Torus quotients of Richardson varieties in the Grassmannian

By

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DECLARATION

I declare that the thesis entitled **"Torus quotients of Richardson varieties in the Grassmannian"** submitted by me for the degree of **Doctor of Philosophy in Mathematics** is the record of academic work carried out by me under the guidance of Professor S Senthamarai Kannan and this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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CERTIFICATE

I certify that the thesis entitled **"Torus quotients of Richardson varieties in the Grassmannian"** submitted for the degree of **Doctor of Philosophy in Mathematics** by Sarjick Bakshi is the record of research work carried out by his under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent his independent work in a very substantial measure.

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Professor Senthamarai Kannan
Thesis Supervisor.

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Sarjick Bakshi
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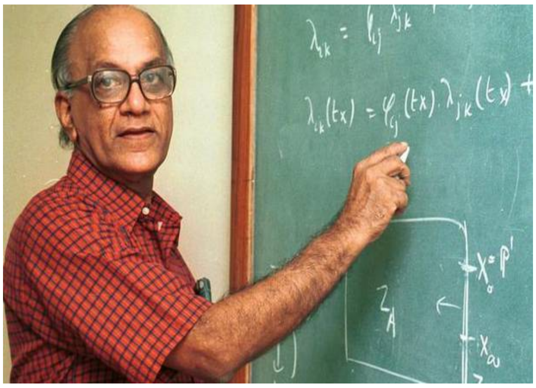
Dedicated to my Parents

In fond memories of Prof. C S Seshadri



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Abstract

We study the GIT quotient of the minimal Schubert variety in the Grassmannian $G_{r,n}$ admitting semistable points for the action of maximal torus T, with respect to the T-linearized line bundle $\mathcal{L}(n\omega_r)$, and show that this is smooth when $gcd(r, n) = 1$. When $n = 7$ and $r = 3$ we study the GIT quotients of all Richardson varieties in the minimal Schubert variety. This builds on previous work by Kumar [Kum08], Kannan and Sardar [KP09b], Kannan and Pattanayak [KP09a], and recent work of Kannan et al. [KPPU18]. It is known that the GIT quotient of $G_{2,n}$ is projectively normal. We give a different combinatorial proof.

Let $r < n$ be positive integers and further suppose r and n are coprime. We study the GIT quotient of Schubert varieties $X(w)$ in the Grassmannian $G_{r,n}$, admitting semistable points for the action of T with respect to the T-linearized line bundle $\mathcal{L}$. We give necessary and sufficient combinatorial conditions for the GIT quotient $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ to be smooth.

Contents

Chapter 1

Introduction

The study of moduli spaces is fundamental for many classification problems related to algebraic geometry. One of the main tools to study moduli spaces is Geometric Invariant Theory (GIT), developed by David Mumford in 1960's. GIT helps us in constructing quotients that preserve algebraic geometric structures. The GIT quotients of the Grassmannian variety and its subvarieties lead to many interesting geometric problems. In this thesis we study the GIT quotient of the Grassmannian variety and its subvarieties.

1.1 Background and History

Let G be a simply connected semi-simple algebraic group over $\mathbb{C}$. Let T be a maximal torus of G. Let B be a Borel subgroup of G containing T. We denote by B^- the Borel subgroup of G opposite to B determined by T. Let Q be a parabolic subgroup of G containing B. Then G/Q is a projective variety (see [Jan07]). Let $\mathcal{L}$ be a T-linearized ample line bundle on G/Q. A point $p \in G/Q$ is said to be semistable with respect to the T-linearized line bundle $\mathcal{L}$ if there is a T-invariant section of a positive power of $\mathcal{L}$ which does not vanish at p. We denote by $(G/Q)^{ss}_T(\mathcal{L})$ the set of all semistable points with respect to $\mathcal{L}$. A point in $(G/Q)^{ss}_T(\mathcal{L})$ is said to be stable if its T-orbit is closed in $(G/Q)^{ss}_T(\mathcal{L})$ and its stabilizer in T is finite. Let $(G/Q)^{s}_T(\mathcal{L})$ denote the set of all stable points with respect to $\mathcal{L}$. These definitions are motivated by the question of understanding the GIT quotient of G/Q with respect to the T-linearized bundle $\mathcal{L}$.

Let $X(T)$ denote the group of characters of T. In the root system R of (G, T) let R^+ denote the set of positive roots with respect to B. Let $S = \{\alpha_1, \ldots, \alpha_l\} \subseteq R^+$ be the set of simple roots and let $\{\omega_1, \ldots, \omega_l\}$ be the fundamental weights. Let U (respectively, U^-) be the unipotent radical of B (respectively, B^-). For each $\alpha \in R^+$, let U_α (respectively, $U^-_{-\alpha}$) be the additive one-dimensional subgroup of U (respectively, U^-) corresponding to the root α (respectively, $-\alpha$) normalized by T.

Let $N_G(T)$ denote the normalizer of T in G. The Weyl group W of G is defined to be the quotient $N_G(T)/T$, and for every $\alpha \in R$ there is a corresponding reflection $s_\alpha \in W$. W is generated by s_α, α running over simple roots in S. This also defines a length function l and the Bruhat order on W.

For a subset $I \subseteq S$ let $W^I = \{w \in W | w(\alpha) > 0, \alpha \in I\}$ and W_I be the subgroup of W generated by s_α, $\alpha \in I$. Then every $w \in W$ can be uniquely expressed as $w = w^I w_I$, with $w^I \in W^I$ and $w_I \in W_I$. For $w \in W$, let $n_w \in N_G(T)$ be a representative of w. We denote by P_I the parabolic subgroup of G generated by B and n_w, $w \in W_I$. Then W_I is the Weyl group of the parabolic subgroup P_I. Sometimes we use the notation W^{P_I}(respectively, W_{P_I}) instead of W^I(respectively, W_I). When $I = S \setminus \{\alpha_r\}$, we denote the corresponding maximal parabolic subgroup of G by $P_{\hat{\alpha}_r}$.

The quotient space G/P is a homogenous space for the left action of G. The T-fixed points in G/P are $e_w = wP/P$ with $w \in W^P$. The B-orbit C_w of e_w is called a Bruhat cell, and it is an affine space of dimension $l(w)$. The closure of C_w in G/P is the Schubert variety $X(w)$. The opposite Bruhat cell C^w is the B^- orbit of e_w, and its closure, denoted by X^w, is the opposite Schubert variety. A Richardson variety in G/P is defined to be the intersection $X(w) \cap X^v$, and it is denoted as X^v_w. For a T-linearized line bundle $\mathcal{L}$ on a Schubert variety (respectively, Richardson variety) in G/P we define the notion of semistable and stable points as before. We use the notation $X(w)^{ss}_T(\mathcal{L})$ (respectively, $(X^v_w)^{ss}_T(\mathcal{L})$) to denote the semistable points and $X(w)^s_T(\mathcal{L})$ (respectively, $(X^v_w)^s_T(\mathcal{L})$) to denote the stable points for the T-linearized line bundle $\mathcal{L}$ on the Schubert variety (respectively, Richardson variety).

Every character λ of P defines a G-linearized line bundle on G/P. We denote the line bundle by $\mathcal{L}(\lambda)$. Furthermore, $\mathcal{L}(\lambda)$ is generated by global sections if and only if λ is a dominant weight (see [Jan07, Part II, Proposition 2.6]).

When $G = SL(n, \mathbb{C})$ and $P = P_{\hat{\alpha}_r}$, G/P is the Grassmannian parametrizing r-dimensional subspaces of $\mathbb{C}^n$. We denote it by $G_{r,n}$. The Grassmannian $G_{r,n}$ comes with the Plücker embedding $G_{r,n} \hookrightarrow \mathbb{P}(\bigwedge^r \mathbb{C}^n)$ sending each r-dimensional subspace to its r-th exterior wedge product (see [Ful97]). The pull back of $\mathcal{O}(1)$ from the projective space to $G_{r,n}$ is an ample generator of the Picard group of $G_{r,n}$ and corresponds to the T-linearized line bundle $\mathcal{L}(\omega_r)$. Gelfand and MacPherson [GM82] considered the GIT quotient of the Grassmannian and showed that the GIT quotient of n generic points in $\mathbb{P}^{r-1}$ by the diagonal action of $PGL(r, \mathbb{C})$ is isomorphic to the GIT quotient of $G_{r,n}$ with respect to the T-linearized line bundle $\mathcal{L}(n\omega_r)$. They showed that the torus action gives rise to a moment map from $G_{r,n}$ to $\mathbb{R}^n$, with the property that the image of each orbit is a convex polyhedron. This was extended by Gelfand et al. in [GGMS87]. In their paper the authors proposed three natural ways to stratify the Grassmannian - the first stratification is motivated by the equivalence of the torus quotient with the configuration of points in $\mathbb{P}^{r-1}$, the second is motivated by the moment map above, and the third is motivated by the geometry of intersections of Schubert cells in the Grassmannian. The authors show that no matter which definition is used to stratify the Grassmannian, the strata are the same.

In this case, there is an isomorphism between W and S_n, the group of permutations on n symbols, with s_{α_i}, $1 \leqslant i \leqslant n-1$ mapping to the transposition $(i, i+1)$. We sometimes use the one line permutation notation $(w(1), w(2), \ldots, w(n))$ to denote $w \in W$. $W_P = S_r \times S_{n-r}$, so the minimal length coset representatives of W^P can be identified with $\{w \in W | w(1) < w(2) < \ldots < w(r), w(r+1) < w(r+2) < \ldots < w(n)\}$. Let $I(r,n) = \{(i_1, i_2, ..i_r) | 1 \leqslant i_1 < i_2 \cdots < i_r \leqslant n\}$. Then there is a natural identification of W^P with $I(r,n)$ sending w to $(w(1), w(2), \ldots, w(r))$.

Hausmann and Knutson [HK97] used the stratification from [GGMS87] to study the GIT quotient of $G_{2,n}$ and related the resulting GIT quotient to the moduli space of polygons in $\mathbb{R}^3$.

Using the Hilbert-Mumford criterion, Skorobogatov [Sko93] gave combinatorial conditions determining when a point in $G_{r,n}$ is semistable with respect to the T-linearized bundle $\mathcal{L}(\omega_r)$. As a corollary he showed that when r and n are coprime semistability is the same as stability.

Independently, for a general G, Kannan (see [Kan98] and [Kan99]) gave a description of parabolic subgroups Q of G for which there exists an ample line bundle $\mathcal{L}$ on G/Q such that $(G/Q)^{ss}_T(\mathcal{L})$ is the same as $(G/Q)^{s}_T(\mathcal{L})$. In particular, in the case when $G = SL(n, \mathbb{C})$ and $Q = P_{\hat{\alpha}_r}$, Kannan showed that $(G_{r,n})^{s}_T(\mathcal{L}(\omega_r))$ is the same as $(G_{r,n})^{ss}_T(\mathcal{L}(\omega_r))$ if and only if r and n are coprime.

In the type A case when $G = SL(n, \mathbb{C})$ and Q is a parabolic subgroup, Howard [How05] considered the problem of determining which line bundles on G/Q descend to ample line bundles of the GIT quotient of G/Q by T. For a line bundle which descends to an ample line bundle on the quotient, by the Gelfand-MacPherson correspondence, the smallest power of the descent bundle that is very ample would give an upper bound on the degree in which the ring of invariants of n-points spanning projective space $\mathbb{P}^{r-1}$ is generated. Howard showed that when $\mathcal{L}(\lambda)$ is a very ample line bundle on G/Q (so the character of T extends to Q and to no larger subgroup of G) and $H^0(G/Q, \mathcal{L}(\lambda))^T$ is non-zero, the line bundle descends to the quotient (see [How05, Proposition 2.3, Theorem 2.3]). He extended these results to the case when the T-linearization of $\mathcal{L}(\lambda)$ is twisted by μ, a character of T. He proved that the line bundle $\mathcal{L}(\lambda)$ twisted by μ descends to the GIT quotient provided the μ-weight space of $H^0(G/Q, \mathcal{L}(\lambda))$ is non-zero, and this is so when $\lambda - \mu$ is in the root lattice and μ is in the convex hull of the Weyl orbit of λ. This was extended to other algebraic groups by Kumar [Kum08, Theorem 3.10]. It is known due to the work of Kumar [Kum08] that the line bundle $\mathcal{L}(n\omega_r)$ descends to the GIT quotient of $G_{r,n}$ with respect to the maximal torus T consisting of diagonal matrices in $SL(n,\mathbb{C})$. Thus, the line bundle $\mathcal{L}(n\omega_r)$ gives an embedding of the quotient variety $T\backslash\backslash(G_{r,n})^{ss}_T(\mathcal{L}(n\omega_r))$ in a projective space $\mathrm{Proj}(\bigoplus_{d\geqslant 0} H^0(G/P, \mathcal{L}(n\omega_r)^{\otimes d})^T)$.

1.2 Our results and organisation of the thesis

We discuss the preliminaries and background required for the thesis in Chapter 2. In §2.1, we recall the basic definitions from Geometric Invariant Theory. We recall some basic facts about finite dimensional representations of $SL(n,\mathbb{C})$ in §2.2. In chapter 3, we introduce the core objects of interest which is the Grassmannian variety and its two important classes of subvarieites namely the Schubert variety and the Richardson variety. We study the Standard monomial theory of Grassmannian in §3.2. In §3.3 we recall Deodhar decomposition of a Richardson variety which we need when computing the GIT quotient of certain Richardson varieties. And in §3.4 we discuss the singular locus of Schubert varieties in the Grassmannian.

In Chapter 4 we move to study the GIT quotient of a minimal dimensional Schubert variety in the Grassmannian. We can study many projective varieties like projective spaces, rational normal scrolls by realising them as the GIT quotient of a minimal dimensional Schubert variety in the Grassmannian. This also helps to understand conditions under which a torus quotient of a Schubert variety is smooth. This is discussed in Chapter 6.

Kannan, Sardar and Pattanayak (see [KP09a], [KPPU18],[KP09b]) studied the GIT quotients of minimal dimensional Schubert varieties in the Grassmannian admitting semistable points. We know from [KP09b] that there is a unique minimal Schubert Variety $X(w_{r,n})$ in $G_{r,n}$ admitting semistable points with respect to the line bundle $\mathcal{L}(n\omega_r)$. In this thesis we give an explicit calculation of $w_{r,n}$. Using our methods, we also find the smallest Richardson variety in $G_{r,n}$ admitting semistable points. We obtain:

Proposition 1.1 (4.4)**.** *Let r and n be coprime. Then $w_{r,n} = (a_1, a_2, \dots, a_r)$ where a_i is the smallest integer such that $a_i \cdot r \geqslant i \cdot n$.*

An understanding of the GIT quotient in the case $\gcd(r,n) \neq 1$ is difficult since stability is different from semistability. So we assume that $\gcd(r,n) = 1$. Under this assumption Skorobogotov (see [Sko93]) and Kannan(see [Kan98]) showed that the quotient variety $T\backslash\backslash(G_{r,n})^{ss}_T(\mathcal{L})$ is smooth. We study the smoothness for minimal dimensional Schubert varieties admitting semistable points under the same hypothesis. In this thesis we show

Theorem 1.2. *(4.1.1) Let r and n be coprime. Then the GIT quotient $T\backslash\backslash X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r))$ is smooth.*

Kannan and Pattanayak [KP09a] extended the results of [KP09b] to the cases when G is of Dynkin type B, C, D and when P is a maximal parabolic subgroup of G. Then $G/P_{\hat{\alpha}_r}$ has an ample line bundle $\mathcal{L}(\omega_r)$. Kannan and Pattanayak gave a combinatorial description of all minimal Schubert varieties in G/B admitting semistable points with respect to $\mathcal{L}(\lambda)$ for any regular dominant character λ of B.

The geometry of the GIT quotient of $T\backslash\backslash X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r))$ is not well understood. In this thesis we study this geometry using Deodhar decompositions of Richardson varieties in $X(w_{3,7})$ by computing the various Deodhar strata and analyzing their quotients. We show:

Theorem 1.3. *(4.2) The polarized variety $T\backslash\backslash X(w_{3,7})^{ss}_T(\mathcal{L}(7\omega_3))$ is projectively normal.*

We give an explicit description of the coordinate ring R of $T\backslash\backslash X(w_{3,7})^{ss}_T(\mathcal{L}(7\omega_3))$ in terms of generators and relations. Let $\mathcal{I}$ be the two sided ideal generated by the following relations in the polynomial ring $\mathbb{C}[Y_1, Y_2, \ldots, Y_7]$.

$$Y_1Y_4 = Y_2Y_3 - Y_2Y_7 + Y_1Y_7, \tag{1.1a}$$
$$Y_1Y_5 = Y_3^2 - Y_3Y_7, \tag{1.1b}$$
$$Y_1Y_6 = Y_3Y_4 - Y_4Y_7, \tag{1.1c}$$
$$Y_2Y_5 = Y_3Y_4 - Y_3Y_7, \tag{1.1d}$$
$$Y_2Y_6 = Y_4^2 - Y_4Y_7, \tag{1.1e}$$
$$Y_3Y_6 = Y_4Y_5. \tag{1.1f}$$

We show that the diamond lemma of ring theory holds for this reduction system (see [Ber78]). We show:

Theorem 1.4. *(4.17) $\mathbb{C}[Y_1, Y_2, \ldots, Y_7]/\mathcal{I} \simeq R$.*

By employing Deodhar decomposition, we obtain the following varieties as quotients of Richardson varieties.

(i) Let $v = s_2s_4s_3$. Then $T\backslash\backslash(X^v_{w_{3,7}})^{ss}_T(\mathcal{L}(7\omega_3))$ is a point.

(ii) Let $v = s_2s_3$. Then $T\backslash\backslash(X^v_{w_{3,7}})^{ss}_T(\mathcal{L}(7\omega_3))$ is isomorphic to $\mathbb{P}^1$ and the descent of $\mathcal{L}(7\omega_3)$ is $\mathcal{O}(1)$.

(iii) Let $v = s_4s_3$. Then $T\backslash\backslash(X^v_{w_{3,7}})^{ss}_T(\mathcal{L}(7\omega_3))$ is isomorphic to $\mathbb{P}^1$ and the descent of $\mathcal{L}(7\omega_3)$ to the GIT quotient is $\mathcal{O}(2)$.

(iv) Let $v = s_3$. Then $T\backslash\backslash(X^{v}_{w_{3,7}})^{ss}_{T}(\mathcal{L}(7\omega_3))$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and the descent of the line bundle to the GIT quotient is $\mathcal{O}(2) \boxtimes \mathcal{O}(1)$.

Using projective normality 4.2 we finally show:

Theorem 1.5. *(4.3) The polarized variety* $T\backslash\backslash(X^{id}_{w_{3,7}})^{ss}_{T}(\mathcal{L}(7\omega_3))$ *is a rational normal scroll.*

We also prove results of a general nature which apply to the GIT quotient of $X(w_{r,n})$ using standard monomial theory.

Proposition 1.6. *(4.1.1) Let r and n be coprime. Let $v_{r,n}$ be such that $X^{v_{r,n}}_{w_{r,n}}$ is the smallest Richardson variety in $X(w_{r,n})$ admitting semistable points. Then $v_{r,n} = (1, a_1, \ldots, a_{r-1})$ with a_i defined as the smallest integer satisfying $a_i r \geqslant i \cdot n$ (as in Proposition 4.4).*

Consider the Weyl group element $c_{r,n} = w_{r,n} v^{-1}_{r,n}$.

Proposition 1.7. *(4.1.1) $c_{r,n}$ is a Coxeter element.*

Theorem 1.8. *(4.8) $T\backslash\backslash(X^{v_{r,n}}_{w_{r,n}})^{ss}_{T}(\mathcal{L}(n\omega_r))$ is a point.*

Let $\Gamma_{r,n}$ denote the unique semistandard Young tableau with $v_{r,n}$ appearing as the first column and $w_{r,n}$ appearing as the last column. The explicit construction of such a tableau has been discussed in the proof of 4.4.

Theorem 1.9. *(4.9) Let $v \in W^{S\setminus\{\alpha_r\}}$ be such that $v < v_{r,n}$. Then, $T\backslash\backslash(X^{v}_{w_{r,n}})^{ss}_{T}(\mathcal{L}(n\omega_r))$ is isomorphic to $\mathbb{P}^1$ if and only if $v = s_\alpha v_{r,n}$ where $s_\alpha = (a_{i-1}, a_i)$ for some $i = 2, \ldots, r-1$. The descent of $\mathcal{L}(n\omega_r)$ to $T\backslash\backslash(X^{v}_{w_{r,n}})^{ss}_{T}(\mathcal{L}(n\omega_r))$ is $\mathcal{O}_{\mathbb{P}^1}(n_i)$ where n_i is the number of times $a_i - 1$ appears in the i-th row of the tableau $\Gamma_{r,n}$.*

Recall that the line bundle $\mathcal{L}(n\omega_r)$ gives an embedding of the quotient variety $T\backslash\backslash(G_{r,n})^{ss}_{T}(\mathcal{L}(n\omega_r))$ in a projective space $\mathrm{Proj}(\bigoplus_{d\geqslant 0} H^0(G/P, \mathcal{L}(n\omega_r)^{\otimes d})^T)$. However, in general it is still unknown whether this embedding gives rise to a projectively normal embedding and appears to be a surprisingly difficult question. The main hindrance is in understanding whether the ring $R = \bigoplus_{d\geqslant 0} H^0(G/P, \mathcal{L}(n\omega_r)^{\otimes d})^T$ is generated in least degree. The least degree generators of the ring R for even n and $r = 2$ have been studied by Howard, Milson, Snowden and Vakil (see, [HMSV05], [Kem93]) using graph theoretic methods. Recently, A.Nayek, S.K Pattanayak and S.Jindal gave another proof of the projective normality of $T\backslash\backslash(G_{2,n})^{ss}_{T}(\mathcal{L}(n\omega_2))$ using graph theoretic methods (see, [NPJ20]). In this thesis, Chapter 5 we have shown that the ring R is generated in least degree, for odd n and $r = 2$, using the combinatorics of Young tableaux obtained from standard monomial theory and straightening relations of Plücker coordinates. More precisely, in Chapter 5 we prove the following theorem.

