ and $1 \leq t \leq i_u$. The multi-branched chain space is defined as

$$Ch_k(\ ) = \left\{ (\forall_t^0, \forall_t^1, \ldots, \forall_t^k) \in (S^1)^{\sum_{u=0}^{k} i_u} : \ _1(\forall_t^0 + \forall_t^u) = a_u \right\} \Big/ \mathbb{Z}_2,$$

where $_1$ is a projection onto the X-axis.

The elements of $Ch_k(\ )$ look like k-branched chains as drawn in Figure 6.1 considered up to the involution across the X-axis.

Figure 6.1: A configuration of multi-branched chain

We state the following theorem without proof.

Theorem 6.2. The space $Ch_k(\ )$ is a closed, smooth manifold of dimension $\sum_{u=0}^{k} i_u - k$. Moreover, $Ch_k(\ )$ is a small cover over a truncated cube of dimension $\sum_{u=0}^{k} i_u - k$.

The following are natural questions.

Question 6.3. What is an analogue of the short code that will help classify the diffeomorphism type of these space?

Question 6.4. How to express topological invariants such as homology, cohomology etc. in terms of the associated combinatorial data?

Question 6.5. What is a characterization of truncated cubes that arise as the quotient polytopes of multi-branched chain spaces?

Question 6.6. For which length vectors the corresponding multi-branched chain space is aspherical?

Question 6.7. Is every small cover over a truncated cube a multi-branched chain space?

6.3 Topological complexity

The topological complexity is an important homotopy invariant in the field of topological robotics. In general, it is hard to compute the exact value of this invariant. So people try to approximate it by upper and lower bounds.

Don Davis computed various bounds on the topological complexity of planar polygon spaces. In some cases, he computed the exact value. Here we list some of his results.

Theorem 6.8 ([17, Theorem 1.2]). Let $= (1, \ldots, 1, r)$ be an n-length vector. If r is a real number such that $1 \le r < n - 3$ and $n - r$ is not an odd integer, then

$$2n - 6 \le TC(\overline{M}) \le 2n - 5.$$

Theorem 6.9 ([16]). Let k be the largest size of the gee of. If $n \geq 2k + 3$, then

$$2n - 6 \le TC(\overline{M}) \le 2n - 5.$$

Question 6.10. For which classes of length vectors stated in Theorem 6.8 and Theorem 6.9, $TC(\overline{M})$ takes the exact value?

Davis computed the bounds on $zcl(M)$ in terms of the size of largest gee.

Theorem 6.11 ([15]). 1. If s is the largest cardinality of the gees of length vector. Then $zcl(\overline{M}) \le 2s + 2$.

2. If S and S^0 are gees of of, not necessarily distinct, and there is an inequality of multisets $S \cup S^0 \geq [k]$, then

$$
zcl(\overline{M}) \geq \begin{cases} k+2, & \text{if } k \equiv (n-3) \bmod 2, \\ k+1, & \text{if } k \not\equiv (n-3) \bmod 2. \end{cases}
$$

Question 6.12. Can we reduce the gap between lower and upper bounds of $zcl(\overline{M})$?

6.4 Bott-type manifolds

In Section 3.4, we have shown that the aspherical chain spaces are small covers over simple polytope $P_i \times I^{n-4}$, for $i = 4, 5, 6$. In particular, the chain space corresponding to the short code $\langle\{1, 2, \ldots, m-3, m\}\rangle$ is also a real Bott manifold. In fact, we have a tower of S^1-bundles. The bundles at each stage are given by a Bott matrix.

$$
Ch(\ell) \xrightarrow{\tilde{\pi}_1} Ch(\ell(I_1)) \xrightarrow{\tilde{\pi}_2} Ch(\ell(I_2)) \xrightarrow{\tilde{\pi}_3} \cdots \xrightarrow{\tilde{\pi}_{m-3}} Ch(\ell(I_{m-3})) \xrightarrow{\tilde{\pi}_{m-2}} \{*\}
$$

where $\ell(I_j) = (\ell_{j+1}, \ldots, \ell_{m-1}, \ell_m + \sum_{i=1}^{j} \ell_i)$ and $I_j = \{1, 2, \ldots, j\}$ $1 \leq j \leq m-3$.

Similarly, other two aspherical chain spaces corresponding to short codes $\langle\{1, 2, \ldots, m-4, m-2, m\}\rangle$ and $\langle\{1, 2, \ldots, m-4, m-1, m\}\rangle$ can be costructed as a towers of S^1-bundles over non-orientable surfaces of genus 3 and 4, respectively.

$$
Ch(\ell) \xrightarrow{\tilde{\pi}_1} Ch(\ell(I_1)) \xrightarrow{\tilde{\pi}_2} Ch(\ell(I_2)) \xrightarrow{\tilde{\pi}_3} \cdots \xrightarrow{\tilde{\pi}_{m-3}} Ch(\ell(I_{m-4})),
$$

where $\ell(I_j) = (\ell_{j+1}, \ldots, \ell_{m-1}, \ell_m + \sum_{i=1}^{j} \ell_i)$ and $I_j = \{1, 2, \ldots, j\}$ $1 \leq j \leq m-4$.

We call the towers of S^1-bundles starting with the non-orientable surfaces as Bott-type manifolds. It is easy to see that these manifolds are small covers over $P_i \times I^n$ where P_i is an i-gon. Real Bott manifolds have been extensively studied by many mathematicians. As far as we know the Bott-type manifolds haven't been considered yet. We would like to explore the topological and combinatorial aspects of Bott-type manifolds. More precisely, we would like to solve the following questions.

Question 6.13. How many Bott-type manifolds exist, up to diffeomorphism, over $P_i \times I^n$?

Question 6.14. Can we characterize Bott-type manifolds, up to diffeomorphism, in terms of some combinatorial data?

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