On Internal Tensor Product of Modules over Schur Algebra

By

Shraddha Srivastava

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

to

Chennai Mathematical Institute
January 2018

Plot No-H1, SIPCOT IT Park,
Siruseri, Kelambakkam,
Tamilnadu - 603103
India

CHENNAI
MATHEMATICAL
INSTITUTE

Shraddha Srivastava

Plot No.H1, SIPCOT IT Park
Padur Post, Siruseri
Tamil Nadu, India-603 103
E-mail: shraddha@cmi.ac.in

DECLARATION

I declare that the thesis entitled **"On Internal Tensor Product of Modules over Schur Algebra"** submitted by me for the degree of **Doctor of Philosophy in Mathematics** is the record of academic work carried out by me under the guidance of Professor Upendra Kulkarni and this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

Shraddha Srivastava

Chennai Mathematical Institute
April, 2017.

CHENNAI
MATHEMATICAL
INSTITUTE

Professor Upendra Kulkarni

Plot No.H1, SIPCOT IT Park
Padur Post, Siruseri
Tamil Nadu, India-603 103
E-mail: upendra@cmi.ac.in

CERTIFICATE

I certify that the thesis entitled **"On Internal Tensor Product of Modules over Schur Algebra"** submitted for the degree of **Doctor of Philosophy in Mathematics** by Shraddha Srivastava is the record of research work carried out by her under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent her independent work in a very substantial measure.

Chennai Mathematical Institute
Date: April, 2017.

Professor Upendra Kulkarni
Thesis Supervisor.

Acknowledgements

I am grateful to my supervisor Upendra Kulkarni for his constant encouragement and guidance throughout my PhD. I am very thankful to him for introducing me to the representation theory of Schur algebra and the language of category theory. I am also very thankful to K.V. Subrahmanyam for being friendly, supportive and helpful in many ways.

I thank M. Prest for giving an opportunity to participate in a workshop and conference at University of Manchester. I also had occasion to visit W. Kallen, H. Krause and A. Touzé. I thank them for warm hospitality. I also thank them, Karin Erdmann and Oded Yacobi for encouragement and fruitful discussions.

I thank V. Balaji, S. Kannan and Krishna for their courses on representations of algebraic groups. I thank A. Prasad for many fruitful discussions and also sharing his joint work with T. Geetha on alternating Schur algebra. I also acknowledge the contributions of Pramathanath Sastry, R. Sridharan, K. N. Raghavan and D. S. Nagaraj who were always very helpful to me during the course of my PhD. I thank Kavita and Priyavrat for many valuable mathematical discussions.

It is my pleasure to thank my teachers at IIT Bombay namely Debraj Chakraborty, J. K. Verma and my teachers at University of Allahabad namely K. K. Azad, B. K. Sharma. I thank S. Kumaresan as attending his MTTS program was very beneficial.

I thank Sarjick, Abhishek, Naveen, Praveen, Rajib, Anbu, Athira, Arghya(IMSc), Chary and Prakash for creating very friendly atmosphere and without them my journey of PhD would have been really difficult. I thank Muthu for helping me in LaTeX. I thank Himalaya and Chary for their careful reading of preliminary version of this thesis.

I would like to thank CMI administration staff S. Sripathy, Rajeshwari Nair, V. Vijaylaksmi, Ranjini Girish for their help.

I was fully funded by NBHM research fellowship and also partially by an Infosys-grant.

Finally, I am indebted to my parents and my sisters Poornima and Jyotsna for their tremendous support and care.

Shraddha Srivastava
CMI, April 2017.

Dedicated to my Parents

Abstract

For modules over Schur algebra, H. Krause defined a new “internal tensor product" using the language of strict polynomial functors. We show that over an arbitrary commutative base ring k, the Schur functor carries this internal tensor product to the usual Kronecker tensor product of symmetric group representations. This is true even at the level of derived categories. The new tensor product is a substantial enrichment of the Kronecker tensor product. E.g. in modular representation theory it brings in homological phenomena not visible on the symmetric group side. We calculate the internal tensor product over any k in several interesting cases involving classical functors and the Weyl functors.

We extend the classical double centralizer property to any subgroup G of the symmetric group S_d. The natural analogue of Schur algebra in this case is the centralizer algebra $\mathrm{End}_G(V^{\otimes d})$ where V is the vector space k^n. We show there exists an internal tensor product of modules over this new centralizer algebra $\mathrm{End}_G(V^{\otimes d})$ analogous to the case of Schur algebra. We also show that there exists an internal tensor product on modules of hyperoctahedral Schur algebra associated to Weyl group of type **B**. In particular, we show that the Schur algebra, these new centralizer algebras and hyperoctahedral Schur algebra are sesquialgebras.