Theorem 1.10 (5.1)**.** *Let n be odd. Then $T\backslash\backslash(G_{2,n})^{ss}_T(\mathcal{L}(n\omega_2))$ is projectively normal.*

We also obtain the following corollaries for Schubert varieties and Richardson varieties in $G_{2,n}$.

Corollary 1.11. *The GIT quotient of a Schubert variety in $G_{2,n}$ is projectively normal with respect to the descent line bundle.*

Corollary 1.12. *The GIT quotient of a Richardson variety in $G_{2,n}$ is projectively normal with respect to the descent line bundle.*

In chapter 6, we use 4.1.1 and give a combinatorial criterion for when the GIT quotient of a Schubert variety in the Grassmannian, $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth. We show:

Theorem 1.13. *(6.3)Let $w = (b_1, b_2, \ldots, b_r) \in I(r, n)$ with $b_i \geqslant a_i$ for all $1 \leqslant i \leqslant r$. Let $X(v_1), \ldots, X(v_k)$, be the k components in the singular locus of $X(w)$. Then the following are equivalent*

(1) $T\backslash\backslash X(w)^{ss}_T(\mathcal{L}(n\omega_r))$ *is smooth.*

(2) *For all i, we have $v_i \ngeqslant w_{r,n}$.*

(3) *Whenever $b_j \geqslant b_{j-1} + 2$ we have $a_j \geqslant b_{j-1} + 1$.*

In other words the GIT quotient is smooth precisely when the semistable locus does not intersect the singular locus, and there is a simple combinatorial criterion describing when this happens.

Chapter 2

Preliminaries

2.1 Preliminaries on GIT

We briefly review the definitions and preliminaries which we will be needing in this thesis here. For the basics of algebraic groups we direct the reader to [Hum12b], [Jan07]. We have mostly followed [LR07, Chapter 9] and [New78] for basics regarding Geometric Invariant Theory (GIT).

Definition 2.1. An *action* of an algebraic group on a variety X is a morphism

$$\sigma : G \times X \to X$$

such that for all $g, g' \in G, x \in X$, $\sigma(g, \sigma(g', x)) = \sigma(gg', x)$ and $\sigma(e, x) = x$. where e denotes the identity of G.

For $x \in X$, we denote the orbit $\mathcal{O}(x)$ of x the subset $\{gx | \ g \in G\}$. The stabiliser G_x of x is the closed subgroup $\{g \in G | gx = x\}$.

Let $R^G = \{r \in R | g.r = r \text{ for all } g \in G\}$ denote the ring of invariants. We recall from [New78, Theorem 3.4] that R^G is finitely generated for a reductive algebraic group G.

Definition 2.2. Let G be a group acting on an algebraic variety X. A pair (Y, ϕ) is called an categorical quotient if it satisfies the following

(i) Y is an algebraic variety.

(ii) ϕ is a morphism of varieties from X to Y.

(iii) ϕ is G-invariant, i.e ϕ is constant on the G-orbits.

(iv) Whenever we have $f : X \to Z$ which is constant on the G-orbits, there exists a unique $g : Y \to Z$ such that $f = g \circ \phi$.

Let G be a reductive group acting on an affine variety X. Let $R = \mathbb{C}[X]$ denote the ring of regular functions on X. We will denote X as Spec(R) and $Y = \mathrm{Spec}(R^G)$. Then there is an action of G on R as

$$(g.f)(x) = f(g^{-1}.x)$$

for all $g \in G, f \in R$ and $x \in X$. It can be checked that $X//G = Y$ defined as above is a categorical quotient.

To define the notion of quotients to non-affine varieties we need to introduce the notion of good quotients and a geometric quotient. We recall the definitions from [LR07, Chapter 9]

Definition 2.3. Let G be a group acting on an algebraic variety X. A pair (Y, ϕ) is called an *good quotient* if it satisfies the following:

(i) $\phi : X \to Y$ is a surjective morphism.

(ii) ϕ is G-invariant.

(iii) ϕ is an affine morphism i.e the inverse image of an open affine set is again affine open set.

(iv) If $W \subset X$ is G-stable and closed, then $\phi(W)$ is closed.

(v) If W_1, W_2 are two disjoint G-stable closed subsets of X, then $\phi(W_1) \cap \phi(W_2) = \emptyset$.

(vi) Given open set $U \subset Y$, the map

$$\phi^* : \mathbb{C}[U] \longrightarrow \mathbb{C}[\phi^{-1}(U)]^G$$

is an isomorphism.

Definition 2.4. Let G be a group acting on an algebraic variety X. A pair (Y, ϕ) is called an *geometric quotient* if it is a good quotient and Y is the orbit space X/G.

Let G be as above acting on a projective variety X in $\mathbb{P}^n$. Let $\pi : \mathcal{L} \to X$ be an ample line bundle.

Definition 2.5. We say $\mathcal{L}$ is G- linearised if there is a G-action

$$\Phi : G \times \mathcal{L} \to \mathcal{L}$$

on $\mathcal{L}$ such that

(i) $\pi : \mathcal{L} \to X$ is G-equivariant. That is the following diagram commutes :

$$\begin{tikzcd} G \times \mathcal{L} \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] \\ G \times X \arrow[r] & X \end{tikzcd}$$

(ii) For $x \in X$ let $\mathcal{L}_x$ denote the fiber of x. Then the action is linear on fibers, i.e for $g \in G$ and $x \in X$ the map $\Phi_x : \mathcal{L}_x \to \mathcal{L}_{gx}$ is linear.

We can now define the notion of semistable and stable points with respect to a G-linearised line bundle $\mathcal{L}$. Since we will be only interested in projective varieties we will restrict our attention to them. Recall from [MFK94], [New78].

Definition 2.6. Let X be a projective variety. Let G be a reductive group acting on X and $\mathcal{L}$ be a G-linearised line bundle on X. A point $x \in X$ is called *semistable* if for some positive integer r, there exist G-invariant section f of $\mathcal{L}^r$ such that $f(x) \neq 0$ and X_f is affine.

Definition 2.7. A semistable point x is called *stable* if the action of G on X_f is closed and dim $\mathcal{O}(x) =$ dim G.

We will denote by $X^{ss}_G(\mathcal{L})$ (respectively, $X^s_G(\mathcal{L})$) the semistable (respectively, stable) locus of X with respect to the G-linearised line bundle $\mathcal{L}$.

Once we have a linearisation on $\mathcal{L}$ we automatically get a linearisation on $\mathcal{L}^{\otimes n}$: Denote by $H^0(X, \mathcal{L}^{\otimes n})^G$ the space of G-invariant sections of $\mathcal{L}^{\otimes n}$. Let s denote a section of $\mathcal{L}^{\otimes n}$. Then the action of G on $H^0(X, \mathcal{L}^{\otimes n})$ is defined by

$$(g.s)(x) = g.s(g^{-1}.x).$$

Then the GIT (Geometric Invariant Theory) quotient of X with respect to the G-linearised line bundle $\mathcal{L}$ is given by

$$G \backslash\backslash (X^{ss})_G(\mathcal{L}) = \mathrm{Proj}(\bigoplus_{n \geqslant 0} H^0(X, \mathcal{L}^{\otimes n})^G).$$

We now recall the definition of projective normality of a variety from [Har13]. Let X be a projective variety and $\phi : X \subset \mathbb{P}^n$ be an embedding. Let $S(X)$ denote the homogeneous coordinate ring of X (see §2 [Har13]). Let $\mathcal{L} = \phi^*\mathcal{O}(1)$. Let $S'(X) = \oplus_{n \geqslant 0} H^0(X, \mathcal{L}^{\otimes n})$.

Definition 2.8. A projective variety $X \subset \mathbb{P}^n$ is projectively normal with respect to the given embedding if $S(X)$ is normal.

Recall from [Har13, Exercise 5.14], X is projectively normal for an embedding given by line bundle $\mathcal{L}$ if and only if X is normal and $S'(X)$ is generated in degree one as a $\mathbb{C}$- algebra. This equivalent characterisation is the one which we will use when we show projective normality of GIT quotient of a Grassmannian $G_{2,n}$ (see 5), and that of a minimal dimensional Schubert varieties (see 4) in $G_{3,7}$.

We will need another geometric property which is intrinsic of the variety called smoothness in Chapter 6 where we study the GIT quotients of Schubert varieties of a Grassmannian $G_{r,n}$ for r and n coprime and classify the Schubert varities that admits smooth quotients. We recall the definition from [Har13, §5].

Definition 2.9. Let X be any variety. X is *nonsingular* or *smooth* at a point $p \in X$ if the local ring $\mathcal{O}_{p,X}$ is a regular local ring. We say that the variety X is *nonsingular* or *smooth* if it is smooth at every point. X is *singular* if it is not a smooth variety.

2.2 Preliminaries on representation theory of $SL(n, \mathbb{C})$

We briefly recall some basics from $G = SL(n, \mathbb{C})$ representation theory. Most of the facts presented here follows from the general fact that $SL(n, \mathbb{C})$ is a semisimple algebraic group. However, since in this thesis we will not be dealing with generalities, we present it only for $SL(n, \mathbb{C})$. There are many references for the same. To cite a few are [Hum12b], [Hum12a], [Jan07], [LR07]. The reference which is most useful in our context is [LR07].

Let $G = SL(n, \mathbb{C})$ be the group of complex $n \times n$ matrices with determinant 1. Let T denote the group of diagonal matrices in G. Let B the subgroup of upper triangular matrices in G and B^- the subgroup of lower triangular matrices in G. The unipotent subgroup U is the subgroup of B with diagonal entries 1, and U^- is the unipotent subgroup of B^- with diagonal entries 1. Let $\mathcal{G}$ denote the category of linear algebraic groups. Let $X(T) = \mathrm{Hom}_{\mathcal{G}}(T, \mathbb{C}^*)$ denote the group of characters and $Y(T) = \mathrm{Hom}_{\mathcal{G}}(\mathbb{C}^*, T)$ denote the set of cocharacters. Let $\lambda \in X(T)$ and $\mu \in Y(T)$. For $t \in \mathbb{C}^*$. We have $\lambda \circ \mu(t) = t^m$ for some integer m. Let $m = \langle \lambda, \mu \rangle$. We recall that

$$\langle , \rangle : X(T) \times Y(T) \to \mathbb{Z}$$

$$(\lambda, \mu) \mapsto \langle \lambda, \mu \rangle$$

is a perfect pairing (see, [Hum12b]). So $Y(T) \simeq \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$.

We know that the Lie algebra $\mathfrak{g}$ of G is $\mathfrak{sl}(n)$ which is the vector space formed by traceless $n \times n$ complex matrix. The Lie bracket is given by

$$[X, Y] = XY - YX$$

Analogously, let $\mathfrak{h}$ denote the subalgebra of diagonal matrices, $\mathfrak{b}$ the subalgebra of upper triangular matrices in $\mathfrak{g}$ and $\mathfrak{b}^-$ the subalgebra of lower triangular matrices in $\mathfrak{g}$. The nilpotent subalgebra $\mathfrak{n}$ is the subgroup of $\mathfrak{b}$ with diagonal entries 0, and $\mathfrak{n}^-$ is the subalgebra of $\mathfrak{b}^-$ with diagonal entries 0. For $i \neq j$, let $E_{i,j}$ denote the matrix with ij-th entry 1 and other entries are zero. Let $h = \mathrm{diag}(h_1, h_2, \ldots, h_n) \in \mathfrak{h}$. Then

$$[h, E_{ij}] = (h_i - h_j)E_{ij}.$$

Define

$$\epsilon_i - \epsilon_j : \mathfrak{h} \to \mathbb{C}$$

$$h \mapsto h_i - h_j.$$

Let

$$R = \{\epsilon_i - \epsilon_j | 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n, i \neq j\};$$

$$R^+ = \{\epsilon_i - \epsilon_j | 1 \leqslant i < j \leqslant n\};$$

$$S = \{\epsilon_i - \epsilon_{i+1} | 1 \leqslant i \leqslant n - 1.\}$$

Let $V = \{(x_1, x_2, \ldots, x_{n-1}) | \sum x_i = 0\}$. V is a $n-1$ dimensional subspace of $\mathbb{R}^n$ which gets identified with $X(T) \otimes \mathbb{R}$. We know there is a standard inner product in V given by $(.,.)$ (see [Hum12a, §9.1]). Define a reflection relative to a nonzero vector $\alpha \in V$ to be the linear transformation given by

$$s_\alpha(v) = v - \frac{2(v, \alpha)}{(\alpha, \alpha)}\alpha$$

We can check R satisfies the following properties.

- R is finite, spans V and doesnot contain 0.
- If $\alpha \in R$, the only multiples of $\alpha \in R$ are $\pm\alpha$.
- For each $\alpha \in R$, R is stable under s_α.
- For $\alpha, \beta \in R$, $\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer.

R is called the set of *roots*. The subset S forms a basis of V and every element of R is either is nonnegative integral linear combination of S or a nonpositive integral linear combination of elements of S. So S is called a set of *simple roots*. The set R^+ is the set of elements of R which are nonnegative $\mathbb{Z}$ linear combinations of elements of S. So R^+ is called the set of *positive roots*. Let $\omega_1, \omega_2, \ldots, \omega_{n-1}$ be the dual basis relative to the inner product in V: $\frac{2(\omega_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{i,j}$, where $\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}$. The weight ω_i is called a *fundamental weight* relative to S. The fundamental weight ω_i is given by

$$\omega_i = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_i - \frac{i}{n}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n).$$

Note that $\epsilon_i - \epsilon_j$ may be considered as a character of T which sends $\mathrm{diag}(t_1, t_2, \ldots t_n)$ in T to $t_i {t_j}^{-1}$. The fundamental weight ω_i will denote the character of T which sends $\mathrm{diag}(t_1, t_2, \ldots t_n)$ in T to $t_1.t_2.\ldots.t_i$.

We recall the Cartan decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{h} \bigoplus (\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha)$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$.

We know the Weyl group W for a general reductive group G with respect to T is defined as $N_G(T)/T$ where $N_G(T)$ is the normaliser of the maximal torus. In our setup i.e $G = SL(n, \mathbb{C})$, $N_G(T)$ gets identified with the set of monomial matrices in G i.e an element of $N_G(T)$ is a matrix X in G such that in each row and column of X there is exactly one non-zero element. So the Weyl group W gets identified with the symmetric group in n letters S_n.

For each simple root α_i we have a morphism $\phi_i : SL(2, \mathbb{C}) \to G$, with ϕ_i sending $M \in SL(2, \mathbb{C})$ to the $n \times n$ matrix having M in rows and columns $i, i+1$, with the other diagonal entries being 1 and the remaining entries zero. We use the following notation:

$$x_i(m) = \phi_i \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \quad y_i(p) = \phi_i \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \quad \alpha_i^\vee(t) = \phi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad \dot{s}_{\alpha_i} = \phi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

To each root $\alpha = \epsilon_i - \epsilon_j$ we can associate a one dimensional unipotent subgroup U_α called the root subgroup with entries are $I + cE_{i,j}$, $c \in \mathbb{C}$, $E_{i,j}$ denote the elementary matrix. Let $\mathbb{G}_a$ denote the additive group $\mathbb{C}$. There is a isomorphism

$$\theta_\alpha : \mathbb{G}_a \to U_\alpha$$

such that for all $t \in T$, $x \in \mathbb{G}_a$, $t\theta_\alpha(x)t^{-1} = \theta_\alpha(\alpha(t)x)$. (see, [LW90, §2.3]).

Recall [Hum12b, Theorem 28.3] the Bruhat decomposition for G is given by

$$G = \bigsqcup_{w \in W} BwB.$$

Definition 2.10. A *parabolic subgroup* Q of G is a closed subgroup of G such that G/Q is a projective variety.

A subgroup Q is parabolic if and only if it contains a Borel subgroup of G.(see, [Hum12b, Corollary 21.3 B]). For each subset $I \subset S$, let W_I be the subgroup of W generated by s_α with $\alpha \in I$. Let

$$P_I = \bigsqcup_{w \in W_I} BwB.$$

Clearly P_I is a parabolic subgroup of G. Conversely, let Q be a parabolic subgroup of G containing B. We can associate a subset of I of S such that $Q = P_I$ (see, [Hum12b, Theorem 29.3]). Since we have the Borel fixed, we will alternatively write W_Q instead of W_I and the set I as S_Q. $W^Q = W/W_Q$ will be called the set of *minimal representatives of* W/W_Q and is defined as :

$$W^Q = \{w \in W | w(\alpha) > 0 \text{ for all } \alpha \in S_Q.\}$$

For $w \in W$ let $e_{w,Q}$ denote the coset wQ in W/W_Q. Then the set of T-fixed points in G/Q for the left multiplication action of G is precisely $\{e_{w,Q} | w \in W^Q\}$.

Definition 2.11. For $w \in W^Q$, the Zariski closure of $Be_{w,Q}$ in G/Q is called the *Schubert variety* in G/Q associated to wW_Q and is denoted by $X_Q(w)$.

So the Schubert varieties are indexed by the set of minimal representatives of W/W_Q. We now recall the extended version of Bruhat decomposition (see, [LW90]).

$$G/Q = \bigsqcup_{w \in W^Q} Be_{w,Q}$$

and

$$X_Q(y) = \bigsqcup_{w \in W^Q, e_{w,Q} \in X_Q(y)} Be_{w,Q}.$$

Definition 2.12. For $w \in W^Q$ the *length* of w is defined to be the minimum l such that there is an expression $w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \dots s_{\alpha_{i_l}}$ where s_{α_i} is a simple reflection for every i. We denote the length of w by $l(w)$. An expression $w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \dots s_{\alpha_{i_k}}$ for which $k = l(w)$ is called a *reduced expression* for w. Recall from [LR07, §3.3] that the length of a word $w \in W^Q$ can also be determined as $l(w) = \dim X_Q(w)$.

There is a partial order on W^Q called the Bruhat order induced by the partial order on the set of Schubert varieties given by inclusion: For $w_1, w_2 \in W^Q, w_1 \leqslant w_2$ if and only if $X_Q(w_1) \subset X_Q(w_2)$ (see, [LR07, §3.3]).

For a $\lambda \in X(T)$ we can associate a line bundle as follows: Let $X = G/B$ and $\pi : G \to X$ denote the projection map. Let $\lambda \in X(T)$. λ can be extended to a character of B by declaring it 1 in the unipotent radical. So now we have $\lambda : B \to GL(\mathbb{C}_\lambda)$. Note $\mathbb{C}_\lambda$ as a vector space is $\mathbb{C}$ and to distinguish it as a B-representation we write $\mathbb{C}_\lambda$. Let $\mathcal{L}_\lambda$ denote the line bundle whose total space is given by $E = \frac{G \times \mathbb{C}_\lambda}{\sim}$, where the equivalence $\sim$ is defined by $(gb, \lambda(b^{-1}x)) \sim (g, x), g \in G, b \in B, x \in \mathbb{C}_\lambda$. Since $G = SL(n, \mathbb{C})$. By Chevalley's Theorem (see, [LW90, §2.8]) the map

$$\mathcal{L} : X(T) \to \mathrm{Pic}(G/B) \qquad (2.2)$$

$$\lambda \mapsto \mathcal{L}_\lambda$$

is an isomorphism of abelian groups.

We know a basis of $X(T)$ is given by the fundamental weights $\omega_i, 1 \leqslant i \leqslant n-1$. For $\lambda = \sum_{1 \leqslant i \leqslant n} a_i \omega_i$ the line bundle in the image is given by $\mathcal{L}_\lambda = \bigotimes_{1 \leqslant i \leqslant n} \mathcal{L}_{\omega_i}^{\otimes a_i}$. Note $a_i = \langle \lambda, \alpha_i \rangle$. We say λ *regular dominant* if $a_i > 0$ for all i. Recall from [Jan07, Prop 4.4, Part II] $\mathcal{L}_\lambda$ is ample if and only if λ is regular dominant.

Chapter 3

The Grassmannian

3.1 Preliminaries on Grassmannian

In this chapter we briefly introduce the Grassmannian variety and two important classes of subvarieties of the Grassmannian namely Schubert varieties and Richardson varieties. In §3.2 we then describe the standard monomial theory for the Schubert varieties in the Grassmannian. In §3.3 we recall the Deodhar decompositions of the Richardson varieties and then briefly mention the standard monomial theory for Richardson varieties. And finally in §3.4 we discuss the singular loci of Schubert varieties in the Grassmannian. We continue to use the notations from §2.2.

Definition 3.1. Let $1 \leqslant r \leqslant n$. The *Grassmannian variety* is the space of all r- dimensional subspaces of a n- dimensional complex vector space W and is denoted by $G_{r,n}$.

For $V \in G_{r,n}$ fix a basis $\{v_1, \dots v_r\}$ of V and define the map

$$\mathcal{P} : G_{r,n} \longrightarrow \mathbb{P}(\bigwedge^r W)$$

$$V \mapsto [v_1 \wedge v_2 \dots \wedge v_r].$$

Suppose $\{u_1, \dots u_r\}$ be another basis of V. Let M denote the $r \times r$ matrix that takes $\{v_1, \dots v_r\}$ to $\{u_1, \dots u_r\}$. Then we know $v_1 \wedge v_2 \dots \wedge v_r = \det(M) u_1 \wedge u_2 \dots \wedge u_r$. So the map $\mathcal{P}$ is well defined and is called the *Plücker map*. We recall from [LR07, Theorem 4.1.2.1] that the map $\mathcal{P}$ is injective, and hence gives an embedding which is called the *Plücker embedding*.

Define the set

$$I(r, n) := \{(i_1, i_2, \dots, i_r) | 1 \leqslant i_1 < i_2 \dots . < i_r \leqslant n\}.$$

Define a partial order $\leqslant$ on $I(r,n)$ as $\underline{i} \leqslant \underline{j}$ if and only if $i_t \leqslant j_t$ for all $1 \leqslant t \leqslant r$. This order is called the Bruhat order on $I(r,n)$.

