Automorphism groups of Schubert varieties and rigidity of Bott-Samelson-Demazure-Hansen varieties

by

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A thesis submitted to the

Board of studies in mathematical Sciences

In partial fulfilment of requirements

for the degree of

Doctor of Philosophy

of

CHENNAI MATHEMATICAL INSTITUTE

Chennai Mathematical Institute
December, 2019

Declaration

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted as a whole or in part for a degree / diploma at this or any other Institution / University.

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Certificate

I certify that the thesis entitled **"Automorphism groups of Schubert varieties and rigidity of Bott-Samelson-Demazure-Hansen varieties"** submitted for the degree of Doctor of Philosophy in Mathematics by Pinakinath Saha is record of research work carried out by him during the period from 1st August 2015 to December 2019 under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent his independent work in a very substantial measure.

Chennai Mathematical Institute
Date: December, 2019

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Acknowledgement

First of all I would like to express my gratefulness to my advisor Prof. S.Senthamarai Kannan for giving me a wonderful opportunity to work under his supervision. I am very grateful to him for introduceing and teaching to me algebraic groups, Lie algebras, and representation theory from geometric point of view in a very interesting and naive way.

I would like to thank Prof. D.S. Nagraj and Prof. K.V. Subrahmanyam for giving various helpful comments. I would like to thank Prof. T.R. Ramadas, Prof. V. Balaji, Prof. P. Sastry, Prof. K. Hanumanthu, Prof. S. Shankar, Prof C. D'Cruz, Prof. M. Kummini, Prof. U. Kulkarni for various helpful discussions. I thank all my friends in CMI especially Sourav, Kuldeep, Praveen, Abhishek, Sarjick, Abhijeet, Navnath, Manu, Aditya, Dharm Veer, Malay, Jagadish, Nirmal. I would like to thank the office staff of the Chennai Mathematical Institute. I would also like to thank all the academic staff at the Chennai Mathematical Institute.

I would like to thank NBHM (National Board for Higher Mathematics) and Chennai Mathematical Institute for supporting me financially throughout the Ph.D. program.

Dedicated to my advisor
Professor S. Senthamarai Kannan

Contents

Synopsis

This thesis consists of two parts, first part of which deals with the study of the connection between parabolic subgroups of G and the automorphism groups of Schubert varieties. In the second part we study the rigidity of Bott-Samelson-Demazure-Hansen varieties corresponding to a reduced expression of the longest element in terms of the combinatorial description of Coxeter elements.

0.0.1 Parabolic subgroups and automorphism groups of Schubert varieties

Let X be a projective variety over $\mathbb{C}$. It is known that the connected component of the group of all algebraic automorphisms of X containing identity element is an algebraic group (see [MO67, Theorem 3.7, p.17]). On the other hand, M. Brion proved every connected algebraic group H over $\mathbb{C}$ is the connected automorphism group of some normal projective variety X (see [Bri14, Theorem 1]). Let G be a simple algebraic group of adjoint type over $\mathbb{C}$. Let T be a maximal torus of G, and let R be the set of roots with respect to T. Let $R^+ \subset R$ be a set of positive roots. Let B^+ be the Borel subgroup of G containing T, corresponding to R^+. Let B be the Borel subgroup of G opposite to B^+ determined by T. Note that B is the Borel subgroup of G containing T, corresponding to the set of roots $R^- := -R^+$. We use the notation $\beta < 0$ for $\beta \in R^-$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of all simple roots in R^+, where n is the rank of G. Let $W := N_G(T)/T$ be the Weyl group of G with respect to T. For simplicity of notation, the simple reflection s_{α_i} in W corresponding to a simple root α_i is denoted by s_i. For $w \in W$, let $X(w) := \overline{BwB/B}$ denote the Schubert variety in G/B corresponding to w. In [Dem77], M. Demazure, studied the automorphism group of the homogeneous space G/P. If (G, P) is not exceptional, the connected component of $Aut(G/P)$ containing identity is G. The Lie algebra of G may be identified with the Lie algebra of global vector fields $H^0(G/P, T_{G/P})$. Let $Aut^0(X(w))$ denote the connected component, containing the identity element of the group of all algebraic automorphisms of $X(w)$. Let α_0 denote the highest root of G with respect to T and B^+. For the left action of G on G/B, let P_w denote the stabiliser of $X(w)$ in G. In [Kan16, p.772,Theorem 4.2(2)], S. S. Kannan, has proved that if G is simply laced and $X(w)$ is smooth, then we have $P_w = Aut^0(X(w))$ if and only if $w^{-1}(\alpha_0) < 0$. Therefore it is a natural question to ask whether given any parabolic subgroup P of G containing B properly, is there a Schubert variety $X(w)$ in G/B such that $P = Aut^0(X(w))$? If $P = B$, there is no such Schubert variety in G/B. The primary goal of the first part of this thesis is to prove the following:

Theorem 0.0.1. *If G is a simple algebraic group of adjoint type over $\mathbb{C}$ and P be a parabolic subgroup of G containing B properly, then there is an element $w \in W$ such that $P = Aut^0(X(w))$.*

Proof strategy: Assume P is a parabolic subgroup of G containing B properly, then we have $P = P_I$ for some non empty subset I of S. We divide the proof in two cases: If G and P are such that (G, P) is not one of the following:

(1) G is of type B_n and $P = P_{\alpha_n}$, the minimal parabolic corresponding to α_n.

(2) G is of type C_n and $P = P_{\alpha_1,}$ the minimal parabolic corresponding to α_1.

(3) G is of type G_2 and $P = P_{\alpha_1}$, the minimal parabolic corresponding to α_1.

Then we find out a $w \in W$ such that: $w^{-1}(\alpha_0) < 0$, the stabilizer of $X(w)$ for the natural left action of G on G/B is P. The natural restriction map $\pi : G/B \longrightarrow G/P'$ to $X(w)$ is birational and surjective. Here $P' = P_{J'}$ and $J' = -w_0(J)$ and $J = S \setminus I$. Further by using [BK05, Theorem 3.3.4(a), p.96] and [BK05, Lemma 3.3.3(b), p.95] we have, $\pi_*(\mathcal{O}_{X(w)}) = \mathcal{O}_{G/P'}$. Then by using [Bri11, Corollary 2.2., p.45], [Akh95, Theorem 2, p.75], and [Kan16, Theorem 4.2(2), p.772] we conclude $P = Aut^0(X(w))$. If G and P are such that (G, P) is one of the above type, then we give an explicit description of w satisfying for which $P = Aut^0(X(w))$.

We prove some partial results for Schubert varieties in partial flag varieties of type A_n. Let $G = PSL(n+1, \mathbb{C})$ and $W(r) = \{w \in W^{S\setminus\{\alpha_r\}} : w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r)$, where $1 \leq a_1 < a_2 < \cdots < a_r = n\}$. For $1 \leq r \leq n$ and $w \in W^{S\setminus\{\alpha_r\}}$, we denote the Schubert variety corresponding to w in the Grassmannian $G/P_{\hat{\alpha}_r}$, by $X_{P_{\hat{\alpha}_r}}(w)$.

Proposition 0.0.2. *Let* $w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r) \in W(r)$. *Let* $J'(w) := \{i \in \{1, 2, \ldots, r-1\} : a_{i+1} - a_i \geq 2\}$, $J''(w) = \{1 + a_i : i \in J'(w)\}$ *and* $J(w) = \{\alpha_j : j \in \{1, \ldots, n\} \setminus J''(w)\}$. *Then we have* $P_{J(w)} = Aut^0(X_{P_{\hat{\alpha}_r}}(w))$.

If Q is the maximal parabolic subgroup of $PSL(n+1, \mathbb{C})$ corresponding to the simple root α_1 or α_n, then G/Q is the projective space $\mathbb{P}^n$. The Schubert varieties in $\mathbb{P}^n$ are $\mathbb{P}^i$ $(0 \leq i \leq n)$. Further, $\mathbb{P}^n$ is the only Schubert variety in $\mathbb{P}^n$ for which the action of B is faithful. On the other hand, we have $Aut^0(\mathbb{P}^n) = PSL(n+1, \mathbb{C})$. Therefore the answer to the above question is negative if we consider partial flag varieties.

0.0.2 Rigidity of Bott-Samelson-Demazure-Hansen variety

Let G, T, B, and W be as discussed in previous subsection. Recall that for $w \in W$, $X(w) := \overline{BwB/B}$ denote the Schubert variety in G/B corresponding to w. Given a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ of w, with the corresponding tuple $\underline{i} := (i_1, \ldots, i_r)$, we denote by $Z(w, \underline{i})$ the desingularization of the Schubert variety $X(w)$ corresponding to $\underline{i}$, which is now known as Bott-Samelson-Demazure-Hansen variety. This was first introduced by Bott and Samelson in a differential geometric and topological context (see [BS58]). Demazure in [Dem74] and Hansen in [Han73] independently adapted the construction in algebro-geometric situation, which explains the reason for the name. For the sake of simplicity, we will denote any Bott-Samelson-Demazure-Hansen variety by a BSDH-variety. The construction of the BSDH-variety $Z(w, \underline{i})$ depends on the choice of the reduced expression $\underline{i}$ of w. In [CKP15], the automorphism groups of these

varieties were studied. There, the following vanishing results of the tangent bundle $T_{Z(w,\underline{i})}$ on $Z(w,\underline{i})$ were proved (see [CKP15, Section 3]):
(1) $H^j(Z(w,\underline{i}), T_{Z(w,\underline{i})}) = 0$ for all $j \geq 2$.
(2) If G is simply laced, then $H^j(Z(w,\underline{i}), T_{Z(w,\underline{i})}) = 0$ for all $j \geq 1$.

As a consequence, it follows that the BSDH-varieties are rigid for simply laced groups and their deformations are unobstructed in general (see [CKP15, Section 3]). The above vanishing result is independent of the choice of the reduced expression $\underline{i}$ of w. While computing the first cohomology module $H^1(Z(w,\underline{i}), T_{Z(w,\underline{i})})$ for non simply laced group, the authors in [CKP15] observed that this cohomology module very much depends upon the choice of a reduced expression $\underline{i}$ of w.

It is a natural to ask for which reduced expressions $\underline{i}$ of w, the cohomology module $H^1(Z(w,\underline{i}), T_{Z(w,\underline{i})})$ does vanish ?

We introduce some notation that we use to state a partial answer of this question.

Recall that a Coxeter element is an element of the Weyl group having a reduced expression of the form $s_{i_1}s_{i_2}\cdots s_{i_n}$ such that $i_j \neq i_l$ whenever $j \neq l$ (see [Hum95, p.56, Section 4.4]). Let c be a Coxeter element of W. Then there exists a decreasing sequence $n \geq a_1 > a_2 > \cdots > a_k = 1$ of positive integers such that $c = [a_1, n][a_2, a_1 - 1]\cdots[a_k, a_{k-1} - 1]$, where $[i, j]$ for $i \leq j$ denotes $s_i s_{i+1}\cdots s_j$. Note that for $n \geq 3$, there is an isomorphism between the Weyl group of B_n and the Weyl group of C_n sending $s_i \mapsto s_i$ for $(1 \leq i \leq n)$. Moreover, in type B_n or C_n $(n \geq 3)$ we have $w_0 = c^n$ and for any sequence $\underline{i}^r (1 \leq r \leq n)$ of reduced expressions of c the sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^n)$ is a reduced expression of w_0 (see [CK17, Lemma 4.2, p.441]).

In [CK17] authors give a partial answer to the question for $w = w_0$ when $G = PSp(2n, \mathbb{C})$.

Theorem 0.0.3. *Let $G = PSp(2n, \mathbb{C})(n \geq 3)$. Then, $H^j(Z(w_0,\underline{i}), T(w_0,\underline{i})) = 0$ for all $j \geq 1$ if and only if $a_1 \neq n-1$ and $a_2 \leq n-2$.*

A partial answer is given to this question in [KS(I)19] for $w = w_0$ when $G = PSO(2n+1, \mathbb{C})$. Moreover, in type B_n or C_n $(n \geq 3)$ we have $w_0 = c^n$ and for any sequence $\underline{i}^r (1 \leq r \leq n)$ of reduced expressions of c the sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^n)$ is a reduced expression of w_0 (see [CK17, Lemma 4.2, p.441]).

Theorem 0.0.4. *Let $G = PSO(2n+1, \mathbb{C})$ $(n \geq 3)$. Then, $H^j(Z(w_0,\underline{i}), T_{(w_0,\underline{i})}) = 0$ for all $j \geq 1$ if and only if $c = \prod_{j=1}^{k}[a_j, a_{j-1} - 1]$, where $a_0 := n+1$ and $a_2 \neq n-1$.*

Proof strategy: Although for $n \geq 3$ the Weyl group W of type B_n and C_n are the same, geometry of BSDH varieties are different. In C_n, each simple root other than α_n is short root. Therefore, by [Kan16, Corollary 5.6, p.778], we have $H^1(w, \alpha_j) = 0$ for any $w \in W$ and for any $j \neq n$. On the other hand, in type B_n the only simple short root is α_n. So, we can't apply [Kan16, Corollary 5.6, p.778]. Hence we first study the cohomology modules $H^1(w, \alpha_j)$ ($w \in W$, $j \neq n-1$). We prove that $H^1(w, \alpha_j) = 0$ for any $w \in W$ and for $j \neq n-1$. Then we adapt the proof of [CK17].

The similar question is considered in [KS(II)19] for $w = w_0$ when G is of type F_4, G_2. In type F_4 we have $w_0 = c^6$ and for any sequence $\underline{i}^r (1 \leq r \leq 6)$ of reduced expressions of c the sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^6)$ is a reduced expression of w_0. In [KS(II)19] we prove the following theorems:

Theorem 0.0.5. *Assume that G is of type F_4. Then, $H^j(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ for all $j \geq 1$ if and only if $c = \prod_{j=1}^{k} [a_j, a_{j-1} - 1]$, where $a_0 := 5$, $a_1 \neq 3$ or $a_2 \neq 2$.*

Proof strategy: is similar to that of Theorem 0.0.4.

In type G_2 we have $w_0 = c^3$ and for any sequence $\underline{i}^r (1 \leq r \leq 3)$ of reduced expressions of c the sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \underline{i}^3)$ is a reduced expression of w_0.

Theorem 0.0.6. *Assume that G is of type G_2. Then, $H^1(Z(w_0, \underline{i_r}), T_{(w_0, \underline{i_r})}) \neq 0$ for $r = 1, 2$.*

Proof strategy: is similar to that of Theorem 0.0.4.

0.0.3 General layout of the thesis

This thesis consists of four chapters. The organization of the thesis is as follows: In Chapter 1, we recall some basic notation and preliminaries on algebraic groups, flag varieties, Schubert varieties, and BSDH varieties, which are going to be used in subsequent chapters. In Chapter 2, we prove Theorem 0.0.1 and Proposition 0.0.2. In Chapter 3, we prove Theorem 0.0.4. In Chapter 4, we prove Theorem 0.0.5, and Theorem 0.0.6.

Chapter 1

Preliminaries

In this chapter, we review basic definitions and results on algebraic groups, Lie algebras, Schubert varieties which will be used in the rest of the thesis. The main sources for this chapter are [Hum72],[Hum75],[Jan03],[Bor91], and [Spr98].

1.1 Algebraic groups and Lie algebras

1.1.1 Algebraic groups

Throughout this thesis, we assume that all algebraic groups are affine algebraic group over $\mathbb{C}$.

Definition 1.1.1. *Algebraic group. An algebraic group is an algebraic variety G together with:*

(id): an element $e \in G$;

(mult): a morphism $\mu : G \times G \longrightarrow G$, denoted $(x, y) \mapsto xy$;

(inv): a morphism $i : G \longrightarrow G$, denoted $x \mapsto x^{-1}$,

with respect to which G is a group.

A morphism of algebraic groups is a morphism of algebraic varieties which is also a homomorphism of groups. The connected component of G containing e is an algebraic group and will be denoted by G°. Any algebraic group is smooth as a variety and G° is a normal subgroup of finite index in G whose cosets are connected, as well as irreducible, components of G. Further, any closed subgroup of finite index contains G°.

A rational representation of G is morphism of algebraic groups $G \longrightarrow GL(V)$, for some finite dimensional vector space V over $\mathbb{C}$.

A maximal closed connected normal solvable (respectively, unipotent) subgroup of G is called the radical (respectively, unipotent radical) of G and we denote it by $R(G)$ (respectively, $(R_u(G))$). $R(G)$ (respectively, $R_u(G)$) is connected component of the intersection of all Borel subgroups (respectively, unipotent parts of $R(G)$) in G.

An algebraic group is called semisimple (respectively, reductive) if $R(G) = e$ (respectively, $R_u(G) = e$). For example, any torus T is a reductive algebraic group, which is not semisimple. In particular, any semisimple algebraic group is reductive, but converse need not be true.

Let H be a closed subgroup of G. Denote by G/H, the set of all left cosets of H in G. One would like to know whether the set G/H is endowed with a structure of an algebraic variety such that the natural map

$$\pi : G \longrightarrow G/H$$

is a morphism satisfying a suitable universal mapping property.

The following theorem of Chevalley gives an affirmative answer to this question:

Theorem 1.1.2. *(C. Chevalley): Let G be an algebraic group, H be a closed subgroup of G. Then there is a rational representation $\phi : G \longrightarrow GL(V)$ and a one dimensional subspace L of V such that $H = \{x \in G : \phi(x)L = L\}$.*

Proof. See [Hum75, p. 80, Theorem 11.2]. □

Let $\mathbb{P}(V)$ be the projective space corresponding to the vector space V. Then, action of G on V via ρ (as above) induces an action of G on $\mathbb{P}(V)$. Further, the action map $G \times \mathbb{P}(V) \longrightarrow \mathbb{P}(V)$ is a morphism. Also, there exists a pint $[v] \in \mathbb{P}(V)$ such that the stabilizer of $[v]$ in G coincides with H. The orbit $G \cdot [v]$ is open in its closure and has a structure of quasi-projective variety with an algebraic transitive G-action. The orbit map $G \longrightarrow \mathbb{P}(V)$, $g \mapsto g \cdot [v]$ defines a bijection $G/H \longrightarrow G \cdot [v]$, and induces a structure of a quasi-projective variety on G/H such that the natural map $G \longrightarrow G/H$ is a morphism of varieties. In fact, we have:

Corollary 1.1.3.

1. *The set G/H admits a unique structure of a quasi-projective algebraic variety such that the natural map $G \longrightarrow G/H$ is a homomorphism of algebraic varieties.*
2. *In addition, if H is a closed normal subgroup of G, then the quotient group G/H has a unique structure of an affine algebraic group such that the natural map $G \longrightarrow G/H$ is a morphism of algebraic groups.*

Theorem 1.1.4. *(Borel's fixed point theorem): Let G be a connected solvable algebraic group, and let X be a nonempty complete variety on which G acts. Then G has a fixed point in X.*

Proof. See [Hum75, p. 134, Theorem 21.2]. □

Definition 1.1.5. *A Borel subgroup of G is a closed connected solvable subgroup of G properly included in no other.*

Theorem 1.1.6. *Let G be connected. Let B be a Borel subgroup of G. Then, G/B is a projective variety, and all other Borel subgroups of G are conjugate in G.*

Proof. See [Hum75, p. 134, Theorem 21.3]. □

Corollary 1.1.7. *The maximal tori (repectively, maximal connected unipotent subgroups) of G are those of the Borel subgroups of G, and are all conjugate.*

Proof. See [Hum75, p. 135, Corollary A]. □

We call the common dimension of the maximal tori of G the rank of G.

Definition 1.1.8. *A closed subgroup H of G is called parabolic subgroup if G/H is a complete variety.*

Corollary 1.1.9. *A closed subgroup H of G is parabolic subgroup if and only if G/P is projective variety.*

Corollary 1.1.10.

1. *A closed subgroup of G is parabolic if and only if it contains a Borel subgroup.*
2. *In particular, a connected subgroup H of G is a Borel subgroup if and only if H is solvable and G/H is projective.*

Corollary 1.1.11. *Let $\phi : G \longrightarrow G'$ be an epimorphism of connected algebraic groups. Let H be a Borel subgroup (respectively, parabolic subgroup, maximal torus, maximal connected unipotent subgroup) of G. Then $\phi(H)$ is a subgroup of the same type in G' and all such subgroups of G' are obtained in this way.*

1.2 Lie Algebra of an Algebraic Group

Let G be an algebraic group. G acts on $K[G]$ by the left (respectively, right) translations, $(\lambda_x f)(y) = f(x^{-1}y)$ (respectively, $\rho_x f(y) = f(yx)$) for $x, y \in G$.

Let $Der(K[G])$ be the set of all derivations of $K[G]$. Note that $Der(K[G])$ admits a Lie algebra structure. Let $\mathcal{L}(G)$ be the space of all left invariant derivations of $K[G]$ (i.e. $\mathcal{L} = \{\delta \in Der(K[G]) : \delta\lambda_x = \lambda_x\delta$, for all $x \in G\}$). Note that $\mathcal{L}(G)$ is a Lie subalgebra of $Der(K[G])$. We call $\mathcal{L}(G)$, the Lie algebra of G.

Theorem 1.2.1. *Let G be an algebraic group. Then,*

1. *The Lie algebra $\mathcal{L}(G)$ of G is isomorphic to $T_e(G)$, the tangent space of G at the identity element e of G.*

2. *Let $\mathfrak{g} = \mathcal{L}(G)$, and $\mathfrak{g}' = \mathcal{L}(G')$. If $\phi : G \longrightarrow G'$ is a morphism of algebraic groups, then the induced map $d\phi_e : \mathfrak{g} \longrightarrow \mathfrak{g}'$ is a homomorphism of Lie algebras.*

Proof. See [Hum75, p. 65, Theorem 9.1]. □

Theorem 1.2.2. *Let $\phi : G \longrightarrow G'$ be a surjective homomorphism of algebraic groups.*

1. *$dim\ G = dim\ G' + dim\ Ker\ \phi$.*

2. *ϕ is an isomorphism if and only if ϕ and $d\phi_e$ are bijective.*

Proof. See [Spr98, p.87, Corollary 5.3.3]. □

1.3 Abstract root system

Here we summarize, without proof, some basic properties of root systems. For further details we refer to Bourbaki [Bou02, 1, Ch. VI], Humphreys [Hum72, Chap. III], Humphreys [Hum75, Appendix].

Let E be a finite dimensional vector space over $\mathbb{R}$. Define a reflection, relative to a non zero $\alpha \in E$, to be a linear transformation which send α to $-\alpha$ and fixes point wise a subspace of codimension 1. Such a transformation is clearly its own inverse, but is not uniquely determined by α. Nevertheless, if R is a finite set of nonzero vectors spanning E, and a reflection s_α relative to $\alpha \in R$ maps R into itself, the s_α is uniquely determined by α.

An abstract root system in the real vector space E is a subset R of E satisfying:

(R1) R is a finite, spans E, and does not contain 0.

(R2) If $\alpha \in R$, the only multiple of α in R are $\pm\alpha$.

(R3) If $\alpha \in R$, there exists a reflection s_α relative to α which leaves R stable.

(R4) If $\alpha, \beta \in R$, then $s_\alpha(\beta) - \beta$ is an integral multiple of α.

The elements of R are called roots. If R' is an abstract root system in E', then R' is said to be isomorphic to R if there exists an isomorphism of vector spaces from E' onto E which maps R' to R and preserves the integers which occur in (R4).

Thanks to s_α in (R3) is uniquely determined by α, so (R4) is unambiguous. $dimE = n$ is called the rank of R. Let R be a root system in E. Since R spans E, and is finite, the group $W(R) \subset GL(E)$ generated by the $\{s_\alpha : \alpha \in R\}$ is also finite. It is called the Weyl group of R and abbreviated as W. There is an inner product

(α,β) on E relative to which W consists of orthogonal transformations. The formula for s_α becomes: $s_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha$, where $\langle\beta,\alpha\rangle = 2(\beta,\alpha)/(\alpha,\alpha)$.

A subset $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ of R is called a base if S is a basis of E, relative to which each $\alpha \in R$ has a unique expression $\alpha = \sum_{i=1}^{n} c_i\alpha_i$, where the c_i's are integers of same sign. Bases exists, W permutes the collection of all bases simply transitively, and every root lies in at least one base. Bases correspond $1-1$ to Weyl chambers in E, which are connected components in the complement of union of hyperplanes orthogonal to roots. Elements of a base S are called simple roots. The roots which are non negative (respectively, non positive) combinations of S comprise the set R^+ (respectively, R^-) of positive roots (resp. negative) roots. If $\alpha,\beta \in R^+$ but $\alpha + \beta \notin R$, then $r\alpha + s\beta \notin R$ for all $r, s \in \mathbb{N}$. Let S be a base of R. Then W is generated by $\{s_\alpha : \alpha \in S\}$. The length $l(w)$ of $w \in W$ (relative to S) is defined to be the smallest r for which $w = s_1 s_2 \cdots s_r$ (s_i is the reflection relative to some simple root α_i). Then $l(w)$ is the number of positive roots α for which $w(\alpha)$ is negative. In particular, s_α permutes $R^+ \setminus \{\alpha\}$ for every $\alpha \in S$.

Let S be a base of R, S' be a subset of S. The roots lying in the subspace E' of E spanned by S' form an abstract root system in E' having S' as a base its Weyl group is identified with the subgroup of W generated by $\{s_\alpha : \alpha \in S'\}$.

R is called irreducible if it (or equivalently, a base S) cannot be partitioned into union of two mutually orthogonal proper subsets. Every root system is the disjoint union of (uniquely determined) irreducible root systems in suitable subspaces of E. Up to isomorphism, the irreducible root systems correspond $1-1$ to the following Dynkin diagrams:

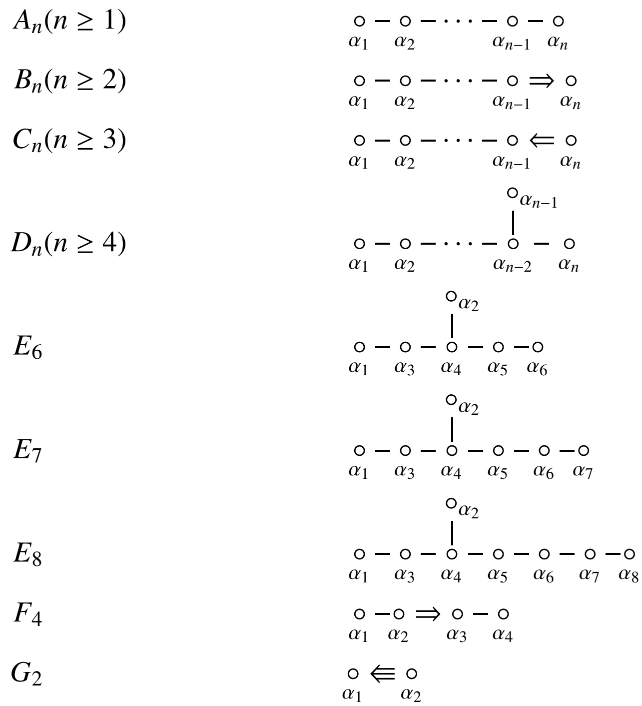

Figure 1.1: Classification of the Dynkin diagrams of the irreducible root systems

A vector $\lambda \in E$ is called an abstract weight provided all $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ $(\alpha \in R)$. These vectors form a lattice Λ, in which the lattice Λ_r spanned by R is a subgroup of finite index. If $S = \{\alpha_1, ..., \alpha_n\}$ is a base of R, Λ has a corresponding basis of fundamental dominant weights $\{\omega_1, ..., \omega_n\}$ for which $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$(Kroneker delta).

Given $S = \{\alpha_1, ..., \alpha_n\}$ and $\{\omega_1, ..., \omega_n\}$ as in the previous, call $\lambda = \sum_{i=1}^{n} m_i \omega_i$ dominant if $m_i \in \mathbb{Z}^+$ for all $1 \leq i \leq n$. Each $\lambda \in \Lambda$ is W-conjugate to one and only one dominant weight. E is partially ordered by $\lambda \geq \mu$ if $\lambda - \mu$ is a non negative integral linear combination of simple roots(This ordering depends on S). If $\lambda \in \Lambda$ is dominant, then $\lambda \geq \sigma(\lambda)$ for all $\sigma \in W$.

The vectors $\check{\alpha} = 2\alpha/(\alpha, \alpha)$ ($\alpha \in R$) form a root system in E, called the dual root system of R. The Cartan integer $\langle \check{\alpha}, \check{\beta} \rangle$ is equal to $\langle \beta, \alpha \rangle$. The Weyl group of $\check{R}$ is canonically isomorphic to W (via $\sigma_{\check{\alpha}} \mapsto \sigma_\alpha$). If E is identified with its dual space by means of inner product, $\check{\alpha}(\alpha \in S)$ becomes identified with the (unique) linear function on E taking value 2 on α and value 0 at all fundamental dominant weights ω_β corresponding to the simple root $\beta \neq \alpha$ (The uniqueness of this linear function follows from the fact the α along with these ω_β form a basis of E).

1.4 Root system associated to a semisimple algebraic group

From now onwards unless otherwise mentioned we assume G to be a connected semisimple algebraic group and T be a fix maximal torus of G.

Definition 1.4.1. *A homomorphism $\chi : T \longrightarrow \mathbb{G}_m$ of algebraic groups is said to be a character T, where $\mathbb{G}_m$ is the multiplicative group $\mathbb{C}^{\times}$.*

Definition 1.4.2. *A homomorphism $\lambda : \mathbb{G}_m \longrightarrow T$ of algebraic groups is said to be a one-parameter subgroup T, where $\mathbb{G}_m$ is the multiplicative group $\mathbb{C}^{\times}$.*

Let $X(T) := Hom(T, \mathbb{G}_m)$ be the group of all characters of T. Let $Y(T) := Hom(\mathbb{G}_m, T)$ be the group of all one-parameter subgroups of T. These groups are free Abelian groups of rank $n = dim(T) = rank(G)$. There is a non-degenerate pairing $\langle -, - \rangle : Y(T) \times X(T) \longrightarrow \mathbb{Z}$, sending the pair (χ, λ) to the integer $r = \langle \chi, \lambda \rangle$ which satisfy : $\chi(\lambda(t)) = t^r$ for all $t \in \mathbb{G}_m$.

Let V be a finite dimensional T-module. Then, we have a decomposition

$$V = \bigoplus_{\chi \in X(T)} V_\chi$$

where $V_\chi = \{v \in V : t \cdot v = \chi(t)v \text{ for all } t \in T\}$. The spaces V_χ are called the weight spaces with respect to T, and $\chi \in X(T)$ is called a weight of the T-module V if $V_\chi \neq 0$.

Definition 1.4.3. Adjoint representation of G : *Let $\mathfrak{g}$ be the Lie algebra of G. For any $g \in G$, $Int(g) : G \longrightarrow G$ is the automorphism of the algebraic group G defined by $Int(g)(h) = ghg^{-1}$ for all $h \in G$. This induces an isomorphism $Ad(g) : \mathfrak{g} \longrightarrow \mathfrak{g}$ of Lie algebras. Hence, we get the morphism of algebraic groups $Ad : G \longrightarrow GL(\mathfrak{g})$, called the adjoint representation of G.*

By the above discussion, if we restrict the adjoint representation of G to the maximal torus T, we have the following decomposition of $\mathfrak{g}$, called Cartan decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in X(T)} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : Ad(t)(x) = \alpha(t)x \text{ for all } t \in T\}$. We denote the set of non zero weights of $\mathfrak{g}$ by R. The weight space $\mathfrak{g}_\alpha$ for $\alpha \in R$ are called root spaces and α's are called roots relative to the maximal torus T.

Recal that for any $\lambda \in X(T)$ and $\phi \in Y(T)$, we have $\lambda \circ \phi : a \mapsto a^{\langle \lambda, \phi \rangle}$ for all $a \in \mathbb{G}_m$. Note that the pairing $\langle -, - \rangle : X(T) \times Y(T) \longrightarrow \mathbb{Z}$ is bilinear and induces an isomorphism $Y(T) \simeq Hom_{\mathbb{Z}}(X(T), \mathbb{Z})$. Therefore, we can identify $X(T)$ with $Y(\check{T})$. Using this identification we get a W-invariant non-degenerate bilinear form on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem 1.4.4. *The set R forms an abstract root system by viewing inside the vector space $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.*

Proof. See [Hum75, 163, Theorem 27.1]. □

Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a base of the root system R. Let R^+ be the set of all positive roots in R. Let B be the Borel subgroup of G containing T associated to R^+ (see [Bor91, p. 168, Proposition 13.8]). Note that $\mathfrak{g}$ is semisimple Lie algebra. For each $\alpha \in R$, there exists a non zero vector x_α of the one dimensional root space $\mathfrak{g}_\alpha$ such that the set $\{x_\alpha : \alpha \in R\} \cup \{h_\alpha : \alpha \in S\}$ forms a Chevalley basis for $\mathfrak{g}$. By a theorem of Serre, $\mathfrak{g}$ is generated by Chevalley basis with some relations (see [Hum72, p. 99, Theorem 18.3]).

Theorem 1.4.5. *Let $Z_\alpha = C_G(T_\alpha)$ where $T_\alpha = (ker\alpha)^\circ$, $\alpha \in R$.*

1. *There exists a unique connected T-stable subgroup U_α of G having Lie algebra $\mathfrak{g}_\alpha$ and $U_\alpha \subset Z_\alpha$.*
2. *If $n \in N$ represents $w \in W$, then $nU_\alpha n^{-1} = U_{w(\alpha)}$.*
3. *There exists an isomorphism $\epsilon_\alpha : \mathbb{G}_a \longrightarrow U_\alpha$ such that for all $t \in T$, and $x \in \mathbb{G}_a$, $t\epsilon_\alpha(x)t^{-1} = \epsilon_\alpha(\alpha(t)x)$.*
4. *G is generated by the groups $U_\alpha(\alpha \in R)$, along with T.*

Proof. See [Hum75, p 161, Theorem 26.3]. □

Theorem 1.4.6. *Let S be a base of R. Then G is generated by T along with all $U_\alpha(\pm\alpha \in S)$.*

Proof. See [Hum75, p 166, Theorem 27.3]. □

Theorem 1.4.7. *(Bruhat Decomposition). Let $N_G(T)$ be the normaliser of T in G. Then, G is a disjoint union of the double cosets BwB, as w running over a set of representatives in $N_G(T)$ of the Weyl group W. i.e.*

$$G = \bigcup_{w \in W} BwB.$$

Proof. See [Hum75, p. 172, Theorem 28.3]. □

Remark 1.4.8. *Note that $BwB = Bw'B$ if and only if $w = w'$ in W.*

Corollary 1.4.9. *If $\alpha \in S$, and $w \in W$, then $s_\alpha Bw \subset BwB \cup Bs_\alpha wB$.*

Corollary 1.4.10. *Let B' be any Borel subgroup of G. Then $B \cap B'$ includes a maximal torus of G.*

Corollary 1.4.11. *For each $w \in W$, fix a coset representative $n_w \in N$. Then each element $x \in G$ can be written in the form $x = u'n_w tu$, where $w \in W$, $t \in T$, $u \in U$, $u' \in U_w$ are all determined uniquely by x.*

Let U^- be the unipotent subgroup of G generated by $U_{-\alpha}$ where $\alpha \in R^+$.

Corollary 1.4.12. *The product map $\pi : U^- \times B \longrightarrow G$ defines an isomorphism of $U^- \times B$ onto an open subset Ω of G. The open subset Ω is called the big cell of G.*

1.5 Root system of parabolic subgroups

Let P be a parabolic subgroup of G containing B. Let $R(P)$ be the radical of P, let $R_u(P)$ be the unipotent radical of P. Let R_P^+ be the subset of R^+ defined by $R^+ \setminus R_P^+ = \{\alpha \in R^+ : U_\alpha \subset R_u(P)\}$. Let $R_P^- = -R_P^+$, $R_P = R_P^+ \cup R_P^-$ and $S_P = S \cap R_P$. Then R_P is a subroot system of R called the root system associated to the parabolic subgroup P, with S_P as a set of simple roots and R_P^+(respectively, R_P^-) as the set of positive roots of R_P relative to S_P.

On the other hand, given a subset J of S, let $R_J^+ := \{\sum\limits_{\beta \in J} a_\beta \beta : a_\beta \in \mathbb{Z}_{\geq 0}\} \cap R^+$. Now define the subgroup P_J of G generated by B and $U_{-\alpha}$, $\alpha \in R_J^+$. Note that P_J is a parabolic subgroup of G containing B such that $S_P = J$. Thus, we have the following theorem:

Theorem 1.5.1. *Each Parabolic subgroup of G is conjugate to one and only one subgroup $P_I = BW_IB$, where $I \subset S$.*

Proof. See [Hum75, p. 184, Theorem 30.1(a)]. □

Remark 1.5.2.

1. *If $P = B$, then $S_P = \emptyset$.*
2. *If $P = G$, then $S_P = S$.*

1.6 The Weyl Group of a Parabolic subgroup

The natural symmetry group attached to a root system R is its Weyl group W, the finite subgroup of $GL(E)$ generated by all reflections s_α with $\alpha \in R$. Note that W is generated by just the simple reflections s_α with $\alpha \in S$, where S is a base of R. Given a parabolic subgroup P of G, let W_P be the subgroup of W generated by $\{s_\alpha : \alpha \in S_P\}$. W_P is called the Weyl group of P. Note that $W_P \simeq N_P(T)/T$, where $N_P(T)$ is the normalizer of T in P. In each coset $wW_P \in W/W_P$, there exists a unique element of minimal length . Let W^P be the set of all minimal length representatives of W/W_P. We have

$$W^P = \{w \in W : l(ww') = l(w) + l(w'), \text{ for all } w' \in W_P\}.$$

In other words, each element $w \in W$ can be written uniquely as $w = uv$, where $u \in W^P, v \in W_P$ such that $l(w) = l(u) + l(v)$. The set W^P can also be characterized as

$$W^P = \{w \in W : w(\alpha) > 0, \text{ for all } \alpha \in S_P\}.$$

If P is the parabolic subgroup corresponding to a subset I of S, then W_P (respectively, W^P) is also denoted by W_I (respectively, W^I).

Abstractly, W is a finite Coxeter group, having generators $s_\alpha(\alpha \in S)$ and defining relation of the form $(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$. Moreover, W satisfies the crystallographic restriction $m(\alpha,\beta) \in \{2, 3, 4, 6\}$ when $\alpha \neq \beta$. Further, The Weyl group of $R^\vee$ is naturally isomorphic to W. Moreover, $w(\alpha)^\vee = w(\alpha^\vee)$ when $\alpha \in R$ and $w \in W$.

Write $l(w) = r$ if $w = s_{i_1} s_{i_2} \cdots s_{i_{r-1}} s_{i_r}$ where each s_{i_j} is simple reflection associated to the simple roots α_{i_j} such that there is no proper subsequence $(j_1, ..., j_k)$ of $(i_1, ..., i_r)$ with $w = s_{j_1} s_{j_2} \cdots s_{j_k}$. Standard facts about the length function on W include:

1. The number of $\alpha \in R^+$ for which $w(\alpha) < 0$ is precisely $l(w)$. In particular, when $\alpha \in S$ we have $s_\alpha(\beta) > 0$ for all $\beta \neq \alpha$ in R^+.

2. If $w \in W$, then $l(w) = l(w^{-1})$. Thus $l(w) = |R^+ \cap w(R^-)|$.

3. There is a unique element $w_0 \in W$ of maximum length $|R^+|$, sending R^+ to R^-. Moreover, $l(w_0 w) = l(w_0) - l(w)$ for all $w \in W$.

4. If $\alpha > 0$ and $w \in W$ satisfy $l(ws_\alpha) > l(w)$, then $w(\alpha) > 0$, while $l(ws_\alpha) < l(w)$ implies $w(\alpha) < 0$. It follows that $l(s_\alpha w) > l(w)$ if and only if $w^{-1}(\alpha) > 0$.

1.7 Chevalley - Bruhat Ordering of W

There is a subtle and very useful way to partially order W. In the case of Weyl groups the ordering arises first in the work of Chevalley and others on inclusions among closures of Bruhat cells for a semisimple group having Lie algebra $\mathfrak{g}$.

The set of all reflections is $\{s_\beta : \beta \in R^+\}$. Note that we have $\{s_\beta : \beta \in R^+\} = \bigcup_{w \in W} \{w s_\alpha w^{-1} : \alpha \in S\}$. For $w, w' \in W$ and $\beta \in R^+$ write $w' \xrightarrow{s_\beta} w$ if $w = s_\beta w'$ and $l(w') < l(w)$. In turn, write $w' \to w$ if $w' \xrightarrow{s_\beta} w$ for some $\beta \in R^+$. Extend this relation to a partial ordering of W by defining $w' < w$ to mean that $w' = w_0 \to w_1 \to \ldots \to w_m = w$ for some $w_1, ..., w_{m-1}$.

The identity element of W is the unique minimal element for this ordering. It is clear from the definition that $w' < w$ implies $l(w') < l(w)$.

Proposition 1.7.1. *The Bruhat ordering of W satisfies:*

1. *$w' \leq w$ if and only if w' occurs as a subexpression in one reduced expression $s_{i_1}...s_{i_r}$ (with $\alpha_{i_j} \in S$) for w. Here a subexpression is a product $s_{j_1}...s_{j_k}$ with $(j_1, ..., j_k)$ is a subsequence of $(i_1, ..., i_r)$.*

2. *Adjacent elements in Bruhat ordering differ in length by 1.*

3. *If $w' < w$ and $\alpha \in S$, then $w's_\alpha \leq w$ or $w's_\alpha \leq ws_\alpha$.*

4. *If $l(w_1) + 2 = l(w_2)$, the number of elements $w \in W$ satisfying $w_1 < w < w_2$ is 0 or 2.*

5. *If $I \subset S$, then the Bruhat ordering of the Coxeter group W_I is induced by the Bruhat ordering of W.*

Proof. See [Hum08, p. 6, Proposition 0.4]. □

1.7.1 Levi decomposition

The subgroup of G generated by T and $\{U_\alpha : \pm\alpha \in S_P\}$ is called the Levi subgroup of P and it is denoted by L_P. Note that P is a semidirect product of $R_u(P)$ and L_P called the Levi decomposition of P. There is a bijection between the set of all maximal parabolic subgroups of G containing B and the of all simple roots in S. Namely, given $\alpha \in S$, the parabolic subgroup P such that $S_P = S \setminus \{\alpha\}$ is a maximal parabolic subgroup and conversely. We denote the maximal parabolic subgroup P such that $S_P = S \setminus \{\alpha_i\}$ by $P_{\hat{\alpha}_i}$.

Theorem 1.7.2. *Any parabolic subgroup P of G has a Levi decomposition $P = LR_u(P)$, any two Levi factors are conjugate by an element of $R_u(P)$.*

Proof. See [Hum75, p.185, Theorem 30.2]. □

1.8 Representations of algebraic groups

In this section, we briefly summarize rational representation of G. Now onwards by a G-module V we mean a rational G-module.

Definition 1.8.1. *Let V be a G-module, let λ be a character of B. A non zero vector $v \in V$ is said to be a maximal weight vector of weight λ if $b \cdot v = \lambda(b)v$ for all $b \in B$.*

Remark 1.8.2. *If V is a non zero finite dimensional G-module, then maximal weight vector exists.*

Theorem 1.8.3. *Let V be an irreducible G-module. Then, we have*

(1) There is a unique B-stable one-dimensional subspace of V, spanned by a maximal weight vector of weight λ for some $\lambda \in X(T)^+$.

(2) *The dimension of the weight space V_λ is* 1.

(3) *The weights μ of V satisfy $\mu \leq \lambda$.*

(4) *W permutes the set of all weights of V, and $dim(V_\mu) = dim(V_{w(\mu)})$ for $w \in W$, and all $\mu \in X(T)$.*

(5) *If V' is another irreducible G-module of highest weight λ then V is isomorphic to V'.*

(6) *Let $\lambda \in X(T)$ be dominant. Then there exists an irreducible G-module of highest weight λ, and we denote it by $V(\lambda)$.*

Proof. See [Hum75, p.190, Theorem 31.3] for the proofs of (1) − (5), and [Hum75, p. 192, Theorem 31.4] for proof of (6). □

Theorem 1.8.4. *There is a one-one correspondence between $X(T)^+$ and the isomorphism classes of finite dimensional irreducible G-modules given by $\lambda \mapsto V(\lambda)$.*

Proof. See [Jan03, p. 200, Proposition 2.4]. □

1.9 Flag varieties and Schubert varieties

In this section, we recall some basics on Schubert varieties and its geometric properties.

1.9.1 Flag varieties

Let V be a vector space of dimension m over $\mathbb{C}$.

Definition 1.9.1. *We denote the Grassmannian of all r-dimensional subspaces of V, by $Gr(r, V)$. It is a smooth projective variety of dimension $r(m-r)$. A full flag in V is a chain $\{0\} = V_0 \subset V_1 \subset ... \subset V_m = V$ of subspaces of V with $dim(V_i) = i$ for $0 \leq i \leq m$.*

Let $\mathfrak{F}(V)$ be the set of all full flags in V. It is easy to see that the set $\mathfrak{F}(V)$ admits a natural structure of a projective variety. The variety $\mathfrak{F}(V)$ is called flag variety. Note that the flag variety $\mathfrak{F}(V)$ is isomorphic to the homogeneous space $SL(m, \mathbb{C})/B$, where B is the set of all upper triangular matrices in $SL(m, \mathbb{C})$.