Contents

Chapter 1

Introduction

1.1 Some history

A correspondence between polynomial representations of degree d of the general linear group $GL_n(\mathbb{C})$ and representations of the symmetric group S_d over $\mathbb{C}$ was first given by Schur in his thesis in 1901. Later J. A. Green showed this correspondence remains true over any infinite field k in [Gre80]. S_d acts on the right of $(k^n)^{\otimes d}$ by permuting the tensor factors. Also, $GL_n(k)$ acts on $(k^n)^{\otimes d}$ diagonally where k^n is the natural representation of $GL_n(k)$. In 1927, Schur showed that the commuting actions of $GL_n(k)$ and S_d on $(k^n)^{\otimes d}$ generate the centralizers of each other. This is called the classical Schur-Weyl duality. This connection could also be understood in terms of the Schur algebra $S_k(n, d)$ which is isomorphic to $\mathrm{End}_{S_d}((k^n)^{\otimes d})$. Over an arbitrary commutative ring k with unity, the category of left $S_k(n, d)$ modules, $S_k(n, d)$-Mod, is equivalent to the category of polynomial representations of degree d of general linear group scheme $GL(k^n)$ [Jan03]. For $n \geqslant d$, the classical Schur-Weyl duality is the isomorphism $\mathrm{End}_{S_k(n,d)}((k^n)^{\otimes d}) \simeq (kS_d)^{op}$. Weyl had called this phenomenon a double centralizer property. For $n \geqslant d$, the Schur functor is $\mathrm{Sch} := \mathrm{Hom}_{S_k(n,d)}((k^n)^{\otimes d}, -) : S_k(n, d)\text{-Mod} \to kS_d\text{-Mod}$. The functor Sch is an equivalence if k is a field of characteristic 0.

It is well known from Akin, Buchsbaum and Weyman [ABW82] that the constructions of $S_k(n, d)$-modules, e.g. Weyl and Dual Weyl modules, are functorial in k^n and these were defined in characteristic free manner. It would be desirable to have modules over Schur algebras in a unified manner, i.e., it could be a functor from a category $\mathcal{C}$ to the category of k-modules k-Mod which on evaluating at k^n becomes a $S_k(n, d)$-module. This can be achieved by strict polynomial functors introduced by Friedlander and Suslin in [FS97]. In order to describe these, first define a category $\Gamma^d P_k$ whose objects are finitely generated projective modules over k and morphism space $\mathrm{Hom}_{\Gamma^d P_k}(V, W)$ is $\mathrm{Hom}_{S_d}(V^{\otimes d}, W^{\otimes d})$. The category $\Gamma^d P_k$ is called the divided power category. A strict polynomial functor is a k-linear covariant functor from $\Gamma^d P_k$ to k-Mod. The category of strict polynomial functors is the functor category $\mathrm{Fct}(\Gamma^d P_k, k\text{-Mod})$ and we denote this category by $\mathrm{Rep}\Gamma_k^d$. By definition of strict polynomial functor X, $ev_{k^n}(X) := X(k^n)$ is a left $S_k(n,d)$-module. In fact,

Theorem 1.1. *[FS97, Theorem 3.2] If $n \geqslant d$, $ev_{k^n} : \mathrm{Rep}\Gamma_k^d \to S_k(n,d)\text{-Mod}$ is an equivalence.*

As $\Gamma^d P_k$ is closed under tensor products over k and k-linear homomorphisms, Krause discovered that via Day convolution/Kan extension, these operations extend to the functor category $\mathrm{Rep}\Gamma_k^d$ to give an internal tensor product $(\underline{\otimes})$ and an internal Hom $(\mathbb{H})$. This internal tensor product was used to give Koszul and Serre duality of strict polynomial functors. A different expression of same internal Hom was studied by Chałupnik [Cha09] and Touzé [Tou13] before Krause [Kra13] to give Ringel duality of strict polynomial functors.

We mention some of the known applications of strict polynomial functors in the literature. In [FS97], strict polynomial functors were used to show the finite generation of the full cohomology ring of a finite group scheme over a field of positive characteristic. Therefore one can apply geometric methods to study representations of a finite group scheme. Also, Touzé has given a functorial viewpoint of cohomology of classical algebraic group schemes using strict polynomial functors in [Tou10]. In [TvdK10], Kallen and Touzé then proved that full cohomology ring of a rational representation of reductive algebraic group scheme over a field of positive characteristic is finitely generated. This is a generalization of Hilbert 14^{th} problem. In [Tou13], Touzé proved that the Ringel dual functor of certain strict polynomial functors yields the computations of homology of Eilenberg-Mac Lane spaces.

1.2 Aim of the thesis

The thesis is divided into three parts and we describe each part in the following sections. Section 1.2.1 is a joint work with U. Kulkarni and K. V. Subrahmanyam [KSS16].

1.2.1 Relating tensor structures on modules over Schur algebra and symmetric group

Let kS_d be the group algebra of S_d. The Kronecker product of two kS_d-modules, i.e., tensor product $(\otimes)$ over k, is a kS_d-module by using the diagonal embedding $S_d \hookrightarrow S_d \times S_d$. Similarly, k-linear homomorphism (Hom) of two kS_d-modules is a kS_d-module and we call it the Kronecker hom. However, the tensor product over k of two polynomial representations of degree d of $GL(k^n)$ (equivalently $S_k(n,d)$-modules) via the diagonal action becomes a polynomial representation of degree $2d$ of $GL(k^n)$ (equivalently a $S_k(n,2d)$-module). But recall that Sch is an equivalence of categories when k is a field of characteristic 0. So there exists an internal tensor product of $S_k(n,d)$-modules induced from the Kronecker product via Sch in characteristic 0.

A natural question is whether there exists an internal tensor product of $S_k(n,d)$-modules which will correspond to the Kronecker product of kS_d-modules via Sch over any commutative ring with unity k? Using the equivalence between $S_k(n,d)$-Mod and $\mathrm{Rep}\Gamma_k^d$, this question can be formulated as follows. Does there exist an internal tensor product on $\mathrm{Rep}\Gamma_k^d$ which will correspond to the Kronecker product of kS_d-modules via Sch over any k? We prove that internal tensor product ($\underline{\otimes}$) on $\mathrm{Rep}\Gamma_k^d$ given by Krause is an answer to this question.