Let $\{e_1, e_2, \dots e_n\}$ denote the standard basis of W. We know

$$\{e_{\underline{i}} = e_{i_1} \wedge e_{i_2} \dots \wedge e_{i_k}\}_{\tau \in I(k,n)}$$

forms a basis of $\bigwedge^r W$. The Plücker coordinates p_ϕ are the basis of $(\bigwedge^r W)^*$ dual to $\{e_\tau\}_{\tau \in I(k,n)}$. That is

$$p_{\underline{j}}(e_{\underline{i}}) = \begin{cases} 1 & \text{if } \underline{i} = \underline{j}; \\ 0 & \text{otherwise.} \end{cases}$$

Let V be a r dimensional subspace of W, let $g \in G$. Then $gV = \{gv | v \in V\}$ is again a r-dimensional subspace of W. This defines a transitive action of G on $G_{r,n}$. Let U be the subspace spanned by $\{e_1, \dots e_r\}$. The stabiliser of U is the subgroup $P = \begin{bmatrix} * & * \\ 0_{n-r,r} & * \end{bmatrix}$. So $G_{r,n}$ becomes a homogeneous space G/P. Note from 2.2 that P is a maximal parabolic subgroup that corresponds to the subset $S \setminus \{\alpha_r\}$. Then the subgroup W_P gets identified with $S_r \times S_{n-r}$. And the set of *minimal length coset representatives* of W/W_P are

$$W^P = \{(w_1, \dots, w_n) \in S_n | w_1 < w_2 < \dots < w_r; w_{r+1} < \dots < w_n\}.$$

So we note that W^P gets identified with the set $I_{r,n} : w = (w_1, \dots w_n) \mapsto (w_1, \dots w_r)$.

Sometimes we would like view the elements $(w_1, w_2, \dots, w_r)$ as a column of numbers. In which case we denote it by $[w_1, w_2, \dots, w_r]$.

The T-fixed points are given by e_w for $w \in I(r,n)$. The B-orbit through e_w is the Schubert cell and its Zariski closure in G/P is the Schubert variety corresponding to w and is denoted by $X(w)$. For $w = (i_1, i_2, \dots, i_r)$ we have

$$X(w) = \{U \in G_{r,n} | \dim(U \cap V_{i_t}) \geqslant t, 1 \leqslant t \leqslant d\},$$

where V_i is the subspace of V spanned by $\{e_1, e_2, \dots, e_i\}$. Once we have the above definition the Bruhat order can be also given as $v \leqslant w$ if and only if $X(v) \subset X(w)$. Note that $G_{r,n}$ is the Schubert variety $X(w_0^P)$, where $w_0^P = (n-r+1, n-r+2, \dots, n)$.

We recall that given $(b_1, \dots, b_r) \in I(r,n)$, one reduced expression for the Weyl group element in W^P corresponding to this is $(s_{b_1-1} \cdots s_1) \dots (s_{b_r-1} \cdots s_r)$ where a bracket is assumed to be empty is if $b_i - 1$ is less than i.

3.2 Standard monomial theory for the Grassmannian

In this section, we will describe the standard monomial theory for Schubert varieties in Grassmannian. We know that the Picard group of the Grassmannian $\mathrm{Pic}(G_{r,n})$ is a rank one, free abelian group generated by the line bundle $\mathcal{L}_{\omega_r} = \mathcal{P}^*(\mathcal{O}(1))$ (see 2.2). This line bundle also gives an embedding of a Schubert variety $X(w)$ in the Grassmannian. Let $H^0(X(w), \mathcal{L}_{\omega_r})$ be the global section for the line bundle $\mathcal{L}_{\omega_r}$. Let $\mathbb{C}[X(w)]$ be the homogeneous coordinate ring of $X(w)$ for this projective embedding i.e $\mathbb{C}[X(w)] = \bigoplus_{d \geqslant 0} H^0(X(w), \mathcal{L}_{\omega_r}^{\otimes d})$. Standard monomial theory explicitly constructs a nice basis for the space $H^0(X(w), \mathcal{L}_{\omega_r}^{\otimes d})$. Let $\tau \in I(r, n)$, and p_τ be a Plücker coordinate.

Definition 3.2. A monomial $p_{\underline{\tau}}$ of degree d is an expression of the form $p_{\tau_1} p_{\tau_2} \dots p_{\tau_d}$.

We associate to each monomial $p_{\underline{\tau}} = p_{\tau_1} \cdots p_{\tau_d}$ a Young tableau $T_{\underline{\tau}}$ of shape $\underbrace{(d, d, \dots, d)}_{r \text{ times}}$ whose i-th column is filled with $\tau_i = [\tau_i(1), \tau_i(2), \dots, \tau_i(r)]$(see [Ses, Chapter 1]).

Definition 3.3. A monomial $p_{\tau_1} p_{\tau_2} \dots p_{\tau_d}$ is said to be *standard* if

$$\tau_1 \leqslant \tau_2 \dots \leqslant \tau_d.$$

Such a monomial is said to be *standard on* $X(w)$ if in addition we have

$$\tau_d \leqslant w.$$

In our convention we say a Young tableau $\mathcal{Y}$ is semistandard if the rows of $\mathcal{Y}$ are weakly increasing and the columns are strictly increasing. Note that the tableau $T_{\underline{\tau}}$ associated with the monomial $p_{\underline{\tau}} = p_{\tau_1} \cdots p_{\tau_d}$ is semistandard if and only if $p_{\underline{\tau}}$ is a standard monomial.

Recall from [LR07],

Theorem 3.4. *[LR07, Theorem 4.3.3.2] Standard monomials on $X(w)$ of degree d forms a basis of $H^0(X(w), \mathcal{L}_{\omega_r}^{\otimes d})$.*

Each standard monomial (more generally any monomial $p_{\tau_1} p_{\tau_2} \dots p_{\tau_d}$) is a T-weight vector. Let $a(i)$ denote the number of times integer i appears in the tableau. Then we have $\mathrm{diag}(t_1, t_2, \dots, t_n).p_{\tau_1} p_{\tau_2} \cdots p_{\tau_d} = \prod_i t_i^{a(i)} p_{\tau_1} \cdots p_{\tau_d}$.

Remark 3.5. Let $\pi_w^d : H^0(G_{r,n}, \mathcal{L}_{\omega_r}^{\otimes d}) \longrightarrow H^0(X(w), \mathcal{L}_{\omega_r}^{\otimes d})$ denote the restriction map. We recall from [LR07, §4.3.4] that π_w^d is surjective and the kernel of π_w^1 has a basis given by $\{p_\tau | \tau \not\leq w\}$. The Schubert variety $X(w)$ as a closed subvariety of $G_{r,n}$ can also be defined as the vanishing of p_τ, $\tau \not\leq w, \tau \in I(r, n)$. Using 3.4 we can conclude that the kernel of π_w^d has a basis given standard monomials $p_{\tau_1}p_{\tau_2}\dots p_{\tau_d}$ with $\tau_1 \leqslant \tau_2 \dots \leqslant \tau_d$ and $\tau_d \nleqslant w$. We can give a similar description for Schubert subvarieties of Schubert varieties in the Grassmannian: Let $v \in I(r,n)$ be such that $v \leqslant w$ in the Bruhat order, so that $X(v)$ is a closed subvariety of $X(w)$. Denote $\pi^d_{w,v} : H^0(X(w), \mathcal{L}^{\otimes d}_{\omega_r}) \longrightarrow H^0(X(v), \mathcal{L}^{\otimes d}_{\omega_r})$ the restriction map. Then $\pi^d_{w,v}$ is a surjection and the kernel of $\pi^1_{w,v}$ has a basis given by $\{p_\tau | \tau \nleqslant v, \tau \leqslant w\}$. The Schubert variety $X(v)$ as a closed subvariety of $X(w)$ can also defined as the vanishing of p_τ, $\tau \nleqslant v, \tau \leqslant w, \tau \in I(r,n)$. Using 3.4 we can conclude that the kernel of $\pi^d_{w,v}$ has a basis given standard monomials $p_{\tau_1}p_{\tau_2}\dots p_{\tau_d}$ with $\tau_1 \leqslant \tau_2 \dots \leqslant \tau_d$ and $\tau_d \nleqslant v, \tau \leqslant w$.

We recall the degree lexicographic order on rectangular $r \times m$ Young tableau. Recall that as per this order a monomial $p = p_{\tau_1}\dots p_{\tau_m}$ corresponding to a $r \times m$ tableau is bigger than a monomial $q = q_{\mu_1}\dots q_{\mu_{m'}}$,corresponding to another $r \times m'$ tableau if $m > m'$ or, if $m = m'$, then for the smallest i such that $\tau_i \neq \mu_i$ it is the case that $\tau_i > \mu_i$ in the usual lexicographic order on words of length r. We denote this by $p >_{lex} q$ or $q <_{lex} p$

Let $\tau = (i_1, i_2, \dots, i_r)$ and $\mu = (j_1, j_2, \dots, j_r)$. We say the Plücker coordinates p_τ and p_μ are not comparable or τ and μ are not comparable iff there is a $t \geqslant 1$ and a $s > t$ such that $i_t < j_t$ and $i_s > j_s$ or there is a $t \geqslant 1$ and a $s > t$ such that $i_t > j_t$ and $i_s < j_s$. We now recall the straightening procedure which is used to describe the product $p_\tau p_\mu$ as a linear combination of standard monomials.

Following notations from [HL18, §2.3]. Let $(a_1, a_2, \dots a_r)\uparrow$ denote the tuple $(a_1, \dots, a_r)$ arranged in increasing order. Let $\tau = (i_1, i_2, \dots, i_r)$ and $\mu = (j_1, j_2, \dots, j_r)$ be two noncomparable words such that $\tau <_{lex} \mu$. Let t be smallest integer such that $i_t > j_t$. Let $Sh(\tau, \mu)$ be the set of all permutations σ of $\{i_t, i_{t+1}, \dots, i_r, j_1, j_2, \dots, j_t\}$ with $\sigma(i_t) < \sigma(i_{t+1}) < \dots < \sigma(i_r)$ and $\sigma(j_1) < \sigma(j_2) < \dots < \sigma(j_t)$. Let $\sigma(\tau) = (i_1, i_2, \dots, i_{t-1}, \sigma(i_t), \sigma(i_{t+1}), \dots \sigma(i_r))\uparrow$ and $\sigma(\mu) = (\sigma(j_1), \sigma(j_2), \dots \sigma(j_t), j_{t+1}, \dots j_r)\uparrow$. Then we have,

$$p_\tau p_\mu = \sum_{\sigma \in Sh(\tau,\mu)} \pm p_{\sigma(\tau)} p_{\sigma(\mu)}. \tag{3.1}$$

The sign of the permutation can be easily deduced by noting the sign of the shuffle permutation. For this thesis we will not need the sign so we ignore.

We will crucially use the fact which follows from [Ses, Lemma 1.3.5] that in equation 3.1

$$p_\tau p_\mu >_{lex} p_{\sigma(\tau)} p_{\sigma(\mu)},$$

for all σ.

Example 3.6. We will explain the above definitions and notations in a simple example of $G_{2,4}$, the Grassmannian variety of planes in $\mathbb{C}^4$. The homogeneous coordinate ring $\mathbb{C}[G_{2,4}]$ is generated as an algebra by the following Plücker coordinates: $p_{(1,2)}, p_{(1,3)}, p_{(2,3)}, p_{(1,4)}, p_{(2,4)}$ and $p_{(3,4)}$. The straightening law here is :

$$p_{(1,4)} \cdot p_{(2,3)} = p_{(1,3)} \cdot p_{(2,4)} - p_{(1,2)} \cdot p_{(3,4)}$$

which in terms of tableau can also be described as:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

The degree d part of $\mathbb{C}[G_{2,4}]$ is $H^0(G_{2,4}, \mathcal{L}_{\omega_2}^{\otimes d})$ has a basis consisting of standard monomials of degree d in p_τ. For example, the two semistandard Young tableau appearing as summands in the right hand side of the above expression corresponds to degree 2 standard monomials. However the monomial appearing in the left hand side of the above equation is not standard monomial.

Example 3.7. We can consider the $G_{3,6}$, and consider the monomials p_τ and p_μ where $\tau = (1,4,5)$ and $\mu = (2,3,6)$. We have $\tau <_{lex} \mu$ however they are incomparable, if we straighten we obtain:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 & 6 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}$$

3.3 Deodhar decomposition to compute quotients of Richardson varieties

In chapter 4 we will study the GIT quotients of Richardson varieties in the Grassmannian. In Section 4.1 we proved some results on quotients of Richardson varieties. A natural strategy to understand the GIT quotient is to take a stratification of a Richardson variety, understand what the GIT quotient of each strata is, and also understand how the GIT quotients of these strata patch up. Such a stratification of the open cell of a Richardson variety was given by Deodhar [Deo85]. This was to be our starting point. Working with small examples we believed that the restriction of a T-invariant section to the open cell would be a homogenous polynomial and that this would lead us to discover the equations defining the GIT quotient of a Richardson variety. However we soon realized that sections may not restrict to homogenous polynomials on the open cell, that the issue is more subtle. We have necessary conditions which guarantee when sections restrict to homogenous polynomials on the open cell. This is Lemma 4.23. To state the Lemma and also the proof we need to introduce the Deodhar decomposition and some more notation and theorems about Deodhar decomposition of Richardson varieties on the Grassmannian. We do that in the next Subsection 3.3.1. We use the Deodhar decomposition to study the GIT quotients of Richardson varieties in $X(w_{3,7})$ in Section 4.3. Although all these results follow from the results in Section 4.1 we prove them again since this can be done by explicit calculations. Finally we show that the GIT quotient of $X(w_{3,7})$ is a rational normal scroll. We were unable to complete this proof using only information about the GIT quotients of Richardson varieties in $X(w_{3,7})$. Instead we show that the equations defining the GIT quotient is a determinantal variety.

3.3.1 Deodhar decomposition

In [Deo85] Deodhar considered the intersection in G/B of the open cell in a Schubert variety with the open cell of an opposite Schubert variety. For $v, w \in W$, define the Richardson strata[1]

$$R_w^v = (BwB/B) \cap (B^- vB/B)$$

Note that this is not the same as the definition of a Richardson variety (see for example [BL03]). Recall that for $v, w \in W$ a Richardson variety X_w^v in G/B is defined to be the intersection of $X(w) \cap X^v$. Since both $X(w)$ and X^v contain the intersection of $(BwB/B) \cap (B^- vB/B)$ it is clear that $R_w^v \subseteq X_w^v$. And so Richardson strata is empty if $v \nleqslant w$ and the closure of R_w^v is X_w^v. Let $X_{w_1}^{v_1}$ and $X_{w_2}^{v_2}$ be two Richardson varieties. Then $X_{w_1}^{v_1} \subset X_{w_2}^{v_2}$ if and only if $v_2 \leqslant v_1 \leqslant w_1 \leqslant w_2$ in the Bruhat order.

In [Deo85] Deodhar gave a refined decomposition of a Richardson strata in G/B into disjoint locally closed subvarieties of a Schubert variety. We follow the notation from Marsh and Reitsch[MR04], and Kodama and Williams [KW13]. The definitions and examples are taken verbatim from [KW13] since it is their notation and set up that we use in our proofs.

Fix a reduced decomposition $\mathbf{w} = s_{i_1} s_{i_2} \cdots s_{i_m}$. We define a subexpression $\mathbf{v}$ of $\mathbf{w}$ to be a word obtained from the reduced expression $\mathbf{w}$ by replacing some of the factors with 1. For example, consider a reduced expression in S_4, say $s_3s_2s_1s_3s_2s_3$. Then $s_3s_21s_3s_21$ is a subexpression of $s_3s_2s_1s_3s_2s_3$. Given a subexpression $\mathbf{v}$, we set $\mathbf{v}_{(k)}$ to be the product of the leftmost k factors of $\mathbf{v}$, if $k \geqslant 1$, and set $\mathbf{v}_{(0)} = 1$. The following definition was given in [MR04] and was inspired from Deodhar's paper [Deo85].

[1] this terminology is not standard. What we have called strata is sometimes called a Richardson variety

Definition 3.8. Given a subexpression $\mathbf{v}$ of a reduced expression $\mathbf{w} = s_{i_1}s_{i_2}\cdots s_{i_m}$, we define

$$\begin{aligned} J^{\circ}_{\mathbf{v}} &:= \{k \in \{1,...,m\} | \mathbf{v}_{(k-1)} < \mathbf{v}_{(k)}\} \\ J^{\Box}_{\mathbf{v}} &:= \{k \in \{1,...,m\} | \mathbf{v}_{(k-1)} = \mathbf{v}_{(k)}\} \\ J^{\bullet}_{\mathbf{v}} &:= \{k \in \{1,...,m\} | \mathbf{v}_{(k-1)} > \mathbf{v}_{(k)}\} \end{aligned}$$

The expression $\mathbf{v}$ is called non-decreasing if $\mathbf{v}_{(j-1)} \leqslant \mathbf{v}_{(j)}$ for all $j = 1, \ldots, m$, and in this case $J^{\bullet}_{\mathbf{v}} = \emptyset$.

The following definition is from [Deo85, Definition 2.3].

Definition 3.9. (Distinguished subexpressions). A subexpression $\mathbf{v}$ of $\mathbf{w}$ is called *distinguished* if we have

$$\mathbf{v}_{(j)} \leqslant \mathbf{v}_{(j-1)}s_{i_j} \ \forall\, j \in \{1, \ldots, m\}$$

In other words, if right multiplication by s_{i_j} decreases the length of $\mathbf{v}_{(j-1)}$, then in a distinguished subexpression we must have $\mathbf{v}_{(j)} = \mathbf{v}_{(j-1)}s_{i_j}$.

We write $\mathbf{v} \prec \mathbf{w}$ if $\mathbf{v}$ is a distinguished subexpression of $\mathbf{w}$.

Definition 3.10. (Positive distinguished subexpressions). We call a subexpression $\mathbf{v}$ of $\mathbf{w}$ a *positive distinguished subexpression* (or a PDS for short) if $\mathbf{v}_{(j-1)} < \mathbf{v}_{(j-1)}s_{i_j}$, for all $j \in \{1, ..., m\}$.

Reitsch and Marsh [MR04] proved

Lemma 3.11. *Given $v \leqslant w$ and a reduced expression $\mathbf{w} = s_{i_1}\cdots s_{i_m}$ for w, there is a unique PDS $\mathbf{v}^{+}$ for v in $\mathbf{w}$.*

We now describe the Deodhar decomposition of the Richardson strata. Marsh and Rietsch [MR04] gave explicit parameterizations for each Deodhar component, identifying each one with a subset in the group. Much of this appears implicitly in Deodhar's paper, but we refer to [MR04] for our exposition because these statements are made explicit there and the authors make references to Deodhar's paper wherever needed.

Definition 3.12. [MR04, Definition 5.1] Let $\mathbf{w} = s_{i_1}\cdots s_{i_m}$ be a reduced expression for w, and let $\mathbf{v}$ be a distinguished subexpression. Define a subset $\mathbf{G}^{\mathbf{v}}_{\mathbf{w}}$ in G by

$$\mathbf{G}^{\mathbf{v}}_{\mathbf{w}} := \left\{ g = g_1 g_2 \cdots g_m \middle| \begin{cases} g_l = x_{i_l}(m_l)s_{i_l} & \text{if } l \in J^{\bullet}_{\mathbf{v}}, \\ g_l = y_{i_l}(p_l) & \text{if } l \in J^{\Box}_{\mathbf{v}}; \\ g_l = s_{i_l} & \text{if } l \in J^{\circ}_{\mathbf{v}} \end{cases} \right\}$$

From[MR04, Theorem 4,2] there is an isomorphism from $\mathbb{C}^{*|J_{\mathbf{v}}^{\square}|} \times \mathbb{C}^{|J_{\mathbf{v}}^{\bullet}|}$ to $\mathbf{G}_{\mathbf{w}}^{\mathbf{v}}$.

Definition 3.13. (Deodhar Component) The Deodhar component $\mathcal{R}_{\mathbf{w}}^{\mathbf{v}}$ is the image of $\mathbf{G}_{\mathbf{w}}^{\mathbf{v}}$ under the map $\mathbf{G}_{\mathbf{w}}^{\mathbf{v}} \subseteq U^{-}vB \cap BwB \to G/B$, sending g to gB.

Then from [Deo85, Theorem 1.1] one has [Deo85, Corollary 1.2], also from Deodhar.

Theorem 3.14. $R_{w}^{v} = \bigsqcup_{\mathbf{v} \prec \mathbf{w}} \mathcal{R}_{\mathbf{w}}^{\mathbf{v}}$ *the union taken over all distinguished subexpressions* $\mathbf{v}$ *such that* $\mathbf{v}_{(m)} = v$. *The component* $\mathcal{R}_{\mathbf{w}}^{\mathbf{v}^{+}}$ *is open in* R_{w}^{v}.

Naturally when one is talking of the Deodhar decomposition of a Richardson strata in $G/P_{\hat{\alpha}_r}$, one can take the projections of the components in G/B into G/P. In [KW13, Proposition 4.16] the authors show that the Deodhar components of a Richardson strata in $G/P_{\hat{\alpha}_r}$ are independent of $\mathbf{w}$ and only depends upon w. This follows from the observation that any two reduced decompositions $\mathbf{w}$ and $\mathbf{w}'$ of w are related by a sequence of commuting transpositions $s_i s_j = s_j s_i$.

