Seshadri Constants on Algebraic Surfaces

By

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DECLARATION

I declare that the thesis entitled **"Seshadri Constants on Algebraic Surfaces"** submitted by me for the degree of **Doctor of Philosophy in Mathematics** is the record of academic work carried out by me under the guidance of Professor Krishna Hanumanthu and this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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CERTIFICATE

I certify that the thesis entitled **"Seshadri Constants on Algebraic Surfaces"** submitted for the degree of **Doctor of Philosophy in Mathematics** by Praveen Kumar Roy is the record of research work carried out by him under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent his independent work in a very substantial measure.

Chennai Mathematical Institute
Date: June, 2019.

Professor Krishna Hanumanthu
Thesis Supervisor.

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Dedicated to my Parents

Abstract

For an ample line bundle L on a projective variety X, J.-P. Demailly defined an invariant of L which studies the local geometry/positivity of L around a point $x \in X$ [11]. He used Seshadri's criterion of ampleness for a line bundle to define the *Seshadri constant* at a point. Since then this area attracted significant attention of many researchers exploring this area. Computing and bounding Seshadri constants of ample line bundles on varieties became an active area of research.

In this thesis we worked on computing and bounding the Seshadri constants on hyperelliptic surfaces. Hyperelliptic surfaces are nonsingular minimal surfaces of Kodaira dimension 0 and irregularity 1. They are realised as finite group quotients of a product of two elliptic curves. Their classification is well understood. We show that for all ample line bundles on hyperelliptic surfaces, global Seshadri constants are rational (except for one case in which we give a partial answer) and we compute them precisely in some cases. We also give some bounds extending the already known bounds. We extend the work of Łucja Farnik [15] which motivated our study of Seshadri constants on hyperelliptic surfaces.

We also discuss surfaces of general type in this thesis. Surfaces of general type are surfaces with maximum Kodaira dimension, i.e., 2. Motivated by the main theorem of Thomas Bauer and Tomasz Szemberg in [6], we prove a result about the multi-point Seshadri constant on a surface of general type. Next, we consider X, a surface of general type of the form $C \times C$. Here C is a general member of the moduli of smooth curves of genus g, where $g \geqslant 2$. In this case, we prove that the global Seshadri constant of an ample line bundle, under some restrictions, is rational. We also compute single point Seshadri constant precisely in this case for the canonical line bundle.

Contents

Chapter 1

Introduction

1.1 Preliminaries

1.1.1 Basics

Throughout this thesis we will be working over the field $\mathbb{C}$ of complex numbers and all varieties are defined over $\mathbb{C}$.

Let X be a projective algebraic variety. A *prime divisor* on X is a co-dimension one subvariety of X. A divisor on X is an element of the free abelian group generated by the set of all prime divisors of X. We write a divisor $D = \sum n_i D_i$, where D_i are prime divisors, n_i are integers, and only finitely many n_i are different from zero. If all $n_i \geqslant 0$, then we say that D is an *effective divisor*.

Let X be a reduced and irreducible complex projective variety, and denote by $\mathcal{M}_X = \mathbb{C}(X)$ the (constant) *sheaf* of rational functions on X. It contains the structure sheaf $\mathcal{O}_X$ as a subsheaf and so there is an inclusion ${\mathcal{O}^*}_X \subset {\mathcal{M}^*}_X$ of sheaves of multiplicative abelian groups. We have the following exact sequence

$$0 \longrightarrow {\mathcal{O}^*}_X \longrightarrow {\mathcal{M}^*}_X \longrightarrow {\mathcal{M}^*}_X/{\mathcal{O}^*}_X \longrightarrow 0, \tag{1.1}$$

where ${\mathcal{M}^*}_X/{\mathcal{O}^*}_X$ is the quotient sheaf on X and its global sections are known as *Cartier divisors* on X. We denote by $\mathrm{Div}(X)$ the group of all such global sections of ${\mathcal{M}^*}_X/{\mathcal{O}^*}_X$. The short exact sequence in (1.1) gives rise to a surjective, connecting homomorphism in the long exact sequence of cohomology:

$$\mathrm{Div}(X) \longrightarrow H^1(X, {\mathcal{O}^*}_X) \cong \mathrm{Pic}(X) \tag{1.2}$$

given by,

$$D \mapsto \mathcal{O}_X(D). \tag{1.3}$$

Here $\mathrm{Pic}(X)$ denotes the set of all line bundles on X. It is known that if X is projective then this map is surjective. Proof for the same can be seen in [28, Example 1.1.5]. Kernal of this map is precisely the set of all *principal divisors* of X. The group $\mathrm{Div}(X)$ modulo principal divisor is known as *divisor class group*, denoted as $\mathrm{Cl}(X)$. Then one obtains $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$ [23]. Therefore from here onwards we will not differentiate between a divisor and the corresponding line bundle.

Given a divisor D on a smooth projective variety X we associate a *linear system* denoted by $|D|$ which consists of all effective divisors on X linearly equivalent to D. It has a structure of projective variety, as it can be viewed as the projective space corresponding to the vector space of global sections of $\mathcal{O}_X(D)$ [23].

Before getting immersed in the theory of positivity of an ample line bundle, we wish to define the intersection product of two divisors on a smooth projective surface. Here, by a *surface* we mean a 2-dimensional projective algebraic variety. To this end, let X be a nonsingular surface. Then we have the following:

Theorem 1.1. *There is a unique pairing Div(X) $\times$ Div(X) $\longrightarrow \mathbb{Z}$, given by*

$$(C, D) \mapsto C \cdot D$$

for any two divisors C and D in X such that

1. *if* C *and* D *are nonsingular curves meeting transversally, then* $C \cdot D = \#(C \cap D)$, *the number of intersection points of* C *with* D.
2. *it is symmetric:* $C \cdot D = D \cdot C$,
3. *it is additive:* $(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D$, *and*
4. *it depends only on the linear equivalence classes: if* $C_1 \sim C_2$ *then* $C_1 \cdot D = C_2 \cdot D$

Proof. We refer to [23, Theorem 1.1]. □

Let X be a smooth complex projective variety of dimension n and let L be a line bundle on X.

Definition 1.2 (Ample and very ample line bundles and divisors)**.**

1. L is *very ample* if there is a closed embedding of X in $\mathbb{P}^n$, for some $n \in \mathbb{N}$ such that $L = \mathcal{O}_X(1)$,

2. L is *ample* if $L^{\otimes m}$ is very ample for some $m > 0$.

We say L is *nef* if $L \cdot C \geqslant 0$ for all curve $C \subset X$. The following are very useful characterisations of ampleness of a line bundle.

Theorem 1.3 (Nakai-Moishezon-Kleiman [28]). *Let X be a projective variety and L be a line bundle on X. Then L is ample if and only if*

$$L^{\dim V} \cdot V > 0$$

for every positive dimensional irreducible subvariety $V \subset X$ (including the irreducible components of X).

Theorem 1.4 (Seshadri's criterion for ampleness [24]). *Let X be a projective variety and L be a line bundle on X. Then, L is ample if and only if for every $x \in X$, there exists a positive number $\varepsilon > 0$ such that*

$$\frac{L \cdot C}{\text{mult}_x C} \geqslant \varepsilon,$$

where $x \in X$ is an arbitrary point and $C \subset X$ is any reduced and irreducible curve containing x.

It is worth mentioning here that, having positive intersection with every curve is not enough to guarantee the ampleness of a line bundle. That is, $L \cdot C > 0$ for all curves $C \subset X$ is not sufficient for ampleness of the line bundle L. Mumford gave an example of a surface X, and a line bundle L such that $L \cdot C > 0$ for all curves $C \subset X$, but L is not ample (see [28, Example 1.5.2]).

1.1.2 Seshadri constants

Definition 1.5 (Seshadri constant at a point). Let X be a smooth projective variety and L be a nef line bundle on X. Let $x \in X$ be a point, then the *Seshadri constant of L at x* is defined as

$$\varepsilon(X, L, x) := \inf_{x \in C} \frac{L \cdot C}{\text{mult}_x C},$$

where the infimum is taken over all reduced and irreducible curves $C \subset X$ passing through x.

The Seshadri criterion of ampleness now takes the form: L is ample if and only if $\varepsilon(X, L, x) > 0$, for all $x \in X$.

Definition 1.6 (Seshadri curve [5])**.** We say a curve $C \subset X$ is a Seshadri curve, if it computes the Seshadri constants of L at a point $x \in X$. In other words, if

$$\varepsilon(X, L, x) = \frac{L \cdot C}{\mathrm{mult}_x C}.$$

Definition 1.5 extends naturally to define Seshadri constant of L at an arbitrary subscheme $Z \subset X$. See [5] for more details. To this end, let $f : Y \to X$ be the blowup of X along Z and let E be the exceptional devisor.

Definition 1.7 (Seshadri constants at a subscheme)**.** The Seshadri constant of L at Z is the real number

$$\varepsilon(X, L, Z) := \sup\{\lambda : f^*L - \lambda E \text{ is ample on } Y\}.$$

Remark 1.8. If Z is a point, then the two definitions agree.

Definition 1.9 (Multi-point Seshadri constants)**.** Let X be a smooth projective variety and L be a nef line bundle on X. Let $r \geqslant 1$ be an integer and $x_1, x_2, ..., x_r \in X$ are r distinct points of X. Then the multi-point Seshadri constant $\varepsilon(X, L, x_1, x_2, ..., x_r)$, is defined as

$$\varepsilon(X, L, x_1, x_2, ..., x_r) := \inf_{C \cap \{x_1, x_2, ..., x_r\} \neq \phi} \frac{L \cdot C}{\sum_{i=1}^{i=r} m_i},$$

where the infimum is taken over all reduced and irreducible curve $C \subset X$ passing through the points $x_1, x_2, ..., x_r$ having multiplicities $m_1, m_2, ..., m_r$ with at least one $m_i > 0$.

Higher dimensional Seshadri constants can be defined analogously which instead of curve, take into account higher dimensional subvarieties passing through the point $x \in X$ [5].

Definition 1.10 (Seshadri constants via higher dimensional subvarieties)**.** Let X be a smooth projective variety, L a nef line bundle on X and $x \in X$ is a point. Then

$$\varepsilon_d(X, L, x) := \inf \left(\frac{L^d \cdot V}{\text{mult}_x V} \right)^{\frac{1}{d}}$$

is the d-dimensional Seshadri constant of L at $x \in X$, where the infimum is taken over all d-dimensional subvarieties $V \subset X$ such that $x \in V$.

Remark 1.11. $\varepsilon(X, L, x) = \varepsilon_1(X, L, x)$.

For a very ample line bundle L on X, $\varepsilon(X, L, x) \geqslant 1$ for all $x \in X$ and if L is ample then we know that mL is very ample for $m >> 0$. This gives $\varepsilon(X, L, x) = \frac{1}{m}\varepsilon(X, mL, x) \geqslant \frac{1}{m}$, for every $x \in X$. To see an obvious bound of the Seshadri constants of an ample line bundle, let L be an ample line bundle on a projective variety of dimension n and let $\varepsilon := \varepsilon(X, L, x)$ then since $f^*L - \varepsilon E$ is nef

$$\begin{aligned} 0 &\leqslant (f^*L - \varepsilon E)^n \\ \Rightarrow 0 &\leqslant L^n - \varepsilon^n \\ \Rightarrow \varepsilon^n &\leqslant L^n \\ \Rightarrow \varepsilon &\leqslant \sqrt[n]{L^n}. \end{aligned}$$

Definition 1.12 (Seshadri constant of a line bundle, a point, a variety)**.** Given X and L as above, we define

1. $\varepsilon(X, L) := \inf\limits_{x \in X} \varepsilon(X, L, x)$, to be the global Seshadri constant of the line bundle L.
2. $\varepsilon(X; x) := \inf\limits_{L \text{ ample}} \varepsilon(X, L, x)$, to be the Seshadri constant of the point $x \in X$.
3. $\varepsilon(X) := \inf\limits_{L \text{ ample}} \varepsilon(X, L)$, to be the Seshadri constant of the variety X.
4. $\varepsilon(X, L, r) := \max\limits_{x_1, x_2, ..., x_r \in X} \varepsilon(X, L, x_1, x_2, ..., x_r)$.

Therefore $\varepsilon(X, L, x) \leqslant \sqrt[n]{L^n}$, and hence $\varepsilon(X, L, 1) \leqslant \sqrt[n]{L^n}$. This gives the following inequalities:

$$0 < \varepsilon(X, L) \leqslant \varepsilon(X, L, x) \leqslant \varepsilon(X, L, 1) \leqslant \sqrt[n]{L^n}.$$

In accordance with definition (1.9), it is known that $\varepsilon(X, L, r)$ is attained at *very general* set of points $x_1, x_2, ..., x_r \in X$ [33]. This means that $\varepsilon(X, L, r) = \varepsilon(X, L, x_1, x_2, ..., x_r)$

for all $(x_1, x_2, ..., x_r)$ outside some countable union of proper Zariski closed sets in $X^r := X \times X \times ... \times X$. Let n be the dimension of X and $x_1, ..., x_r \in X$ are any r-points then the following is a well-known upper bound for Seshadri constants.

$$\varepsilon(X, L, x_1, x_2, ..., x_r) \leqslant \sqrt[n]{\frac{L^n}{r}}.$$

The proof of this follows from the same technique as in the case of single point Seshadri constant.

On a surface X, it is known that $\varepsilon(X, L, 1) = \varepsilon(X, L, x)$ for a very general point $x \in X$ (i.e., for all points of X which lie outside a countable union of proper Zariski closed subsets of X). While at the other end of the interval $(0, \sqrt{L^2})$, $\varepsilon(X, L)$ is computed at some special points of X. In other words, some information about the special curves of X is required to compute $\varepsilon(X, L)$. For $\varepsilon(X, L, 1)$ one needs to know more about geometry of curves on X, i.e., about $\overline{NE}(X)$. Thus this gives different dynamics in the study of Seshadri constants of an ample line bundle on smooth projective surfaces.

Some interesting questions related to Seshadri constants that people study are:

Q 1 Rationality of Seshadri constants.

Q 2 Computing $\varepsilon(X, L, x_1, x_2, ..., x_r), \varepsilon(X, L, r), \varepsilon(X, L)$.

Q 3 Bounding $\varepsilon(X, L, x_1, x_2, ..., x_r), \varepsilon(X, L, r), \varepsilon(X, L)$.

Rationality of Seshadri constants is an interesting problem to study on surfaces. It is known that if $\varepsilon(X, L, x) < \sqrt{L^2}$, then $\varepsilon(X, L, x) = \frac{L \cdot C}{\text{mult}_x C}$ for some curve C and in that case we say that it is *sub-maximal* [6]. So sub-maximal Seshadri constants are always rational. On the other hand a maximal Seshadri constant is irrational if L^2 is not a square and in that case we have

$$0 < \varepsilon(X, L) = \varepsilon(X, L, x) = \varepsilon(X, L, 1) = \sqrt{L^2}.$$

Question 1.13. *(Some open questions about Seshadri constants)*
(a) Is there an example of an irrational Seshadri constant ?
(b) Is there an example of X with $\varepsilon(X) = 0$?

No example of an irrational Seshadri constant is known. In other words no example of a triple (X, L, x) is known for which $\varepsilon(X, L, x) \notin \mathbb{Q}$. Nevertheless people have explored the possibility of existence of irrational Seshadri constants. Some results in that direction, on blow ups of general points in $\mathbb{P}^2$ can be seen in [12], [16] and [20].

Towards the second question, it is known that Seshadri constants can be arbitrarily small. More precisely, given $\delta > 0$, Miranda has shown a way to construct an example of a surface X, a line bundle L and a point $x \in X$, such that $\varepsilon(X, L, x) < \delta$. We include the example of Miranda here, but first we prove the following:

Lemma 1.14. *Let $\Gamma \subset \mathbb{P}^2$ be an integral curve of degree $d \geqslant 3$. Then there exists an integral curve $\Gamma' \subset \mathbb{P}^2$ which intersects Γ transversally such that the linear system of plane curves spanned by Γ and Γ' contains only integral curves.*

Proof. Let $C \subset \mathbb{P}^2$ be a curve defined by a form f of degree d. Note that the space of degree d curves in $\mathbb{P}^2$ is of dimension $\binom{d+2}{2} - 1$. Then C is not integral, precisely when there exists polynomials f_1, f_2 such that $f = f_1 f_2$. Therefore, the space of all non-integral degree d curves has dimension bounded above by $\binom{e+2}{2} + \binom{d-e+2}{2} - 2 = \left(\frac{d^2+3d}{2}\right) + e^2 - de$, where $e \in \{1, 2, 3, ..., d-1\}$. This is largest when $e = 1$ and in that case it is equal to $\left(\frac{d^2+3d}{2}\right) + 1 - d$. So the space of all non-integral curves of degree d has codimension at least one in the space of all degree d curves. Therefore a general line through Γ avoids this subspace, i.e., there is a dense open set of integral curves Γ' such that the pencil spanned by Γ and Γ' contains only integral curves.