Theorem 1.2 (Theorem 5.3)**.** *For* X *and* Y *in* $\mathrm{Rep}\Gamma_k^d$*. There are the natural isomorphisms*

$$\mathrm{Sch}(X\underline{\otimes}Y) \simeq \mathrm{Sch}(X)\otimes \mathrm{Sch}(Y)^{1} \quad \text{and} \quad \mathrm{Sch}(\mathbb{H}(X,Y)) \simeq \mathrm{Hom}(\mathrm{Sch}(X),\mathrm{Sch}(Y)).$$

The Kronecker product and the Kronecker hom have the left derived functor ($\overset{\mathbf{L}}{\otimes}$) and the right derived functor ($\mathbf{R}\mathrm{Hom}$) respectively on the unbounded derived category $D(kS_d\text{-Mod})$. Similarly, the internal tensor product and internal hom have the left derived functor ($\overset{\mathbf{L}}{\underline{\otimes}}$) and the right derived functor ($\mathbf{R}\mathbb{H}$) on the unbounded derived category $D(\mathrm{Rep}\Gamma_k^d)$. We prove that Theorem 1.2 remains true when we pass to the unbounded derived categories.

Theorem 1.3 (Theorem 5.9)**.** *For* $X^\bullet$ *and* $Y^\bullet$ *in* $D(\mathrm{Rep}\Gamma_k^d)$*. There are the natural isomorphisms*

$$\mathrm{Sch}(X^\bullet\overset{\mathbf{L}}{\underline{\otimes}}Y^\bullet) \simeq \mathrm{Sch}(X^\bullet)\overset{\mathbf{L}}{\otimes}\mathrm{Sch}(Y^\bullet) \quad \text{and} \quad \mathrm{Sch}(\mathbf{R}\mathbb{H}(X,Y)) \simeq \mathbf{R}\mathrm{Hom}(\mathrm{Sch}(X^\bullet),\mathrm{Sch}(Y^\bullet)).$$

Let k be a field. Then the Kronecker product is exact. But if $\mathrm{char}\, k = p > 0$ then $\underline{\otimes}$ is not exact in general. Every finitely generated strict polynomial functors over k has a composition series whose composition factors are simple strict polynomial functors. The simple strict polynomial functors of degree d are indexed by the partitions of d. The functor Sch being an exact functor preserves (co)homologies. A theorem of Clausen, James ([Gre80, Theorem 6.4b]) says that Sch vanishes on a simple polynomial functor correponding to a partition λ if and only if λ is a non p-restricted partition. So as an application of Theorem 1.3 along with this theorem of Clausen, James we get that the higher derived internal tensor products and higher derived internal Hom of finitely generated strict polynomial functors over k have only non p-restricted composition factors.

The functor Sch also has both the left adjoint $\mathcal{L}$ and the right adjoint $\mathcal{R}$ [DEN04]. The left derived (resp. right derived) functor $\mathbf{L}\mathcal{L}$ (resp. $\mathbf{R}\mathcal{R}$) exists on the unbounded derived category. It is well known that the derived functors of the adjoints of Sch contain valuable information

[1]This is also proved by Acquilino and Reischuk in [AR15] but our proof is different from theirs.

in relating modular representation theories of $GL(k^n)$ and S_d [DEN04]. In [Rei16], Reischuk showed that $\mathcal{L}$ takes the Kronecker product to the internal tensor product and $\mathcal{R}$ takes the Kronecker hom to the internal Hom up to duality in the first slot. We give a simple proof of these and also observe that condition up to duality is not necessary. We also show the compatibilities of internal structures under adjoints of Sch are true even at the level of unbounded derived categories.

Corollary 1.4 (Corollaries 5.5 and 5.11). *For* M *and* N *in* kS_d-Mod *and* X *in* $\mathrm{Rep}\Gamma^d_k$ *such that* $\mathrm{Sch}(X) \simeq M$. *We have following isomorphisms*

$$\mathcal{L}(M \otimes N) \simeq X \underline{\otimes} \mathcal{L}(N) \quad \text{and} \quad \mathcal{R}(\mathrm{Hom}(M, N)) \simeq \mathbb{H}(X, \mathcal{R}(N)).$$

For $M^\bullet$ *and* $N^\bullet$ *in* $D(kS_d\text{-Mod})$ *and* $X^\bullet$ *in* $D(\mathrm{Rep}\Gamma^d_k)$ *such that* $\mathrm{Sch}(X^\bullet) \simeq M^\bullet$. *We have the following isomorphisms*

$$\mathbf{L}\mathcal{L}(M^\bullet \overset{\mathbf{L}}{\otimes} N^\bullet) \simeq X^\bullet \overset{\mathbf{L}}{\underline{\otimes}} \mathbf{L}\mathcal{L}(N^\bullet) \quad \text{and} \quad \mathbf{R}\mathcal{R}(\mathbf{R}\mathrm{Hom}(M^\bullet, N^\bullet)) \simeq \mathbf{R}\mathbb{H}(X^\bullet, \mathbf{R}\mathcal{R}(N^\bullet)).$$

In light of the connection with the representation theory of S_d, it is interesting to compute/describe $\underline{\otimes}$ in various cases. Some of important strict polynomial functors are the classical exponential functors from [ABW82], d^{th} divided power functor (Γ^d), d^{th} symmetric power functor (Sym^d) and d^{th} exterior power functor $(\wedge^d)$. We have computed $\underline{\otimes}$ involving these exponential functors.