Let X_{w}^{v} be a Richardson variety in Grassamnnian $G_{r,n}$ for $v, w \in I(r, n)$. We note that for X_{w}^{v} to be non-empty we need that $v < w$. We now recall the standard monomial theory for Richardson varieties in Grassmannian. In [KL02], Lakshmibai and Kreiman have obtained a standard monomial basis for the Richardson varieties in Grassmannian. Then using the basis they obtain a basis for the tangent space and a criteria for smoothness for X_{w}^{v} at any T-fixed point e_{τ}. Using the recursive formula, they show that the multiplicity of X_{w}^{v} at e_{τ} is the product of multiplicity of X^{v} at e_{τ} and the multiplicity of X_{w} at e_{τ} and generalised the Rosenthal-Zelevinsky determinantal formula for multiplicities at T-fixed points of Schubert varietes to the case of Richardson varieties. A geometric construction of a standard monomial basis for the homogeneous coordinate ring associated with any ample line bundle on any flag varieties which is compatible with the Schubert varieties and Richardson varieties has been carried out by Lakshmibai and Brion in [BL03]. Lakshmibai and Littelmann in [LL03] gave a geometric interpretation of the standard monomial theory. They constructed nice filtrations of the vanishing ideal of the boundary of the varieties above. They establish a relation between equivariant K-theory and standard monomial theory, in particular, the computation of the coefficients of the classes of structure sheaves appearing in the product of the class of the structure sheaf of a Schubert variety with the class of a line bundle. In this thesis we only deal with the standard monomial basis for Richardson varieties in Grassmannian. We recall the following definition:

Definition 3.15. A monomial $p_{\tau_1} p_{\tau_2} \dots p_{\tau_d}$ is said to be *standard on* X_{w}^{v} if

$$v \leqslant \tau_1 \leqslant \tau_2 \dots \leqslant \tau_d \leqslant w.$$

The standard monomial theorem for Richardson varieties is also along the lines of standard monomial theorem for the Schubert varieties. We recall,

Theorem 3.16. *[KL02, Theorem 4.2.1] Standard monomials on X_w^v of degree d forms a basis of* $H^0(X_w^v, \mathcal{L}_{\omega_r}^{\otimes d})$.

Remark 3.17. As in 3.5 we also have a similar remark for the Richardson varieties in the Grassmannian: Let $(\pi_w^v)(d) : H^0(G_{r,n}, \mathcal{L}_{\omega_r}^{\otimes d}) \longrightarrow H^0(X_w^v, \mathcal{L}_{\omega_r}^{\otimes d})$ denote the restriction map. We recall from [KL02, §3] that $(\pi_w^v)(d)$ is surjective and the kernel of $(\pi_w^v)(1)$ has a basis given by $\{p_\tau | w \not\geq \tau \text{ or } \tau \not\geq v\}$. The Richardson variety X_w^v as a closed subvariety of $G_{r,n}$ can also be defined as the vanishing of p_τ, $w \not\geq \tau$ or $\tau \not\geq v, \tau \in I(r,n)$.

3.4 Singular locus of Schubert varieties in Grassmannian

The singular loci of Schubert varieties in miniscule G/P were determined by Lakshmibai and Weyman [LW90]. There is another description of the singular locus of Schubert varieties $X(w)$ in terms of the stabiliser parabolic subgroup of $X(w)$, due to Brion and Polo [BP99]. They proved the following theorem.

Theorem 3.18. *Let $w \in I(r,n)$. Let $P_w = \{g \in G | gX(w) = X(w)\}$, the stabilizer of $X(w)$ in G. The smooth locus of the Schubert variety $X(w)$ is $X(w)_{sm} = P_w w P/P \subseteq X(w) \subseteq G_{r,n}$.*

We recall the following proposition from [LMS74].

Proposition 3.19. *Let $w = (b_1, b_2, \dots, b_r) \in I(r,n)$. Define*

$$J'(w) := \{j \in [1, \dots, n-1] \big| \exists m \textit{ with } j = b_m, j+1 \neq b_{m+1}\}.$$

Let $J(w) := \{1, 2, \dots, n-1\} \setminus J'(w)$. Then $P_w = P_J$ where $J = \{\alpha_j | j \in J(w)\}$.

We need some more notation to describe the work in [LW90]. Let $w = (b_1, b_2, \dots, b_r)$. Associate to w the increasing sequence $\mathbf{w} = (\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_r})$ where $\mathbf{b_i} = b_i - i$, so $0 \leqslant \mathbf{b_1} \leqslant \mathbf{b_2} \leqslant \dots \leqslant \mathbf{b_r} \leqslant n-r$. Clearly we have a bijective correspondence between $I(r,n)$ and non-decreasing r length sequences in $0 \leqslant \mathbf{b_1} \leqslant \mathbf{b_2} \leqslant \dots \leqslant \mathbf{b_r} \leqslant n-r$. An increasing sequence gives us a Young diagram, $Y(\mathbf{w})$, in an $r \times n-r$ rectangle with the i-th row having $\mathbf{b_i}$ boxes[2]. We call this the Young diagram corresponding to the Schubert variety $X(w)$.

Recall the following Theorem from [LW90][3].

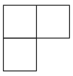
[2] rows are numbered $1, \dots, r$ from bottom to top.
[3] the notation we use is different from theirs, they work with non-increasing sequences

Theorem 3.20 (Theorem 5.3 [LW90])**.** *Let $X(w)$ be a Schubert variety in the Grassmannian. Let $\mathbf{w} = (\mathbf{p_1}^{q_1}, \ldots, \mathbf{p_k}^{q_k}) = (\underbrace{\mathbf{p_1}, \ldots \mathbf{p_1}}_{q_1 \text{ times}} \ldots, \underbrace{\mathbf{p_k}, \ldots \mathbf{p_k}}_{q_k \text{ times}})$ be the non-zero parts of the increasing sequence $\mathbf{w}$ with $1 \leqslant \mathbf{p_1} < \mathbf{p_2} \ldots < \mathbf{p_k} \leqslant n - r$. The singular locus $X(w)$ consists of $k-1$ components. The components are given by the Schubert varieties corresponding to the Young diagrams $Y(\mathbf{w_1}), \ldots Y(\mathbf{w_{k-1}})$, where the sequences $\mathbf{w_i}$ are given by*

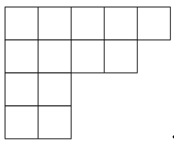
$$\mathbf{w_i} = (\mathbf{p_1}^{q_1}, \ldots, \mathbf{p_{i-1}}^{q_{i-1}}, (\mathbf{p_i} - \mathbf{1})^{q_i + 1}, \mathbf{p_{i+1}}^{q_{i+1} - 1}, \mathbf{p_{i+2}}^{q_{i+2}}, \ldots, \mathbf{p_r}^{q_r}),$$

for $1 \leqslant i \leqslant r - 1$ and $1 \leqslant \mathbf{p_i} < \mathbf{p_{i+1}}$.

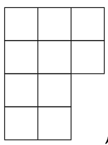
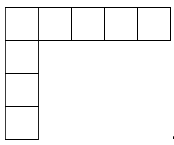
An inner corner in a Young diagram is a box that, if it is removed, still gives us the Young diagram of an non-decreasing sequence. So an easy to remember description of the irreducible components of the singular locus of $X(w)$ is as follows :- they are the Schubert varieties in correspondence with Young diagram $Y(\mathbf{w_i})$ obtained from $Y(\mathbf{w})$ by removing the hook from the i-th inner box to the $i+1$-st inner box.

Example 3.21. We continue with the $G_{2,4}$ as in 3.6. We note that the Schubert varieties are also indexed by the set $I(2,4)$. So the Schubert varieties are $X((1,2)), X((1,3)), X((2,3)), X((1,4)), X((2,4))$ and $X((3,4))$. The Young diagram associated to the word (2,4) is . So the Schubert variety $X((2,4))$ is not smooth. However, the Young diagram associated to the other words are rectangular, hence they all are smooth. So the only Schubert variety which is not smooth inside $G_{2,4}$ is $X((2,4))$.

Example 3.22. We give another example which describes the singular locus of a Schubert variety. Let us consider the Schubert variety $X((3,4,7,9))$ in $G_{4,9}$. The Young diagram corresponding to this Schubert variety looks like


.

The singular locus obtained by removing the hooks has Schubert varieties $X((3,4,6,7)), X((2,3,4,9))$, whose Young diagrams are given by the following tableaux


,

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Chapter 4

Minimal dimensional Schubert varities admitting semistable points

In this chapter we will study the Minimal dimensional Schubert varieties in the Grassmannian admitting semistable points. The geometry and the combinatorics of these varieties has been studied by [KP09b]. The conditions of Schubert varieties admitting semistable point is one of the crucial tool used in the study of these varieties (see [Kan98], [Kan99]). They also help in classification of Schubert varieties for which the torus quotient is a smooth quotient. We will deal with this in detail in Chapter 6. The main reference for this chapter is [BSKS20]. Then we move on to the study of Richardson varieties in minimal dimensional Schubert varieties admitting semistable point. The conditions of Richardson varieties admitting semistable points is explored in [KPPU18] where they mainly studied the GIT quotient of Richardson varieties in Grassmannian. Using Deodhar decomposition we will explicitly study Richardson varieties in a minimal dimensional Schubert variety admitting semistable points.

We will use the conventions as setup in Chapter 3 : We will denote $G_{r,n}$ the Grassmannian variety of r dimensional subspaces of $\mathbb{C}^n$. The set of *minimal length coset representatives* of W/W_P are

$$W^P = \{(w_1, \ldots, w_n) \in S_n | w_1 < w_2 < \cdots < w_r; w_{r+1} < \cdots < w_n\}$$

Note also we will use the fact that W^P gets identified with the set $I_{r,n} : w = (w_1, \ldots w_n) \mapsto (w_1, \ldots w_r)$.

Also recall that we associated with each monomial $p_{\underline{\tau}} = p_{\tau_1} \cdots p_{\tau_d}$ a Young tableau $T_{\underline{\tau}}$ of shape $\underbrace{(d, d, \ldots, d)}_{r \text{ times}}$ whose i-th column is filled with $\tau_i = [\tau_i(1), \tau_i(2), \ldots, \tau_i(r)]$. Let $a(i)$

denote the number of times integer i appears in the tableau. Then we have

$$\mathrm{diag}(t_1, t_2, \ldots, t_n).p_{\tau_1} p_{\tau_2} \cdots p_{\tau_d} = \prod_i t_i^{a(i)} p_{\tau_1} \cdots p_{\tau_d}.$$

Let $H^0(X(w), \mathcal{L}(\omega_r)^{\otimes d})^T$ denote set of T-fixed points of $H^0(X(w), \mathcal{L}(\omega_r)^{\otimes d})$. Also note that the elements of $H^0(X(w), \mathcal{L}(\omega_r)^{\otimes d})$ are also called the zero weight vectors.

Lemma 4.1. *If a monomial $p_{\underline{\tau}}$ is a zero weight vector in $H^0(X(w), \mathcal{L}(\omega_r)^{\otimes d})$, then all $a(i)$'s appear the same number of times in the Young tableau $T_{\underline{\tau}}$.*

Proof. Since $G = SL(n, \mathbb{C})$ and $\mathrm{diag}(t_1, t_2, \ldots, t_n).p_{\tau_1} p_{\tau_2} \cdots p_{\tau_d} = \prod_i t_i^{a(i)} p_{\tau_1} \cdots p_{\tau_d}$. For the monomial to have weight zero we need $\prod_i t_i^{a(i)} = 1$. Since $t_1 \cdots t_n = 1$, a standard monomial is a zero weight vector iff all $a(i)$'s appear the same number of times in the Young tableau. $\square$

We recall some lemmas and propositions which have appeared earlier. We state them nevertheless since they are required in the rest of the paper. Some of these are folklore.

The following lemma appears in [Kum08], [Kan98].

Lemma 4.2. *Let r and n be coprime. Let $v \neq 0$ be a zero weight vector in $H^0(X(w), \mathcal{L}(\omega_r)^{\otimes d})$. Then n divides d.*

Proof. Since 0 is a weight, $d\omega_r$ is in the root lattice. So n divides d. $\square$

Remark 4.3. The lemma 4.2 can also be deduced directly from lemma 4. A semistandard basis $H^0(X(w), \mathcal{L}(\omega_r)^{\otimes d})$ is indexed by semistandard tableau of shape $\underbrace{(d, d, \ldots, d)}_{r \text{ times}}$. Since r and n are coprime the necessary condition that all entries appear equal number of times in the tableau with d columns is n divides d.

Recall from [KP09b], that there is a unique minimal Schubert Variety $X(w_{r,n})$ in $G_{r,n}$ admitting semistable points with respect to the line bundle $\mathcal{L}(n\omega_r)$. For completeness we explicitly calculate $w_{r,n}$.

Proposition 4.4. *Let r and n be coprime. Then $w_{r,n} = (a_1, a_2, \ldots, a_r)$ where a_i is the smallest integer such that $a_i \cdot r \geqslant i \cdot n$.*

Proof. Clearly $w_{r,n} > \mathrm{id}$ since $X(\mathrm{id})$ is a point. Let α be a simple root with $s_\alpha w_{r,n} < w_{r,n}$. Note, $s_\alpha w_{r,n} \in W^{S \setminus \{\alpha_r\}}$. Recall from 3.5 we have a surjection $\pi^n_{w_{r,n}, s_\alpha w_{r,n}}$:

$H^0(X(w_{r,n}), \mathcal{L}(n\omega_r)) \to H^0(X(s_\alpha w_{r,n}), \mathcal{L}(n\omega_r))$. Let K denote its kernel. So we have a short exact sequence

$$0 \to K \to H^0(X(w_{r,n}), \mathcal{L}(n\omega_r)) \to H^0(X(s_\alpha w_{r,n}), \mathcal{L}(n\omega_r)) \to 0.$$

From the minimality of $w_{r,n}$ we get $K^T \to H^0(X(w_{r,n}), \mathcal{L}(n\omega_r))^T$ an isomorphism. Recall from 3.5 that the kernel of $\pi^n_{w,v}$ has a basis given standard monomials $p_{\tau_1}p_{\tau_2}\dots p_{\tau_n}$ with $\tau_1 \leqslant \tau_2 \dots \leqslant \tau_n$ and $\tau_n \nleq s_\alpha w_{r,n}, \tau_n \leqslant w_{r,n}$. This forces $\tau_n = w_{r,n}$. Now if we choose a standard monomial $p_{\underline{\tau}} = p_{\tau_1} \cdots p_{\tau_n}$ in $H^0(X(w_{r,n}), \mathcal{L}(n\omega_r))^T$ then we have $\tau_n = w_{r,n}$, since the elements in the kernel are precisely those with a term $p_{w_{r,n}}$.

To construct such a standard monomial, we need a filling of the associated tableau $T_{\underline{\tau}}$ with rn boxes such that each i, $1 \leqslant i \leqslant n$ appears exactly r-times and the last column is as small as possible in the Bruhat order. Clearly, the filling which results in the smallest element in the Bruhat order appearing as the last column is the one in which the tableau is filled from left to right and top to bottom with numbers $1, 2, \dots, n$, in order, with each appearing exactly r times - so the first entry of the last column is the least integer a_1 such that $ra_1 \geqslant n$ and, in general, the i-th entry in the last column is the smallest integer a_i such that $r \cdot a_i \geqslant in$, completing the proof. $\square$

The tableau constructed in the proof of Proposition 4.4 will be used repeatedly in the paper. We denote it by $\Gamma_{r,n}$. The figure below gives $\Gamma_{3,8}$.

$$\Gamma_{3,8} = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ \hline 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 \\ \hline 6 & 6 & 7 & 7 & 7 & 8 & 8 & 8 \\ \hline \end{array}$$

4.1 GIT quotients of Richardson varieties in $X(w_{r,n})$

The results in this section pertain to GIT quotients of Richardson varieties in $G/P_{\hat{\alpha}_r}$ with respect to the T-linearized line bundle $\mathcal{L}(n\omega_r)$.

4.1.1 GIT quotients of Richardson varieties

Theorem 4.5. *Let r and n be coprime. Then the GIT quotient $T\backslash\backslash X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r))$ is smooth.*

Proof. $X(w_{r,n})$ is the minimal Schubert variety admitting semistable points with respect to $\mathcal{L}(n\omega_r)$. So $X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r)) \cap BwP_{\hat{\alpha}_r}/P_{\hat{\alpha}_r} = \phi$ for all $w < w_{r,n}$. From the Bruhat decomposition it follows that $X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r)) \subseteq Bw_{r,n}P_{\hat{\alpha}_r}/P_{\hat{\alpha}_r}$. Thus, $X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r))$ is a smooth open subset of $X(w_{r,n})$. Since r and n are coprime we have $X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r)) = X(w_{r,n})^{s}_T(\mathcal{L}(n\omega_r))$ (see [Kan14]). Let $G_{ad} = G/Z(G)$ be the adjoint group of G. Let $\pi : G \to G_{ad}$ be the natural homomorphism and $T_{ad} = \pi(T)$. Note that $\mathcal{L}(n\omega_r)$ is also T_{ad}-linearized. Therefore, $X(w_{r,n})^{ss}_{T_{ad}}(\mathcal{L}(n\omega_r)) = X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r)) = X(w_{r,n})^{s}_T(\mathcal{L}(n\omega_r)) = X(w_{r,n})^{s}_{T_{ad}}(\mathcal{L}(n\omega_r))$. Hence for any point $x \in X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r))$ the orbit $T_{ad}.x$ is closed in $X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r))$ and the stabiliser of x is finite. By [Kan14, Lemma 3.2] and the proof of example 3.3 in that paper, the stabiliser of every point of $X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r))$ in T_{ad} is trivial. Therefore the GIT quotient $T\backslash\backslash X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r))$ is a geometric quotient. Since $X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r))$ is smooth, $T\backslash\backslash X(w_{r,n})^{ss}_T(\mathcal{L}(n\omega_r))$ is also smooth. □

Recall that a Richardson variety X^v_w in $G_{r,n}$ is the intersection of the Schubert variety $X(w)$ in $G_{r,n}$ with the opposite Schubert variety X^v in $G_{r,n}$.

In [KPPU18, Proposition 3.1] the authors give a characterisation of the smallest Richardson variety in $G_{r,n}$ admitting semistable points. From the proof of Proposition 4.4 we obtain an explicit characterization.

Proposition 4.6. *Let r and n be coprime. Let $v_{r,n}$ be such that $X^{v_{r,n}}_{w_{r,n}}$ is the smallest Richardson variety in $X(w_{r,n})$ admitting semistable points. Then $v_{r,n} = (1, a_1, \dots, a_{r-1})$ with the a_i defined as the smallest integer satisfying $a_i r \geqslant i \cdot n$ (as in Proposition 4.4).*

Proof. Let $v_{r,n} = (b_1, \dots, b_r)$. Since $X^{v_{r,n}}_{w_{r,n}}$ has a semistable point, $H^0(X^{v_{r,n}}_{w_{r,n}}, \mathcal{L}(n\omega_r))^T$ is non-zero. Now $H^0(X^{v_{r,n}}_{w_{r,n}}, \mathcal{L}(n\omega_r))$ has a standard monomial basis $p_{\tau_1} \dots p_{\tau_n}$ with $\tau_1 \leqslant \tau_2 \cdots \leqslant \tau_n$ (see [BL03]). We identify this basis with semistandard Young tableau having columns $\tau_1, \tau_2, \dots, \tau_n$ as before. It follows from this identification that there is a semistandard Young tableau with r rows and n-columns in which each integer $1 \leqslant k \leqslant n$ appears exactly r times. From Proposition 4.4 and [BL03, Proposition 6] we have $\tau_n = w_{r,n}$ and $v_{r,n} \leqslant \tau_1$. Since every semistandard Young tableau has each integer in $\{1, \dots, n\}$ appearing r times and the first entry of τ_1 is always 1, b_1 must be 1. Since r, n are coprime, from the definition of a_1 it is immediate that all a_1's cannot be in the first row. For the same reason the a_i's cannot all appear in the first i rows. So a_i must appears in a row j where $j > i$. Hence $b_i \leqslant a_{i-1}$. Note that the first column of the Young tableau $\Gamma_{r,n}$ from Proposition 4.4 is $v = (1, a_1, \dots, a_{r-1})$. From [BL03, Proposition 6], the T-invariant $\Gamma_{r,n}$ is non-zero on $X^v_{w_{r,n}}$. Hence $v_{r,n} = v = (1, a_1, \dots, a_{r-1})$.

□

Consider the Weyl group element $c_{r,n} = w_{r,n} v_{r,n}^{-1}$.

Claim 4.7. $c_{r,n}$ is a Coxeter element.

Proof. We have a reduced expression $w_{r,n} = (s_{a_1-1}\cdots s_1)(s_{a_2-1}\cdots s_2)\cdots(s_{a_r-1}\cdots s_r)$ and $v_{r,n} = (s_{a_1-1}\cdots s_2)(s_{a_2-1}\cdots s_3)\cdots(s_{a_{r-1}-1}\cdots s_r)$. Then

$$w_{r,n}v_{r,n}^{-1} = (s_{a_1-1}\cdots s_1)(s_{a_2-1}\cdots s_{a_1})(s_{a_3-1}\cdots s_{a_2})\cdots(s_{a_r-1}\cdots s_{a_{r-1}}).$$

This is a Coxeter element.

$\square$

We now consider the GIT quotients of Richardson varieties in $X_{w_{r,n}}$.

Theorem 4.8. *$T\backslash\backslash(X_{w_{r,n}}^{v_{r,n}})_T^{ss}(\mathcal{L}(n\omega_r))$ is a point.*

Proof. Since $\mathrm{dim}X_{w_{r,n}}^{v_{r,n}} = l(w_{r,n}) - l(v_{r,n}) = l(c_{r,n}) = n-1 = \mathrm{dim}T$ and $(X_{w_{r,n}}^{v_{r,n}})_T^{ss}(\mathcal{L}(n\omega_r)) = (X_{w_{r,n}}^{v_{r,n}})_T^{s}(\mathcal{L}(n\omega_r))$, so the dimension of the GIT quotient is 0. Since $T\backslash\backslash(X_{w_{r,n}}^{v_{r,n}})_T^{ss}(\mathcal{L}(n\omega_r))$ is irreducible, $T\backslash\backslash(X_{w_{r,n}}^{v_{r,n}})_T^{ss}(\mathcal{L}(n\omega_r))$ is a point. Alternatively, there is a unique standard monomial $p_{\tau_1}p_{\tau_2}\dots p_{\tau_n}$ of weight zero with $\tau_1 = [1, a_1, a_2, \dots, a_{r-1}]$ and $\tau_n = [a_1, a_2, \dots, a_r]$ (the corresponding Young tableau being $\Gamma_{r,n}$). $\square$

Theorem 4.9. *Let $v \in W^{S\setminus\{\alpha_r\}}$ be such that $v < v_{r,n}$. Then, $T\backslash\backslash(X_{w_{r,n}}^{v})_T^{ss}(\mathcal{L}(n\omega_r))$ is isomorphic to $\mathbb{P}^1$ if and only if $v = s_\alpha v_{r,n}$ where $s_\alpha = (a_i-1, a_i)$ for some $i = 1, 2, \dots, r-1$. The descent of $\mathcal{L}(n\omega_r)$ to $T\backslash\backslash(X_{w_{r,n}}^{v})_T^{ss}(\mathcal{L}(n\omega_r))$ is $\mathcal{O}_{\mathbb{P}^1}(n_i)$ where n_i is the number of times a_i-1 appears in the i-th row of the tableau $\Gamma_{r,n}$.*

Proof. We start with the only if part. Since $X_{w_{r,n}}^{v}$ is normal, $(X_{w_{r,n}}^{v})_T^{ss}(\mathcal{L}(n\omega_r))$ is normal and hence $T\backslash\backslash(X_{w_{r,n}}^{v})_T^{ss}(\mathcal{L}(n\omega_r))$ is normal. Since $\dim(T\backslash\backslash(X_{w_{r,n}}^{v})_T^{ss}(\mathcal{L}(n\omega_r))) = 1$, the GIT quotient $T\backslash\backslash(X_{w_{r,n}}^{v})_T^{ss}(\mathcal{L}(n\omega_r))$ is a smooth, rational projective curve. Hence $T\backslash\backslash(X_{w_{r,n}}^{v})_T^{ss}(\mathcal{L}(n\omega_r))$ is isomorphic to $\mathbb{P}^1$.