Now Bertini's theorem says that there is a dense open subspace of degree d curves consisting of integral curves which meet Γ transversally [23]. □

Theorem 1.15. (**Miranda's example**) *Let $\delta > 0$ be an arbitrary number then there exists a smooth surface X and a point $x \in X$ and an ample line bundle L on X such that*

$$\varepsilon(X, L, x) < \delta.$$

Proof. To find such X, let's start with $\mathbb{P}^2$ and choose $m \in \mathbb{Z}$ a positive integer, such that $m > \frac{1}{\delta}$. Let Γ be a degree d curve in $\mathbb{P}^2$ such that there exists a point $x \in \Gamma$ for which $\mathrm{mult}_x \Gamma = m$. This can be ensured by taking degree d to be large enough. We then choose another degree d curve $\Gamma' \subset \mathbb{P}^2$, such that Γ and Γ' meet transversely (by Lemma (1.14)).

Consider the pencil generated by Γ and Γ':

$$\langle \Gamma, \Gamma' \rangle = \{\lambda\Gamma + \lambda'\Gamma' | \lambda, \lambda' \in \mathbb{C} \text{ not both of them zero simultaneously}\}.$$

By choosing d large enough, if necessary, we can ensure that all the curves in the above pencil are irreducible. This pencil determines a rational morphism from $\mathbb{P}^2$ to $\mathbb{P}^1$ with indeterminacy locus equal to $\Gamma \cap \Gamma' = \{p_1, p_2, ..., p_{d^2}\}$. By blowing up these points the indeterminacy can be resolved to get a morphism.

Now let

$$\mu : X = Bl_{\Gamma \cap \Gamma'}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$$

be the surface obtained by blowing up the base points of that pencil. Thus X admits a mapping

$$\pi : X \longrightarrow \mathbb{P}^1.$$

Let $C, C' \subset X$ be the strict transforms of Γ and Γ'. Let E be an exceptional curve on X corresponding to the blow up map π. Consider $L = aC + E$ for $a \geqslant 2$. We show that L is ample. The various intersection products are: $C^2 = 0$, $E^2 = -1$ and $C \cdot E = 1$. Therefore we have $L^2 = 2a - 1 > 0$, $L \cdot C = 1$ and $L \cdot E = a - 1 > 0$. To show L is ample, we show that $L \cdot D > 0$ for every curve $D \subset X$.

If D is inside a fibre of π then D must lie inside a curve which belongs to the pencil generated by C and C' and since every member of the pencil is an integral curve (D being itself an integral curve), should belong to $\langle C, C' \rangle$ and hence D is numerically equivalent to C. Therefore $L \cdot D = 1 > 0$.

Now assume D is not inside any fibre, so D dominates $\mathbb{P}^1$ and hence $D \cdot C > 0$. This gives, $L \cdot D = aC \cdot D + E \cdot D > 0$ and hence L is ample. Finally, notice that

$$\varepsilon(X, L, x) \leqslant \frac{L \cdot C}{\text{mult}_x C} = \frac{1}{m} < \delta.$$

□

For an ample line bundle L on a smooth complex projective surface X, Ein and Lazarsfeld proved that $\varepsilon(X, L, 1) \geqslant 1$. We include the proof here.

Proposition 1.16. *[14, Proposition 5.2.3] Let X be a smooth projective surface and L be an ample line bundle on X, then*

$$\varepsilon(X, L, x) \geqslant 1,$$

for all except, perhaps countably many points $x \in X$.

Proof. We prove the proposition by showing that the set

$$\{(C, x) : C \subset X \text{ is reduced and irreducible curve with } \operatorname{mult}_x C > (L \cdot C)\},$$

consists of countably many algebraic families. On contrary, suppose not. Then, there exists a family (C_t, x_t) of reduced and irreducible curves in X with point x_t parametrised by a smooth curve T, such that $\operatorname{mult}_{x_t} C_t > (L \cdot C_t)$. Let $C = C_{t*}$ and $x = x_{t*}$, where $t* \in T$ is a general point. Set $m := \operatorname{mult}_x C$. This determines a deformation of C and hence gives a section

$$\rho\left(\frac{\partial}{\partial t}\right) \in H^0(C, \mathcal{O}_C(C)).$$

Since the deformation preserves the m-fold points of C, $\rho\left(\frac{\partial}{\partial t}\right)$ vanishes upto order at least $m-1$ at x, therefore

$$C^2 \geqslant m(m-1).$$

But then, Hodge Index Theorem and $L \cdot C \leqslant m-1$ (by assumption) gives

$$m(m-1) \leqslant L^2 \cdot C^2 \leqslant (L \cdot C)^2 < (m-1)^2, \Rightarrow m < m-1.$$

Which is absurd. □

In view of this lower bound, a lot of research is aimed at finding good lower bounds for the Seshadri constants of ample line bundles, primarily when X is a surface. There has been extensive work on computing or finding lower bounds for Seshadri constants on surfaces, mainly in the single point case $(r = 1)$. The multi-point case $(r \geqslant 2)$ is also of interest and there are results in this case, in various situations. Some results for multi-point Seshadri constants can be found in [21].

The above proposition continues to hold even after weakening the assumption of ample line bundle to be big and nef. We include the statement here.

Proposition 1.17. *Let X be a smooth projective surface and L be a nef and big line bundle on X. Then*

$$\varepsilon(X, L, 1) \geqslant 1.$$

The same result is conjectured to be true in arbitrary dimension [28, Conjecture 5.2.4].

Conjecture: Let X be any projective variety and L be any nef and big line bundle on X. Then

$$\varepsilon(X, L, 1) \geqslant 1.$$

However the best known result in this direction is the following proposition proved by Ein, Küchle and Lazarsfeld [13].

Proposition 1.18. *Let X be an irreducible projective variety of dimension n and let L be a big and nef line bundle on X. Then*

$$\varepsilon(X, L, x) \geqslant \frac{1}{n},$$

for any very general point $x \in X$.

Let X be a surface and let L be an ample line bundle on X. One of the crucial ideas in finding lower bounds is the observation that, if a Seshadri constant $\varepsilon(X, L, x_1, x_2, ..., x_r)$ is sub- maximal (i.e., $\varepsilon(X, L, x_1, x_2, ..., x_r) \leqslant \sqrt{L^2/r}$), then there is actually an irreducible and reduced curve C which passes through at least one of the points x_i, such that $\varepsilon(X, L, x_1, x_2, ..., x_r) = \frac{L \cdot C}{\sum_{i=1}^{i=r} \text{mult}_{x_i} C}$. Such curves are called Seshadri curves. See [6, Proposition 1.1], for a proof of their existence for sub-maximal single-point Seshadri constants which generalizes easily to the multi-point case.

Let X be a smooth complex projective surface and L be an ample line bundle on X. The *Nagata-Biran-Szemberg conjecture* predicts that for $r \geqslant k_0^2 L^2$, where k_0 is the smallest integer such that the linear system $|k_0 L|$ contains smooth non-rational curves, the Seshadri constants $\varepsilon(X, L, r)$ is maximal; i.e., $\varepsilon(X, L, r) = \sqrt{\frac{L^2}{r}}$. When $(X, L) = (\mathbb{P}^2, \mathcal{O}(1))$, we obtain the celebrated *Nagata conjecture* [28] (note $k_0 = 3$ here).

1.1.3 SHGH Conjecture

There is a more general conjecture called the SHGH (*Segre-Harbourne-Gimigliano-Hirschowitz*) conjecture about linear systems of plane curves, which implies Nagata conjecture. We describe it below.

Let $\pi: X \to \mathbb{P}^2$ be a blow-up of $\mathbb{P}^2$ at r general points $p_1, p_2, ..., p_r \in \mathbb{P}^2$. Consider the linear system $\mathcal{L}_d(p_1^{m_1}, p_2^{m_2}, ..., p_r^{m_r})$ of plane curves in $\mathbb{P}^2$ of degree d passing through $p_1, p_2, ..., p_r$ with multiplicities at least $m_1, m_2, ..., m_r$ respectively. Let $\mathcal{L}$ denote the linear system on X which is the pullback of $\mathcal{L}_d(p_1^{m_1}, p_2^{m_2}, ..., p_r^{m_r})$ via π. Let K_X denote the canonical bundle on X. One then defines:

$$\mathcal{V}(\mathcal{L}) := \mathcal{X}(\mathcal{L}) - 1 = \frac{\mathcal{L} \cdot (\mathcal{L} - K_X)}{2},$$

to be the *virtual dimension* of $\mathcal{L}$ and

$$e(\mathcal{L}) := \max\{\mathcal{V}(\mathcal{L}), -1\},$$

to be the *expected dimension* of $\mathcal{L}$. We call a system $\mathcal{L}$ to be *non-special* if $\dim(\mathcal{L}) = e(\mathcal{L})$, otherwise we call it *special* i.e., when virtual dimension is more than expected dimension). It is not hard to see

$$\mathcal{L} \text{ is non-special} \quad \Leftrightarrow \quad h^0(X, L) \cdot h^1(X, L) = 0$$

★ One way speciality can arise is, if the linear system $\mathcal{L}$ has a multiple of (-1) curve in its base locus.

SHGH says ★ is the only possible way speciality can arise. An equivalent formulation of SHGH is:

SHGH Conjecture: Suppose $\mathcal{L}$ is a linear system on X which is *non empty* and *reduced* (i.e., general member of $\mathcal{L}$ is reduced), then $\mathcal{L}$ is non-special.

Seshadri constants are well connected to SHGH conjecture. Let X_r denote the blow-up of $\mathbb{P}^2$ at r very general points of $\mathbb{P}^2$. In [12], the authors have shown that, for $r > 9$ the SHGH conjecture on X_r implies that certain ample line bundles L on X_{r-1} have irrational Seshadri constants $\varepsilon(X_{r-1}, L, x)$, when x is a very general point of X_{r-1}. In a recent work [20], the authors have shown that less is required to draw the same conclusion. Namely, they assume that a prime divisor C on the blow up Y of X at a point $x \in X$ satisfying $C^2 = -1$, also satisfy $C^2 = K_X \cdot C = -1$. This assumption known as (-1) conjecture is a consequence of SHGH conjecture but not known to be equivalent to full SHGH conjecture. They have also raised the question, if an even weaker assumption viz., Nagata's Conjecture, is sufficient to draw the same conclusion.

Although the Seshadri constant is much explored in the case of surfaces, lots of problems on surfaces remain open for higher dimensional projective varieties. Researchers have considered some interesting questions in higher dimensional projective varieties.

When L is ample, J.-P. Demailly has given a nice interpretation of Seshadri constants in terms of separation of jets. We start with a definition.

Definition 1.19 (Separation of jets). Let L be a line bundle on a projective variety X and fix a smooth point $x \in X$ with maximal ideal sheaf $m_x \subset \mathcal{O}_X$. One says that $|L|$ *separates s-jets at x* if the natural map

$$H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_X/m_x^{s+1}) =: \mathcal{J}_x^s(L)$$

taking sections of L to their s-jets is surjective.

Note that, $|L|$ separates 0-jets if and only if L is free at x. In that case, $|L|$ separates 1-jets if and only if the derivative $d_x\phi_{|L|}$ at x of the rational map $\phi_{|L|} : X \dashrightarrow \mathbb{P}(H^0(X, L))$ is injective. Next we define an invariant measuring the order of jets a linear series separates.

Definition 1.20. With X and L as above, denote by $s(X, L, x)$, the largest natural number s such that the linear system $|L|$ separates s-jets at x (if x is a base-point of $|L|$ then put $s(X, L, x) = -1$).

If L is ample then one expects that $|kL|$ separates higher jets as k increases. Demailly observed that the Seshadri constant of L at x controls the rate of growth of $s(X, kL, x)$ as a function of k.

Theorem 1.21 (Growth of jets separation [28]). *Let X be an irreducible projective variety and L be an ample line bundle on X. Let $x \in X$ be a smooth point. Then*

$$\varepsilon(X, L, x) = \lim_{k \to \infty} \frac{s(X, kL, x)}{k}.$$

Seshadri constants and separation of jets are also related in other ways, as the following theorem suggests.

Theorem 1.22 (Seshadri constants and adjoint bundles [28]). *Let X be a smooth projective variety of dimension n and L be a big and nef line bundle on X. Let $x \in X$ be a point. Then*

1. $\varepsilon(X, L, x) > n + s \Rightarrow K_X + L$ *separates s-jets at* x.

2. $\varepsilon(X, L, x) > 2n \Rightarrow$ *the rational map* $\phi_{|K_X+L|} : X \dashrightarrow \mathbb{P}(H^0(X, K_X + L))$ *is birational onto its image.*

3. $\varepsilon(X, L, x) > 2n$ *for every* $x \in X \Rightarrow K_X + L$ *is very ample.*

The proof of above theorem crucially uses the *Kawamata-Viehweg vanishing theorem* [28]. But the result also holds if the ground field is of positive characteristic [30, 31]. For that we first define Seshadri constants in positive characteristic. Using the idea of jet separation, M. Mustaţă and K. Schwede defined a Frobenius version of Seshadri constants. It is defined in a way which takes advantage of the Frobenius morphism. We first fix some notation.

Let X be a projective variety defined over an algebraically closed field k of characteristic $p > 0$. Let $\mathcal{F} : X \longrightarrow X$ be the Frobenius morphism which acts as an identity on X and sends the section f of $\mathcal{O}_X$ to f^p. We denote by $\mathcal{F}^e$, for $e \geqslant 1$, to be $\mathcal{F} \circ \mathcal{F} \circ ... \circ \mathcal{F}$ (e-times). If $\mathcal{J}$ is a sheaf of ideal on X then we denote by $\mathcal{J}^{[p^e]}$ to be the inverse image of $\mathcal{J}$ by $\mathcal{F}^e$: If $\mathcal{J}$ is locally generated by $(h_i)_{i\in I}$, then $\mathcal{J}^{[p^e]}$ is locally generated by $(h_i^{p^e})_{i\in I}$.

Let Z be the set of smooth, closed points of X with the reduced scheme structure. Denote by $\mathcal{I}_Z$ to be the ideal sheaf of Z. Given a line bundle L on X and a positive integer e, we say that L generates p^e-Frobenius jets at Z if the following restriction map is surjective

$$H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_X/\mathcal{I}_Z^{[p^e]}) =: \mathcal{I}_Z^{[p^e]}(L).$$

Let $s_F(L^{\otimes m}; Z)$ be the largest integer e such that $L^{\otimes m}$ separates p^e-Frobenius jets at Z. If there is no such e, then we put $s_F(L^{\otimes m}; Z) = 0$.

Definition 1.23 (Frobenius-Seshadri Constant). Let X, L and Z are as above in the discussion, then the *Frobenius-Seshadri constant of* L *at* Z is

$$\varepsilon_F(X, L, Z) := \sup_{m \geqslant 1} \frac{p^{s_F(L^{\otimes m}; Z)} - 1}{m}.$$

It can be easily shown from the definition that the following relation between the two versions of Seshadri constants holds:

$$\frac{\varepsilon(X, L, x)}{n} \leqslant \varepsilon_F(L, x) \leqslant \varepsilon(X, L, x).$$

The Frobenius-Seshadri constants satisfy some of the basic properties of the usual Seshadri constants. For example, $\varepsilon_F(X, L, x)$ depends only on the numerical class of L and $\varepsilon_F(X, L^{\otimes m}, Z) = m \cdot \varepsilon_F(X, L, Z)$. More details can be seen in [31]. In [31], authors have proved similar results like (2) and (3) of Theorem (1.22), in positive characteristic.

In [30], the Frobenius-Seshadri constant is further generalised to higher-order variants of the Frobenius-Seshadri constant, which mix both ordinary and Frobenius powers of m_x. Using the new definition the author generalises Theorem (1.22).

While most of the research related to Seshadri constants is being focused in the case of dimension 2, there are some results in higher dimension as well, arising basically out of 2-dimensional cases. For example proposition (1.18) have been improved in case of some specific classes of varieties. More precisely, if X is a threefold, Cascini and Nakamaye obtained the bound $\varepsilon(X, L, 1) > 1/2$ [10]. When X is a Fano variety of dimension at least 3 such that its anti canonical divisor is globally generated, then $\varepsilon(X, L) \geqslant 1/(n-2)$, except when X is del Pezzo threefold of degree 1 (see [29]). In [26], Ito gives a method to estimate Seshadri constants on toric varieties at any point. Also by using toric degeneration they obtain some new computations or estimations of Seshadri constants on non-toric varieties.