Proposition 1.5 (Proposition 4.16). $\mathrm{Sym}^d \underline{\otimes} \mathrm{Sym}^d \simeq \mathrm{Sym}^d$, $\mathrm{Sym}^d \underline{\otimes} \wedge^d \simeq \wedge^d$ *if* 2 *is unit in* k, $\mathrm{Sym}^d \underline{\otimes} \wedge^d \simeq \mathrm{Sym}^d$ *if* $2 = 0$ *in* k.

Other important strict polynomial functors are the Weyl functors $(\Delta(\lambda))$ and dual Weyl functors $(\nabla(\lambda))$ associated to partitions λ of d [ABW82]. (Weyl and dual Weyl functors respectively correspond to dual Specht and Specht module of S_d via Sch.) The strict polynomial functor Γ^λ is used in order to define $\Delta(\lambda)$ and corresponds via Sch to the permutation module of S_d associated to a partition λ of d. The collection $\{\Gamma^\lambda\}$, λ a partition of d, also forms a class of projective generator in $\mathrm{Rep}\Gamma^d_k$ from [Kra13]. In order to describe internal tensor product involving Γ^λ, $\Delta(\lambda)$ and $\nabla(\lambda)$ we first recall from [Tou14] an important construction of strict polynomial functors. The upper parametrization of X in $\mathrm{Rep}\Gamma^d_k$ by an object V in $\Gamma^d P_k$ is defined by $X^V(W) := X(V^* \otimes W)$. Note that $X^V(W)$ is $GL(V^*) \times GL(W)$-module. It is very well known that weights of a polynomial representation of degree d of $GL(k^n)$ are indexed by the set $(\Lambda(n, d))$ consisting of sequences of length n of non-negative integers adding to d. If $V = k^n$ then λ-weight space of $X^V(W)$ with respect to GL_n-action is still functorial in the variable W. So we define $X^\lambda(W) = \lambda$-weight space of $X^{k^n}(W)$ and note that X^λ is a strict polynomial functor of degree d [KSS16]. Relevance of this definition is the following lemma.

Lemma 1.6 (Lemma 4.13). *For an object X in $\mathrm{Rep}\Gamma_k^d$, $X\underline{\otimes}\Gamma^\lambda \simeq X^\lambda$.*

By taking X equal to Γ^μ or $\Delta(\mu)$ in Lemma 1.6 we solve some questions posed by Krause. To describe the results we first need some notations. Let S denote the set of $n \times m$ matrices with non-negative integer entries, row sums μ and column sums λ. Every such matrix S naturally gives us a $\nu \in \Lambda(mn, d)$.

Theorem 1.7 (Propositions 4.14 and 4.25). *For λ and μ partitions of d, we have*

1. $\Gamma^\lambda\underline{\otimes}\Gamma^\mu \simeq \bigoplus_{\nu\in S}\Gamma^\nu$ [2].
2. $\Delta(\lambda)\underline{\otimes}\Gamma^\mu$ *has a Weyl filtration independent of ground ring and multiplicities of composition factors are given by products of sums of Littlewood-Richardson coefficients.*

In [Kra13], Krause proved that $\wedge^d\underline{\otimes}\Delta(\lambda)$ is isomorphic to $\nabla(\lambda')$ where λ' is the conjugate of λ. Moreover, he showed that $H^i(\wedge^d\overset{\mathbf{L}}{\underline{\otimes}}\Delta(\lambda)) = 0$ for $i \neq 0$. Using this vanishing result along with part 2 of Theorem 1.7 we give a description of $\nabla(\lambda)\underline{\otimes}\Gamma^\mu$. We also generalize this vanishing result to arbitrary Weyl functors..

Proposition 1.8 (Corollary 4.26 and Proposition 4.28). *For λ and μ partitions of d, we have*

1. $\nabla(\lambda)\underline{\otimes}\Gamma^\mu$ *has a dual Weyl filtration independent of ground ring and multiplicities of composition factors are given by products of sums of Littlewood-Richardson coefficients.*
2. *For $i \neq 0$, $H^i(\Delta(\lambda)\overset{\mathbf{L}}{\underline{\otimes}}\Delta(\mu)) = 0$.*

We end this section by studying the higher derived internal tensor product $\mathbb{Z}$.

Corollary 1.9 (Proposition 4.33). *For X and Y in $\mathrm{Rep}\Gamma_{\mathbb{Z}}^d$ such that they take values in the category of finitely generated projective modules, $H^i(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^m)$ is a finite abelian group where $i > 0$ and $m \in \mathbb{N}$.*

1.2.2 Sesquialgebra structures of Schur algebras for Weyl groups of types A and B

Recall the equivalence (1.1) between $\mathrm{Rep}\Gamma_k^d$ and $S_k(n,d)$-Mod. It is natural to ask what does extra structure internal tensor product $(\underline{\otimes})$ on $\mathrm{Rep}\Gamma_k^d$ put on $S_k(n,d)$? It follows from [Hov11] that answer is the notion of sesquialgebra structure on an unital associative algebra defined by Tang, Weinstein and Zhu in [TWZ07]. In order to put a sesquialgebra structure on an

[2] This is also proved in [AR15] by a completely different method.

unital associative algebra A, we require an $(A, A \otimes A)$-bimodule $\triangle$ and an (A, k)-bimodule ϵ which satisfies certain compatibility conditions [TWZ07]. In the case of Schur algebra $S_k(n, d)$, we observe that $\triangle = \mathrm{Hom}_{S_d}((k^n \otimes k^n)^{\otimes d}, (k^n)^{\otimes d})$ and $\epsilon = \mathrm{Hom}_{S_d}(k, (k^n)^{\otimes d})$. Also, a sesquialgebra structure on A defines an internal tensor product on A-Mod as follows, $M \underline{\otimes} N := \triangle \otimes_{A \otimes A} (M \otimes N)$ for objects M and N in A-Mod.