More generally, let G, T, and B be as in the previous section. Let $\mathfrak{B}$ be the set of all Borel subgroups of G. The variety G/B can be identified with $\mathfrak{B}$. In fact, we have the following:

Proposition 1.9.2. *The set $\mathfrak{B}$ is endowed with a structure of a variety such that it is isomorphic to the homogeneous space G/B.*

Proof. See [Hum75, p. 145]. □

1.9.2 Bruhat decomposition of G relative to P

For $w \in W$, let n_w be a lift of w in $N_G(T)$. Observe that the double coset Bn_wP in G depends only on the set wW_P in W but not n_w. We write BwP for Bn_wP and we call it the open Bruhat cell in G associated to wW_P. The Zariski closure is called closed Bruhat cell in G associated to wW_P. The Bruhat decomposition of G relative to P is

$$G = \coprod_{w \in W^P} BwP.$$

Note that when $P = B$ we get the Bruhat decomposition of G that we described before.

For $w \in W^P$, the B-orbit $C_P(w) = BwP/P$ in G/P is a locally closed subset of G/P, called Schubert cell or Bruhat cell. The Zariski closure of $C_P(w)$ is called Schubert variety associated to w, and is denoted by $X_P(w)$. Thus, the Schubert varieties in G/P are indexed by W^P.

Note that if $P = B$, then $W_P = \{id\}$, and the Schubert varieties in G/B are indexed by elements of W. We denote the Schubert variety corresponding to $w \in W$ by $X(w)$.

Dimension of $X_P(w)$: If $P = B$, then for $w \in W$, we have

$$C_B(w) \simeq \prod_{\alpha \in R^+(w^{-1})} U_\alpha,$$

where $R^+(w^{-1}) := \{\alpha \in R^+ : w^{-1}(\alpha) < 0\}$. Since $|R(w^{-1})| = l(w)$, $C_B(w)$ is isomorphic to the affine space $\mathbb{C}^{l(w)}$. Hence we have

$$dim(X(w)) = dim(C_B(w)) = l(w).$$

Recall that from section 1.6, for a general parabolic P, consider $w \in W/W_P$ and denote the unique representative for $w \in W^P$ (respectively, $W^{P,max}$) by w^P (respectively, $w^{P,max}$). Now under the canonical projection $\pi_P : G/B \longrightarrow G/P$, $X(w^P)$ maps birationally onto $X_P(w)$. Hence we obtain

$$dim X_P(w) = dim X(w^P) = l(w^P).$$

Note that $G/B = X(w_0)$, w_0 being the longest element in W. The cell $C_B(w_0)$ is the unique cell of maximal dimension $(= l(w_0) = |R^+|)$, it is affine open dense in G/B, called the big cell of G/B.

There is a partial order on W^P, known as Bruhat-Chevalley order, induced by the partial order on the set of Schubert varieties given by inclusion, namely for $w_1, w_2 \in W^P$, $w_1 \leq w_2$ if and only if $X(w_1) \subseteq X(w_2)$.

The Bruhat decomposition of G/P and $X_P(w)$ are induced by the Bruhat decomposition of G/B.

$$G/P = \coprod_{w \in W^P} BwP/P$$

and

$$X_P(w) = \coprod_{w' \in W^P : w' \leq w} Bw'P/P.$$

1.10 Picard group of G/B

Let $R^+ \subset R$ be a set of positive roots. Let B^+ be the Borel subgroup of G containing T, corresponding to R^+. Let B be the Borel subgroup of G opposite to B^+ determined by T. Let U be the unipotent radical of B. Then, we have

$$U \simeq \prod_{\alpha \in R^+} U_{-\alpha},$$

where the product is taken in some order.

In the rest of the thesis B^+ corresponds to the set of positive roots and B corresponds to the set of negative roots as above.

Let $\tilde{G}$ be a simply connected covering of G and let $\tilde{B}$ and $\tilde{T}$ be the Borel subgroup and maximal torus of $\tilde{G}$ corresponding to B and T in G.

Recall that the root system in $\tilde{G}$ with respect to $\tilde{T}$ is same as the root system in G with respect to T. $X(\tilde{T}) \otimes \mathbb{Q} = X(T) \otimes \mathbb{Q}$ generated by fundamental weights $\{\omega_i : 1 \leq i \leq n\}$. Further, $\tilde{G}/\tilde{B}$ is isomorphic to G/B. In fact, $\tilde{G}/\tilde{P}$ is isomorphic to G/P for any parabolic subgroup of G, $\tilde{P}$ being corresponding parabolic subgroup in $\tilde{G}$. Note that the character group $X(\tilde{T})$ coincides with the weight lattice Λ. Since $X(\tilde{B}_u)$ is trivial every character λ of $\tilde{T}$ extends to a unique character of $\tilde{B}$. Hence, we have $X(\tilde{T}) = X(\tilde{B})$.

The canonical map $G \longrightarrow G/B$ is a principal B bundle (see [Jan03, p. 183, section 1.10]). Let $\lambda \in X(B)$. Set $G \times_B \mathbb{C}_{-\lambda} := G \times \mathbb{C}_{-\lambda}/\sim$ where $\sim$ is the equivalence relation defined by $(g, x) \sim (gb, \lambda(b^{-1})x)$ $g \in G, b \in B, x \in \mathbb{C}_{-\lambda}$. $G \times_B \mathbb{C}_{-\lambda}$ is the total space of a line bundle over G/B and we denote this line bundle by $\mathcal{L}(\lambda)$. Let $Pic(G/B)$ be the Picard group of G/B which is by definition, the group of isomorphism classes of line bundles on G/B. Thus we have a group homomorphism

$$\mathcal{L} : X(T) \longrightarrow Pic(G/B),$$
$$\lambda \mapsto \mathcal{L}(\lambda).$$

We have the following theorem due to Chevalley

Theorem 1.10.1. *(C. Chevalley): The map $\mathcal{L}$ is an isomorphism if G is simply connected.*

Proof. See [KKV89, p. 82, Corllary 3.3]. □

On the other hand, consider the prime divisor $X(w_0s_i)$, $1 \leq i \leq n$ on G/B. Let $\mathcal{L}_i = O_{G/B}(X(w_0s_i))$ be the line bundle defined by $X(w_0s_i)$, $1 \leq i \leq n$. Recall that the Picard group $Pic(G/B)$ is a free abelian group generated by the $\mathcal{L}_i'$s and under the isomorphism $\mathcal{L} : X(T) \longrightarrow Pic(G/B)$, we have $\mathcal{L}(\omega_i) = \mathcal{L}_i$, for $1 \leq i \leq n$. Thus for $\lambda = \sum_{i=1}^{n} \langle \lambda, \alpha_i \rangle \omega_i$, we have $\mathcal{L}(\lambda) = \bigotimes_{i=1}^{n} \mathcal{L}_i^{\otimes \langle \lambda, \alpha_i \rangle}$.

For a general parabolic P, any $\lambda \in X(T)$ can not be lifted to a character of P always. To be a character of P the weight λ must be orthogonal to the positive roots of P. Therefore, λ must be an integral linear combination of the fundamental weights, $\omega_1, ..., \omega_r$ dual to the simple roots in $S \setminus \{S_P\}$. We call $\omega_1, ..., \omega_r$ the fundamental weights of P and the stabilizer $\Lambda_P \subset \Lambda$ they generate the weight of P.

A line bundle $\mathcal{L}$ on an algebraic variety X is very ample if there exists an immersion $i : X \hookrightarrow \mathbb{P}^n$ such that $i^*(O_{\mathbb{P}^n}(1)) = \mathcal{L}$. A line bundle $\mathcal{L}$ on X is said to be numerically effective, if the degree of the restriction to any irreducible reduced algebraic curve in X is non-negative.

In the following theorem we recall some of the well known facts about the line bundles on G/P.

Theorem 1.10.2. *Let G is a simple algebraic group. Let $X = G/P$. Let $\omega_1, ..., \omega_r$ be the fundamental weights of P and let $\lambda \in \Lambda_P$. Then*

1. *$Pic(X) \simeq \Lambda_P$. In particular, $Pic(X) = \mathbb{Z}$ if P is a maximal parabolic subgroup of G.*
2. *$\mathcal{L}(\lambda)$ is numerically effective (nef) if and only if λ is dominant.*
3. *$\mathcal{L}(\lambda)$ is ample if and only if $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in S \setminus S_P$.*
4. *$\mathcal{L}(\lambda)$ is ample if and only if it is very ample.*

Proof. See [Jan03, Part II, Chapter 4], [BL00, p. 14]. □

As we have described above, let E denotes the total space of the line bundle $\mathcal{L}(\lambda)$ over G/B. Let $\sigma : E \longrightarrow G/B$ be the natural map $\sigma([g, c]) = gB$.

Let

$$M_\lambda = \{f \in k[G] : f(gb) = \lambda(b^{-1})f(g), \forall g \in G, b \in G\}.$$

Then M_λ can be identified with the space of global sections $H^0(G/B, \mathcal{L}(\lambda)) := \{s : G/B \longrightarrow E : \sigma \circ s = id_{G/B}\}$. This identification preserves the respective G-module structures.

1.11 Cohomology of Line bundles on Schubert varieties

We recall some results on cohomology of line bundles. We start this section by stating the Borel-Weil theorem. Let G, T, and B be as in section 1.10.

Theorem 1.11.1. *(Borel-Weil): Let G be simply connected and $\lambda \in X(B)$.*

1. $H^0(G/B, \mathcal{L}(\lambda)) \neq 0$ *if and only if λ is dominant.*
2. *If λ is dominant,* $H^j(G/B, \mathcal{L}(\lambda)) = 0$ *for all $j \geq 1$.*

Let $\lambda \in X(T)$. We define the dot action by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is the half sum of positive roots.

Theorem 1.11.2. *(Borel-Weil-Bott): Let G be simply connected and $\lambda \in X(B)$.*

1. *If $\lambda + \rho$ is singular (i.e. there is a $\beta \in R^+$ such that $\langle \lambda + \rho, \beta \rangle = 0$). Then, we have*

$$H^j(G/B, \mathcal{L}(\lambda)) = 0 \text{ for all } j.$$

2. *Let $\lambda + \rho$ is non singular,*

(i) $H^{l(w)}(G/B, \mathcal{L}(\lambda)) = H^0(G/B, \mathcal{L}(w \cdot \lambda))$ *for $w \cdot \lambda$ is dominant.*

(ii) $H^j(G/B, \mathcal{L}(\lambda)) = 0$ *for $j \neq l(w)$.*

Proof. See [Dem76, p. 272]. □

Theorem 1.11.3. *If $\lambda \in X(T)$ is dominant, then $H^0(G/B, \mathcal{L}(\lambda)) = V(\lambda)^*$.*

Proof. See [Dem76, p. 272]. □

Let $X(T)^+$ be the set of all dominant characters of T.

Corollary 1.11.4. *The map $\lambda \longrightarrow H^0(G/B, \mathcal{L}(\lambda))$ gives a bijection between $X(T)^+$ and the set of all finite dimensional irreducible representations of G.*

The following theorem gives the vanishing of cohomology of line bundles on Schubert varieties.

Theorem 1.11.5. *Let $\lambda \in X(T)^+$. Then we have:*

(1) $H^i(X(w), \mathcal{L}(\lambda)) = 0$ *for all $i \geq 1$.*

(2) The restriction map $H^0(G/B, \mathcal{L}(\lambda)) \longrightarrow H^0(X(w), \mathcal{L}(\lambda))$ is surjective.

The above theorem first proved by Demazure over algebraically closed field and characteristic zero in [Dem76]. In positive characteristic $p > 0$, it was proved by many authors:

1. H.H. Andersen proved by using "Characteristic p methods" (for example, see [And85]).
2. Mehta, Ramanan and Ramanathan proved by using Frobenius splitting methods (see [MR85] and [BK05]).
3. Lakshmibai, Musili and Seshadri proved by using standard monomial theory (see [BLPM12]).

For non-dominant weights, the vanishing results of line bundle on Schubert varieties in characteristic zero has been studied in [BKS04]. In the case of Kac-Moody setting, the cohomology of line bundles on Schubert variety has been studied in [Kan07].

We recall the following theorem of Bott on cohomology of tangent bundle $T_{G/B}$ of G/B form [Bot57].

Theorem 1.11.6. *(R. Bott):*

1. $H^i(G/B, T_{G/B}) = 0$ *for all* $i \geq 1$.
2. $H^0(G/B, T_{G/B})$ *is the adjoint representation* $\mathfrak{g}$ *of* G.

Now onwards we assume G is simple and α_0 is the highest root. By abuse of notation, we denote the restriction $T_{G/B}$ to $X(w)$ by $T_{G/B}$. Now, we state the following theorem from [Kan16] (see [Kan16, Theorem 4.1,Theorem 4.2, Theorem 6.5, and Theorem 6.6]). Though the following theorem is stated for smooth Schubert variety, it is true in general for any Schubert variety due to the theorem of Matsumura and Oort (see [MO67]).

Theorem 1.11.7. *(S. S. Kannan): Let $w \in W$. Then, we have*

1. $H^i(X(w), T_{G/B}) = 0$ *for every* $i \geq 1$.
2. *The adjoint representation $\mathfrak{g}$ of G is a B-submodule of $H^0(X(w), T_{G/B})$ if and only if $w^{-1}(\alpha_0) < 0$.*
3. *If G is simply laced, $H^0(X(w), T_{G/B})$ is the adjoint representation $\mathfrak{g}$ of G if and only if $w^{-1}(\alpha_0) < 0$.*
4. *Assume that G is simply laced and $X(w)$ smooth Schubert variety. Let $Aut^0(X(w))$ be the connected component of the automorphism group of $X(w)$ containing the identity automorphism. Let P_w denote the stabilizer of $X(w)$ in G. Let $\phi_w : P_w \longrightarrow Aut^0(X(w))$ be the homomorphism induced by the action of P_w on $X(w)$. Then, we have*

(i) $\phi_w : P_w \longrightarrow Aut^0(X(w))$ *is surjective.*

(ii) $\phi_w : P_w \longrightarrow Aut^0(X(w))$ *is an isomorphism if and only if* $w^{-1}(\alpha_0) < 0$.

5. *Assume that G is not simply laced and X(w) smooth Schubert variety. Let* $Aut^0(X(w))$, P_w, *and* ϕ_w *be as above. Then, the homomorphism* $\phi_w : P_w \longrightarrow Aut^0(w)$ *of algebraic groups is injective if and only if* $w^{-1}(\alpha_0) < 0$.

1.12 Geometry of Schubert varieties and Bott-Samelson-Demazure-Hansen varieties

In this section, we briefly recall some geometric properties of Schubert varieties.

Schubert varieties are non singular in co-dimension one (that is the singular locus has co-dimension at least 2), (arithmetically) normal, (arithmetically) Cohen-macaulay and have rational singularities (see [And85],[BK05], [BL00], [BLPM12] and [Ses07]). Note that in general, Scubert varieties need not be smooth.

For $w \in W$, recall that $X(w) := \overline{BwB/B}$ denotes the Schubert variety in G/B corresponding to w. Given a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ of w, with the corresponding tuple $\underline{i} := (i_1, i_2, ..., i_r)$, we denote by $Z(w, \underline{i})$ the desingularization of the Schubert variety $X(w)$, which is now known as Bott-Samelson-Demazure-Hansen variety. This was first introduced by Bott and Samelson in a differential geometric and topological context (see [BS58]). Demazure in [Dem74] and Hansen in [Han73] independently adapted the construction in algebro-geometric situation, which explains the reason for the name. For the sake of simplicity, we will denote a Bott-Samelson-Demazure-Hansen variety by BSDH-variety.

For a simple root $\alpha \in S$, we denote by P_α the minimal parabolic subgroup of G generated by B and n_α, a lift of s_α in $N_G(T)$.

We recall that the BSDH-variety corresponds to a reduced expression $\underline{i}$ of $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ defined by

$$Z(w, \underline{i}) = \frac{P_{\alpha_1} \times P_{\alpha_2} \times \cdots \times P_{\alpha_{i_r}}}{B \times B \times \cdots \times B},$$

where the action of $B \times B \times \cdots \times B$ on $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}$ is given by

$(p_1, p_2, \ldots, p_r)(b_1, b_2 \ldots, b_r) = (p_1 \cdot b_1, b_1^{-1} \cdot p_2 \cdot b_2, \ldots, b_{r-1}^{-1} \cdot p_r \cdot b_r)$, $p_j \in P_{\alpha_{i_j}}$, $b_j \in B$ and $\underline{i} = (i_1, i_2, \ldots, i_r)$ (see [Dem74, Definition 1, p.73], [BK05, Definition 2.2.1, p.64]).

For each reduced expression $\underline{i}$ of w, $Z(w, \underline{i})$ is a smooth projective variety and the orbit map

$$P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots P_{\alpha_{i_r}} \longrightarrow Z(w, \underline{i})$$

is locally trivial principal B^r bundle.

Define a morphism

$$\phi_w : Z(w, \underline{i}) \longrightarrow G/B$$

by

$$[p_1, p_2, ..., p_r] \mapsto p_1 p_2 \cdots p_r B.$$

This morphism can be seen as follows:

Let $m : P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}} \longrightarrow G$ be the multiplication map given by $(p_1, p_2, ..., p_r) \mapsto p_1 p_2 \cdots p_r$.

Now consider the following commutative diagram:

$$\begin{array}{ccc} P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}} & \xrightarrow{m} & G \\ \downarrow & & \downarrow \\ Z(w, \underline{i}) = P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}} / B \times B \times \cdots \times B & \xrightarrow{\phi_w} & G/B \end{array}$$

Since $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is a reduced expression of w, we have

$$BwB = Bs_{i_1} Bs_{i_2} B \cdots Bs_{i_r} B.$$

Since $Bs_{i_j}B$ is open in $P_{\alpha_{i_j}}$, $Z^0(w, \underline{i}) := (Bs_{i_1} Bs_{i_2} B \cdots Bs_{i_r} B)/B^r$ is open in $Z(w, \underline{i})$. Observe that the image $\phi_w(Z(w, \underline{i}))$ of $Z(w, \underline{i})$ in G/B is $X(w)$ and $Z^0(w, \underline{i})$ is isomorphic to the Schubert cell BwB/B.

Hence, ϕ_w is a birational surjective morphism from $Z(w, \underline{i})$ to $X(w)$.

Let $f_r : Z(w, \underline{i}) \longrightarrow Z(ws_{i_r}, \underline{i'})$ be the map induced by the projection

$$P_{i_1} \times P_{i_2} \times \cdots \times P_{i_r} \longrightarrow P_{i_1} \times P_{i_2} \times \cdots \times P_{i_{r-1}}$$

where $\underline{i'} = (i_1, i_2, ..., i_{r-1})$. We first note that f_r is a $P_{\alpha_{i_r}}/B \simeq \mathbb{P}^1$-fibration. In fact we have, the following commutative diagram:

$$\begin{array}{ccc} Z(ws_{\alpha_{i_r}}, \underline{i'}) \times_{G/P_{\alpha_{i_r}}} G/B & \longrightarrow & G/B \\ \downarrow f_r & & \downarrow \\ Z(ws_{\alpha_{i_r}}, \underline{i'}) & \longrightarrow & G/P_{\alpha_{i_r}} \end{array}$$

Then we observe that f_r is a $P_{\alpha_{i_r}}/B \simeq \mathbb{P}^1$ fibration.

Let $\sigma_r : P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots P_{\alpha_{i_{r-1}}} \longrightarrow P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots P_{\alpha_{i_r}}$ be the inclusion map. It induces a closed immersion $Z(ws_{\alpha_{i_r}}, \underline{i'}) \longrightarrow Z(w, \underline{i})$ and for convenience of the notation we also denote it by σ_r.

Let L_α denote the Levi subgroup of P_α containing T for $\alpha \in S$. We denote by B_α the intersection of L_α and B. Then L_α is the product of T and a homomorphic image G_α of $SL(2, \mathbb{C})$ via the homomorphism $\psi : SL(2, \mathbb{C}) \longrightarrow L_\alpha$ (see [Jan03, II, 1.3]).

1.12.1 Homogeneous vector bundles and its cohomology modules

Let $B'_\alpha := B_\alpha \cap G_\alpha \subset L_\alpha$. We note that the morphism $G_\alpha/B_\alpha \longrightarrow L_\alpha/B_\alpha$ induced by the inclusion is an isomorphism. Since $L_\alpha/B_\alpha \hookrightarrow P_\alpha/B$ is an isomorphism, to compute the cohomology modules $H^i(P_\alpha/B, \mathcal{L}(V))$ for any B-module V, we treat V as a B_α-module and we compute $H^i(L_\alpha/B_\alpha, \mathcal{L}(V))$. Here $\mathcal{L}(V)$ is the homogeneous vector bundle on P_α/B associated to the B-module V. For a B-module V, let $\mathcal{L}(w, V)$ denote the restriction of the associated homogeneous vector bundle on G/B to $X(w)$. By abuse of notation we denote the pull back of $\mathcal{L}(w, V)$ via ϕ_w to $Z(w, \underline{i})$ also by $\mathcal{L}(w, V)$, when there is no cause for confusion. Then we have

Lemma 1.12.1. *There is an isomorphism of B-linearized sheaves:*

(1) $R^j f_{r*}\mathcal{L}(w, V) = \mathcal{L}(ws_{i_r}, H^j(P_{\alpha_{i_r}}/B, \mathcal{L}(w, V)|_{(P_{\alpha_{i_r}}/B)}))$ *for all* $j \geq 0$.

(2) $\phi_{w*}\mathcal{O}_{Z(w,\underline{i})} = \mathcal{O}_{X(w)}$.

(3) For any locally free sheaf $\mathcal{F}$ *on* $X(w)$, *we have* $H^i(X(w), \mathcal{F}) \simeq H^i(Z(w, \underline{i}), \phi_w{}^*\mathcal{F})$, $i \geq 0$.

We use the following ascending 1-step construction as a basic tool in computing cohomology modules.

For $w \in W$, let $l(w)$ denote the length of w. Let γ be a simple root such that $l(w) = l(s_\gamma w) + 1$. Let $Z(w, \underline{i})$ be a BSDH-variety corresponding to a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$, where $\alpha_{i_1} = \gamma$. Then we have an induced morphism

$$g : Z(w, \underline{i}) \longrightarrow P_\gamma/B \simeq \mathbb{P}^1$$

with fibres $Z(s_\gamma w, \underline{i'})$, where $\underline{i'} = (i_2, i_3, ..., i_r)$.

By an application of Leray spectral sequence together with the fact that base is $\mathbb{P}^1$, we obtain for $j \geq 0$ and every B-module V, the following short exact sequence of P_γ-modules:

$$0 \longrightarrow H^1(P_\gamma/B, R^{j-1}g_*\mathcal{L}(w, V)) \longrightarrow H^j(Z(w, \underline{i}), \mathcal{L}(w, V)) \longrightarrow H^0(P_\gamma/B, R^j g_*\mathcal{L}(w, V)) \longrightarrow 0.$$

Since for any B-module V, the vector bundle $\mathcal{L}(w, V)$ on $Z(w, \underline{i})$ is the pull back of the homogeneous vector bundle from $X(w)$, we conclude that

$$H^j(Z(w,\underline{i}),\mathcal{L}(w,V)) \simeq H^j(X(w),\mathcal{L}(w,V))$$

(see [BK05, Theorem 3.3.4(b)]), and are independent of the choice of the reduced expression $\underline{i}$. Hence we denote $H^j(Z(w,\underline{i}),\mathcal{L}(w,V))$ by $H^j(w,V)$. For a character λ of B, we denote the one dimensional B-module corresponding to λ by $\mathbb{C}_\lambda$. Further, we denote the cohomology modules $H^j(Z(w,\underline{i}),\mathcal{L}(w,\mathbb{C}_\lambda))$ by $H^j(w,\lambda)$.

Rewriting the above short exact sequence using this simple notation, we have the following:

$$0 \longrightarrow H^1(s_\gamma, H^{j-1}(s_\gamma w, V)) \longrightarrow H^j(w,V) \longrightarrow H^0(s_\gamma, H^j(w,V)) \longrightarrow 0.$$

The B-modules V we deal with, satisfy $R^j g_* \mathcal{L}(w,V) = 0$ for all $j \geq 2$. Moreover, we use only the following two special cases of the above short exact sequence, which we denote by SES.

(1) For $j = 0$, we have $H^0(w,V) \simeq H^0(s_\gamma, H^0(s_\gamma w, V))$.

(2) For $j = 1$, we have $0 \longrightarrow H^1(s_\gamma, H^0(s_\gamma w, V)) \longrightarrow H^1(w,V) \longrightarrow H^0(s_\gamma, H^1(w,V)) \longrightarrow 0$.

The construction of BSDH variety depends on the choice of a reduced expression $\underline{i}$ of w. The BSDH-varieties corresponding to two different choice of reduced expressions need not be isomorphic (see [CKP15]).

1.13 Cohomology of the Tangent bundle of BSDH-variety

Assume G is simple. Let T, B be as above.

In this chapter, we survey some results related to cohomology of tangent bundle on BSDH-varieties that we are going to use in later chapters.

1.13.1 Cohomology of line bundles

Now, recall the following results due to Demazure ([Dem76, page 271]) on a short exact sequence of B-modules :

Lemma 1.13.1. *(M. Demazure): Let α be a simple root and $\lambda \in X(T)$ be such that $\langle \lambda, \alpha \rangle \geq 0$. Let ev denote the evaluation map $H^0(s_\alpha, \lambda) \longrightarrow \mathbb{C}_\lambda$. Then we have*

(1) If $\langle \lambda, \alpha \rangle = 0$, then $H^0(s_\alpha, \lambda) \simeq \mathbb{C}_\lambda$.

(2) If $\langle \lambda, \alpha \rangle \geq 1$, then $\mathbb{C}_{s_\alpha(\lambda)} \hookrightarrow H^0(s_\alpha, \lambda)$, and there is a short exact sequence of B-modules:

$$0 \to H^0(s_\alpha, \lambda - \alpha) \longrightarrow H^0(s_\alpha, \lambda)/\mathbb{C}_{s_\alpha(\lambda)} \longrightarrow \mathbb{C}_\lambda \to 0.$$

Further more, $H^0(s_\alpha, \lambda - \alpha) = 0$ *when* $\langle \lambda, \alpha \rangle = 1$.

(3) *Let* $m = \langle \lambda, \alpha \rangle$. *As a B-module,* $H^0(s_\alpha, \lambda)$ *has a composition series*

$$0 \subseteq V_m \subseteq V_{m-1} \subseteq \cdots \subseteq V_0 = H^0(s_\alpha, \lambda)$$

such that $V_i/V_{i+1} \simeq \mathbb{C}_{\lambda - i\alpha}$ *for* $i = 0, 1, \ldots, m-1$ *and* $V_m = \mathbb{C}_{s_\alpha(\lambda)}$.

As a consequence of exact sequences of Lemma 1.13.1, we can prove the following.

Let $w \in W$, α be a simple root, and set $v = ws_\alpha$.

Lemma 1.13.2. *If* $l(w) = l(v) + 1$, *then we have*

1. *If* $\langle \lambda, \alpha \rangle \geq 0$, *then* $H^j(w, \lambda) = H^j(v, H^0(s_\alpha, \lambda))$ *for all* $j \geq 0$.
2. *If* $\langle \lambda, \alpha \rangle \geq 0$, *then* $H^j(w, \lambda) = H^{j+1}(w, s_\alpha \cdot \lambda)$ *for all* $j \geq 0$.
3. *If* $\langle \lambda, \alpha \rangle \leq -2$, *then* $H^{j+1}(w, \lambda) = H^j(w, s_\alpha \cdot \lambda)$ *for all* $j \geq 0$.
4. *If* $\langle \lambda, \alpha \rangle = -1$, *then* $H^j(w, \lambda)$ *vanishes for every* $j \geq 0$.

The following consequence of Lemma 1.13.2 will be used to compute the cohomology modules in the later chapters. Recall that the Levi subgroup of $P_\alpha (\alpha \in S)$ containing T is denoted by L_α and the subgroup $L_\alpha \cap B$ by B_α. Let $\pi : \tilde{G} \longrightarrow G$ be the universal cover. Let $\tilde{L}_\alpha$ (respectively, $\tilde{B_\alpha}$) be the inverse image of L_α (respectively, of B_α).

Lemma 1.13.3. *Let V be an irreducible* L_α*-module. Let* λ *be a character of* B_α. *Then we have*

1. *As* L_α*-modules,* $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) \simeq V \otimes H^j(L_\alpha/B_\alpha, \mathbb{C}_\lambda)$.
2. *If* $\langle \lambda, \alpha \rangle \geq 0$, *then* $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda)$ *is isomorphic as an* L_α*-module to the tensor product of V and* $H^0(L_\alpha/B_\alpha, \mathbb{C}_\lambda)$. *Further, we have* $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ *for every* $j \geq 1$.
3. *If* $\langle \lambda, \alpha \rangle \leq -2$, *then* $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$, *and* $H^1(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda)$ *is isomorphic to the tensor product of V and* $H^0(L_\alpha/B_\alpha, \mathbb{C}_{s_\alpha \cdot \lambda})$.
4. *If* $\langle \lambda, \alpha \rangle = -1$, *then* $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ *for every* $j \geq 0$.

Proof. Proof (1). By [Jan03, Proposition 4.8, p.53, I] and [Jan03, Proposition 5.12, p.77, I], for all $j \geq 0$, we have the following isomorphism of L_α-modules:

$$H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) \simeq V \otimes H^j(L_\alpha/B_\alpha, \mathbb{C}_\lambda).$$

Proof of (2), (3) and (4) follows from Lemma 1.13.2 by taking $w = s_\alpha$ and the fact that $L_\alpha/B_\alpha \simeq P_\alpha/B$. ◻

Recall the structure of indecomposable modules over B_α and $\widetilde{B}_\alpha$ (see [BKS04, Corollary 9.1, p.130]).

Lemma 1.13.4. *1. Any finite dimensional indecomposable $\widetilde{B}_\alpha$-module V is isomorphic to $V' \otimes \mathbb{C}_\lambda$ for some irreducible representation V' of $\widetilde{L}_\alpha$ and for some character λ of $\widetilde{B}_\alpha$.*

2. Any finite dimensional indecomposable B_α-module V is isomorphic to $V' \otimes \mathbb{C}_\lambda$ for some irreducible representation V' of $\widetilde{L}_\alpha$ and for some character λ of $\widetilde{B}_\alpha$.

Proof. Proof of (1) follows from [BKS04, p.130, Corollary 9.1].

Proof of (2) follows from the fact that every B_α-module can be viewed as $\tilde{B}_\alpha$-module via natural homomorphism. ◻

Now, we prove the following:

Corollary 1.13.5. *Let $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression for w such that $\langle \alpha_{i_j}, \alpha_{i_r}\rangle = 0$ for every $j = 1, 2, ..., r-1$. Then, $H^0(w, \alpha_{i_r})$, is isomorphic to $H^0(s_{i_r}, \alpha_{i_r})(\simeq sl_{2,\alpha_{i_r}})$.*

Proof. Since $L_{\alpha_{i_r}}/B_{\alpha_{i_r}} \hookrightarrow P_{\alpha_{i_r}/B}$ is an isomorphism, we have

$$sl_{2,\alpha_{i_r}} \simeq H^0(L_{\alpha_{i_r}/B_{\alpha_{i_r}}}, \alpha_{i_r}) \simeq H^0(s_{i_r}, \alpha_{i_r}).$$

We note that $sl_{2,\alpha_{i_r}}$, gets a natural B-module structure via the above isomorphism $sl_{2,\alpha_{i_r}} \simeq H^0(s_{\alpha_{i_r}}, \alpha_{i_r})$.

Let $v = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$. If $l(v) = 0$, then $w = s_{i_r}$, and we are done. Otherwise, Let $v' = s_{i_2}\cdots s_{i_{r-1}}$. By induction on $l(v)$, we have

$$H^0(v', \alpha_{i_r}) = H^0(s_{i_r}, \alpha_{i_r}).$$

By SES, we have

$$H^0(w, \alpha_{i_r}) = H^0(s_{i_1}, H^0(v', \alpha_{i_r})) = H^0(s_{i_1}, H^0(s_{i_r}, \alpha_{i_r})).$$

Since $\langle \alpha_{i_r}, \alpha_{i_1}\rangle = 0$ and $\langle -\alpha_{i_r}, \alpha_{i_1}\rangle = 0$, by Lemma 1.13.4 $H^0(s_{i_r}, \alpha_{i_r})$ is the $B_{\alpha_{i_1}}$-module restriction of the three dimensional $L_{\alpha_{i_1}}$-module. Hence, the vector bundle $\mathcal{L}(s_{i_1}, H^0(s_{i_r}, \alpha_{i_r}))$ on $X(s_{i_1}) \simeq \mathbb{P}^1$ is the trivial bundle of rank 3. Thus, we have

$$H^0(s_{i_1}, H^0(s_{i_r}, \alpha_{i_r})) = H^0(s_{i_r}, \alpha_{i_r}).$$

◻

We recall the following vanishing results from [Kan16] (see [Kan16, Corollary 3.6] and [Kan16, Corollary 6.4]).

Lemma 1.13.6. *(S. S. Kannan): Let $w \in W$, and $\alpha \in R^+$. Then we have*

1. $H^j(w, \alpha) = 0$ *for all* $j \geq 2$.
2. *If G is simply laced,* $H^j(w, \alpha) = 0$ *for all* $j \geq 1$.

1.14 Vanishing of the Higher Cohomology of the Tangent Bundle of Z(w,i̲)

In this section, we prove that a BSDH variety has unobstructed deformations and it has no deformations whenever the group G is simply laced. We recall that the BSDH-variety corresponding to a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is denoted by $Z(w, \underline{i})$ and we denote the tangent bundle of $Z(w, \underline{i})$ by $T_{(w,\underline{i})}$, where $\underline{i} = (i_1, i_2, ..., i_r)$.

Let $v = s_{i_1} s_{i_2} \cdots s_{i_{r-1}}$ and $\underline{i'} = (i_1, i_2, ..., i_{r-1})$. Note that $l(v) = l(w) - 1$. Consider the fibration $f_r : Z(w, \underline{i}) \longrightarrow Z(v, \underline{i'})$ as in section 1.12. One can easily see that this fibration is the fibre product of $\pi_r : G/B \longrightarrow G/P_{\alpha_{i_r}}$ and $\pi_r \phi_v : Z(v, \underline{i'}) \longrightarrow G/P_{\alpha_{i_r}}$; namely, we have the following commutative diagram:

$$\begin{array}{ccc} Z(w,\underline{i}) = Z(v,\underline{i'}) \times_{G/P_{\alpha_{i_r}}} G/B & \xrightarrow{\phi_w} & G/B \\ \downarrow f_r & & \downarrow \pi_r \\ Z(v,\underline{i'}) & \xrightarrow{\pi_r \circ \phi_v} & G/P_{\alpha_{i_r}} \end{array}$$

The relative tangent bundle of π_r is the line bundle $\mathcal{L}(w_0, \alpha_{i_r})$. Hence the relative tangent bundle of f_r is $\phi_w^* \mathcal{L}(w_0, \alpha_{i_r})$. By taking the differentials of this smooth fibration f_r we obtain the following exact sequence:

$$0 \longrightarrow \phi_w^* \mathcal{L}(w_0, \alpha_{i_r}) \longrightarrow T_{(w,\underline{i})} \longrightarrow {f_r}^* T_{(v,\underline{i'})} \longrightarrow 0. \qquad (rel)$$

B. N. Chary, S. S. Kannan, and A. J. Parameswaran used the above short exact sequence (rel) and Lemma 1.13.6 to prove the following:

Theorem 1.14.1. *(B. N. Chary, S. S. Kannan, A. J. Parameswaran): Let $w \in W$, $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression for w and let $\underline{i} = (i_1, i_2, \ldots, i_r)$. Then, we have*

1. $H^j(Z(w, \underline{i}), T_{(w,\underline{i})}) = 0.$ *for all* $j \geq 2$.
2. *If G is simply laced,* $H^j(Z(w, \underline{i}), T_{(w,\underline{i})}) = 0$ *for all* $j \geq 1$.

Proof. We start by proving (2). We first recall the following isomorphism (see [BK05, Theorem 3.3.4(b)]):

$$H^j(Z(w,\underline{i}), \phi_w^*\mathcal{L}(w_0,\alpha_{i_r})) \simeq H^j(X(w), \mathcal{L}(w_0,\alpha_{i_r})) = H^j(w,\alpha_{i_r}) \text{ for all } j \geq 0.$$

Let $v = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$ and $\underline{i'} = (i_1, i_2, \ldots, i_{r-1})$. Since $f_r : Z(w,\underline{i}) \longrightarrow Z(v,\underline{i'})$ is a smooth fibration with fibre $\mathbb{P}^1$, by using [Har77, p.288, Corollary 12.9] and [Jan03, p369, Section 14.6(3)]we have
$H^j(Z(w,\underline{i}), f_r^*(T_{(v,\underline{i'})})) = H^j(Z(v,\underline{i'}), T_{(v,\underline{i'})})$ for every $j \geq 0$.

By considering the long exact sequence associated to the short exact sequence (*rel*) and using above arguments, we have the following long exact sequence of B-modules:

$$0 \longrightarrow H^0(w,\alpha_{i_r}) \to H^0(Z(w,\underline{i}), T_{(w,\underline{i})}) \to H^0(Z(v,\underline{i'}), T_{(v,\underline{i'})}) \to H^1(w,\alpha_{i_r}) \to H^1(Z(w,\underline{i}), T_{(w,\underline{i})}) \to$$
$$H^1(Z(v,\underline{i'}), T_{(v,\underline{i'})}) \to H^2(w,\alpha_{i_r}) \to H^2(Z(w,\underline{i}), T_{(w,\underline{i})}) \to H^2(Z(v,\underline{i'}), T_{(v,\underline{i'})}) \to H^3(w,\alpha_{i_r}) \to \cdots.$$

Since G is simply laced, by Lemma 1.13.6(2), we have $H^j(w,\alpha_{i_r}) = 0$ for every $j \geq 1$. Thus we have $H^j(Z(w,\underline{i}), T_{(w,\underline{i})}) = H^j(Z(v,\underline{i'}, T_{(v,\underline{i'})}))$ for every $j \geq 1$. Now the proof follows by induction on $l(w)$. Proof of (1) is similar by using Lemma 1.13.6(1). □

Note: The long exact sequence associated to the short exact sequence (rel) which is considered in the proof of the Theorem 1.14.1 will be used frequently in the future. We call this *LES*.

Theorem 1.14.1(1) yields $H^2(Z(w,\underline{i}), T_{(w,\underline{i})}) = 0$. Hence, we see that $Z(w,\underline{i})$ has unobstructed deformations. That is, $Z(w,\underline{i})$ admits a smooth versal deformation (see [Huy05, p.273, lines 19-21]).

If in addition G is simply laced, Theorem 1.14.1(2) yields $H^1(Z(w,\underline{i}), T_{(w,\underline{i})}) = 0$. Using [Huy05, p.272, Proposition 6.2.10], we see that $Z(w,\underline{i})$ has no deformations. That is, a BSDH variety for a simply laced group G is rigid.

Chapter 2

Parabolic subgroups and Automorphism groups of Schubert varieties

2.1 Notation and Result

Throughout this chapter we assume that G is simple adjoint type. Let T, B be as in previous chapter.

Our main goal in this chapter is to prove the following results:

Theorem 2.1.1. *(S. S. Kannan, P. Saha): Let P be a parabolic subgroup of G containing B properly. Then there is an element $w \in W$ such that $P = Aut^0(X(w))$ (see Theorem 0.0.1).*

Let $G = PSL(n+1, \mathbb{C})$. For $1 \leq r \leq n$ and $w \in W^{S\setminus\{\alpha_r\}}$, we denote the Schubert variety corresponding to w in the Grassmannian $G/P_{\hat{\alpha}_r}$, by $X_{P_{\hat{\alpha}_r}}(w)$.

Proposition 2.1.2. *(S. S. Kannan, P. Saha): Let $w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r) \in W(r)$. Let $J'(w) := \{i \in \{1, 2, \ldots, r-1\} : a_{i+1} - a_i \geq 2\}$, $J''(w) = \{1 + a_i : i \in J'(w)\}$ and $J(w) = \{\alpha_j : j \in \{1, \ldots, n\} \setminus J''(w)\}$. Then we have $P_{J(w)} = Aut^0(X_{P_{\hat{\alpha}_r}}(w))$ (see Proposition 0.0.2).*

For more precise statement see Proposition 2.5.2.

2.2 Important results going to be used in the proof of the Main Theorem

In this section, we gather some important results that are going to be used in the proof of the main theorem.

Let P be a parabolic subgroup of a semisimple algebraic group G of adjoint type. M. Demazure studied the automorphisms of the partial flag variety G/P. The following is proved by M. Demazure in [Dem77].

Theorem 2.2.1. *(M. Demazure: see [Dem77], [Akh95, Theorem 2, p.75]): Let G be a simple complex Lie group with trivial center and let $X := G/P$ be a flag manifold. Then, the connected automorphism group $Aut^0(X)$ coincides with G. Provided (G, P) is not any of the following:*

(1). G is of type B_n, $P = P_{\hat{\alpha}_r}$, $Aut^0(X)$ is of type D_{n+1}, $n \geq 3$.

(2). G is of type C_n, $P = P_{\hat{\alpha}_1}$, $Aut^0(X)$ is of type A_{2n-1}, $n \geq 2$.

(3). G is of type G_2, $P = P_{\hat{\alpha}_1}$, $Aut^0(X)$ is of type B_3.

In order to make this chapter self contained, we give a sketch of the proof of the Theorem 1.11.7(4),(5) for general Schubert variety in the following theorem.

Theorem 2.2.2. *Let $\phi_w : P_w \longrightarrow Aut^0(X(w))$ be the homomorphism induced by the action of P_w on $X(w)$. Then, we have $\phi_w : P_w \longrightarrow Aut^0(X(w))$ is an injective if and only if $w^{-1}(\alpha_0) < 0$.*

Sketch of the proof: Let T_w denote the tangent sheaf of $X(w)$. By [MO67, Theorem 3.7, p.17], we see that $Aut^0(X(w))$ is an algebraic group. Further, by [MO67, Lemma 3.4, p.13], it follows that the Lie algebra of $Aut^0(X(w))$ is isomorphic to the space of all global sections $H^0(X(w), T_w)$. Since T_w is a subsheaf of $T_{G/B}|_{X(w)} = \mathcal{L}(\mathfrak{g}/\mathfrak{b})$, we have an injective homomorphism $i : H^0(X(w), T_w) \hookrightarrow H^0(X(w), \mathcal{L}(\mathfrak{g}/\mathfrak{b}))$, where $T_{G/B} = \mathcal{L}(\mathfrak{g}/\mathfrak{b})$ is the tangent bundle on G/B. We note that ϕ_w induces a homomorphism $\psi_w : \mathfrak{p}_w \longrightarrow H^0(X(w), T_w)$ of Lie algebras, where $\mathfrak{p}_w$ is the Lie algebra of P_w.

Assume $\phi_w : P_w \longrightarrow Aut^0(X(w))$ is injective. Then the induced homomorphism $\psi_w : \mathfrak{p}_w \longrightarrow H^0(X(w), T_w) \subseteq H^0(X(w), T_{G/B}|_{X(w)})$ is injective. In particular, the $-\alpha_0$-weight space $H^0(X(w), T_{G/B}|_{X(w)})_{-\alpha_0}$ is non-zero. We have a short exact sequence $(0) \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{b} \longrightarrow (0)$ of B-modules. Applying $H^0(w, -)$ to this short exact sequence of B-modules, we obtain the following long exact sequence of B-modules:

$$\cdots \longrightarrow H^i(X(w), \mathcal{L}(\mathfrak{b})) \longrightarrow H^i(X(w), \mathcal{L}(\mathfrak{g})) \longrightarrow H^i(X(w), \mathcal{L}(\mathfrak{g}/\mathfrak{b})) \longrightarrow H^{i+1}(X(w), \mathcal{L}(\mathfrak{b})) \longrightarrow \cdots$$

On the otherhand, by [Kan16, Lemma 2.5(2), p.766], we have $H^i(X(w), \mathcal{L}(\mathfrak{g})) = (0)$ for every $i \geq 1$. Thus we have an exact sequence

$$(0) \longrightarrow H^0(X(w), \mathcal{L}(\mathfrak{b})) \longrightarrow H^0(X(w), \mathcal{L}(\mathfrak{g})) \longrightarrow H^0(X(w), \mathcal{L}(\mathfrak{g}/\mathfrak{b})) \longrightarrow H^1(X(w), \mathcal{L}(\mathfrak{b})) \longrightarrow (0)$$

of B-modules.

Moreover, by [CKP15, Lemma 4.2, p.675], we have $H^1(X(w), \mathcal{L}(\mathfrak{b}))_{-\alpha_0} = 0$. Therefore, we have a short exact sequence

$$(0) \longrightarrow H^0(X(w), \mathcal{L}(\mathfrak{b}))_{-\alpha_0} \longrightarrow H^0(X(w), \mathcal{L}(\mathfrak{g}))_{-\alpha_0} \longrightarrow H^0(X(w), \mathcal{L}(\mathfrak{g}/\mathfrak{b}))_{-\alpha_0} \longrightarrow (0)$$

of T-modules.

Note that by [Kan16, Lemma 2.5(1), p.766], we have $H^0(X(w), \mathcal{L}(\mathfrak{g})) = \mathfrak{g}$. Hence, we have a short exact sequence of T-modules

$$(0) \longrightarrow H^0(X(w), \mathcal{L}(\mathfrak{b}))_{-\alpha_0} \longrightarrow \mathfrak{g}_{-\alpha_0} \longrightarrow H^0(X(w), \mathcal{L}(\mathfrak{g}/\mathfrak{b}))_{-\alpha_0} \longrightarrow (0).$$

Since $\mathfrak{g}_{-\alpha_0}$ is one dimensional, $H^0(X(w), \mathcal{L}(\mathfrak{b}))_{-\alpha_0}$ is zero if and only if $H^0(X(w), \mathcal{L}(\mathfrak{g}/\mathfrak{b}))_{-\alpha_0}$ is non-zero.