If $T\backslash\backslash(X_{w_{r,n}}^{v})_T^{ss}(\mathcal{L}(n\omega_r))$ is isomorphic to $\mathbb{P}^1$ we get $l(v) = l(v_{r,n}) - 1$. Also $v < v_{r,n}$ and $v \in W^{S\setminus\{\alpha_r\}}$. So $v = (s_{b_i}\cdots s_i)\cdots(s_{b_r}\cdots s_r)$ for some i, $1 \leqslant i \leqslant r$, and for some $1 \leqslant b_i < b_{i+1}\cdots < b_r \leqslant n-1$ (see the discussion preceding Lemma 4.2). Since $v_{r,n} = (s_{a_1-1}\cdots s_2)(s_{a_2-1}\cdots s_3)\cdots(s_{a_{r-1}-1}\cdots s_r)$, $v = s_\alpha v_{r,n}$ only when $s_\alpha = (a_i-1, a_i)$, $1 \leqslant i \leqslant r-1$.

We start with a zero weight standard monomial basis for $H^0(X_{w_{r,n}}^{v}, \mathcal{L}(n\omega_r))$. Let $v = s_\alpha v_{r,n}$ with $s_\alpha = (a_i-1, a_i)$ for some fixed i. We have $v_{r,n} = (1, a_1, \dots, a_i, \dots, a_{r-1})$ and $v = (1, a_1, \dots, a_i-1, \dots, a_{r-1})$. The $i+1$-st entry of $v_{r,n}$ is a_i and that of v is a_i-1 and the rest of the entries are equal. We need to count the number of semistandard tableau of shape $n, n, \dots, n$ (r rows) with first column v. Because the tableau is semistandard, the positions of all integers other than a_i-1 and a_i are fixed. So the number of such tableaux depends only on the number of a_i-1 in the i-th row. a_i-1 appears n_i times in the i-th row of $\Gamma_{r,n}$. It is easy to see that for every j in $\{0, \cdots, n_i\}$ there is a semistandard tableau with $a_i - 1$ appearing j times and a_i appearing $n_i - j$ times in row i. So we have $n_i + 1$ linearly independent sections of the descent line bundle on the GIT quotient. This completes the proof.

$\square$

4.2 Projective normality of the GIT quotient of $X(w_{3,7})$

In this section we will work with $G = SL(7, \mathbb{C})$. We use the same notation as before. We study the GIT quotient of the Schubert variety $X(w_{3,7})$ with respect to T-linearized line bundle $\mathcal{L}(7\omega_3)$. From [Kum08, Theorem 3.10] we know that this line bundle descends to the line bundle $\mathcal{L}(7\omega_3)$ on the GIT quotient $T\backslash\backslash X(w_{3,7})^{ss}_T(\mathcal{L}(7\omega_3))$.

Theorem 4.10. *The polarized variety* $T\backslash\backslash X(w_{3,7})^{ss}_T(\mathcal{L}(7\omega_3))$ *is projectively normal.*

Remark 4.11. Let $S(m) = H^0(X(w_{3,7}), \mathcal{L}(7\omega_3)^{\otimes m})$ be the global sections of the line bundle $\mathcal{L}(7\omega_3)$ on $X(w_{3,7})$ and let $R(m) = H^0(X(w_{3,7}), \mathcal{L}(7\omega_3)^{\otimes m})^T$ denote the invariant subspace with respect to action of T. The GIT quotient is precisely $\mathrm{Proj}(\oplus_m R(m))$ (see [Dol03, Proposition 8.1]). Since the polarized variety $(X(w_{3,7}), \mathcal{L}(7\omega_3))$ is projectively normal, we have a surjective map $S(1)^{\otimes m} \longrightarrow S(m)$ (see [LR07]) and an induced map $\phi : R(1)^{\otimes m} \to R(m)$. Now the GIT quotient is smooth and it is normal. Therefore to show projective normality of the GIT quotient all we need to show is that ϕ is surjective.

From Lemma 4.4 we get $w_{3,7} = [3,5,7]$. As before we identify the standard monomial basis of $H^0(X(w_{3,7}), \mathcal{L}(7m\omega_3))^T$ with semistandard Young tableaux. These tableaux have 3 rows and $7m$ columns with each integer from $\{1, \ldots, 7\}$ appearing exactly $3m$ times - furthermore the last column is $[3,5,7]$.

To aid in the proof of projective normality we list the semistandard Young tableau basis of $R(1)$ and we also write down a semistandard tableau of shape $[14, 14, 14]$ from $R(2)$ which will play a role in the proof. Henceforth, we will use the notation y_i for both the tableau y_i it defines and also the standard monomial associated it to. Set

$$y_1 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&2&3\\\hline 3&3&4&4&4&5&5\\\hline 5&6&6&6&7&7&7\\\hline\end{array}\,,\quad y_2 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&2&3\\\hline 3&3&4&4&5&5&5\\\hline 4&6&6&6&7&7&7\\\hline\end{array}\,,\quad y_3 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3\\\hline 2&3&4&4&4&5&5\\\hline 5&6&6&6&7&7&7\\\hline\end{array}\,,$$

$$y_4 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 & 4 & 5 & 5 & 5 \\ \hline 4 & 6 & 6 & 6 & 7 & 7 & 7 \\ \hline \end{array}, \quad y_5 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 2 & 4 & 4 & 4 & 5 & 5 \\ \hline 5 & 6 & 6 & 6 & 7 & 7 & 7 \\ \hline \end{array}, \quad y_6 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 2 & 4 & 4 & 5 & 5 & 5 \\ \hline 4 & 6 & 6 & 6 & 7 & 7 & 7 \\ \hline \end{array},$$

$$y_7 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 & 5 & 5 & 5 \\ \hline 3 & 6 & 6 & 6 & 7 & 7 & 7 \\ \hline \end{array}, \quad z = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 5 \\ \hline 3 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 \\ \hline \end{array}.$$

We first make some simple observations.

Observation 4.12. *Every semistandard tableau basis element of* $H^0(X(w_{3,7}), \mathcal{L}(7m\omega_3))^T$ *begins with one of the following columns -* $[1,2,3], [1,2,4][1,2,5], [1,3,4], [1,3,5]$, *and ends with the column* $[3,5,7]$.

Proof. We already noted above that the last column of every semistandard tableau basis element of $H^0(X(w_{3,7}), \mathcal{L}(7m\omega_3))^T$ is $[3,5,7]$.

Clearly, semistandardness forces that in the first row the leftmost $3m$ entries are filled with 1, and that in the last row the rightmost 6m entries are filled with 3m $6'$s followed by $3m$ 7's. So clearly the last entry of the first column cannot be 6 or 7 otherwise we will have more 6's or 7's than permitted. The second entry of the first column cannot be 5, otherwise the entire second row will have only 5's, a contradiction to the number of 5's present. The second entry of the first column cannot be 4 for a similar reason - in that case the second row will only have 4's and 5's forcing at least one of them to occur more than 3m times, a contradiction. This completes the proof. □

Observation 4.13. *No semistandard tableau basis element of* $H^0(X(w_{3,7}), \mathcal{L}(7m\omega_3))^T$ *has the following columns:* $[1,2,7]$, $[1,3,7]$, $[1,4,7]$,$[1,5,6]$,$[1,5,7]$, $[2,3,4]$, $[2,3,5]$, $[2,3,6]$, $[2,3,7]$, $[2,4,5]$,$[2,5,6]$,$[3,4,6]$, $[3,5,6]$. *The only columns containing a 6 are columns* $[1,2,6]$, $[1,3,6]$, $[1,4,6]$ *and* $[2,4,6]$. *There are exactly* m *columns with* $[2,4,6]$ *and at least* m *columns with* $[1,4,6]$. *The only columns containing a 7 are* $[2,4,7]$, $[2,5,7]$, $[3,4,7]$ *and* $[3,5,7]$ *and there are at least* $2m$ *occurrences of columns* $[2,5,7]$ *and* $[3,5,7]$.

Proof. If there is a column with $[1,x,7]$, x among $2,3,4,5,6$, standardness forces that the entries in the first row to the left of this column are all 1's and the entries in the third row to its right are all 7. Then no matter where this column appears either the number of 1's or the number of 7's is incorrect.

It follows from the previous paragraph that all the 6's must be in the bottom row. If there is a column with $[1,5,6]$, then standardness forces the subsequent columns to all have a 5 in the second row and the columns preceding it to have a 1 in the topmost row. Then no matter where this column appears either the number of 1's or the number of 5's is not $3m$.

If $[2,3,4]$ occurs it is necessarily in column $3m+1$ appearing immediately after the occurrence of all the columns containing 1 because it is lexicographically least among columns beginning with 2. But the entry in the bottom row position in column $3m+1$ cannot be a 4 since that position is occupied by 6.

The same argument shows that $[2,3,5]$, $[2,4,5]$ cannot occur. If $[2,3,6]$ or $[2,3,7]$ is present, the first row to the right of this column and the second row to the left of this column contain only 2,3's yielding a total of $7m$ entries with 2 and 3, a contradiction.

If $[2,5,6]$ is present all columns to the right of this column will have a 5 in the second row by standardness. But then all the $3m$ columns containing 7 will be of the type $[x,5,7]$, for some x. But then the number of 5's is at least $3m+1$, a contradiction.

If the column $[3,4,6]$ is present then the top row to its right is filled with 3's. So every column containing 7 in the bottom has 3 as its topmost element. So the number of 3's is at least $3m+1$, a contradiction.

Now suppose the column $[3,5,6]$ is present. If it is in the left half of the tableau, standardness will forces the number of 5's to be more than what is allowed. If it is in the right half of the tableau then all entries in the second row to the right of this column are filled with 5's. So all the columns containing 7 in fact contain both 5 and 7. Again, the number of 5's is more than $3m+1$, a contradiction.

The above argument shows that the column appearing immediately after all the columns containing a 1 is column $[2,4,6]$. It appears before the $3m$ columns containing a 7. Since the tableau has no lexicographically larger column containing a 6, this column repeats till the appearance of a 7. So it occurs exactly m times.

Now the remaining $2m$ columns containing a 6 in the last row occur to the left of column number $3m+1$ which has a $[2,4,6]$. Suppose there are less than m columns with $[1,4,6]$ in the tableau. Since $[1,4,6]$ appears to left of the column numbered $3m+1$, all the entries in the second row to the left of first column labeled $[1,4,6]$ must have 2 or 3. So there are at least $2m+1$ 2's and 3's in the second row. Now there are at least $4m$ locations in the first row to the right of last 1 which have only 2 or 3. So the total number of 2's and 3s is at least $6m+1$, a contradiction. So we conclude that at least m rows to the left of column numbered $3m+1$ contain $[1,4,6]$.

We cannot have a 5 in the first row. Since we can have a 5 in the third row only in positions $\{1,2,\ldots,m\}$, and the only columns having a 5 in the second row are $[2,5,7]$ and $[3,5,7]$ it follows that we need at least $3m-m$ columns with $[2,5,7]$ and $[3,5,7]$. $\square$

Lemma 4.14. *Let $m \geqslant 2$. Every semistandard basis element of $R(m)$ is a product of a y_i and an element of $R(m-1)$, or is a product of z and an element of $R(m-2)$.*

Proof. Let f be a semistandard basis element of $R(m)$. The proof follows a case by case analysis.

a The first column of f is $[1,2,3]$. By Observation 4.13 above we have at least m columns with $[1,4,6]$ and exactly m columns labeled $[2,4,6]$. Furthermore we can have at most $m-1$ 5's in the last row of f. So we have at least $2m+1$ columns in f with $[2,5,7]$ and $[3,5,7]$, since these are the only columns containing 5 in the second row. The last column of f is a $[3,5,7]$. If the remaining 2m, columns were all $[2,5,7]$, using Observation 4.13, the total number of 2's is at least $m+1+2m$, a contradiction. (the m 2's from columns with $[2,4,6]$, and one from the first column having $[1,2,3]$. It follows that there are at least two columns with $[3,5,7]$.

1. Suppose f has at least one column with $[2,5,7]$. Then we have one $[1,2,3]$, at least two $[1,4,6]$'s and one $[2,4,6]$, one $[2,5,7]$ and two $[3,5,7]$'s. So the tableau y_7 appears as a subtableau. The complement of this subtableau in f is an element of $R(m-1)$. So f is a product of y_7 and an element from $R(m-1)$ as required.
2. f has no $[2,5,7]$. So we have at least $2m+1$ columns in f having $[3,5,7]$'s. Now the remaining 7's can be made up from $[3,5,7]$'s or $[2,4,7]$'s or $[3,4,7]$'s. These cannot all come from $[3,5,7]$ and $[3,4,7]$ since the number of 3's in that case would be more than $3m+1$. So there is at least one $[2,4,7]$. Note that there are also at most $m-1$ $[2,4,7]$'s, $[3,4,7]$'s, and additional $[3,5,7]$'s in from column numbers $4m+1$ to $5m-1$. Now the number of 2's in row 1 is at most $m+m-1$ (from the columns with $[2,4,6]$, and at most $m-1$ columns with $[2,4,7]$). So we need at least $m+1$ 2's in the second row. In this case then the second row of columns 1 to column $m+1$ contains only 2. In particular $[1,2,6]$ is present in f. Since there are at most $3m-1$ 5's in the second row of f (since we know there is a $[2,4,7]$), there is at least one 5 in the bottom row of f in position $\{1,2,\ldots,m\}$, forcing a $[1,2,5]$ in f.

   Now the total number of 4's and 5's is $6m$. The total number of 4's and 5's in the last row is at most $m-1$. We have m 4's from the $[2,4,6]$. The total number of 4's and 5's from the columns containing $[2,4,7]$, $[3,4,7]$ and $[3,5,7]$ is at most $3m$. All of these can account for a total of $5m-1$ 4's and 5's coming from these columns. Since we can have no more 5's in the second row, the deficit $m+1$ needed must be made from 4's in the second row, in fact occurring in columns numbered $m+2$ to $3m$. So we have at least 3 columns in f with $[1,4,6]$'s.

   Taking stock, in f we have one $[1,2,3]$, a $[1,2,5]$, a $[1,2,6]$, 3 columns with $[1,4,6]$, 2 columns with $[2,4,6]$, one $[2,4,7]$ and at least 5 $[3,5,7]$'s. So we see that the tableau indexing the basis vector z is a subtableau of f, and the complement of this subtableau in f is an element of $R(m-2)$. f is a product of z and an element from $R(m-2)$.

b The first column of f is $[1,2,4]$. In this case there are at most $m-1$ 5's in the last row of f and so there should be at least $2m+1$ columns in f with $[2,5,7]$ and $[3,5,7]$.

Notice that the $3m$ 6's cannot all come from columns with $[1,4,6]$ and $[2,4,6]$ alone. If that were the case we will have $3m$ 4's from these columns, and an additional 4 from the first column, a contradiction. So at least one of the columns in f with a 6 has to be $[1,2,6]$ or $[1,3,6]$.

1. Suppose a $[1,2,6]$ is present in f. Then it has to be in column $m+1$ of f. Then we have 2's in the second row of f in columns 1 to columns $m+1$ by semistandardness. From Observation 4.13 we have m 2's from the columns with $[2,4,6]$, so we can have at most $m-1$ columns with $[2,5,7]$. This means there are at least $2m+1-(m-1)=m+2$ columns with $[3,5,7]$, so we have at least 4 columns with $[3,5,7]$. But this means we have a $[1,2,4]$, a $[1,2,6]$, a $[1,4,6]$, a $[2,4,6]$, and three $[3,5,7]$, i.e. the tableau indexing the basis element y_6 is a subtableau in this case.
2. Now suppose we do not have a $[1,2,6]$ in f but have a $[1,3,6]$. We claim a $[2,5,7]$ must appear. Notice that we can have at most $m-1$ $[2,4,7]$'s since we already have $2m+1$ 7's. Now there are at most m 2's in the second row of f. But this means we have at most $m+m-1+m<3m$ 2's in f, a contradiction. So we may assume we have at least one $[2,5,7]$ in f. We claim that we have at least 2 $[3,5,7]$'s in f, for otherwise we have $2m$ $[2,5,7]$'s. But we have more 2's than allowed since we have m 2's from the $[2,4,6]$ and a 2 also from the $[1,2,4]$. So we conclude we have a $[1,2,4], [1,3,6], [1,4,6], [2,4,6], [2,5,7], [3,5,7], [3,5,7]$. So y_4 is a subtableau and we are done in this case.

c If the first column in f is $[1,2,5]$. If there are no $[1,2,6]$ or $[1,3,6]$ in f then column $m+1$ must be $[1,4,6]$ and the first m elements in the third row must be all 5's. But then the second and third row together have more than $6m$ 4's and 5's, a contradiction. So either $[1,2,6]$ or $[1,3,6]$, or both, are present in f.

1. Suppose first that f has a $[1,2,6]$. Then the second row of f has at least $m+1$ 2's, and since we already have m 2's from the $[2,4,6]$'s we can have at most $m-1$ $[2,5,7]$'s. This forces at least $m+1$ columns of f to be $[3,5,7]$'s.

   If f has a column with $[3,4,7]$ we see the tableau indexing y_5 is present as a subtableau of f. If f has no $[3,4,7]$ - we count the number of 4's - we have m from columns $[2,4,6]$. We can have at most $2m-1$ columns with$[1,4,6]$ since 2's occupies positions 1 up to $m+1$ in the second row. To make up the requisite 4's we need to have at least one $[2,4,7]$. But then the number of 3's in the first row of the tableau from $[3,5,7]$ is at most $3m-1$, so to make up the requisite 3's, there must be a $[1,3,6]$. In which case we see that the tableau indexing y_3 is present as a subtableau of f.

2. Suppose we only have a $[1,3,6]$ in f and no $[1,2,6]$. Now the total number of columns in f with $[1,2,5]$ and $[1,3,5]$ is m. We have exactly $2m$ $[2,5,7]$'s and $[3,5,7]$'s put together. If f has no $[2,4,7]$, the remaining m 7's come from $[3,4,7]$. But then the tableau cannot have a $[2,5,7]$ otherwise semistandardness will be violated. So we only have $2m$ columns of f with $[3,5,7]$. However this means we have $3m+1$ 3's, a contradiction. So we may assume that we have a $[2,4,7]$. We show then that there are at least two $[3,5,7]$'s, so the tableau indexing y_3 is a subtableau. If we have only one $[3,5,7]$ we would have $2m-1$ $[2,5,7]$'s. But we already have one $[2,4,7]$, one $[1,2,5]$ and m [2,4,6]'s, a contradiction to the number of allowed 2's.

d If the first column of f is $[1,3,5]$. Then all the 2's occur in the first row of f and in columns $3m+1$ to $6m$ and there are m 3's in the last m columns of the first row. Since there are no 3's in the last row, the remaining $2m$ 3's must occur in the second row. Since 6 occurs in the last row in positions $m+1$ to $4m$, and a 1 occurs in the first row in columns 1 to $3m$, it follows that $[1,3,6]$ is a column in f. Now all the 4's occur in the second row, starting at position $2m+1$ and ending at position $5m$, after which we only have 5's in the second row. Since the 2's in the first row of f occur in positions $3m+1$ to $6m$ and the 7's occur in the bottom row in position $4m+1$ to $7m$ it follows that there is a column containing $[2,4,7]$ and a column containing $[2,5,7]$. So y_1 is a subtableau of f.

e In case the first column is $[1,3,4]$, all the 2's occur in the first row, and so we have m 3's in the first row appearing in the columns $[3,4,7]$ and $[3,5,7]$. So f has at least m columns with $[2,5,7]$. Now there are $2m$ 3's in the second row and these must occur in positions 1 to $2m$. Since the last row has 6 in columns $m+1$ to $4m$ and the first row has a 1 in columns 1 to $3m$, it follows that there is a column with filling $[1,3,6]$ in the given tableau. It follows that the tableau indexing y_2 is a subtableau of f. We are done by induction.

$\square$

Remark 4.15. Let $\tau_1=[2,5,7]$, $\tau_2=[3,4,7]$, $\tau_3=[2,4,7]$, $\tau_4=[3,5,7]$, $\tau_5=[2,3,7]$, $\tau_6=[4,5,7]$. Consider the product of the Plücker coordinates $p_{\tau_1}p_{\tau_2}$. The straightening law gives us $p_{\tau_1}p_{\tau_2}=p_{\tau_3}p_{\tau_4}-p_{\tau_5}p_{\tau_6}$. On the Schubert variety $X(w_{3,7})$, $p_{\tau_6}=0$. So, on $X(w_{3,7})$, $p_{\tau_1}p_{\tau_2}=p_{\tau_3}p_{\tau_4}$. As a result $y_5y_7=z$.

Theorem 4.16. *The ring* $R = \oplus_{m \geqslant 0} H^0(X(w_{3,7}), \mathcal{L}(7m\omega_3))^T$ *is generated in degree 1.*

Proof. We continue to use the notation $y_1, \ldots, y_1, z$ for the semistandard tableau basis elements and the monomials they index.