It is well known that double centralizer property between kS_d and $\mathrm{End}_{S_d}((k^n)^{\otimes d})$ could be proved by using an idempotent in $\mathrm{End}_{S_d}((k^n)^{\otimes d})$. We give a simple proof of this fact without doing any explicit calculations in the centralizer algebra itself unlike for example in [Gre80, Equations (6.1d) and (6.4f)] and [Mar09, Theorem 4.1]. Notably our proof makes it very transparent that for a subgroup G of S_d acting on $(k^n)^{\otimes d}$ by restriction, the double centralizer property remains true between the group algebra kG and the centralizer algebra $\mathrm{End}_G((k^n)^{\otimes d})$. If $n \geqslant d$, we show that $\mathrm{End}_G((k^n)^{\otimes d})$-Mod is equivalent to the functor category $\mathrm{Fct}(\mathcal{C}, k\text{-Mod})$ where $\mathcal{C}$ is a k-linear category and has an internal tensor product. In particular, we show that like Schur algebra this new centralizer algebra $\mathrm{End}_G((k^n)^{\otimes d})$ has a sesquialgebra structure with a preantipode. There is also a functor Sch_G from $\mathrm{Fct}(\mathcal{C}, k\text{-Mod})$ to kG-Mod which is analogous to Sch. Similar to Theorem 1.2, we show that Sch_G takes internal tensor product on $\mathrm{Fct}(\mathcal{C}, k\text{-Mod})$ to the Kronecker product of kG-modules.

Theorem 1.10 (Example 3.3 and Corollary 6.10). *For $n \geqslant d$ and $A = \mathrm{End}_G((k^n)^{\otimes d})$,*

1. $\mathrm{End}_A((k^n)^{\otimes d}) \simeq (kG)^{op}$.
2. *A is a sesquialgebra with a preantipode.*

A Schur algebra of the Weyl group of type **B**, $H(2n, d)$, was introduced by R. M. Green in [Gre97] and it is also called the hyperoctahedral Schur algebra. More precisely, $H(2n, d)$ is the centralizer algebra $\mathrm{End}_{B_d}((k^{2n})^{\otimes d})$ where $B_d = (\mathbb{Z}/2\mathbb{Z})^d \rtimes S_d$. Like in the case of Schur algebra of the Weyl group of type **A**, we give a simple proof of double centralizer property between kB_d and $H(2n, d)$ (Example 3.4). Moreover, as in type **A** case, we show if 2 is a unit in k and $n \geqslant d$, we show that $H(2n, d)$-Mod is equivalent to the functor category $\mathrm{Fct}(\mathcal{C}, k\text{-Mod})$ where $\mathcal{C}$ is a k-linear category and has an internal tensor product.

Theorem 1.11 (Corollary 6.12). *If 2 is unit in k and $n \geqslant d$, $H(2n, d)$ is a sesquialgebra with a preantipode.*

1.2.3 Coherent functors and strict polynomial functors

Observe that $\Gamma^d P_k$ is a full subcategory of the category of left kS_d-modules which are finitely generated over k, kS_d-mod. So we have an explicit functor $j^* : \mathrm{Fct}(kS_d\text{-mod}, k\text{-Mod}) \to$

$\mathrm{Rep}\Gamma_k^d$. Following Auslander [Aus66a], the coherent functors on kS_d-mod are the objects in $\mathrm{Fct}(kS_d\text{-mod}, k\text{-Mod})$ presented by the representable functors. We denote the category of coherent functors by $\mathcal{C}(S_d)$. The functor j^* restricted to $\mathcal{C}(S_d)$ takes values in the full subcategory $\mathrm{rep}\Gamma_k^d$ of $\mathrm{Rep}\Gamma_k^d$ whose objects take values in the category of finitely generated projective modules. A connection between coherent functors and strict polynomial functors over a field k was given by Franjou and Pirashivli in [FP08], namely the restriction of j^* to $\mathcal{C}(S_d)$ is a quotient functor.

The external tensor product of representations of S_{d_1} and S_{d_2} induces an external tensor product $(\odot)$ on $\mathcal{C}(S_{d_1+d_2})$ [FP08]. Also, if X_1 and X_2 are strict polynomial functors of degree d_1 and degree d_2 respectively then $(X_1 \boxtimes X_2)(V) := X_1(V) \otimes X_2(V)$, for object V in $\Gamma^d P_k$, defines a strict polynomial functor of degree $d_1 + d_2$. So the bifunctor $\boxtimes$ is an external tensor product on $\mathrm{Rep}\Gamma_k^{d_1+d_2}$ [FP08]. In [FP08], it was shown that $j^*(F \odot G) \simeq j^*(F) \boxtimes j^*(G)$ where F and G are objects of $\mathcal{C}(S_{d_1})$ and $\mathcal{C}(S_{d_2})$ respectively. We observe that the Kronecker product of kS_d-modules induces an internal tensor product $(\underline{\odot})$ on $\mathrm{Fct}(kS_d\text{-mod}, k\text{-Mod})$ via Day convolution. We compare $(\underline{\odot})$ with $(\underline{\otimes})$ via j^*.