Now we proceed the steps of the proof of [Kan16, Theorem 4.1(2), p.771] to conclude that $H^0(X(w), \mathcal{L}(\mathfrak{g}/\mathfrak{b}))_{-\alpha_0}$ is non-zero if and only if $w^{-1}(\alpha_0)$ is a negative root.

Conversely, if $w^{-1}(\alpha_0) < 0$, then by the above discussion we have $H^0(X(w), \mathcal{L}(\mathfrak{g}/\mathfrak{b}))_{-\alpha_0} \neq 0$. Note that since $w^{-1}(\alpha_0) < 0$, we have $w \neq 1$. Therefore, the action of P_w on $X(w)$ is non trivial. Hence, the homomorphism $\psi_w : \mathfrak{p}_w \longrightarrow H^0(X(w), T_w)$ of B-modules is non-zero. Therefore, the B-stable line $H^0(X(w), \mathcal{L}(\mathfrak{g}/\mathfrak{b}))_{-\alpha_0}$ is in the image $\psi_w(\mathfrak{p}_w) \subset H^0(X(w), T_w) \subset H^0(X(w), \mathcal{L}(\mathfrak{g}/\mathfrak{b}))$. Hence, we have $\mathfrak{g}_{-\alpha_0} \bigcap ker(\psi_w) = 0$. Thus, $\psi_w : \mathfrak{p}_w \longrightarrow H^0(X(w), T_w)$ is injective. Since the base field is $\mathbb{C}$, it follows that ϕ_w is separable. Hence, kernel of ψ_w is the Lie algebra of the kernel of ϕ_w. Therefore, $\phi_w : \mathfrak{p}_w \longrightarrow Aut^0(X(w))$ is injective.

2.3 Proof of theorem 2.1.1 except in three cases

In this section we prove Theorem 2.1.1 in all cases except in three cases:

(1) G is of type B_n and $P = P_{\alpha_n}$ is the minimal parabolic subgroup of G corresponding to α_n.

(2) G is of type C_n and $P = P_{\alpha_1}$ is the minimal parabolic subgroup of G corresponding to α_1.

(3) G is of type G_2 and $P = P_{\alpha_1}$ is the minimal parabolic subgroup of G corresponding to α_1.

The three cases left will be treated by Proposition 2.4.1.

Proof. Let P be a parabolic subgroup of G containing B properly. If $P = G$, then we take $w = w_0$, the longest element w_0 of W. In this case, by Theorem 2.2.1, we have the following:

$$Aut^0(X(w_0)) = Aut^0(G/B) = G.$$

Now we assume that P is any proper parabolic subgroup of G such that $B \subsetneq P \subsetneq G$. Since $B \subsetneq P \subsetneq G$, there is a subset $\emptyset \neq I \subsetneq S$ such that $P = P_I$. Consider $J = S \setminus I$. Hence, there exist unique elements $w_0^J \in W^J$ and $w_{0,J} \in W_J$ such that $w_0 = w_0^J \cdot w_{0,J}$. Consider the natural left action of G on G/B. Take $w = (w_0^J)^{-1}$. Then P is the stabilizer of $X(w)$, since $R^+(w^{-1}) \cap S = I$. The natural action of P on $X(w)$ induces a homomorphism,

$$\phi_w : P \longrightarrow Aut^0(X(w))$$

of algebraic groups.

Since $w^{-1}(\alpha_0) < 0$, by Theorem 2.2.2, $\phi_w : P \longrightarrow Aut^0(X(w))$ is injective.

Let $J' := -w_0(J)$, and $P' := P_{J'}$. Consider the natural morphism $\pi : G/B \longrightarrow G/P'$. We denote the restriction of π to $X(w)$ also by π. Then $\pi : X(w) \longrightarrow G/P'$ is a birational morphism. Therefore by [BK05, Theorem 3.3.4(a), p.96] and [BK05, Lemma 3.3.3(b), p.95] we have,

$$\pi_*(\mathcal{O}_{X(w)}) = \mathcal{O}_{G/P'}.$$

Thus from [Bri11, Corollary 2.2., p.45], π induces a homomorphism of algebraic groups

$$\pi_* : Aut^0(X(w)) \longrightarrow Aut^0(G/P').$$

Since π is birational, $\pi_* : Aut^0(X(w)) \longrightarrow Aut^0(G/P')$ is injective.

If G is of type B_n, C_n or G_2, then $w_0 = -id$ (see [Bou02, p.216, p.217, p.233]). If G is of type B_n and $P = P_{\alpha_n}$, then $I = \{\alpha_n\}$. Therefore $J' = -w_0(J) = J = S \setminus \{\alpha_n\}$ and $P' = P_{\hat{\alpha_n}}$. Thus (G, P') is one of the three types as in the statement of Theorem 2.2.1. If G is of the type C_n and $P = P_{\alpha_1}$, then $(G, P') = (G, P_{\hat{\alpha_1}})$ is one of the three types as in the statement of Theorem 2.2.1. If G is of type G_2 and $P = P_{\alpha_1}$ then $(G, P') = (G, P_{\hat{\alpha_1}}) = (G, P_{\alpha_2})$ is one of the three types as in the statement of Theorem 2.2.1. Similarly, we can see that if (G, P') is one of the three types as in Theorem 2.2.1, then (G, P) is one of the three types as in the statement of Proposition 2.4.1.

Case 1: G is not of type B_n, C_n and G_2. Then for any parabolic subgroup P of G, (G, P) is not one of the three types as in Proposition 2.4.1. Therefore (G, P') is not one of the three exceptional types as in the statement of Theorem 2.2.1.

Case 2: $G = B_n$ or C_n or G_2 and (G, P) is not one of the three types as in the statement of Proposition 2.4.1. In these cases $w_0 = -id$ and $J' = -w_0(J) = J = S \setminus I$. Therefore (G, P') is not one of the three exceptional types as in the statement of Theorem 2.2.1. Thus (G, P) is not one of the three types as in the statement of Proposition 2.4.1 if and only if (G, P') is not one of the three exceptional types as in the statement of Theorem 2.2.1. Hence, we have $Aut^0(G/P') = G$. Therefore $Aut^0(X(w))$ is a parabolic subgroup of G containing P. Since P is the stabilizer of $X(w)$, we have $P = Aut^0(X(w))$. Now, the proof follows from the proofs of Case 1 and Case 2. $\square$

2.4 Proof of theorem 2.1.1 in three left cases

To complete the proof of Theorem 2.1.1, it is sufficient to prove the following proposition. By (G, P) we mean G is a simple algebraic group of adjoint type over $\mathbb{C}$ and P is a parabolic subgroup of G containing B.

Proposition 2.4.1. *(S. S. Kannan, P. Saha): Let (G, P) be one of the following types:*

(1) G is of type B_n and $P = P_{\alpha_n}$ is the minimal parabolic subgroup of G corresponding to α_n.

(2) G is of type C_n and $P = P_{\alpha_1}$ is the minimal parabolic subgroup of G corresponding to α_1.

(3) G is of type G_2 and $P = P_{\alpha_1}$ is the minimal parabolic subgroup of G corresponding to α_1.

Then, there exists an element $w \in W$ such that $P = Aut^0(X(w))$.

Proof. Let $T_{X(w)}$ be the tangent sheaf of $X(w)$. Let $T_{G/B}$ be the restriction of the tangent bundle to $X(w)$. Then $T_{X(w)}$ is a subsheaf of $T_{G/B}$ on $X(w)$. By [MO67, Lemma 3.4, p.13] we have
$Lie(Aut^0(X(w)) = H^0(X(w), T_{X(w)}) \subset H^0(X(w), T_{G/B}) = H^0(w, \mathfrak{g}/\mathfrak{b})$.

As in the strategy of proof in Section 2.3, it is sufficient to prove that for all the three types (G, P) as above, there is an element $w \in W$ such that

(i) P is the stabilizer of $X(w)$ in G.

(ii) $w^{-1}(\alpha_0) < 0$.

(iii) $H^0(w, \mathfrak{g}/\mathfrak{b}) = \mathfrak{g}$.

For instance, let $\phi_w : P \longrightarrow Aut^0(X(w))$ be the natural homomorphism induced by the action of P on $X(w)$.

Since $w^{-1}(\alpha_0) < 0$, $\phi_w : P \longrightarrow Aut^0(X(w))$ is injective. Since $H^0(w, \mathfrak{g}/\mathfrak{b}) = \mathfrak{g}$, we have $H^0(X(w), T_{X(w)}) \subseteq \mathfrak{g}$. Therefore $Aut^0(X(w))$ is a closed subgroup of G containing P. Since P is the stabilizer of $X(w)$ in G, we have $P = Aut^0(X(w))$.

We first make a note about statement (ii) and statement (iii). Let $w \in W$ be such that $w^{-1}(\alpha_0) < 0$. To prove that $H^0(w, \mathfrak{g}/\mathfrak{b}) = \mathfrak{g}$, it is sufficient to prove that for any negative root β, the dimension of the weight space $H^0(w, \mathfrak{g}/\mathfrak{b})_\beta$ is one.

Proof of this note :

The restriction of the natural map $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{b}$ to $\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ is an isomorphism of T-modules and hence, we have $\mathfrak{g}/\mathfrak{b} = \bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha$. Since s_i permutes all positive roots other than α_i for every $1 \leq i \leq n$, every indecomposable

B_{α_i}-summand V of $\mathfrak{g}/\mathfrak{b}$ with highest weight, a positive root different from α_i is indeed an $\hat{L}_{\alpha_i}$-module and hence for every $\alpha \in R^+ \setminus S$, the dimension of the weight space $H^0(s_i, \mathfrak{g}/\mathfrak{b})_\alpha$ is one. Using this argument and by induction on length of w we see that the dimension of the weight space $H^0(w, \mathfrak{g}/\mathfrak{b})_\alpha$ is one for every $\alpha \in R^+ \setminus S$. Further, since $(\mathfrak{g}/\mathfrak{b})_\alpha$ is one dimensional for every simple root α, each fundamental coweight $h(\alpha_i)$ $(1 \leq i \leq n)$ appears exactly once. Hence, it is sufficient to prove that for any negative root β the dimension of the weight space $H^0(w, \mathfrak{g}/\mathfrak{b})_\beta$ is one.

We prove the existence of an element $w \in W$ satisfying the first two conditions and that the dimension of the weight space $H^0(w, \mathfrak{g}/\mathfrak{b})_\beta$ is one for any negative root β in all the three cases separately.

Case 1: Assume that G is of type B_n and $P = P_n$. For every $1 \leq r \leq n-1$, let $v_r = s_n s_{n-1} \cdots s_r$. Take $w = v_1 v_2 \cdots v_{n-1}$. It is easy to see that P_n is the stabilizer of $X(w)$.

In this case $\alpha_0 = \omega_2$. So, we have $v_1^{-1}(\alpha_0) = \alpha_2 + 2(\sum_{i=3}^{n} \alpha_i)$. This is the highest root of type B_{n-1} corresponding to the root system whose set of simple roots is $S \setminus \{\alpha_1\}$. By induction on rank of G, we have $w^{-1}(\alpha_0) = (v_2 \cdots v_{n-1})^{-1}(\alpha_2 + 2(\sum_{i=3}^{n} \alpha_i)) < 0$.

Now, if $v \in W$ is of minimal length such that the dimension of $H^0(v, \mathfrak{g}/\mathfrak{b})_\beta$ is at least two for some negative root β, then $\beta = -(\sum_{j=i}^{n} \alpha_j)$ for some $1 \leq i \leq n-1$.

Justification of the above statement: Clearly for any such v, $l(v) > 1$. Choose $\gamma \in S$ such that $l(s_\gamma v) = l(v) - 1$. Let $u = s_\gamma v$.

Then we have $\dim H^0(s_\gamma, H^0(u, \mathfrak{g}/\mathfrak{b}))_\beta \geq 2$.

If $\langle \beta, \gamma \rangle = 1$, then there exists an indecomposable B_γ-summand V of $H^0(u, \mathfrak{g}/\mathfrak{b})$ such that $H^0(u, V)_\beta \neq 0$. In this case, either $V = \mathbb{C}_\beta \oplus \mathbb{C}_{\beta-\gamma}$ or $V = \mathbb{C}_\beta$.

So we have $\dim H^0(s_\gamma, H^0(u, \mathfrak{g}/\mathfrak{b}))_\beta = 1$.

If $\langle \beta, \gamma \rangle = -1$, we have, either $V = \mathbb{C}_\beta \oplus \mathbb{C}_{\beta+\gamma}$ or $V = \mathbb{C}_{\beta+\gamma}$.

So we have $\dim H^0(s_\gamma, H^0(u, \mathfrak{g}/\mathfrak{b}))_\beta = 1$.

If $\langle \beta, \gamma \rangle = 2$ then there exists a unique indecomposable B_γ-summand V of $H^0(u, \mathfrak{g}/\mathfrak{b})$ with highest weght β.

Therefore $\dim H^0(s_\gamma, H^0(u, \mathfrak{g}/\mathfrak{b}))_\beta = 1$.

If $\langle \beta, \gamma \rangle = -2$ then there exists a unique indecomposable B_γ-summand of $H^0(u, \mathfrak{g}/\mathfrak{b})$ with highest weight $\beta + 2\gamma$. Therefore $\dim H^0(s_\gamma, H^0(u, \mathfrak{g}/\mathfrak{b}))_\beta = 1$.

Following the case by case analysis as above, we conclude that $\langle \beta, \gamma \rangle = 0$ and there is a unique indecomposable B_γ-summand V of $H^0(u, \mathfrak{g}/\mathfrak{b})$ such that $V = \mathbb{C}_{\beta+\gamma} \oplus \mathbb{C}_\beta$. In particular, we have $\beta + \gamma \in R^-$. Since G is of type B_n, we have $\gamma = \alpha_n$ and $\beta = -(\sum_{j=i}^{n} \alpha_j)$ for some $1 \leq i \leq n-1$.

By induction on the rank of G, we may assume that $H^0(v_2v_3\cdots v_{n-1},\mathfrak{g}/\mathfrak{b})_{-(\sum\limits_{j=i}^{n}\alpha_j)}$ is one dimensional for every $2\le i\le n-1$. Also $H^0(v_2v_3\cdots v_{n-1},\mathfrak{g}/\mathfrak{b})_{-(\sum\limits_{j=1}^{n}\alpha_j)}=0$.

Since $\langle\sum\limits_{j=i}^{n}\alpha_j,\alpha_1\rangle=0$ for every $3\le i\le n-1$, the restriction of the evaluation map

$H^0(s_1v_2v_3\cdots v_{n-1},\mathfrak{g}/\mathfrak{b})_{-(\sum\limits_{j=i}^{n}\alpha_j)}\longrightarrow H^0(v_2v_3\cdots v_{n-1},\mathfrak{g}/\mathfrak{b})_{-(\sum\limits_{j=i}^{n}\alpha_j)}$

is an isomorphism for every $3\le i\le n-1$ (see Lemma 1.13.1 and Lemma 1.13.2).

Since $\langle-(\sum\limits_{j=2}^{n}\alpha_j),\alpha_1\rangle=1$, we have

$H^0(s_1v_2v_3\cdots v_{n-1},\mathfrak{g}/\mathfrak{b})_{-(\sum\limits_{j=i}^{n}\alpha_j)}=H^0(s_1,H^0(v_2v_3\cdots v_{n-1},\mathfrak{g}/\mathfrak{b}))_{-(\sum\limits_{j=i}^{n}\alpha_j)}$

is one dimensional for every $i=1,2$ (see Lemma 1.13.1 and Lemma 1.13.2).

Now, it is easy to see that for every $2\le r\le n$ the evaluation map

$H^0(s_rs_{r-1}\cdots s_2,H^0(s_1v_2v_3\cdots v_{n-1},\mathfrak{g}/\mathfrak{b}))_{-(\sum\limits_{j=i}^{n}\alpha_j)}\longrightarrow H^0(s_1v_2v_3\cdots v_{n-1},\mathfrak{g}/\mathfrak{b})_{-(\sum\limits_{j=i}^{n}\alpha_j)}$

is an isomorphism for every $1\le i\le n$ by induction on r and using Lemma 1.13.1, Lemma 1.13.2. Thus, the space $H^0(w,\mathfrak{g}/\mathfrak{b})_\alpha$ is one dimensional for every negative root α.

Case 2: Assume that G is of type C_n ($n\ge3$) and $P=P_1$. Take $w=s_1s_2\cdots s_n$. In this case we have $\alpha_0=2\omega_1$, and $w^{-1}(\alpha_0)=-\alpha_n$. Further, the stabilizer of $X(w)$ in G is P_1.

First note that

$H^0(s_n,\mathfrak{g}/\mathfrak{b})=(\bigoplus\limits_{\alpha\in R^+}\mathbb{C}_\alpha)\oplus\mathbb{C}_{h(\alpha_n)}\oplus\mathbb{C}_{-\alpha_n}$ (see Lemma 1.13.1 and Lemma 1.13.2).

Further, we have

$H^0(s_{n-1}s_n,\mathfrak{g}/\mathfrak{b})=H^0(s_{n-1},H^0(s_n,\mathfrak{g}/\mathfrak{b}))=(\bigoplus\limits_{\alpha\in R^+}\mathbb{C}_\alpha)\oplus\mathbb{C}_{h(\alpha_n)}\oplus\mathbb{C}_{-\alpha_n}\oplus\mathbb{C}_{h(\alpha_{n-1})}$
$\oplus\mathbb{C}_{-\alpha_{n-1}}\oplus\mathbb{C}_{-(\alpha_{n-1}+\alpha_n)}\oplus\mathbb{C}_{-(2\alpha_{n-1}+\alpha_n)}$ (see Lemma 1.13.1 and Lemma 1.13.2).

By using Lemma 1.13.1, Lemma 1.13.2 and the descending induction on $1\le r\le n-1$, we see that

$$H^0(s_r\cdots s_{n-1}s_n,\mathfrak{g}/\mathfrak{b})=(\bigoplus_{\alpha\in R^+}\mathbb{C}_\alpha)\oplus(\bigoplus_{i=r}^{n}\mathbb{C}_{h(\alpha_i)})\oplus\mathbb{C}_{-\mu}$$

where μ runs over all positive roots in $\sum_{i=r}^{n}\mathbb{Z}_{\ge0}\alpha_i$. Thus, we have $H^0(w,\mathfrak{g}/\mathfrak{b})=\mathfrak{g}$.

Case 3: Assume that G is of type G_2 and $P=P_1$. Take $w=s_1s_2s_1s_2$. Here, we follow the convention in [Hum72]. In this case, we have $\alpha_0=3\alpha_1+2\alpha_2$. Further, $w^{-1}(\alpha_0)=-\alpha_2$.

First note that $H^0(s_2, \mathfrak{g}/\mathfrak{b}) = (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus \mathbb{C}_{h(\alpha_2)} \oplus \mathbb{C}_{-\alpha_2}$ (see Lemma 1.13.1 and Lemma 1.13.2).

$H^0(s_1, H^0(s_2, \mathfrak{g}/\mathfrak{b})) = (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus \mathbb{C}_{h(\alpha_2)} \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{h(\alpha_1)} \oplus \mathbb{C}_{-\alpha_1} \oplus (\bigoplus_{i=1}^{3} \mathbb{C}_{-(\alpha_2 + i\alpha_1)})$ (see Lemma 1.13.1 and Lemma 1.13.2).

Therefore we have

$H^0(s_1 s_2, \mathfrak{g}/\mathfrak{b}) = (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus \mathbb{C}_{h(\alpha_2)} \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{h(\alpha_1)} \oplus \mathbb{C}_{-\alpha_1} \oplus (\bigoplus_{i=1}^{3} \mathbb{C}_{-(\alpha_2 + i\alpha_1)})$.

$H^0(s_2, H^0(s_1 s_2, \mathfrak{g}/\mathfrak{b})) = (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus \mathbb{C}_{h(\alpha_2)} \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{h(\alpha_1)} \oplus \mathbb{C}_{-\alpha_1} \oplus (\bigoplus_{i=1}^{3} \mathbb{C}_{-(\alpha_2 + i\alpha_1)}) \oplus \mathbb{C}_{-(3\alpha_1 + 2\alpha_2)} = \mathfrak{g}$ (see Lemma 1.13.1 and Lemma 1.13.2).

Therefore we have

$H^0(s_2 s_1 s_2, \mathfrak{g}/\mathfrak{b}) = (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus \mathbb{C}_{h(\alpha_2)} \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{h(\alpha_1)} \oplus \mathbb{C}_{-\alpha_1} \oplus (\bigoplus_{i=1}^{3} \mathbb{C}_{-(\alpha_2 + i\alpha_1)}) \oplus \mathbb{C}_{-(3\alpha_1 + 2\alpha_2)}$.

Thus, we have $H^0(w, \mathfrak{g}/\mathfrak{b}) = H^0(s_1, \mathfrak{g}) = \mathfrak{g}$.

□

Example 2.4.2. *Let $G = PSL(3, \mathbb{C})$. In this case, B is the set of invertible lower triangular matrices, $P_{\alpha_1} = Aut^0(X(s_1 s_2))$ and $X(s_1 s_2)$ is smooth.*

Remark 2.4.3. *In Theorem 2.1.1, for a given parabolic subgroup P of G containing B properly, the Schubert variety $X(w)$ for which $P = Aut^0(X(w))$ is not necessarily smooth. For example, take $G = PSL(4, \mathbb{C})$, and $P_{\alpha_2} = Aut^0(X(s_2 s_1 s_3 s_2))$. Note that $X(s_2 s_1 s_3 s_2)$ is not smooth (see [LS90, Theorem 2.2, p.48]).*

2.5 Automorphism groups of Schubert varieties in partial flag varieties of type A_n

In this section, we discuss about parabolic subgroups of $G = PSL(n+1, \mathbb{C})$ and connected component, containing identity element of the group of all algebraic automorphisms of Schubert varieties in the Grassmannian $G/P_{\hat{\alpha}_r}$, where $1 \leq r \leq n$ and $P_{\hat{\alpha}_r} = P_{S \setminus \{\alpha_r\}}$.

Lemma 2.5.1. *Let $G = PSL(n+1, \mathbb{C})$. Let $1 \leq r \leq n$ and $w \in W^{S \setminus \{\alpha_r\}}$. Then $w^{-1}(\alpha_0) < 0$ if and only if there exists an increasing sequence $1 \leq a_1 < a_2 < \cdots < a_r = n$ of positive integers such that $w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r)$.*

Proof. Note that $\alpha_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Let $w \in W^{S \setminus \{\alpha_r\}}$ be such that $w \neq id$. Then there exists an integer $1 \leq i \leq r$ and an increasing sequence of positive integers $i \leq a_i < a_{i+1} < \cdots < a_r \leq n$ such that

$w = (s_{a_i} \cdots s_i)(s_{a_{i+1}} \cdots s_{i+1}) \cdots (s_{a_r} \cdots s_r)$. Now, it is easy to see that $w^{-1}(\alpha_0) < 0$ if and only if $i = 1$ and $a_r = n$. □

Let $W(r) = \{w \in W^{S \setminus \{\alpha_r\}} : w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r)$, where $1 \leq a_1 < a_2 < \cdots < a_r = n\}$. For $w \in W^{S \setminus \{\alpha_r\}}$, we denote the Schubert variety in the Grassmannian $G/P_{\hat{\alpha}_r}$ corresponding to w by $X_{P_{\hat{\alpha}_r}}(w)$.

Proposition 2.5.2. *(S. S. Kannan, P. Saha): Let* $w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r) \in W(r)$. *Let* $J'(w) := \{i \in \{1, 2, \ldots, r-1\} : a_{i+1} - a_i \geq 2\}$, $J''(w) = \{1 + a_i : i \in J'(w)\}$ *and* $J(w) = \{\alpha_j : j \in \{1, \ldots, n\} \setminus J''(w)\}$. *Then we have* $P_{J(w)} = Aut^0(X_{P_{\hat{\alpha}_r}}(w))$.

Proof. Let P_w be the stabilizer of $X_{P_{\hat{\alpha}_r}}(w)$ in G. First we show that $P_w = P_{J(w)}$. If $a_{i+1} - a_i \geq 2$ for some $1 \leq i \leq r-1$ then $s_{1+a_i} w > w$, and $s_{1+a_i} w \in W^{S \setminus \{\alpha_r\}}$. Hence s_{1+a_i} is not in the Weyl group of P_w. Therefore P_w is a subgroup of $P_{J(w)}$. Let $R(P_{\hat{\alpha}_r}) = R \cap (\sum_{\alpha \in S \setminus \{\alpha_r\}} \mathbb{Z}\alpha)$. Further, it is easy to see that for $\alpha \in J(w)$ either we have $w^{-1}(\alpha) < 0$ or $w^{-1}(\alpha) \in R(P_{\hat{\alpha}_r})$. Therefore $P_{J(w)} \subseteq P_w$.

Let $\psi_w : P_{J(w)} \longrightarrow Aut^0(X_{P_{\hat{\alpha}_r}}(w))$ be the natural homomorphism induced by action of $P_{J(w)}$ on $X_{P_{\hat{\alpha}_r}}(w)$.

Since $w \in W(r)$, $w^{-1}(\alpha_0) < 0$ (see Lemma 2.5.1). Therefore $\psi_w : P_{J(w)} \longrightarrow Aut^0(X_{P_{\hat{\alpha}_r}}(w))$ is injective.

Let $\mathfrak{p}_{\hat{\alpha}_r}$ be the Lie algebra of $P_{\hat{\alpha}_r}$. Since G is simply laced, the restriction map $H^0(w_{0,r}, \mathfrak{g}/\mathfrak{p}_{\hat{\alpha}_r}) \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p}_{\hat{\alpha}_r})$ is surjective, where $w_{0,r} \in W^{S \setminus \{\alpha_r\}}$ is the minimal representative of w_0 (see [Kan16, Lemma 3.5(3), p.770]).

Further, since $w^{-1}(\alpha_0) < 0$, $H^0(w_{0,r}, \mathfrak{g}/\mathfrak{p}_{\hat{\alpha}_r}) = \mathfrak{g} \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p}_{\hat{\alpha}_r})$ is an isomorphism.

Therefore we have $H^0(X_{P_{\hat{\alpha}_r}}(w), T_{X_{P_{\hat{\alpha}_r}}(w)}) \subseteq \mathfrak{g}$. Hence $Aut^0(X_{P_{\hat{\alpha}_r}}(w))$ is a closed subgroup of G containing $P_{J(w)}$. Thus we have $P_{J(w)} = Aut^0(X_{P_{\hat{\alpha}_r}}(w))$. □

Corollary 2.5.3. *Let* $B \subsetneq P$ *be a parabolic subgroup of* G *and* $w \in W^{S \setminus \{\alpha_r\}}$ *such that* $P = Aut^0(X_{P_{\hat{\alpha}_r}}(w))$. *Then we have* $P = P_{J(w)}$.

Corollary 2.5.4. *(1) If* $P \neq G$, *then there is no element* $w \in W^{S \setminus \{\alpha_1\}}$ *such that* $P = Aut^0(X_{P_{\hat{\alpha}_1}}(w))$.

(2) If $P \neq G$, *then there is no element* $w \in W^{S \setminus \{\alpha_n\}}$ *such that* $P = Aut^0(X_{P_{\hat{\alpha}_n}}(w))$.

Proof. Proof of (1): The Schubert varieties in $G/P_{\hat{\alpha}_1}$ are projective space $\mathbb{P}^i$ $(0 \leq i \leq n)$. Therefore the automorphism groups of these Schubert varieties are $PSL(i+1, \mathbb{C})$ $(0 \leq i \leq n)$. Further, the map ϕ_w is injective for only one w.

Proof of (2): Proof of (2) is similar to that of (1).

□

Chapter 3

Rigidity of Bott-Samelson-Demazure-Hansen variety

In this chapter, we will assume that $G = PSO(2n+1, \mathbb{C})(n \geq 3)$[1]. Note that w_0 is equal to $-id$. We recall the following Proposition from [YZ08, Proposition 1.3, p.858].

Proposition 3.0.1. *(S. W. Yang, A. Zelevinsky): Let $c \in W$ be a Coxeter element, let ω_i be the fundamental weight corresponding to the simple root α_i. Then there exists a least positive integer $h(i,c)$ such that $c^{h(i,c)}(\omega_i) = w_0(\omega_i)$.*

Lemma 3.0.2. *Let $c \in W$ be a Coxeter element. Then we have*

(1) $w_0 = c^n$.

(2) For any sequence $\underline{i}^r (1 \leq r \leq n)$ of reduced expressions of c; the sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^n)$ is a reduced expression of w_0.

Proof. Note that for $n \geq 3$, there is an isomorphism of Weyl group of B_n and Weyl group of C_n sending $s_i \mapsto s_i$ for $(1 \leq i \leq n)$. Proof of the lemma holds in the case of type C_n for $(n \geq 3)$ (see [CK17, Lemma 4.2, p.441]). Therefore lemma holds for type B_n $(n \geq 3)$.

□

Lemma 3.0.3. *Let $n \geq a_1 > a_2 > \cdots > a_{r-1} > a_r \geq 1$ be a decreasing sequence of integers. Then, $w = (\prod_{j=a_1}^{n} s_j)(\prod_{j=a_2}^{n} s_j)\cdots(\prod_{j=a_{r-1}}^{n} s_j)(\prod_{j=a_r}^{n-1} s_j)$ is a reduced expression of w.*

[1]Note that $PSO(2n+1, \mathbb{C}) = SO(2n+1, \mathbb{C})$.

Proof. Note that for $n \geq 3$, there is an isomorphism of Weyl group of B_n and Weyl group of C_n sending $s_i \mapsto s_i$ for $(1 \leq i \leq n)$. Proof of the lemma holds in the case of type C_n for $(n \geq 3)$ (see [CK17, Lemma 4.3,p.441]). Therefore lemma holds for type B_n $(n \geq 3)$. $\square$

Let c be a Coxeter element in W. We take a reduced expression

$c = [a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$, where $[i, j] = s_i s_{i+1} \cdots s_j$ for $i \leq j$ and $n \geq a_1 > a_2 > \cdots > a_k = 1$.

Then we have following:

Lemma 3.0.4. *(1) For all* $1 \leq i \leq k - 1$,

$$c^i = (\prod_{l_1=1}^{i} [a_{l_1}, n])(\prod_{l_2=i+1}^{k} [a_{l_2}, a_{l_2-i} - 1])(\prod_{l_3=1}^{i-1} [a_{l_k}, a_{k-i+l_3} - 1]).$$

(2) For all $k \leq j \leq n$,

$$c^j = (\prod_{l_1=1}^{k-1} [a_{l_1}, n])([a_k, n]^{j+1-k})(\prod_{l_2=1}^{k-1} [a_k, a_{l_2} - 1]).$$

(3) The expressions of c^i *for* $1 \leq i \leq n$ *as in* (1) *and* (2) *are reduced.*

Proof. Note that for $n \geq 3$, there is an isomorphism of Weyl group of B_n and Weyl group of C_n sending $s_i \mapsto s_i$ for $(1 \leq i \leq n)$. Proof of the lemma holds in the case of type C_n for $(n \geq 3)$ (see [CK17, Lemma 4.4, p.442]). Therefore lemma holds for type B_n $(n \geq 3)$. $\square$

3.1 Cohomology modules $H^1(w, \alpha_j)$ where $j \neq n - 1$ and $w \in W$

In this section, we prove that $H^1(w, \alpha_j) = 0$ for every $w \in W$ and $j \neq n - 1$.

Lemma 3.1.1. *Let* $v \in W$ *and* $\alpha \in S$. *Then* $H^1(s_j, H^0(v, \alpha)) = 0$ *for* $j \neq n$.

Proof. By [Kan16, Corollary 5.6, p.778] we have $H^1(w, \alpha_n) = 0$. Therefore, we may assume that α is a long simple root. If $H^1(s_j, H^0(v, \alpha))_\mu \neq 0$, then there exists an indecomposable $\tilde{L}_{\alpha_j}$-summand V of $H^0(v, \alpha)$ such that $H^1(s_j, V)_\mu \neq 0$. By Lemma 2.4, we have $V \simeq V' \otimes \mathbb{C}_\lambda$ for some character λ of $\tilde{B}_{\alpha_j}$ and for some irreducible $\tilde{L}_{\alpha_j}$-module V'. Since $H^1(s_j, V)_\mu \neq 0$ from Lemma 1.13.3(3) we have $\langle \lambda, \alpha_j \rangle \leq -2$. Since α is a long root, there exists $w \in W$ such that $w(\alpha) = \alpha_0$. Thus $H^0(v, \alpha) \subseteq H^0(vw, \alpha_0)$. Again, since α_0 is highest long root, $H^0(w_0, \alpha_0) = \mathfrak{g} \longrightarrow H^0(vw, \alpha_0)$ is surjective. Let μ' be the lowest weight of V. Then by the above argument μ' is a root. Therefore we have $\mu' = \mu_1 + \lambda$, where μ_1 is the lowest weight of V'. Hence, we have $\langle \mu', \alpha_j \rangle \leq -2$. Since α_j is a long root and μ' is a root, we have $\langle \mu', \alpha_j \rangle = -1, 0, 1$. This is a contradiction. Thus we have $H^1(s_j, H^0(v, \alpha))_\mu = 0$. $\square$

Lemma 3.1.2. *Let $v \in W$ and $\alpha_j \in S$ be such that $j \neq n-1$. Then we have $H^1(s_k, H^0(v, \alpha_j)) = 0$, for every $k = 1, 2, \ldots, n$.*

Proof. **Step 1:** $H^0(v, \alpha_j)_{-(\alpha_{n-1}+2\alpha_n)} = 0$.
Case 1:
Assume that $j = n$, choose an element $u \in W$ of minimal length such that $u^{-1}(\alpha_n) = \beta_0$, the highest short root. Then we have $H^0(v, \alpha_j) \subseteq H^0(vu, \beta_0)$.

Since β_0 is dominant weight the natural restriction map

$$H^0(w_0, \beta_0) \longrightarrow H^0(vu, \beta_0)$$

is surjective.

Hence $H^0(v, \alpha_j)_\mu \neq 0$ implies either $\mu = 0$ or μ is a short root.
Therefore, we have $H^0(v, \alpha_j)_{-(\alpha_{n-1}+2\alpha_n)} = 0$.
Case 2:
Assume that $1 \leq j \leq n-2$. Note that if $H^0(v, \alpha_j)_\mu \neq 0$ then either $\mu = \alpha_j, 0$ or $\mu \leq -\alpha_j$ (see [CK17, Corollary 4.5, p.678]).

Hence $H^0(v, \alpha_j)_{-(\alpha_{n-1}+2\alpha_n)} = 0$.

Step 2: If $H^0(v, \alpha_j)_{-(\beta_i+2\alpha_n)} \neq 0$ for some $1 \leq i \leq n-2$, then $H^0(v, \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$.

Proof of Step 2: If $v = id$, we are done. So choose $1 \leq t \leq n$ such that $l(s_t v) = l(v) - 1$. Let $v' = s_t v$. Then $H^0(v, \alpha_j) = H^0(s_t, H^0(v', \alpha_j))$.
Case 1: Assume that $t = n$. In this case $\langle -(\beta_i + 2\alpha_n), \alpha_t \rangle = -2$. If $H^0(v, \alpha_j)_{-(\beta_i+2\alpha_n)} \neq 0$, then there is an indecomposable B_{α_n}-summand V of $H^0(v', \alpha_j)$ with highest weight $-\beta_i$. Since $\langle -\beta_i, \alpha_n \rangle = 2$, we have $H^0(s_t, V)_{-(\beta_i+\alpha_n)} \neq 0$. Therefore we have $H^0(v, \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$.
Case 2: Assume that $t = n-1$. In this case $\langle -(\beta_i + 2\alpha_n), \alpha_t \rangle = 1$. If $H^0(v, \alpha_j)_{-(\beta_i+2\alpha_n)} \neq 0$, then there is an indecomposable B_{α_n}-summand V of $H^0(v', \alpha_j)$ with highest weight $-(\beta_i + 2\alpha_n)$. Thus by induction hypothesis we have $H^0(v', \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$. Since $\langle -(\beta_i + \alpha_n), \alpha_t \rangle = 0$, we have $H^0(v, \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$.
Case 3: Assume that $1 \leq t \leq n-2$. In this case $\langle -(\beta_i + 2\alpha_n), \alpha_t \rangle = -1, 0$ or 1.
Assume that $i = t$. Then we have $\langle -(\beta_i + 2\alpha_n), \alpha_t \rangle = -1$. If further $H^0(v, \alpha_j)_{-(\beta_i+2\alpha_n)} \neq 0$, then there is an indecomposable B_{α_t}-summand V of $H^0(v', \alpha_j)$ with highest weight $-(\beta_{i+1} + 2\alpha_n)$. Therefore we have $H^0(v', \alpha_j)_{-(\beta_{i+1}+2\alpha_n)} \neq 0$. It is clear from Step: 1 that $t + 1 \leq n-2$. Therefore by induction $H^0(v', \alpha_j)_{-(\beta_{t+1}+\alpha_n)} \neq 0$. Since $\langle -(\beta_{t+1} + \alpha_n), \alpha_t \rangle = 1$, we have $H^0(v, \alpha_j)_{-(\beta_t+\alpha_n)} \neq 0$.

Assume that $1 \leq t \leq i-2$ or $i+1 \leq t \leq n-2$. Then we have $\langle -(\beta_i + 2\alpha_n), \alpha_t \rangle = 0$. Thus $H^0(v, \alpha_j)_{-(\beta_t+2\alpha_n)} = H^0(v', \alpha_j)_{-(\beta_t+2\alpha_n)} \neq 0$. Therefore by induction $H^0(v, \alpha_j)_{-(\beta_t+\alpha_n)} \neq 0$. Since

$\langle -(\beta_i + \alpha_n), \alpha_t \rangle = 0$, we have $H^0(v, \alpha_j)_{-(\beta_i+2\alpha_n)} \neq 0$.

Assume that $i = t + 1$. Since $\langle -(\beta_i + \alpha_n), \alpha_t \rangle = 1$, then there is an indecomposable B_{α_t}-summand V of $H^0(v', \alpha_j)$ such that: $V = \mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_{i-1}+2\alpha_n)}$ or $V = \mathbb{C}_{-(\beta_i+2\alpha_n)}$. Then we have $H^0(v', \alpha_j)_{-(\beta_t+2\alpha_n)} \neq 0$. Therefore by induction $H^0(v', \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$. Since $\langle -(\beta_t + \alpha_n), \alpha_t \rangle = 1$, we have $H^0(v, \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$ Hence the proof of Step 2.

Proof of Lemma :

Case 1: Assume that $k \neq n$. Then by Lemma 3.1.1 we have $H^1(s_k, H^0(v, \alpha_j)) = 0$.

Case 2: Assume that $k = n$. By Step 1 we see that $H^0(v, \alpha_j)_{-(\alpha_{n-1}+2\alpha_n)} = 0$. Note that if β is a root such that $H^0(v, \alpha_j)_\beta \neq 0$ and $\langle \beta, \alpha_n \rangle = -2$, then we have $\beta = -(\beta_i + 2\alpha_n)$ for some $1 \leq i \leq n-2$.

By Step 2 if $H^0(v, \alpha_j)_{-(\beta_i+2\alpha_n)} \neq 0$ for some $1 \leq i \leq n-2$, then $H^0(v, \alpha_j)_{-(\beta_i+\alpha_n)} \neq 0$. Therefore $\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}$ is an indecomposable B_{α_n}-summand of $H^0(v, \alpha_j)$. By Lemma 1.13.4, $\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}$ is isomorphic to $V \otimes \mathbb{C}_{-\omega_n}$, where V is an irreducible $\tilde{L}_{\alpha_n}$-module. Therefore by Lemma 1.13.3(4) we have $H^1(s_k, \mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}) = 0$. Thus our result follows. □

Lemma 3.1.3. *Let w be an element of W and α_j be an element of S such that $j \neq n-1$. Then $H^1(w, \alpha_j) = 0$.*

Proof. We will prove by induction on length of w. If length of w is 0, then $w = id$. Thus it follows trivially. Now suppose $w \in W$ such that $l(w) \geq 1$. Then there exists a simple root $\alpha \in S$ such that $l(s_\alpha w) = l(w) - 1$. Then using SES:

$$0 \longrightarrow H^1(s_\alpha, H^0(s_\alpha w, \alpha_j)) \longrightarrow H^1(w, \alpha_j) \longrightarrow H^0(s_\alpha, H^1(s_\alpha w, \alpha_j)) \longrightarrow 0.$$

From the above SES using induction hypothesis and Lemma 3.1.2, we get $H^1(w, \alpha_j)) = 0$ for $j \neq n-1$.

□

3.2 Cohomology module H^0 of the relative tangent bundle

In this section we describe the weights of H^0 of the relative tangent bundle.

Notation:

Let c be a Coxeter element of W. We take a reduced expression $c = [a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$, where $[i, j] = s_i s_{i+1} \cdots s_j$ for $i \leq j$ and $n \geq a_1 > a_2 > \cdots > a_k = 1$.

Let $\beta_i = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-1}$ for all $1 \leq i \leq n-1$.

For $1 \leq r \leq k$, let $n \geq a_1 > a_2 > a_3 > \cdots > a_r \geq 1$ be a decreasing sequence of integers.

Let $w_r = (\prod_{j=a_1}^{n} s_j)(\prod_{j=a_2}^{n} s_j)(\prod_{j=a_3}^{n} s_j)\cdots(\prod_{j=a_{r-1}}^{n} s_j)(\prod_{j=a_r}^{n-1} s_j)$

and let

$\tau_r = (\prod_{j=a_1}^{n} s_j)(\prod_{j=a_2}^{n} s_j)(\prod_{j=a_3}^{n} s_j)\cdots(\prod_{j=a_{r-1}}^{n} s_j)(\prod_{j=a_r}^{n-2} s_j)$.

Note that $l(w_r) = l(\tau_r) + 1$.

Lemma 3.2.1. *Assume that $r \geq 3$.*

(1) *Let $v = s_{a_{r-1}} s_{a_{r-1}+1} \cdots s_n s_{a_r} s_{a_r+1} \cdots s_{n-1}$. Then we have*

$$H^0(v, \alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1} (\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})})$$
$$\bigoplus_{i=a_{r-1}}^{n-2} \mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}$$

(2) *Let $v' = s_1 \cdots s_n s_1 \cdots s_{n-1}$. Then we have*

$$H^0(v', \alpha_{n-1}) = \bigoplus_{i=1}^{n-2} (\mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}$$

Proof. Proof of (1): Let $u = s_{a_r} s_{a_r+1} \cdots s_{n-1}$. By using SES we have

$H^0(u, \alpha_{n-1}) = \mathbb{C}h(\alpha_{n-1}) \oplus (\bigoplus_{j=a_r}^{n-1} \mathbb{C}_{-\beta_j})$. Since $\langle -\beta_j, \alpha_n \rangle = 2$ for all $a_r \leq j \leq n-1$,

by using SES we have

$$H^0(s_n u, \alpha_{n-1}) = \mathbb{C}h(\alpha_{n-1}) \oplus (\bigoplus_{j=a_r}^{n-1} \mathbb{C}_{-\beta_j} \oplus \mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)}). \qquad (3.2.1.1)$$

Since $\mathbb{C}h(\alpha_{n-1}) \oplus \mathbb{C}_{-\alpha_{n-1}}$ is an indecomposable $B_{\alpha_{n-1}}$-module,by Lemma 1.13.4, we have

$\mathbb{C}h(\alpha_{n-1}) \oplus \mathbb{C}_{-\alpha_{n-1}} = V \otimes \mathbb{C}_{-\omega_{n-1}}$, where V is the two dimensional irreducible representation of $\tilde{L}_{\alpha_{n-1}}$.

Therefore by Lemma 1.13.3(4) we have

$H^0(\tilde{L}_{\alpha_{n-1}}/\tilde{B}_{\alpha_{n-1}}, \mathbb{C}h(\alpha_{n-1}) \oplus \mathbb{C}_{-\alpha_{n-1}}) = 0$.

Moreover, we have $\langle -(\beta_{n-1}+\alpha_n), \beta_{n-1} \rangle = -1$, $\langle -(\beta_{n-1}+2\alpha_n), \beta_{n-1} \rangle = 0$, $\langle -\beta_j, \beta_{n-1} \rangle = -1$, $\langle -(\beta_j+\alpha_n), \beta_{n-1} \rangle = 0$ and $\langle -(\beta_j+2\alpha_n), \beta_{n-1} \rangle = 1$ for all $a_r \leq j \leq n-2$.

Therefore we have

$$H^0(s_{n-1} s_n u, \alpha_{n-1}) = (\bigoplus_{j=a_r}^{n-2} \mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})}) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}. \qquad (3.2.1.2)$$

Claim: For $a_{r-1} \leq k \leq n-2$

$$H^0(s_ks_{k+1}\cdots s_nu,\alpha_{n-1}) = \bigoplus_{j=a_r}^{k-1}(\mathbb{C}_{-(\beta_j+\alpha_n)}\oplus\mathbb{C}_{-(\beta_j+2\alpha_n)}\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_k)})$$
$$\bigoplus_{j=k}^{n-2}(\mathbb{C}_{-(\beta_j+2\alpha_n)}\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{j+1})})\oplus\mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Proof of the claim: We will prove by descending induction on k.