The notion of Seshadri constant can be generalized to ample vector bundles. Let X be a smooth complex projective variety of dimension n and $\mathcal{E}$ be an ample vector bundle on X of rank $r+1$. Let $\mathbb{P}(\mathcal{E})$ be the projective space corresponding to $\mathcal{E}$ and $p : \mathbb{P}(\mathcal{E}) \longrightarrow X$ be the natural projection. Put $\xi := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be the tautological line bundle on $\mathbb{P}(\mathcal{E})$. Fix a point $x \in X$ and consider $\pi_x := \tilde{X} \longrightarrow X$ be the blow up of X at x. We get the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}(\pi_x^*\mathcal{E}) & \xrightarrow{\tilde{\pi_x}} & \mathbb{P}(\mathcal{E}) \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{X} & \xrightarrow{\pi_x} & X \end{array}$$

Let $\tilde{\xi_x} := \mathcal{O}_{\mathbb{P}(\pi_x^*\mathcal{E})}(1)$ denotes the tautological line bundle on $\mathbb{P}(\pi_x^*\mathcal{E})$ and $F_x := p^{-1}(x)$ be the fibre of p over $x \in X$. Let $\tilde{E_x} := \tilde{\pi_x}^{-1}(F_x)$, then the Seshadri constant of $\mathcal{E}$ at x is defined as:

$$\varepsilon(\mathcal{E}, x) := \sup\{\lambda > 0 \mid \tilde{\xi_x} - \lambda\tilde{E_x} \text{ is } nef\}.$$

Just as in the case of ample line bundle, the question about the Seshadri constant at any point being at least one, is interesting for ample vector bundles as well. In [8] the authors have shown that the Seshadri constant of a very ample vector bundle, at any point x of X, is at least one. They also conjecture that the same result holds under a weaker hypothesis of $\mathcal{E}$ being ample and generated, as in the case of line bundle. While it is true for $\dim(X) = 1$, for $\dim(X) \geqslant 2$ it is however not the case. As in [19], the author gives an example of an ample and generated vector bundle $\mathcal{E}$ on a K3 surface X such that

$$\varepsilon(\mathcal{E}, x) \leqslant \sqrt{\frac{2}{\operatorname{rank}(\mathcal{E})}},$$

for all $x \in X$. The author also states the condition under which $\varepsilon(\mathcal{E}, 1) > 1$ can be expected. Apart from this they also give the bounds on Seshadri constant, in their main theorem, of various other line bundles on X.

1.1.4 Hyperelliptic surfaces

Hyperelliptic surfaces are minimal surfaces of Kodaira dimension 0 and irregularity 1. They are realized as finite group quotients of products of two elliptic curves. These surfaces have been classified and are known to belong to one of the seven different types. They all have Picard rank 2 and the free group $\mathrm{Num}(X)$ of divisors modulo numerical equivalence is well-understood. See [7] for more details.

Definition 1.24. A hyperelliptic surface X is a minimal smooth surface with Kodaira dimension $\kappa(X) = 0$ satisfying $h^1(X, \mathcal{O}_X) = 1$ and $h^2(X, \mathcal{O}_X) = 0$.

Hyperelliptic surfaces are also known as *bielliptic surfaces* (cf. [7, 35]). We recall below some key properties of hyperelliptic surfaces that we use repeatedly. More details can be found in [7, 35]. We follow the notation in [15, 35].

There is an alternate characterization of hyperelliptic surfaces. A smooth surface X is hyperelliptic if and only if $X \cong (A \times B)/G$, where A and B are elliptic curves and G is a finite group of translation of A acting on B in such a way that $B/G \cong \mathbb{P}^1$.

We have the following diagram:

$$\begin{array}{ccc} X \cong (A \times B)/G & \xrightarrow{\Phi} & A/G \\ \Psi \downarrow & & \\ B/G \cong \mathbb{P}^1 & & \end{array}$$

In the above diagram Φ and Ψ are natural projections. The fibres of Φ are all smooth and isomorphic to B. The fibres of Ψ are all multiples of smooth elliptic curves, and all but finitely many of them are smooth and isomorphic to A. Further the singular fibres of Ψ are all multiples of smooth elliptic curves.

Hyperelliptic surfaces were classified more than a hundred years ago by G. Bagnera and M. de Franchis by analyzing the group G and its action on B. They showed that every hyperelliptic surface is of one of the seven types listed in the table below; see [7, V1.20].

Every hyperelliptic surface has Picard rank 2. Serrano [35] has described a basis for the free group Num(X) of divisors modulo numerical equivalence for each of the seven types of hyperelliptic surfaces. For each type, Serrano also lists the multiplicities $m_1, \dots, m_s$ of the singular fibres of Ψ, where s is the number of singular fibres.

Theorem 1.25. *[35, Theorem 1.4]. Let $X \cong (A \times B)/G$ be a hyperelliptic surface. A basis for the group Num(X) of divisors modulo numerical equivalence and the multiplicities of the singular fibres of $\Psi : X \to B/G$ in each type are given in the following table.*

Type of X	G	$m_1, m_2, \dots, m_s$	*Basis of Num*(X)
1	$\mathbb{Z}_2$	$2,2,2,2$	$A/2$, B
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2,2,2,2$	$A/2$, $B/2$
3	$\mathbb{Z}_4$	$2,4,4$	$A/4$, B
4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2,4,4$	$A/4, B/2$
5	$\mathbb{Z}_3$	$3,3,3$	$A/3, B$
6	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$3,3,3$	$A/3, B/3$
7	$\mathbb{Z}_6$	$2,3,6$	$A/6, B$

Let X be a hyperelliptic surface. Let $\mu = \operatorname{lcm}(m_1, m_2, \dots, m_s)$ and let $\gamma = |G|$. By Serrano's theorem, a basis of Num(X) is given by A/μ, $(\mu/\gamma)B$.

Notation: We say that L *is a line bundle of type* (a, b) on X if L is numerically equivalent to $a.A/\mu + b.(\mu/\gamma)B$. If L is of type (a, b), we write $L \equiv (a, b)$.

We note the following properties of line bundles on X.

1. $A^2 = 0, B^2 = 0, A \cdot B = \gamma$.
2. A divisor $b.(\mu/\gamma)B \equiv (0, b)$ is effective if and only if $b(\mu/\gamma) \in \mathbb{N}$ ([1, Proposition 5.2]).
3. A line bundle of type (a, b) is ample if and only if $a > 0$ and $b > 0$ ([35, Lemma 1.3]).
4. If C is an irreducible and reduced curve on X and $x \in C$ is a point of multiplicity m, then $C^2 \geqslant m^2 - m$.

The inequality in (4) follows from the genus formula, and the facts that the canonical divisor is numerically trivial on a hyperelliptic surface and that there are no rational curves on a hyperelliptic surface.

We also use the following important lower bound on self-intersection of a curve C passing through a very general point. See [2, 14, 27, 41], for instance.

Theorem 1.26. *Let X be a hyperelliptic surface and let C be an irreducible and reduced curve on X. Suppose that C passes through a very general point $x \in X$ with multiplicity $m \geqslant 2$. Then $C^2 \geqslant m^2 - m + 2$.*

Let X be a hyperelliptic surface and L be an ample line bundle on X. We first consider the problem of computing $\varepsilon(X, L)$ in subsection 2.1.1. In our main result Theorem 2.3 in this subsection, we show that $\varepsilon(X, L) \in \mathbb{Q}$ provided X is not of type 6. This partially answers [36, Question 1.6], which asks if $\varepsilon(X, L)$ is always rational for any pair (X, L). Some affirmative answers to this question are known ([3, 4, 18, 37, 39]), but it is open in general. In other results in this subsection, we also explicitly compute $\varepsilon(X, L)$ in some cases.

In subsection 2.1.2, we study Seshadri constants of L at a very general point x. One of our main results, Theorem 2.11, says that $\varepsilon(X, L, 1) \geqslant (0.93)\sqrt{L^2}$, or $\varepsilon(X, L, 1)$ is equal to one of two easily computable natural numbers. Let L be of numerical type (a, b). Then depending on how a and b are related to each other, we either explicitly compute $\varepsilon(X, L, 1)$ or show that $\varepsilon(X, L, 1) \geqslant (0.93)\sqrt{L^2}$. We have such a result for each of the seven types of hyperelliptic surfaces. There are several results in the literature giving lower bounds for $\varepsilon(X, L, 1)$ when X is an arbitrary surface and L is any ample line bundle. In Remark 2.19, we compare our bound $(0.93)\sqrt{L^2}$ with some existing bounds and note that it is often better.

1.1.5 Surface of general type

Surface of general type are surfaces of Kodaira dimension 2 (see 3.1). These are the most abundant surfaces in the nature.

In chapter 3, we study Seshadri constants on surfaces of general type. Let X be a surface of general type and K_X be the canonical line bundle on X. Motivated by the Theorem 1 of [6], we consider the problem of studying multi-point Seshadri constant $\varepsilon(X, K_X, x_1, x_2, ..., x_r)$ at any r-points $x_1, x_2, ..., x_r \in X$. In theorem 3.12, we prove the result about multi-point Seshadri constant. Later in this chapter, in section 3.3, we study the Seshadri constants on surfaces of general type of the form $C \times C$, where C is a general member of the moduli of smooth curves of genus g with $g \geqslant 2$. In theorem 3.13, we prove that the global Seshadri constant of an ample line bundle, under some conditions, is a rational number. We then compute single point and double point Seshadri constant of canonical line bundle in theorem 3.15 and in 3.16 respectively. Finally, in 3.18, we prove a similar result like [21, Theorem 3.8] for the canonical line bundle on $C \times C$.

1.2 Aim of the thesis

The thesis is divided into two parts and we describe each part in the following sections. Section 1.2.1 is a joint work with Krishna Hanumanthu [22].

1.2.1 Seshadri constants on hyperelliptic surfaces

We prove the following main theorems.

Theorem 1.27. *Let X be a hyperelliptic surface of type different from 6 and let L be an ample line bundle on X. Then $\varepsilon(X, L)$ is rational.*

This answers the question about the rationality of an ample line bundle on hyperelliptic surfaces except in the case of type 6 hyperelliptic surface, in which case we give a partial result.

Theorem 1.28. *Let X be a hyperelliptic surface of type 6 and let $L \equiv (a, b)$ be an ample line bundle on X such that b is not in the interval $(2a, 9a/2)$. Then $\varepsilon(X, L) \in \mathbb{Q}$.*

Following is our main theorem for $\varepsilon(X, L, 1)$.

Theorem 1.29. *Let X be a hyperelliptic surface and let L be an ample line bundle on X. If $\varepsilon(X, L, 1) < (0.93)\sqrt{L^2}$, then $\varepsilon(X, L, 1) = \min(L \cdot A, L \cdot B)$.*

1.2.2 Seshadri constants on surfaces of general type

In this part, we first consider the canonical line bundle on surface of general type and prove the following theorem for the multi-point Seshadri constant.

Theorem 1.30. *Let X be a surface of general type and K_X be canonical line bundle on X. If K_X is nef and big and $x_1, x_2, ..., x_r \in X$ be $r \geqslant 2$ points then,*

1. $\varepsilon(X, K_X, x_1, x_2, ..., x_r) = 0 \Leftrightarrow$ *at least one of x_i lies on one of the finitely many (-2) curves.*
2. *If $0 < \varepsilon(X, K_X, x_1, x_2, ..., x_r) < \frac{1}{r}$, then*

$$\varepsilon(X, K_X, x_1, x_2, ..., x_r) = \begin{cases} \frac{1}{r+1} \text{ or } \frac{2}{5} & \text{if} \quad r = 2, \\ \frac{1}{r+1} \text{ or } \frac{1}{r+2} & \text{if} \quad 3 \leqslant r < 10, \\ \frac{1}{r+1} \text{ or } \frac{1}{r+2} \text{ or } \frac{1}{r+3} & \text{if} \quad r \geqslant 10. \end{cases}$$

We then consider surface of general type of the form $C \times C$, where C is a general curve of genus at least two. Let F_1 and F_2 be the fibres corresponding to the two projections $C \times C \longrightarrow C$ and let δ be the diagonal of $C \times C$. We prove the following main theorem regarding the rationality of the Seshadri constant of an ample line bundle on X (see 3.3).

Theorem 1.31. *Let $X = C \times C$, where C is a general member of moduli of smooth curves of genus g with $g \geqslant 2$. Let $L \equiv a_1F_1 + a_2F_2 + a_3\delta$ be an ample line bundle satisfying any of the following conditions on a_1, a_2 and a_3.*

1. $a_3 = 0$,

2. $a_3 > 0$, $a_1 \leqslant a_2$ *and* $a_1^2 + a_3^2 < 2a_1a_2$,

3. $a_3 > 0$, $a_2 \leqslant a_1$ *and* $a_2^2 + a_3^2 < 2a_1a_2$,

4. $a_3 < 0$ *and* $a_2 \geqslant \left(\frac{2gk^2+2k+1}{2(k+1)}\right) \cdot a_1$ *where* $k = \lceil \frac{|a_3|/a_1}{1-|a_3|/a_1} \rceil$ *or*

5. $a_3 < 0$ *and* $a_1 \geqslant \left(\frac{2gl^2+2l+1}{2(l+1)}\right) \cdot a_2$ *where* $l = \lceil \frac{|a_3|/a_2}{1-|a_3|/a_2} \rceil$.

Then $\varepsilon(X, L) \in \mathbb{Q}$.

1.2.3 Seshadri constants on blow up of $\mathbb{P}^2$

Finally, we consider the blow up of $\mathbb{P}^2$ and motivated by [21, Theorem 3.8], we prove the following theorem.

Theorem 1.32. *Let* $f : X \longrightarrow \mathbb{P}^2$ *be a blow up of* $\mathbb{P}^2$ *at* r *very general points. Let* $L \equiv aH - \sum_{i=1}^{i=r} b_iE_i$ *be an ample line bundle on* X*, where* H *is the pull back of a general line in* $\mathbb{P}^2$ *and* E_i *denotes the exceptional curve corresponding to* f*. Then for* $r \geqslant \max\{L^2+1, 4\}$*, we have*

$$\varepsilon(X, L, r) \geqslant \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}.$$

1.3 Strategy of proofs

The crucial idea behind the proof of rationality of Seshadri constants is to exhibit a curve $C \subset X$, for which the ratio $\frac{L \cdot C}{\mathrm{mult}_x C} < \sqrt{L^2}$. Then it follows from the following theorem that the Seshadri constant is a rational number.

Theorem 1.33. *Let* X *be a smooth projective surface and* L *be an ample line bundle on* X*. If there exists a point* $x \in X$ *such that* $\varepsilon(X, L, x) < \sqrt{L^2}$*, then* $\varepsilon(X, L) \in \mathbb{Q}$*.*

Proof. Suppose that $\alpha = \varepsilon(X, L)$; Then, by definition of $\varepsilon(X, L)$, there exists a sequence (x_n) of points of X such that $\varepsilon(X, L, x_n) \mapsto \alpha$. Further by invoking the definition of $\varepsilon(X, L, x_n)$ we see that, there are sequence (C_n) of curves in X such that $\frac{L \cdot C_n}{\mathrm{mult}_{x_n} C_n} \mapsto \alpha$.

We will show that the set $\{\mathrm{mult}_{x_n} C_n : n \in \mathbb{N}\}$ is a finite set. This will show that the sequence $\frac{L \cdot C_n}{\mathrm{mult}_{x_n} C_n}$ of rational numbers stabilises, which in turn shows that α is a rational number.

Let D be any divisor on X. By Serre duality $h^2(X, \mathcal{L}(D)) = h^0(X, \mathcal{L}(K_X - D))$. If $h^0(X, \mathcal{L}(K_X - D)) > 0$, $K_X - D$ is effective, hence $L \cdot (K_X - D) > 0$. i.e., $q_0 := L \cdot K_X > L \cdot D$. Thus if we choose D such that $L \cdot D > q_0$, the vanishing of $h^2(X, \mathcal{L}(D))$ can be ensured. Therefore by Reimann-Roch we get $h^0(nL) > (n^2L^2)/2$, for $n > q_0$.

Choose a rational number β such that, $\alpha < \beta < \sqrt{L^2}$ and call $\varepsilon = \varepsilon(X, L, x)$. Since a point of multiplicities m imposes $\frac{m+1}{2}$ conditions, there is curve $D \in |dL|$ with multiplicity m if $d^2L^2 \geqslant m^2 + m$, which can be ensured by assuming $d \geqslant \frac{m}{\sqrt{L^2}} + 1$. Taking m large enough, the ratio d/m can be made arbitrarily close to $1/\sqrt{L^2}$. Hence we can take $d >> q_0$ so that $\frac{L \cdot D}{\text{mult}_x D} < \sqrt{L^2} + \delta$, where $0 < \delta < \frac{L^2 - \sqrt{L^2}\beta}{\beta}$.

Let D_n be a sequence of divisors in $|dL|$ satisfying $\frac{L \cdot D_n}{\text{mult}_{x_n} D_n} < \sqrt{L^2} + \delta$ then C_n must be a component of D_n, else we have by Bezout's theorem:

$$dL \cdot C_n = D_n \cdot C_n \geqslant \text{mult}_{x_n} D_n \cdot \text{mult}_{x_n} C_n > \frac{dL^2}{\sqrt{L^2} + \delta}$$

$$\Rightarrow \frac{L \cdot C_n}{\text{mult}_{x_n} C_n} > \frac{L^2}{\sqrt{L^2} + \delta} > \beta.$$

which is a contradiction. Therefore C_n is a component of D_n which further gives $L \cdot C_n \leqslant L \cdot D_n = dL^2$. Again by Bezout's theorem we obtain $\text{mult}_{x_n} C_n$ is bounded universally. This completes the theorem. □

In the case of hyperelliptic surfaces we show that that the fibers of the two projections ϕ and ψ do that job. In other words, we show that $\frac{L \cdot F_i}{\text{mult}_x F_i} < \sqrt{L^2}$ for some $x \in F_i$, where F_i denotes the fibres.