Proposition 1.12 (Propositions 7.3 and 7.5)**.** *The category $\mathcal{C}(S_d)$ is closed under $(\underline{\odot})$. Moreover, $j^* : \mathcal{C}(S_d) \to \mathrm{rep}\Gamma_k^d$ is a lax monoidal functor.*

1.3 Strategy of proofs

From [Kra13, Proposition 2.4], recall that the representable functors $\Gamma^{d,V}$ are the building blocks in order to define $\underline{\otimes}$ and $\mathbb{H}$. The key points in proving that the compatibility of internal structures under Sch (Theorem 1.2) are our calculation (Proposition 5.1) $\mathrm{Sch}(\Gamma^{d,V}) = (V^{\otimes d})^*$ and the fact that Sch preserves both arbitrary colimits and limits.

For providing isomorphism between the left derived (resp. right derived) functors on unbounded derived categories we use homotopy colimits (resp. homotopy limits) introduced by Bökstedt and Neeman [BN93]. We first prove the isomorphism in Theorem 1.3 for bounded above (resp. bounded below) complexes and then use that every complex can be written as homotopy colimit (resp. homotopy limit) of its bounded above (resp. bounded below) truncations of unbounded complexes.

As a consequence of Theorem 1.2 (resp. Theorem 1.3), we prove, Corollary 1.4, the compatibility of internal structures under $\mathcal{L}$ and $\mathcal{R}$ (resp. under $\mathbf{L}\mathcal{L}$ and $\mathbf{R}\mathcal{R}$) by using Yoneda lemma.

For the computations (Proposition 1.5) of internal tensor products of exponential strict polynomial functors, we use their presentations from [ABW82] and the right exactness of

$\underline{\otimes}$. To calculate $\underline{\otimes}$ of the morphisms involved in these presentations we use effectively the Yoneda lemma.

Lemma 1.6 gives insight in proving both the parts of Theorem 1.7. Then, part 1 is the (λ, μ)-weight space of certain $GL(k^m) \times GL(k^n)$-module and part 2 is obtained by using an explicit Weyl filtration of a specific skew Weyl functor from [AB85].

For proving vanishing of higher derived internal tensor between two Weyl functors (part 2 of Proposition 1.8) we use [Kra13, Lemma 4.5] and vanishing of extension groups between Weyl and dual Weyl functors.

We first prove a form of universal coefficient theorem (Proposition 4.31) for the higher derived internal tensors for the base change $\mathbb{Z} \to k$. Then by taking $k = \mathbb{Q}$, we get the desired Proposition 1.9.

In both the cases, G acting on $(k^n)^{\otimes d}$ (part 1 of Proposition 1.10) and B_d acting on $(k^{2n})^{\otimes d}$, actions of groups on modules are arising from their actions on some specific sets. Our Theorem 3.1 says that there always exists an idempotent in the centralizer algebra to set-up the double centralizer property if the cardinality of an orbit is equal to the order of the group.

From [TWZ07], if A is a sesquialgebra then there exists an internal tensor product of A-modules. From [Hov11] converse is also true, that is if there exists an internal tensor product of A-modules which preserve colimit in each slot then A is a sesquialgebra. The sesquialgebra structures of the new centralizer algebra $\mathrm{End}_G((k^n)^{\otimes d})$ (part 2 of Theorem 1.10) and $H(2n, d)$ (Theorem 1.11) are exhibited by showing that their category of modules are equivalent to the functor categories of certain monoidal categories (Theorems 6.1 and 6.4).

To prove that $\mathcal{C}(S_d)$ is closed under $(\odot)$ we use the fact that it is an abelian category [Har98]. Our key observation is that the left adjoint of j^* is a monoidal functor. Then j^* is a lax monoidal functor follows from the fact the right adjoint of a monoidal functor is lax monoidal (Proposition 1.12).

1.4 Organization of the thesis

In Chapter 2, we give preliminaries on the representations of S_d and the polynomial representations of the general group scheme $GL(k^n)$.

In Chapter 3, we give a proof of the double centralizer properties in the cases of Schur algebras of Weyl groups for types **A** and **B**.

In Chapter 4, we define strict polynomial functors and their internal tensor products. We give some of our computations of $(\underline{\otimes})$. We also study higher derived internal tensor products.

In Chapter 5, we show that Sch sends internal tensor product and internal Hom to the Kronecker product and the Kronecker hom functorially. We also show the left adjoint (resp. right adjoint) of Sch takes the Kronecker product (resp. Kronecker hom) to the internal tensor (resp. internal Hom). We show that the results about preservation of these internal structures hold at the level of unbounded derived category. We give an application to some Kronecker multiplicities in characteristic 0.

In Chapter 6, we give sesquialgebra structures of $S_k(n, d)$ and $H(2n, d)$. In particular, we express internal tensor products of $S_k(n, d)$-modules and $H(2n, d)$-modules directly without passing to functor categories.

In Chapter 7, we compare the internal tensor products of coherent functors on kS_d-mod with the internal tensor product of the strict polynomial functors.

In Chapter 8, we give some basics of functor categories, monoidal categories and monoidal functors. We also collect notions of homotopy colimits and limits in an unbounded derived category.

Chapter 2

Preliminaries

The first and second sections are about the necessary terminologies in the representation theory of symmetric group and the polynomial representations of the general linear group scheme which will be needed in the later part of this thesis.