The proof is by induction on m, the base case $m = 1$ being obvious. Now assume $m \geqslant 2$. Given any semistandard basis element of $H^0(X(w_{3,7}), \mathcal{L}(7m\omega_3))^T$, Lemma 4.14 shows that it can be written as a product of one of the y_i's and a semistandard basis element of $R(m-1)$, or as a product of z in $R(2)$, and a semistandard basis element of $R(m-2)$. Because of Remark 4.15 we have $z = y_5 y_7$. So we can replace z by $y_5 y_7$. It follows by induction that every basis element of $R(m)$ is in the algebra generated by the y_i's. □

It follows that there is a surjective ring homomorphism $\Phi : \mathbb{C}[Y_1, Y_2, \ldots, Y_7] \to R$, sending Y_i to y_i.

Now let $\mathcal{I}$ be the two sided ideal generated by the following relations in $\mathbb{C}[Y_1, Y_2, \ldots, Y_7]$.

$$Y_1 Y_4 = Y_2 Y_3 - Y_2 Y_7 + Y_1 Y_7, \tag{4.1a}$$
$$Y_1 Y_5 = Y_3^2 - Y_3 Y_7, \tag{4.1b}$$
$$Y_1 Y_6 = Y_3 Y_4 - Y_4 Y_7, \tag{4.1c}$$
$$Y_2 Y_5 = Y_3 Y_4 - Y_3 Y_7, \tag{4.1d}$$
$$Y_2 Y_6 = Y_4^2 - Y_4 Y_7, \tag{4.1e}$$
$$Y_3 Y_6 = Y_4 Y_5. \tag{4.1f}$$

Theorem 4.17. *The map* Φ *induces an isomorphism* $\tilde{\Phi} : \mathbb{C}[Y_1, Y_2, \ldots, Y_7]/\mathcal{I} \simeq R$.

Proof. By explicit calculations one can check that the above relations hold with Y_i replaced by y_i; they are in the kernel of $\tilde{\Phi}$. We omit these calculations. To complete the proof we show we can use the above relations as a reduction system. The process consists of replacing a monomial M in the Y_i's which is divisible by a term L_i on the left hand side of one of the reduction rules $L_i = R_i$, by $(M/L_i)R_i$. Here R_i is the right hand side of $L_i = R_i$.

We show that the diamond lemma of ring theory holds for this reduction system (see [Ber78]). What this implies is that any monomial in the Y_i's reduces, after applying these reductions (in any order, when multiple reduction rules apply) to a unique expression in the Y_i's, in which no term is divisible by a term appearing on the left hand side of the above reduction system.

We prove that the diamond lemma holds for this reduction system by looking at the reduction of the minimal overlapping ambiguities $Y_1 Y_2 Y_5, Y_1 Y_2 Y_6, Y_1 Y_3 Y_6$ and $Y_2 Y_3 Y_6$. We show in each case that the final expression is unambiguous. It follows that any relation among Y_i's is in the two sided ideal $\mathcal{I}$ generated by the above relations. This proves that the map $\tilde{\Phi}$ constructed above is injective.

To complete the proof we look at the reductions of overlapping ambiguities.

$Y_1Y_2Y_5$ - using rule (4.1b) above we get $Y_2(Y_3^2 - Y_3Y_7) = Y_2Y_3^2 - Y_2Y_3Y_7$, which cannot be reduced further. On the other hand using rule (4.1d) above we get $Y_1(Y_3Y_4 - Y_3Y_7)$. Now this can be further reduced using rule (4.1a) and we get $Y_3Y_1Y_7 + Y_3Y_2Y_3 - Y_3Y_2Y_7 - Y_1Y_3Y_7$, and this is equal to $Y_2Y_3^2 - Y_2Y_3Y_7$. The reduction is unique in this case.

Likewise, one can show that $Y_1Y_2Y_6$ reduces to the unique expression $Y_2Y_3Y_4 - Y_2Y_4Y_7$, $Y_2Y_5Y_6$ reduces to $Y_4^2Y_5 - Y_4Y_5Y_7$ and $Y_2Y_3Y_6$ reduces to $Y_3Y_4^2 - Y_3^2Y_4Y_7$, completing the proof. □

4.3 Quotients of Deodhar components in $X(w_{3,7})$

Let us fix a reduced decomposition $\mathbf{w_{3,7}} = s_2s_1s_4s_3s_6s_5s_2s_4s_3$ for the Weyl group element $w_{3,7}$ with $X(w_{3,7})$ being the minimal Schubert variety in $G_{3,7}$ admitting semistable points. In this section we describe the GIT quotients of Richardson varieties in $X(w_{3,7})$ by computing the various Deodhar strata in this Schubert variety and analyzing their quotients. Note that 4.19 (respectively, 4.20 and 4.21) can also be obtained as a special case of 4.8 (respectively, 4.9). It will be useful to recall Definition 3.12 and the notation developed in Subsection 3.3.1.

We begin with a corollary to Theorem 4.10.

Corollary 4.18. *The GIT quotient of Richardson varieties in $X(w_{3,7})$ is projectively normal with respect to the descent of the* T*-linearized line bundle $\mathcal{L}(7\omega_3)$.*

Proof. Let $X^v_{w_{3,7}}$ be a Richardson variety in $X(w_{3,7})$. From the proof of [BL03, Proposition 1] it follows that $H^0(X(w_{3,7}), \mathcal{L}(\omega_3)^{\otimes m}) \to H^0(X^v_{w_{3,7}}, \mathcal{L}(\omega_3)^{\otimes m})$ is surjective. Since T is linearly reductive it follows that the map $H^0(X(w_{3,7}), \mathcal{L}(\omega_3)^{\otimes m})^T \to H^0(X^v_{w_{3,7}}, \mathcal{L}(\omega_3)^{\otimes m})^T$ is also surjective. From Theorem 4.10 we know that the polarized variety $T\backslash\backslash X(w_{3,7})^{ss}_T(\mathcal{L}(7\omega_3))$ is projectively normal. Since $T\backslash\backslash(X^v_{w_{3,7}})^{ss}_T(\,\mathcal{L}(7\omega_3))$ is normal it follows that the GIT quotient of $X^v_{w_{3,7}}$ is projectively normal with respect to the descent line bundle. □

Lemma 4.19. *Let $v = v_{3,7} = s_2s_4s_3$. Then $T\backslash\backslash(X^v_{w_{3,7}})^{ss}_T(\,\mathcal{L}(7\omega_3))$ is a point.*

Proof. The only torus-invariant section of $H^0(X(w_{3,7}), \mathcal{L}(7\omega_3))$ which is non-zero on $X^v_{w_{3,7}}$ is the section y_1. Consider the Deodhar component of $X^v_{w_{3,7}}$ corresponding to the subexpression $\mathbf{v} = 111111s_2s_4s_3$. From Definition 3.12, every matrix in $\mathbf{G}^{\mathbf{v}}_{\mathbf{w_{3,7}}}$ is a product of matrices

$y_2(p_1)y_1(p_2)y_4(p_3)y_3(p_4)y_6(p_5)y_5(p_6)s_2s_4s_3$. We use Equation 2.1, § 2.2 to obtain each term of the product. Multiplying these terms we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_2 & 0 & 0 & 1 & 0 & 0 & 0 \\ p_1p_2 & 1 & 0 & p_1 & 0 & 0 & 0 \\ 0 & p_4 & 0 & 0 & 1 & 0 & 0 \\ 0 & p_3p_4 & 1 & 0 & p_3 & 0 & 0 \\ 0 & 0 & p_6 & 0 & 0 & 1 & 0 \\ 0 & 0 & p_5p_6 & 0 & 0 & p_5 & 1 \end{pmatrix}$$

The corresponding point in $G_{3,7}$ is the three dimensional subspace spanned by the first three columns of the above matrix. Denote this submatrix by M. The Plücker coordinates of the embedding of M in projective space are the determinants of the the 3×3 submatrices of M. The section y_1 evaluated on such a matrix M in $\mathbf{G}^{\mathbf{v}}_{\mathbf{w}_{3,7}}$ is the product of the 3×3 determinants of M whose rows are indexed by the columns in y_1. For the above matrix this evaluates to $p_1p_2^4p_3^2p_4^5p_5^3p_6^6$. Using the reduced expression for $\mathbf{w}_{3,7}$, note that the weight of this monomial is $\alpha_2 + 4\alpha_1 + 2\alpha_4 + 5\alpha_3 + 3\alpha_6 + 6\alpha_5$. □

Lemma 4.20. *Let $v = s_2s_3$. Then $T\backslash\backslash(X^v_{w_{3,7}})^{ss}_T(\mathcal{L}(7\omega_3))$ is isomorphic to $\mathbb{P}^1$ and the descent of $\mathcal{L}(7\omega_3)$ is $\mathcal{O}(1)$.*

Proof. On the open Deodhar component corresponding to the reduced subexpression $\mathbf{v} = 11111s_21s_3$ the only non-zero T-invariant standard monomials of shape $7\omega_3$ are y_1, y_2. Using a calculation as in the proof of Lemma 4.19 we see that these are algebraically independent. The lemma follows now from Corollary 4.18. □

Lemma 4.21. *Let $v = s_4s_3$. Then $T\backslash\backslash(X^v_{w_{3,7}})^{ss}_T(\mathcal{L}(7\omega_3))$ is isomorphic to $\mathbb{P}^1$ and the descent of $\mathcal{L}(7\omega_3)$ to the GIT quotient is $\mathcal{O}(2)$.*

Proof. The three non-zero sections on the open Deodhar cell $\mathbf{G}^{\mathbf{v}}_{\mathbf{w}_{3,7}}$ corresponding to the subexpression $\mathbf{v} = 1111111s_4s_3$ are y_1, y_3, y_5. Let $p = p_1p_2^4p_3^2p_4^5p_5^3p_6^6p_7^5$. Let $X = (p_1 + p_7), Y = p_1$. Calculating as in the proof of Lemma 4.19, one checks that on the open Deodhar cell $\mathbf{G}^{\mathbf{v}}_{\mathbf{w}_{3,7}}$, y_1 evaluates to pX^2, y_3 to pXY and y_5 to pY^2. So p is nowhere vanishing on the Deodhar cell. The lemma follows from Corollary 4.18. □

Lemma 4.22. *Let $v = s_3$. Then $T\backslash\backslash(X^v_{w_{3,7}})^{ss}_T(\mathcal{L}(7\omega_3))$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and the descent of the line bundle to the GIT quotient is $\mathcal{O}(2) \boxtimes \mathcal{O}(1)$.*

Proof. We use the distinguished subexpression $\mathbf{v} = 1111111s_3$. Let $A = p_3, B = p_3 + p_8$. Let $X = (p_1 + p_7), Y = p_1$. Note that p_3 and p_8 are algebraically independent and so A, B are algebraically independent. Since p_1 and p_7 are algebraically independent so are X, Y.

Let $p = p_1p_2^4p_3^2p_4^5p_5^3p_6^6p_7^5p_8^6$. Calculating as in the proof of Lemma 4.19, we see that y_1 evaluates to pBX^2 on $\mathbf{G}^{\mathbf{v}}_{\mathbf{w}_{3,7}}$. The section y_5 evaluates to pBY^2, y_3 evaluates to $pXYB$, y_2 evaluates to pAX^2, y_6 evaluates to pAY^2 and y_4 evaluates to $pXYA$.

So upto a multiple of p, the sections $y_2, y_4, y_6, y_1, y_3, y_5$ can be respectively written as $(X^2A, XYA, Y^2A, X^2B, XYB, Y^2B)$. Using Corollary 4.18 it follows that the GIT quotient is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded as $\mathcal{O}(2) \boxtimes \mathcal{O}(1)$. $\square$

In the next lemma we give conditions guaranteeing when a section of the line bundle $\mathcal{L}(n\omega_r)$ on X^v_w restricts to a homogenous polynomial on the Richardson strata in X^v_w.

Lemma 4.23. *Let $u \in W, v \in W^{S\setminus\{\alpha_r\}}$ be such that $w = uv \in W^{S\setminus\{\alpha_r\}}$ and $l(uv) = l(u) + l(v)$. Fix a reduced expression $u = s_{i_1}\cdots s_{i_k}$ and a reduced expression $v = s_{i_{k+1}}\cdots s_{i_m}$ such that $\mathbf{w} = s_{i_1}\cdots s_{i_k}.s_{i_{k+1}}\cdots s_{i_m}$ is a reduced expression for w. Consider $\mathbf{v} = 1\cdots 1 s_{i_{k+1}}\cdots s_{i_m}$, a distinguished subexpression of $\mathbf{w}$. $\mathcal{R}^{\mathbf{v}}_{\mathbf{w}}$ is the unique open Deodhar component of R^v_w. The restriction of any section $s \in H^0(X^v_w, \mathcal{L}(n\omega_r))$ to R^v_w is a homogeneous polynomial in $p_1, p_2, \cdots, p_k$ having degree equal to the height of $v(n\omega_r)$.*

Proof. Note that $\mathbf{v}$ is the unique positive distinguished subexpression for v in $\mathbf{w}$ and so $\mathcal{R}^{\mathbf{v}}_{\mathbf{w}}$ is the unique open Deodhar component of R^v_w.

Matrices in $\mathbf{G}^{\mathbf{v}}_{\mathbf{w}}$ are of the form $y_{i_1}(p_1)y_{i_2}(p_2)\ldots y_{i_k}(p_k)s_{i_{k+1}}\cdots s_{i_m}$. From this identification we see that the section s restricted to this Deodhar component is $s_{|\mathcal{R}^{\mathbf{v}}_{w}} = \sum_{\mathbf{m}} a_{\mathbf{m}} p_1^{m_1}\ldots p_k^{m_k}$ where $\mathbf{m} = (m_1, \ldots, m_t)$. If $a_{\mathbf{m}} \neq 0$ then $wt(s) = wt(p_1^{m_1}\ldots p_k^{m_t}) = v(n\omega_r)$. In particular, $deg(p_1^{m_1}\ldots x_k^{m_k})$ is equal to the height of $v(n\omega_r))$. $\square$

Finally we prove the main theorem of this section.

Theorem 4.24. *The polarized variety $T\backslash\backslash X(w_{3,7})^{ss}_T(\mathcal{L}(7\omega_3))$ is a rational normal scroll.*

Proof. The relations (4.1a)-(4.1f) given before Theorem 4.17 describe the homogenous ideal defining the polarized variety. These defining relations can be written succinctly in a matrix form

$$\operatorname{rank}\begin{pmatrix} Y_1 & Y_3 & Y_4 & Y_2 \\ Y_3 - Y_7 & Y_5 & Y_6 & Y_4 - Y_7 \end{pmatrix} \leqslant 1.$$

For example the minor corresponding to the first two columns above gives us $Y_1Y_5 = Y_3^2 - Y_3Y_7$, which is (4.1b), and the minor corresponding to columns 1 and 3 gives relation

(4.1c) shown there. The polarized variety is defined by the vanishing of 2×2 determinants of a generic 4×2 matrix. Such a variety is called a determinantal variety and it is known that it is a rational normal scroll (see [Rei96]). $\square$

Chapter 5

Projective normality of the GIT quotient of $G_{2,n}$

In this chapter we study the GIT quotient of $G_{2,n}$ with respect to the T-linearized line bundle $\mathcal{L}(n\omega_2)$ for n odd. The main reference for this chapter is [BSKS20]. It is also one of the most important example in geometric invariant theory (see [MFK94]). The quotient can be thought of as the configuration space of n ordered points in the projective space. It is not clear to us whether this result extends to GIT quotients of higher rank Grassmannians. To the best of our knowledge this question is open. As mentioned earlier, this line bundle descends to the quotient, and it is well known that the polarized variety $T\backslash\backslash(G_{2,n})^{ss}_T(\mathcal{L}(n\omega_2))$ is projectively normal (see [HMSV05, Theorem 3.7], [Kem93]). Recently another proof of projective normality of $T\backslash\backslash(G_{2,n})^{ss}_T(\mathcal{L}(n\omega_2))$ using graph theoretic methods has also been given by A.Nayek, S.K Pattanayak, S.Jindal (see, [NPJ20]). We give an alternate combinatorial proof in the $G_{2,n}$ case. The main ingredient we use for the study of this quotient is Standard monomial theory. We believe that it is this kind of combinatorics which will be required to settle the general question. The main result we prove in this chapter is

Theorem 5.1. *(5.1) $T\backslash\backslash(G_{2,n})^{ss}_T(\mathcal{L}(n\omega_2))$ is projectively normal.*

5.1 Projective normality of $T\backslash\backslash(G_{2,n})^{ss}_T(\mathcal{L}(n\omega_2))$

We follow the strategy outlined in Remark 4.11. Defining $R(m)$ to be $H^0(G_{2,n}, \mathcal{L}(n\omega_2)^{\otimes m})^T$ we show that $R(1)^{\otimes m} \to R(m)$ is surjective.

Let $p_{\underline{\tau}} = p_{\tau_1}p_{\tau_2}\dots p_{\tau_{mn}}$ be a standard monomial in $R(m)$ and let $T_{\underline{\tau}}$ be the tableau associated to this monomial.

Denote the columns of $T_{\underline{\tau}}$ by $C_1, C_2, \cdots, C_{mn}$ with $C_i = [a_i, b_i]$. Our idea is to extract from the tableau $T_{\underline{\tau}}$ a semistandard Young subtableau $T_{\underline{\mu}}$, with each integer $1, 2, \ldots, n$ appearing exactly 2 times. Then the monomial $p_{\underline{\mu}}$ corresponding to this subtableau would be a zero weight vector in $R(1)$, and the monomial corresponding to the remaining columns in $T_{\underline{\tau}}$ would be a monomial $p_{\underline{\nu}} \in R(m-1)$. If we were to succeed in doing this, we could write $p_{\underline{\tau}}$ as a product of $p_{\underline{\mu}}$ and $p_{\underline{\nu}}$, and we would be done by induction on m. Since we were unable to do this directly we use straightening laws on tableaux to show that $p_{\underline{\tau}}$ can be written as a sum of products of elements in $R(1)$.

Let $p_{\underline{\mu}} := p_{\tau_1} p_{\tau_{m+1}} \ldots p_{\tau_{mn-m+1}}$ and $p_{\underline{\nu}} = \widehat{p_{\tau_1}} p_{\tau_2} . p_{\tau_3} \cdots p_{\tau_m} \widehat{p_{\tau_{m+1}}} \cdots p_{\tau_{mn}}$. Here $\widehat{p}$ indicates that the corresponding term is omitted. Clearly $p_{\underline{\tau}} = p_{\underline{\mu}} p_{\underline{\nu}}$.

Let $T_{\underline{\mu}}$ and $T_{\underline{\nu}}$ denote the corresponding tableaux.

Definition 5.2. An integer i is *defected* if i appears an odd number of times in $T_{\underline{\mu}}$. Denote the set of defected integers by $\mathcal{D}$.

Lemma 5.3. *All integers in* $\{1, 2, \ldots, n\}$ *occur in* $T_{\underline{\mu}}$.

Proof. Every integer j has to appear at least m times in one of the rows of $T_{\underline{\tau}}$. Since $T_{\underline{\tau}}$ is semistandard j appears consecutively in that row, so there is a column C_i with $i \equiv 1 \pmod{m}$ containing j. $\square$

Lemma 5.4. *There are even number of defected integers.*

Proof. $T_{\underline{\mu}}$ has $2n$ boxes and all the integers appear in $T_{\underline{\mu}}$. Each integer which is not defected appears twice. The number of times a defected integer appears is odd, so there are an even number of defected integers. $\square$

Before we prove the next lemma we set up some notation and make some observations.

Let f_i (respectively, l_i) be such that C_{f_i}(respectively, C_{l_i}) is the column in which i appears for the first (respectively, last) time in the bottom row of $T_{\underline{\tau}}$. Similarly define f^i and l^i with respect to occurrences of i in the top row.

Observation 5.5. $f_i \equiv x + 1 \pmod{m}$ *if and only if* $f^i \equiv m - x + 1 \pmod{m}$. *In particular* $f_i \equiv 1 \pmod{m}$ *if and only if* $f^i \equiv 1 \pmod{m}$, *and in this case* i *appears at least 2 times in* $T_{\underline{\mu}}$.

Proof. Each integer less than i appears $2m$ times and occurs in the top row in columns before column f^i and in the bottom row in columns before column f_i. The total number of positions for numbers from 1 to $i-1$ is therefore a multiple of m. If f_i is $am + 1 + x$, then the number of boxes to the left of this column in the bottom row is $am+x$. So f^i must $bm+m-x+1$ for some b so that the number of positions for integers 1 to $i-1$ is $bm+m-x$ as needed.

The last statement follows since $T_{\underline{\mu}}$ is constructed by taking only columns numbered $1 \pmod m$ in $T_{\underline{\tau}}$ □

We know that the number of defected integers is an even number, say $2l$, for $1 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$. Let $\mathcal{D} = \{i_1, i_2, \cdots, i_{2l}\}$ denote this set with $i_1 < i_2 \cdots < i_{2l}$.

Lemma 5.6. *Let $1 \leqslant j \leqslant 2l$. In $T_{\underline{\mu}}$, i_j appears 3 times if j is odd and i_j appears once if j is even.*

Proof. We show that two consecutive defected integers cannot appear in $T_{\underline{\mu}}$ 3 times and they cannot appear once. Then we show that the first integer which is defected appears 3 times.

Let us assume that some integer i_j which is defected appears 3 times. W.l.o.g we may assume that it appears 2 times in the top row and appears once in the bottom row. Assume that the next defected integer i_{j+1} also appears 3 times. We prove it in the case when i_{j+1} appears 2 times in the top row and once in the bottom row. The proof in the other case is similar.