By hypothesis we have

$$H^0(s_{k+1}\cdots s_nu,\alpha_{n-1}) = \bigoplus_{j=a_r}^{k}(\mathbb{C}_{-(\beta_j+\alpha_n)}\oplus\mathbb{C}_{-(\beta_j+2\alpha_n)}\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{k+1})})$$
$$\bigoplus_{j=k+1}^{n-2}(\mathbb{C}_{-(\beta_j+2\alpha_n)}\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{j+1})})\oplus\mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}. \quad (3.2.1.3)$$

Let $V = \bigoplus_{j=a_r}^{k-1}(\mathbb{C}_{-(\beta_j+\alpha_n)}\oplus\mathbb{C}_{-(\beta_j+2\alpha_n)}\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{k+1})})$ and
$V' = \bigoplus_{j=k+2}^{n-2}(\mathbb{C}_{-(\beta_j+2\alpha_n)}\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{j+1})})\oplus\mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}$.

Then roots $\{(\beta_j+\alpha_n),(\beta_j+2\alpha_n),(\beta_j+2\alpha_n+\beta_{n-1}),\ldots,(\beta_j+2\alpha_n+\beta_{k+2}) : a_r\le j\le k-1\}$,
$\{(\beta_j+2\alpha_n),(\beta_j+2\alpha_n+\beta_{n-1}),\ldots,(\beta_j+2\alpha_n+\beta_{j+1}) : k+2\le j\le n-2\}$ and $-(\beta_{n-1}+2\alpha_n)$ are orthogonal to α_k.

Therefore V, V' are direct sums of irreducible $\tilde{L}_{\alpha_k}-$modules. By Lemma 1.13.3(2), we have

$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k},V)=V$, $H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k},V')=V'$ and

$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k},\mathbb{C}_{-(\beta_{n-1}+2\alpha_n)})=\mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}$.

Further the remaining roots of (3.2.1.3) are
$\{-(\beta_k+\alpha_n),-(\beta_k+2\alpha_n),-(\beta_k+2\alpha_n+\beta_{n-1}),\ldots,-(\beta_k+2\alpha_n+\beta_{k+2}),-(\beta_k+2\alpha_n+\beta_{k+1})\}$,
$\{-(\beta_{k+1}+2\alpha_n),-(\beta_{k+1}+2\alpha_n+\beta_{n-1}),\ldots,-(\beta_k+2\alpha_n+\beta_{k+2})\}$ and $\{-(\beta_j+2\alpha_n+\beta_{k+1}) : a_r\le j\le k\}$.

Since $\langle-(\beta_k+\alpha_n),\alpha_k\rangle=-1$, by Lemma 1.13.3(4) we have

$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k},\mathbb{C}_{-(\beta_k+\alpha_n)})=0$.

Since $\mathbb{C}_{-(\beta_k+2\alpha_n)}\oplus\mathbb{C}_{-(\beta_{k+1}+2\alpha_n)}$ is the irreducible two dimensional $\tilde{L}_{\alpha_k}$-module, by Lemma 1.13.3(2) we have

$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k},\mathbb{C}_{-(\beta_k+2\alpha_n)}\oplus\mathbb{C}_{-(\beta_{k+1}+2\alpha_n)})=\mathbb{C}_{-(\beta_k+2\alpha_n)}\oplus\mathbb{C}_{-(\beta_{k+1}+2\alpha_n)}$.

Similarly for each $k+2\le j\le n-1$, $\mathbb{C}_{-(\beta_k+2\alpha_n+\beta_j)}\oplus\mathbb{C}_{-(\beta_{k+1}+2\alpha_n+\beta_j)}$ is the irreducible two dimensional $\tilde{L}_{\alpha_k}-$module. Therefore by Lemma 1.13.3(2) we have

$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k},\mathbb{C}_{-(\beta_k+2\alpha_n+\beta_j)}\oplus\mathbb{C}_{-(\beta_{k+1}+2\alpha_n+\beta_j)})=\mathbb{C}_{-(\beta_k+2\alpha_n+\beta_j)}\oplus\mathbb{C}_{-(\beta_{k+1}+2\alpha_n+\beta_j)}$ for each $k+2\le j\le n-1$.

Moreover, $\langle-(\beta_j+2\alpha_n+\beta_{k+1}),\alpha_k\rangle=1$ for all $a_r\le j\le k-1$. Therefore by Lemma 1.13.3(2) we have

$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k},\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{k+1})})=\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{k+1})}\oplus\mathbb{C}_{-(\beta_j+2\alpha_n+\beta_k)}$ for all $a_r\le j\le k-1$.

Since $\langle -(\beta_k + 2\alpha_n + \beta_{k+1}), \alpha_k \rangle = 0$, by Lemma 1.13.3(2) we have

$H^0(\tilde{L}_{\alpha_k}/\tilde{B}_{\alpha_k}, \mathbb{C}_{-(\beta_k+2\alpha_n+\beta_{k+1})}) = \mathbb{C}_{-(\beta_k+2\alpha_n+\beta_{k+1})}$.

From the above discussion, we have

$$H^0(s_k s_{k+1} \cdots s_n u, \alpha_{n-1}) = \bigoplus_{j=a_r}^{k-1} (\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_k)})$$
$$\bigoplus_{j=k}^{n-2} (\mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{j+1})}) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Therefore the claim follows.

The claim implies

$$H^0(v, \alpha_{n-1}) = H^0(s_{a_{r-1}} s_{a_r+1} \cdots s_n v'_r, \alpha_{n-1}) = \bigoplus_{j=a_r}^{a_{r-1}-1} (\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{a_{r-1}})})$$
$$\bigoplus_{j=a_{r-1}}^{n-2} (\mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{j+1})}) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Proof of (2): By (1) we have

$$H^0(s_2 s_3 \cdots s_n s_1 s_2 \cdots s_{n-1}, \alpha_{n-1}) = (\mathbb{C}_{-(\beta_1+\alpha_n)} \oplus \mathbb{C}_{-(\beta_1+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_1+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_1+2\alpha_n+\beta_2)})$$
$$\bigoplus_{j=2}^{n-2} (\mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{j+1})}) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}. \quad (4.1.4)$$

The roots of (4.1.4) are $\{-(\beta_1 + \alpha_n), -(\beta_1 + 2\alpha_n), -(\beta_1 + 2\alpha_n + \beta_{n-1}), \ldots, -(\beta_1 + 2\alpha_n + \beta_3), -(\beta_1 + 2\alpha_n + \beta_2)\}$, $\{-(\beta_2 + 2\alpha_n), -(\beta_2 + 2\alpha_n + \beta_{n-1}), \ldots, -(\beta_1 + 2\alpha_n + \beta_3)\}$ and $-(\beta_1 + 2\alpha_n + \beta_2)$.

Since $-(\beta_{n-1} + 2\alpha_n)$ is orthogonal to α_1, by Lemma 1.13.3(2) we have

$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\beta_{n-1}+\alpha_n)}) = \mathbb{C}_{-(\beta_{n-1}+\alpha_n)}$.

Since $\langle -(\beta_1 + \alpha_n), \alpha_1 \rangle = -1$, by Lemma 1.13.3(4) we have

$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\beta_1+\alpha_n)}) = 0$.

Since $\mathbb{C}_{-(\beta_1+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_2+2\alpha_n)}$ is the irreducible two dimensional $\tilde{L}_{\alpha_1}$-module, by Lemma 1.13.3(2) we have

$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\beta_1+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_2+2\alpha_n)}) = \mathbb{C}_{-(\beta_1+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_2+2\alpha_n)}$.

Similarly for each $3 \leq j \leq n-1$, $\mathbb{C}_{-(\beta_1+2\alpha_n+\beta_j)} \oplus \mathbb{C}_{-(\beta_2+2\alpha_n+\beta_j)}$ is the irreducible two dimensional $\tilde{L}_{\alpha_1}$−module. Therefore by Lemma 1.13.3(2) we have

$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\beta_1+2\alpha_n+\beta_j)} \oplus \mathbb{C}_{-(\beta_2+2\alpha_n+\beta_j)}) = \mathbb{C}_{-(\beta_1+2\alpha_n+\beta_j)} \oplus \mathbb{C}_{-(\beta_2+2\alpha_n+\beta_j)}$ for each $3 \leq j \leq n-1$.

Since $\langle -(\beta_1 + 2\alpha_n + \beta_2), \alpha_1 \rangle = 0$, by Lemma 1.13.3(2) we have

$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\beta_1+2\alpha_n+\beta_2)}) = \mathbb{C}_{-(\beta_1+2\alpha_n+\beta_2)}$.

From the above discussion, we have

$$H^0(v', \alpha_{n-1}) = \bigoplus_{i=1}^{n-2} (\mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}. \qquad \square$$

Lemma 3.2.2. *Let $3 \leq r \leq k$ and let $v = s_{a_{r-1}} s_{a_{r-1}+1} \cdots s_n s_{a_r} s_{a_r+1} \cdots s_{n-1}$. Then we have*

(1) $H^0(s_{a_{r-2}} \cdots s_n v, \alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1} (\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})})$

$\bigoplus_{i=a_{r-1}}^{a_{r-2}-2} (\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})})$.

(2) $H^0(w_r, \alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1} (\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})})$

$\bigoplus_{i=a_{r-1}}^{a_{r-2}-2} (\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})})$.

(3) Let $u_1 = (s_{a_1} \cdots s_n)(s_{a_2} \cdots s_n) \cdots (s_{a_{k-1}} \cdots s_n)v'$, where v' is defined as in Lemma 3.2.1. Then we have

$$H^0(u_1, \alpha_{n-1}) = \bigoplus_{i=1}^{a_{k-1}-2} (\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{k-1}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}).$$

Proof. Proof of (1): Since $r \geq 3$, we have $a_{r-1} < n$. By Lemma 3.2.1(1), we have

$$H^0(v, \alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1} (\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})})$$

$$\bigoplus_{i=a_{r-1}}^{n-2} (\mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}) \oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Since $\{-(\beta_i + 2\alpha_n + \beta_{n-1}), \ldots, -(\beta_i + 2\alpha_n + \beta_{a_{r-1}}) : a_r \leq i \leq a_{r-1} - 1\}$ are orthogonal to α_n, by Lemma 1.13.3(2), we have

$H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_t)}) = \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_t)}$ for all $a_r \leq i \leq a_{r-1} - 1$ and $a_{r-1} \leq t \leq n - 1$.

Since $\{-(\beta_i + 2\alpha_n + \beta_{n-1}), \ldots, -(\beta_i + 2\alpha_n + \beta_{i+1}) : a_{r-1} \leq i \leq n - 2\}$ are orthogonal to α_n, by Lemma 1.13.3(2) we have

$H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_l)}) = \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_l)}$ for all $i + 1 \leq l \leq n - 1$, where $a_{r-1} \leq i \leq n - 2$.

Since $\langle -(\beta_i + 2\alpha_n), \alpha_n \rangle = -2$ for all $a_r \leq i \leq n - 1$, by Lemma 1.13.3(3) we have

$H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}) = 0$ for all $a_{r-1} \leq i \leq n - 1$.

Moreover, for each $a_r \leq i \leq a_{r-1} - 1$, $\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}$ is an indecomposable two dimensional B_{α_n}-module. Therefore by Lemma 1.13.4, we have $\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)} = V_i \otimes \mathbb{C}_{-\omega_n}$, where V_i is the irreducible two dimensional representation of $\tilde{L}_{\alpha_n}$. By Lemma 1.13.3(4) we have

$H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}) = 0$ for each $a_r \leq i \leq a_{r-1} - 1$.

From the above discussion, we have

$$H^0(s_nv_r,\alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1}(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})})\bigoplus_{i=a_{r-1}}^{n-2}(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}).$$

Since $\langle-(\beta_i+2\alpha_n+\beta_{n-1}),\alpha_{n-1}\rangle = -1$ for each $a_r\le i\le n-2$, by Lemma 1.13.3(4) we have
$H^0(\tilde{L}_{\alpha_{n-1}}/\tilde{B}_{\alpha_{n-1}},\mathbb{C}_{-(\beta_i+\alpha_n+\beta_{n-1})}) = 0$ for each $a_r\le i\le n-2$.

Moreover, $\{-(\beta_i+2\alpha_n+\beta_{n-2}),\ldots,-(\beta_i+2\alpha_n+\beta_{a_{r-1}}) : a_r\le i\le a_{r-1}-1\}$,
$\{-(\beta_i+2\alpha_n+\beta_{n-2}),\ldots,-(\beta_i+2\alpha_n+\beta_{i+1}) : a_{r-1}\le i\le n-3\}$, and $\beta_{n-1}+2\alpha_n$ are orthogonal to α_{n-1}.

Therefore we have

$$H^0(s_{n-1}s_nv,\alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1}(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-2})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})})\bigoplus_{i=a_{r-1}}^{n-3}(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-2})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}).$$

Proceeding recursively, we have

$$H^0(s_{a_{r-2}}s_{a_{r-2}+1}\cdots s_nv,\alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1}(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})})\bigoplus$$
$$\bigoplus_{i=a_{r-1}}^{a_{r-2}-2}(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}).$$

Proof of (2):

Since $\{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1}),\ldots,-(\beta_i+2\alpha_n+\beta_{a_{r-1}}) : a_r\le i\le a_{r-1}-1\}$,
$\{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1}),\ldots,-(\beta_i+2\alpha_n+\beta_{i+1}) : a_{r-1}\le i\le a_{r-2}-2\}$ are orthogonal to α_j for all $a_{r-3}\le j\le n$, by Lemma 1.13.3(2) we have

$$H^0(s_{a_{r-3}}\cdots s_ns_{a_{r-2}}\cdots s_nv,\alpha_{n-1}) = H^0(s_{a_{r-2}}\cdots s_nv,\alpha_{n-1}).$$

Proceeding recursively we have

$$H^0(w_r,\alpha_{n-1}) = H^0(s_{a_{r-2}}\cdots s_nv,\alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1}(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})})$$
$$\bigoplus_{i=a_{r-1}}^{a_{r-2}-2}(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-2}-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}).$$

Proof of (3):

By Lemma 3.2.1(2) we have

$$H^0(v',\alpha_{n-1}) = \bigoplus_{i=1}^{n-2}(\mathbb{C}_{-(\beta_i+2\alpha_n)}\oplus\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})}\oplus\cdots\oplus\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})})\oplus\mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}.$$

Since $\{-(\beta_i+2\alpha_n+\beta_{n-1}),\ldots,-(\beta_i+2\alpha_n+\beta_{i+1}) : 1\le i\le n-2\}$ are orthogonal to α_n, by Lemma 1.13.3(2) we have

$H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n},\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_l)}) = \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_l)}$ for all $i+1\le l\le n-1$, where $1\le i\le n-2$.

Since $\langle-(\beta_i+2\alpha_n),\alpha_n\rangle = -2$ for all $1\le i\le n-1$, by Lemma 1.13.3(3) we have

$H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}) = 0$ for all $1 \leq i \leq n-1$.

From the above discussion, we have

$$H^0(s_n v', \alpha_{n-1}) = \bigoplus_{i=1}^{n-2}(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}).$$

Since $\langle -(\beta_i + 2\alpha_n + \beta_{n-1}), \alpha_{n-1}\rangle = -1$ for each $1 \leq i \leq n-2$, by Lemma 1.13.3(4) we have $H^0(\tilde{L}_{\alpha_{n-1}}/\tilde{B}_{\alpha_{n-1}}, \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})}) = 0$ for each $1 \leq i \leq n-2$.

Moreover, $\{-(\beta_i + 2\alpha_n + \beta_{n-2}), \ldots, -(\beta_i + 2\alpha_n + \beta_{i+1}) : 1 \leq i \leq n-3\}$, and $\beta_{n-1} + 2\alpha_n$ are orthogonal to α_{n-1}.

Therefore we have

$$H^0(s_{n-1}s_n v', \alpha_{n-1}) = \bigoplus_{i=1}^{n-3}(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-2})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}).$$

Proceeding recursively we have

$$H^0(s_{a_{k-1}}s_{a_{k-1}+1}\cdots s_n v', \alpha_{n-1}) = \bigoplus_{i=1}^{a_{k-1}-2}(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{k-1}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}).$$

Since $\{-(\beta_i + 2\alpha_n + \beta_{a_{k-2}-1}), \ldots, -(\beta_i + 2\alpha_n + \beta_{i+1}) : 1 \leq i \leq a_{k-1} - 2\}$ are orthogonal to α_j for all $a_{k-2} \leq j \leq n$, we have

$$H^0(s_{a_{k-2}}\cdots s_n s_{a_{k-1}}\cdots s_n v', \alpha_{n-1}) = H^0(s_{a_{k-1}}\cdots s_n v', \alpha_{n-1}).$$

Proceeding recursively we have

$$H^0(u_1, \alpha_{n-1}) = \bigoplus_{i=1}^{a_{k-1}-2}(\mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{k-1}-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}).$$ □

Lemma 3.2.3. *Let $3 \leq r \leq k$. Then $H^0(w_{r-2}s_n s_{a_{r-1}}s_{a_{r-1}+1}\cdots s_{n+2-r}, \alpha_{n+2-r})_\mu \neq 0$ if μ is of the form $\mu = -(\beta_j + \alpha_n)$ for some $a_{r-1} \leq j \leq a_{r-2} - 1$.*

Proof. By applying SES repeatedly, it is easy to see that

$$H^0(s_n s_{a_{r-1}}s_{a_{r-1}+1}\cdots s_{n+2-r}, \alpha_{n+2-r}) = \mathbb{C}h(\alpha_{n+2-r}) \oplus (\bigoplus_{j=a_{r-1}}^{n+2-r}\mathbb{C}_{-\gamma_{j,n+2-r}}),$$ where $\gamma_{j,j'} = (\alpha_j + \cdots + \alpha_{j'})$ for $j' \geq j$.

Let $V_1 = H^0(s_n s_{a_{r-1}}s_{a_{r-1}+1}\cdots s_{n+2-r}, \alpha_{n+2-r})$. We next calculate $H^0(s_{a_{r-2}}\cdots s_{n-1}, V_1)$. Since $n \geq a_1 > a_2 > \cdots > a_k = 1$, we have $a_i \leq n + 1 - i$ for all $1 \leq i \leq k$. Assume $l \geq n + 4 - r$, then $\langle \gamma_{j,n+2-r}, \alpha_l\rangle = 0$ for all $a_{r-1} \leq j \leq n + 2 - r$. By Lemma 1.13.3(2) we have $H^0(\tilde{L}_{\alpha_l}/\tilde{B}_{\alpha_l}, V_1) = V_1$ for all $l \geq n + 4 - r$.

Therefore we have $H^0(s_{a_{r-2}}\cdots s_{n-1}, V_1) = H^0(s_{a_{r-2}}\cdots s_{n+2-r}s_{n+3-r}, V_1)$.

Note that, since $\langle -\gamma_{j,n+2-r}, \alpha_{n+3-r}\rangle = 1$ for all $a_{r-1} \leq j \leq n + 2 - r$, by Lemma 1.13.3(2) we have

$$H^0(s_{n+3-r}, V_1) = \mathbb{C}h(\alpha_{n+2-r}) \oplus (\bigoplus_{j=a_{r-1}}^{n+2-r}(\mathbb{C}_{-\gamma_{j,n+2-r}} \oplus \mathbb{C}_{-\gamma_{j,n+3-r}})).$$

Since $\mathbb{C}h(\alpha_{n+2-r}) \oplus \mathbb{C}_{-\gamma_{n+2-r,n+2-r}}$ is an indecomposable two dimensional $B_{\alpha_{n+2-r}}$-module, by Lemma 1.13.4,

$\mathbb{C}h(\alpha_{n+2-r}) \oplus \mathbb{C}_{-\gamma_{n+2-r,n+2-r}} = V \otimes \mathbb{C}_{-\omega_{n+2-r}}$, where V is the irreducible two dimensional $\tilde{L}_{\alpha_{n+2-r}}$-module.

Since $\langle \gamma_{j,n+2-r}, \alpha_{n+2-r} \rangle = -1$, for all $a_{r-1} \leq j \leq n+1-r$, by Lemma 1.13.3(4) we have

$H^0(\tilde{L}_{\alpha_{n+2-r}}/\tilde{B}_{\alpha_{n+2-r}}, \mathbb{C}_{\gamma_{j,n+2-r}}) = 0$ for all $a_{r-1} \leq j \leq n+1-r$.

From the above discussion, we have

$H^0(s_{n+2-r}, H^0(s_{n+3-r}, V_1)) = \bigoplus_{j=a_{r-1}}^{n+1-r} \mathbb{C}_{-\gamma_{j,n+3-r}}$.

Since $\langle \gamma_{n+1-r,n+3-r}, \alpha_{n+1-r} \rangle = -1$, by Lemma 1.13.3(4), we have

$H^0(\tilde{L}_{\alpha_{n+1-r}}/\tilde{B}_{\alpha_{n+1-r}}, \mathbb{C}_{\gamma_{n+1-r,n+3-r}}) = 0$.

Moreover, $\langle \gamma_{j,n+3-r}, \alpha_{n+1-r} \rangle = 0$ for all $a_{r-1} \leq j \leq n-r$. Therefore we have

$H^0(s_{n+1-r}, H^0(s_{n+2-r}, H^0(s_{n+3-r}, V_1))) = \bigoplus_{j=a_{r-1}}^{n-r} \mathbb{C}_{-\gamma_{j,n+3-r}}$.

Proceeding recursively we have

$H^0(s_{a_{r-2}} \cdots s_{n-1}, V_1) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-\gamma_{j,n+3-r}}$ and $H^0(s_n s_{a_{r-2}} \cdots s_{n-1}, V_1) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-\gamma_{j,n+3-r}}$.

Let $V_2 = H^0(s_n s_{a_{r-2}} \cdots s_{n-1}, V_1)$. Similarly, we have $H^0(s_{a_{r-3}} \cdots s_{n-1}, V_2) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-\gamma_{j,n+4-r}}$ and

$H^0(s_n s_{a_{r-3}} \cdots s_{n-1}, V_2) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-\gamma_{j,n+4-r}}$.

Proceeding recursively we have

$V_{r-2} = H^0(s_{a_2} \cdots s_{n-1}, V_{r-3}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-\gamma_{j,n-1}}$,

where $V_{r-3} = H^0((s_n s_3 \cdots s_n) \cdots (s_{a_{r-2}} \cdots s_{n+2-r}), \alpha_{n+2-r})$.

Note that $\gamma_{j,n-1} = \beta_j$. Since $\langle -\alpha_{n-1}, \alpha_n \rangle = 2$, by Lemma 1.13.3(2) and Lemma 1.13.4, we have

$$H^0(s_n, V_{r-2}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} (\mathbb{C}_{-\beta_j} \oplus \mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)}).$$

Moreover, $\langle -\beta_j, \alpha_{n-1} \rangle = -1$, $\langle -(\beta_j + \alpha_n), \alpha_{n-1} \rangle = 0$ and $\langle -(\beta_j + 2\alpha_n), \alpha_{n-1} \rangle = 1$ for all $a_{r-1} \leq j \leq a_{r-2} - 1$.

Therefore by Lemma 1.13.3(2), Lemma 1.13.3(4) and Lemma 1.13.4 we have

$$H^0(s_{n-1}, H^0(s_n, V_{r-2})) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} (\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})}).$$

Proceeding recursively we have

$$H^0(s_{a_1}, H^0(s_{a_1+1}\cdots s_n, V_{r-2})) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1}(\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{a_1})}).$$

Hence the proof of the lemma follows. □

Lemma 3.2.4. *Let $3 \le r \le k$. Then $H^0(w_{r-1}s_n, \alpha_n)_\mu \neq 0$ if and only if μ is of the form $\mu = -(\beta_j + \alpha_n)$, for some $a_{r-1} \le j \le a_{r-2} - 1$.*

Proof. By applying SES repeatedly, it is easy to see that

$$H^0(s_{a_{r-1}}\cdots s_{n-1}s_n, \alpha_n) = \bigoplus_{j=a_{r-1}}^{n-1}\mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Let $V_1 = H^0(s_{a_{r-1}}\cdots s_{n-1}s_n, \alpha_n)$. Since $\langle -(\beta_j + \alpha_n), \alpha_n\rangle = 0$, we have $H^0(s_n, V_1) = V_1$.

Since $\langle -(\beta_{n-1} + \alpha_n), \alpha_{n-1}\rangle = -1$ and $\langle -(\beta_j + \alpha_n, \alpha_{n-1})\rangle = 0$, for all $a_{r-1} \le j \le n-2$, by Lemma 1.13.3(2), Lemma 1.13.3(4) and Lemma 1.13.4 we have

$$H^0(s_{n-1}, V_1) = \bigoplus_{j=a_{r-1}}^{n-2}\mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Proceeding recursively we have

$$H^0(s_{a_{r-2}}\cdots s_{n-1}s_n, V_1) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1}\mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Since $n \ge a_1 > a_2 > \cdots > a_k = 1$, we see that $\langle -(\beta_j + \alpha_n), \alpha_t\rangle = 0$, for all $a_{r-1} \le j \le a_{r-2} - 1$ and for all $a_{r-3} \le t \le n$, therefore by Lemma 1.13.3(2) and Lemma 1.13.4, we have

$$H^0(s_{a_{r-3}}\cdots s_n s_{a_{r-2}}\cdots s_n s_{a_{r-1}}\cdots s_{n-1}s_n, \alpha_n) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1}\mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Since $n \ge a_1 > a_2 > \cdots > a_k = 1$, we see that $\langle -(\beta_j + \alpha_n), \alpha_t\rangle = 0$ for all $a_{r-1} \le j \le a_{r-2} - 1$ and for all $a_{r-4} \le t \le n$. By Lemma 1.13.3(2) and Lemma 1.13.4, we have

$$H^0(s_{a_{r-4}}\cdots s_n s_{a_{r-3}}\cdots s_n s_{a_{r-2}}\cdots s_n s_{a_{r-1}}\cdots s_{n-1}s_n, \alpha_n) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1}\mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Proceeding recursively we have

$$H^0(w_{r-1}s_n, \alpha_n) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Hence the lemma follows. □

Lemma 3.2.5. *If μ is of the form $\mu = -(\beta_j + \alpha_n)$ for some $1 \leq j \leq a_{k-1} - 1$, then we have $H^0(w_{k-1}s_n s_1 s_2 \cdots s_{n+1-k}, \alpha_{n+1-k})_\mu \neq 0$.*

Proof. By applying SES repeatedly, it is easy to see that

$H^0(s_n s_{a_1} s_2 \cdots s_{n+1-k}, \alpha_{n+1-k}) = \mathbb{C}h(\alpha_{n+1-k}) \oplus (\bigoplus_{j=1}^{n+1-k} \mathbb{C}_{-\gamma_{j,n+1-k}})$,

where $\gamma_{j,j'} = (\alpha_j + \cdots + \alpha_{j'})$ for $j' \geq j$.

Let $V_1 = H^0(s_n s_1 s_2 \cdots s_{n+1-k}, \alpha_{n+1-k})$. We next calculate $H^0(s_{a_{k-1}} \cdots s_{n-1}, V_1)$.

Since $n \geq a_1 > a_2 > \cdots > a_k = 1$, we have $a_i \leq n + 1 - i$ for all $1 \leq i \leq k$. Moreover, $\langle \gamma_{j,n+1-k}, \alpha_l \rangle = 0$ for all $1 \leq j \leq n + 1 - k$ and for all $l \geq n + 3 - k$.

Therefore by using Lemma 1.13.3(2) and Lemma 1.13.4, we have

$$H^0(s_{a_{k-1}} \cdots s_{n-1}, V_1) = H^0(s_{a_{k-1}} \cdots s_{n+1-k}s_{n+2-k}, V_1).$$

Since $\langle -\gamma_{j,n+1-k}, \alpha_{n+2-k} \rangle = 1$ for all $1 \leq j \leq n + 1 - k$, by using Lemma 1.13.3(2) and Lemma 1.13.4, we have

$H^0(s_{n+2-k}, V_1) = \mathbb{C}h(\alpha_{n+1-k}) \oplus (\bigoplus_{j=1}^{n+1-k} (\mathbb{C}_{-\gamma_{j,n+1-k}} \oplus \mathbb{C}_{-\gamma_{j,n+2-k}}))$.

Since $\mathbb{C}h(\alpha_{n+1-k}) \oplus \mathbb{C}_{-\gamma_{n+1-k,n+1-k}}$ is an indecomposable two dimensional $B_{\alpha_{n+1-k}}$-module, by Lemma 1.13.4 we have $\mathbb{C}h(\alpha_{n+1-k}) \oplus \mathbb{C}_{-\gamma_{n+1-k,n+1-k}} = V \otimes \mathbb{C}_{-\omega_{n+1-k}}$, where V is the irreducible two dimensional $\tilde{L}_{\alpha_{n+1-k}}$-module.

By Lemma 1.13.3(4) we have

$H^0(\tilde{L}_{\alpha_{n+1-k}}/\tilde{B}_{\alpha_{n+1-k}}, \mathbb{C}h(\alpha_{n+1-k}) \oplus \mathbb{C}_{-\gamma_{n+1-k,n+1-k}}) = 0$.

Since $\langle \gamma_{j,n+1-k}, \alpha_{n+1-k} \rangle = -1$ for all $1 \leq j \leq n - k$, by Lemma 1.13.3(4) we have

$H^0(\tilde{L}_{\alpha_{n+1-k}}/\tilde{B}_{\alpha_{n+1-k}}, \mathbb{C}_{\gamma_{j,n+1-k}}) = 0$ for all $1 \leq j \leq n - k$.

From the above discussion, we have

$H^0(s_{n+1-k}, H^0(s_{n+2-k}, V_1)) = \bigoplus_{j=1}^{n-k} \mathbb{C}_{-\gamma_{j,n+2-k}}$.

Since $\langle \gamma_{j,n+2-k}, \alpha_{n-k} \rangle = 0$ for all $1 \leq j \leq n - k - 1$, and $\langle \gamma_{n-k,n+2-k}, \alpha_{n-k} \rangle = -1$, by using Lemma 1.13.3(2), Lemma 1.13.3(2)(4) and Lemma 1.13.4, we have

$$H^0(s_{n-k}, H^0(s_{n+1-k}, H^0(s_{n+2-k}, V_1))) = \bigoplus_{j=1}^{n-k-1} \mathbb{C}_{-\gamma_{j,n+2-k}}.$$

Proceeding recursively we have

$$H^0(s_n s_{a_{k-1}} \cdots s_{n-1}, V_1) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-\gamma_{j,n+2-k}}.$$

Let $V_2 = H^0(s_n s_{a_{k-1}} \cdots s_{n-1}, V_1)$. Then similarly, we have

$$H^0(s_{a_{k-2}} \cdots s_{n-1}, V_2) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-\gamma_{j,n+3-k}}.$$

Proceeding recursively we have

$$V_{k-1} = H^0(s_{a_2} \cdots s_{n-1}, V_{k-2}) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-\gamma_{j,n-1}}$$

where $V_{k-2} = H^0((s_n s_3 \cdots s_n) \cdots (s_{a_{k-1}} \cdots s_{n+1-k}), \alpha_{n+1-k})$.

Note that $\gamma_{j,n-1} = \beta_j$. Since $\langle -\alpha_{n-1}, \alpha_n \rangle = 2$, by using Lemma 1.13.3(2) and Lemma 1.13.4 we have

$H^0(s_n, V_{k-1}) = \bigoplus_{j=1}^{a_{k-1}-1} (\mathbb{C}_{-\beta_j} \oplus \mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)})$.

Moreover, $\langle -\beta_j, \alpha_{n-1} \rangle = -1$, $\langle -(\beta_j + \alpha_n), \alpha_{n-1} \rangle = 0$ and $\langle -(\beta_j + 2\alpha_n), \alpha_{n-1} \rangle = 1$, for all $1 \leq j \leq a_{k-1} - 1$.

Therefore by using Lemma 1.13.3(2), Lemma 1.13.3(4) and Lemma 1.13.4 we have

$$H^0(s_{n-1}, H^0(s_n, V_{k-1})) = \bigoplus_{j=1}^{a_{k-1}-1} (\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})}).$$

Proceeding recursively we have

$H^0(s_{a_1}, H^0(s_{a_1+1} \cdots s_n, V_{k-1})) = \bigoplus_{j=1}^{a_{k-1}-1} (\mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_j+2\alpha_n+\beta_{a_1})})$.

Hence the proof of the lemma follows.

□

Lemma 3.2.6. $H^0(w_k s_n, \alpha_n)_\mu \neq 0$ *if and only if* μ *is of the form* $\mu = -(\beta_j + \alpha_n)$ *for some* $1 \leq j \leq a_{k-1} - 1$.

Proof. By applying SES repeatedly it is easy to see that

$$H^0(s_1 \cdots s_{n-1}s_n, \alpha_n) = \bigoplus_{j=1}^{n-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Let $V_1 = H^0(s_1 \cdots s_{n-1}s_n, \alpha_n)$. Since $\langle -(\beta_j + \alpha_n), \alpha_n\rangle = 0$, for all $1 \leq j \leq n$, by Lemma 1.13.3(2) and Lemma 1.13.4 we have $H^0(s_n, V_1) = V_1$.

Moreover, $\langle -(\beta_{n-1} + \alpha_n), \alpha_{n-1}\rangle = -1$ and $\langle -(\beta_j + \alpha_n, \alpha_{n-1})\rangle = 0$ for all $1 \leq j \leq n-2$. Therefore by using Lemma 1.13.3(4), Lemma 1.13.3(2) and Lemma 1.13.4, we have

$$H^0(s_{n-1}, V_1) = \bigoplus_{j=1}^{n-2} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Proceeding recursively we have

$$H^0(s_{a_{k-1}} \cdots s_{n-1}s_n, V_1) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Since $n \geq a_1 > a_2 > \cdots > a_k = 1$, we have $\langle -(\beta_j + \alpha_n), \alpha_t\rangle = 0$ for all $1 \leq j \leq a_{k-1} - 1$ and for all $a_{k-2} \leq t \leq n$. By using Lemma 1.13.3(2), Lemma 1.13.4 we have

$$H^0(s_{a_{k-2}} \cdots s_n s_{a_{k-1}} \cdots s_n s_1 \cdots s_{n-1}s_n, \alpha_n) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Similarly, since $n \geq a_1 > a_2 > \cdots > a_k = 1$, we have $\langle -(\beta_j + \alpha_n), \alpha_t\rangle = 0$ for all $1 \leq j \leq a_{k-1} - 1$ and for all $a_{k-3} \leq t \leq n$, therefore by using Lemma 1.13.3(2) and Lemma 1.13.4 we have

$$H^0(s_{a_{k-3}} \cdots s_n s_{a_{k-2}} \cdots s_n s_{a_{k-1}} \cdots s_n s_1 \cdots s_{n-1}s_n, \alpha_n) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Proceeding recursively we have

$$H^0(w_k s_n, \alpha_n) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Hence the proof of the lemma follows. □

3.3 Cohomology module H^1 of the relative tangent bundle

In this section, we describe the weights of H^1 of the relative tangent bundle. Let $n \geq a_1 > a_2 > \ldots > a_{k-1} > a_k = 1$ be a decreasing sequence of integers such that $k \geq 3$. Fix $3 \leq r \leq k$.

Lemma 3.3.1. *Let $v_r = s_n s_{a_r} \cdots s_{n-2}$, $v_{r-1} = s_{a_{r-1}} \cdots s_{n-1}$ and $v_{r-2} = s_{a_{r-2}} \cdots s_{n-1} s_n$. Then we have*

(1) $H^1(v_{r-1} v_r s_{n-1}, \alpha_{n-1}) = 0$.

(2) Let $w = v_{r-2} v_{r-1} v_r s_{n-1}$. $H^1(w, \alpha_{n-1})_\mu \neq 0$ if and only if μ is of the form $\mu = -(\beta_t + \alpha_n)$ for some $a_{r-1} \leq t \leq a_{r-2} - 1$. In such a case, dim$(H^1(w, \alpha_{n-1})_\mu) = 1$.

Proof. Proof of (1): Note that $H^0(s_{n-1}, \alpha_{n-1}) = \mathbb{C}_{-\alpha_{n-1}} \oplus \mathbb{C}h(\alpha_{n-1}) \oplus \mathbb{C}_{\alpha_{n-1}}$ (see [CKP15, Corollary 2.5]). (3.3.1.1)

Since $\langle \alpha_{n-1}, \alpha_{n-1} \rangle = 2$, we have $H^1(s_{n-1}, \alpha_{n-1}) = 0$. Since $\langle \alpha_{n-1}, \alpha_{n-2} \rangle = -1$, we have $H^1(s_{n-2}, H^0(s_{n-1}, \alpha_{n-1})) = 0$.

Therefore by using SES we have $H^1(s_{n-2} s_{n-1}, \alpha_{n-1}) = 0$. (3.3.1.2)

Since $\langle -\alpha_{n-1}, \alpha_{n-2} \rangle = 1$, by using (3.3.1.1) we have

$H^0(s_{n-2} s_{n-1}, \alpha_{n-1}) = \mathbb{C}h(\alpha_{n-1}) \oplus \mathbb{C}_{-\alpha_{n-1}} \oplus \mathbb{C}_{-\beta_{n-2}}$.

Since $\langle -\alpha_{n-1}, \alpha_{n-3} \rangle = 0$ and $\langle -\beta_{n-2}, \alpha_{n-3} \rangle = 1$ we have

$H^1(s_{n-3}, H^0(s_{n-2} s_{n-1}, \alpha_{n-1})) = 0$. (3.3.1.3)

Therefore by using SES together with (3.3.1.2), (3.3.1.3) we have

$H^1(s_{n-3} s_{n-2} s_{n-1}, \alpha_{n-1}) = 0$. (3.3.1.4)

Proceeding in this way we have $H^1(s_{a_r} \cdots s_{n-2} s_{n-1}, \alpha_{n-1}) = 0$ (3.3.1.5)

and $H^0(s_{a_r} \cdots s_{n-2} s_{n-1}, \alpha_{n-1}) = \mathbb{C}h(\alpha_{n-1}) \oplus \bigoplus_{j=a_r}^{n-1} \mathbb{C}_{-\beta_j}$. (3.3.1.6)

Since $\langle -\beta_j, \alpha_n \rangle > 0$ for all $a_r \leq j \leq n-1$, by using (3.3.1.6) we have

$H^1(s_n, H^0(s_{a_r} \cdots s_{n-2} s_{n-1}, \alpha_{n-1})) = 0$. (3.3.1.7)

Therefore by using SES, (3.3.1.5) and (3.3.1.7) together we have

$H^1(v_r s_{n-1}, \alpha_{n-1}) = 0$. (3.3.1.8)

In the proof of Lemma 3.2.1(1) we notice that

$$H^0(v_r s_{n-1}, \alpha_{n-1}) = \mathbb{C}h(\alpha_{n-1}) \oplus (\bigoplus_{j=a_r}^{n-1} \mathbb{C}_{\beta_j} \oplus \mathbb{C}_{-(\beta_j+\alpha_n)} \oplus \mathbb{C}_{-(\beta_j+2\alpha_n)}).$$

Thus we have

$$H^1(s_{n-1}, H^0(v_r s_{n-1}, \alpha_{n-1})) = 0 \text{ (see lines from (3.2.1.1) to (3.2.1.2))}. \quad (3.3.1.9)$$

Proceeding by similar arguments and using (3.3.1.8), (3.3.1.9) we have

$$H^1(v_{r-1} v_r s_{n-1}, \alpha_{n-1}) = 0.$$

Proof of (2):

From the Lemma 3.2.1(1) we have

$$H^0(v_{r-1} v_r s_{n-1}, \alpha_{n-1}) = \bigoplus_{i=a_r}^{a_{r-1}-1} (\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{a_{r-1}})})$$

$$\bigoplus_{i=a_{r-1}}^{n-2} (\mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}) \oplus \mathbb{C}_{-(\alpha_{n-1}+2\alpha_n)}.$$

Notice that for each $a_r \leq i \leq a_{r-1} - 1$, $\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}$ forms an indecomposable two dimensional B_{α_n}-module.

Since $\langle -(\beta_i + 2\alpha_n), \alpha_n \rangle = -2$, by Lemma 1.13.4 we have

$\mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)} = V_i \otimes \mathbb{C}_{-\omega_n}$, where V_i is the irreducible two dimensional $\tilde{L}_{\alpha_n}$-module.

By Lemma 1.13.3(4) we have

$H^j(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n)}) = 0$ for all $j \geq 0$ and for all $a_r \leq i \leq a_{r-1} - 1$.

Since $\langle -(\beta_i + 2\alpha_n + \beta_t), \alpha_n \rangle = 0$ for each $a_r \leq i \leq a_{r-1} - 1$, and $a_{r-1} \leq t \leq n - 1$, we have

$H^1(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_t)}) = 0$ for all $a_r \leq i \leq a_{r-1} - 1$ and $a_{r-1} \leq t \leq n - 1$.

Moreover, we have $\langle -(\beta_i + 2\alpha_n + \beta_{i+1}), \alpha_n \rangle = 0$ for each $a_{r-1} \leq i \leq n - 2$ and $\langle -(\beta_i + 2\alpha_n), \alpha_n \rangle = -2$ for each $a_{r-1} \leq i \leq n - 1$.

Therefore we have

$H^1(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})}) = 0$ for all $a_{r-1} \leq i \leq n - 2$.

By Lemma 1.13.3(3) we have

$H^1(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, \mathbb{C}_{-(\beta_i+2\alpha_n)}) = H^0(\tilde{L}_{\alpha_n}/\tilde{B}_{\alpha_n}, s_{n-1} \cdot -(\beta_i + 2\alpha_n)) = \mathbb{C}_{-(\beta_i+\alpha_n)}$ for all $a_{r-1} \leq i \leq n - 1$.

From the above discussion, we have

$$H^1(s_n, H^0(v_{r-1} v_r s_{n-1}, \alpha_{n-1})) = \bigoplus_{i=a_{r-1}}^{n-1} \mathbb{C}_{-(\beta_i+\alpha_n)}. \quad (3.3.1.10)$$

By (1) and using SES we have

$H^1(s_n, H^0(v_{r-1}v_r s_{n-1}, \alpha_{n-1})) = H^1(s_n v_{r-1} v_r s_{n-1}, \alpha_{n-1}).$ (3.3.1.11)

From (3.3.1.10) and (3.3.1.11) we have

$$H^1(s_n v_{r-1} v_r s_{n-1}, \alpha_{n-1}) = \bigoplus_{i=a_{r-1}}^{n-1} \mathbb{C}_{-(\beta_i+\alpha_n)}. \quad (3.3.1.12)$$

By Lemma 3.1.1 we have

$H^1(s_{n-1}, H^0(s_n v_{r-1} v_r s_{n-1}, \alpha_{n-1})) = 0.$ (3.3.1.13)

Let $v = v_{r-1}v_r s_{n-1}$. Therefore by using SES and (3.3.1.13) we have

$H^1(s_{n-1}s_n v, \alpha_{n-1}) = H^0(s_{n-1}, H^1(s_n v, \alpha_{n-1})).$

Since $\langle -(\beta_{n-1}+\alpha_n), \alpha_{n-1}\rangle = -1$ and $\langle -(\beta_i+\alpha_n), \alpha_{n-1}\rangle = 0$ for all $a_{r-1} \leq i \leq n-2$, by using (3.3.1.12) we have

$H^1(s_{n-1}s_n v, \alpha_{n-1}) = \bigoplus_{i=a_{r-1}}^{n-2} \mathbb{C}_{-(\beta_i+\alpha_n)}.$

Proceeding recursively we have

$H^1(v_{r-2}v_{r-1}v_r s_{n-1}, \alpha_{n-1}) = H^1(s_{a_{r-2}} \cdots s_{n-1}s_n v, \alpha_{n-1}) = \bigoplus_{i=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_i+\alpha_n)}.$ □

Recall that $w_r = [a_1, n][a_2, n] \cdots [a_{r-1}, n][a_r, n-1]$, where $1 \leq r \leq k$ and $n \geq a_1 > a_2 > \ldots > a_{k-1} > a_k = 1$.

Lemma 3.3.2. *(1)* $H^1(w_1, \alpha_{n-1}) = 0$.

(2) If $a_2 \neq n-1$, *then* $H^1(w_2, \alpha_{n-1}) = 0$.

(3) Let $3 \leq r \leq k$. *Then,* $H^1(w_r, \alpha_{n-1})_\mu \neq 0$ *if and only if* μ =-$(\beta_j + \alpha_n)$ *for some* j *such that* $a_{r-1} \leq j \leq a_{r-2} - 1$. *In such case dim*$(H^1(w_r, \alpha_{n-1}))_\mu = 1$.

Proof. Proof of (1): Follows from proof of Lemma 3.1.1 and using SES.

Proof of (2): By proof of (1), we have

$H^1(s_{a_2} \cdots s_{n-1}, \alpha_{n-1}) = 0.$ (3.3.2.1)

Since $a_2 \neq n-1$, $H^0(s_{a_2} \cdots s_{n-1}, \alpha_{n-1}) = \mathbb{C}h(\alpha_{n-1}) \oplus (\bigoplus_{j=a_2}^{n-1} \mathbb{C}_{-(\beta_j)})$.

Since $\langle -\beta_j, \alpha_n\rangle \geq 1$ for all $a_2 \leq j \leq n-1$, we have

$H^1(s_n, H^0(s_{a_2} \cdots s_{n-1}, \alpha_{n-1})) = 0.$ (3.3.2.2)

By SES, (3.3.2.1), (3.3.2.2) we have $H^1(s_n s_{a_2} \cdots s_{n-1}, \alpha_{n-1}) = 0$.