2.1 Short introduction to the representation theory of symmetric group

For positive integers n and d, define the sets

$$\begin{aligned} \Lambda(n,d) &= \{\lambda = (\lambda_1,\dots,\lambda_n\} \,|\, \lambda_i \in \mathbb{N}\cup\{0\} \text{ and } \sum_{i=1}^{i=n}\lambda_i = d\}, \\ \Lambda^+(n,d) &= \{\lambda \in \Lambda(n,d) \,|\, \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n > 0\} \text{ and} \\ [m] &= \{1,2\dots,m\}. \end{aligned}$$

Given $\lambda \in \Lambda^+(n,d)$, a filling of a Young diagram of the shape λ by numbers from $[d]$ without any repetition is called a λ-tableau or Young tableau of shape λ. The symmetric group S_d acts on the set of λ-tableaux. The row stabilizer (resp. column stabilizer) of a λ-tableau, t, is the subgroup of S_d keeping the rows (resp. the columns) of t fixed set-wise. We denote the row stabilizer (resp. the column stabilizer) of a λ-tableau, t, by R_t (resp. C_t). Our main source for this section is [Jam78].

2.1.1 Permutation modules and Specht Modules

Definition 2.1. Given two λ-tableaux t_1 and t_2 we write $t_1 \sim t_2$ if there exist $\pi \in R_{t_1}$ such that $\pi t_1 = t_2$. This defines an equivalence relation on the set of λ-tableaux. An equivalence class containing t is called a λ-tabloid and we denote it by $\{t\}$. The free k-module spanned by all the λ-tabloids is called permutation module and denoted by M^λ.

If $t_1 \sim t_2$ then $\sigma.t_1 \sim \sigma.t_2$ for every $\sigma \in S_d$. Thus M^λ is a representation of S_d.

Remark 2.2. Let J_λ be the set $\{f : [d] \to [n] \mid \lambda_i = |f^{-1}(i)| \ \forall\ 1 \leqslant i \leqslant n\}$. Note that the set of λ-tabloids is in bijection with the set J_λ.

The following proposition is well-known in the literature for example see [Jam78, Page 22].

Proposition 2.3. *The permutation module M^λ is isomorphic to $\mathrm{Ind}_{S_\lambda}^{S_d}(k)$ where k is the trivial representation of S_d and S_λ is the Young subgroup of S_d.*

Definition 2.4. For a λ-tableau t, a λ-polytabloid is $e_t := \kappa_t\{t\}$ where $\kappa_t = \sum_{\pi \in C_t} (\mathrm{sgn}\ \pi)\pi$.

Definition 2.5. The submodule of M^λ spanned by all λ-polytabloids is called Specht module and denoted by S^λ.

The action of S_d on the set polytabloids of shape λ makes Specht module S^λ a representation of S_d.

If char $k = 0$, the representations of S_d are completely reducible or the group algebra kS_d is a semisimple algebra. In this case, the irreducible representations of S_d are indexed by all partitions of d.
On the other hand, if char $k = p > 0$ and $p \leqslant d$ then kS_d is not semisimple. Given $\lambda \in \Lambda^+(n, d)$ is said to be p-restricted if $(\lambda_i - \lambda_{i+1}) < p$ for every $1 \leqslant i \leqslant (n-1)$ and $\lambda_n < p$. In this case, it is a theorem of Clausen, James that the irreducible representations of S_d are indexed by p-restricted partitions of d, for example see [Mar09, Theorem 4.2.3].

2.2 Representations of a group scheme

2.2.1 Group Scheme

Definition 2.6. A functor G from Alg_k to Grp is called a k-group functor where Grp denotes the category of groups and Alg_k denotes the category of unital associative, commutative algebras over k.

The additive group functor G_a sends A to the group $(A, +)$ where A is an object of Alg_k. The space $\mathrm{Hom}(G, G_a)$ is the set of all natural transformations from G to G_a and is denoted by $k[G]$. Note that $k[G]$ is an object of Alg_k.

Definition 2.7. A group functor G is called an affine group scheme if $k[G]$ is finitely generated algebra and $G = \mathrm{Hom}_{\mathrm{Alg}_k}(k[G], -)$.

A group functor G being an affine group scheme is equivalent to $k[G]$ being a commutative Hopf algebra, see [Jan03, Section 2.3].

Example 2.8. The following are examples of affine groups schemes where $k[T_{ij}]$ denotes the polynomial ring in n^2 variables and A is an object of Alg_k.

1. The additive group scheme $G_a = \mathrm{Hom}_{k\text{-alg}}(k[T], -)$. The comultiplication map for the Hopf algebra $k[T]$ is given by

$$\triangle\colon k[T] \to k[T] \otimes k[T] \quad \text{where} \quad \triangle\,(T) = T \otimes 1 + 1 \otimes T.$$

2. The multiplicative group scheme G_m sends A to $(A^\times, .)$ where $A^\times$ denotes the group of units in A. Then $G_m = \mathrm{Hom}_{k\text{-alg}}(k[T, T^{-1}], -)$. The comultiplication map for the Hopf algebra $k[T, T^{-1}]$ is given by

$$\triangle\colon k[T, T^{-1}] \to k[T, T^{-1}] \otimes k[T, T^{-1}] \quad \text{where} \quad \triangle\,(T) = T \otimes T.$$