Proof of theorem 2.11 follows by showing that, if $C \equiv (\alpha, \beta)$ where $\alpha \neq 0, \beta \neq 0$ is a reduced and irreducible curve in X passing through a general point with multiplicity $m \geqslant 1$, then $\frac{L \cdot C}{m} \geqslant (0.93)\sqrt{L^2}$. This is proved by using Xu type lemma [41] and Hodge Index Theorem.

We include the statement and proof of Xu type lemma [41].

Theorem 1.34. *Let X be a smooth complex projective surface. Let T be smooth variety and consider a family $\{x_t \in E_t\}_{t \in T}$ consisting of a curve $E_t \in X$ through a very general point*

$x_t \in X$ *such that the* $\mathrm{mult}_{x_t} E_t > m$ *for any* $t \in T$ *and for any* $m \geqslant 2$.
If the central fibre E_0 *is reduced and irreducible and the family is non-trivial then,*

$$E_0^2 \geqslant m(m-1) + \mathrm{gon}(\tilde{E}_0).$$

Proof. Consider the blowing-up $f : X' \mapsto X$ of X at the point $x_0 \in X$ where F denotes the exceptional divisor.Let E_0' be the strict transform of E_0 so that $E_0' = f*(E_0) - kF$ where $k = \mathrm{multi}_{x_0} E_0 \geqslant m$.

Since each x_t is a point of multiplicity at least 2, the variety T is of dimension at least two. Upon choosing a subfamily of $\{x_t \in E_t\}_{t\in T}$ we may assume that the dimension of T is two. Let $(t_1, t_2) \in \mathbb{C}^2$ denotes the local coordinate near the point $t = 0 \in T$.

Consider the image of $\frac{d}{dt_1}$ and $\frac{d}{dt_2}$ under the *kodaira-spencer map*

$$\rho : T_0 T \mapsto H^1(E_0, \mathcal{T}_{E_0})$$

Write $s_1 = \rho\left(\frac{d}{dt_1}\right)$ and $s_2 = \rho\left(\frac{d}{dt_2}\right) \in H^0(E_0, \mathcal{O}_{E_0}(E_0))$. these sections induces two non-zero sections $s_1', s_2' \in H^0(E_0, f*(\mathcal{O}_{E_0}(E_0)) \otimes \mathcal{O}_{X'}((1-m)F|_{E_0'})$.

Using these two we define a map $\phi : E_0' \to \mathbb{P}^1$ which can be extended to map $\tilde{\phi} : \tilde{E}_0 \to \mathbb{P}^1$, hence

$$E_0^2 = \deg \mathcal{O}_{E_0}(E_0) = \deg f*(\mathcal{O}_{E_0}(E_0))|_{E_0'} \geqslant m(m-1)(F \cdot E_0') + \deg\phi \geqslant m(m-1) + \mathrm{gon}(\tilde{E}_0).$$

□

1.4 Organization of the thesis

This thesis is divided into two chapters. In chapter 2, we give detailed proofs of the results about Seshadri constants on hyperelliptic surfaces. While in chapter 3, we discuss the multi-point Seshadri constant on surfaces of general type and some results about Seshadri constants on surfaces of general type X of the form $C \times C$, where C is a general member of the moduli of smooth curves of genus $g \geqslant 2$.

Chapter 2

Seshadri constants on hyperelliptic surfaces

In this chapter, we address some of the problems stated in **Q1**, **Q2** and **Q3** of Chapter 1, in the case of hyperelliptic surfaces. Our primary motivation is [15], where several results on Seshadri constants on hyperelliptic surfaces are proved. The results in this chapter are published in the Proc. of AMS [22].

When the surface X is clear from the context, we denote Seshadri constants simply by $\varepsilon(L, x), \varepsilon(L)$, or $\varepsilon(L, 1)$.

2.1 Seshadri constants

In this section we first consider $\varepsilon(L)$ and then prove our results on $\varepsilon(L, 1)$.

2.1.1 Results about $\varepsilon(L)$.

Theorem 2.1. *Let X be a hyperelliptic surface of odd type (i.e., of type $1, 3, 5$, or 7). Let $L \equiv (a, b)$ be an ample line bundle on X. Then $\varepsilon(L) = \min\{a, b\}$.*

Proof. We first prove that $\varepsilon(L, x) \geqslant \min\{a, b\}$ for any $x \in X$. We then show that equality holds for a suitable x.

Note that since X is a hyperelliptic surface of odd type, $\mu = \gamma$. Hence B is given by $(0, 1)$ in $\mathrm{Num}(X)$. On the other hand, A is given by $(2, 0), (4, 0), (3, 0)$ and $(6, 0)$ in types $1, 3, 5$ and 7, respectively.

Since the fibres of $\Phi : X \to A/G$ cover X, are smooth and are isomorphic to B, there is a smooth curve which is numerically equivalent to $(0, 1)$ that contains any given point x. Similarly, the fibres of $\Psi : X \to B/G$ cover X, but they are not all smooth. The smooth fibres of Ψ are isomorphic to A and singular fibres are multiples of smooth fibres. The number of singular fibres and their multiplicities are completely determined by the type of X. See the table in Theorem 1.25.

Now let $x \in X$ be an arbitrary point. Let C be a reduced and irreducible curve on X passing through x with multiplicity $m \geqslant 1$. We consider three possibilities for C. First, it is a fibre of Φ; second, it is a fibre of Ψ; and third, it is different from the fibres of Φ and Ψ.

If C is a fibre of Φ, then C is smooth and is isomorphic to B and is numerically equivalent to $(0, 1)$. In this case, $m = 1$. So the Seshadri ratio is $L \cdot C = a$.

If C is a fibre of Ψ, then C is not necessarily smooth. Numerically, C is given by $(\mu, 0)$. The multiplicity m is determined by the table in Theorem 1.25. For instance, if X has type 1, then $m = 1$, or 2. Or, if X has type 3, then $m = 1, 2$, or 4. In this case, the Seshadri ratio is $\frac{L \cdot C}{m} = \frac{\mu b}{m}$.

Now let C be different from the fibres of Ψ and Φ. Let C be represented by (α, β) in $\mathrm{Num}(X)$. We use Bezout's theorem to bound the values of α and β. Since x is a point of a smooth fibre $(0, 1)$, we have $C \cdot (0, 1) = \alpha \geqslant m$. On the other hand, the fibre of Ψ containing x may not be smooth. In this case, Bezout's theorem gives $C \cdot (\mu, 0) = \mu\beta \geqslant mn$, where n is the multiplicity of the the fibre of Ψ containing x. Thus we have $\frac{L \cdot C}{m} = \frac{a\beta + b\alpha}{m} \geqslant b + \frac{an}{\mu}$.

Since $\mu \geqslant m$ and $n \geqslant 1$, we conclude that the Seshadri ratio $\frac{L \cdot C}{m} \geqslant \min(a, b, b + \frac{a}{\mu})$ for any reduced irreducible curve C passing through x. Hence $\varepsilon(L, x) \geqslant \min(a, b, b + \frac{a}{\mu}) \geqslant \min(a, b)$.

Now let x be a point on a singular fibre of Ψ with the maximum possible multiplicity. For instance, if X has type 7, x is any point on a fibre of Ψ of multiplicity 6. Then, in the notation above, $m = n = \mu$. So $\varepsilon(L, x) = \min(a, b, a + b) = \min(a, b)$. This completes the proof of the theorem. □

Remark 2.2. The result in Theorem 2.1 is proved for hyperelliptic surfaces of type 1 in [15, Theorem 3.4] and our proof essentially follows from the arguments given by Farnik. In fact, Farnik gives a precise value for $\varepsilon(L, x)$ for any x and any ample line bundle L on a hyperelliptic surface of type 1.

Our next result partially answers [36, Question 1.6] for hyperelliptic surfaces. This question asks if $\varepsilon(X, L)$ is rational for any surface X and any ample line bundle L on X. So far an affirmative answer to this question has been found in some cases.

The case of quartic surfaces $X \subset \mathbb{P}^3$ and $L = \mathcal{O}_X(1)$ is considered in [3, Theoerem]. It is proved that $\varepsilon(X, L) = 1, 4/3$ or 2, depending on certain geometric properties of X. In [4, Theorem A.1], it is proved that $\varepsilon(X, L)$ is rational if X is an abelian surface and L is any ample line bundle on X. The same result is shown for Enriques surfaces in [39, Theorem 3.3]. Finally, [18, 37] study minimal ruled surfaces. Such surfaces are geometrically ruled over a smooth curve C and one attaches a certain invariant $e \in \mathbb{Z}$ to them. If $e \geqslant 0$, then [37, Theorem 3.27] and [18, Theorem 4.14] show that $\varepsilon(X, L) \in \mathbb{Q}$ for any ample line bundle L on X.

Theorem 2.3. *Let X be a hyperelliptic surface of type different from 6 and let L be an ample line bundle on X. Then $\varepsilon(L)$ is rational.*

Proof. Let X be a hyperelliptic surface of type different from 6 and let $L \equiv (a, b)$ be an ample line bundle on X. If X has odd type then the assertion follows from Theorem 2.1.

X is of type 2: If $2a = b$, then $L^2 = 2ab$ is a perfect square and $\varepsilon(L)$ is a rational number (for instance, see [36, Corollary 1.8]).

If $b < 2a$, let x be a point on a singular fibre of Ψ. This fibre is numerically equivalent to $(2, 0)$ and x is a point of multiplicity 2 on it. So $\varepsilon(L, x) \leqslant \frac{(a,b)\cdot(2,0)}{2} = b < \sqrt{2ab} = \sqrt{L^2}$. It follows by [36, Lemma 1.7] that $\varepsilon(L) \in \mathbb{Q}$. On the other hand, if $b > 2a$, then let $x \in X$ be any point. Then x belongs to a fibre of Φ. Note that all the fibres of Φ are smooth and are numerically equivalent to $(0, 2)$. So $\varepsilon(L, x) \leqslant \frac{(a,b)\cdot(0,2)}{1} = 2a < \sqrt{L^2}$. Again it follows that $\varepsilon(L) \in \mathbb{Q}$. Note that in fact $\varepsilon(L, 1) \leqslant 2a < \sqrt{L^2}$, if $b > 2a$.

X is of type 4: As in the above case, if $2a = b$, then $\varepsilon(L) \in \mathbb{Q}$.

If $b < 2a$, let x be a point on a fibre of Ψ of multiplicity 4. Then $\varepsilon(L, x) \leqslant b < \sqrt{L^2}$. On the other hand, let $b > 2a$ and let x be any point. Consider the fibre of Φ

containing x. This fibre is smooth and numerically equivalent to $(0,2)$. Again as before, $\varepsilon(L,x) \leqslant 2a < \sqrt{L^2}$. $\square$

We have the following result for type 6 hyperelliptic surfaces.

Theorem 2.4. *Let X be a hyperelliptic surface of type 6 and let $L \equiv (a,b)$ be an ample line bundle on X such that b is not in the interval $(2a, 9a/2)$. Then $\varepsilon(L) \in \mathbb{Q}$.*

Proof. If $b = 2a$ or $b = 9a/2$, then $L^2 = 2ab$ is a square and $\varepsilon(L) \in \mathbb{Q}$. So we assume that either $b < 2a$ or $b > 9a/2$.

If $b < 2a$, choose a point x on a singular fibre of Ψ. Then the fibre is represented numerically by $(3,0)$ and the multiplicity of the fibre at x is 3. So $\varepsilon(L,x) \leqslant \frac{(a,b)\cdot(3,0)}{3} = b < \sqrt{2ab}$. If $b > 9a/2$, then choose any point x and consider a fibre of Φ containing it. We have $\varepsilon(L,x) \leqslant \frac{(a,b)\cdot(0,3)}{1} = 3a < \sqrt{2ab}$. Thus $\varepsilon(L) \in \mathbb{Q}$. $\square$

Discussion 2.5. Let X be any surface, and let L be ample on X. If $x \in X$, then an easy upper bound for $\varepsilon(L,x)$ is given by $\frac{L\cdot C}{\mathrm{mult}_x C}$, where C is a curve containing x, *provided* this ratio is smaller than $\sqrt{L^2}$.

Of course, there are no such curves if $\varepsilon(X,L) = \sqrt{L^2}$. We note below why there are no obvious examples of such curves if X is a hyperelliptic surface of type 6 and $L \equiv (a,b)$ with $b \in (2a, 9a/2)$.

According to Theorems 2.3 and 2.4, if X is a hyperelliptic surface of type different from 6, or if X has type 6, but $L \equiv (a,b)$ with $b \notin (2a, 9a/2)$, then $\frac{L\cdot C}{\mathrm{mult}_x C} < \sqrt{L^2}$, for a suitable x and a suitable fibre C of Ψ or Φ. This in turn allows us to conclude $\varepsilon(X,L) \in \mathbb{Q}$ in these cases. It is also clear from the proof of Theorem 2.4 that if X has type 6 and $L \equiv (a,b)$ with $b \in (2a, 9a/2)$, then $\frac{L\cdot C}{\mathrm{mult}_x C} \geqslant \sqrt{L^2}$, for *any* fibre C of Ψ or Φ.

In general, for a surface X and an ample line bundle L on X, in order to conclude that $\varepsilon(X,L) \in \mathbb{Q}$, we must establish the existence of a suitable pair $x \in C$ for which $\frac{L\cdot C}{\mathrm{mult}_x C} < \sqrt{L^2}$. If X is a hyperelliptic surface of type 6 and $L \equiv (a,b)$ with $b \in (2a, 9a/2)$, there are no obvious candidates for such a pair. One needs more specific information about singular curves on such a surface.

We do however give a lower bound for $\varepsilon(L,x)$ for any x in the following proposition.

Proposition 2.6. *Let X be a hyperelliptic surface of type 6 and let $L \equiv (a,b)$ be an ample line bundle with $b \in (2a, 9a/2)$. Then $\varepsilon(L,x) \geqslant (0.7)\sqrt{L^2}$ for all $x \in X$.*

Proof. If $\varepsilon(L,x) < (0.7)\sqrt{L^2}$ for some $x \in X$, then $\varepsilon(L,x) = \frac{L \cdot C}{\text{mult}_x C}$ for an irreducible and reduced curve $C \equiv (\alpha, \beta)$ containing x. Let $m = \text{mult}_x C$. If $m = 1$, then $L \cdot C < \sqrt{L^2}$. Then the Hodge Index Theorem gives $L^2.C^2 \leqslant (L \cdot C)^2 < L^2$. So $C^2 = 2\alpha\beta < 1$. Thus $\alpha = 0$ or $\beta = 0$. Then C is a fibre of Φ or Ψ. But this is not possible, as mentioned in Discussion 2.5.

So assume $m \geqslant 2$. We know $C^2 \geqslant m^2 - m$ (see Section 1.1.4). Applying the Hodge Index Theorem again, we get $m^2 - m < (0.7)^2 m^2$, which gives $(0.51)m^2 - m < 0$. But this is not possible for $m \geqslant 2$. □

We use the idea in the above proof again in Theorem 2.9.

Proposition 2.7. *Let X be a hyperelliptic surface of even type. Let $L \equiv (a,b)$ be an ample line bundle on X satisfying the following:*

1. $b \leqslant a$ *if X is of type 2;*
2. $2b \leqslant a$ *if X is of type 4 or 6.*

Then $\varepsilon(L) = b$.

Proof. First let X be of type 2. If x is a point on a singular fibre of Ψ, then as in the proof of Theorem 2.3, $\varepsilon(L,x) \leqslant b$.

Now let $x \in X$ be an arbitrary point. Then x is in a fibre of Φ which is represented by $(0,2)$. The Seshadri ratio for this fibre is $\frac{L\cdot(0,2)}{1} = 2a \geqslant b$. The Seshadri ratio for any fibre of Ψ containing x is at least b. On the other hand, let $C \equiv (\alpha, \beta)$ be an irreducible and reduced curve, different from the fibres of Ψ or Φ, passing through x with multiplicity $m \geqslant 1$. Then (as in the proof of Theorem 2.1) Bezout's theorem gives $2\alpha \geqslant m$ and $2\beta \geqslant m$. So $\frac{L \cdot C}{m} = \frac{a\beta + b\alpha}{m} \geqslant \frac{a+b}{2} \geqslant b$. In other words, $\varepsilon(X,L,x) = \inf \frac{L \cdot C}{\text{mult}_x C} \geqslant b$.

Thus $\varepsilon(L,x) \geqslant b$ for all $x \in X$ and $\varepsilon(L,x) \leqslant b$ if x is on a singular fibre of Ψ. It follows that $\varepsilon(L) = b$.

Now let X be of type 4 or 6. As in the above case, if x is on a singular fibre of Ψ, then $\varepsilon(L,x) \leqslant b$ (when X is of type 4, we take the point x on a fibre of multiplicity 4).

Now let $x \in X$ be arbitrary and let $C \equiv (\alpha, \beta)$ be an irreducible and reduced curve, different from the fibres of Ψ or Φ, passing through x with multiplicity $m \geqslant 1$. Then we have $4\beta \geqslant m$ and $2\alpha \geqslant m$ when X is of type 4 and $3\beta \geqslant m$ and $3\alpha \geqslant m$ when X is of type 6. In either case, $\frac{L \cdot C}{m} = \frac{a\beta + b\alpha}{m} \geqslant b$. As above, we conclude that $\varepsilon(L) = b$. □

Proposition 2.8. *Let X be a hyperelliptic surface of even type. Let $L \equiv (a, b)$ be an ample line bundle on X. Then the following statements hold:*

1. *Let X be of type 2. If $b \geqslant 3a$, then $\varepsilon(L, x) = 2a$ for all $x \in X$.*
2. *Let X be of type 4. If $b \geqslant 7a/2$, then $\varepsilon(L, x) = 2a$ for all $x \in X$.*
3. *Let X be of type 6. If $b \geqslant 8a$, then $\varepsilon(L, x) = 3a$ for all $x \in X$.*

Proof. The proof is similar to the proof of Proposition 2.7, so we will only give a brief sketch.