By using Lemma 3.1.1 and SES repeatedly, we see that

$H^1(w_2, \alpha_{n-1}) = H^1(s_{a_1} \cdots s_n s_{a_2} \cdots s_{n-1}, \alpha_{n-1}) = 0.$

Proof of (3):

Let $v_r = s_n s_{a_r} \cdots s_{n-2}$, $v_{r-1} = s_{a_{r-1}} \cdots s_{n-1}$ and $v_{r-2} = s_{a_{r-2}} \cdots s_{n-1} s_n$. Then by Lemma 3.3.1(2) we have

$$H^1(v_{r-2}v_{r-1}v_r s_{n-1}, \alpha_{n-1}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

By Lemma 3.2.2(1) if $(H^0(v_{r-2}v_{r-1}v_r s_{n-1}, \alpha_{n-1}))_\mu \neq 0$ then $\langle \mu, \alpha_n \rangle \geq 0$.

Therefore we have $H^1(s_n, H^0(v_{r-2}v_{r-1}v_r s_{n-1}, \alpha_{n-1})) = 0.$ (3.3.2.3)

By using SES and (3.3.2.3) we have

$H^1(s_n v_{r-2}v_{r-1}v_r s_{n-1}, \alpha_{n-1}) = H^0(s_n, H^1(v_{r-2}v_{r-1}v_r, \alpha_{n-1})).$ (3.3.2.4)

Since $\langle -(\beta_j + \alpha_n), \alpha_n \rangle = 0$ for all $a_{r-1} \leq j \leq a_{r-2} - 1$, we have from (3.3.2.4)

$H^1(s_n v_{r-2}v_{r-1}v_r s_{n-1}, \alpha_{n-1}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$

By Lemma 3.1.1, we have $H^1(s_{n-1}, H^0(s_n v_{r-2}v_{r-1}v_r s_{n-1}, \alpha_{n-1})) = 0.$ (3.3.2.5)

Therefore by using SES together with (3.3.2.5) we have

$H^1(s_{n-1}s_n v_{r-2}v_{r-1}v_r, \alpha_{n-1}) = H^0(s_{n-1}, H^1(s_n v_{r-2}v_{r-1}v_r s_{n-1}, \alpha_{n-1})).$ (3.3.2.6)

Since $\langle -(\beta_j + \alpha_n), \alpha_{n-1} \rangle = 0$ for all $a_{r-1} \leq j \leq a_{r-2} - 1$, using (3.3.2.6) we have

$H^1(s_{n-1}s_n v_{r-2}v_{r-1}v_r s_{n-1}, \alpha_{n-1}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$

Proceeding recursively we have

$$H^1(s_{a_{r-3}} \cdots s_{n-1}s_n v_{r-2}v_{r-1}v_r s_{n-1}, \alpha_{n-1}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Since $\langle -(\beta_j + \alpha_n), \alpha_t \rangle$ for all $a_{r-1} \leq j \leq a_{r-2} - 1$ and $a_{r-4} \leq t \leq n$, using similar arguments as above we have

$$H^1(w_r, \alpha_{n-1}) = \bigoplus_{j=a_{r-1}}^{a_{r-2}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

□

Corollary 3.3.3. *Let* $3 \leq r \leq k$. *If* $H^1(w_r, \alpha_{n-1})_\mu \neq 0$, *then*

$H^0(w_{r-2}s_ns_{a_{r-1}} \cdots s_{n+2-r}, \alpha_{n+2-r})_\mu \neq 0$.

Proof. Corollary follows from Lemma 3.2.3 and Lemma 3.3.2(2). □

Corollary 3.3.4. *Let* $3 \leq r \leq k$. *If* $H^1(w_r, \alpha_{n-1})_\mu \neq 0$, *then* $H^0(w_{r-1}s_n, \alpha_n)_\mu \neq 0$.

Proof. Corollary follows from Lemma 3.2.4 and Lemma 3.3.2(2). □

Lemma 3.3.5. *Let* $u_1 = w_ks_n[a_k, n-1]$. *Then* $H^1(u_1, \alpha_{n-1})_\mu \neq 0$ *if and only if* μ *is of the form,* $\mu = -(\beta_j + \alpha_n)$ *for some* $1 \leq j \leq a_{k-1} - 1$.

Proof. Let $v_{k+1} = s_ns_1 \cdots s_{n-2}$, $v_k = s_1 \cdots s_{n-1}$ and $v_{k-1} = s_{a_{k-1}} \cdots s_{n-1}s_n$.

Step 1:

$$H^1(v_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1}) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Proof of Step 1: By Lemma 3.2.1(2), we have

$H^0(s_1s_2 \cdots s_ns_1 \cdots s_{n-1}, \alpha_{n-1}) = \bigoplus_{i=1}^{n-2}(\mathbb{C}_{-(\beta_i+2\alpha_n)} \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{n-1})} \oplus \cdots \oplus \mathbb{C}_{-(\beta_i+2\alpha_n+\beta_{i+1})})$

$\oplus \mathbb{C}_{-(\beta_{n-1}+2\alpha_n)}$.

By Lemma 3.3.2(1), $H^1(s_2 \cdots s_ns_1 \cdots s_{n-1}, \alpha_{n-1}) = 0$. Therefore by SES, we have

$H^1(v_kv_{k+1}s_{n-1}, \alpha_{n-1}) = H^1(s_1, H^0(s_2 \cdots s_ns_1 \cdots s_{n-1}, \alpha_{n-1})) = 0$ (by Lemma 3.1.1). (3.3.5.1)

By SES and (3.3.5.1) we have

$H^1(s_nv_kv_{k+1}s_{n-1}, \alpha_{n-1}) = H^1(s_n, H^0(v_kv_{k+1}s_{n-1}, \alpha_{n-1})) = \bigoplus_{i=1}^{n-1} \mathbb{C}_{-(\beta_i+\alpha_n)}$.

By Lemma 3.1.1, we have $H^1(s_{n-1}, H^0(s_nv_kv_{k+1}s_{n-1}, \alpha_{n-1})) = 0$. Therefore by SES, we have $H^1(s_{n-1}s_nv_kv_{k+1}s_{n-1}, \alpha_{n-1}) = H^0(s_{n-1}, H^1(s_nv_kv_{k+1}s_{n-1}, \alpha_{n-1}))$.

Since $\langle -(\beta_{n-1} + \alpha_n), \alpha_{n-1} \rangle = -1$ and $\langle -(\beta_i + \alpha_n), \alpha_{n-1} \rangle = 0$ for all $1 \leq i \leq n-2$, we have $H^1(s_{n-1}s_nv_kv_{k+1}s_{n-1}, \alpha_{n-1}) = \bigoplus_{i=1}^{n-2} \mathbb{C}_{-(\beta_i+\alpha_n)}$.

Proceeding in this way recursively, we see that $H^1(v_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1}) = \bigoplus_{i=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_i+\alpha_n)}$. Hence Step 1 follows.

By Lemma 3.2.2, we have $H^1(s_n, H^0(v_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1})) = 0.$ (3.3.5.2)

Therefore by using SES and (3.3.5.2) we have

$H^1(s_nv_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1}) = H^0(s_n, H^1(v_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1})).$ (3.3.5.3)

Since $\langle -(\beta_j + \alpha_n), \alpha_n \rangle = 0$ for all $1 \leq j \leq a_{k-1} - 1$, by using (3.3.5.3) we have

$H^1(s_nv_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1}) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$

By Lemma 3.1.1, we have $H^1(s_{n-1}, H^0(s_nv_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1})) = 0.$ (3.3.5.4)

Therefore by using SES and (3.3.5.4) we have

$H^1(s_{n-1}s_nv_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1}) = H^0(s_{n-1}, H^1(s_nv_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1})).$ (3.3.5.5)

Since $\langle -(\beta_j + \alpha_n), \alpha_{n-1} \rangle = 0$ for all $1 \leq j \leq a_{k-1} - 1$, by using (3.3.5.5) we have

$H^1(s_{n-1}s_nv_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1}) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$

Proceeding recursively we have

$$H^1(s_{a_{k-2}} \cdots s_{n-1}s_nv_{k-1}v_kv_{k+1}s_{n-1}, \alpha_{n-1}) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

Using the similar arguments as above we have

$$H^1(u_1, \alpha_{n-1}) = \bigoplus_{j=1}^{a_{k-1}-1} \mathbb{C}_{-(\beta_j+\alpha_n)}.$$

□

Corollary 3.3.6. *If* $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, *then* $H^0(w_{k-1}s_ns_1s_2 \cdots s_{n+1-k}, \alpha_{n+1-k})_\mu \neq 0.$

Proof. Corollary follows from Lemma 3.2.5 and Lemma 3.3.5. □

Corollary 3.3.7. *If* $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, *then* $H^0(w_ks_n, \alpha_n)_\mu \neq 0.$

Proof. Corollary follows from Lemma 3.2.6 and Lemma 3.3.5. □

Let $3 \leq r \leq k$. Let $M_r := \{\mu \in X(T) : H^1(w_r, \alpha_{n-1})_\mu \neq 0\}$ and $M_0 := \{\mu \in X(T) : H^1(u_1, \alpha_{n-1})_\mu \neq 0\}$. Then we have

Corollary 3.3.8.

(1) $M_r \cap M_{r'} = \emptyset$ *whenever* $r \neq r'$.

(2) $M_0 \cap M_r = \emptyset$ *for every* $1 \leq r \leq k$.

Proof. Proof of (1) and (2) follow from Lemma 3.3.2 and Lemma 3.3.5. □

Lemma 3.3.9. *Let* $c = s_1 s_2 \cdots s_n$. *Let* $T_r = c^{r-1} s_1 s_2 \cdots s_{n-1}$, *for all* $2 \leq r \leq n$. *Then,* $T_r(\alpha_j) < 0$ *for all* $n + 1 - r \leq j \leq n - 1$.

Proof. Assume $r = 2$. Then we see that $T_r(\alpha_{n-1}) = -\alpha_0$. We assume that for $2 < l < n$, we have $T_l(\alpha_j) < 0$ for $n + 1 - l \leq j \leq n - 1$.

Note that $T_{l+1} = T_l s_n s_1 \cdots s_{n-1}$. Then for all $n - l \leq i \leq n - 3$, we have $s_1 \cdots s_{n-1}(\alpha_i) = \alpha_{i+1}$. Since $n - (i + 1) \geq 2$ and $n + 1 - l \leq i + 1 \leq n - 1$, we have $T_{l+1}(\alpha_i) = T_l(\alpha_{i+1}) < 0$ (by assumption). Since $s_1 \cdots s_{n-1}(\alpha_{n-2}) = \alpha_{n-1}$, we have $T_{l+1}(\alpha_{n-2}) = T_l(s_n(-\alpha_{n-1}))$.

Since $s_n(\alpha_{n-1}) = \alpha_{n-1} + 2\alpha_n$, we have $T_l s_n(\alpha_{n-1}) = T_l(\alpha_{n-1} + 2\alpha_n)$. As $s_1 \cdots s_{n-1}(\alpha_{n-1} + 2\alpha_n) = \beta_1 + 2\alpha_n$, we have $T_l(\alpha_{n-1} + 2\alpha_n) = T_{l-1} s_n(\beta_1 + 2\alpha_n)$.

Since $s_n s_1 \cdots s_{n-1} s_n(\beta_1 + 2\alpha_n) = -\alpha_1$, we have $T_{l-1} s_n(\beta_1 + 2\alpha_n) = T_{l-2}(-\alpha_1)$.

Since $s_n s_1 \cdots s_{n-1}(-\alpha_1) = -\alpha_2$, we have $T_{l-2}(-\alpha_1) = T_{l-3}(-\alpha_2)$. Therefore by recursively we have $T_{l-3}(-\alpha_2) = -\alpha_{l-1}$. Hence $T_{l+1}(\alpha_{n-2}) = -\alpha_{l-1} < 0$. Also it is clear that $T_{l+1}(\alpha_{n-1}) < 0$. Therefore we have $T_{l+1}(\alpha_j) < 0$ for all $n - l \leq j \leq n - 1$. Hence the result follows. □

Lemma 3.3.10. *Let* $u, v \in W$, *let* $v := (\prod_{j=1}^{n} s_j)^{l-1} s_1 s_2 \cdots s_{n-1}$ *for some positive integer* $l \leq n$, *such that* $l(uv) = l(u) + l(v)$. *Let* $w = uv$. *If* $l \geq 3$, *then* $H^i(w, \alpha_{n-1}) = 0$ *for all* $i \geq 0$.

Proof. We note that by SES, we have

$$H^0(w, \alpha_{n-1}) = H^0(u, H^0(v, \alpha_{n-1})).$$

We show that $H^0(v, \alpha_{n-1}) = 0$.

By Lemma 3.3.9 we have for each $1 \leq r \leq n - 1$, $c^{r-1} s_1 \cdots s_{n-1}(\alpha_j) < 0$ for all $n + 1 - r \leq j \leq n - 1$. In particular, we have $l(v s_{n-2}) = l(v) - 1$.

Therefore, by Lemma 1.13.2(4) and using SES, we have

$$H^0(v, \alpha_{n-1}) = H^0(v s_{n-2}, H^0(s_{n-2}, \alpha_{n-1})) = 0. \tag{3.3.10.1}$$

Next we show that $H^1(w, \alpha_{n-1}) = H^0(u, H^1(v, \alpha_{n-1}))$.

We will prove by induction $l(u)$. If $l(u) = 0$ then it follows trivially. Next suppose that $l(u) > 1$. Then there exists a simple root γ such that $l(s_\gamma u) = l(u) - 1$. By using SES and (3.3.10.1) we have $H^0(s_\gamma uv, \alpha_{n-1}) = 0$. Again, by induction hypotheses $H^1(s_\gamma uv, \alpha_{n-1}) = H^0(s_\gamma u, H^1(v, \alpha_{n-1}))$.

Therefore by SES, we have

$$H^1(w, \alpha_{n-1}) = H^0(s_\gamma, H^1(s_\gamma uv, \alpha_{n-1})) = H^0(s_\gamma, H^0(s_\gamma u, H^1(v, \alpha_{n-1}))) = H^0(u, H^1(v, \alpha_{n-1})).$$

$H^1(v, \alpha_{n-1}) = 0$, follows from the fact that $l(vs_{n-2}) = l(v) - 1$ and Lemma 1.13.2(4). Thus, we have $H^1(w, \alpha_{n-1}) = 0$. Therefore by [Kan16, Corollary 6.4, p.780] we have $H^i(w, \alpha_{n-1}) = 0$ for all $i \geq 0$. ◻

3.4 Cohomology modules of the tangent bundle of $Z(w, i)$

Let $w \in W$ and let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression for w and let $\underline{i} = (i_1, i_2, \ldots, i_r)$. Let $\tau = s_{i_1} s_{i_2} \cdots s_{i_{r-1}}$ and $\underline{i'} = (i_1, i_2, \ldots, i_{r-1})$.

Recall the following long exact sequence of B-modules from the proof of Theorem 1.14.1:

$$0 \to H^0(w, \alpha_{i_r}) \to H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \to H^0(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \to$$
$$H^1(w, \alpha_{i_r}) \to H^1(Z(w, \underline{i}), T_{(w,\underline{i})}) \to H^1(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \to H^2(w, \alpha_{i_r}) \to$$
$$H^2(Z(w, \underline{i}), T_{(w,\underline{i})}) \to H^2(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \to H^3(w, \alpha_{i_r}) \to \cdots$$

By Lemma 1.13.6, we have $H^j(w, \alpha_{i_r}) = 0$ for every $j \geq 2$. Thus we have the following exact sequence of B-modules:

$$0 \to H^0(w, \alpha_{i_r}) \to H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \to H^0(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \to$$
$$H^1(w, \alpha_{i_r}) \to H^1(Z(w, \underline{i}), T_{(w,\underline{i})}) \to H^1(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \to 0$$

Let $w_0 = s_{j_1} s_{j_2} \cdots s_{j_N}$ be a reduced expression of w_0 such that $\underline{i} = (j_1, j_2, \ldots, j_r)$ and let and $\underline{j} = (j_1, j_2, \ldots, j_N)$.

Lemma 3.4.1. *The natural homomorphism*

$$f : H^1(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})})$$

of B- modules is surjective.

Proof. (see [CK17, lemma 3.3.17, p.459]). ◻

Lemma 3.4.2. *Let $J = S \setminus \{\alpha_{n-1}\}$. Let $v \in W_J$ and $u \in W$ be such that $l(uv) = l(u) + l(v)$. Let $u = s_{i_1} \cdots s_{i_r}$ and $v = s_{i_{r+1}} \cdots s_{i_t}$ be reduced expressions of u and v respectively. Let $\underline{i} = (i_1, i_2, \ldots, i_r)$ and $\underline{j} = (i_1, i_2, \ldots, i_r, i_{r+1}, \ldots, i_t)$. Then we have*

(1) The natural homomorphism

$$H^0(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^0(Z(u, \underline{i}), T_{(u,\underline{i})})$$

of B-modules is surjective.

(2) The natural homomorphism

$$H^1(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^1(Z(u, \underline{i}), T_{(u,\underline{i})})$$

of B-modules is an isomorphism.

Proof. Let $r + 1 \leq l \leq t$. Let $v_l = us_{i_{r+1}} \cdots s_{i_l}$ and $\underline{i}_l = (\underline{i}, i_{r+1}, \ldots, i_l)$.

Proof of (1): By Lemma 3.1.3 we have $H^1(v_t, \alpha_{i_t}) = 0$. Therefore, using LES the natural homomorphism

$$H^0(Z(v_t, \underline{i}_t), T_{(v_t,\underline{i}_t)}) \longrightarrow H^0(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})})$$

is surjective.

By induction on $l(v)$, the natural homomorphism

$$H^0(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})}) \longrightarrow H^0(Z(u, \underline{i}), T_{(u,\underline{i})})$$

is surjective. Hence we conclude that the natural homomorphism

$$H^0(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^0(Z(u, \underline{i}), T_{(u,\underline{i})})$$

is surjective.

Proof of (2): We will prove by induction on $l(v)$. By LES, we have the following exact sequence of B-modules:

$$0 \longrightarrow H^0(uv, \alpha_{i_t}) \longrightarrow H^0(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^0(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})}) \longrightarrow H^1(uv, \alpha_{i_t}) \longrightarrow$$
$$H^1(Z(uv, \underline{j}), T_{(uv,\underline{j})}) \longrightarrow H^1(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1},\underline{i}_{t-1})}) \longrightarrow 0.$$

By induction on $l(v)$, the natural homomorphism

$$H^1(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1}, \underline{i}_{t-1})}) \longrightarrow H^1(Z(u, \underline{i}), T_{(u, \underline{i})})$$

is an isomorphism.

By Lemma 3.1.3, $H^1(uv, \alpha_{i_t}) = 0$. Therefore, by the above exact sequence we have $H^1(Z(uv, \underline{j}), T_{(uv, \underline{j})}) \longrightarrow H^1(Z(v_{t-1}, \underline{i}_{t-1}), T_{(v_{t-1}, \underline{i}_{t-1})})$ is an isomorphism. Hence we conclude that the homomorphism

$$H^1(Z(uv, \underline{j}), T_{(uv, \underline{j})}) \longrightarrow H^1(Z(u, \underline{i}), T_{(u, \underline{i})})$$

of B-modules is an isomorphism. □

Recall that by Lemma 3.0.2(1) and Lemma 3.0.4(2) we have

$$w_0 = (\prod_{l_1=1}^{k-1} [a_{l_1}, n])([a_k, n]^{n+1-k})(\prod_{l_2=1}^{k-1} [a_k, a_{l_2} - 1])$$

is a reduced expression for w_0. Let $\underline{i}$ be the tuple corresponding to this reduced of w_0. Let $u_1 = w_k s_n [a_k, n-1]$ and $\underline{i}_1$ be the tuple corresponding to the reduced expression $\prod_{l_1=1}^{k} [a_{l_1}, n]([a_k, n-1])$. Note that $a_k = 1$. With this notation, we have

Lemma 3.4.3. *(1) The natural homomorphism*

$$H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})$$

of B-modules is an isomorphism.

(2) The natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})$$

of B-modules is an isomorphism.

Proof. For $1 \leq n-k$, let $u_j = w_k s_n [a_k, n]^{j-1} s_1 s_2 \cdots s_{n-1}$ and $\underline{i}_j$ be the tuple corresponding to the reduced expression $u_j = (\prod_{l_1=1}^{k} [a_{l_1}, n])([a_k, n])^{j-1}[a_k, n-1]$ (see Lemma 3.0.4(2)). Note that $w_0 = u_{n-k} s_n (\prod_{l_2=1}^{k-1} [a_k, a_{l_2} - 1])$.

Case 1: $a_1 \neq n$. In this case, we have $s_n(\prod_{l_2=1}^{k-1} [a_k, a_{l_2-1}]) \in W_J$, where $J = S \setminus \{\alpha_{n-1}\}$.

Then by Lemma 3.4.2, the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_{n-k}, \underline{i}_{n-k}), T_{(u_{n-k},\underline{i}_{n-k})}) \tag{3.4.3.1}$$

is surjectie and the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_{n-k}, \underline{i}_{n-k}), T_{(u_{n-k},\underline{i}_{n-k})})$$

of B-modules is an isomorphism. (3.4.3.2)

If $j \geq 2$, then by Lemma 3.3.10, we have $H^1(u_j, \alpha_{n-1}) = 0$. Let $u'_j = u_j s_{n-1}$ and let $\underline{i'}_j$ be the partial subsequence of u'_j such that $\underline{i}_j = (\underline{i'}_j, n-1)$. Hence by LES , we observe that the natural homomorphism

$$H^0(Z(u_j, \underline{i}_j), T_{(u_j,\underline{i}_j)}) \longrightarrow H^0(Z(u'_j, \underline{i'}_j), T_{(u'_j,\underline{i'}_j)}) \tag{3.4.3.3}$$

is surjective and

$$H^1(Z(u_j, \underline{i}_j), T_{(u_j,\underline{i}_j)}) \longrightarrow H^1(Z(u'_j, \underline{i'}_j), T_{(u'_j,\underline{i'}_j)}) \tag{3.4.3.4}$$

is an isomorphism.

Therefore by Lemma 3.4.2 we have the natural homomorphism

$$H^0(Z(u'_j, \underline{i'}_j), T_{(u'_j,\underline{i'}_j)}) \longrightarrow H^0(Z(u_{j-1}, \underline{i}_{j-1}), T_{(u_{j-1},\underline{i}_{j-1})}) \tag{3.4.3.5}$$

is surjective and

$$H^1(Z(u'_j, \underline{i'}_j), T_{(u'_j,\underline{i'}_j)}) \longrightarrow H^1(Z(u_{j-1}, \underline{i}_{j-1}), T_{(u_{j-1},\underline{i}_{j-1})}) \tag{3.4.3.6}$$

is an isomorphism. Therefore, by combining (3.4.3.3), (3.4.3.5) together and (3.4.3.4), (3.4.3.6) together we see that the homomorphism

$$H^0(Z(u_j, \underline{i}_j), T_{(u_j,\underline{i}_j)}) \longrightarrow H^0(Z(u_{j-1}, \underline{i}_{j-1}), T_{(u_{j-1},\underline{i}_{j-1})})$$

is surjective and

$$H^1(Z(u_j, \underline{i}_j), T_{(u_j,\underline{i}_j)}) \longrightarrow H^1(Z(u_{j-1}, \underline{i}_{j-1}), T_{(u_{j-1},\underline{i}_{j-1})})$$

is an isomorphism.

Proceeding recursively we get that the homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})$$

of B-modules is surjective and homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})$$

of B-modules is an isomorphism.

Further since $u_1^{-1}(\alpha_0) < 0$, by [CKP15, lemma 3.3.8, p.667], we have $H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})_{-\alpha_0} \neq 0$. By [CKP15, Theorem 7.1], $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a parabolic subalgebra of $\mathfrak{g}$ and hence there is a unique B-stable line in $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$, namely $\mathfrak{g}_{-\alpha_0}$. Thereore we conclude that the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})$$

of B-modules is an isomorphism.

Case 2 : $a_1 = n$

Then by Lemma 3.4.2, the natural morphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_{n+1-k}, \underline{i}_{n+1-k}), T_{(u_{n+1-k},\underline{i}_{n+1-k})})$$

is surjectie and the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_{n+1-k}, \underline{i}_{n+1-k}), T_{(u_{n+1-k},\underline{i}_{n+1-k})})$$

of B-modules is an isomorphism.

If $j \geq 2$, then by Lemma 3.3.10, we have $H^1(u_j, \alpha_{n-1}) = 0$. Hence by LES, for each $2 \leq j \leq n+1-k$, we observe that the natural homomorphism

$$H^0(Z(u_j, \underline{i}_j), T_{(u_j,\underline{i}_j)}) \longrightarrow H^0(Z(u_{j-1}, \underline{i}_{j-1}), T_{(u_{j-1},\underline{i}_{j-1})})$$

is surjective and

$$H^1(Z(u_j, \underline{i}_j), T_{(u_j,\underline{i}_j)}) \longrightarrow H^1(Z(u_{j-1}, \underline{i}_{j-1}), T_{(u_{j-1},\underline{i}_{j-1})})$$

is an isomorphism.

Therefore, the homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})$$

of B-modules is surjective and the homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})$$

of B-modules is an isomorphism.

Further since $u_1^{-1}(\alpha_0) < 0$, by [CKP15, lemma 3.3.8, p.667], we have $H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})_{-\alpha_0} \neq 0$. By [CKP15, Theorem 7.1], $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a parabolic subalgebra of $\mathfrak{g}$ and hence there is a unique B-stable line in $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$, namely $\mathfrak{g}_{-\alpha_0}$. Therefore we conclude that the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})$$

of B-modules is an isomorphism. □

The following is a useful Corollary.

Corollary 3.4.4. *If* $\mu \in X(T) \setminus \{0\}$, *then, we have*

$$dim(H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})_\mu) \leq 1.$$

Proof. By [CKP15, Theorem 7.1], $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a parabolic subalgebra of $\mathfrak{g}$. By Lemma 3.4.3(1) we have

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \simeq H^0(Z(u_1, \underline{i_1}), T_{(u_1,\underline{i_1})})$$

(as B-modules).

Hence for any $\mu \in X(T) \setminus \{0\}$, we have

$$dim(H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)})_\mu) \leq 1.$$

□

Let $u'_1 = w_k s_n[a_k, n-2]$ *and let* $\underline{i'}_1$ *be the tuple corresponding to the reduced expression* $\prod_{l_1=1}^{k}[a_{l_1}, n][a_k, n-2]$. *Let* $3 \leq r \leq k$, *and let* $\underline{j}_r = (a_1, \ldots, n; a_2, \ldots, n; \ldots; a_{r-1}, \ldots, n; a_r, \ldots, n-1)$ *and* $\underline{j'}_r = (a_1, \ldots, n; a_2, \ldots, n; a_r, \ldots, n-2)$.

We now prove

Lemma 3.4.5. *Let* $\mu \in X(T) \setminus \{0\}$.

(1) *If* $H^1(u_1, \alpha_{n-1})_\mu = 0$, *then dim*$(H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)})_\mu) \leq 1$.

(2) *If* $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, *then dim*$(H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)})_\mu) = 2$, *and the natural homomorphism*

$$H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)})_\mu \longrightarrow H^1(u_1, \alpha_{n-1})_\mu$$

is surjective.

Proof. By LES, we have the following long exact sequence of B-modules:

$$0 \longrightarrow H^0(u_1, \alpha_{n-1}) \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)}) \longrightarrow$$

$$H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)}) \longrightarrow H^1(u_1, \alpha_{n-1}) \longrightarrow \cdots \qquad (3.4.5.1)$$

Proof of (1): Assume that $H^1(u_1, \alpha_{n-1})_\mu = 0$ and $\mu \in X(T) \setminus \{0\}$, then by the above exact sequence the natural homoorphism (of T-modules) $H^0(Z(u_1, \underline{i}_1), T_{(u'_1, \underline{i}_1)})_\mu \longrightarrow H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)})_\mu$ is surjective. By Corollary 3.4.4, we have $\dim(H^0(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)})_\mu) \leq 1$.

Proof of (2): Assume that $H^1(u_1, \alpha_{n-1})_\mu \neq 0$. Then by Lemma 3.3.5, μ is of the form $-(\beta_j + \alpha_n)$ for some $1 \leq j \leq a_{k-1} - 1$ and $\dim(H^1(u_1, \alpha_{n-1})_\mu)=1$. Hence by using Corollary 3.4.4, we see that if $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, then

$$dim(H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)})_\mu) \leq 2. \qquad (3.4.5.2)$$

Let $\underline{j''}_k = (\underline{j}_k, n)$ be the tuple corresponding to the reduced expression $w_k s_n = \prod_{l_1=1}^{k}[a_{l_1}, n]$. Then by Lemma 3.4.2 the natural homomorphism

$$H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1, \underline{i'}_1)}) \longrightarrow H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})$$

is surjective. By (3.4.5.2), we have $\dim(H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})_\mu) \leq 2.$ (3.4.5.3)

Again by the Lemma 3.4.2 the natural homomorphism

$$H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)}) \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)}) \tag{3.4.5.4}$$

is surjective.

Recall that $u'_t = u_k s_{n-1}$. By LES we have the following long exact sequence of B-modules:

$$0 \longrightarrow H^0(w_k, \alpha_{n-1}) \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)}) \longrightarrow$$
$$H^0(Z(u'_k, \underline{i'}_k), T_{(u'_k, \underline{i'}_k)}) \longrightarrow H^1(w_k, \alpha_{n-1}) \longrightarrow \cdots$$

Since $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, by Corollary 3.3.8 we have $H^1(w_k, \alpha_{n-1})_\mu = 0$. Therefore we have an exact sequence

$$0 \longrightarrow H^0(w_k, \alpha_{n-1})_\mu \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)})_\mu \longrightarrow$$
$$H^0(Z(\tau_k, \underline{j'}_k), T_{(\tau_k, \underline{j'}_k)})_\mu \longrightarrow 0.$$

Let $\sigma_k = w_{k-1} s_n s_1 \cdots s_{n+1-k}$ and $\underline{j^*}_k = (\underline{j}_{k-1}, n, 1, 2, \ldots, n+1-k)$ be the tuple corresponding to this reduced expression of σ_k. Then by using Lemma 3.4.2, the natural homomorphism

$$H^0(Z(\tau_k, \underline{j'}_k), T_{(\tau_k, \underline{j'}_k)}) \longrightarrow H^0(Z(\sigma_k, \underline{j^*}_k), T_{(\sigma_k, \underline{j^*}_k)}) \tag{3.4.5.5}$$

is surjective.

Therefore the natural map

$$H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)})_\mu \longrightarrow H^0(Z(\sigma_k, \underline{j^*}_k), T_{(\sigma_k, \underline{j^*}_k)})_\mu \tag{3.4.5.6}$$

is surjective.

Since $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, by Corollary 3.3.6 we have $H^0(\sigma_k, \alpha_{n+1-k})_\mu \neq 0$. Therefore, $H^0(Z(\sigma_k, \underline{j^*}_k), T_{(\sigma_k, \underline{j^*}_k)})_\mu \neq 0$. Thus from (3.4.5.6) we have $H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)})_\mu \neq 0$. Then we have an exact sequence of T-modules

$$0 \longrightarrow H^0(w_k s_n, \alpha_n)_\mu \longrightarrow H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})_\mu \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k, \underline{j}_k)})_\mu \longrightarrow 0$$

Since $H^1(u_1, \alpha_{n-1})_\mu \neq 0$, by Corollary 3.3.7 we have $H^0(w_k s_n, \alpha_n)_\mu \neq 0$.

Therefore $\dim(H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})_\mu) \geq 2$. On the other hand by (3.4.5.3) we have $\dim(H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})_\mu) \leq 2$. Hence we have $\dim(H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n, \underline{j''}_k)})_\mu) = 2$.

Since the natural homomorphism

$$H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1,\underline{i'}_1)}) \longrightarrow H^0(Z(w_k s_n, \underline{j''}_k), T_{(w_k s_n,\underline{j''}_k)})$$

is surjective, we have dim($H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1,\underline{i'}_1)_\mu}$))= 2. (3.4.5.7)

It is clear from (3.4.5.1),(3.4.5.7), and Corollary 3.4.4, that

$$H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1,\underline{i'}_1)})_\mu \longrightarrow H^1(u_1, \alpha_{n-1})_\mu$$

is surjective.

□

Corollary 3.4.6. *The natural homomorphism*

$$H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1,\underline{i'}_1)}) \longrightarrow H^1(u_1, \alpha_{n-1})$$

is surjective.

Proof. Note that by Lemma 3.3.5, if $H^1(u_1, \alpha_{n-1})_\mu \neq 0$ then $\mu \in X(T) \setminus \{0\}$. Now the proof follows from Lemma 3.4.5. □

Lemma 3.4.7. *(1) If $H^1(w_m, \alpha_{n-1})_\mu = 0$ for all $r \leq m \leq k$ and $H^1(u_1, \alpha_{n-1})_\mu = 0$, then dim($H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r,\underline{j'}_r)})_\mu$) ≤ 1.*

(2) Let $3 \leq r \leq k$. If $H^1(w_r, \alpha_{n-1})_\mu \neq 0$, then dim($H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r,\underline{j'}_r)})_\mu$) $= 2$, and the natural homomorphism

$$H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r,\underline{j'}_r)})_\mu \longrightarrow H^1(w_r, \alpha_{n-1})_\mu$$

is surjective.

Proof. Proof of (1): If $H^1(u_1, \alpha_{n-1})_\mu = 0$, then by Lemma 3.4.5, we have

dim($H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1,\underline{i'}_1)})_\mu$)$\leq 1$. By Lemma 3.4.2, the natural homomorphism

$$H^0(Z(u'_1, \underline{i'}_1), T_{(u'_1,\underline{i'}_1)})_\mu \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)})_\mu$$

is surjective.

If $H^1(w_m, \alpha_{n-1})_\mu = 0$ for all $r \leq m \leq k$, by using LES, we see that the natural homomorphism

$$H^0(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)})_\mu \longrightarrow H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r,\underline{j'}_r)})_\mu$$

is surjective.

Therefore, we have $\dim(H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r,\underline{j'}_r)})_\mu) \leq 1$.

Proof of (2): Assume that $H^1(w_r, \alpha_{n-1})_\mu \neq 0$. Then by Corollary 3.3.8, we have $H^1(w_m, \alpha_{n-1})_\mu = 0$ for all $r+1 \leq m \leq k$ and $H^1(u_1, \alpha_{n-1})_\mu = 0$. Then by (1), we have

$$\dim(H^0(Z(\tau_{r+1}, \underline{j'}_{r+1}), T_{(\tau_{r+1},\underline{j'}_{r+1})})_\mu) \leq 1.$$

By Lemma 3.4.2, the natural homomorphism

$$H^0(Z(\tau_{r+1}, \underline{j'}_{r+1}), T_{(\tau_{r+1},\underline{j'}_{r+1})})_\mu \longrightarrow H^0(Z(w_r, \underline{j}_r), T_{(w_r,\underline{j}_r)})_\mu$$

is surjective. Hence, we have $\dim(H^0(Z(w_r, \underline{j}_r), T_{(w_r,\underline{j}_r)})_\mu) \leq 1$. (3.4.7.1)

By LES we have the following long exact sequence of B-modules:

$$0 \longrightarrow H^0(w_r, \alpha_{n-1}) \longrightarrow H^0(Z(w_r, \underline{j}_r), T_{(w_r,\underline{j}_r)}) \longrightarrow$$
$$H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r,\underline{j'}_r)}) \longrightarrow H^1(w_r, \alpha_{n-1}) \longrightarrow \cdots$$

Since $\dim(H^0(Z(w_r, \underline{j}_r), T_{(w_r,\underline{j}_r)})_\mu) \leq 1$ and $\dim(H^1(w_r, \alpha_{n-1})_\mu) = 1$, we see that

$\dim(H^0(Z(\tau_r, \underline{j'}_r))_\mu) \leq 2$. (3.4.7.2)

Let $\underline{j''}_{r-1} = (\underline{j}_{r-1}, n)$ be the tuple corresponding to the reduced expression $w_{r-1}s_n = \prod_{l_1=1}^{r-1}[a_{l_1}, n]$. Then by Lemma 3.4.2 the natural homomorphism

$$H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r,\underline{j'}_r)}) \longrightarrow H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n,\underline{j''}_{r-1})})$$

is surjective. By (3.4.7.2), we have $\dim(H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n,\underline{j''}_{r-1})_\mu})) \leq 2$. (3.4.7.3)

Again by the Lemma 3.4.2 the natural homomorphism

$$H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n,\underline{j''}_{r-1})}) \longrightarrow H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1},\underline{j}_{r-1})}) \quad (3.4.7.4)$$

is surjective.

By LES we have the following long exact sequence of B-modules:

$$0 \longrightarrow H^0(w_{r-1}, \alpha_{n-1}) \longrightarrow H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1},\underline{j}_{r-1})}) \longrightarrow$$
$$H^0(Z(\tau_{r-1}, \underline{j'}_{r-1}), T_{(\tau_{r-1},\underline{j'}_{r-1})}) \longrightarrow H^1(w_{r-1}, \alpha_{n-1}) \longrightarrow \cdots$$

Since $H^1(w_r, \alpha_{n-1})_\mu \neq 0$, then by Corollary 3.3.8 we have $H^1(w_{r-1}, \alpha_{n-1})_\mu = 0$. Therefore we have an exact sequence

$$0 \longrightarrow H^0(w_{r-1}, \alpha_{n-1})_\mu \longrightarrow H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})})_\mu \longrightarrow$$
$$H^0(Z(\tau_{r-1}, \underline{j'}_{r-1}), T_{(\tau_{r-1}, \underline{j'}_{r-1})})_\mu \longrightarrow 0 \qquad (3.4.7.5)$$

Let $\sigma_{r-1} = w_{r-2}s_n s_{a_{r-1}\cdots s_{n+2-r}}$, and let $\underline{j^*}_{r-1} = (\underline{j}_{r-2}, n, a_{r-1}, a_{r-1}+1, \ldots, n+2-r)$ be the reduced expression of σ_{r-1}.

Then by using Lemma 3.4.2 the natural homomorphism

$$H^0(Z(\tau_{r-1}, \underline{j'}_{r-1}), T_{(\tau_{r-1}, \underline{j'}_{r-1})}) \longrightarrow H^0(Z(\sigma_{r-1}, \underline{j^*}_{r-1}), T_{(\sigma_{r-1}, \underline{j^*}_{r-1})}) \qquad (3.4.7.6)$$

is surjective.

Therefore, by (3.4.7.5) and (3.4.7.6) we have

$$H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})})_\mu \longrightarrow H^0(Z(\sigma_{r-1}, \underline{j^*}_{r-1}), T_{(\sigma_{r-1}, \underline{j^*}_{r-1})})_\mu \qquad (3.4.7.7)$$

is surjective. Since $H^1(w_r, \alpha_{n-1})_\mu \neq 0$, by Corollary 3.3.3 we have $H^0(\sigma_{r-1}, \alpha_{n+2-r})_\mu \neq 0$. Therefore, $H^0(Z(\sigma_{r-1}, \underline{j^*}_{r-1}), T_{\sigma_{r-1}, \underline{j^*_{r-1}}})_\mu \neq 0$.

Thus from (3.4.7.7) we have $H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})})_\mu \neq 0$. Then we have an exact sequence of T-modules

$$0 \longrightarrow H^0(w_{r-1}s_n, \alpha_n)_\mu \longrightarrow H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n, \underline{j''}_{r-1})})_\mu \longrightarrow H^0(Z(w_{r-1}, \underline{j}_{r-1}), T_{(w_{r-1}, \underline{j}_{r-1})})_\mu \longrightarrow 0$$

Since $H^1(w_r, \alpha_{n-1})_\mu \neq 0$, by Corollary 3.3.4 we have $H^0(w_{r-1}s_n, \alpha_n)_\mu \neq 0$.

Therefore $\dim(H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n, \underline{j''}_{r-1})_\mu})) \geq 2$.

Hence, by (3.4.7.3) we have $\dim(H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n, \underline{j''}_{r-1})})_\mu) = 2$.

Since the natural homomorphism

$$H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)}) \longrightarrow H^0(Z(w_{r-1}s_n, \underline{j''}_{r-1}), T_{(w_{r-1}s_n, \underline{j''}_{r-1})})$$

is surjective, we have $\dim(H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)})_\mu) = 2$.

Therefore by (3.4.7.1),

$$H^0(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)})_\mu \longrightarrow H^1(w_r, \alpha_{n-1})_\mu$$

is surjective. □

3.5 Main theorem

In this section we prove the main theorem.

Recall that $G = PSO(2n+1, \mathbb{C}) (n \geq 3)$, and let c be a Coxeter element of W. Recall that there exists a decreasing sequence $n \geq a_1 > a_2 > \cdots > a_k = 1$ of positive integers such that $c = [a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$, where $[i, j]$ for $i \leq j$ denotes $s_i s_{i+1} \cdots s_j$.

Let $\underline{i} = (\underline{i}^1, \underline{i}^2, \ldots, \underline{i}^n)$ be a sequence corresponding to a reduced expression of w_0, where $\underline{i}^r$ $(1 \leq r \leq n)$ is a sequence of reduced expressions of c (see Lemma 3.0.4).

Theorem 3.5.1. *(S. S. Kannan, P. Saha): $H^j(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ for all $j \geq 1$ if and only if $a_2 \neq n-1$.*

Proof. From [CKP15, Proposition 3.1, p. 673], we have $H^j(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ for all $j \geq 2$. It is enough to prove the following: $H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ if and only if c is of the form $[a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$ with $a_2 \neq n - 1$.

Proof of ($\Longrightarrow$): If $a_2 = n - 1$, then $a_1 = n$ and $c = s_n s_{n-1} v$, where $v \in W_J$ and $J = S \setminus \{\alpha_{n-1}, \alpha_n\}$. Let $u = s_n s_{n-1}$. Then $c = uv$. Let $\underline{j} = (n, n-1)$ be the sequence corresponding to u. Then using LES, we have:

$$0 \longrightarrow H^0(u, \alpha_{n-1}) \longrightarrow H^0(Z(u, \underline{j}), T_{(u, \underline{j})}) \longrightarrow H^0(s_n, \alpha_n) \longrightarrow$$

$$H^1(u, \alpha_{n-1}) \xrightarrow{f} H^1(Z(u, \underline{j}), T_{(u, \underline{j})}) \longrightarrow 0.$$

We see that $H^1(s_n s_{n-1}, \alpha_{n-1}) = \mathbb{C}_{\alpha_n + \alpha_{n-1}}$ and $H^0(s_n, \alpha_n)_{\alpha_n + \alpha_{n-1}} = 0$. Hence f is non zero homomorphism. Hence $H^1(Z(u, \underline{j}), T_{(u, \underline{j})})) \neq 0$. By Lemma 3.4.1, the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \longrightarrow H^1(Z(u, \underline{j}), T_{(u, \underline{j})})$$

is surjective.

Hence we have

$$H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \neq 0.$$

Proof of ($\Longleftarrow$) : Assume that $a_2 \neq n - 1$. Recall that $w_k = [a_1, n] \cdots [a_{k-1}, n][a_k, n-1]$, $u_1 = w_k s_n [a_k, n-1]$. By Lemma 3.4.3(2), the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1, \underline{i}_1)}) \qquad (3.5.1.1)$$

of B-modules is an isomorphism.

By using LES, we have the following exact sequence of B-modules:

$$\cdots \longrightarrow H^0(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)}) \longrightarrow H^0(Z(u_1', \underline{i'}_1), T_{(u_1',\underline{i'}_1)}) \xrightarrow{h_1}$$
$$H^1(u_1, \alpha_{n-1}) \longrightarrow H^1(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)}) \longrightarrow H^1(Z(u_1', \underline{i'}_1), T_{(u_1',\underline{i'}_1)}) \longrightarrow 0.$$

By Corollary 3.4.6, we see that the natural homomorphism $h_1 : H^0(Z(u_1', \underline{i'}_1), T_{(u_1',\underline{i'}_1)}) \longrightarrow H^1(u_1, \alpha_{n-1})$ is surjective. Therefore, the natural homomorphism

$$H^1(Z(u_1, \underline{i}_1), T_{(u_1,\underline{i}_1)}) \longrightarrow H^1(Z(u_1', \underline{i'}_1), T_{(u_1',\underline{i'}_1)}) \tag{3.5.1.2}$$

is an isomorphism.

By Lemma 3.4.2(2), the natural homomorphism

$$H^1(Z(u_1', \underline{i'}_1), T_{(u_1',\underline{i'}_1)}) \longrightarrow H^1(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \tag{3.5.1.3}$$

is an isomorphism.

Therefore, by (3.5.1.1), (3.5.1.2) and (3.5.1.3) the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \tag{3.5.1.4}$$

is an isomorphism.

Recall that $\tau_k = [a_1, n][a_2, n] \cdots [a_{k-1}, n][a_k, n-2]$. By using LES, we have the following exact sequence of B-modules:

$$\cdots \longrightarrow H^0(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \longrightarrow H^0(Z(\tau_k, \underline{j'}_k), T_{(\tau_k,\underline{j'}_k)}) \xrightarrow{h_2}$$
$$H^1(w_k, \alpha_{n-1}) \longrightarrow H^1(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \xrightarrow{h_3} H^1(Z(\tau_k, \underline{j'}_k), T_{(\tau_k,\underline{j'}_k)}) \longrightarrow 0.$$

By Lemma 3.4.7(2), we see that the map $h_2 : H^0(Z(\tau_k, \underline{j'}_k), T_{(\tau_k,\underline{j'}_k)}) \longrightarrow H^1(w_k, \alpha_{n-1})$ is surjective.

Therefore, the map $h_3 : H^1(Z(w_k, \underline{j}_k), T_{(w_k,\underline{j}_k)}) \longrightarrow H^1(Z(\tau_k, \underline{j'}_k), T_{(\tau_k,\underline{j'}_k)})$ (3.5.1.5)

is an isomorphism.