Assume that the positions of i_j (respectively, i_{j+1}) in $T_{\underline{\tau}}$ which contribute to its two occurrences in the top row of $T_{\underline{\mu}}$ are $(a-1)m+1, am+1$ (respectively, $bm+1, (b+1)m+1$). Likewise, assume that the positions of i_j, (respectively, i_{j+1}) in $T_{\underline{\tau}}$ contributing to the bottom row in $T_{\underline{\mu}}$ are $cm+1$ (respectively, $dm+1$). Clearly $c < a-1$ and $d < b$. Let x be the number of i_j to the right of position $am+1$ in the top row of $T_{\underline{\tau}}$ and z be the number of i_j to the right of $cm+1$ in the bottom row of $T_{\underline{\tau}}$. Similarly let y denote the number of i_{j+1} to the left of position $bm+1$ in the top row and w be the number of i_{j+1} to the left of position $dm+1$ in the bottom row of $T_{\underline{\tau}}$. Clearly $x+z \leqslant m-2$ and $y+w \leqslant m-2$.

Now $i_{j+1} = i_j + 1$ is not possible. Because the number of i_{j+1} in the top row is then at least $2m-x$ and the number of i_{j+1} in the bottom row is at least $m-z$, a contradiction to the number of i_{j+1} in $T_{\underline{\tau}}$ since $x+z \leqslant m-2$.

So let us assume that $i_{j+1} > i_j + 1$. Now there are $i_{j+1} - i_j - 1$ integers in between i_j and i_{j+1} which are not defected. Hence in $T_{\underline{\tau}}$ each of these integers occurs in exactly two positions which are in positions $1 \pmod m$. Hence the number of positions which are $1 \pmod m$ between the positions $am+1, bm+1$ and between $cm+1, dm+1$ is exactly $2(i_{j+1} - i_j - 1)$. But this count is also equal to $(b-a-1)+(d-c-1)$. Hence $b+d-a-c-2 = 2(i_{j+1} - i_j - 1)$. Or $b+d-a-c = 2(i_{j+1} - i_j)$. The total number of positions available for integers in the range i_j+1 to i_{j+1} is exactly $bm-am-x-y-1+dm-cm-z-w-1$ which is $m(b+d-a-c-2)-(x+y+z+w)$. Since each integer in this range appears exactly $2m$ times, and since $b+d-a-c = 2((i_{j+1} - i_j)$ it follows that $x+y+z+w$ is 0 modulo

$2m$. Since $x+z \leqslant m-2$ and $y+w \leqslant m-2$ this is only possible if x, y, z, w are all zero. Then the positions $am+1+1$ to bm and $cm+1+1$ to dm are available for the integers $i_j+1, \ldots, i_{j+1}-1$. This is $(b+d-a-c)m-2$ positions in all, which is also $2(i_{j+1}-i_j)m-2$ positions. But this is more positions than are required, since we have $i_{j+1}-i_j-1$ numbers each occurring $2m$ times - we require only $2m(i_{j+1}-i_j-1)$ positions. We conclude that if i_j appears 3 times i_{j+1} cannot appear 3 times.

Now we show that if i_j appears with defect one then i_{j+1} appears with defect three. Without loss of generality, assume that i_j appears in the top row in a column numbered $1 \pmod m$. So we know that i_j appears less than m times in the bottom row.

Assume that f^{i_j} is $am+x+1$ for some $1 \leqslant x \leqslant m-1$. Then f_{i_j} is $bm+m-x+1$ for some b. Now since i_j does not occur in a column numbered $1 \pmod m$ in the bottom row, it follows that the number of i_j in the bottom is at most x, so the number of i_j in the top row is at least $2m-x$. But since there is only one occurrence of i_j in a column numbered $1 \pmod m$, there are at most $2m-x$ occurrences of i_j in the top row. It follows that the top row has exactly $2m-x$ occurrences of i_j and the bottom row has exactly x occurrences of i_j. So $l^{i_j} = l_{i_j} = 0 \pmod m$. Hence each integers between i_j and i_{j+1} which is not defected starts at a position which is $1 \pmod m$ on the top and ends at a $0 \pmod m$ position in the top and bottom rows (if it occurs in them). So $f_{i_{j+1}}$ is forced to be $1 \pmod m$ and this implies that $f^{i_{j+1}}$ is also $1 \pmod m$ by Observation 5.5. Since it is defected it occurs once more in a column numbered $1 \pmod m$.

To complete the proof we show that the first defected integer occurs 3 times. Suppose that i_1 occurs only once in $T_{\underline{\mu}}$. Then it occurs strictly more than m times in the top or bottom row of $T_{\underline{\tau}}$. W.l.o.g it occurs strictly more then m times in the top row of $T_{\underline{\tau}}$, and say it occurs in column $am+1$. Suppose i_1 makes its first appearance in $T_{\underline{\tau}}$ in column $(a-1)m+1+j$ for $0 < j \leqslant m-1$. Since it occurs only once in a column numbered $1 \pmod m$, the total number of occurrences of i_1 in the top row of $T_{\underline{\tau}}$ is at most $2m-j$. So it occurs in the bottom row of $T_{\underline{\tau}}$ as well. Now suppose its first occurrence in the bottom row of $T_{\underline{\tau}}$ is in column $bm+1+k$, for $0 < k \leqslant m-1$. Since all integers less than i_1 occur $2m$ times in $T_{\underline{\tau}}$ it follows that $j+k = 0 \pmod m$. But $j+k < 2m$ and so $j+k = m$. Now each integer less than i_1 is not defected and so appears twice in $T_{\underline{\tau}}$ in columns numbered $1 \pmod m$. The number of such columns available is $a-1+b$ and since this has to be even, $a+b$ must be odd. But then the total number of positions available for integers less than i_1 in $T_{\underline{\tau}}$ is $(a-1+b)m+j+k$ which is $(a+b)m$, an odd multiple of m. But each integer less than i_1 appears $2m$ times in $T_{\underline{\tau}}$ so, together, they occupy an even number of positions, a contradiction. $\square$

Lemma 5.7. *If j is odd, i_j appears in the top row and in the bottom row of $T_{\underline{\mu}}$.*

Proof. From Lemma 5.6 i_j appears three times in $T_{\underline{\mu}}$. If all of them appear consecutively in $T_{\underline{\mu}}$ then the number of i_j in $T_{\underline{\tau}}$ would be greater than $2m$, a contradiction. $\square$

Notation 5.8. Let $T_{\underline{\tau}}^k$ be the subtableau of $T_{\underline{\tau}}$ having m columns starting with $C_{(k-1)m+1}$ and ending with C_{km}. We denote the first column of $T_{\underline{\tau}}^k$ by $T_{\underline{\tau}}^k[1]$ and the last column as $T_{\underline{\tau}}^k[m]$.

For j odd, define $l(j)$ to be that k for which $(k-1)m+1 \leqslant l_{i_j} \leqslant km$. So $T_{\underline{\tau}}^{l(j)}$ is the subtableau containing the last occurrence of i_j in the bottom i.e containing $C_{l_{i_j}}$ as one of its m columns. For j even let $f(j)$ be that k for which $(k-1)m+1 \leqslant f_{i_j} \leqslant km$. So $T_{\underline{\tau}}^{f(j)}$ is the subtableau containing the first occurrence of i_j in the bottom row i.e containing $C_{f_{i_j}}$ as one of its m columns.

For j odd, let $S_{\underline{\tau},j}$ denote the subtableau with columns $T_{\underline{\tau}}^{l(j)}[1], T_{\underline{\tau}}^{l(j)}[m], T_{\underline{\tau}}^{l(j)+1}[1], T_{\underline{\tau}}^{l(j)+1}[m], \ldots, T_{\underline{\tau}}^{f(j+1)}[1]C_{f_{i_{j+1}}}$. Note that this tableau contains an even number of columns since $T_{\underline{\tau}}^{f(j+1)}[1]$ is different from $C_{f_{i_{j+1}}}$ - by definition i_{j+1} appears only once in $T_{\underline{\mu}}$ and so its first occurrence cannot be in a column numbered 1 $\pmod m$ in $T_{\underline{\tau}}$ by Observation 5.5.

We denote by $S_{\underline{\tau},j}[k]$ the 2×2 subtableau of $S_{\underline{\tau},j}$ containing columns $2k-1$ and $2k$. To simplify notation we mostly omit the $\underline{\tau}$ and just denote this by $S_j[k]$ when $\underline{\tau}$ is clear from the context. Let $S_{\underline{\tau},j}[k] = \begin{array}{|c|c|}\hline p & q \\ \hline r & s \\ \hline\end{array}$.

We set $S_{\underline{\tau},j}[k](1) = p$, $S_{\underline{\tau},j}[k](2) = q$, $S_{\underline{\tau},j}[k](3) = r$ and $S_{\underline{\tau},j}[k](4) = s$.

Example 5.9. We illustrate the above notation using the monomial in $H^0(G_{2,13}, \mathcal{L}(13\omega_2)^{\otimes 2})^T$ given by the following semistandard tableau T_τ.

1	1	1	1	2	2	2	2	3	3	3	3	4	7	7	8	8	9	9	9	10	10	10	11	11	12
4	4	4	5	5	5	5	6	6	6	6	7	7	8	8	9	10	11	11	12	12	12	13	13	13	13

For this tableau we have,

$T_\mu =$

1	1	2	2	3	3	4	7	8	9	10	10	11
4	4	5	5	6	6	7	8	10	11	12	13	13

,

$T_\nu =$

1	1	2	2	3	3	7	8	9	9	10	11	12
4	5	5	6	6	7	8	9	11	12	12	13	13

.

$\mathcal{D} = \{4, 9, 10, 12\}$ and $l_4 = 3, f_9 = 16, l_{10} = 17, f_{12} = 20$. We have $l(1) = 2, f(2) = 8, l(3) = 9, f(4) = 10$. Furthermore,

$S_{\tau,1} =$

1	1	2	2	2	2	3	3	3	3	4	7	7	8
4	5	5	5	5	6	6	6	6	7	7	8	8	9

, $S_{\tau,2} =$

8	9	9	9
10	11	11	12

.

For the above example

$$S_{\tau,1}[1] = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 & 5 \\ \hline\end{array}, \quad S_{\tau,1}[2] = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 5 & 5 \\ \hline\end{array}, \quad S_{\tau,1}[3] = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 5 & 6 \\ \hline\end{array}, \quad S_{\tau,1}[4] = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 6 & 6 \\ \hline\end{array},$$

$$S_{\tau,1}[5] = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 6 & 7 \\ \hline\end{array}, \quad S_{\tau,1}[6] = \begin{array}{|c|c|}\hline 4 & 7 \\ \hline 7 & 8 \\ \hline\end{array}, \quad S_{\tau,1}[7] = \begin{array}{|c|c|}\hline 7 & 8 \\ \hline 8 & 9 \\ \hline\end{array},$$

$$S_{\tau,2}[1] = \begin{array}{|c|c|}\hline 8 & 9 \\ \hline 10 & 11 \\ \hline\end{array}, \quad S_{\tau,2}[2] = \begin{array}{|c|c|}\hline 9 & 9 \\ \hline 11 & 12 \\ \hline\end{array}.$$

We will use the degree lexicographic order on rectangular $2 \times m$ Young tableau. Recall that as per this order a monomial $p = p_{\tau_1} \dots p_{\tau_m}$ corresponding to a $2 \times m$ tableau is bigger than a monomial $q = q_{\mu_1} \dots q_{\mu_{m'}}$ corresponding to another $2 \times m'$ tableau if $m > m'$ or, if $m = m'$, then for the smallest i such that $\tau_i \neq \mu_i$ it is the case that $\tau_i > \mu_i$ in the usual lexicographic order on words of length 2.

Now we fix a j which is odd and look at the subtableau S_j defined above for this j. Suppose S_j has $2t$ columns.

Lemma 5.10. *For* $1 \leqslant k < t$ *we have* $S_j[k](4) = S_j[k+1](3)$. *If there exists a* k *such that* $i_j \leqslant S_j[k](1) < i_{j+1}$ *then* $S_j[k](2) = S_j[k+1](1)$.

Proof. We prove the first statement by contradiction. Suppose $S_j[k](4) \neq S_j[k+1](3)$ for some k. Then $f_{S_j[k+1](3)} \equiv 1 \pmod m$. So we have $f^{S_j[k+1](3)} \equiv 1 \pmod m$ from Observation 5.5. If the number of times $S_j[k+1](3)$ appears in row 1 or row 2 is not m then $S_j[k+1](3)$ would occur 3 times in $T_{\underline{\mu}}$, a contradiction to the fact that $S_j[k+1](3)$ is not defected. So $S_j[k+1](3)$ appears m times in row 1 and m times in row 2. As we iterate over k, this is true of all the $S_j[k+1](3)$'s - they appear m times in the bottom row starting with a column numbered $1 \pmod m$ and also m times in the top row starting with a column numbered $1 \pmod m$. When $k = f(j+1) - 1$ this means $f_{i_{j+1}} \equiv 1 \pmod m$ and, as argued above, $f^{i_{j+1}} \equiv 1 \pmod m$. Since i_{j+1} is defected it has to occur 3 times which is a contradiction to lemma 5.6, since j is odd.

The proof of the second statement is similar and is omitted. □

Lemma 5.11. *Let* j *be odd and suppose that* S_j *has* $2t_j$ *columns for some* t_j. *Then for* $1 \leqslant k \leqslant t_j$ *we have* $S_j[k](3) > S_j[k](2)$.

Proof. We prove the lemma for $j = 1$. We first show this for $k = 1$. Assume $S_1[1](3) < S_1[1](2)$. Consider the tableau $T_{\underline{\mu}}$. The column $[S_1[1](1), S_1[1](3)]$ is a column numbered $1 \pmod m$ in $T_{\underline{\tau}}$ and so this column appears in $T_{\underline{\mu}}$. If $S_1[1](3) < S_1[1](2)$, then all occurrences of $S_1[1](3)$ in

$T_{\underline{\mu}}$ appear in this column and to the left. The total number of positions in the boxes to the left of this column (including this column) in $T_{\underline{\mu}}$ is an even number. But $S_1[1](3)$ appears 3 times in these boxes since i_1 has defect 3, and each other integer appears an even number of times since they are not defected. This is a contradiction.

Now we show this for $k > 1$. Note that the column $[S_1[k](1), S_1[k](3)]$ occurs in $T_{\underline{\mu}}$ since it is a column numbered 1 (mod m) in $T_{\underline{\tau}}$. If $S_1[k](3) > S_1[k](2)$ then all occurrences of $S_1[k](3)$ in $T_{\underline{\mu}}$ are in this column and to its left. This is true for $S_1[k](1)$ too. Since $S_1[k](1)$ and $S_1[k](3)$ are not defected, they appear twice. The total number of positions to the left of (and including this) column $[S_1[k](1), S_1[k](3)]$ in $T_{\underline{\mu}}$ is even. This is a contradiction since i_1 appears 3 times and all the other numbers appear twice.

For j odd and bigger than 1, the proof is similar. Recall that the first column of $S_{\underline{\tau},j}$ is column $T_{\underline{\tau}}^{l(j)}[1]$ and this appears in $T_{\underline{\mu}}$. The only point to note is that in $T_{\underline{\mu}}$, the columns strictly to left of the column $T_{\underline{\tau}}^{l(j)}[1]$ contain all occurrences of the previous i_k, $k < j$ and the sum of the occurrences of these i_k, $k < j$ is even. So too is the sum of occurrences of the remaining integers since they are not defected. The argument then proceeds as in the $j = 1$ case. $\square$

Proposition 5.12. *The map* $R(1)^{\otimes m} \to R(m)$ *is surjective.*

Proof. The proof is by induction. For $p_{\underline{\tau}}$ in $R(m)$, we show that there exists $p_{\underline{\mu}} \in R(1)$ and $p_{\underline{\nu}} \in R(m-1)$ and $p_{\underline{\tau}^j} \in R(m)$ such that $p_{\underline{\tau}} = p_{\underline{\nu}} p_{\underline{\gamma}} + \sum_j p_{\underline{\tau}^j}$ with $p_{\underline{\tau}^j} < p_{\underline{\tau}}$ in lexicographic order. An induction based on degree lexicographic order on monomials completes the proof.

The base case - the lexicographically smallest monomial $p_{\underline{\tau}}$ in $R(m)$, is the one corresponding to the semistandard Young tableau filled with $[1,2]$ in the first $2m$ columns and then $[3,4]$ and so on. Fix $0 \leqslant j \leqslant m-1$. Consider the monomial $p_{\underline{\tau}_j}$ associated to the subtableau of this tableau consisting of columns $1+j, m+1+j, 2m+1+j, \ldots, (n-1)m+1+j$. This is a zero weight vector in $R(1)$. The product of $p_{\underline{\tau}_j}$ is the lexicographically smallest monomial $p_{\underline{\tau}}$ in $R(m)$.

In general starting with $p_{\underline{\tau}}$ we construct $p_{\underline{\mu}}$ and $p_{\underline{\nu}}$ as given before Definition 5.2, by taking for $T_{\underline{\mu}}$ the subtableau with columns $1, m+1, \ldots, (n-1)m+1$. If $p_{\underline{\mu}}$ is a zero weight vector (i.e in the corresponding tableau no integer is defected) we are done. The monomial $p_{\underline{\tau}}$ is the product of a zero weight vector in $R(1)$ and an element in $R(m-1)$ and we are done by induction on degree.

Otherwise, proceeding as above we have defected integers $\{i_1, i_2, \ldots, i_{2l}\}$. Corresponding to the integer i_j in $\{i_1, i_3, \ldots, i_{2l-1}\}$, we have subtableaux S_j and Lemma 5.11 holds. For j odd let the number of columns in S_j be $2t_j$.

Case 1 : Suppose for all j and $1 \leqslant k \leqslant t_j$ it is the case that $S_j[k](3) > S_j[k](2)$.

In this case we do the following operation : We swap $S_j[k](3)$ and $S_j[k](4)$ and keep $S_j[k](1)$, $S_j[k](2)$ fixed for all j odd and for all $1 \leqslant k \leqslant t_j$. We get a new Young tableau, call it S'_j. We modify the original tableau corresponding to $p_{\underline{\tau}}$ by replacing the columns which were previously used to get S_j by the corresponding columns of S'_j. We do this for every j.

Denote the new monomial computed by this tableau by $p_{\underline{\tau'}}$ and denote by $p_{\underline{\mu'}}$ the monomial obtained from this tableau by selecting columns numbered $1, m+1, 2m+1, \ldots, (n-1)m+1$. It is clear that $T_{\underline{\mu'}}$ is semistandard. Furthermore for every j odd, one of the i_j's which appeared in a column numbered $1 \pmod m$ in $T_{\underline{\mu}}$ appears now in a column numbered $0 \pmod m$, and so its count in $T_{\underline{\mu'}}$ is one less than in $T_{\underline{\mu}}$. So i_j is not defected in $T_{\underline{\mu'}}$. For this same j the last exchange is done between $S_j[t_j](3)$ and $S_j[t_j](4)$ and this is i_{j+1}. So this i_{j+1} now occurs in a column numbered $1 \pmod m$ in $T_{\underline{\tau'}}$, and so the count of i_{j+1} in $T_{\underline{\mu'}}$ is one more than in $T_{\underline{\mu}}$. So i_{j+1} is not defected in $T_{\underline{\mu'}}$. From Lemma 5.10, $S_j[k](4) = S_j[k+1](3)$ for all k, so the counts of the remaining integers in $T_{\underline{\mu'}}$ are the same as their count in $T_{\underline{\mu}}$, so these continue to be not defected. This is true for every j. So no integer is defected in $T_{\underline{\mu'}}$ and the corresponding monomial is a zero weight vector in $R(1)$. So $p_{\underline{\tau'}}$ is a product of a zero weight monomial in $R(1)$ and an element of $R(m-1)$.

To finish the proof in this case we compare $p_{\underline{\tau}}$ with $p_{\underline{\tau'}}$. Let us denote the set of columns of $T_{\underline{\tau}}$ not in any S_j by Q and the monomial computed by them as y. If S_j has $2t_j$ columns the monomial computed by it is a product of the t_j monomials computed by the 2×2 subtableaux $S_j[k]$, $p_{S_j[k]} := p_{(S_j[k](1), S_j[k](3))} p_{(S_j[k](2), S_j[k](4))}$. We have

$$p_{\underline{\tau}} = y \cdot \Pi_{j=1}^{j=l} \Pi_{k=1}^{k=t_j} p_{S_j[k]}, \tag{5.1}$$

$$p_{\underline{\tau'}} = y \cdot \Pi_{j=1}^{j=l} \Pi_{k=1}^{k=t_j} p_{S'_j[k]}. \tag{5.2}$$

From the straightening laws the following relation holds between the tableaux $S_j[k]$ and $S'_j[k]$.

$$\begin{array}{|c|c|} \hline p & q \\ \hline r & s \\ \hline \end{array} = \begin{array}{|c|c|} \hline p & q \\ \hline s & r \\ \hline \end{array} \pm \begin{array}{|c|c|} \hline p & r \\ \hline q & s \\ \hline \end{array} \tag{5.3}$$

Recall that in the equation above $S_j[k]$ is the tableau on the left hand side of the equation and $S'_j[k]$ is the first tableau on the right hand side.

Plugging this into Equation 5.1 above we see that $p_{\underline{\tau}}$ is the sum of $p_{\underline{\tau'}}$ and sums of products of monomials obtained from $p_{\underline{\tau}}$ by replacing at least one of the terms $p_{S_j[k]}$ in its expression by $p'_j[k]$, the monomial computed by the second tableau on the right hand side of Equation 5.3. However from the hypothesis of this case $r > q$. So the second tableau on the right is lexicographically smaller than the tableau $S_j[k]$. So the $2 \times mn$ tableau corresponding to each additional term obtained by plugging Equation 5.3 into Equation 5.1 is lexicographically smaller than $T_{\underline{\tau}}$. It is possible that this tableau is not semistandard and needs to be straightened into a sum of semistandard tableaux. But each such tableau $T_{\underline{\tau''}}$, will be lexicographically smaller than the (non semistandard) tableau we started with (see, [Ses, Lemma 1.3.5]). We proved above that $p_{\tau'}$ is the product of $p_{\underline{\mu'}} \in R(1)$ and a monomial $p_{\underline{\nu'}} \in R(m-1)$. We have

$$p_{\underline{\tau}} = p_{\underline{\mu'}} p_{\underline{\nu'}} + \sum_s p_{\underline{\tau''_s}},$$

with the sum being over tableaux which are smaller than τ in lexicographic order. By induction on lexicographic order each of these is in the image of $R(1)^{\otimes m}$. By induction on degree $p_{\underline{\nu'}}$ is in the image of $R(1)^{\otimes(m-1)}$. So we are done.