First let X be of type 2. Let $x \in X$ be any point. Since a fibre of Φ contains x, we have $\varepsilon(L, x) \leqslant \frac{L \cdot (0,2)}{1} = 2a$. If x is on a singular fibre of Ψ, then the corresponding Seshadri ratio is $\frac{L \cdot (2,0)}{2} = b \geqslant 2a$. If x is on a smooth fibre of Ψ, then the corresponding Seshadri ratio is $\frac{L \cdot (2,0)}{1} = 2b \geqslant 2a$.

Now let $C \equiv (\alpha, \beta)$ be an irreducible and reduced curve, different from the fibres of Ψ or Φ, passing through x with multiplicity $m \geqslant 1$. Bezout's theorem gives $2\alpha \geqslant m$ and $2\beta \geqslant m$. So $\frac{L \cdot C}{m} = \frac{a\beta + b\alpha}{m} \geqslant \frac{a+b}{2} \geqslant 2a$, by hypothesis. So we conclude that $\varepsilon(L, x) = 2a$ for all $x \in X$.

The proof for types 4 and 6 is similar. □

2.1.2 Results about $\varepsilon(L, 1)$.

Theorem 2.9. *Let X be a hyperelliptic surface and let L be an ample line bundle on X. Suppose that $C \equiv (\alpha, \beta)$ is an irreducible, reduced curve with $\alpha \neq 0$, $\beta \neq 0$ and which passes through a very general point with multiplicity $m \geqslant 1$. Then $\frac{L \cdot C}{m} \geqslant (0.93)\sqrt{L^2}$.*

Proof. First, let $m = 1$. If $L \cdot C < (0.93)\sqrt{L^2}$, then the Hodge Index Theorem gives $C^2 < (0.93)^2$. So $C^2 = 2\alpha\beta = 0$, which violates the hypothesis on C.

So assume $m \geqslant 2$. Then we have $C^2 \geqslant m^2 - m + 2$, by Theorem 1.26. Again, the Hodge Index Theorem gives $m^2 - m + 2 < (0.93)^2 m^2$. So m satisfies the quadratic relation $(0.13)m^2 - m + 2 < 0$. But it is easy to see that the quadratic expression $(0.13)m^2 - m + 2$ is always positive, since it grows as m goes to ∞ or $-\infty$ and its discriminant is $1 - 8(0.13) = -0.04 < 0$. □

Remark 2.10. In the above proof, we used the fact that the quadratic $(1-\delta^2)m^2 - m + 2$ is positive for all $m \geqslant 1$, where $\delta = 0.93$. In order to get a better bound in Theorem 2.9, we have to increase δ. But this forces the above quadratic to become negative for some m. For instance, if $\delta = 0.94$, then the quadratic $(1 - 0.94^2)m^2 - m + 2$ is negative for $m = 4, 5$. Similarly, for $\delta = 0.99$, the quadratic $(1 - 0.99^2)m^2 - m + 2$ is negative for $2 \leqslant m \leqslant 48$. As δ approaches 1, the set $\{m \mid (1-\delta^2)m^2 - m + 2 < 0\}$ keeps increasing.

As δ approaches 1, more precise information about $L \cdot C$ for curves passing through very general points will be required to prove the inequality $\frac{L \cdot C}{m} \geqslant \delta\sqrt{L^2}$. This may be possible to do for specific line bundles L.

As a corollary to Theorem 2.9, we obtain our main theorem about $\varepsilon(L, 1)$ for ample line bundles on hyperelliptic surfaces.

Theorem 2.11. *Let X be a hyperelliptic surface and let L be an ample line bundle on X. If $\varepsilon(L,1) < (0.93)\sqrt{L^2}$, then $\varepsilon(L, 1) = \min(L \cdot A, L \cdot B)$.*

Proof. If $\varepsilon(L, 1) \geqslant (0.93)\sqrt{L^2}$, then there is nothing to prove. Otherwise, we have $\varepsilon(L, 1) = \frac{L \cdot C}{m}$, where C is a reduced and irreducible curve passing through a very general point with multiplicity m. Let $C \equiv (\alpha, \beta)$. By Theorem 2.9, either $\alpha = 0$ or $\beta = 0$. In other words, C is a fibre of Φ or Ψ. Since x is a very general point, we may assume that it does not lie on any of the finitely many singular fibres of Ψ. Thus C is smooth and isomorphic to B or A. Hence $\varepsilon(L, 1) = \min(L \cdot A, L \cdot B)$. □

We next consider different types of hyperelliptic surfaces and prove specific results about $\varepsilon(L, 1)$.

Theorem 2.12. *Let X be a hyperelliptic surface of type 1. Let $L \equiv (a, b)$ be an ample line bundle on X. Then $\varepsilon(L, 1) = \min(a, 2b)$.*

Proof. We repeat the proof that is already essentially given in [15, Theorem 3.4]. This proof illustrates the special property of type 1 hyperelliptic surfaces in the sense that the Seshadri constants are always computed by the fibres of Φ or Ψ.

Note that when X has type 1, a fibre B of Φ is given by $(0,1)$ and a smooth fibre A of Ψ is given by $(2,0)$. So $L \cdot A = 2b$ and $L \cdot B = a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L,1) = \varepsilon(L,x) \leqslant \min(a, 2b)$.

Now if $C \equiv (\alpha, \beta)$ is a curve with $\alpha\beta \neq 0$ and it passes through a very general point with multiplicity m, then Bezout's theorem gives $\alpha \geqslant m$ and $\beta \geqslant m/2$. Thus $\frac{L \cdot C}{m} = \frac{a\beta + b\alpha}{m} \geqslant \frac{a}{2} + b \geqslant \min(a, 2b)$. It follows that $\varepsilon(L,1) = \min(a, 2b)$. □

Theorem 2.13. *Let X be a hyperelliptic surface of type 2. Let $L \equiv (a,b)$ be an ample line bundle on X. Then the following statements hold:*

1. *If* $\frac{2\min(a,b)}{(0.93)^2} \leqslant \max(a,b)$, *then* $\varepsilon(L,1) = 2\min(a,b)$.
2. *If* $\frac{2\min(a,b)}{(0.93)^2} \geqslant \max(a,b)$, *then* $\varepsilon(L,1) \geqslant (0.93)\sqrt{L^2}$.

Proof. Note that when X has type 2, a fibre B of Φ is given by $(0,2)$ and a smooth fibre A of Ψ is given by $(2,0)$. So $L \cdot A = 2b$ and $L \cdot B = 2a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L,1) = \varepsilon(L,x) \leqslant \min(2a, 2b)$. Also, by Theorem 2.11, $\varepsilon(L,1) \geqslant (0.93)\sqrt{L^2}$ or $\varepsilon(L,1) = \min(2a,2b)$. Note that $L^2 = 2ab$.

We have

$$\begin{aligned} \frac{2\min(a,b)}{(0.93)^2} &\leqslant \max(a,b) \\ \Leftrightarrow 4(\min(a,b))^2 &\leqslant (0.93)^2(2ab) \\ \Leftrightarrow 2\min(a,b) &\leqslant (0.93)\sqrt{2ab} \\ \Rightarrow \varepsilon(L,1) &= \min(2a,2b). \end{aligned}$$

On the other hand,

$$\frac{2\min(a,b)}{(0.93)^2} \geqslant \max(a,b) \Leftrightarrow 2\min(a,b) \geqslant (0.93)\sqrt{2ab} \Rightarrow \varepsilon(L,1) \geqslant (0.93)\sqrt{2ab}.$$ □

Theorem 2.14. *Let X be a hyperelliptic surface of type 3. Let $L \equiv (a,b)$ be an ample line bundle on X. Then the following statements hold:*

1. *If* $b \leqslant \frac{a(0.93)^2}{8}$, *then* $\varepsilon(L,1) = 4b$.
2. *If* $\frac{a(0.93)^2}{8} \leqslant b \leqslant \frac{a}{2(0.93)^2}$, *then* $\varepsilon(L,1) \geqslant (0.93)\sqrt{L^2}$.

3. *If* $b \geqslant \frac{a}{2(0.93)^2}$, *then* $\varepsilon(L, 1) = a$.

Proof. Note that when X has type 3, a fibre B of Φ is given by $(0, 1)$ and a smooth fibre A of Ψ is given by $(4, 0)$. So $L \cdot A = 4b$ and $L \cdot B = a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L, 1) = \varepsilon(L, x) \leqslant \min(a, 4b)$. Also, by Theorem 2.11, $\varepsilon(L, 1) \geqslant (0.93)\sqrt{L^2}$ or $\varepsilon(L, 1) = \min(a, 4b)$.

If $b \leqslant \frac{a(0.93)^2}{8}$, then clearly $4b \leqslant a$. Further $b \leqslant \frac{a(0.93)^2}{8} \Leftrightarrow 4b \leqslant (0.93)\sqrt{2ab}$. So $\varepsilon(L, 1) = 4b$.

If $b \geqslant \frac{a}{2(0.93)^2}$, then clearly $a \leqslant 4b$. Further $b \geqslant \frac{a}{2(0.93)^2} \Leftrightarrow a \leqslant (0.93)\sqrt{2ab}$. So $\varepsilon(L, 1) = a$.

Finally, if $\frac{a(0.93)^2}{8} \leqslant b \leqslant \frac{a}{2(0.93)^2}$, then $a \geqslant (0.93)\sqrt{2ab}$ and $4b \geqslant (0.93)\sqrt{2ab}$. So $\varepsilon(L, 1) \geqslant (0.93)\sqrt{2ab}$. □

Theorem 2.15. *Let* X *be a hyperelliptic surface of type* 4. *Let* $L \equiv (a, b)$ *be an ample line bundle on* X. *Then the following statements hold:*

1. *If* $b \leqslant \frac{a(0.93)^2}{8}$, *then* $\varepsilon(L, 1) = 4b$.
2. *If* $\frac{a(0.93)^2}{8} \leqslant b \leqslant \frac{2a}{(0.93)^2}$, *then* $\varepsilon(L, 1) \geqslant (0.93)\sqrt{L^2}$.
3. *If* $b \geqslant \frac{2a}{(0.93)^2}$, *then* $\varepsilon(L, 1) = 2a$.

Proof. Note that when X has type 4, a fibre B of Φ is given by $(0, 2)$ and a smooth fibre A of Ψ is given by $(4, 0)$. So $L \cdot A = 4b$ and $L \cdot B = 2a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L, 1) = \varepsilon(L, x) \leqslant \min(2a, 4b)$. Also, by Theorem 2.11, $\varepsilon(L, 1) \geqslant (0.93)\sqrt{L^2}$ or $\varepsilon(L, 1) = \min(2a, 4b)$.

If $b \leqslant \frac{a(0.93)^2}{8}$, then clearly $4b \leqslant 2a$. Further $b \leqslant \frac{a(0.93)^2}{8} \Leftrightarrow 4b \leqslant (0.93)\sqrt{2ab}$. So $\varepsilon(L, 1) = 4b$.

If $b \geqslant \frac{2a}{(0.93)^2}$, then clearly $2a \leqslant 4b$. Further $b \geqslant \frac{2a}{(0.93)^2} \Leftrightarrow 2a \leqslant (0.93)\sqrt{2ab}$. So $\varepsilon(L, 1) = 2a$.

Finally, if $\frac{a(0.93)^2}{8} \leqslant b \leqslant \frac{2a}{(0.93)^2}$, then $2a \geqslant (0.93)\sqrt{2ab}$ and $4b \geqslant (0.93)\sqrt{2ab}$. So $\varepsilon(L, 1) \geqslant (0.93)\sqrt{2ab}$. □

Theorem 2.16. *Let* X *be a hyperelliptic surface of type* 5. *Let* $L \equiv (a, b)$ *be an ample line bundle on* X. *Then the following statements hold:*

1. *If* $b \leqslant \frac{2a(0.93)^2}{9}$, *then* $\varepsilon(L,1) = 3b$.

2. *If* $\frac{2a(0.93)^2}{9} \leqslant b \leqslant \frac{a}{2(0.93)^2}$, *then* $\varepsilon(L,1) \geqslant (0.93)\sqrt{L^2}$.

3. *If* $b \geqslant \frac{a}{2(0.93)^2}$, *then* $\varepsilon(L,1) = a$.

Proof. Note that when X has type 5, a fibre B of Φ is given by $(0,1)$ and a smooth fibre A of Ψ is given by $(3,0)$. So $L \cdot A = 3b$ and $L \cdot B = a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L,1) = \varepsilon(L,x) \leqslant \min(a,3b)$. Also, by Theorem 2.11, $\varepsilon(L,1) \geqslant (0.93)\sqrt{L^2}$ or $\varepsilon(L,1) = \min(a,3b)$.

If $b \leqslant \frac{2a(0.93)^2}{9}$, then clearly $3b \leqslant a$. Further $b \leqslant \frac{2a(0.93)^2}{9} \Leftrightarrow 3b \leqslant (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) = 3b$.

If $b \geqslant \frac{a}{2(0.93)^2}$, then clearly $a \leqslant 3b$. Further $b \geqslant \frac{a}{2(0.93)^2} \Leftrightarrow a \leqslant (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) = a$.

Finally, if $\frac{2a(0.93)^2}{9} \leqslant b \leqslant \frac{a}{2(0.93)^2}$, then $a \geqslant (0.93)\sqrt{2ab}$ and $3b \geqslant (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) \geqslant (0.93)\sqrt{2ab}$. ∎

Theorem 2.17. *Let X be a hyperelliptic surface of type 6. Let $L \equiv (a,b)$ be an ample line bundle on X. Then the following statements hold:*

1. *If* $\frac{9\min(a,b)}{2(0.93)^2} \leqslant \max(a,b)$, *then* $\varepsilon(L,1) = 3\min(a,b)$.

2. *If* $\frac{9\min(a,b)}{2(0.93)^2} \geqslant \max(a,b)$, *then* $\varepsilon(L,1) \geqslant (0.93)\sqrt{L^2}$.

Proof. Note that when X has type 6, a fibre B of Φ is given by $(0,3)$ and a smooth fibre A of Ψ is given by $(3,0)$. So $L \cdot A = 3b$ and $L \cdot B = 3a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L,1) = \varepsilon(L,x) \leqslant \min(3a,3b)$. Also, by Theorem 2.11, $\varepsilon(L,1) \geqslant (0.93)\sqrt{L^2}$ or $\varepsilon(L,1) = \min(3a,3b)$.

We have

$$\begin{aligned} \frac{9\min(a,b)}{2(0.93)^2} &\leqslant \max(a,b) \\ \Leftrightarrow 9(\min(a,b))^2 &\leqslant (0.93)^2(2ab) \\ \Leftrightarrow 3\min(a,b) &\leqslant (0.93)\sqrt{2ab} \\ \Rightarrow \varepsilon(L,1) &= \min(3a,3b). \end{aligned}$$

On the other hand,

$$\frac{9\min(a,b)}{2(0.93)^2} \geqslant \max(a,b) \Leftrightarrow 3\min(a,b) \geqslant (0.93)\sqrt{2ab} \Rightarrow \varepsilon(L,1) \geqslant (0.93)\sqrt{2ab}.$$ □

Theorem 2.18. *Let* X *be a hyperelliptic surface of type 7. Let* $L \equiv (a,b)$ *be an ample line bundle on* X*. Then the following statements hold:*

1. *If* $b \leqslant \frac{a(0.93)^2}{18}$*, then* $\varepsilon(L,1) = 6b$.
2. *If* $\frac{a(0.93)^2}{18} \leqslant b \leqslant \frac{a}{2(0.93)^2}$*, then* $\varepsilon(L,1) \geqslant (0.93)\sqrt{L^2}$.
3. *If* $b \geqslant \frac{a}{2(0.93)^2}$*, then* $\varepsilon(L,1) = a$.

Proof. Note that when X has type 7, a fibre B of Φ is given by $(0,1)$ and a smooth fibre A of Ψ is given by $(6,0)$. So $L \cdot A = 6b$ and $L \cdot B = a$. Since a very general point $x \in X$ always belongs to a fibre B and a smooth fibre A, we have $\varepsilon(L,1) = \varepsilon(L,x) \leqslant \min(a,6b)$. Also, by Theorem 2.11, $\varepsilon(L,1) \geqslant (0.93)\sqrt{L^2}$ or $\varepsilon(L,1) = \min(a,6b)$.

If $b \leqslant \frac{a(0.93)^2}{18}$, then clearly $6b \leqslant a$. Further $b \leqslant \frac{a(0.93)^2}{18} \Leftrightarrow 6b \leqslant (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) = 6b$.

If $b \geqslant \frac{a}{2(0.93)^2}$, then clearly $a \leqslant 6b$. Further $b \geqslant \frac{a}{2(0.93)^2} \Leftrightarrow a \leqslant (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) = a$.