By using Lemma 3.4.2(2) we see that the natural map

$$H^1(Z(\tau_k, \underline{j'}_k), T_{(\tau_k, \underline{j'}_k)}) \longrightarrow H^1(Z(w_{k-1}, \underline{j}_{k-1}), T_{(w_{k-1}, \underline{j}_{k-1})}) \tag{3.5.1.6}$$

is an isomorphism.

By using LES, we have the following exact sequence of B-modules:

$$\cdots \longrightarrow H^0(Z(w_{k-1}, \underline{j}_{k-1}), T_{(w_{k-1}, \underline{j}_{k-1})}) \longrightarrow H^0(Z(\tau_{k-1}, \underline{j'}_{k-1}), T_{(\tau_{k-1}, \underline{j'}_{k-1})}) \to$$
$$H^1(w_{k-1}, \alpha_{n-1}) \longrightarrow H^1(Z(w_{k-1}, \underline{j}_{k-1}), T_{(w_{k-1}, \underline{j}_{k-1})}) \to H^1(Z(\tau_{k-1}, \underline{j'}_{k-1}), T_{(\tau_{k-1}, \underline{j'}_{k-1})}) \longrightarrow 0.$$

By Lemma 3.4.7(2), we see that the natural map $H^0(Z(\tau_{k-1}, \underline{j'}_{k-1}), T_{(\tau_{k-1}, \underline{j'}_{k-1})}) \longrightarrow H^1(w_{k-1}, \alpha_{n-1})$ is surjective.

Therefore, the natural map

$$H^1(Z(w_{k-1}, \underline{j}_{k-1}), T_{(w_{k-1}, \underline{j}_{k-1})}) \longrightarrow H^1(Z(\tau_{k-1}, \underline{j'}_{k-1}), T_{(\tau_{k-1}, \underline{j'}_{k-1})}) \tag{3.5.1.7}$$

is an isomorphism.

By using Lemma 3.4.2(2) and 3.4.7(2) repeatedly, we see that the natural map

$$H^1(Z(\tau_{k-1}, \underline{j'}_{k-1}), T_{(\tau_{k-1}, \underline{j'}_{k-1})}) \longrightarrow H^1(Z(\tau_r, \underline{j'}_r), T_{(\tau_r, \underline{j'}_r)}) \tag{3.5.1.8}$$

is an isomorphism for all $3 \leq r \leq k-2$.

Therefore by (3.5.1.4), (3.5.1.5), (3.5.1.6), (3.5.1.7), (3.5.1.8) we have the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \longrightarrow H^1(Z(\tau_3, \underline{j'}_3), T_{(\tau_3, \underline{j'}_3)}) \tag{3.5.1.9}$$

is an isomorphism.

Again from Lemma 3.4.2(2), we see that the natural map

$$H^1(Z(\tau_3, \underline{j'}_3), T_{(\tau_3, \underline{j'}_3)}) \longrightarrow H^1(Z(w_2, \underline{j}_2), T_{(w_2, \underline{j}_2)})$$

is an isomorphism.

Since $a_2 \neq n-1$, by Lemma 3.3.2(2) we have $H^1(w_2, \alpha_{n-1}) = 0$. Note that by Lemma 3.3.2(1) we have $H^1(w_1, \alpha_{n-1}) = 0$.

By using Lemma 3.1.3 and using LES, we have $H^1(Z(w_2, \underline{j}_2), T_{(w_2, \underline{j}_2)}) = 0$. Hence we conclude that

$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$. This completes the proof of the theorem.

□

Corollary 3.5.2. *Let c be a Coxeter element such that c is of the form $[a_1, n][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$ with $a_2 \neq n - 1$ and $a_k = 1$. Let $(w_0, \underline{i})$ be a reduced expression of w_0 in terms of c as in Theorem 3.5.1. Then, $Z(w_0, \underline{i})$ has no deformations.*

Proof. By Theorem 3.5.1 and by [CKP15, Proposition 3.1, p.673], we have $H^i(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$ for all $i > 0$. Hence, by [Huy05, Proposition 6.2.10, p.272], we see that $Z(w_0, \underline{i})$ has no deformations. □

Remark 3.5.3. *Theorem 7.1 does not hold for $PSO(5, \mathbb{C})$.*

Proof. We take $c = s_1s_2$. Here $a_2 \neq 1$. Further, we have $w_0 = c^2 = s_1s_2s_1s_2$. Let $\underline{i} = (1, 2, 1, 2)$. It is easy to see by using SES repeatedly that $H^1(s_1s_2s_1, \alpha_1) = \mathbb{C}_{\alpha_1+\alpha_2} \oplus \mathbb{C}_{\alpha_2}$. Further, note that $H^0(Z(s_1s_2, (1, 2)), T_{(s_1s_2,(1,2))})_{\alpha_1+\alpha_2} = 0$ (see [CKP15, Proposition 6.3(1), p.688]). Hence by using LES we have $H^1(Z(s_1s_2s_1, (1, 2, 1)), T_{(s_1s_2s_1,(1,2,1))}) \neq 0$.

Therefore by using Lemma 3.4.1 we have $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \neq 0$. □

Chapter 4

Rigidity of Bott-Samelson-Demazure-Hansen variety for F_4 and G_2

4.1 Reduced expressions

In this chapter till the end of the section 4.5 we will assume that G is of type F_4. Note that w_0 is equal to $-identity$. We recall the following Proposition from [YZ08, Proposition 1.3, p.858]. We use the notation as in [YZ08].

Proposition 4.1.1. *Let $c \in W$ be a Coxeter element, let ω_i be the fundamental weight corresponding to the simple root α_i. Then there exists a least positive integer $h(i,c)$ such that $c^{h(i,c)}(\omega_i) = w_0(\omega_i)$.*

Now we can deduce the following:

Lemma 4.1.2. *Let $c \in W$ be a Coxeter element. Then, we have*

(1) $w_0 = c^6$.

(2) For any sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \cdots, \underline{i}^6)$ of reduced expressions of c; the sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \cdots, \underline{i}^6)$ is a reduced expression of w_0.

Proof. Proof of (1): Let $\eta : S \longrightarrow S$ be the involution of S defined by $i \to i^*$, where i^* is given by $\omega_{i^*} = -w_0(\omega_i)$. Since G is of type F_4, $w_0 = -identity$, and hence $\omega_{i^*} = \omega_i$ for every i. Therefore, we have $i = i^*$ for every i. Let h be the Coxeter number. By [YZ08, Proposition 1.7], we have $h(i,c) + h(i^*,c) = h$. Since $h = 2|R^+|/4$ (see [Hum95, Proposition 3.18]) and $i = i^*$, we have $h(i,c) = h/2 = 6$, as $|R^+| = 24$. By Proposition 4.1.1, we have $c^6(\omega_i) = -\omega_i$ for all $1 \leq i \leq 4$. Since $\{\omega_i : 1 \leq i \leq 4\}$ forms an $\mathbb{R}$-basis of $X(T) \otimes \mathbb{R}$, it follows that $c^6 = -identity$. Hence, we have $w_0 = c^6$. The assertion (2) follows from the fact that $l(c) = 4$ and $l(w_0) = |R^+| = 24$. (see [Hum92, p.66, Table 1]). □

Lemma 4.1.3. *Let $v \in W$ and $\alpha \in S$. Then $H^1(s_j, H^0(v, \alpha)) = 0$ for $j = 1, 2$.*

Proof. If $H^1(s_j, H^0(v, \alpha))_\mu \neq 0$, then there exists an indecomposable $\tilde{L}_{\alpha_j}$-summand V of $H^0(v, \alpha)$ such that $H^1(s_j, V)_\mu \neq 0$. By Lemma 2.4, we have $V \simeq V' \otimes \mathbb{C}_\lambda$ for some character λ of $\tilde{B}_{\alpha_j}$ and for some irreducible $\tilde{L}_{\alpha_j}$-module V'. Since $H^1(s_j, V)_\mu \neq 0$, from Lemma 1.13.3(3) we have $\langle \lambda, \alpha_j \rangle \leq -2$. If α is a short root, then $H^1(w, \alpha) = 0$ for all $w \in W$ (see [Kan16, Corollary 5.6, p.778]). Hence we may assume that α is a long root. Then there exists $w \in W$ such that $w(\alpha) = \alpha_0$. Thus $H^0(v, \alpha) \subseteq H^0(vw, \alpha_0)$. Again, since α_0 is highest long root, $H^0(w_0, \alpha_0) = \mathfrak{g} \longrightarrow H^0(vw, \alpha_0)$ is surjective. Let μ' be the lowest weight of V. Then by the above argument μ' is a root. Therefore we have $\mu' = \mu_1 + \lambda$, where μ_1 is the lowest weight of V'. Hence, we have $\langle \mu', \alpha_j \rangle \leq -2$. Since α_j is a long root and μ' is a root, we have $\langle \mu', \alpha_j \rangle = -1, 0, 1$. This is a contradiction. Thus we have $H^1(s_j, H^0(v, \alpha))_\mu = 0$. □

4.2 Cohomology modules $H^0(w, \alpha_i)$

Let $w_r = (s_1 s_2 s_3 s_4)^r s_1 s_2$ for $1 \leq r \leq 5$. In this section we compute various cohomology modules $H^0(w, \alpha_i)$ for some elements $w \in W$ and $i = 2, 3$.

Lemma 4.2.1.

(1) $H^0(w_3, \alpha_2) = 0$.

(2) $H^0(w_r, \alpha_2) = 0$ *for* $r = 4, 5$.

Proof. We have $w_3 = [1, 4]^3 12$. By using SES we have

$H^0(s_1 s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2)}$.

Since $\langle \alpha_2, \alpha_4 \rangle = 0$, by using SES we have

$$H^0(s_4 s_1 s_2, \alpha_2) = H^0(s_1 s_2, \alpha_2).$$

Since $\langle -\alpha_2, \alpha_3 \rangle = 2$ and $\langle -(\alpha_1 + \alpha_2), \alpha_3 \rangle = 2$, then by using SES we have

$$H^0(s_3 s_4 s_1 s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus (\mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3)}) \oplus (\mathbb{C}_{-(\alpha_1+\alpha_2)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}). \quad (4.2.1.1)$$

Since $\mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2}$ is indecomposable two dimensional $\tilde{B}_{\alpha_2}$-module, by Lemma 1.13.4 we have $\mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} = V \otimes \mathbb{C}_{-\omega_2}$, where V is the standard two dimensional irreducible $\tilde{L}_{\alpha_2}$-module.

Thus by Lemma 1.13.3(4), we have $H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2}) = 0$.

Since $\langle -(\alpha_2 + \alpha_3), \alpha_2 \rangle = -1$, $\langle -(\alpha_1 + \alpha_2), \alpha_2 \rangle = -1$, by Lemma 1.13.2(4) we have $H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = 0$ and $H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1+\alpha_2)}) = 0$.

Since $\langle -(\alpha_2 + 2\alpha_3), \alpha_2 \rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_2 \rangle = 0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+2\alpha_3)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3)}$$

and

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3), \alpha_2 \rangle = 1$, we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$

Thus we have

$$H^0(s_2 s_3 s_4 s_1 s_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}. \tag{4.2.1.2}$$

Since $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_1 \rangle = -1$, by using Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = 0,$$

and

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}) = 0.$$

Since $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_1 \rangle = 0$, by using Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$

Since $\langle -(\alpha_2 + 2\alpha_3), \alpha_1 \rangle = 1$, by using Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_2+2\alpha_3)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}.$$

Therefore we have

$$H^0(w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}. \tag{4.2.1.3}$$

By using SES we have

$$H^0(w_3,\alpha_2) = H^0([1,4]^2, H^0(w_1,\alpha_2)).$$

Note that the computations of the module $H^0([1,4]^2, H^0(w_1,\alpha_2))$ is independent of the choice of a reduced expression of $[1,4]^2$. We consider the reduced expression $s_2s_1s_2s_3s_2s_3s_4s_3$, of $[1,4]^2$ to compute $H^0([1,4]^2, H^0(w_1,\alpha_2))$.

Since $\langle -(\alpha_2+2\alpha_3), \alpha_3\rangle = -2$, $\langle -(\alpha_1+\alpha_2+2\alpha_3), \alpha_3\rangle = -2$, by using Lemma 1.13.3(3) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+2\alpha_3)}) = 0$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}) = 0.$$

Since $\langle -(\alpha_1+2\alpha_2+2\alpha_3), \alpha_3\rangle = 0$, by using Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$

Thus from the above discussion we have

$$H^0(s_3w_1,\alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$

Since $\langle -(\alpha_1+2\alpha_2+2\alpha_3), \alpha_4\rangle = 2$, by using SES and Lemma 1.13.3(2) we have

$$H^0(s_4s_3w_1,\alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}.$$

Since $\langle -(\alpha_1+2\alpha_2+2\alpha_3), \alpha_3\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = 1$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4), \alpha_3\rangle = 2$, by using Lemma 1.13.3(2) we have

$$H^0(s_3s_4s_3w_1,\alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}.$$

Since $\langle -(\alpha_1+2\alpha_2+2\alpha_3), \alpha_2\rangle = -1$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_2\rangle = -1$, $\langle -(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4), \alpha_2\rangle = -1$, $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_2\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4), \alpha_2\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4), \alpha_2\rangle = 1$, by using Lemma 1.13.3(2), Lemma 1.13.3(4) we have

$$H^0(s_2s_3s_4s_3w_1,\alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}$ is two dimensional indecomposable $\tilde{B}_{\alpha_3}$-module, thus by Lemma 1.13.4(1) we have

$\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} = V \otimes \mathbb{C}_{-\omega_3}$, where V is the standard two dimensional irreducible $\tilde{L}_{\alpha_3}$-module.

Thus by Lemma 1.13.3(4) we have

$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}) = 0.$

Since $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_3 \rangle = -1$, and $\langle -(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_3 \rangle = 0$, by Lemma 1.13.3(2), Lemma 1.13.3(4) we have $H^0(s_3s_2s_3s_4s_3w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}$.

Since $\langle -(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_2 \rangle = -1$, by Lemma 1.13.3(4) we have

$H^0(s_2s_3s_2s_3s_4s_3w_1, \alpha_2) = 0.$

Thus by using SES and Lemma 1.13.3(2) we have $H^0(s_1s_2s_3s_2s_3s_4s_3w_1, \alpha_2) = 0$.

Again by using SES and Lemma 1.13.3(2) we have $H^0(w_3, \alpha_2) = H^0([1, 4]^2, H^0(w_1, \alpha_2)) = 0$.

Proof of (2) follows from (1). ◻

Recall that $\omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. Now onwards we replace $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ by ω_4.

Lemma 4.2.2.

(1) $H^0(w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4}$.

(2) $H^0(w_3s_3, \alpha_3) = \mathbb{C}_{-\omega_4}$.

Proof. Proof of (1): Using SES we have $H^0(s_3, \alpha_3) = \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{\alpha_3}$. Since $\langle \alpha_3, \alpha_2 \rangle = -1$, by using SES and Lemma 1.13.3 we have

$$H^0(s_2s_3, \alpha_3) = \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_3+\alpha_2)}.$$

Further, since $\langle \alpha_3, \alpha_1 \rangle = 0$ and $\langle -(\alpha_3 + \alpha_2), \alpha_1 \rangle = 1$, by using SES and Lemma 1.13.3 we have

$$H^0(s_1s_2s_3, \alpha_3) = \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Note that the computations of the module $H^0([1, 4]^2, H^0(s_1s_2s_3, \alpha_3))$ is independent of the choice of a reduced expression of $[1, 4]^2$. We consider the reduced expression $s_1s_2s_1s_3s_2s_3s_4s_3$ of $[1, 4]^2$ to compute $H^0([1, 4]^2, H^0(s_1s_2s_3, \alpha_3))$.

Since $\mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}$ is two dimensional $\tilde{B}_{\alpha_3}$-module, by Lemma 1.13.4(1) we have

$$\mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} = V \otimes \mathbb{C}_{-\omega_3}$$

where V is the standard two dimensional irreducible $\tilde{L}_{\alpha_3}$-module.

Thus by using Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}) = 0.$$

Since $\langle -(\alpha_2 + \alpha_3), \alpha_3 \rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_3 \rangle = 0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)}$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Thus from the above discussion we have

$$H^0(s_3 s_1 s_2 s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $\langle -(\alpha_2 + \alpha_3), \alpha_4 \rangle = 1$, $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_4 \rangle = 1$, by using Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Thus from the above discussion we have

$$H^0(s_4 s_3 s_1 s_2 s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_2 + \alpha_3), \alpha_3 \rangle = 0$, and $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_3 \rangle = 0$, by using Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)}$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $\langle -(\alpha_2 + \alpha_3 + \alpha_4), \alpha_3 \rangle = 1$, and $\langle -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \alpha_3 \rangle = 1$, by using Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus from the above discussion we have

$$H^0(s_3s_4s_3s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_2 + \alpha_3), \alpha_2 \rangle = -1$, $\langle -(\alpha_2 + \alpha_3 + \alpha_4), \alpha_2 \rangle = -1$, by using Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = 0$$

and

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = 0.$$

Since $\langle -(\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2 \rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_2 \rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \alpha_2 \rangle = 0$, and $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2 \rangle = 1$, by using Lemma 1.13.3(2) we have

$$H^0(s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_3}$-module, $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_3 \rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3 \rangle = 1$, and $\langle -(\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3 \rangle = -1$, by using similar arguments as above and using Lemma 1.13.3(2), Lemma 1.13.3(4) we have

$$H^0(s_3s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus$$
$$\oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_1 \rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_1 \rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_1 \rangle = -1$, $\langle -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \alpha_1 \rangle = -1$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_1 \rangle = -1$, by using similar arguments as above and using Lemma 1.13.3(2), Lemma 1.13.3(4) we have

$$H^0(s_1s_3s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_2 \rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2 \rangle = -1$, by using Lemma 1.13.3(2), Lemma 1.13.3(4) we have

$$H^0(s_2s_1s_3s_2s_3s_4s_3s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_1 \rangle = 0$, by using Lemma 1.13.3(2) we have

$$H^0(s_1 s_2 s_1 s_3 s_2 s_3 s_4 s_3 s_1 s_2 s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Thus we have

$$H^0(w_2 s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} = \mathbb{C}_{-\omega_4+\alpha_4}.$$

Proof of (2): By the proof of (1) we have

$$H^0(w_2 s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4}.$$

Since $\langle -\omega_4 + \alpha_4, \alpha_4 \rangle = 1$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-\omega_4+\alpha_4}) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Therefore we have

$$H^0(s_4 w_2 s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Since $\langle -\omega_4 + \alpha_4, \alpha_3 \rangle = -1$, by Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-\omega_4+\alpha_4}) = 0.$$

Since $\langle -\omega_4, \alpha_3 \rangle = 0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-\omega_4}) = \mathbb{C}_{-\omega_4}.$$

Thus from above discussion we have

$$H^0(s_3 s_4 w_2 s_3, \alpha_3) = \mathbb{C}_{-\omega_4}.$$

Since α_1, α_2 are orthogonal to ω_4, by Lemma 1.13.3(2) we have

$$H^0(w_3 s_3, \alpha_3) = \mathbb{C}_{-\omega_4}.$$

□

Corollary 4.2.3.

(1) $H^0(s_4 w_1 s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}$.

(2) $H^0(s_4 w_2 s_3, \alpha_3) = \mathbb{C}_{-\omega_4} \oplus \mathbb{C}_{-\omega_4+\alpha_4}$.

(3) $H^0(s_4w_rs_3, \alpha_3) = 0$ *for* $r = 3, 4, 5$.

Proof. Proof of (1): we have

$$H^0(s_1s_2s_3, \alpha_3) = \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $\langle -\alpha_3, \alpha_4 \rangle = 1$, $\langle -(\alpha_2 + \alpha_3), \alpha_4 \rangle = 1$, and $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_4 \rangle = 1$, by using SES and Lemma 1.13.3(2) we have

$$H^0(s_4s_1s_2s_3, \alpha_3) = \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}$ is two dimensional $\tilde{B}_{\alpha_3}$-module, by Lemma 1.13.4(1) we have

$$\mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} = V \otimes \mathbb{C}_{-\omega_3}$$

where V is the standard two dimensional $\tilde{L}_{\alpha_3}$-module.

Thus by using Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}) = 0.$$

Since $\langle -(\alpha_3 + \alpha_4), \alpha_3 \rangle = -1$, by Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_3+\alpha_4)}) = 0.$$

Since $\langle -(\alpha_2 + \alpha_3), \alpha_3 \rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_3 \rangle = 0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)}$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $\langle -(\alpha_2 + \alpha_3 + \alpha_4), \alpha_3 \rangle = 1$, $\langle -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \alpha_3 \rangle = 1$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus combining the above discussion we have

$$H^0(s_3s_4s_1s_2s_3,\alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}. \tag{4.3.1}$$

Since $\langle -(\alpha_2+\alpha_3), \alpha_2\rangle = -1$, $\langle -(\alpha_2+\alpha_3+\alpha_4), \alpha_2\rangle = -1$, by Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = 0$$

and

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = 0.$$

Since $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_2\rangle = 0$, $\langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_2\rangle = 0$, and $\langle -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), \alpha_2\rangle = 0$, by Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}$$

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), \alpha_2\rangle = 1$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_2}/\tilde{B}_{\alpha_2}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus combining the above discussion we have

$$H^0(s_2s_3s_4s_1s_2s_3,\alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_1\rangle = -1$ and $\langle -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), \alpha_1\rangle = -1$, by Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}) = 0$$

and

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = 0.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_1}$-module, by using Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_1 \rangle = 0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_1}/\tilde{B}_{\alpha_1}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus combining the above discussion we have

$$H^0(w_1s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4 \rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4 \rangle = 0$, and $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4 \rangle = 0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$$

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Therefore we have

$$H^0(s_4w_1s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Proof of (2): By Lemma 4.2.2(1) we have

$$H^0(w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4}.$$

Since $\langle -\omega_4 + \alpha_4, \alpha_4 \rangle = 1$, by using SES and Lemma 1.13.3(2) we have

$$H^0(s_4w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Proof of (3): By the Lemma 4.2.2(3) we have

$$H^0(w_3s_3, \alpha_3) = \mathbb{C}_{-\omega_4}.$$

□

Since $\langle -\omega_4, \alpha_4 \rangle = -1$, by Lemma 1.13.3(4) we have

$$H^0(s_4w_3s_3, \alpha_3) = 0.$$

By using SES repeatedly we have

$$H^0(s_4w_rs_3, \alpha_3) = 0 \text{ for } r = 4, 5.$$

Corollary 4.2.4.

(1) $H^0(s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$.

(2) $H^0(s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4}$.

(3) $H^0(s_3s_4w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4}$.

Proof. Proof of (1): Proof follows from (4.3.1).

Proof of (2): Proof follows from the Corollary 4.2.3(1).

Proof of (3): Proof follows from the Corollary 4.2.3(2). □

Corollary 4.2.5.

(1) $H^0(s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}$.

(2) $H^0(s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4}$.

(3) $H^1(s_2s_3s_4w_2s_3, \alpha_3) = \mathbb{C}_{-\omega_4}$.

Proof. Proof of (1): Proof follows from Corollary 4.2.4(1).

Proof of (2): Proof follows from Corollary 4.2.4(2).

Proof of (3): Proof follows from Corollary 4.2.4(3).

□

Corollary 4.2.6.

(1) $H^0(s_4s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$.

(2) $H^0(s_4s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$.

Proof. Proof of (1): By the Corollary 4.2.4(1) we have

$$H^0(s_3s_4s_1s_2s_3,\alpha_3)=\mathbb{C}_{-(\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_4}$-module, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4},\mathbb{C}_{-(\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)})=\mathbb{C}_{-(\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}.$$

Also, since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_4}$-module, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4},\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)})=\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Since $\langle-(\alpha_2+2\alpha_3+\alpha_4),\alpha_4\rangle=0$ and $\langle-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4),\alpha_4\rangle=0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4},\mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)})=\mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4},\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)})=\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus combining the above discussion we have

$$H^0(s_4s_3s_4s_1s_2s_3,\alpha_3)=\mathbb{C}_{-(\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Proof of (2): By Corollary 4.2.4(2) we have

$$H^0(s_3s_4w_1s_3,\alpha_3)=\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\langle-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4),\alpha_4\rangle=0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4},\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)})=\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Further, since $\langle-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4),\alpha_4\rangle=1$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4},\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)})=\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}.$$

Thus combining the above discussion we have

$$H^0(s_4s_3s_4w_1s_3,\alpha_3)=\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-\omega_4+\alpha_4}\oplus\mathbb{C}_{-\omega_4},$$

since $\omega_4=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$. ∎

Corollary 4.2.7. *(1)* $H^0(s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}$.

(2) $H^0(s_4s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$.

Proof. Proof of (1): By Corollary 4.2.5(1) we have

$$H^0(s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_4}$-module, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Moreover, Since $\langle -(\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4 \rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4 \rangle = 0$ and $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4 \rangle = 0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$$

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus combining the above discussion we have

$$H^0(s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Proof of (2): By Corollary 4.2.5(2) we have

$$H^0(s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4}.$$

Since $\langle -\omega_4 + \alpha_4, \alpha_4 \rangle = 1$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-\omega_4+\alpha_4}) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Thus we have

$$H^0(s_4s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

□

Corollary 4.2.8. *(1)*
$H^0(s_3s_4s_2s_3s_4s_1s_2s_3,\alpha_3)=\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-\omega_4+\alpha_4}.$

(2) $H^0(s_3s_4s_2s_3s_4w_1s_3,\alpha_3)=\mathbb{C}_{-\omega_4}.$

Proof. Proof of (1): By Corollary 4.2.7(1) we have

$$H^0(s_4s_2s_3s_4s_1s_2s_3,\alpha_3)=\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\langle-(\alpha_2+2\alpha_3+\alpha_4),\alpha_3\rangle=-1$, by Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3},\mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)})=0.$$

Since $\langle-(\alpha_1+\alpha_2+\alpha_3),\alpha_3\rangle=0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3},\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)})=\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_3}$-module, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3},\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)})=\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\langle-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4),\alpha_3\rangle=1$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3},\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)})=\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Thus combining the above discussion we have

$$H^0(s_4s_2s_3s_4s_1s_2s_3,\alpha_3)=\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Proof of (2): By Corollary 4.2.7(2) we have

$$H^0(s_4s_2s_3s_4w_1s_3,\alpha_3)=\mathbb{C}_{-\omega_4+\alpha_4}\oplus\mathbb{C}_{-\omega_4}.$$

Since $\langle-\omega_4+\alpha_4,\alpha_3\rangle=-1$, by Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3},\mathbb{C}_{-\omega_4+\alpha_4})=0.$$

Further, since $\langle-\omega_4,\alpha_3\rangle=0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_{\alpha_3}, \mathbb{C}_{-\omega_4}) = \mathbb{C}_{-\omega_4}.$$

Thus from the above discussion we have

$$H^0(s_3s_4s_2s_3s_4w_1s_3, \alpha_3) = \mathbb{C}_{-\omega_4}.$$

□

Corollary 4.2.9.

(1) $H^0(s_4s_3s_4s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$.

(2) $H^0(s_4s_3s_4s_2s_3s_4s_1s_2s_3, \alpha_3)$=$\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$.

Proof. Proof of (1):It is easy to see that

$$H^0(s_3, \alpha_3) = \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{\alpha_3}.$$

Since $\langle -\alpha_3, \alpha_2 \rangle = 1$, by using Lemma 1.13.3(2), Lemma 1.13.3(4) we have

$$H^0(s_2s_3, \alpha_3) = \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)}.$$

Since $\langle -\alpha_3, \alpha_4 \rangle = 1$, by using Lemma 1.13.3(2) we have

$$H^0(s_4s_2s_3, \alpha_3) = \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} \oplus \mathbb{C}_{-(\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}$ is the two dimensional indecomposable $\tilde{B}_{\alpha_3}$-module, by Lemma 1.13.4(1) we have

$$\mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3} = V \otimes \mathbb{C}_{-\omega_3}$$

where V is the standard two dimensional irreducible $\tilde{L}_{\alpha_3}$- module. Thus by Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_3, \mathbb{C}h(\alpha_3) \oplus \mathbb{C}_{-\alpha_3}) = 0.$$

Also, since $\langle -(\alpha_3 + \alpha_4), \alpha_3 \rangle = -1$, by Lemma 1.13.3(4) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_3, \mathbb{C}_{-(\alpha_3+\alpha_4)}) = 0.$$

Since $\langle -(\alpha_2 + \alpha_3), \alpha_3 \rangle = 0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_3, \mathbb{C}_{-(\alpha_2+\alpha_3)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)}.$$

Since $\langle -(\alpha_2+\alpha_3+\alpha_4), \alpha_3\rangle = 1$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_3}/\tilde{B}_3, \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus combining the above discussion we have

$$H^0(s_3s_4s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_4}$-module, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_4, \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}.$$

Further, since $\langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$, by Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_4, \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Therefore we have

$$H^0(s_4s_3s_4s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Proof of (2): By Corollary 4.2.8(1) we have

$$H^0(s_4s_2s_3s_4s_1s_2s_3, \alpha_3)=\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$, $\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}$ are the two dimensional irreducible $\tilde{L}_{\alpha_3}$-modules and $\langle -(\alpha_1+\alpha_2+\alpha_3), \alpha_3\rangle = 0$, by Lemma 1.13.3(2) we have

$$H^0(s_3s_4s_2s_3s_4s_1s_2s_3, \alpha_3)=\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_4}$-module, by using Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Moreover, since $\langle -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$ and $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_4\rangle = 0$, by using Lemma 1.13.3(2) we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$$

and

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_4 \rangle = 1$, by Lemma 1.13.3 we have

$$H^0(\tilde{L}_{\alpha_4}/\tilde{B}_{\alpha_4}, \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}.$$

Therefore combing the above discussion we have

$$H^0(s_4s_3s_4s_2s_3s_4s_1s_2s_3, \alpha_3) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$$

since $\omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. □

4.3 Computations of relative tangent bundles $H^1(w, \alpha_2)$

In this section we compute cohomology modules $H^1(w, \alpha_2)$ corresponding to some special Weyl group elements.

Lemma 4.3.1.

(1) $H^1(w_r, \alpha_2) = 0$ *for* $r = 1, 2, 5$.

(2) $H^1(w_3, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4}$.

(3) $H^1(w_4, \alpha_2) = \mathbb{C}_{-\omega_4}$.

Proof. It is easy to see $H^1(s_2, \alpha_2) = 0$.

Note that we have

$$H^0(s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{\alpha_2}.$$

Since $\langle -\alpha_2, \alpha_1 \rangle = 1$, by using Lemma 1.13.3(2), Lemma 1.13.3(4) we have

$$H^1(s_1, H^0(s_2, \alpha_2)) = 0.$$

Since $H^1(s_2, \alpha_2) = 0$, by using Lemma 1.13.3(1) we have

$$H^0(s_1, H^1(s_2, \alpha_2)) = 0.$$

Thus by using SES and the above discussion we have

$$H^1(s_1s_2, \alpha_2) = 0.$$

By using SES and Lemma 1.13.3(2) we have

$$H^0(s_1s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2)}.$$

Since $\langle \alpha_2, \alpha_4 \rangle = 0$, by using Lemma 1.13.3(2) we have

$$H^1(s_4, H^0(s_1s_2, \alpha_2)) = 0$$

and

$$H^0(s_4s_1s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2)}. \quad (4.3.1.1)$$

Again, since $H^1(s_1s_2, \alpha_2) = 0$, by using Lemma 1.13.3(1) we have

$$H^0(s_4, H^1(s_1s_2, \alpha_2)) = 0.$$

Thus by using SES and the above discussion we have

$$H^1(s_4s_1s_2, \alpha_2) = 0. \quad (4.3.1.2)$$

Since $\langle -\alpha_2, \alpha_3 \rangle = 2$, $\langle -(\alpha_1 + \alpha_2), \alpha_3 \rangle = 2$, by using (4.3.1.1), and Lemma 1.13.3(2) we have

$$H^1(s_3, H^0(s_4s_1s_2, \alpha_2)) = 0.$$

Further, by (4.3.1.2) we have

$$H^0(s_3, H^1(s_4s_1s_2, \alpha_2)) = 0.$$

Thus by using SES we have

$$H^1(s_3s_4s_1s_2, \alpha_2) = 0. \quad (4.3.1.3)$$

Therefore we have

$$H^0(s_2, H^1(s_3s_4s_1s_2, \alpha_2)) = 0.$$

By using Lemma 4.1.3 we have

$$H^1(s_2, H^0(s_3s_4s_1s_2, \alpha_2)) = 0.$$

Thus by SES we have

$$H^1(s_2s_3s_4s_1s_2, \alpha_2) = 0. \tag{4.3.1.4}$$

Therefore we have

$$H^0(s_1, H^1(s_2s_3s_4s_1s_2, \alpha_2)) = 0.$$

By Lemma 4.1.3 we have

$$H^1(s_1, H^0(s_2s_3s_4s_1s_2, \alpha_2)) = 0.$$

Therefore by using SES we have

$$H^1(w_1, \alpha_2) = 0.$$

Since $H^1(w_1, \alpha_2) = 0$, we have

$$H^0(s_4, H^1(w_1, \alpha_2)) = 0.$$

Recall that by (4.2.1.3) we have

$$H^0(w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}.$$

Since $\langle -(\alpha_2 + 2\alpha_3), \alpha_4 \rangle = 2$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3), \alpha_4 \rangle = 2$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_4 \rangle = 2$, by using Lemma 1.13.3(2) we have

$$H^1(s_4, H^0(w_1, \alpha_2)) = 0.$$

Thus by using SES and the above discussion we have

$$H^1(s_4w_1, \alpha_2) = 0 \tag{4.3.1.5}$$

and

$$H^0(s_4w_1, \alpha_2) = (\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}) \oplus (\mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)}) \oplus (\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)}).$$

Since $H^1(s_4w_1, \alpha_2) = 0$, we have

$$H^0(s_3, H^1(s_4w_1, \alpha_2)) = 0.$$

Since $\langle -(\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_3\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_3\rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_3\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\rangle = 1$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_3\rangle = 2$, $\langle -(\alpha_2 + 2\alpha_3), \alpha_3\rangle = -2$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3), \alpha_3\rangle = -2$, $\langle -(\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\rangle = -1$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\rangle = -1$, by using Lemma 1.13.3 we have

$$H^1(s_3, H^0(s_4w_1, \alpha_2)) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

Thus by using SES and the above discussion we have

$$H^1(s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}, \tag{4.3.1.6}$$

and

$$H^0(s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}. \tag{4.3.1.7}$$

Since $H^1(s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}$, by using Lemma 1.13.3 we have

$$H^0(s_2, H^1(s_3s_4w_1, \alpha_2)) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}.$$

By Lemma 4.1.3 we have

$$H^1(s_2, H^0(s_3s_4w_1, \alpha_2)) = 0.$$

Thus using SES and above discussion we have

$$H^1(s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}. \tag{4.3.1.8}$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_2}$-module and $\langle -(\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_2\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_2\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4), \alpha_2\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_2\rangle = 1$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_2\rangle = -1$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_2\rangle = -1$, by using SES and Lemma 1.13.3 we have

$$H^0(s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \tag{4.3.1.9}$$

Since $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_1\rangle = -1$, by using Lemma 1.13.3(4) we have

$$H^0(s_1, H^1(s_2s_3s_4w_1, \alpha_2)) = 0.$$

Further, by Lemma 4.1.3 we have

$$H^1(s_1, H^0(s_2s_3s_4w_1, \alpha_2)) = 0.$$

Thus using SES we have

$$H^1(w_2, \alpha_2) = 0.$$

Since $\mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_1}$-module and $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_1 \rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_1 \rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4), \alpha_1 \rangle = 0$, and $\langle -(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_1 \rangle = 0$, $\langle -(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_1 \rangle = 1$, by using SES and Lemma 1.13.3 we have

$$H^0(w_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}.$$

Since $H^1(w_2, \alpha_2) = 0$, we have

$$H^0(s_4, H^1(w_2, \alpha_2)) = 0.$$

Since $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_4}$-module and $\langle -(\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_4 \rangle = 0$, $\langle -(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_4 \rangle = 0$, $\langle -(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_4 \rangle = 0$, $\langle -(\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_4 \rangle = -2$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_4 \rangle = -2$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_4 \rangle = -2$, by using SES and Lemma 1.13.3 we have

$$H^1(s_4, H^0(w_2, \alpha_2)) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Thus from the above discussion we have

$$H^1(s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \tag{4.3.1.10}$$

and

$$H^0(s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}.$$

Since $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = 1$, $\langle -(\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = -1$, and $\langle -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), \alpha_3\rangle = -1$, by using Lemma 1.13.3 we have

$$H^0(s_3, H^1(s_4w_2, \alpha_2)) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\langle -(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_3\rangle = 0$, $\langle -(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_3\rangle = 0$, $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_3\rangle = -1$, and $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} = V \otimes \mathbb{C}_{-\omega_3}$(where V is the standard two dimensional irreducible $\tilde{L}_{\alpha_3}$-module), by Lemma 1.13.3 we have

$$H^1(s_3, H^0(s_4w_2, \alpha_2)) = 0.$$

and

$$H^0(s_3s_4w_2, \alpha_2)=\mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \tag{4.3.1.11}$$

Therefore we have

$$H^1(s_3s_4w_2, \alpha_2)=\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}. \tag{4.3.1.12}$$

Since $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_2\rangle = 0$, and $\langle -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), \alpha_2\rangle = -1$, by using Lemma 1.13.3 we have

$$H^0(s_2, H^1(s_3s_4w_2, \alpha_2)) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

By Lemma 4.1.3 we have

$$H^1(s_2, H^0(s_3s_4w_2, \alpha_2)) = 0.$$

Thus from the above discussion we have

$$H^1(s_2s_3s_4w_2, \alpha_2)=\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}=\mathbb{C}_{-\omega_4+\alpha_4}. \tag{4.3.1.13}$$

Since $\langle -(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_2\rangle = 0$ and $\langle -(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4), \alpha_2\rangle = -1$, by using SES and Lemma 1.13.3 we have

$$H^0(s_2s_3s_4w_2, \alpha_2)=\mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \tag{4.3.1.14}$$

Since $\langle -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), \alpha_1\rangle = 0$, by using Lemma 1.13.3(2) we have

$$H^0(s_1, H^1(s_2s_3s_4w_2, \alpha_2)) = \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

By Lemma 4.1.3 we have

$$H^1(s_1, H^0(s_2s_3s_4w_2, \alpha_2)) = 0.$$

Thus from the above discussion we have

$$H^1(w_3, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4}$$

since $\omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. This proves (2).

Since we have $H^0(w_3, \alpha_2) = 0$ (see Lemma 4.2.1), by using SES we have

$$H^1(s_4w_3, \alpha_2) = H^0(s_4, H^1(w_3, \alpha_2)).$$

Since $\langle -\omega_4 + \alpha_4, \alpha_4 \rangle = 1$, by using Lemma 1.13.3(2) we have

$$H^1(s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}. \tag{4.3.1.15}$$

Since we have $H^0(w_3, \alpha_2) = 0$ (see Lemma 4.2.1), by using SES we have

$$H^1(s_3s_4w_3, \alpha_2) = H^0(s_3, H^1(s_4w_3, \alpha_2)).$$

Since $\langle -\omega_4, \alpha_3 \rangle = 0$ and $\langle -\omega_4 + \alpha_4, \alpha_3 \rangle = -1$, by using Lemma 1.13.3(2), Lemma 1.13.3(4) we have

$$H^1(s_3s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4}. \tag{4.3.1.16}$$

Since we have $H^0(w_3, \alpha_2) = 0$ (see Lemma 4.2.1) and α_1, α_2 are orthogonal to ω_4, by Lemma 1.13.3(2) we have

$$H^1(w_4, \alpha_2) = \mathbb{C}_{-\omega_4}.$$

This gives the proof of (3).

Since we have $H^0(w_3, \alpha_2) = 0$ (see Lemma 4.2.1) and $\langle -\omega_4, \alpha_4 \rangle = -1$, by using Lemma 1.13.3(4) we have

$$H^1(s_4w_4, \alpha_2) = 0.$$

Since by Lemma 4.2.1 we have $H^0(w_3, \alpha_2) = 0$, and $H^1(s_4w_4, \alpha_2) = 0$, by using SES repeatedly we have

$$H^1(w_5, \alpha_2) = 0.$$

This completes the proof of (1). □

Corollary 4.3.2.

(1) $H^1(s_4s_1s_2, \alpha_2) = 0$.

(2) $H^1(s_4w_r, \alpha_2) = 0$ *for* $r = 1, 4, 5$.

(3) $H^1(s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}$.

(4) $H^1(s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4} \oplus \mathbb{C}_{-\omega_4+\alpha_4}$.

Proof. Proof of (1) follows from (4.3.1.2).

Proof of (2) for $r = 1$, proof follows from (4.3.1.5). For $r = 4, 5$ proof follows by using SES, Lemma 4.3.1 and Lemma 4.2.1.

Proof of (3) follows from (4.3.1.10).

Proof of (4) follows from (4.3.1.15).

□

Corollary 4.3.3.

(1) $H^1(s_3s_4s_1s_2, \alpha_2) = 0$.

(2) $H^1(s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}$.

(3) $H^1(s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4}$.

(4) $H^1(s_3s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4}$.

(5) $H^1(s_3s_4w_r, \alpha_2) = 0$ *for* $r = 4, 5$.

Proof. Proof of (1) follows from (4.3.1.3).

Proof of (2) follows from (4.3.1.6).

Proof of (3) follows from (4.3.1.12).

Proof of (4) follows from (4.3.1.16).

Proof of (5): By Lemma 4.2.1 we have $H^0(w_r, \alpha_2) = 0$ for $r = 4, 5$. Therefore $H^0(s_4w_r, \alpha_2) = 0$ for $r = 4, 5$. Hence we have $H^1(s_3, H^0(s_4w_r, \alpha_2)) = 0$ for $r = 4, 5$. On the other hand, by Corollary 4.3.2(2) we have $H^1(s_4w_r, \alpha_2) = 0$ for $r = 4, 5$. Therefore $H^0(s_3, H^1(s_4w_r, \alpha_2)) = 0$ for $r = 4, 5$. Thus by SES we have $H^1(s_3s_4w_r, \alpha_2) = 0$ for $r = 4, 5$. □

Corollary 4.3.4.

(1) $H^1(s_2s_3s_4s_1s_2, \alpha_2) = 0$.

(2) $H^1(s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}$.

(3) $H^1(s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4}$.

(4) $H^1(s_2s_3s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4}$.

(5) $H^1(s_2s_3s_4w_4, \alpha_2) = 0$.

Proof. Proof of (1) follows from (4.3.1.4).

Proof of (2) follows from (4.3.1.8).

Proof of (3) follows from (4.3.1.13).

Proof of (4): By Lemma 4.2.1 we have $H^0(w_3, \alpha_2) = 0$. Therefore $H^0(s_3s_4w_3, \alpha_2) = 0$. Hence we have $H^1(s_2, H^0(s_3s_4w_3, \alpha_2)) = 0$. On the other hand, by Corollary 4.3.3(4) we have $H^1(s_3s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4}$. Since ω_4 is orthogonal to α_2, by Lemma1.13.3(2) we have $H^0(s_2, H^1(s_3s_4w_3, \alpha_2)) = \mathbb{C}_{-\omega_4}$. Thus by SES we have $H^1(s_2s_3s_4w_3, \alpha_2) = \mathbb{C}_{-\omega_4}$.

Proof of (5): By Lemma 4.2.1(2) we have $H^0(w_4, \alpha_2) = 0$.

Therefore $H^0(s_3s_4w_4, \alpha_2) = 0$. Hence we have $H^1(s_2, H^0(s_3s_4w_4, \alpha_2)) = 0$. On the other hand, by Corollary 4.3.3(5) we have $H^1(s_3s_4w_4, \alpha_2) = 0$. Therefore $H^0(s_2, H^1(s_3s_4w_4, \alpha_2)) = 0$. Thus by SES we have $H^1(s_2s_3s_4w_4, \alpha_2) = 0$. ◻

Corollary 4.3.5.

(1) $H^1(s_4s_3s_4s_1s_2, \alpha_2) = 0$.

(2) $H^1(s_4s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$.

(3) $H^1(s_4s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$.

(4) $H^1(s_4s_3s_4w_r, \alpha_2) = 0$ *for* $r = 3, 4$.

Proof. Proof of (1): By (4.2.1.1) if $H^0(s_3s_4s_1s_2, \alpha_2)_\mu \neq 0$, then we have $\langle \mu, \alpha_4 \rangle \geq 0$. Thus using Lemma 1.13.3(3) we have

$$H^1(s_4, H^0(s_3s_4s_1s_2, \alpha_2)) = 0.$$

On the other hand, by using Corollary 4.3.3(1) we have

$$H^0(s_4, H^1(s_3s_4s_1s_2, \alpha_2)) = 0.$$

Hence we have $H^1(s_4s_3s_4s_1s_2, \alpha_2) = 0$.

Proof of (2): By (4.3.1.7) the $\tilde{B}_{\alpha_4}$-indecomposable summands V of $H^0(s_3s_4w_1, \alpha_2)$ for which $H^1(s_4, V) \neq 0$ are $\mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)}$ and $\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)}$. Thus using Lemma 1.13.3(3) we have

$$H^1(s_4, H^0(s_3s_4w_1, \alpha_2)) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

On the other hand, by using Corollary 4.3.3(2) and Lemma 1.13.3(2) we have

$$H^0(s_4, H^1(s_3s_4w_1, \alpha_2)) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}.$$

Hence we have

$$H^1(s_4s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}.$$

Proof of (3): By (4.3.1.11) we have if $H^0(s_3s_4w_2, \alpha_2)_\mu \neq 0$, then $\langle \mu, \alpha_4 \rangle = 0$. Thus using Lemma 1.13.3(3) we have

$$H^1(s_4, H^0(s_3s_4w_2, \alpha_2)) = 0.$$

On the other hand, by using Corollary 4.3.3(3) and Lemma 1.13.3(2) we have

$$H^0(s_4, H^1(s_3s_4w_2, \alpha_2)) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}.$$

Hence we have

$$H^1(s_4s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Proof of (4): By Lemma 4.2.1 we have $H^0(w_r, \alpha_2) = 0$ for $r = 3, 4$.