Case 2 : For j in which the conditions of Case 1 hold we do exactly as in that case. Let j be such that $S_j[k](3) = S_j[k](2)$ for some $1 \leqslant k < t_j$. For each such j we do the following. First note that for such a j, i_j appears in S_j as $S_j[k](1)$ for some $1 < k < t_j$, since it has defect 3. Let z be the set of all elements $i_j \leqslant z < i_{j+1}$ with $z = S_j[k](3)$ in some subtableau $S_j[k]$ with $S_j[k](3) = S_j[k](2)$. Order this set as $\{z_1, z_2, \cdots, z_e\}$ such that $i_j \leqslant z_1 < z_2 < z_3 \cdots < z_e < i_{j+1}$, and let k_i denote the index for which $z_i = S_j[k_i](3) = S_j[k_i](2)$ - clearly $z_s \neq z_t$ for $s \neq t$ and $z_e < i_{j+1}$ and $k_i \geqslant k$. Let x_i, y_i denote $S_j[k_i](1)$ and $S_j[k_i](4)$. For $k < k_i < t_j$ it follows from Lemma 5.10 that $S_j[k_i - 1](4) = z_i$, $S_j[k_i + 1](1) = z_i$, $S_j[k_i - 1](2) = x_i$ and $S_j[k_i + 1](3) = y_i$. We have two subcases.

i e is odd: In this case we first swap $S_j[l](1)$ and $S_j(l)(2)$ for all $k \leqslant l \leqslant k_1 - 1$. Then swap the two columns in S_{k_1}. And swap $S_j[l](3)$ and $S_j[l](4)$ for all $k_1 + 1 \leqslant l \leqslant k_2 - 1$. Do nothing with $S_j[k_2]$. Instead start with z_2 which appears in $S_j[k_2+1](1)$ and $S_j[k_2+1](2)$ and repeat these steps. Since e is odd, the last set of swaps will happen in the bottom row starting from $y_e = S_j[k_e + 1](3)$ up to $i_{j+1} = S_j[t_j](4)$.

ii e is even: In this case we swap $S_j[l](3)$ and $S_j(l)(4)$ for all $1 \leqslant l \leqslant k_1 - 1$. Do nothing with $S_j[k_1]$. Instead swap $S_j[l](1)$ and $S_j[l](2)$ for all $k_1 + 1 \leqslant l \leqslant k_2 - 1$ and then swap the two columns of $S_j[k_2]$. And repeat the procedure from the y_2 which appears as $S_{k_2+1}[l](3)$. Since e is even it can be checked that the last swaps will happen in the bottom row from $y_e = S_j[k_e + 1](3)$ to $i_{j+1} = S_j[t_j](4)$.

After these round of swaps, we can use straightening as we did in case 1 above, to complete the proof. The last set of swaps take place in the bottom row starting with an element occurring in a $1 \pmod m$ position and ending with the first occurrence of i_{j+1} in the bottom row - this is true in both cases. In both cases the first set of swaps start with i_j occurring in a $1 \pmod m$ position and end with an element occurring in a position 0

(mod m). It can be checked that if we form tableau $S_{j'}$ as we did in Case 1 above, the number of i_j has reduced and the number of i_{j+1} has increased. The number of occurrences of the intermediate numbers does not change because of the column swaps performed. Furthermore, the other set of swaps between elements in the top row and elements in the bottom row in an $S_j[l]$ take place in those l wherein $S_j[l](3) > S_j[l](2)$. One checks as in Case 1 above that straightening introduces new zero weight tableaux, but all of them are lexicographically smaller than the tableau we start with. This completes the proof.

$\square$

Example 5.13. We continue with the example in 5.9. We reproduce T_τ below. For the subtableau $S_{\tau,1}$ of T_τ we perform the operations suggested in Case 2 above, and for the subtableau $S_{\tau,2}$ we perform the operations as in Case 1. So we can write the monomial given by

1	1	1	1	2	2	2	2	3	3	3	3	4	7	7	8	8	9	9	9	10	10	10	11	11	12
4	4	4	5	5	5	5	6	6	6	6	7	7	8	8	9	10	11	11	12	12	12	13	13	13	13

as a sum of the monomials given by the following two tableaux,

1	1	1	1	2	2	2	2	3	3	3	3	4	7	8	7	8	9	9	9	10	10	10	11	11	12
4	4	5	4	5	5	6	5	6	6	7	6	7	8	9	8	11	10	12	11	12	12	13	13	13	13

,

1	1	1	1	2	2	2	2	3	3	3	3	4	7	7	8	8	10	9	9	10	10	10	11	11	12
4	4	4	5	5	5	5	6	6	6	6	7	7	8	8	9	9	11	11	12	12	12	13	13	13	13

.

The first term of the sum can be written as a product of two zero weight vectors. The odd numbered columns give rise to the zero weight vector represented by the tableau

1	1	2	2	3	3	4	8	8	9	10	10	11
4	5	5	6	6	7	7	9	11	12	12	13	13

.

The even numbered columns give rise to the zero weight vector represented by the tableau

1	1	2	2	3	3	7	7	9	9	10	11	12
4	4	5	5	6	6	8	8	10	11	12	13	13

.

The tableau corresponding to the second summand is nonstandard but lexicographically smaller than T_τ.

Theorem 5.14. $T\backslash\backslash(G_{2,n})^{ss}_T(\mathcal{L}(n\omega_2))$ *is projectively normal.*

Proof. Now $T\backslash\backslash(G_{2,n})^{ss}_T(\mathcal{L}(n\omega_2))$ is normal. R_1-generation follows from Proposition 5.1. The theorem follows. $\square$

Corollary 5.15. *The GIT quotient of a Schubert variety in* $G_{2,n}$ *is projectively normal with respect to the descent line bundle.*

Proof. T is a linearly reductive group. For a Schubert variety $X(w)$ in $G(2,n)$ the map $H^0(G_{2,n}, \mathcal{L}(n\omega_2)^{\otimes m})^T \longrightarrow H^0(X(w), \mathcal{L}(n\omega_2)^{\otimes m})^T$ is surjective. Since $X(w)^{ss}_T(\mathcal{L}(n\omega_2))$ is normal the corollary follows. $\square$

We have an analogue of Corollary 4.18. The proof is similar and is omitted.

Corollary 5.16. *The GIT quotient of a Richardson variety in* $G_{2,n}$ *is projectively normal with respect to the descent line bundle.*

Chapter 6

Smooth torus quotients of Schubert varieties in the Grassmannian

We denote $\mathcal{L}$ for $\mathcal{L}(n\omega_r)$ throughout in this chapter. We study the GIT quotients $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ of Schubert varieties in the Grassmannian $G_{r,n}$ where r and n are coprime and give a combinatorial criteria to describe the Schubert varieties whose quotients are smooth. Skorobogotov (see [Sko93]) and Kannan(see [Kan98]) has studied the quotient variety $T\backslash\backslash(G_{r,n})^{ss}_T(\mathcal{L})$ is smooth and shown that $T\backslash\backslash(G_{r,n})^{ss}_T(\mathcal{L})$ is smooth when r and n are coprime. Our current exposer builds on ideas from [Kan14, §3]. The main reference for this chapter is [BKS19]. The main result of this chapter is:

Theorem 6.1. *Let $w_{r,n} = (a_1, a_2, \dots, a_r)$ be such that $X(w_{r,n})$ denote the unique minimal Schubert variety admitting semistable point. Let $w = (b_1, b_2, \dots, b_r) \in I(r, n)$ with $b_i \geqslant a_i$ for all $1 \leqslant i \leqslant r$. Let $X(v_1), \dots, X(v_k)$ be the k components in the singular locus of $X(w)$. Then the following are equivalent*

(1) *$T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth.*

(2) *For all i, we have $v_i \ngeqslant w_{r,n}$.*

(3) *Whenever $b_j \geqslant b_{j-1} + 2$ we have $a_j \geqslant b_{j-1} + 1$.*

We note that 6.1 is a generalisation of 4.5.

6.1 Smoothness of GIT quotients

In this section we first give a criterion for the GIT quotient to be smooth. We then prove the main theorem by showing that if the combinatorial conditions in the statement of the main theorem hold this criterion is met. We assume that r and n coprime.

Proposition 6.2. *Let $w \in I(r,n)$. $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth if and only if $X(w)^{ss} \subseteq X(w)_{sm}$.*

Proof. Assume that $X(w)^{ss} \subseteq X(w)_{sm}$. Since $gcd(r,n) = 1$, it follows from [Sko93, Corollary 2.5] and [Kan98, Theorem 3.3] that $X(w)^{ss} = X(w)^{s}$. So the stablizer of all semistable points $x \in X(w)^{ss}$ is finite. The proof now follows along the lines described in [Kan14]. Suppose $x \in BvP/P$ for some v. Let $R^{+}(v^{-1})$ denote the set of all positive roots made negative by v^{-1}. And choose a subset $\beta_1, \ldots, \beta_k$ of positive roots in $R^{+}(v^{-1})$ such that $x = u_{\beta_1}(t_1)\ldots, u_{\beta_k}(t_k)vP/P$ with $u_{\beta_j}(t_j)$ in the root subgroup U_{β_j}, $t_j \neq 0$ for $j = 1, \ldots, k$. The isotropy group T_x is $\cap_{i=1}^{i=k}\ker(\beta_j)$. Since this is finite, it follows from [Kan14, Example 3.3] that $T_x = Z(G)$, the center of G. Working with the adjoint group we may assume that the stablizer is trivial. So $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth.

For the converse, first note that since we are in the case $gcd(r,n) = 1$ the quotient is a geometric quotient i.e. it is an orbit space. But then restricted to $X(w)^{ss}$ the quotient is a T-bundle. So smooth points go to smooth points in the quotient and singular points go to singular points. Since the quotient is smooth it follows that each point $x \in X(w)^{ss}$ is smooth in $X(w)^{ss}$. Since $\mathcal{O}_{x,X(w)^{ss}} = \mathcal{O}_{x,X(w)}$, it follows that $X(w)^{ss} \subseteq X(w)_{sm}$. $\square$

We prove the main theorem.

Theorem 6.3. *Let $w = (b_1, b_2, \ldots, b_r) \in I(r,n)$ with $b_i \geqslant a_i$ for all $1 \leqslant i \leqslant r$. Let $X(v_1), \ldots, X(v_k)$ be the k components in the singular locus of $X(w)$. Then the following are equivalent*

(1) $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ *is smooth.*

(2) *For all i, we have $v_i \ngeqslant w_{r,n}$.*

(3) *Whenever $b_j \geqslant b_{j-1} + 2$ we have $a_j \geqslant b_{j-1} + 1$.*

Proof. Since $b_i \geqslant a_i$ for all i we have that $X(w)^{ss}_T(\mathcal{L})$ is non empty. From Proposition 6.2, $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth if and only if $w_i \ngeqslant w_{r,n}$ for all i. Hence the equivalence of (1) and (2).

We prove the equivalence of $(2), (3)$. The components of the singular locus of $X(w)$ are Schubert varieties $X(w_i)$ in correspondence with diagrams obtained from $Y(\mathbf{w})$ by removing hooks. There is a hook at row j of $Y(\mathbf{w})$ if and only if $b_j \geqslant b_{j-1} + 2$. We denote the Schubert variety obtained from w by removing the hook at row j by $X(w_j)$. Let the word corresponding to it in $I(r,n)$ be $(b'_1, b'_2, \ldots, b'_r)$. Now $X(w)$ contains $X(w_{r,n})$. Let t be the smallest integer less than j such that $b_{k+1} = b_k + 1$ for all $t \leqslant k < j$. By definition of w_j we have

$$b'_p = \begin{cases} b_p & 1 \leqslant p \leqslant t-1, \\ b_p - 1 & t \leqslant p \leqslant j-1, \\ b_{j-1} & p = j, \\ b_p & j+1 \leqslant p \leqslant r. \end{cases}$$

Now $X(w)$ contains $X(w_{r,n})$ so $b_p \geqslant a_p$ for $1 \leqslant p \leqslant r$. So $X(w_j)$ does not contain $X(w_{r,n})$ if and only if $a_p \geqslant b'_p + 1 = b_p$, for some $t \leqslant p \leqslant j-1$ or if $a_j \geqslant b'_j + 1 = b_{j-1} + 1$. Now $b_p = b_{p-1} + 1$ for all $t < p \leqslant j-1$, and $a_p \geqslant a_{p-1} + 1$. It follows that if for some $t \leqslant p \leqslant j-1$, $a_p \geqslant b_p$ then $a_{p+1} \geqslant a_p + 1 \geqslant b_p + 1 = b_{p+1}$, and so we conclude that $a_{j-1} \geqslant b_{j-1}$. Then $a_j \geqslant a_{j-1} + 1 \geqslant b_{j-1} + 1$, completing the proof.

An alternate proof of the equivalence of (2), (3) is as follows. First assume that for all j for which $b_j \geqslant b_{j-1} + 2$ we have $a_j \geqslant b_{j-1} + 1$. Now let $u \in W^P$ be such that $w_{r,n} \leqslant u \leqslant w$. Let the one line notation for u be $(b'_1, b'_2, \ldots, b'_r)$. Then $a_i \leqslant b'_i \leqslant b_i$ for all $1 \leqslant i \leqslant r$. Define $u_i = s_{b'_i} s_{b'_i+1} \ldots s_{b_i-1}$ for $1 \leqslant i \leqslant r$. Clearly $u = u_1(s_{b_1-1} \cdots s_1) u_2 (s_{b_2-1} \cdots s_2) \cdots u_r (s_{b_r-1} \cdots s_r)$.

For every $1 < i \leqslant r$, the index of the least simple reflection less than u_i in the Bruhat order is $s_{b'_i}$ and the index of the largest simple reflection less than u_{i-1} in the Bruhat order is $s_{b_{i-1}-1}$. Take any $1 \leqslant i \leqslant r$ for which $b_i \geqslant b_{i-1} + 2$. By our hypothesis we have $b'_i \geqslant b_{i-1} + 1$, so $s_{b_{i-1}} \nleqslant u_i$ and $u_i \in P_w$ from Proposition 3.19. Further u_i and u_{i-1} commute. For each $1 < i \leqslant r$ for which $b_i = b_{i-1} + 1$, $u_i \in P_w$ from Proposition 3.19. Clearly $u_1 \in P_w$. So for all i, $u_i \in P_w$. It is easy to check that $u = u_r u_{r-1} \ldots u_2 u_1 wP/P$. Therefore, by Theorem 2.2, $X(v) \subset X(w)_{sm}$.

Now assume that $b_j \geqslant b_{j-1} + 2$ but $a_j \leqslant b_{j-1}$. iI follows from the definition of $J'(w)$ in Proposition 3.19 that $b_{j-1} \in J'(w)$. Let t be the smallest integer less than j such that $b_{k+1} = b_k + 1$ for all $t \leqslant k < j$. Then w has a reduced expression of the form

$$w = w'' s_{b_t-1} \ldots s_t s_{b_{t+1}-1} \ldots s_{t+1} \ldots s_{b_{j-1}-1} \ldots s_{j-1} s_{b_j-1} \ldots s_j w'.$$

Now consider the Weyl group element

$$u = w'' s_{b_t-2} \ldots s_t s_{b_{t+1}-2} \ldots s_{t+1} \ldots s_{b_{j-1}-2} \ldots s_{j-1} s_{b_{j-1}-1} \ldots s_j w'.$$

Clearly $u \leqslant w$ and u is obtained from w by left multiplying with the reduced word

$$s_{b_{j-1}} s_{b_{j-1}+1} \ldots s_{b_j-2} s_{b_j-1} \ldots s_{b_{t+2}-1} s_{b_{t+1}-1} s_{b_t-1}.$$

In the one line notation $u = (b_1, \dots, b_{t-1}, b_t - 1, \dots, b_{j-1} - 1, b_{j-1}, b_{j+1}, \dots, b_r)$. Note that $J'(u) \subseteq J'(w)$, so $P_w \subseteq P_u$ and therefore P_w stabilises $X(u)$. Since $u < w$, w is not element of $X(u)$. And so $wP/P \notin P_w uP/P$. Hence $uP/P \notin P_w wP/P$. Therefore, by Theorem 3.18, $X(u)$ is in the singular locus of $X(w)$. However, if $a_j \leqslant b_{j-1}$, it can be easily seen that $u \geqslant w_{r,n}$, implying that $X(u)$ contains a semistable point, a contradiction. $\square$

6.2 Examples and non-examples

We illustrate the proof of the main theorem with a simple example.

Example 6.4. Consider the Schubert variety corresponding to $w = (3,5,7,9)$ in $I(4,9)$. The Young diagram associated to w is given by the increasing sequence $\mathbf{w} = (\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5})$. Fill this diagram starting with s_i at the leftmost box in row i, and filling the boxes to the right of this entry in row i with $s_{i+1}, s_{i+2}, \dots$, in order, all the way to the last box in row i. We get the filling

s_4	s_5	s_6	s_7	s_8
s_3	s_4	s_5	s_6	
s_2	s_3	s_4		
s_1	s_2			

Reading the entries in the Young diagram from right to left in each row, and bottom to top yields $s_2s_1s_4s_3s_2s_6s_5s_4s_3s_8s_7s_6s_5s_4$, the element in W^P corresponding to the Schubert variety $(3,5,7,9)$. According to Theorem 3.20 the singular locus of this Schubert variety has three irreducible components given by the sequences $(1,1,4,5)$,$(2,2,2,5)$ and $(2,3,3,3)$. The corresponding Schubert varieties are given by the tuples $(2,3,7,9)$, $(3,4,5,9)$ and $(3,5,6,7)$, respectively. The Weyl group elements corresponding to these varieties are $s_1s_2s_6s_5s_4s_3s_8s_7s_6s_5s_4$, $s_2s_1s_3s_2s_4s_3s_8s_7s_6s_5s_4$ and $s_2s_1s_4s_3s_2s_5s_4s_3s_6s_5s_4$ respectively. Note that these words can be obtained by removing the hooks occupied by $s_2s_3s_4$, $s_4s_5s_6$ and $s_6s_7s_8$, respectively, and reading the entries left in the resulting Young diagrams from bottom to top, and right to left in each row - exactly as we did for w.

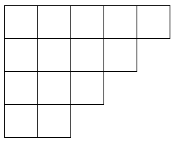
Let us show for example that the Schubert variety corresponding to the Weyl group element $v = s_1s_2s_6s_5s_4s_3s_8s_7s_6s_5s_4$ is not in the smooth locus by showing that it does not satisfy the hypothesis of Theorem 3.18. The stabilizer of $X(w)$ is the parabolic subgroup corresponding to the subset of simple reflections $\{\alpha | s_\alpha w \leqslant w\}$. In this case it can be checked that this is the parabolic subgroup corresponding to $\{\alpha_1, \alpha_2, \alpha_4, \alpha_6, \alpha_8\}$ which is $P_{\hat{\alpha}_3} \cap P_{\hat{\alpha}_5} \cap P_{\hat{\alpha}_7}$. However v is obtained from w by multiplying on the left with $s_3s_4s_2$. And this element is not in $P_{\hat{\alpha}_3} \cap P_{\hat{\alpha}_5} \cap P_{\hat{\alpha}_7}$. It can be similarly shown that the other two components are also not in the smooth locus - the Weyl group elements corresponding to them are obtained from w by multiplying on the left with $s_4s_6s_5$ and $s_6s_8s_7$ respectively and these elements are clearly not in $P_{\hat{\alpha_3}} \cap P_{\hat{\alpha_5}} \cap P_{\hat{\alpha_7}}$.

We conclude with examples of Schubert varieties in $G_{4,9}$ whose GIT quotients are singular, and examples of Schubert varieties whose GIT quotients are smooth.

Example 6.5. We know from 4.4 that $w_{4,9} = (3,5,7,9)$. A reduced expression for the word $w_{4,9}$ is

$$s_2s_1s_4s_3s_2s_6s_5s_4s_3s_8s_7s_6s_5s_4.$$

The Young diagram $Y(\mathbf{w_{4,9}})$ corresponding to $w_{4,9}$ is



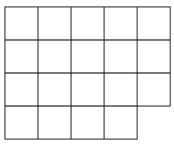
.

Recall from Theorem 3.1 [BSKS20] we have $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth.

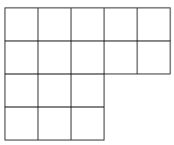
Example 6.6. Let us consider the word $w = (5,7,8,9)$. A reduced expression for w is

$$s_4s_3s_2s_1s_6s_5s_4s_3s_2s_7s_6s_5s_4s_3s_8s_7s_6s_5s_4.$$

The Young diagram $Y(\mathbf{w})$ is



The singular locus $X(w)$, obtained by removing the only hook corresponds the following tableau:

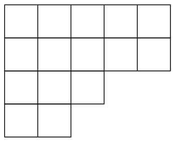


Here $w' = (4,5,8,9)$. Since $w' > w_{4,9}$, $X(w')$ contains semistable points and hence the quotient space $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is not smooth (using 6.3).

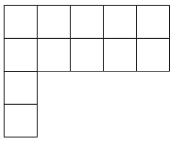
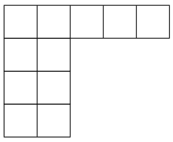
Example 6.7. Consider the word $w = (3,5,8,9)$. A reduced expression for w is

$$s_2 s_1 s_4 s_3 s_2 s_7 s_6 s_5 s_4 s_3 s_8 s_7 s_6 s_5 s_4.$$

The Young diagram $Y(\mathbf{w})$ is



The singular locus obtained by removing the hooks has Schubert varieties $X(w_1), X(w_2)$, whose Young diagrams are given by the following tableaux.

Here $w_1 = (2,3,8,9)$ and $w_2 = (3,4,5,9)$. Note for $i = 1,2$ $w_i \not\geq w_{4,9}$, so neither $X(w_1)$ nor $X(w_2)$ contain semistable points. Hence the quotient space $T\backslash\backslash X(w)^{ss}_T(\mathcal{L})$ is smooth (using Theorem 4.17).

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