Finally, if $\frac{a(0.93)^2}{18} \leqslant b \leqslant \frac{a}{2(0.93)^2}$, then $a \geqslant (0.93)\sqrt{2ab}$ and $6b \geqslant (0.93)\sqrt{2ab}$. So $\varepsilon(L,1) \geqslant (0.93)\sqrt{2ab}$. □

Remark 2.19. We compare the result in Theorem 2.11 with some bounds in the literature. There has been a lot of interest in finding good lower bound for $\varepsilon(L,1)$. See, for instance, [17, 25, 32, 38, 40].

Let X be any surface and let L be an ample line bundle on X. It is known that $\varepsilon(X,L,1) \geqslant \sqrt{\frac{7}{9}}\sqrt{L^2}$, or X is fibred by Seshadri curves, or X is a cubic surface in $\mathbb{P}^3$ and $L = \mathcal{O}_X(1)$; see [38, Corollary 3.3]. Since $\sqrt{\frac{7}{9}}$ is approximately 0.88, the bound we give in Theorem 2.11 is better.

Another recent result in this direction is contained in [17]. Let $d := L^2$ and suppose that d is not a square. Then the equation $y^2 - dx^2 = 1$ is known as *Pell's equation*. If $x = p, y = q$ is a solution to this equation, then [17, Theorem 1.3] shows that $\varepsilon(L,1) \geqslant \frac{p}{q}d$ or $\varepsilon(X,L,1)$ is contained in a finite set $\mathrm{Exc}(d;p,q)$ of rational numbers which are easy to list. Though this bound is often better than $(0.93)\sqrt{L^2}$, the set $\mathrm{Exc}(d;p,q)$ is typically large.

As an example, let X be a hyperelliptic surface of type 6 and let $L \equiv (5,11)$. Then $d = L^2 = 110$ and $\sqrt{L^2} \sim 10.49$. By Theorem 2.17, $\varepsilon(X,L,1) \geqslant (0.93)\sqrt{110} \sim 9.75$. On the other hand, $(2,21)$ is a solution to Pell's equation $y^2 - 110x^2 = 1$. So by [17, Theorem 1.3], $\varepsilon(X,L,1) \geqslant \frac{220}{21} \sim 10.48$, or $\varepsilon(X,L,1) \in \mathrm{Exc}(110;2,21)$. Though 10.48 is a much better approximation to $\sqrt{L^2}$ compared to our 9.75, the exceptional set $\mathrm{Exc}(110;2,21)$ is large and it is not easy in general to lower the number of possibilities. In this case, $\mathrm{Exc}(110;2,21) = \{1,2,\ldots,10\} \cup \{\frac{r}{s} \mid 1 \leqslant \frac{r}{s} < \frac{220}{21}$ and $2 \leqslant s < 21^2 = 441\}$.

We also note that our results give precise values of $\varepsilon(X,L,1)$ in many cases. For example, if X is hyperelliptic of type 6 and $L \equiv (5,b)$ and $b \geqslant 27$, then $\varepsilon(X,L,1) = 15$, by Theorem 2.17.

Chapter 3

Seshadri constants on surfaces of general type.

We divide this chapter into two sections. In the first section we consider arbitrary surfaces of general type and prove a result about multi-point Seshadri constants of the canonical line bundle. In the second section we consider surfaces of general type of the form $C \times C$, where C is a general member of the moduli of smooth curves of genus $g \geqslant 2$ and answer some of the questions mentioned in Chapter (1).

3.1 Surfaces of general type

Let X be smooth complex projective variety and L be a line bundle on X. Consider the linear system $|mL|$ for $m \in \mathbb{N}$. The global sections of mL defines a rational map

$$\phi_{mL} : X \dashrightarrow \mathbb{P}(H^0(X, mL)).$$

Clearly, the $\dim(\phi_{mL}(X)) \leqslant \dim(X)$.

Definition 3.1. $\kappa(X, L) := \max\{\dim(\phi_{mL}(X)) : m \in \mathbb{N}\}$.

Definition 3.2. Given a smooth complex projective variety X with canonical divisor K_X, the *Kodaira dimension* of X is defined as $\kappa(X, K_X)$.

Definition 3.3. A smooth complex algebraic surface X is said to be of *general type* if the Kodaira dimension $\kappa(X) = 2$.

Proposition 3.4. *Let S be a minimal surface of general type. Then $K_S^2 \geqslant 1$.*

Proof. Let H be a general hyperplane section of S. It is clear that nK_S is effective for large enough n, this gives $H \cdot K_S > 0$. Consider the long exact sequence in cohomology corresponding the following short exact sequence

$$0 \longrightarrow \mathcal{O}_S(nK_S - H) \longrightarrow \mathcal{O}_S(nK_S) \longrightarrow \mathcal{O}_H(nK_S) \longrightarrow 0.$$

By using the Reimann-Roch theorem for curves one can see that $h^0(H, \mathcal{O}_H(nK_S))$ grows linearly as a function of n whereas $h^0(\mathcal{O}_S(nK_S))$ grows as n^2 (S being a surface of general type). Therefore, for large n, there is an effective divisor in the liner system $|nK_S - H|$. Since K_S is nef we see that $(nK_S - H) \cdot K_S \geqslant 0$, which gives $nK_S^2 \geqslant K_S \cdot H > 0$. □

We note the following proposition [9, Propositon 1] [34, Proposition 2.1.3].

Proposition 3.5. *Let X be a minimal surface of general type. Then we have:*

1. *The number of (-2)-curves on X is finite. They are independent over $\mathbb{Q}$ and their number is at most equal to $\rho(X) - 1$, where $\rho(X)$ is the picard number of X.*
2. *The intersection form is negative definite on the subspace spanned by (-2)-curves in $NS(X) \otimes \mathbb{Q}$.*

Now we give some examples of surface of general type.

Example 3.6. Let $S_{d_1,d_2,\dots,d_r}$ denote a surface in $\mathbb{P}^{r+2}$ which is a complete intersection of hypersurfaces of degrees $d_1, d_2, \dots, d_r$. By Bertini's theorem, given $d_1, d_2, \dots, d_r \geqslant 1$ with $r \leqslant n$ there exist hypersurfaces H_i in $\mathbb{P}^n$ of degree d_i such that $H_1 \cap H_2 \cap \dots \cap H_r$ is a surface of complete intersection. It's canonical bundle is given by $K_{S_{d_1,d_2,\dots,d_r}} = k \cdot H$ where $k = (\sum d_i - r - 3)$ and H is a hyperplane section. For $r = 1$, $d_1 = 2, 3$ and $r = 2, d_1 = d_2 = 2$ we get $k < 0$ hence Kodaira dimension of S_2, S_3 and $S_{2,2}$ is $-\infty$. Similarly when $k = 0$ we get $S_4, S_{2,3}$ and $S_{2,2,2}$. All other complete intersection surfaces has $K_S^2 > 0$ since $k > 0$ hence these are surfaces of general type.

Example 3.7. Let $S = C \times D$ where C and D are two curves of genus $\geqslant 2$ then S is a surface of general type. Now let p and q denotes the two projection then $K_S = p^*(K_C) \otimes q^*(K_D)$ and the rational map $\phi_{nK_S} : S \dashrightarrow \mathbb{P}^N$ factorises as

$$\phi_{nK_S} : C \times D \overset{(\phi_{nK_C}, \phi_{nK_D})}{\dashrightarrow} \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \overset{s}{\hookrightarrow} \mathbb{P}^N$$

where s is the Segre embedding (defined by $(x_i),(y_j) \mapsto (x_i y_j)$). Since $\kappa(C) = \kappa(D) = 1$ we see that $\kappa(S) = 2$ and hence S is a surface of general type. In fact if S is any surface fibred over a curve of genus at least 2, whose generic fibre is of genus at least two, is a surface of general type.

Example 3.8. Let $f : S' \longrightarrow S$ be a surjective morphism of surfaces, if S is a surface of general type, then so is S'.

Now we give an example of a surface of general type S with $p_g = 0 = q$, where $p_g = h^0(S, \mathcal{O}(K_S))$ and $q = h^1(S, \mathcal{O}_S)$

Example 3.9. Let x, y, z, t denotes the coordinates of $\mathbb{P}^3$ and $S' \subset \mathbb{P}^3$ be quintic defined by $x^5 + y^5 + z^5 + t^5 = 0$. Let σ be an automorphism of S' defined by $\sigma(x, y, z, t) = (x, \zeta y, \zeta^2 z, \zeta^3 t)$ where ζ is a primitive 5th root of unity, then the group $G \cong \mathbb{Z}_5$ generated by σ acts on S' with no fixed points hence $S := S'/G$ is smooth. Also S is a surface of general type with $p_g = q = 0$ [7, Example X.3].

Noether's inequality: Let S be a minimal surface of general type, then $K_S^2 \geqslant 2p_g(S) - 4$. If the equality holds then ϕ_{K_S} is a degree two morphism onto a nondegenerate surface of minimal degree $p_g - 2$ in $\mathbb{P}^{p_g - 1}$.

The following theorem together with Noether's inequality gives a concise geography of surface of general type [34].

Theorem 3.10. *Let S be a surface of general type. Then $\chi(\mathcal{O}_S) \geqslant 1$ and $K_S^2 \leqslant 9\chi$.*

One defines a line bundle L on a smooth complex projective variety X to be *big* if $\kappa(X, L) = \dim(X)$. Therefore a surface of general type is a surface whose canonical divisor is big. The following theorem is a characterisation for a nef line bundle to be big [28].

Theorem 3.11. *Let X be an irreducible projective variety of dimension n and L be a nef line bundle on X. Then L is big if and only if its top self-intersection is positive, i.e., $(L^n) > 0$.*

3.2 Multi-point Seshadri constants

Motivated by [6, Theorem 1], we prove the following theorem:

Theorem 3.12. *Let X be a surface of general type and K_X be canonical line bundle on X. If K_X is nef and big and $x_1, x_2, ..., x_r \in X$ be $r \geqslant 2$ points then we have the following,*

1. $\varepsilon(X, K_X, x_1, x_2, ..., x_r) = 0 \Leftrightarrow$ *at least one of x_i lies on one of the finitely many (-2)-curves on X.*
2. *If $0 < \varepsilon(X, K_X, x_1, x_2, ..., x_r) < \frac{1}{r}$, then*

$$\varepsilon(X, K_X, x_1, x_2, ..., x_r) = \begin{cases} \frac{1}{r+1} \text{ or } \frac{2}{5} & \text{if} \quad r = 2, \\ \frac{1}{r+1} \text{ or } \frac{1}{r+2} & \text{if} \quad 3 \leqslant r < 10, \\ \frac{1}{r+1} \text{ or } \frac{1}{r+2} \text{ or } \frac{1}{r+3} & \text{if} \quad r \geqslant 10. \end{cases}$$

Proof. The proof of (1) uses the same technique as the case of the single point Seshadri constant in [6]. Since K_X is big and nef, its self-intersection is strictly positive, i.e., $K_X^2 > 0$.

$\Rightarrow$: Let $C \subset X$ be a smooth curve passing through at least one of the points $x_1, x_2, ..., x_r \in X$ with multiplicities $m_1, m_2, ..., m_r$, such that $0 = \varepsilon(X, K_X, x_1, x_2, ..., x_r) = K_X \cdot C / \sum_{i=1}^{i=r} m_i$. This gives $K_X \cdot C = 0$. Using the Hodge Index Theorem and the fact that $K_X^2 > 0$, we get $C^2 < 0$. Since K_X is nef, there are no (-1)-curves in X, therefore $C^2 = -2$. Using adjunction formula we conclude that the genus of C is 0, and hence C is a rational curve.

$\Leftarrow$: Conversely, suppose some x_i lies on a (-2)-curve C, then using the adjunction formula and the fact that the arithmetic genus of C is 0, we find that $K_X \cdot C = 0$. Hence $\varepsilon(X, K_X, x_1, x_2, ..., x_r) = \frac{K_X \cdot C}{\sum_{i=1}^{i=r} \text{mult}_{x_i} C} = 0$.

(2) Let $\varepsilon(X, K_X, x_1, x_2, ..., x_r) < \frac{1}{r}$ which in turn is less than $\sqrt{\frac{K_X^2}{r}}$, so that by a generalized statement of [6, Proposition 1.1] for the multi-point case, there exists a reduced and irreducible curve C computing $\varepsilon(X, K_X, x_1, x_2, ..., x_r)$. Let C be a reduced and irreducible curve in X passing through at least one of the points $x_1, x_2, ..., x_r$ with multiplicities $m_1, m_2, ..., m_r$ such that

$$\varepsilon(X, K_X, x_1, x_2, ..., x_r) = \frac{K_X \cdot C}{m}, \tag{3.1}$$

where $m := \sum_{i=1}^{i=r} m_i$. Put $K_X \cdot C = d$. Notice that $\frac{d}{m} < \frac{1}{r}$ which gives $m > dr$. Now, using the positivity of K_X^2 and the Hodge Index Theorem, we get $C^2 \leqslant K_X^2 C^2 \leqslant (K_X \cdot C)^2 = d^2$. Let $\tilde{C}$ be the normalization of C. Then,

$$
\begin{aligned}
0 \leqslant P_a(\tilde{C}) &\leqslant P_a(C) - \sum_{i=1}^{i=r} \frac{m_i(m_i-1)}{2} \\
\Rightarrow \sum_{i=1}^{i=r} \frac{m_i(m_i-1)}{2} &\leqslant P_a(C) = 1 + \frac{1}{2}C^2 + \frac{1}{2}K_X \cdot C \leqslant 1 + \frac{d(d+1)}{2} \\
\Rightarrow \left(\frac{1}{r}m^2 - m\right) &\leqslant \left(\sum_{i=1}^{i=r} {m_i}^2 - \sum_{i=1}^{i=r} m_i\right) < 2 + d^2 + d \\
\Rightarrow \quad m^2 - rm &- r(2+d^2+d) < 0. \qquad (3.2)
\end{aligned}
$$

We see that equation (3.15) implies the inequality (3.2). Therefore, it is enough to find out when the inequality (3.2) holds. We show that the possible choices of d and m satisfying the above conditions are as stated in the statement of the theorem.

Put $\phi_{r,d}(m) := m^2 - rm - r(2+d^2+d)$.

Claim $\phi_{r,d}(m) < 0 \Longrightarrow d = 1, 2$ and $m = r+1, r+2, r+3$ and 5 with some conditions on r.

Since $m > dr$, substituting $m = dr + j$ in $\phi_{r,d}(m)$, we get

$$\phi_{r,d}(dr+j) := r^2d^2 + j^2 + 2drj - r^2d - rd^2 - (d+2)r - rj.$$

$$d = 1 \qquad \phi_{r,1}(r+j) = j^2 + rj - 4r \begin{cases} < 0 & \text{if } j = 1 \text{ and } r \geqslant 2 \\ < 0 & \text{if } j = 2 \text{ and } r \geqslant 3 \\ < 0 & \text{if } j = 3 \text{ and } r \geqslant 10 \\ > 0 & \text{otherwise} \end{cases}$$

$$d = 2 \qquad \phi_{r,2}(2r+j) = 2r^2 + j^2 + 3rj - 8r \begin{cases} < 0 & \text{if } j = 1 \text{ and } r = 2 \\ > 0 & \text{otherwise} \end{cases}$$

$d \geqslant 3 \qquad \phi_{r,d}(rd+j) > 0$ for $r \geqslant 2$.

In order to see the last statement, it is sufficient to show that $\phi_{r,3}(3r+j) > 0$ for $r \geqslant 2$ and the derivative of $\phi_{r,d}$ with respect to d is positive. This will imply that $\phi_{r,d}$ is an increasing function of d and hence positive for all $d \geqslant 3, r \geqslant 2$. The first condition is easily checked. The second condition is also satisfied since differentiating $\phi_{r,d}$ with respect to d gives

$$\phi'_{r,d}(rd+j) = r^2(2d-1) - r(2d+1-2j)$$

which is always positive whenever $r \geqslant 2$. □

3.3 Surface of general type of the form $X = C \times C$.

Let C be a smooth complex projective curve of genus $g \geqslant 2$ and consider the surface $X = C \times C$. Let F_1 and F_2 be fibres corresponding to the two projections $C \times C \longrightarrow C$ and let δ be the diagonal. Assume that C is a general member of the moduli of smooth curves of genus g, where $g \geqslant 2$. Then, it is known that they span $NS(X)$ [28]. Intersections among them is governed by the following formulae:

$$\begin{aligned} (F_1)^2 &= 0, \qquad (F_2)^2 = 0, \\ F_1 \cdot F_2 &= F_1 \cdot \delta = F_2 \cdot \delta = 1, \\ \text{and} \ \delta^2 &= 2 - 2g. \end{aligned}$$

Let K_X be the canonical divisor of X then it is not difficult to see that $K_X \equiv 2(g-1)(F_1 + F_2)$ [23] and therefore $K_X^2 = 8(g-1)^2$.