Therefore $H^0(s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. Hence we have $H^1(s_4, H^0(s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$.

On the other hand, by Corollary 4.3.3(4), Corollary 4.3.3(5), we have $H^0(s_4, H^1(s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. Thus by using SES we have $H^1(s_4s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. □

Corollary 4.3.6.

(1) $H^1(s_4s_2s_3s_4s_1s_2, \alpha_2) = 0$.

(2) $H^1(s_4s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}$.

(3) $H^1(s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$.

(4) $H^1(s_4s_2s_3s_4w_r,\alpha_2)=0$ *for* $r=3,4$.

Proof. Proof of (1): By Lemma 4.1.3 we have

$$H^1(s_2,H^0(s_4s_3s_4s_1s_2,\alpha_2))=0.$$

On the other hand, by using Corollary 4.3.5(1) we have

$$H^0(s_2,H^1(s_4s_3s_4s_1s_2,\alpha_2))=0.$$

Hence we have $H^1(s_4s_2s_3s_4s_1s_2,\alpha_2)=H^1(s_2s_4s_3s_4s_1s_2,\alpha_2)=0$.

Proof of (2): By Corollary 4.3.5(2) we have

$$H^0(s_2,H^1(s_4s_3s_4w_1,\alpha_2))=\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Now the proof of (2) follows from Lemma 4.1.3 and SES.

Proof of (3): By Corollary 4.3.5(3), using SES, and Lemma 1.13.3 we have

$$H^0(s_2,H^1(s_4s_3s_4w_2,\alpha_2))=\mathbb{C}_{-\omega_4+\alpha_4}\oplus\mathbb{C}_{-\omega_4}.$$

Now the proof of (3) follows from Lemma 4.1.3 and SES.

Proof of (4): By Lemma 4.1.3 we have $H^1(s_2,H^0(s_4s_3s_4w_r,\alpha_2))=0$ for $r=3,4$. On the other hand, by Corollary 4.3.5(4) we have $H^0(s_2,H^1(s_4s_3s_4w_r,\alpha_2))=0$ for $r=3,4$. Thus by using SES we have $H^1(s_4s_2s_3s_4w_r,\alpha_2)=0$ for $r=3,4$. □

Lemma 4.3.7.

(1) $H^1(s_3s_4s_2s_3s_4s_1s_2,\alpha_2)=\mathbb{C}_{-(\alpha_2+\alpha_3)}$.

(2) $H^1(s_3s_4s_2s_3s_4w_1,\alpha_2)=\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-\omega_4+\alpha_4}$.

(3) $H^1(s_3s_4s_2s_3s_4w_2,\alpha_2)=\mathbb{C}_{-\omega_4}$.

(4) $H^1(s_3s_4s_2s_3s_4w_r,\alpha_2)=0$ *for* $r=3,4$.

Proof. Proof of (1): Recall from (4.2.1.2) that

$$H^0(s_2s_3s_4s_1s_2,\alpha_2)=\mathbb{C}_{-(\alpha_2+2\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}.$$

Since $\langle-(\alpha_2+2\alpha_3),\alpha_4\rangle=2$, $\langle-(\alpha_1+\alpha_2+\alpha_3),\alpha_4\rangle=1$, $\langle-(\alpha_1+\alpha_2+2\alpha_3),\alpha_4\rangle=2$, and $\langle-(\alpha_1+2\alpha_2+2\alpha_3),\alpha_4\rangle=2$, by using SES and Lemma 1.13.3(2) we have

$$H^0(s_4s_2s_3s_4s_1s_2,\alpha_2)=\mathbb{C}_{-(\alpha_2+2\alpha_3)}\oplus\mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}\oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}. \tag{4.3.7.1}$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)}$ is the indecomposable $\tilde{B}_{\alpha_3}$-module, by using Lemma 1.13.4(1) we have $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} = V\otimes\mathbb{C}_{-\omega_3}$ (where V is the standard two dimensional irreducible $\tilde{L}_{\alpha_3}$-module), and $\langle-(\alpha_2+2\alpha_3),\alpha_3\rangle=-2$, by using SES and Lemma 1.13.3(3) we have

$$H^1(s_3,H^0(s_4s_2s_3s_4s_1s_2,\alpha_2))=\mathbb{C}_{-(\alpha_2+\alpha_3)}.$$

By using SES and Corollary 4.3.6(1) we have

$$H^0(s_3,H^1(s_4s_2s_3s_4s_1s_2,\alpha_2))=0.$$

Thus we have

$$H^1(s_3s_4s_2s_3s_4s_1s_2,\alpha_2)=\mathbb{C}_{-(\alpha_2+\alpha_3)}.$$

Proof of (2): Recall from (4.3.1.9) that

$$H^0(s_2s_3s_4w_1,\alpha_2)=\mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)}\oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_4}$-module, $\langle-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4),\alpha_4\rangle=0$, $\langle-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4),\alpha_4\rangle=0$, $\langle-(\alpha_2+2\alpha_3+2\alpha_4),\alpha_4\rangle=-2$, $\langle-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4),\alpha_4\rangle=-2$, and $\langle-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4),\alpha_4\rangle=-2$, by using Lemma 1.13.3 we have

$$H^0(s_4s_2s_3s_4w_1,\alpha_2)=\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}\oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}. \tag{4.3.7.2}$$

By Corollary 4.3.6(2) we have

$$H^1(s_4s_2s_3s_4w_1,\alpha_2)=\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}$ is indeomposable $\tilde{B}_{\alpha_3}$-module, by Lemma 1.13.4(1) we have

$$\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}=V\otimes\mathbb{C}_{-\omega_3}$$

where V is the standard two dimensional irreducible $\tilde{L}_{\alpha_3}$-module.

Further, since $\langle -(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_3 \rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_3 \rangle = -1$, by using Lemma 1.13.3 we have

$$H^1(s_3, H^0(s_4s_2s_3s_4w_1, \alpha_2)) = 0.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_3}$-module, $\langle -(\alpha_1 + \alpha_2 + \alpha_3), \alpha_3 \rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3 \rangle = 1$, $\langle -(\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3 \rangle = -1$, by using Lemma 1.13.3(2) we have

$$H^0(s_3, H^1(s_4s_2s_3s_4w_1, \alpha_2)) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Thus we have

$$H^1(s_3s_4s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Proof of (3): Recall from (4.3.1.14) that

$$H^0(s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} = \mathbb{C}_{-\omega_1}.$$

Since α_4 is orthogonal to ω_1, by using Lemma 1.13.3(2) we have

$$H^0(s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} = \mathbb{C}_{-\omega_1}.$$

Since α_3 is orthogonal to ω_1, by using Lemma 1.13.3(2) we have

$$H^1(s_3, H^0(s_4s_2s_3s_4w_2, \alpha_2)) = 0.$$

On the other hand, by Corollary 4.3.6(3) we have

$$H^1(s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}.$$

Since $\langle -\omega_4, \alpha_3 \rangle = 0$ and $\langle -\omega_4 + \alpha_4, \alpha_3 \rangle = -1$, by using Lemma 1.13.3 we have

$$H^0(s_3, H^1(s_4s_2s_3s_4w_2, \alpha_2)) = \mathbb{C}_{-\omega_4}.$$

Thus we have

$$H^1(s_3s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_4}.$$

Proof of (4): By Lemma 4.2.1 we have $H^0(w_r, \alpha_2) = 0$ for $r = 3, 4$. Therefore we have

$H^0(s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. Hence we have $H^1(s_3, H^0(s_4s_2s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. On the other hand, Corollary 4.3.6(4) we have $H^0(s_3, H^1(s_4s_2s_3s_4w_r, \alpha_2)) = 0$ for $r = 3, 4$. Thus by using SES we have $H^1(s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 3, 4$. ☐

Lemma 4.3.8.

(1) $H^1(s_4s_3s_4s_2, \alpha_2) = 0$.

(2) $H^1(s_4s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}$.

(3) $H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_2) = \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-\omega_4+\alpha_4} \oplus \mathbb{C}_{-\omega_4}$.

(4) $H^1(s_4s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$ *for* $r = 2, 3$.

Proof. Proof of (1): By using SES it is easy to see that

$$H^0(s_3s_4s_2, \alpha_2) = \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3)}$$

and

$$H^1(s_3s_4s_2, \alpha_2) = \mathbb{C}_{\alpha_2+\alpha_3}.$$

Since $H^0(s_3s_4s_2, \alpha_2)_\mu \neq 0$ implies $\langle \mu, \alpha_4 \rangle \geq 0$, by using Lemma 1.13.3(2) we have

$$H^1(s_4, H^0(s_3s_4s_2, \alpha_2)) = 0.$$

Since $\langle \alpha_2 + \alpha_3, \alpha_4 \rangle = -1$, by using Lemma 1.13.3(4) we have $H^0(s_4, H^1(s_3s_4s_2, \alpha_2)) = 0$. Therefore by using SES we have $H^1(s_4s_3s_4s_2, \alpha_2) = 0$.

Proof of (2): By the Corollary 4.3.7(1) we have

$$H^1(s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)}.$$

Since $\langle -(\alpha_2 + \alpha_3), \alpha_4 \rangle = 1$, by using Lemma 1.13.3(2) we have

$$H^0(s_4, H^1(s_3s_4s_2s_3s_4s_1s_2, \alpha_2)) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)}.$$

Recall from (4.3.7.1) that

$$H^0(s_4s_2s_3s_4s_1s_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_3}$-module, $\langle -(\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_3\rangle = 0$, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_3\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_3\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\rangle = 1$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4), \alpha_3\rangle = 2$, $\langle -(\alpha_2 + 2\alpha_3), \alpha_3\rangle = -2$, $\langle -(\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_3\rangle = -1$, and $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3)} = V \otimes \mathbb{C}_{-\omega_3}$ (where V is the standard twi dimensional irreducible $\tilde{L}_{\alpha_3}$), by using SES and Lemma 1.13.3 we have

$$H^0(s_3s_4s_2s_3s_4s_1s_2, \alpha_2){=}\mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)}{\oplus}\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}{\oplus}\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}{\oplus}\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)}{\oplus}\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}{\oplus} \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}. \quad (4.3.8.1)$$

The $\tilde{B}_{\alpha_4}$-indecomposable summands V of $H^0(s_3s_4s_2s_3s_4s_1s_2, \alpha_2)$ for which $H^1(s_4, V) \neq 0$ is $\mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)}$. Thus by using SES and Lemma 1.13.3(3) we have

$$H^1(s_4, H^0(s_3s_4s_2s_3s_4s_1s_2, \alpha_2)) = \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Therefore by using SES we have

$$H^1(s_4s_3s_4s_2s_3s_4s_1s_2, \alpha_2) = \mathbb{C}_{-(\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_2+2\alpha_3+\alpha_4)}.$$

Proof of (3): Recall that from Corollary 4.3.7(2) we have

$$H^1(s_3s_4s_2s_3s_4w_1, \alpha_2){=}\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$ is the standard two dimensional irreducible $\tilde{L}_{\alpha_4}$-module, $\langle -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4\rangle = 0$, $\langle -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), \alpha_4\rangle = 0$, and $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_4\rangle = 1$, by using SES and Lemma 1.13.3 we have

$$H^0(s_4, H^1(s_3s_4s_2s_3s_4w_1, \alpha_2)){=}\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}.$$

On the other hand, from (4.3.7.2) we have

$$H^0(s_4s_2s_3s_4w_1, \alpha_2){=}\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}.$$

Since $\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)} = V \otimes \mathbb{C}_{-\omega_3}$, where V is the standard two dimensional irreducible $\tilde{L}_{\alpha_3}$-module, $\langle -(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_3\rangle = 0$, and $\langle -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4), \alpha_3\rangle = -1$, by using SES and Lemma 1.13.3 we have

$$H^0(s_3s_4s_2s_3s_4w_1, \alpha_2)=\mathbb{C}_{-(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}.$$

Since $\langle -(\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), \alpha_4\rangle = 0$, by using SES and Lemma 1.13.3 we have

$$H^1(s_4, H^0(s_3s_4s_2s_3s_4w_1, \alpha_2)) = 0.$$

Therefore by SES we have

$$H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_2)=\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)} \oplus \mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}.$$

Proof of (4): For $r = 2$, we recall that from (4.3.1.14) that $H^0(s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_1}$. Since α_4, α_3 are orthogonal to ω_1, by using SES we have $H^0(s_3s_4s_2s_3s_4w_2, \alpha_2) = \mathbb{C}_{-\omega_1}$. Further, using the orthogonality of α_4 and ω_1 we have $H^1(s_4, H^0(s_3s_4s_2s_3s_4w_2, \alpha_2)) = 0$. On the other hand, by Corollary 4.3.7(3) we have $H^0(s_4, H^1(s_3s_4s_2s_3s_4w_2, \alpha_2)) = 0$. Thus we have $H^1(s_4s_3s_4s_2s_3s_4w_2, \alpha_2) = 0$. For $r = 3$, By Lemma 4.2.1 we have $H^0(w_3, \alpha_2) = 0$. Therefore $H^0(s_3s_4s_2s_3s_4w_3, \alpha_2) = 0$. Hence we have $H^1(s_4, H^0(s_3s_4s_2s_3s_4w_3, \alpha_2)) = 0$. On the other hand,by Corollary 4.3.7(4) we have $H^0(s_4, H^1(s_3s_4s_2s_3s_4w_3, \alpha_2)) = 0$. Thus by using SES we have $H^1(s_4s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$. □

We denote $v_r = [1, 4]^r$ for $1 \le r \le 6$ and $\tau_r = [1, 4]^r1$ for $1 \le r \le 5$.

Lemma 4.3.9. *We have*

(1) $H^i(\tau_r, \alpha_1) = 0$ *for all* $i \ge 0, 1 \le r \le 5$.

(2) $H^i(v_r, \alpha_4) = 0$ *for all* $i \ge 0, 2 \le r \le 6$.

Proof. Proof of (1): By [Kan16, Corollary 6.4, p.780] we have

$$H^i(\tau_r, \alpha_1) = 0 \text{ for all } i \ge 2, r \ge 1.$$

Note that $H^i(s_1s_2s_3s_4s_1, \alpha_1) = H^i(s_1s_2s_1, \alpha_1) = H^i(s_2s_1s_2, \alpha_1) = 0$ for $i = 0, 1$ (see Lemma 1.13.3(4)). Now by using SES repeatedly we have the required result.

Proof of (2): By [Kan16, Corollary 6.4, p.780] we have

$$H^i(v_r, \alpha_4) = 0 \text{ for all } i \ge 2, r \ge 1.$$

We note that

$$H^i(s_4s_1s_2s_3s_4, \alpha_4) = H^i(s_1s_2s_4s_3s_4, \alpha_4)=H^i(s_1s_2s_3s_4s_3, \alpha_4) = 0 \quad (4.3.9.1)$$

for $i = 0, 1$ (see Lemma 1.13.3(4)).

Since $2 \leq r \leq 6$, we have $v_r = us_4s_1s_2s_3s_4$ for some $u \in W$ such that $l(v_r) = l(u) + 5$. Thus by using SES repeatedly we have the required result. □

Corollary 4.3.10. *We have the following:*

(1) $H^i(s_4\tau_r, \alpha_1) = 0$ *for* $i \geq 0, 1 \leq r \leq 5$.

$H^i(s_4v_r, \alpha_4) = 0$ *for* $i \geq 0, 1 \leq r \leq 5$.

(2) $H^i(s_3s_4\tau_r, \alpha_1) = 0$ *for* $i \geq 0, 1 \leq r \leq 5$.

$H^i(s_3s_4v_r, \alpha_4) = 0$ *for* $i \geq 0, 1 \leq r \leq 5$.

(3) $H^i(s_2s_3s_4\tau_r, \alpha_1) = 0$ *for* $i \geq 0, 1 \leq r \leq 5$.

$H^i(s_2s_3s_4v_r, \alpha_4) = 0$ *for* $i \geq 0, 1 \leq r \leq 5$.

(4) $H^i(s_4s_3s_4\tau_r, \alpha_1) = 0$ *for* $i \geq 0, 1 \leq r \leq 4$.

$H^i(s_4s_3s_4v_r, \alpha_4) = 0$ *for* $i \geq 0, 1 \leq r \leq 4$.

(5) $H^i(s_4s_2s_3s_4\tau_r, \alpha_1) = 0$ *for* $i \geq 0, 1 \leq r \leq 4$.

$H^i(s_4s_2s_3s_4v_r, \alpha_4) = 0$ *for* $i \geq 0, 1 \leq r \leq 4$.

(6) $H^i(s_4s_3s_4s_2s_3s_4\tau_r, \alpha_1) = 0$ *for* $i \geq 0, 1 \leq r \leq 3$.

$H^i(s_4s_3s_4s_2s_3s_4v_r, \alpha_4) = 0$ *for* $i \geq 0, 1 \leq r \leq 3$.

Proof. Proofs of (1): By using SES and Lemma 4.3.9(1) we have $H^i(s_4\tau_r, \alpha_1) = 0$ for all $1 \leq r \leq 5, i \geq 0$.

By (4.3.9.1) we have $H^i(s_4v_1, \alpha_4) = 0$ for $i \geq 0$. On the other hand by using SES and Lemma 4.3.9(2) we have $H^i(s_4v_r, \alpha_4) = 0$ for $i \geq 0, 2 \leq r \leq 5$. Thus combining we have $H^i(s_4v_r, \alpha_4) = 0$ for $i \geq 0, 1 \leq r \leq 5$.

Proofs of (2), (3), (4), (5), and (6) follow by using SES and (1). □

4.4 Surjectivity of some maps

Let $w \in W$ and let $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression for w and let $\underline{i} = (i_1, i_2, \ldots, i_r)$. Let $\tau = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$ and $\underline{i'} = (i_1, i_2, \ldots, i_{r-1})$.

Recall the following long exact sequence of B-modules from the proof of Theorem 1.14.1 :

$$0 \to H^0(w, \alpha_{i_r}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \longrightarrow$$
$$H^1(w, \alpha_{i_r}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \longrightarrow H^2(w, \alpha_{i_r}) \longrightarrow$$
$$H^2(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^2(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \longrightarrow H^3(w, \alpha_{i_r}) \longrightarrow \cdots$$

By Lemma 1.13.6, we have $H^j(w, \alpha_{i_r}) = 0$ for every $j \geq 2$. Thus we have the following exact sequence of B-modules:

$$0 \to H^0(w, \alpha_{i_r}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^0(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \longrightarrow$$
$$H^1(w, \alpha_{i_r}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})}) \longrightarrow H^1(Z(\tau, \underline{i'}), T_{(\tau,\underline{i'})}) \longrightarrow 0$$

Let $w_0 = s_{j_1} s_{j_2} \cdots s_{j_N}$ be a reduced expression of w_0. Let $w = s_{j_1} s_{j_2} \cdots s_{j_r}$, $\underline{i} = (j_1, j_2, \ldots, j_r)$, and $\underline{j} = (j_1, j_2, \ldots, j_N)$.

Lemma 4.4.1. *The natural homomorphism*

$$f : H^0(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$$

of B-modules is injective if and only if $w^{-1}(\alpha_0) < 0$.

Proof. Suppose $w^{-1}(\alpha_0) < 0$. By [CKP15, Lemma 6.2, p.667], we have $H^0(Z(w, \underline{i}), T_{(w,\underline{i})})_{-\alpha_0} \neq 0$. By [CKP15, Theorem 7.1], $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$ is a parabolic subalgebra of $\mathfrak{g}$ and hence there is a unique B-stable line in $H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})})$, namely $\mathfrak{g}_{-\alpha_0}$. Therefore we conclude that the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \to H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$$

is injective.

Conversely, suppose the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \to H^0(Z(w, \underline{i}), T_{(w,\underline{i})})$$

is injective. Then by [CKP15, Lemma 6.2, p.667], we have $w^{-1}(\alpha_0) < 0$. □

Lemma 4.4.2. *The natural homomorphism*

$$f : H^1(Z(w_0, \underline{j}), T_{(w_0,\underline{j})}) \longrightarrow H^1(Z(w, \underline{i}), T_{(w,\underline{i})})$$

of B-modules is surjective.

Proof. (see [CK17, Lemma 7.1, p.459]). □

For $1 \leq r \leq 5$ let $\underline{j}_r$ be the reduced expression of $\tau_r = [1,4]^r s_1$, $\underline{i}_r = (\underline{j}_r, 2)$ be the reduced expression of $w_r = [1,4]^r s_1 s_2$, and $\underline{l}_r = (\underline{i}_r, 3)$ be the reduced expression of $w_r s_3 = [1,4]^r s_1 s_2 s_3$.

Lemma 4.4.3.

(1) We have $dim H^0(Z(\tau_4, \underline{j}_4), T_{(\tau_4, \underline{j}_4)})_{-\omega_4} = 2$. *Further, the natural map* $H^0(Z(\tau_4, \underline{j}_4), T_{(\tau_4, \underline{j}_4)}) \longrightarrow H^1(w_4, \alpha_2)$ *is surjective.*

(2) We have $dim H^0(Z(\tau_3, \underline{j}_3), T_{(\tau_3, \underline{j}_3)})_{-\omega_4+\alpha_4} = 2$. *Further, the natural map* $H^0(Z(\tau_3, \underline{j}_3), T_{(\tau_3, \underline{j}_3)}) \longrightarrow H^1(w_3, \alpha_2)$ *is surjective.*

Proof. Proof of (1): Since $w_4^{-1}(\alpha_0) < 0$, by Lemma 4.4.1 we conclude that the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \to H^0(Z(w_4, \underline{i_4}), T_{(w_4, \underline{i_4})})$$

is injective.

Since α_3 is a short simple root, by [Kan16, Corollary 5.6, p.778] we have $H^1(w_r s_3, \alpha_3) = 0$ for $r = 4, 5$. On the other hand, by Lemma 4.3.1 we have $H^1(w_5, \alpha_2) = 0$, and by Lemma 4.3.9 $H^1(v_r, \alpha_4) = 0$, and $H^1(\tau_r, \alpha_1) = 0$ for $r = 4, 5$.

Thus from above observations and using LES the natural map

$$H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) \to H^0(Z(w_4, \underline{i_4}), T_{(w_4, \underline{i_4})}) \tag{4.4.3.1}$$

is surjective, hence an isomorphism.

By [CKP15, Theorem 7.1] $H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})})$ is parabolic subalgebra of $\mathfrak{g}$. Hence for any $\mu \in X(T) \setminus \{0\}$, we have

$$\dim H^0(Z(w_0, \underline{i}), T_{(w_0, \underline{i})})_\mu \leq 1.$$

By using LES repeatedly and using Lemma 4.3.9 we have

$$H^0(Z(\tau_4, \underline{j_4}), T_{(\tau_4, \underline{j_4})}) = H^0(Z(w_3 s_3, \underline{l_3}), T_{(l_3, \underline{l_3})}). \tag{4.4.3.2}$$

By using LES and [Kan16, Corollary 5.6, p.778] we have an exact sequence

$$0 \to H^0(w_3 s_3, \alpha_3) \to H^0(Z(w_3 s_3, \underline{l_3}), T_{(w_3 s_3, \underline{l_3})}) \to H^0(Z(w_3, \underline{i_3}), T_{(w_3, \underline{i_3})}) \to 0 \tag{4.4.3.3}$$

of B-modules.

Since $w_3^{-1}(\alpha_0) < 0$, by using Lemma 4.4.1 we conclude that the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \to H^0(Z(w_3, \underline{i_3}), T_{(w_3,\underline{i_3})}) \tag{4.4.3.4}$$

is injective.

Thus by (4.4.3.4) we have $\dim H^0(Z(w_3, \underline{i_3}), T_{(w_3,\underline{i_3})})_{-\omega_4} \geq 1$. Hence by Lemma 4.2.2(2), (4.4.3.3) we have

$$\dim H^0(Z(w_3s_3, \underline{l_3}), T_{(w_3s_3,\underline{l_3})})_{-\omega_4} \geq 2. \tag{4.4.3.5}$$

By (4.4.3.1) we have $\dim H^0(Z(w_4, \underline{i_4}), T_{(w_4,\underline{i_4})})_{-\omega_4} \leq 1$. Therefore by using LES we see that $\dim H^0(Z(\tau_4, \underline{j_4}), T_{(\tau_4,\underline{j_4})})_{-\omega_4} \leq 2$.

Thus by (4.4.3.2), (4.4.3.5) we have $\dim H^0(Z(\tau_4, \underline{j_4}), T_{(\tau_4,\underline{j_4})})_{-\omega_4} = 2$. Therefore by LES the natural map $H^0(Z(\tau_4, \underline{j_4}), T_{(\tau_4,\underline{j_4})})_{-\omega_4} \longrightarrow H^1(w_4, \alpha_2)_{-\omega_4}$ is surjective. Hence by Lemma 4.3.1(3) the natural map $H^0(Z(\tau_4, \underline{j_4}), T_{(\tau_4,\underline{j_4})}) \longrightarrow H^1(w_4, \alpha_2)$ is surjective.

Proof of (2): By using LES repeatedly and using Lemma 4.3.9 we have

$$H^0(Z(\tau_3, \underline{j_3}), T_{(\tau_3,\underline{j_3})}) = H^0(Z(w_2s_3, \underline{l_2}), T_{(w_2s_3,\underline{l_2})}). \tag{4.4.3.6}$$

By using LES and [Kan16, Corollary 5.6, p.778] we have an exact sequence

$$0 \to H^0(w_2s_3, \alpha_3) \to H^0(Z(w_2s_3, \underline{l_2}), T_{(w_2s_3,\underline{l_2})}) \to H^0(Z(w_2, \underline{i_2}), T_{(w_2,\underline{i_2})}) \to 0 \tag{4.4.3.7}$$

of B-modules.

Since $w_2^{-1}(\alpha_0) < 0$, by using Lemma 4.4.1 we conclude that the natural homomorphism

$$H^0(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \to H^0(Z(w_2, \underline{i_2}), T_{(w_2,\underline{i_2})}) \tag{4.4.3.8}$$

is injective.

Thus by (4.4.3.8) we have $\dim H^0(Z(w_2, \underline{i_2}), T_{(w_2,\underline{i_2})})_{-\omega_4+\alpha_4} \geq 1$. Hence by Lemma 4.2.2(1), (4.4.3.7) we have

$$\dim H^0(Z(w_2s_3, \underline{l_2}), T_{(w_2s_3,\underline{l_2})})_{-\omega_4+\alpha_4} \geq 2. \tag{4.4.3.9}$$

By Lemma 4.3.1, we have $H^1(w_4, \alpha_2)_{-\omega_4+\alpha_4} = 0$. Since α_3 is a short simple root, by [Kan16, Corollary 5.6, p.778] we have $H^1(w_3s_3, \alpha_3) = 0$. On the other hand, by Lemma 4.3.9 we have $H^1(v_4, \alpha_4) = 0$ and $H^1(\tau_4, \alpha_1) = 0$. Thus by using LES and from above discussion we have the natural map

$$H^0(Z(w_4, \underline{i_4}), T_{(w_4,\underline{i_4})})_{-\omega_4+\alpha_4} \longrightarrow H^0(Z(w_3, \underline{i_3}), T_{(w_3,\underline{i_3})})_{-\omega_4+\alpha_4}$$

is surjective.

Thus by using (4.4.3.1) and above surjectivity we have $\dim H^0(Z(w_3, \underline{i_3}), T_{(w_3,\underline{i_3})})_{-\omega_4+\alpha_4} \leq 1$. Therefore by using LES we see that $\dim H^0(Z(\tau_3, \underline{j_3}), T_{(\tau_3,\underline{j_3})})_{-\omega_4+\alpha_4} \leq 2$. Thus by (4.4.3.6), (4.4.3.9) we have $\dim H^0(Z(\tau_3, \underline{j}_3), T_{(\tau_3,\underline{j}_3)})_{-\omega_4+\alpha_4} = 2$. Therefore by LES the natural map $H^0(Z(\tau_3, \underline{j}_3), T_{(\tau_3,\underline{j}_3)})_{-\omega_4+\alpha_4} \longrightarrow H^1(w_3, \alpha_2)_{-\omega_4+\alpha_4}$ is surjective. Hence by Lemma 4.3.1(2) the natural map $H^0(Z(\tau_3, \underline{j_3}), T_{(\tau_3,\underline{j_3})}) \longrightarrow H^1(w_3, \alpha_2)$ is surjective. □

Lemma 4.4.4.

(1) Let $\mu = -\omega_4, -\omega_4 + \alpha_4$. Then we have $\dim H^0(Z(s_4\tau_3, (4, \underline{j}_3)), T_{(s_4\tau_3,(4,\underline{j}_3))})_\mu = 2$. Further, the natural map $H^0(Z(s_4\tau_3, (4, \underline{j}_3)), T_{(s_4\tau_3,(4,\underline{j}_3))}) \longrightarrow H^1(s_4w_3, \alpha_2)$ is surjective.

(2) Let $\mu = -(\alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)$. Then we have $\dim H^0(Z(s_4\tau_2, (4, \underline{j}_2)), T_{(s_4\tau_2,(4,\underline{j}_2))})_\mu = 2$. Further, the natural map $H^0(Z(s_4\tau_2, (4, \underline{j}_2)), T_{(s_4\tau_2,(4,\underline{j}_2))}) \longrightarrow H^1(s_4w_2, \alpha_2)$ is surjective.

Proof. Since $(s_4w_3)^{-1}(\alpha_0) < 0$, by Lemma 4.4.1 we conclude that the natural homomorphism

$$H^0(Z(w_0, (4, \underline{l_5})), T_{(w_0,(4,\underline{l_5}))}) \to H^0(Z(s_4w_3, (4, \underline{i_3})), T_{(s_4w_3,(4,\underline{i_3}))})$$

is injective.

Since α_3 is a short simple root, by [Kan16, Corollary 5.6, p.778] we have $H^1(s_4w_rs_3, \alpha_3) = 0$ for $r = 3, 4, 5$. On the other hand, by Corollary 4.3.2 we have $H^1(s_4w_r, \alpha_2) = 0$ for $r = 4, 5$, and by Corollary 4.3.10(1) we have $H^1(s_4v_r, \alpha_4) = 0$ and $H^1(s_4\tau_r, \alpha_1) = 0$ for $r = 4, 5$.

Thus from above observations and using LES the natural map

$$H^0(Z(w_0, (4, \underline{l_5})), T_{(w_0,(4,\underline{l_5}))}) \to H^0(Z(s_4w_3, (4, \underline{i_3})), T_{(s_4w_3,(4,\underline{i_3}))}) \tag{4.4.4.1}$$

is surjective, hence an isomorphism.

Proof of (1): By using LES repeatedly and using Corollary 4.3.10(1) we have

$$H^0(Z(s_4\tau_3, (4, \underline{j_3})), T_{(s_4\tau_3,(4,\underline{j_3}))}) = H^0(Z(s_4w_2s_3, (4, \underline{l_2})), T_{(s_4w_2s_3,(4,\underline{l_2}))}). \tag{4.4.4.2}$$

By using LES and [Kan16, Corollary 5.6, p.778] we have an exact sequence

$$0 \to H^0(s_4w_2s_3,\alpha_3) \to H^0(Z(s_4w_2s_3,(4,\underline{l_2})),T_{(s_4w_2s_3,(4,\underline{l_2}))}) \to H^0(Z(s_4w_2,(4,\underline{i_2})),T_{(s_4w_2,(4,\underline{i_2}))}) \to 0 \quad (4.4.4.3)$$

of B-modules. On the other hand, since $(s_4w_2)^{-1}(\alpha_0) < 0$, by using Lemma 4.4.1, we conclude that the natural homomorphism

$$H^0(Z(w_0,(4,\underline{l_5})),T_{(w_0,(4,\underline{l_5}))}) \to H^0(Z(s_4w_2,(4,\underline{i_2})),T_{(s_4w_2,(4,\underline{i_2}))}) \quad (4.4.4.4)$$

is injective.

Let $\mu = -\omega_4, -\omega_4 + \alpha_4$. Thus by (4.4.4.4), we have $\dim H^0(Z(s_4w_2,(4,\underline{i_2})),T_{(s_4w_2,(4,\underline{i_2}))})_\mu \geq 1$. Hence by (4.4.4.2), by Corollary 4.2.3(2) we have

$$\dim H^0(Z(s_4w_2s_3,(4,\underline{l_2})),T_{(s_4w_2s_3,(4,\underline{l_2}))})_\mu \geq 2. \quad (4.4.4.5)$$

By (4.4.4.1), $\dim H^0(Z(s_4w_3,(4,\underline{i_3})),T_{(s_4w_3,(4,\underline{i_3}))})_\mu \leq 1$.

By using LES we have $\dim H^0(Z(s_4\tau_3,(4,\underline{j_3})),T_{(s_4\tau_3,(4,\underline{j_3}))})_\mu \leq 2$. Thus by (4.4.4.2), (4.4.4.5) we have $\dim H^0(Z(s_4\tau_3,(4,\underline{j_3})),T_{(s_4\tau_3,(4,\underline{j_3}))})_\mu = 2$. Therefore by LES the natural map $H^0(Z(s_4\tau_3,(4,\underline{j_3})),T_{(s_4\tau_3,(4,\underline{j_3}))})_\mu \longrightarrow H^1(s_4w_3,\alpha_2)_\mu$ is surjective. Hence by Corollary 4.3.2(4) the natural map $H^0(Z(s_4\tau_3,(4,\underline{j_3})),T_{(s_4\tau_3,(4,\underline{j_3}))}) \longrightarrow H^1(s_4w_3,\alpha_2)$ is surjective.

Proof of (2): By using LES repeatedly and using Corollary 4.3.10(1) we have

$$H^0(Z(s_4\tau_2,(4,\underline{j_2})),T_{(s_4\tau_2,(4,\underline{j_2}))}) = H^0(Z(s_4w_1s_3,(4,\underline{l_1})),T_{(s_4w_1s_3,(4,\underline{l_1}))}). \quad (4.4.4.6)$$

By using LES and [Kan16, Corollary 5.6, p.778] we have an exact sequence

$$0 \to H^0(s_4w_1s_3,\alpha_3) \to H^0(Z(s_4w_1s_3,(4,\underline{l_1})),T_{(s_4w_1s_3,(4,\underline{l_1}))}) \to H^0(Z(s_4w_1,(4,\underline{i_1})),T_{(s_4w_1,(4,\underline{i_1}))}) \to 0 \quad (4.4.4.7)$$

of B-modules.

Let $\mu = -(\alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4), -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)$. Since $H^1(s_4w_2,\alpha_2)_\mu \neq 0$, by Corollary 4.3.2, (4.3.1.5) the same weight appears in $H^0(s_4w_1,\alpha_2)$, i.e. $H^0(s_4w_1,\alpha_2)_\mu \neq 0$. This implies $H^0(Z(s_4w_1,(4,\underline{i_1})),T_{(s_4w_1,(4,\underline{i_1}))})_\mu \neq 0$.

Thus by (4.4.4.7), Corollary 4.2.3(1) we have

$$\dim H^0(Z(s_4w_1s_3,(4,\underline{l_1})),T_{(s_4w_1s_3,(4,\underline{l_1}))})_\mu \geq 2. \quad (4.4.4.8)$$

Since $H^1(s_4w_2,\alpha_2)_\mu \neq 0$, by Corollary 4.3.2 we have $H^1(s_4w_3,\alpha_2)_\mu = 0$. Since α_3 is a short simple root, by [Kan16, Corollary 5.6, p.778] we have $H^1(s_4w_2s_3,\alpha_3) = 0$. On the other hand, by using Corollary 4.3.10(1) we have $H^1(s_4v_3,\alpha_4) = 0$ and $H^1(s_4\tau_3,\alpha_1) = 0$. Thus by using LES and from above discussion we have the natural map

$$H^0(Z(s_4w_3,(4,\underline{i_3})),T_{(s_4w_3,(4,\underline{i_3}))})_\mu \longrightarrow H^0(Z(s_4w_2,(4,\underline{i_2})),T_{(s_4w_2,(4,\underline{i_2}))})_\mu$$

is surjective.

By (4.4.4.1) and above surjectivity we have $\dim H^0(Z(s_4w_2,(4,\underline{i_2})),T_{(s_4w_2,(4,\underline{i_2}))})_\mu \leq 1$.

By using LES we see that $\dim H^0(Z(s_4\tau_2,(4,\underline{j_2})),T_{(s_4\tau_2,(4,\underline{j_2}))})_\mu \leq 2$. Thus by (4.4.4.6), (4.4.4.8) we have $\dim H^0(Z(s_4\tau_2,(4,\underline{j}_2)),T_{(s_4\tau_2,(4,\underline{j}_2))})_\mu = 2$. Therefore by LES the natural map $H^0(Z(s_4\tau_2,(4,\underline{j}_2)),T_{(s_4\tau_2,(4,\underline{j}_2))})_\mu \longrightarrow H^1(s_4w_2,\alpha_2)_\mu$ is surjective. Hence by Corollary 4.3.2(3) the natural map $H^0(Z(s_4\tau_2,(4,\underline{j_2})),T_{(s_4\tau_2,(4,\underline{j_2}))}) \longrightarrow H^1(s_4w_2,\alpha_2)$ is surjective. □

Lemma 4.4.5.

(1) We have $\dim H^0(Z(s_3s_4\tau_3,(3,4,\underline{j}_3)),T_{(s_3s_4\tau_3,(3,4,\underline{j}_3))})_{-\omega_4} = 2$. *Further, the natural map* $H^0(Z(s_3s_4\tau_3,(3,4,\underline{j}_3)),T_{(s_3s_4\tau_3,(3,4,\underline{j}_3))}) \longrightarrow H^1(s_3s_4w_3,\alpha_2)$ *is surjective.*

(2) Let $\mu = -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)$. *Then we have* $\dim H^0(Z(s_3s_4\tau_2,(3,4,\underline{j}_2)),T_{(s_3s_4\tau_2,(3,4,\underline{j}_2))})_\mu = 2$. *Further, the natural map* $H^0(Z(s_3s_4\tau_2,(3,4,\underline{j}_2)),T_{(s_3s_4\tau_2,(3,4,\underline{j}_2))}) \longrightarrow H^1(s_3s_4w_2,\alpha_2)$ *is surjective.*

(3) Let $\mu = -(\alpha_2+\alpha_3), -(\alpha_1+\alpha_2+\alpha_3)$. *Then we have* $\dim H^0(Z(s_3s_4\tau_1,(3,4,\underline{j}_1)),T_{(s_3s_4\tau_1,(3,4,\underline{j}_1))})_\mu = 2$. *Further, the natural map* $H^0(Z(s_3s_4\tau_1,(3,4,\underline{j}_1)),T_{(s_3s_4\tau_1,(3,4,\underline{j}_1))}) \longrightarrow H^1(s_3s_4w_1,\alpha_2)$ *is surjective.*

Proof. Proofs of Lemma 4.4.5(1), Lemma 4.4.5(2), Lemma 4.4.5(3) are similar to that of Lemma 4.4.4 with using [Kan16, Corollary 4.3.6, p,778], Corollary 4.3.3 and Corollary 4.3.10(2) appropriately. □

Lemma 4.4.6.

(1) We have $\dim H^0(Z(s_2s_3s_4\tau_3,(2,3,4,\underline{j_3})),T_{(s_2s_3s_4\tau_3,(2,3,4,\underline{j_3}))})_{-\omega_4} = 2$. *Further, the natural map* $H^0(Z(s_2s_3s_4\tau_3,(2,3,4,\underline{j_3})),T_{(s_2s_3s_4\tau_3,(2,3,4,\underline{j_3}))}) \longrightarrow H^1(s_2s_3s_4w_3,\alpha_2)$ *is surjective.*

(2) We have $\dim H^0(Z(s_2s_3s_4\tau_2,(2,3,4,\underline{j_2})),T_{(s_2s_3s_4\tau_2,(2,3,4,\underline{j_2}))})_{-\omega_4+\alpha_4} = 2$. *Further, the natural map* $H^0(Z(s_2s_3s_4\tau_2,(2,3,4,\underline{j_2})),T_{(s_2s_3s_4\tau_2,(2,3,4,\underline{j_2}))}) \longrightarrow H^1(s_2s_3s_4w_2,\alpha_2)$ *is surjective.*

(3) We have $\dim H^0(Z(s_2s_3s_4\tau_1,(2,3,4,\underline{j_1})),T_{(s_2s_3s_4\tau_1,(2,3,4,\underline{j_1}))})_{-(\alpha_1+\alpha_2+\alpha_3)} = 2$. *Further, the natural map* $H^0(Z(s_2s_3s_4\tau_1,(2,3,4,\underline{j_1})),T_{(s_2s_3s_4\tau_1,(2,3,4,\underline{j_1}))}) \longrightarrow H^1(s_2s_3s_4w_1,\alpha_2)$ *is surjective.*

Proof. Proofs of Lemma 4.4.6(1), Lemma 4.4.6(2), Lemma 4.4.6(3) are similar to that of Lemma 4.4.4 with using [Kan16, Corollary 4.3.6, p,778], Corollary 4.3.4 and Corollary 4.3.10(3) appropriately. □

Lemma 4.4.7.

(1) Let $\mu = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4), -\omega_4 + \alpha_4, -\omega_4$. *Then we have*

$dimH^0(Z(s_4s_3s_4\tau_2, (4,3,4,\underline{j}_2)), T_{(s_4s_3s_4\tau_2,(4,3,4,\underline{j}_2))})_\mu = 2$. *Further, the natural map*

$H^0(Z(s_4s_3s_4\tau_2, (4,3,4,\underline{j}_2)), T_{(s_4s_3s_4\tau_2,(4,3,4,\underline{j}_2))}) \longrightarrow H^1(s_4s_3s_4w_2, \alpha_2)$ *is surjective.*

(2) Let

$\mu = -(\alpha_2+\alpha_3), -(\alpha_2+\alpha_3+\alpha_4), -(\alpha_1+\alpha_2+\alpha_3), -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), -(\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)$. *Then we have*

$dimH^0(Z(s_4s_3s_4\tau_1, (4,3,4,\underline{j}_1)), T_{(s_4s_3s_4\tau_1,(4,3,4,\underline{j}_1))})_\mu = 2$. *Further, the natural map*

$H^0(Z(s_4s_3s_4\tau_1, (4,3,4,\underline{j}_1)), T_{(s_4s_3s_4\tau_1,(4,3,4,\underline{j}_1))}) \longrightarrow H^1(s_4s_3s_4w_1, \alpha_2)$ *is surjective.*

Proof. Proofs of Lemma 4.4.7(1), Lemma 4.4.7(2), are similar to that of Lemma 4.4.4 with using [Kan16, Corollary 4.3.6, p,778], Corollary 4.3.5 and Corollary 4.3.10(4) appropriately. □

Lemma 4.4.8.

(1) Let $\mu = -\omega_4 + \alpha_4, -\omega_4$. *Then we have*

$dimH^0(Z(s_4s_2s_3s_4\tau_2, (4,2,3,4,\underline{j}_2)), T_{(s_4s_2s_3s_4\tau_2,(4,2,3,4,\underline{j}_2))})_\mu = 2$. *Further, the natural map*

$H^0(Z(s_4s_2s_3s_4\tau_2, (4,2,3,4,\underline{j}_2)), T_{(s_4s_2s_3s_4\tau_2,(4,2,3,4,\underline{j}_2))}) \longrightarrow H^1(s_4s_2s_3s_4w_2, \alpha_2)$ *is surjective.*

(2) Let

$\mu = -(\alpha_1+\alpha_2+\alpha_3), -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), -(\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)$. *Then we have* $dimH^0(Z(s_4s_2s_3s_4\tau_1, \underline{j'}_1), T_{(s_4s_2s_3s_4\tau_1,(4,2,3,4,\underline{j}_1))})_\mu = 2$. *Further, the natural map*

$H^0(Z(s_4s_2s_3s_4\tau_1, (4,2,3,4,\underline{j}_1)), T_{(s_4s_2s_3s_4\tau_1,(4,2,3,4,\underline{j}_1))}) \longrightarrow H^1(s_4s_2s_3s_4w_1, \alpha_2)$ *is surjective.*

Proof. Proofs of Lemma 4.4.8(1), Lemma 4.4.8(2), are similar to that of Lemma 4.4.4 with using [Kan16, Corollary 4.3.6, p,778], Corollary 4.3.6 and Corollary 4.3.10(5) appropriately. □

Lemma 4.4.9. *Let* $\underline{j'_1} = (4,3,4,2,3,4,\underline{j}_1)$ *and* $\underline{j'} = (4,3,4,2,3,4,1)$.

(1) Let $\Lambda = \{-(\alpha_1+\alpha_2+\alpha_3), -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)\}$. *Then we have*

$dimH^0(Z(s_4s_3s_4s_2s_3s_4\tau_1, \underline{j'_1}), T_{(s_4s_3s_4s_2s_3s_4\tau_1,\underline{j'_1})})_\mu = 2$ for all $\mu \in \Lambda$. Further, the natural map $H^0(Z(s_4s_3s_4s_2s_3s_4\tau_1, \underline{j'_1}), T_{(s_4s_3s_4s_2s_3s_4\tau_1,\underline{j'_1})}) \longrightarrow H^1(s_4s_3s_4s_2s_3s_4w_1, \alpha_2)$ is surjective.