We consider X as above and show that the global Seshadri constant of an ample line bundle L on X is a rational. Let L be an ample line bundle on X. Write $L \equiv a_1F_1 + a_2F_2 + a_3\delta$ where $a_1, a_2, a_3 \in \mathbb{Z}$ and "$\equiv$" denotes the *numerical equivalence*. Since L is ample, we have

$$\begin{aligned} L \cdot F_1 &= a_2 + a_3 > 0, \\ L \cdot F_2 &= a_1 + a_3 > 0, \\ L \cdot \delta &= a_1 + a_2 - (2g-2)a_3 > 0, \text{ and} \\ L^2 &= 2(a_1a_2 + a_1a_3 + a_2a_3) - (2g-2)a_3^2 > 0. \end{aligned} \tag{3.3}$$

Note that, if $a_3 = 0$ in above expression for L then, L is ample if and only if $a_1, a_2 > 0$.

3.3.1 Seshadri constants

In this section we first consider $\varepsilon(L)$ and then prove our results on $\varepsilon(L, r)$

3.3.1.1 Results about $\varepsilon(L)$.

In this section we partially answer the question about the rationality of $\varepsilon(L)$ [36, Question 1.6]. In other words, under some conditions on a_1, a_2 and a_3 we address the question of rationality in affirmative. Following is our main theorem.

Theorem 3.13. *Let $X = C \times C$, where C is a general member of moduli of smooth curves of genus $g \geqslant 2$. Let $L \equiv a_1F_1 + a_2F_2 + a_3\delta$ be an ample line bundle satisfying any of the following conditions on a_1, a_2 and a_3.*

1. $a_3 = 0$,
2. $a_3 > 0$, $a_1 \leqslant a_2$ *and* $a_1^2 + a_3^2 < 2a_1a_2$,
3. $a_3 > 0$, $a_2 \leqslant a_1$ *and* $a_2^2 + a_3^2 < 2a_1a_2$,
4. $a_3 < 0$ *and* $a_2 \geqslant \left(\frac{2gk^2+2k+1}{2(k+1)}\right) \cdot a_1$ *where* $k = \lceil \frac{|a_3|/a_1}{1-|a_3|/a_1} \rceil$ *or*
5. $a_3 < 0$ *and* $a_1 \geqslant \left(\frac{2gl^2+2l+1}{2(l+1)}\right) \cdot a_2$ *where* $l = \lceil \frac{|a_3|/a_2}{1-|a_3|/a_2} \rceil$.

Then $\varepsilon(L) \in \mathbb{Q}$.

Proof. (1) Assume $a_3 = 0$, then we have $L \equiv a_1F_1 + a_2F_2$. In this case, we show that either $L \cdot F_2 \leqslant \sqrt{L^2}$ or $L \cdot F_1 \leqslant \sqrt{L^2}$. This is equivalent to show that

$$\begin{aligned} &\text{either} \quad a_1^2 \leqslant 2a_1a_2 \quad \text{or} \quad a_2^2 \leqslant 2a_1a_2, \\ \Leftrightarrow \quad &\text{either} \quad a_1 \leqslant 2a_2 \quad \text{or} \quad a_2 \leqslant 2a_1. \end{aligned} \tag{3.4}$$

Notice that, when $a_1 > 2a_2$, we get

$$2a_1 > a_1 > 2a_2 > a_2,$$

implying that the statement (3.4) always holds.

(2) Let $a_3 > 0$, $a_1 \leqslant a_2$ and $a_1^2 + a_3^2 < 2a_1a_2$. Then, we show that $L \cdot F_2 \leqslant \sqrt{L^2}$. Notice that

$$\begin{aligned}
& L \cdot F_2 \leqslant \sqrt{L^2} \\
\Leftrightarrow\ & (a_1 + a_3)^2 \leqslant L^2 = 2a_1a_2 + 2a_2a_3 + 2a_1a_3 - a_3^2(2g-2) \\
\Leftrightarrow\ & a_1^2 + a_3^2 + 2a_1a_3 \leqslant 2a_1a_2 + 2a_2a_3 + 2a_1a_3 - a_3^2(2g-2) \\
\Leftrightarrow\ & a_1^2 + a_3^2 \leqslant 2a_1a_2 + 2a_2a_3 - a_3^2(2g-2) \\
\Leftrightarrow\ & a_1^2 + a_3^2 + a_3^2(2g-2) \leqslant 2a_1a_2 + 2a_2a_3 \qquad (3.5) \\
\Leftrightarrow\ & a_1^2 + a_3^2(2g-1) \leqslant 2a_1a_2 + 2a_2a_3. \qquad (3.6)
\end{aligned}$$

Now, since L is ample, we have

$$\begin{aligned}
& L \cdot \delta = a_1 + a_2 - a_3(2g-2) > 0 \\
\Rightarrow\ & 2a_2 \geqslant a_1 + a_2 > a_3(2g-2) \\
\Rightarrow\ & a_2 > a_3(g-1) \\
\Rightarrow\ & 2a_2a_3 > a_3^2(2g-2). \qquad (3.7)
\end{aligned}$$

It is easy to see that the equation (3.5) follows from the hypothesis and the equation (3.7).

(3) The proof follows similar to that of (2).

(4) Let $a_3 < 0$ and $a_2 \geqslant \left(\frac{2gk^2+2k+1}{2(k+1)}\right) a_1$, where $k = \lceil \frac{|a_3|/a_1}{1-|a_3|/a_1} \rceil$. We will show that

$$L \cdot F_2 = a_1 + a_3 \leqslant \sqrt{L^2}.$$

It suffices to show that the equation (3.6) holds. Since L is ample, we get

$$L \cdot F_1 = a_2 + a_3 > 0 \quad \text{and} \quad L \cdot F_2 = a_1 + a_3 > 0. \qquad (3.8)$$

This implies that, a_3 can at the very least be $-a_1$, i.e., $a_3 > -a_1 > -a_2$. Thus, there must exist a positive integer k such that

$$a_3 > -\frac{k}{k+1}a_1, \qquad (3.9)$$

since $-(k/k+1)a_1$ converges to $-a_1$. Choose the least such positive k for which the above inequality holds. That is,

$$k := \lceil \frac{|a_3|/a_1}{1-|a_3|/a_1} \rceil.$$

Here, $\lceil x \rceil$ represents the least integer greater than or equal to x. We have the following

$$\begin{aligned} a_1\left(\frac{2gk^2+2k+1}{2(k+1)}\right) &\leqslant a_2 \quad \text{(by hypothesis in (4))} \\ \Leftrightarrow \quad a_1^2\left(\frac{2gk^2+2k+1}{(k+1)^2}\right) &\leqslant 2a_1a_2\left(\frac{1}{k+1}\right) \\ \Leftrightarrow a_1^2+\left(\frac{k}{k+1}\right)^2 a_1^2(2g-1) &\leqslant 2a_1a_2+2a_2\left(-\frac{k}{k+1}\right)a_1. \end{aligned} \tag{3.10}$$

Therefore, the following holds:

$$\begin{aligned} a_1^2+a_3^2(2g-1) &< a_1^2+\left(\frac{k}{k+1}\right)^2 a_1^2(2g-1) \\ &\leqslant 2a_1a_2+2a_2\left(-\frac{k}{k+1}\right)a_1 \\ &< 2a_1a_2+2a_2a_3. \end{aligned}$$

Where the first and last inequalities hold by (3.9) and the fact that $a_3 < 0$, while the second inequality follows from (3.10). Therefore, inequality (3.6) holds.

(5) The proof is similar to that of (4).

□

Example 3.14. We give an example to show the occurrence of case (4). Let $X = C \times C$ be a surface of general type, where C is a smooth curve of genus g. Let $L \equiv a_1F_1 + a_2F_2 + a_3\delta$ be an ample line bundle on X. Assume that $a_2 > a_1$ as in the hypothesis of (4). Therefore, $a_2 > a_1 > |a_3| > 0$.

Note that $k = \lceil \frac{|a_3|}{a_1-|a_3|} \rceil$, and, in general, we have $1 \leqslant k \leqslant |a_3|$. When $a_1 \geqslant 2|a_3|$, we have $k = 1$. Then, the condition on a_2 becomes

$$a_2 \geqslant \left(\frac{2gk^2+2k+1}{2(k+1)}\right)a_1 = \left(\frac{2g+3}{4}\right)a_1.$$

So for an ample line bundle $L \equiv a_1F_1 + a_2F_2 + a_3\delta$ with $a_3 < 0$, $a_1 \geqslant 2|a_3|$, and $a_2 > \left(\frac{2g+3}{4}\right)a_1$, we have $\varepsilon(L) \in \mathbb{Q}$.

For example, fix $g = 2$ and take $a_3 = -10$. Then if $a_1 = 20$, we get the least value of k i.e., 1. In this case, we require $a_2 \geqslant (7/4)a_1 = 35$. But when $a_1 = 11$, we get the highest value of k i.e., 10. So we require $a_2 \geqslant (421/22)a_1 = 382.72$.

Now, for a line bundle of the form $L \equiv aF_1 + bF_2$ with $a, b > 0$ we explicitly compute the Seshadri constants of L at one or two points.

Theorem 3.15. *Let $X = C \times C$, where C is a smooth curve of genus $g \geqslant 2$ and let $L \equiv aF_1 + bF_2$ be an ample line bundle on X. Then $\varepsilon(L, x) = \min\{a, b\}$ for every $x \in X$.*

Proof. Since a fibre numerically equivalent to F_1 and F_2 passes through every point $x \in X$, we get

$$\begin{aligned} \varepsilon(L,x) &\leqslant L \cdot F_1 = b \quad \text{and} \\ \varepsilon(L,x) &\leqslant L \cdot F_2 = a \\ \Rightarrow \quad \varepsilon(L,x) &\leqslant \min\{a,b\}. \end{aligned}$$

Now, let C_1 be any curve in X (not numerically equivalent to F_1 and F_2) passing through x with multiplicity m. Then, by Bézout's theorem we obtain

$$C_1 \cdot F_i \geqslant \mathrm{mult}_x C_1 \cdot \mathrm{mult}_x F_i = m$$

for $i = 1$ and 2. Therefore, notice that

$$\begin{aligned} L \cdot C_1 &= a(C_1 \cdot F_1) + b(C_1 \cdot F_2) \\ &\geqslant \min\{a,b\}(m+m) \\ \Rightarrow \frac{L \cdot C_1}{m} &\geqslant 2\min\{a,b\}. \end{aligned}$$

Hence, we get $\varepsilon(L, x) = \min\{a, b\}$. □

We also have a similar result like above for the Seshadri constant at two points $x_1, x_2 \in X$.

Theorem 3.16. *Let $X = C \times C$, where C is a smooth curve of genus $g \geqslant 2$ and let $L \equiv aF_1 + bF_2$ be an ample line bundle on X. Then*

$$\varepsilon(L, x_1, x_2) = \begin{cases} \min\left\{a, \frac{b}{2}\right\}, & \text{if both } x_1 \text{ and } x_2 \text{ lie on a fixed } F_1, \\ \min\left\{\frac{a}{2}, b\right\}, & \text{if both } x_1 \text{ and } x_2 \text{ lie on a fixed } F_2, \\ \min\{a,b\}, & \text{otherwise.} \end{cases}$$

Proof. Let C_1 be a curve not numerically equivalent to F_1 and F_2 and passing through x_1 and x_2 with multiplicity m_1 and m_2 respectively. Since there is a fibre numerically equivalent to F_1 and F_2 passing through every point of X, by Bézout's theorem we get

$$\begin{aligned} C_1 \cdot F_1 &\geqslant \mathrm{mult}_{x_1} C_1 \cdot \mathrm{mult}_{x_1} F_1 = m_1, \\ C_1 \cdot F_1 &\geqslant \mathrm{mult}_{x_2} C_1 \cdot \mathrm{mult}_{x_2} F_1 = m_2 \quad \text{and} \\ C_1 \cdot F_2 &\geqslant \mathrm{mult}_{x_1} C_1 \cdot \mathrm{mult}_{x_1} F_1 = m_1, \\ C_1 \cdot F_2 &\geqslant \mathrm{mult}_{x_2} C_1 \cdot \mathrm{mult}_{x_2} F_1 = m_2. \end{aligned}$$

This gives $C_1 \cdot F_1 \geqslant m$ and $C_1 \cdot F_2 \geqslant m$ where $m := \max\{m_1, m_2\}$. Now

$$\begin{aligned} L \cdot C_1 &= a(C_1 \cdot F_1) + b(C_1 \cdot F_2) \\ &\geqslant \min\{a, b\}(m + m) \\ \Rightarrow \frac{L \cdot C_1}{m_1 + m_2} &\geqslant \frac{\min\{a, b\}(2m)}{m_1 + m_2} \geqslant \min\{a, b\} \end{aligned}$$

since $2m \geqslant m_1 + m_2$. Now, if both the points x_1 and x_2 lie either on a fibre F_1 or on a fibre F_2, then we have

$$\begin{aligned} \varepsilon(L, x_1, x_2) &\leqslant \frac{L \cdot F_1}{1+1} = \frac{b}{2} \text{ and } \varepsilon(L, x_1, x_2) \leqslant \frac{L \cdot F_2}{1} = a, \quad \text{or} \\ \varepsilon(L, x_1, x_2) &\leqslant \frac{L \cdot F_2}{1+1} = \frac{a}{2} \text{ and } \varepsilon(L, x_1, x_2) \leqslant \frac{L \cdot F_1}{1} = b \\ \Rightarrow \varepsilon(L, x_1, x_2) &\leqslant \min\left\{a, \frac{b}{2}\right\} \text{ or } \varepsilon(L, x_1, x_2) \leqslant \min\left\{\frac{a}{2}, b\right\}. \end{aligned}$$

However, $\min\{a, b\} \geqslant \min\left\{a, \frac{b}{2}\right\}$ and $\min\{a, b\} \geqslant \min\left\{\frac{a}{2}, b\right\}$. Therefore, we get

$$\begin{aligned} \varepsilon(L, x_1, x_2) &= \min\left\{a, \frac{b}{2}\right\} \text{ or} \\ \varepsilon(L, x_1, x_2) &= \min\left\{\frac{a}{2}, b\right\}. \end{aligned}$$

In case both the points x_1 and x_2 do not lie on the same fixed fibre, then we get

$$\begin{aligned} \varepsilon(L, x_1, x_2) &\leqslant \frac{L \cdot F_1}{1} = b \text{ and} \\ \varepsilon(L, x_1, x_2) &\leqslant \frac{L \cdot F_2}{1} = a. \\ \Rightarrow \varepsilon(L, x_1, x_2) &\leqslant \min\{a, b\}. \end{aligned}$$

Hence, we obtain $\varepsilon(L, x_1, x_2) = \min\{a, b\}$. □

Remark 3.17. When X is as in the above two theorems, i.e., of the form $C \times C$, the canonical divisor K_X of X is given by $p_1^*(K_C) \otimes p_2^*(K_C)$ where p_1 and p_2 are the two natural projections from $C \times C \longrightarrow C$. Since $\deg(K_C)$ is $2(g-1)$, K_X is numerically equivalent to $2(g-1)(F_1+F_2)$. Hence the above two theorems apply to K_X.

3.3.1.2 Result about $\varepsilon(L, r)$

Motivated by [21, Theorem 3.8] we prove similar result about the multi-point Seshadri constant of canonical line bundle K_X which is numerically equivalent to $2(g-1)F_1 + 2(g-1)F_2$ on $X = C \times C$.

Theorem 3.18. *Let $X = C \times C$, where C is a general member of the moduli of smooth curves of genus $g \geqslant 2$. Let K_X be the canonical line bundle on X and $r \geqslant K_X^2$ be an integer. Then either*

$$\varepsilon(X, K_X, r) \geqslant \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{K_X^2}{r}}$$

or $\varepsilon(X, K_X, r)$ is computed by a curve C_1 of the form $a(F_1+F_2)$ (for some $a \in \mathbb{N}$) passing through r very general points with multiplicity one each. In other words,

$$\varepsilon(X, K_X, r) = \frac{a(K_X \cdot F_1 + K_X \cdot F_2)}{r}.$$

Proof. Suppose

$$\varepsilon(X, K_X, r) \quad < \quad \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{K_X^2}{r}}.$$

Then, there exists an effective curve $C_1 \subset X$ passing through $s \leqslant r$ very general points with multiplicities one each [21], such that

$$\varepsilon(X, K_X, r) = \frac{K_X \cdot C_1}{s} < \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{K_X^2}{r}}.$$

By [21, Remark 2.4], we get $C_1^2 < s$. Also, since C_1 is a general curve in X passing through $s \leqslant r$ very general points with multiplicities $m_1 \geqslant m_2 \geqslant ... \geqslant m_s > 0$, then by Xu's lemma [41],

$$C_1^2 \geqslant \sum_{i=1}^{i=s} m_i^2 - m_s \geqslant s - 1. \tag{3.11}$$

Thus, we have $C_1^2 = s-1$. We will show that C_1 is numerically equivalent to $a(F_1+F_2)$ for some $a \in \mathbb{N}$.

Case 1 : $s = 1$

In this case, we have $\varepsilon(X, K_X, r) = K_X \cdot C_1 \geqslant 1 > \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{K_X^2}{r}}$ since $r \geqslant K_X^2$. This is a contradiction to our assumption.

Case 2 : $2 \leqslant s \leqslant r-1$

Notice that

$$\begin{aligned} \left(\frac{K_X \cdot C_1}{s}\right)^2 &\geqslant \left(\frac{r+2}{r+3}\right)\left(\frac{K_X^2}{r}\right), \\ \Leftrightarrow r(r+3)(K_X \cdot C_1)^2 &\geqslant s^2(r+2)K_X^2. \end{aligned} \tag{3.12}$$

Using Hodge Index Theorem, we obtain $(K_X \cdot C_1)^2 \geqslant (s-1)K_X^2$ and hence equation (3.14) follows if we prove

$$r(r+3)(s-1) \geqslant s^2(r+2).$$

This is true for $r \geqslant 4$. To see this, it is enough to check the inequality at the maximal possible value of s, i.e., at $s = r-1$:

$$\begin{aligned} r(r+3)(r-2) &\geqslant (r-1)^2(r+2) \\ \Leftrightarrow r^3 + r^2 - 6r &\geqslant r^3 - 3r + 2 \\ \Leftrightarrow r^2 &\geqslant 3r + 2. \end{aligned}$$

This holds for $r \geqslant 4$. By hypothesis $r \geqslant K_X^2 = 8(g-1)^2 \geqslant 8$. So we again arrive at a contradiction to our assumption.

Case 3 : $s = r$

Notice that, the equation (3.14) follows if we prove $(K_X \cdot C_1)^2 \geqslant (C_1^2 + \frac{1}{3})K_X^2 = \left(r - \frac{2}{3}\right) K_X^2$. because we have the following

$$r(r+3)\left(r - \frac{2}{3}\right) \geqslant r^2(r+2)$$

$$\begin{aligned}\Leftrightarrow\ r^3-\frac{2}{3}r^2+3r^2-2r &\geqslant r^3+2r^2\\ \Leftrightarrow\ \frac{r^2}{3} &\geqslant 2r.\end{aligned}$$

However, the last inequality holds for $r \geqslant 6$. Now to see $(K_X \cdot C_1)^2 \geqslant (C_1^2 + \frac{1}{3})K_X^2$, we start by putting $C_1 \equiv a_1F_1 + a_2F_2 + a_3\delta$ for some $a_1, a_2, a_3 \in \mathbb{Z}$ and $L := F_1 + F_2$. We know that $K_X = 2(g-1)L$ [?], so it's enough to show that

$$\begin{aligned}(L\cdot C_1)^2 &\geqslant \left(C_1^2+\frac{1}{3}\right)L^2\\ \Leftrightarrow (a_1+a_2+2a_3)^2 &\geqslant \left(2a_1a_2+2a_2a_3+2a_1a_3-a_3^2(2g-2)+\frac{1}{3}\right)\cdot 2\\ \Leftrightarrow a_1^2+a_2^2+4a_3^2+2a_1a_2+4a_2a_3+4a_1a_3 &\geqslant 4a_1a_2+4a_2a_3+4a_1a_3-4a_3^2(g-1)+\frac{2}{3}\\ \Leftrightarrow a_1^2+a_2^2+4a_3^2g &\geqslant 2a_1a_2+\frac{2}{3}.\end{aligned}$$

This clearly holds when $a_3 \neq 0$. In the case $a_3 = 0$, we see that the equation $a_1^2 + a_2^2 \geqslant 2a_1a_2 + 2/3$ does not hold only when $a := a_1 = a_2$. In the latter case, $C_1 \equiv a(F_1 + F_2)$ is a curve passing through r points with multiplicity one each such that

$$\varepsilon(X, K_X, r) = \frac{a(K_X \cdot F_1 + K_X \cdot F_2)}{r}.$$

□

Finally, we include a similar result like (3.18) for an ample line bundle on the blow up of $\mathbb{P}^2$ at r very general points.

Theorem 3.19. *Let $f : X \longrightarrow \mathbb{P}^2$ be a blow up of $\mathbb{P}^2$ at r very general points. Let $L \equiv aH - \sum_{i=1}^{i=r} b_iE_i$ be an ample line bundle on X, where H is the pull back of a general line in $\mathbb{P}^2$ and E_i denotes the exceptional curve corresponding to f. Then for $r \geqslant \max\{L^2 + 1, 4\}$, we have*

$$\varepsilon(X, L, r) \geqslant \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}.$$

Proof. Suppose on contrary the theorem does not hold. Then, we have

$$\varepsilon(X, L, r) < \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}.$$

Therefore, there exists an effective curve $C_1 \subset X$ passing through $s \leqslant r$ very general points with multiplicities one each [21] such that

$$\varepsilon(X, L, r) = \frac{L \cdot C_1}{s} < \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}.$$

By [21, Remark 2.4], we have $C_1^2 < s$. Now, suppose C' is a general curve in X passing through $s \leqslant r$ very general points with multiplicities $m_1 \geqslant m_2 \geqslant ... \geqslant m_s > 0$. Then, by Xu's lemma [41],

$$C'^2 \geqslant \sum_{i=1}^{i=s} m_i^2 - m_s \geqslant s - 1. \tag{3.13}$$

Thus, we have $C_1^2 = s - 1$.

Case 1 : $s = 1$

In this case, we have $\varepsilon(X, L, r) = L \cdot C_1 \geqslant \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$, since r is greater than $L^2 + 1$. This is a contradiction to our assumption.

Case 2 : $2 \leqslant s \leqslant r - 1$

Notice that

$$\begin{aligned} \left(\frac{L \cdot C_1}{s}\right)^2 &\geqslant \left(\frac{r+2}{r+3}\right)\left(\frac{L^2}{r}\right) \\ \Leftrightarrow \; r(r+3)(L \cdot C_1)^2 &\geqslant s^2(r+2)L^2. \end{aligned} \tag{3.14}$$

Using Hodge Index Theorem, we obtain $(L \cdot C_1)^2 \geqslant (s-1)L^2$ and hence equation (3.14) follows if we prove

$$r(r+3)(s-1) \geqslant s^2(r+2).$$

This is true for $r \geqslant 4$. To see this, it is enough to check the inequality at the maximal possible value of s i.e., at $s = r - 1$:

$$\begin{aligned} r(r+3)(r-2) &\geqslant (r-1)^2(r+2) \\ \Leftrightarrow \; r^3 + r^2 - 6r &\geqslant r^3 - 3r + 2 \\ \Leftrightarrow \; r^2 &\geqslant 3r + 2. \end{aligned}$$

This holds for $r \geqslant 4$. Hence, we arrive at a contradiction to our assumption.

Case 3 : $s = r$

We will prove this case by showing $(L \cdot C_1)^2 \geqslant (C_1^2 + 1)L^2 = rL^2$. Then the result will follow from (3.14).

Let $\pi_r : X_r \longrightarrow X$ denotes the blow up of X at r general points. We can assume that these points do not lie on any exceptional divisor. Let F_j, for $1 \leqslant j \leqslant r$ be the corresponding exceptional curve. Notice then that

$$\begin{aligned} L \cdot C_1 &= \pi_r^*(L) \cdot \pi_r^*(C_1) \\ &= \left(aH - \sum_{i=1}^{i=l} b_i E_i \right) \left(dH - \sum_{j=1}^{j=r} m_j F_j \right) \\ &= ad. \end{aligned}$$

Since $E_i \cdot F_j = 0$ for all $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant r$. Now $C_1^2 = d^2 - \sum_{j=1}^{j=r} m_j^2$ and $L^2 = a^2 - \sum_{i=1}^{i=l} b_i^2$, we get

$$L^2 C_1^2 + L^2 = a^2 d^2 - a^2 \left(\sum_{j=1}^{j=r} m_j^2 \right) - d^2 \left(\sum_{i=1}^{i=l} b_i^2 \right) + \left(\sum_{j=1}^{j=r} m_j^2 \right) \left(\sum_{i=1}^{i=l} b_i^2 \right) + L^2.$$

All we need to show is

$$\begin{aligned} & -a^2 \left(\sum_{j=1}^{j=r} m_j^2 \right) - d^2 \left(\sum_{i=1}^{i=l} b_i^2 \right) + \left(\sum_{j=1}^{j=r} m_j^2 \right) \left(\sum_{i=1}^{i=l} b_i^2 \right) + L^2 < 0 \\ \Leftrightarrow\ & a^2 - \sum_{i=1}^{i=l} b_i^2 - a^2 \left(\sum_{j=1}^{j=r} m_j^2 \right) - d^2 \left(\sum_{i=1}^{i=l} b_i^2 \right) + \left(\sum_{j=1}^{j=r} m_j^2 \right) \left(\sum_{i=1}^{i=l} b_i^2 \right) < 0 \\ \Leftrightarrow\ & a^2 \left(1 - \sum_{j=1}^{j=r} m_j^2 \right) - \sum_{i=1}^{i=l} b_i^2 (d^2 + 1) + \left(\sum_{j=1}^{j=r} m_j^2 \right) \left(\sum_{i=1}^{i=l} b_i^2 \right) < 0. \end{aligned}$$

This will follow, if we, instead show

$$\left(\sum_{i=1}^{i=l} b_i^2 + 1 \right) \left(1 - \sum_{j=1}^{j=r} m_j^2 \right) - \left(\sum_{i=1}^{i=l} b_i^2 \right) (d^2 + 1) + \left(\sum_{j=1}^{j=r} m_j^2 \right) \left(\sum_{i=1}^{i=l} b_i^2 \right) < 0, \quad (3.15)$$

because by ampleness of L, we have $a^2 \geqslant 1 + \sum_{i=1}^{i=l} b_i^2$. Rewriting (3.15), we get

$$1 + \sum_{i=1}^{i=l} b_i^2 - \sum_{j=1}^{j=r} m_j^2 - \left(\sum_{i=1}^{i=l} b_i^2 \right) (d^2 + 1) < 0.$$

Which is clearly true. Hence the theorem. □

Bibliography

[1] Aprodu, Marian, *An Appell-Humbert theorem for hyperelliptic surfaces*, J. Math. Kyoto Univ. 38 (1998), no. 1, 101-121.

[2] Bastianelli, Francesco, *Remarks on the nef cone on symmetric products of curves*, Manuscripta Math. 130 (2009), no. 1, 113-120.

[3] Bauer, Thomas, *Seshadri constants of quartic surfaces*, Math. Ann. 309 (1997), no. 3, 475-481.

[4] Bauer, Thomas, *Seshadri constants and periods of polarized abelian varieties*, With an appendix by the author and Tomasz Szemberg, Math. Ann. 312 (1998), no. 4, 607-623.

[5] Bauer, Thomas; Di Rocco, Sandra; Harbourne, Brian; Kapustka, Michał; Knutsen, Andreas; Syzdek, Wioletta; Szemberg, Tomasz, *A primer on Seshadri constants*, Interactions of classical and numerical algebraic geometry, 33-70, Contemp. Math., 496, Amer. Math. Soc., Providence, RI, 2009.

[6] Bauer, Thomas; Szemberg, Tomasz, *Seshadri constants on surfaces of general type*, Manuscripta Math. 126 (2008), no. 2, 167-175.

[7] Beauville, Arnaud, *Complex algebraic surfaces*, Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid. Second edition. London Mathematical Society Student Texts, 34. Cambridge University Press, Cambridge, 1996.

[8] Beltrametti, M.C.; Schneider, M.; Sommese, A.J., *Applications of the Ein-Lazarsfeld criterion for spannedness of adjoint bundles*, Math Z (1993) 214: 593.

[9] Bombieri, Enrico, *Canonical models of surfaces of general type*, Publications Mathématiques de l'IHÉS, Volume 42 (1973) 171-219.

[10] Cascini, Paolo; Nakamaye, Michael, *Seshadri constant on smooth threefolds*, Adv. Geom. 14 (2014), no. 1, 59–79.

[11] Demailly, Jean-Pierre, *Singular Hermitian metrics on positive line bundles*, Complex algebraic varieties (Bayreuth, 1990), 87-104, Lecture Notes in Math., 1507, Springer, Berlin, 1992.

[12] Dumnicki, M.; Küronya, A.; Maclean, C.; Szemberg, T. *Rationality of Seshadri constants and the Segre-Harbourne-Gimigliano-Hirschowitz conjecture*, Adv. Math. 303 (2016), 1162-1170.

[13] Ein, Lawrence; Küchle, Oliver; Lazarsfeld, Robert, *Local positivity of ample line bundles*, J. Differential Geom. 42 (1995), no. 2, 193-219.

[14] Ein, Lawrence; Lazarsfeld, Robert, *Seshadri constants on smooth surfaces*, Journées de Géométrie Algébrique d'Orsay (Orsay, 1992), Astérisque No. 218 (1993), 177-186.

[15] Farnik, Łucja, *A note on Seshadri constants of line bundles on hyperelliptic surfaces*, Arch. Math. (Basel) 107 (2016), no. 3, 227-237.

[16] Farnik, Łucja; Hanumanthu, Krishna; Huizenga, Jack; Schmitz, David; Szemberg, Tomasz, *Rationality of Seshadri constants on general blow ups of* $\mathbb{P}^2$.

[17] Farnik, Łucja; Szemberg, Tomasz; Szpond, Justyna; Tutaj-Gasińska, Halszka, *Restrictions on Seshadri constants on surfaces*, Taiwanese J. Math. 21 (2017), no. 1, 27-41.

[18] García, Luis Fuentes, *Seshadri constants on ruled surfaces: the rational and the elliptic cases*, Manuscripta Math. 119 (2006), no. 4, 483-505.

[19] Hacon, Christopher, *Remarks on Seshadri constants on vector bundles*. Annales de l'Institut Fourier, Volume 50 (2000) no. 3, p. 767-780.

[20] Hanumanthu, Krishna; Harbourne, Brian, *Single point Seshadri constants on rational surfaces*. J. Algebra 499 (2018), 37-42.

[21] Hanumanthu, Krishna; Mukhopadhyay, Alapan, *Multi-point Seshadri constant on ruled surfaces*. Proc. Amer. Math. Soc. 145 (2017), no. 12, 5145-5155.

[22] Hanumanthu, Krishna; Roy, Praveen Kumar, *Seshadri constants on hyperelliptic surfaces*, Proc. Amer. Math. Soc. 146 (2018), no. 10, 4175–4187.

[23] Hartshorne, Robin, *Algebraic Geometry: Graduate Texts in Mathematics*, New York-Heidelberg, Volume 52 Springer-Verlag, 1977.

[24] Hartshorne, Robin, *Ample subvarieties of algebraic varieties*, Notes written in collaboration with C. Musili. Lecture Notes in Mathematics, Vol. 156 Springer-Verlag, Berlin-New York 1970.

[25] Hwang, Jun-Muk; Keum, JongHae, *Seshadri-exceptional foliations*, Math. Ann. 325 (2003), no. 2, 287-297.

[26] Ito, Atsushi, *Seshadri constant via toric degenerations*, J. Reine Angew. Math. 695 (2014), 151–174.

[27] Knutsen, Andreas Leopold; Syzdek, Wioletta; Szemberg, Tomasz, *Moving curves and Seshadri constants*, Math. Res. Lett. 16 (2009), no. 4, 711-719.

[28] Lazarsfeld, Robert, *Positivity in algebraic geometry 1 Classical settings: line bundles and linear series*. Ergebnisse der Mathematik und ihrer Grenzgebiete.

[29] Lee, Seunghun, *Seshadri constant and Fano manifolds*, Math. Z. 245 (2003), no. 4, 645–656.

[30] Murayama, Takumi, *Frobenius-Seshadri constants and characterisations of projective space*. Math. Res. Lett. 25 (2018), no. 3, 905-936.

[31] Mustaţă, Miercea; Schwede Karl, *A Frobenius variant of Seshadri constants*. Mathematische Annalen 358(2014), 861-878.

[32] Nakamaye, Michael, *Seshadri constants and the geometry of surfaces*, J. Reine Angew. Math. 564 (2003), 205-214.

[33] Oguiso, Keiji, *Seshadri constants in a family of surfaces*, Math. Ann. 323 (2002), no. 4, 625-631.

[34] Pignatelli, Roberto, *Surfaces of general type*. See www.science.unitn.it/~pignatel.

[35] Serrano, Fernando, *Divisors of bielliptic surfaces and embeddings in* $\mathbb{P}^4$, Math. Z. 203 (1990), no. 3, 527-533.

[36] Strycharz-Szemberg, Beata; Szemberg, Tomasz, *Remarks on the Nagata conjecture*, Serdica Math. J. 30 (2004), no. 2-3, 405-430.

[37] Syzdek, Wioletta, *Seshadri constants and geometry of surfaces*, Ph.D thesis, 2005, University of Duisburg-Essen.

[38] Syzdek, Wioletta; Szemberg, Tomasz, *Seshadri fibrations of algebraic surfaces*, Math. Nachr. 283 (2010), no. 6, 902-908.

[39] Szemberg, Tomasz, *On positivity of line bundles on Enriques surfaces*, Trans. Amer. Math. Soc. 353 (2001), no. 12, 4963-4972.

[40] Szemberg, Tomasz; Tutaj-Gasińska, Halszka, *Seshadri fibrations on algebraic surfaces*, Ann. Acad. Pedagog. Crac. Stud. Math. 4 (2004), 225-229.

[41] Xu, Geng, *Curves in* $\mathbb{P}^2$ *and symplectic packings*, Math. Ann. 299 (1994), 609-613.