(2) Let $\Pi = \{-(\alpha_2+\alpha_3), -(\alpha_2+\alpha_3+\alpha_4), -(\alpha_2+2\alpha_3+\alpha_4)\}$. Then we have $dimH^0(Z(s_4s_3s_4s_2s_3s_4s_1, \underline{j'}), T_{(s_4s_3s_4s_2s_3s_4s_1,\underline{j'})})_\mu = 2$ for all $\mu \in \Pi$. Further, the natural map $H^0(Z(s_4s_3s_4s_2s_3s_4s_1, \underline{j'}), T_{(s_4s_3s_4s_2s_3s_4s_1,\underline{j'})}) \longrightarrow H^1(s_4s_3s_4s_2s_3s_4s_1s_2, \alpha_2)$ is surjective.

Proof. Let $u_1 = s_4s_3s_4s_2s_3s_4\tau_1$ and $u = s_4s_3s_4s_2s_3s_4s_1$.

Note that $w_0 = s_4s_3s_4s_2s_3s_4w_3s_3s_1s_2$. Let $\underline{i'}$ be this reduced expression of w_0.

By Lemma 4.2.1(2) and Corollary 4.3.2(2) we have $H^i(s_4w_4, \alpha_2) = 0$ for $i \geq 0$. Since s_4 commutes with s_1, s_2, we have $H^i(s_4w_4, \alpha_2)$=$H^i(s_4w_3s_3s_4s_1s_2, \alpha_2)$=$H^i(s_4w_3s_3s_1s_2, \alpha_2)$ for $i \geq 0$. Thus we have $H^i(s_4w_3s_3s_1s_2, \alpha_2) = 0$ for $i \geq 0$.

Therefore by using SES we have $H^i(s_4s_3s_4s_2s_3s_4w_3s_3s_1s_2, \alpha_2) = 0$ for $i \geq 0$. Since s_3 commutes with s_1 we have $H^i(s_4s_3s_4s_2s_3s_4w_3s_3s_1, \alpha_1) = H^i(s_4s_3s_4s_2s_3s_4w_3s_1, \alpha_1)$ for $i \geq 0$.
$H^i(s_4s_3s_4s_2s_3s_4w_3s_1, \alpha_1) = H^i(s_4s_3s_4s_2s_3s_4[1,4]^3s_2s_1s_2, \alpha_1) = 0$ for $i \geq 0$ (see Lemma 1.13.3(4)). Thus we have $H^i(s_4s_3s_4s_2s_3s_4w_3s_3s_1, \alpha_1) = 0$ for $i \geq 0$. By Lemma 4.3.8(4) we have $H^1(s_4s_3s_4s_2s_3s_4w_r, \alpha_2) = 0$ for $r = 2, 3$. Since α_3 is a short simple root, by [Kan16, Corollary 5.6, p.778] we have $H^1(s_4s_3s_4s_2s_3s_4w_rs_3, \alpha_3) = 0$ for $r = 1, 2, 3$. On the other hand, by using Corollary 4.3.10(6) we have $H^1(s_4s_3s_4s_2s_3s_4v_r, \alpha_4) = 0$, and $H^1(s_4s_3s_4s_2s_3s_4\tau_r, \alpha_1) = 0$ for $r = 2, 3$.

Thus by using LES and the above discussion we have the natural map

$$H^0(Z(w_0, \underline{i'}), T_{(w_0,\underline{i'})}) \to H^0(Z(s_4s_3s_4s_2s_3s_4w_1, (\underline{j'_1}, 2)), T_{(s_4s_3s_4s_2s_3s_4w_1,(\underline{j'_1},2))}). \tag{4.4.9.1}$$

is surjective.

Proof of (1): By using LES repeatedly and Corollary 4.3.10(6) we have

$$H^0(Z(u_1, \underline{j'_1}), T_{(u_1,\underline{j'_1})}) = H^0(Z(us_2s_3, (\underline{j'}, 2, 3)), T_{(us_2s_3,(\underline{j'},2,3))}). \tag{4.4.9.2}$$

By using LES and [Kan16, Corollary 5.6, p.778] we have an exact sequence

$$0 \to H^0(us_2s_3, \alpha_3) \to H^0(Z(us_2s_3), T_{(us_2s_3,(\underline{j'},2,3))}) \to H^0(Z(us_2), T_{(us_2,(\underline{j'},2))}) \to 0. \tag{4.4.9.3}$$

of B-modules.

Let $\Lambda_1 = \{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4), -(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)\}$.

Let $\Lambda_2 = \{-(\alpha_1+\alpha_2+\alpha_3), -(\alpha_1+\alpha_2+\alpha_3+\alpha_4), -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)\}$.

By (4.3.8.1) we have

$H^0(s_3s_4s_2s_3s_4s_1s_2,\alpha_2)=\mathbb{C}_{-(\alpha_2+2\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3)}\oplus$
$\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)}\oplus\mathbb{C}_{-(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)}.$

Thus by using SES we see that $H^0(us_2,\alpha_2)_\mu\neq 0$ for all $\mu\in\Lambda_1$.

By using LES and Lemma 4.3.8(2) we have an exact sequence

$$0\to H^0(us_2,\alpha_2)_\mu\to H^0(Z(us_2,(\underline{j'},2)),T_{(us_2,(\underline{j'},2))})_\mu\to H^0(Z(u,\underline{j'}),T_{(u,(\underline{j'}))})_\mu\to 0$$

for all $\mu\in\Lambda$.

Note that $H^0(u,\alpha_1)=H^0(s_4s_3s_2s_1,\alpha_1)$. Now it is easy to see that $H^0(s_4s_3s_2s_1,\alpha_1)_\mu\neq 0$ for $\mu\in\Lambda_2$. Therefore we have $H^0(Z(u,\underline{j'}),T_{(u,\underline{j'})})_\mu\neq 0$, for all $\mu\in\Lambda_2$. Thus combining above discussion we have
$H^0(Z(us_2,(\underline{j'},2)),T_{(us_2,(\underline{j'},2))})_\mu\neq 0$ for all $\mu\in\Lambda$. (4.4.9.4)

Therefore by using (4.4.9.3), (4.4.9.4) and Corollary 4.2.9(2) we have

$\dim H^0(Z(us_2s_3,(\underline{j'},2,3)),T_{(us_2s_3,(\underline{j'},2,3))})_\mu\geq 2$ for all $\mu\in\Lambda$. (4.4.9.5)

By (4.4.9.1) we have $H^0(Z(s_4s_3s_4s_2s_3s_4w_1,(\underline{j'_1},2)),T_{(s_4s_3s_4s_2s_3s_4w_1,(\underline{j'_1},2))})_\mu\leq 1$ for all $\mu\in\Lambda$.

Therefore by using LES, Lemma 4.3.8(3) we have $\dim H^0(Z(u_1,\underline{j'_1}),T_{(u_1,\underline{j'_1})})_\mu\leq 2$ for all $\mu\in\Lambda$.

Thus by (4.4.9.2), (4.4.9.5) we have $\dim H^0(Z(u_1,\underline{j'_1}),T_{(u_1,\underline{j'_1})})_\mu=2$ for all $\mu\in\Lambda$.

By using LES we have

$H^0(Z(u_1,\underline{j'_1}),T_{(u_1,\underline{j'_1})})_\mu\longrightarrow H^1(s_4s_3s_4s_2s_3s_4w_1,\alpha_2)_\mu$ is surjective for all $\mu\in\Lambda$. Hence by Lemma 4.3.8(3) the natural map $H^0(Z(u_1,\underline{j'_1}),T_{(u_1,\underline{j'_1})})\longrightarrow H^1(s_4s_3s_4s_2s_3s_4w_1,\alpha_2)$ is surjective.

Proof of (2): It is easy to see that $H^1(u,\alpha_1)=H^1(s_4s_3s_2s_1,\alpha_1)=0$ and $H^0(u,\alpha_1)=H^0(s_4s_3s_2s_1,\alpha_1)_\mu=0$ for all $\mu\in\Pi$.

Further, we have $H^i(s_4s_3s_4s_2s_3s_4,\alpha_4)=H^i(s_4s_3s_2s_3s_4s_3,\alpha_3)=0$ for all $i\geq 0$ (see Lemma 1.13.3(4)).

From above discussions and using LES repeatedly we have

$$H^0(Z(u,\underline{j'}),T_{(u,\underline{j'})})_\mu=H^0(Z(s_4s_3s_4s_2s_3,(4,3,4,2,3)),T_{(s_4s_3s_4s_2s_3,(4,3,4,2,3))})_\mu \tag{4.4.9.6}$$

for all $\mu\in\Pi$.

By using LES and [Kan16, Corollary 5.6, p.778] we have an exact sequence

$$0\to H^0(s_4s_3s_4s_2s_3,\alpha_3)\to H^0(Z(s_4s_3s_4s_2s_3),T_{(s_4s_3s_4s_2s_3,(4,3,4,2,3))})\to H^0(Z(s_4s_3s_4s_2),T_{(s_4s_3s_4s_2,(4,3,4,2))})\to 0. \tag{4.4.9.7}$$

It is easy to see that $H^0(s_4s_3s_4s_2,\alpha_2)_\mu \neq 0$ for all $\mu \in \Pi$. Therefore we have

$H^0(Z(s_4s_3s_4s_2),T_{(s_4s_3s_4s_2,(4,3,4,2))})_\mu \neq 0$ for all $\mu \in \Pi$. Thus from (4.4.9.7) and Corollary 4.2.9(1) we have

$\dim H^0(Z(s_4s_3s_4s_2s_3),T_{(s_4s_3s_4s_2s_3,(4,3,4,2,3))})_\mu \geq 2$ for $\mu \in \Pi$. (4.4.9.8)

Since α_3 is a short simple root, by [Kan16, Corollary 5.6, p.778] we have $H^1(us_2s_3,\alpha_3) = 0$.

By using Corollary 4.3.10(6) we have

$H^1(s_4s_3s_4s_2s_3s_4\tau_1,\alpha_1) = 0$ and $H^1(s_4s_3s_4s_2s_3s_4v_1,\alpha_4) = 0$.

By Lemma 4.3.8 we have $H^1(s_4s_3s_4s_2s_3s_4w_1,\alpha_2)_\mu = 0$ for all $\mu \in \Pi$.

Thus combining above discussion we have the natural map

$$H^0(Z(s_4s_3s_4s_2s_3s_4w_1,(\underline{j_1'},2)),T_{(s_4s_3s_4s_2s_3s_4w_1,(\underline{j_1'},2))})_\mu \to H^0(Z(us_2,(\underline{j'},2)),T_{(us_2,(\underline{j'},2))})_\mu,$$

is surjective for all $\mu \in \Pi$.

Now, using (4.4.9.1) and above surjectivity we have $H^0(Z(us_2,(\underline{j'},2)),T_{(us_2,(\underline{j'},2))})_\mu \leq 1$ for all $\mu \in \Pi$. Further, by Lemma 4.3.8(2) $\dim H^1(us_2,\alpha_2)_\mu = 1$ for all $\mu \in \Pi$.

Therefore by using LES

$$0 \longrightarrow H^0(us_2,\alpha_2) \longrightarrow H^0(Z(us_2,(\underline{j'},2)),T_{(us_2,(\underline{j'},2))}) \longrightarrow H^0(Z(u,\underline{j'}),T_{(u,\underline{j'})}) \longrightarrow$$

$$H^1(us_2,\alpha_2) \longrightarrow H^1(Z(us_2,(\underline{j'},2)),T_{(us_2,(\underline{j'},2))}) \longrightarrow H^1(Z(u,\underline{j'}),T_{(u,\underline{j'})}) \longrightarrow 0$$

we have $H^0(Z(u,\underline{j'}),T_{(u,\underline{j'})})_\mu \leq 2$ for all $\mu \in \Pi$.

Therefore by (4.4.9.6), (4.4.9.8) we have $\dim H^0(Z(s_4s_3s_4s_2s_3),T_{(s_4s_3s_4s_2s_3,(4,3,4,2,3))})_\mu = 2$ for all $\mu \in \Pi$.

Therefore $H^0(Z(u,\underline{j'}),T_{(u,\underline{j'})})_\mu \to H^1(us_2,\alpha_2)_\mu$ is surjective for all $\mu \in \Pi$.

Hence by Lemma 4.3.8(2) the natural map $H^0(Z(u,\underline{j'}),T_{(u,\underline{j'})}) \to H^1(us_2,\alpha_2)$ is surjective.

□

4.5 Main theorem

In this section we prove the main theorem. Let c be a Coxeter element of W. Then there exists a decreasing sequence $4 \geq a_1 > a_2 > \cdots > a_k = 1$ of positive integers such that $c = [a_1,4][a_2,a_1-1]\cdots[a_k,a_{k-1}-1]$, where for $i \leq j$ denotes $[i,j] = s_is_{i+1}\cdots s_j$.

Theorem 4.5.1. *(S. S. Kannan, P. Saha): $H^j(Z(w_0,\underline{i}),T_{(w_0,\underline{i})}) = 0$ for all $j \geq 1$ if and only if $a_1 \neq 3$ or $a_2 \neq 2$.*

Proof. From [CKP15, Proposition 3.1, p. 673], we have $H^j(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$ for all $j \geq 2$. It is enough to prove the following: $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = 0$ if and only if c is of the form $[a_1, 4][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$ with $a_1 \neq 3$ or $a_2 \neq 2$.

Proof of ($\Longrightarrow$): If $a_1 = 3$, and $a_2 = 2$, then $c = s_3s_4s_2s_1$, Let $u = s_3s_4s_2$. Then $c = us_1$. Let $\underline{j} = (3, 4, 2)$ be the sequence corresponding to u. Then using LES, we have:

$$0 \to H^0(u, \alpha_2) \to H^0(Z(u, \underline{j}), T_{(u,\underline{j})}) \to H^0(Z(s_3s_4, (3, 4)), T_{(s_3s_4,(3,4))}) \to$$

$$H^1(u, \alpha_2) \xrightarrow{f} H^1(Z(u, \underline{j}), T_{(u,\underline{j})}) \to H^1(Z(s_3s_4, (3, 4)), T_{(s_3s_4,(3,4))}) \to 0.$$

We see that $H^1(u, \alpha_2) = \mathbb{C}_{\alpha_2+\alpha_3}$, $H^0(s_3, \alpha_3)_{\alpha_2+\alpha_3} = 0$, and $H^0(s_3s_4, \alpha_4)_{\alpha_2+\alpha_3} = 0$.

Therefore by LES we have $H^0(Z(s_3s_4, (3, 4)), T_{(s_3s_4,(3,4))})_{\alpha_2+\alpha_3} = 0$. Hence f is non zero homomorphism. Hence $H^1(Z(u, \underline{j}), T_{(u,\underline{j})}) \neq 0$. By Lemma 4.4.2, the natural homomorphism

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \longrightarrow H^1(Z(u, \underline{j}), T_{(u,\underline{j})})$$

is surjective.

Hence we have $H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) \neq 0$.

Proof of ($\Longleftarrow$) : Assume that $a_1 \neq 3$ or $a_2 \neq 2$. We prove the result by studying case by case. Note that by using Lemma 1.13.3(4) we have $H^1(w_0, \alpha_i) = 0$ for $i = 1, 2, 3, 4$. In each of the following cases we use these appropriately.

Case 1: $c = s_1s_2s_3s_4$. Then in this case we have $w_0 = v_6 = [1, 4]^6$. By using LES and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(w_0, \underline{i}), T_{(w_0,\underline{i})}) = H^1(Z(w_5, \underline{i_5}), T_{(w_5,\underline{i_5})}).$$

By using LES, Lemma 4.3.1, Lemma 4.3.9, and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(w_5, \underline{i_5}), T_{(w_5,\underline{i_5})}) = H^1(Z(w_4, \underline{i_4}), T_{(w_4,\underline{i_4})}).$$

By using LES and Lemma 4.4.3(1) we have

$$H^1(Z(w_4, \underline{i_4}), T_{(w_4,\underline{i_4})}) = H^1(Z(\tau_4, \underline{j_4}), T_{(\tau_4,\underline{j_4})}).$$

By using LES, Lemma 4.3.9, and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(\tau_4, \underline{j_4}), T_{(\tau_4, \underline{j_4})}) = H^1(Z(w_3, \underline{i_3}), T_{(w_3, \underline{i_3})}).$$

By using LES and Lemma 4.4.3(2) we have

$$H^1(Z(w_3, \underline{i_3}), T_{(w_3, \underline{i_3})}) = H^1(Z(\tau_3, \underline{j_3}), T_{(\tau_3, \underline{j_3})}).$$

By using LES, Lemma 4.3.9, and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(\tau_3, \underline{j_3}), T_{(\tau_3, \underline{j_3})}) = H^1(Z(w_2, \underline{i_2}), T_{(w_2, \underline{i_2})}).$$

By using LES, Lemma 4.3.1, Lemma 4.3.9, and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(w_2, \underline{i_2}), T_{(w_2, \underline{i_2})}) = H^1(Z(w_1, \underline{i_1}), T_{(w_1, \underline{i_1})}).$$

By using LES, Lemma 4.3.1, Lemma 4.3.9, and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(w_1, \underline{i_1}), T_{(w_1, \underline{i_1})}) = H^1(Z(s_1 s_2, (1, 2)), T_{(s_1 s_2, (1,2))}).$$

We see that $H^1(s_1, \alpha_1) = 0$, $H^1(s_1 s_2, \alpha_2) = 0$. Thus by using LES we have $H^1(Z(s_1 s_2, (1, 2)), T_{(s_1 s_2, (1,2))}) = 0$. Thus combining all we have $H^1(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$.

Case 2: $c = s_4 s_1 s_2 s_3$. Then in this case we have $w_0 = s_4 w_5 s_3$.

By using LES and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(w_0, (4, \underline{l_5})), T_{(w_0, (4, \underline{l_5}))}) = H^1(Z(s_4 w_5, (4, \underline{i_5})), T_{(s_4 w_5, (4, \underline{i_5}))}).$$

By using LES, Corollary 4.3.2, Corollary 4.3.10(1), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(s_4 w_5, (4, \underline{i_5})), T_{(s_4 w_5, (4, \underline{i_5}))}) = H^1(Z(s_4 w_4, (4, \underline{i_4})), T_{(s_4 w_4, (4, \underline{i_4}))}).$$

By using LES, Corollary 4.3.2, Corollary 4.3.10(1), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(s_4 w_4, (4, \underline{i_4})), T_{(s_4 w_4, (4, \underline{i_4}))}) = H^1(Z(s_4 w_3, (4, \underline{i_3})), T_{(s_4 w_3, (4, \underline{i_3}))}).$$

By using LES and Lemma 4.4.4(1) we have

$$H^1(Z(s_4 w_3, (4, \underline{i_3})), T_{(s_4 w_3, (4, \underline{i_3}))}) = H^1(Z(s_4 \tau_3, (4, \underline{j_3})), T_{(s_4 \tau_3, (4, \underline{j_3}))}).$$

By using LES, Corollary 4.3.10(1), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(s_4 \tau_3, (4, \underline{j_3})), T_{(s_4 \tau_3, (4, \underline{j_3}))}) = H^1(Z(s_4 w_2, (4, \underline{i_2})), T_{(s_4 w_2, (4, \underline{i_2}))}).$$

By using LES, Lemma 4.4.4(2) we have

$$H^1(Z(s_4w_2,(4,\underline{i_2})),T_{(s_4w_2,(4,\underline{i_2}))}) = H^1(Z(s_4\tau_2,(4,\underline{j_2})),T_{(s_4\tau_2,(4,\underline{j_2}))}).$$

By using LES, Corollary 4.3.10(1), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(s_4\tau_2,(4,\underline{j_2})),T_{(s_4\tau_2,(4,\underline{j_2}))}) = H^1(Z(s_4w_1,(4,\underline{i_1})),T_{(s_4w_1,(4,\underline{i_1}))}).$$

By using LES, Corollary 4.3.2, Corollary 4.3.10(1), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(s_4w_1,(4,\underline{i_1})),T_{(s_4w_1,(4,\underline{i_1}))}) = H^1(Z(s_4s_1s_2,(4,1,2)),T_{(s_4s_1s_2,(4,1,2))}).$$

We see that $H^1(s_4s_1,\alpha_1) = 0$, $H^1(s_4s_1s_2,\alpha_2) = 0$. Thus by using LES we have

$$H^1(Z(s_4s_1s_2,(4,1,2)),T_{(s_4s_1s_2,(4,1,2))}) = 0.$$

Thus combining all we have $H^1(Z(w_0,(4,\underline{l_5})),T_{(w_0,(4,\underline{l_5}))}) = 0$.

Case 3: $c = s_3s_4s_1s_2$. Then we have $w_0 = s_3s_4w_5$.

By using LES, Corollary 4.3.3, Corollary 4.3.10(2), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(s_3s_4w_5,(3,4,\underline{i_5})),T_{(s_3s_4w_5,(3,4,\underline{i_5}))}) = H^1(Z(s_3s_4w_4,(3,4,\underline{i_4})),T_{(s_3s_4w_4,(3,4,\underline{i_4}))}).$$

By using LES, Corollary 4.3.3, Corollary 4.3.10(2), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(s_3s_4w_4,(3,4,\underline{i_4})),T_{(s_3s_4w_4,(3,4,\underline{i_4}))}) = H^1(Z(s_3s_4w_3,(3,4,\underline{i_3})),T_{(s_3s_4w_3,(3,4,\underline{i_3}))}).$$

By using LES and Lemma 4.4.5(1) we have

$$H^1(Z(s_3s_4w_3,(3,4,\underline{i_3})),T_{(s_3s_4w_3,(3,4,\underline{i_3}))}) = H^1(Z(s_3s_4\tau_3,(3,4,\underline{j_3})),T_{(s_3s_4\tau_3,(3,4,\underline{j_3}))}).$$

By using LES, Corollary 4.3.10(2), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(s_3s_4\tau_3,(3,4,\underline{j_3})),T_{(s_3s_4\tau_3,(3,4,\underline{j_3}))}) = H^1(Z(s_3s_4w_2,(3,4,\underline{i_2})),T_{(s_3s_4w_2,(3,4,\underline{i_2}))}).$$

By using LES, Lemma 4.4.5(2) we have

$$H^1(Z(s_3s_4w_2,(3,4,\underline{i_2})),T_{(s_3s_4w_2,(3,4,\underline{i_2}))}) = H^1(Z(s_3s_4\tau_2,(3,4,\underline{j_2})),T_{(s_3s_4\tau_2,(3,4,\underline{j_2}))}).$$

By using LES, Corollary 4.3.10(2) and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(s_3s_4\tau_2,(3,4,\underline{j_2})),T_{(s_3s_4\tau_2,(3,4,\underline{j_2}))}) = H^1(Z(s_3s_4w_1,(3,4,\underline{i_1})),T_{(s_3s_4w_1,(3,4,\underline{i_1}))}).$$

By using LES, Lemma 4.4.5(3) we have

$$H^1(Z(s_3s_4w_1,(3,4,\underline{i_1})),T_{(s_3s_4w_1,(3,4,\underline{i_1}))})=H^1(Z(s_3s_4\tau_1,(3,4,\underline{j_1})),T_{(s_3s_4\tau_1,(3,4,\underline{j_1}))}).$$

By using LES, Corollary 4.3.10(2), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(s_3s_4\tau_1,(3,4,\underline{j_1})),T_{(s_3s_4\tau_1,(3,4,\underline{j_1}))})=H^1(Z(s_3s_4s_1s_2,(3,4,1,2)),T_{(s_3s_4s_1s_2,(3,4,1,2))}).$$

We see that $H^1(s_3s_4,\alpha_4)=0$ (see [Kan16, Corollary 5.6, p.778]), $H^1(s_3s_4s_1,\alpha_1)=0$, $H^1(s_3s_4s_1s_2,\alpha_2)=0$. Thus by using LES we have

$$H^1(Z(s_3s_4s_1s_2,(3,4,1,2)),T_{(s_3s_4s_1s_2,(3,4,1,2))})=0.$$

Thus combining all we have $H^1(Z(w_0,(3,4,\underline{i_5})),T_{(w_0,(3,4,\underline{i_5}))})=0$.

Case 4: $c=s_2s_3s_4s_1$. Then $w_0=s_2s_3s_4\tau_5$. Let $t_1=s_2s_3s_4$.

By using LES, Corollary 4.3.10(3) and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(w_0,(2,3,4,\underline{j_5})),T_{(w_0,(2,3,4,\underline{j_5}))})=H^1(Z(t_1w_4,(2,3,4,\underline{i_4})),T_{(t_1w_4,(2,3,4,\underline{i_4}))}).$$

By using LES, Corollary 4.3.4, Corollary 4.3.10(3), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_1w_4,(2,3,4,\underline{i_4})),T_{(t_1w_4,(2,3,4,\underline{i_4}))})=H^1(Z(t_1w_3,(2,3,4,\underline{i_3})),T_{(t_1w_3,(2,3,4,\underline{i_3}))}).$$

By using LES and Lemma 4.4.6(1) we have

$$H^1(Z(t_1w_3,(2,3,4,\underline{i_3})),T_{(t_1w_3,(2,3,4,\underline{i_3}))})=H^1(Z(t_1\tau_3,(2,3,4,\underline{j_3})),T_{(t_1\tau_3,(2,3,4,\underline{j_3}))}).$$

By using LES, Corollary 4.3.10(3), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_1\tau_3,(2,3,4,\underline{j_3})),T_{(t_1\tau_3,(2,3,4,\underline{j_3}))})=H^1(Z(t_1w_2,(2,3,4,\underline{i_2})),T_{(t_1w_2,(2,3,4,\underline{i_2}))}).$$

By using LES, Lemma 4.4.6(2) we have

$$H^1(Z(t_1w_2,(2,3,4,\underline{i_2})),T_{(t_1w_2,(2,3,4,\underline{i_2}))})=H^1(Z(t_1\tau_2,(2,3,4,\underline{j_2})),T_{(t_1\tau_2,(2,3,4,\underline{j_2}))}).$$

By using LES, Corollary 4.3.10(3), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_1\tau_2,(2,3,4,\underline{j_2})),T_{(t_1\tau_2,(2,3,4,\underline{j_2}))})=H^1(Z(t_1w_1,(2,3,4,\underline{i_1})),T_{(t_1w_1,(2,3,4,\underline{i_1}))}).$$

By using LES, Lemma 4.4.6(3) we have

$$H^1(Z(t_1w_1,(2,3,4,\underline{i_1})),T_{(t_1w_1,(2,3,4,\underline{i_1}))}) = H^1(Z(t_1\tau_1,(2,3,4,\underline{j_1})),T_{(t_1\tau_1,(2,3,4,\underline{j_1}))}).$$

By using LES, Corollary 4.3.10(3), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_1\tau_1,(2,3,4,\underline{j_1})),T_{(t_1\tau_1,(2,3,4,\underline{j_1}))}) = H^1(Z(t_1s_1s_2,(2,3,4,1,2)),T_{(t_1s_1s_2,(2,3,4,1,2))}).$$

It is easy to see that $H^1(t_1s_1,\alpha_1) = H^1(s_2s_1,\alpha_1) = 0$. We see that $H^1(s_2s_3,\alpha_3) = 0, H^1(t_1,\alpha_4) = 0$ by [Kan16, Corollary 5.6, p.778]. $H^1(t_1s_1s_2,\alpha_2) = 0$ by Corollay 4.3.4.

Thus by using LES we have

$$H^1(Z(t_1s_1s_2,(2,3,4,1,2)),T_{(t_1s_1s_2,(2,3,4,1,2))}) = 0.$$

Thus combining all we have $H^1(Z(w_0,(2,3,4,\underline{j_5})),T_{(w_0,(2,3,4,\underline{j_5}))}) = 0$.

Case 5: $c = s_4s_3s_1s_2$. In this case we have $w_0 = s_4s_3s_4w_4s_3s_1s_2$. Let $t_2 = s_4s_3s_4$. Since s_3 commutes with s_1, we have $H^i(t_2w_4s_3s_1,\alpha_1) = H^i(t_2w_3s_3s_4s_2s_1s_2,\alpha_1) = 0$ for $i \geq 0$ (see Lemma 1.13.3(4)).

Thus by using LES, Corollary 4.3.10(4) and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(w_0,(4,3,4,\underline{i_4},3,1,2)),T_{(w_0,(4,3,4,\underline{i_4},3,1,2))}) = H^1(Z(t_2w_4,(4,3,4,\underline{i_4})),T_{(t_2w_4,(4,3,4,\underline{i_4}))}).$$

By using LES, Corollary 4.3.5, Corollary 4.3.10(4), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_2w_4,(4,3,4,\underline{i_4})),T_{(t_2w_4,(4,3,4,\underline{i_4}))}) = H^1(Z(t_2w_3,(4,3,4,\underline{i_3})),T_{(t_2w_3,(4,3,4,\underline{i_3}))}).$$

By using LES, Corollary 4.3.5, Corollary 4.3.10(4), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_2w_3,(4,3,4,\underline{i_3})),T_{(t_2w_3,(4,3,4,\underline{i_3}))}) = H^1(Z(t_2w_2,(4,3,4,\underline{i_2})),T_{(t_2w_2,(4,3,4,\underline{i_2}))}).$$

By using LES and Lemma 4.4.7(1) we have

$$H^1(Z(t_2w_2,(4,3,4,\underline{i_2})),T_{(t_2w_2,(4,3,4,\underline{i_2}))}) = H^1(Z(t_2\tau_2,(4,3,4,\underline{j_2})),T_{(t_2\tau_2,(4,3,4,\underline{j_2}))}).$$

By using LES, Corollary 4.3.10(4), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_2\tau_2,(4,3,4,\underline{j_2})),T_{(t_1\tau_2,(4,3,4,\underline{j_2}))}) = H^1(Z(t_2w_1,(4,3,4,\underline{i_1})),T_{(t_2w_1,(4,3,4,\underline{i_1}))}).$$

By using LES, Lemma 4.4.7(2) we have

$$H^1(Z(t_2w_1,(4,3,4,\underline{i_1})),T_{(t_2w_1,(4,3,4,\underline{i_1}))}) = H^1(Z(t_2\tau_1,(4,3,4,\underline{j_1})),T_{(t_2\tau_1,(4,3,4,\underline{j_1}))}).$$

By using LES, Corollary 4.3.10(4) and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_2\tau_1,(4,3,4,\underline{j_1})),T_{(t_2\tau_1,(4,3,4,\underline{j_1}))}) = H^1(Z(t_2s_1s_2,(4,3,4,1,2)),T_{(t_2s_1s_2,(4,3,4,1,2))}).$$

We see that $H^1(s_4s_3,\alpha_3)=0, H^1(t_2,\alpha_4)=0$ by [Kan16, Corollary 5.6, p.778]. Since s_3, s_4 commutes with s_1 we have $H^1(t_2s_1,\alpha_1)=H^1(s_1,\alpha_1)=0$. By Corollary 4.3.5 we have $H^1(t_2s_1s_2,\alpha_2)=0$.

Thus by using LES we have $H^1(Z(t_2s_1s_2,(4,3,4,1,2)),T_{(t_2s_1s_2,(4,3,4,1,2))})=0$. Thus combining all we have $H^1(Z(w_0,(4,3,4,\underline{i_4},3,1,2)),T_{(w_0,(4,3,4,\underline{i_4},3,1,2))})=0$.

Case 6: $c=s_4s_2s_3s_1$. In this case we have $w_0=s_4s_2s_3s_4w_4s_3s_1$. Let $t_3=s_4s_2s_3s_4$. By using LES and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(w_0,(4,2,3,4,\underline{i_4},3,1)),T_{(w_0,(4,2,3,4,\underline{i_4},3,1))}) = H^1(Z(t_3w_4,(4,2,3,4,\underline{i_4})),T_{(t_3w_4,(4,2,3,4,\underline{i_4}))}).$$

By using LES, Corollary 4.3.6, Corollary 4.3.10(5), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_3w_4,(4,2,3,4,\underline{i_4})),T_{(t_3w_4,(4,2,3,4,\underline{i_4}))}) = H^1(Z(t_3w_3,(4,2,3,4,\underline{i_3})),T_{(t_3w_3,(4,2,3,4,\underline{i_3}))}).$$

By using LES, Corollary 4.3.6, Corollary 4.3.10(5), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_3w_3,(4,2,3,4,\underline{i_3})),T_{(t_3w_3,(4,2,3,4,\underline{i_3})}) = H^1(Z(t_3w_2,(4,2,3,4,\underline{i_2})),T_{(t_3w_2,(4,2,3,4,\underline{i_2}))}).$$

By using LES and Lemma 4.4.8(1) we have

$$H^1(Z(t_3w_2,(4,2,3,4,\underline{i_2})),T_{(t_3w_2,(4,2,3,4,\underline{i_2}))}) = H^1(Z(t_3\tau_2,(4,2,3,4,\underline{j_2})),T_{(t_3\tau_2,(4,2,3,4,\underline{j_2}))}).$$

By using Corollary 4.3.10(5) and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_3\tau_2,(4,2,3,4,\underline{j_2})),T_{(t_3\tau_2,(4,2,3,4,\underline{j_2}))}) = H^1(Z(t_3w_1,(4,2,3,4,\underline{i_1})),T_{(t_3w_1,(4,2,3,4,\underline{i_1}))}).$$

By using LES, Lemma 4.4.8(2) we have

$$H^1(Z(t_3w_1,(4,2,3,4,\underline{i_1})),T_{(t_3w_1,(4,2,3,4,\underline{i_1}))}) = H^1(Z(t_3\tau_1,(4,2,3,4,\underline{j_1})),T_{(t_3\tau_1,(4,2,3,4,\underline{j_1}))}).$$

By using LES, Corollary 4.3.10(5), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_3\tau_1,(4,2,3,4,\underline{j_1})),T_{(t_3\tau_1,(4,2,3,4,\underline{j_1}))}) = H^1(Z(t_3s_1s_2,(4,2,3,4,1,2)),T_{(t_3s_1s_2,(4,2,3,4,1,2))}).$$

We see that $H^1(s_4s_2,\alpha_2)=0$, $H^1(t_3s_1,\alpha_1)=0$. Further, by using [Kan16, Corollary 5.6, p.778] we have $H^1(s_4s_2s_3,\alpha_3)=0, H^1(t_3,\alpha_4)=0$. By Corollary 4.3.6 we have $H^1(t_3s_1s_2,\alpha_2)=0$.

Therefore by using LES we have $H^1(Z(t_3s_1s_2,(4,2,3,4,1,2)),\mathcal{T}_{(t_3s_1s_2,(4,2,3,4,1,2))})=0$. Thus combining all we have $H^1(Z(w_0,(4,2,3,4,\underline{i_4},3,1)),\mathcal{T}_{(w_0,(4,2,3,4,\underline{i_4},3,1))})=0$.

Case 7: $c=s_4s_3s_2s_1$. In this case we have $w_0=s_4s_3s_4s_2s_3s_4w_3s_3s_1s_2s_1$. Let $t_4=s_4s_3s_4s_2s_3s_4$. Let $\underline{i'}=(4,3,4,2,3,4,\underline{l_3},1,2,1)$. Recall that $l_r=(i_r,3)$. Let $\underline{i'_r}=(4,3,4,2,3,4,\underline{i_r})$ be the reduced expressions of t_4w_r for $r=1,2,3$. Let $\underline{j'_r}=(4,3,4,2,3,4,\underline{j_r})$ be the reduced expressions of $t_4\tau_r$ for $r=1,2,3$. Let $\underline{j'}=(4,3,4,2,3,4,1)$ be the reduced expression of t_4s_1.

By Lemma 4.2.1(2) and Corollary 4.3.2(2) we have $H^i(s_4w_4,\alpha_2)=0$ for $i\geq 0$. Since s_4 commutes with s_1,s_2, we have $H^i(s_4w_4,\alpha_4)$=$H^i(s_4w_3s_3s_4s_1s_2,\alpha_2)$=$H^i(s_4w_3s_3s_1s_2,\alpha_2)$ for $i\geq 0$. Thus we have $H^i(s_4w_3s_3s_1s_2,\alpha_2)=0$ for $i\geq 0$. Therefore by using SES we have $H^i(t_4w_3s_3s_1s_2,\alpha_2)=0$ for $i\geq 0$. Since s_3 commutes with s_1 we have $H^i(t_4w_3s_3s_1,\alpha_1)=H^i(t_4w_3s_1,\alpha_1)$ for $i\geq 0$. $H^i(t_4w_3s_1,\alpha_1)=H^i(t_4[1,4]^3s_2s_1s_2,\alpha_1)=0$ for $i\geq 0$ (see Lemma 1.13.3(4)). Thus we have $H^i(t_4w_3s_3s_1,\alpha_1)=0$ for $i\geq 0$. Thus by using LES, above discussion, and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(w_0,\underline{i'}),\mathcal{T}_{(w_0,\underline{i'})})=H^1(Z(t_4w_3,\underline{i'_3}),\mathcal{T}_{(t_4w_3,\underline{i'_3})}).$$

By using LES, Lemma 4.3.8(4), Corollary 4.3.10(6), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_4w_3,\underline{i'_3}),\mathcal{T}_{(t_4w_3,\underline{i'_3})})=H^1(Z(t_4w_2,\underline{i'_2}),\mathcal{T}_{(t_4w_2,\underline{i'_2})}).$$

By using LES, Lemma 4.3.8(4), Corollary 4.3.10(6), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_4w_2,\underline{i'_2}),\mathcal{T}_{(t_4w_2,\underline{i'_2})})=H^1(Z(t_4w_1,\underline{i'_1}),\mathcal{T}_{(t_4w_1,\underline{i'_1})}).$$

By using LES and Lemma 4.4.9(1) we have

$$H^1(Z(t_4w_1,\underline{i'_1}),\mathcal{T}_{(t_1w_1,\underline{i'_1})})=H^1(Z(t_4\tau_1,\underline{j'_1}),\mathcal{T}_{(t_4\tau_1,\underline{j'_1})}).$$

By using LES, Corollary 4.3.10(6), and [Kan16, Corollary 5.6, p.778] we have

$$H^1(Z(t_4\tau_1,\underline{j'_1}),\mathcal{T}_{(t_4\tau_1,\underline{j'_1})})=H^1(Z(t_4s_1s_2,(\underline{j'},2)),\mathcal{T}_{(t_4s_1s_2,(\underline{j'},2))}).$$

By using LES and Lemma 4.4.9(2) we have

$$H^1(Z(t_4s_1s_2,(\underline{j'},2)),\mathcal{T}_{(t_4s_1s_2,(\underline{j'},2))})=H^1(Z(t_4s_1,\underline{j'}),\mathcal{T}_{(t_4s_1,\underline{j'})}).$$

By [Kan16, Corollary 5.6, p.778] we see that $H^1(s_4s_3,\alpha_3)=0$, $H^1(s_4s_3s_4,\alpha_4)=0$, $H^1(s_4s_3s_4s_2s_3,\alpha_3)=0$, and $H^1(t_4,\alpha_4)=0$. By Lemma 4.3.8(1) we have $H^1(s_4s_3s_4s_2,\alpha_2)=0$.

Since s_3, s_4 commute with s_1, we have $H^1(t_4s_1, \alpha_1) = H^1(s_4s_3s_2s_1, \alpha_1)$. It is easy to see by using SES that $H^1(s_4s_3s_2s_1, \alpha_1) = 0$. Thus we have $H^1(t_4s_1, \alpha_1) = 0$. Therefore by using LES we have $H^1(Z(t_4s_1, \underline{j'}), T_{(t_4s_1, \underline{j'})}) = 0$. Thus combining all we have $H^1(Z(w_0, \underline{i'}), T_{(w_0, \underline{i'})}) = 0$. □

Corollary 4.5.2. *Let c be a Coxeter element such that c is of the form $[a_1, 4][a_2, a_1 - 1] \cdots [a_k, a_{k-1} - 1]$ with $a_1 \neq 3$ or $a_2 \neq 2$ and $a_k = 1$. Let $(w_0, \underline{i})$ be a reduced expression of w_0 in terms of c as in Theorem 4.5.1. Then, $Z(w_0, \underline{i})$ has no deformations.*

Proof. By Theorem 4.5.1 and by [CKP15, Proposition 3.1, p.673], we have $H^i(Z(w_0, \underline{i}), T_{(w_0, \underline{i})}) = 0$ for all $i > 0$. Hence, by [Huy05, Proposition 6.2.10, p.272], we see that $Z(w_0, \underline{i})$ has no deformations. □

4.6 Non rigidity for G_2

In this section we will assume that G is of type G_2. Note that w_0 is equal to $-identity$. We recall the following proposition from [YZ08, Proposition 1.3, p.858]. We use Proposition 4.1.1 and the notation as in [YZ08] to deduce the following:

Lemma 4.6.1. *Let $c \in W$ be a Coxeter element. Then, we have*

(1) $w_0 = c^3$.

(2) For any sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \underline{i}^3)$ of reduced expressions of c; the sequence $\underline{i} = (\underline{i}^1, \underline{i}^2, \underline{i}^3)$ is a reduced expression of w_0.

Proof. Proof of (1): Let $\eta : S \longrightarrow S$ be the involution of S defined by $i \to i^*$, where i^* is given by $\omega_{i^*} = -w_0(\omega_i)$. Since G is of type G_2, $w_0 = -identity$. Therefore, we have $i = i^*$ for every i. Let h be the Coxeter number. By [YZ08, Proposition 1.7], we have $h(i, c) + h(i^*, c) = h$. Since $h = 2|R^+|/2$ (see [Hum92, Proposition 3.18]) and $i = i^*$, we have $h(i, c) = h/2 = 3$, as $|R^+| = 6$. By Proposition 4.1.1, we have $c^6(\omega_i) = -\omega_i$ for all $i = 1, 2$. Since $\{\omega_i : i = 1, 2\}$ forms an $\mathbb{R}$-basis of $X(T) \otimes \mathbb{R}$, it follows that $c^3 = -identity$. Hence, we have $w_0 = c^3$. The assertion (2) follows from the fact that $l(c) = 2$ and $l(w_0) = |R^+| = 6$. (see [Hum72, p.66, Table 1]). □

Let c be a coxeter element of W. Then $c = s_1s_2$ or $c = s_2s_1$. Then from Lemma 4.6.1 we have $w_0 = s_1s_2s_1s_2s_1s_2$, or $w_0 = s_2s_1s_2s_1s_2s_1$ according as $c = s_1s_2$ or $c = s_2s_1$.

Let $\underline{i_1}$ (respectively, $\underline{i_2}$) be the the reduced expression of $w_0 = s_1s_2s_1s_2s_1s_2$ (respectively, $w_0 = s_2s_1s_2s_1s_2s_1$). Then we have

Theorem 4.6.2. *(S. S. Kannan, P. Saha): $H^1(Z(w_0, \underline{i_r}), T_{(w_0, \underline{i_r})}) \neq 0$ for $r = 1, 2$.*

Proof. Let $c = s_1 s_2$. Let $\underline{i} = (1, 2)$ be the sequence corresponding to c. Then using LES, we have:

$$0 \longrightarrow H^0(c, \alpha_2) \longrightarrow H^0(Z(c, \underline{i}), T_{(c,\underline{i})}) \longrightarrow H^0(s_1, \alpha_1) \longrightarrow$$
$$H^1(c, \alpha_2) \xrightarrow{g} H^1(Z(c, \underline{i}), T_{(c,\underline{i})}) \longrightarrow 0.$$

By using SES, we see that $H^1(s_1 s_2, \alpha_2) = \mathbb{C}_{\alpha_2+\alpha_1} \oplus \mathbb{C}_{\alpha_2+2\alpha_1}$. Now $H^0(s_1, \alpha_1)_{\alpha_2+\alpha_1} = 0$. Hence g is a non zero homomorphism. Hence $H^1(Z(c, \underline{i}), T_{(c,\underline{i})})) \neq 0$. By Lemma 4.4.2, the natural homomorphism

$$H^1(Z(w_0, \underline{i_1})) \longrightarrow H^1(Z(c, \underline{i}), T_{(c,\underline{i})})$$

is surjective.

Hence we have $H^1(Z(w_0, \underline{i_1}), T_{(w_0,\underline{i_1})}) \neq 0$.

Let $c = s_2 s_1$, $u = s_2 s_1 s_2$. Let $\underline{j} = (2, 1, 2)$ be the sequence corresponding to u. Then using LES, we have:

$$0 \longrightarrow H^0(u, \alpha_2) \longrightarrow H^0(Z(u, \underline{j}), T_{(u,\underline{j})}) \longrightarrow H^0(Z(s_2 s_1, (2, 1)), T_{(s_2 s_1,(2,1))}) \longrightarrow$$
$$H^1(u, \alpha_2) \xrightarrow{h} H^1(Z(u, \underline{j}), T_{(u,\underline{j})}) \longrightarrow H^1(Z(s_2 s_1, (2, 1)), T_{(s_2 s_1,(2,1))}) \to 0.$$

We see that $H^1(u, \alpha_2) = \mathbb{C}_{\alpha_1} \oplus \mathbb{C}_{\alpha_2+\alpha_1} \oplus \mathbb{C}_{\alpha_2+2\alpha_1}$, $H^0(s_1, \alpha_1)_{\alpha_2+\alpha_1} = 0$, and $H^0(s_2 s_1, \alpha_1)_{\alpha_2+\alpha_1} = 0$.

Therefore by LES we have $H^0(Z(s_2 s_1, (2, 1)), T_{(s_2 s_1,(2,1))})_{\alpha_2+\alpha_1} = 0$. Hence h is a non zero homomorphism. Hence $H^1(Z(u, \underline{j}), T_{(u,\underline{j})})) \neq 0$. By Lemma 4.4.2, the natural homomorphism

$$H^1(Z(w_0, \underline{i_2})) \longrightarrow H^1(Z(u, \underline{j}), T_{(u,\underline{j})})$$

is surjective.

Hence we have $H^1(Z(w_0, \underline{i_2}), T_{(w_0,\underline{i_2})}) \neq 0$. □

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