3. The general linear group scheme GL_n sends A to the group of $n \times n$ invertible matrices with entries in A. Then $GL_n = \mathrm{Hom}_{k\text{-alg}}(k[T_{ij}][\frac{1}{\det(T_{ij})}], -)$. The comultiplication map for the Hopf algebra $k[T_{ij}][\frac{1}{\det(T_{ij})}]$ is given by

$$\triangle\colon k[T_{ij}][\tfrac{1}{\det(T_{ij})}] \to k[T_{ij}][\tfrac{1}{\det(T_{ij})}] \otimes k[T_{ij}][\tfrac{1}{\det(T_{ij})}] \quad \text{where} \quad \triangle\,(T_{ij}) = \sum_{k=1}^{n} T_{ik} \otimes T_{kj}.$$

It will be useful to generalize the definition of GL_n. Let k-Mod be the category of left k-modules and M be an object of k-Mod. Define a k-functor from $\mathrm{Alg}_k \to k$-Mod, by $M_a(A) = M \otimes A$. One defines general linear group scheme $GL(M)(A) := \mathrm{Aut}_A(M \otimes A)$ where $\mathrm{Aut}_A(M \otimes A)$ is the group of left A-algebra automorphisms of $(M \otimes A)$. When $M = k^n$ we get $GL(M) = GL_n$.

2.2.2 Representations of a group scheme

Definition 2.9. A representation of a group scheme G is a k-module M together with a homomorphism of k-group functors $G \to GL(M)$.

Above definition of a representation of G is equivalent to G acting on M_a. This means the group $G(A)$ acts on $M_a(A) = M \otimes A$ through A-linear maps and if there is a k-algebra homomorphism $A \xrightarrow{f} B$, the following diagram commutes:

$$\begin{array}{ccc} G(A) \times (M \otimes A) & \longrightarrow & (M \otimes A) \\ {\scriptstyle G(f) \times M_a(f)}\downarrow & & {\scriptstyle M_a(f)}\downarrow \\ G(B) \times (M \otimes B) & \longrightarrow & (M \otimes B) \end{array}$$

We note that M being a representation of an affine group scheme G is equivalent to M being a comodule over the Hopf algebra $k[G]$, see [Jan03, Sections 2.7 and 2.8].

2.2.3 Weight spaces of representations of a group scheme

Definition 2.10. Let G be a group functor. A morphism of group functors $G \to G_m$ is called a character of G. The set of characters is denoted as $X(G)$.

Definition 2.11. Let M be a representation of G and $\chi \in X(G)$. The k-submodule,

$$M_\chi = \{m \in M \mid g.(m \otimes 1) = \chi_A(g)(m \otimes 1),\ \forall\ g \in G(A) \text{ and } \forall\ k\text{-algebras } A\}$$

of M is called the χ-weight space of M.

2.2.4 Polynomial representations of GL_n

Definition 2.12. The polynomial algebra $k[T_{ij}]$ is a subalgebra of $k[T_{ij}][\frac{1}{\det(T_{ij})}]$. The algebra $k[T_{ij}]$ is a graded algebra. We denote d^{th} homogeneous piece of $k[T_{ij}]$ by $k[T_{ij}]_d$ and it is stable under $\triangle$ as given in the part 3 of Example 2.8.

Let M be representation of GL_n. We have comodule map

$$M \xrightarrow{\triangle_M} M \otimes k[T_{ij}][\frac{1}{\det(T_{ij})}].$$

We say that M is a polynomial representation of GL_n, if the map $\triangle_M$ factors through the inclusion $M \otimes k[T_{ij}] \to M \otimes k[T_{ij}][\frac{1}{\det(T_{ij})}]$. Moreover, if $\triangle_M$ factors through the inclusion $M \otimes k[T_{ij}]_d \to M \otimes k[T_{ij}][\frac{1}{\det(T_{ij})}]$, we say M is a polynomial representation of degree d. We denote the category of polynomial representation of degree d of GL_n by $\mathrm{Pol}_n(d)$.

We first describe some important constructions from the multilinear algebra, before giving some examples of polynomial representations of the general linear group scheme GL_n.

Given a k-module V, the symmetric group S_d acts on $V^{\otimes d}$ by permuting its factors. Let $T(V) := \bigoplus_{d \geqslant 0} V^{\otimes d}$ be the tensor algebra of V. The algebra $T(V)$ is a graded algebra.

Definition 2.13. The space of symmetric tensors $(V^{\otimes d})^{S_d}$ is called the d^{th} divided power and it is denoted by $\Gamma^d(V)$. The direct sum $\Gamma(V) = \bigoplus_{i=0} \Gamma^i(V)$ is a graded algebra and called as the divided power algebra. For $a \in \Gamma^d(V)$ and $b \in \Gamma^e(V)$, the multiplication of a and b is $\sum_{\sigma \in S_{d+e}/(S_d \times S_e)} \sigma.(a \otimes b)$ where $S_{d+e}/S_d \times S_e$ denotes the set of left cosets of $S_d \times S_e$ in S_{d+e}.

The original definition of the divided powers is different from Definition 2.13. However, since we restrict ourselves to the case where V is an object of P_k, the category of finitely generated projective modules over k, the two definitions are the same, see [Bou81, IV.5, Exercise 8].

Definition 2.14. The Schur algebra $S_k(n,d)$ is the algebra $\mathrm{End}_{S_d}((k^n)^{\otimes d})$, see [Gre80, Theorem 2.6c]. We also have that $S_k(n,d)$ is isomorphic to $\Gamma_k^d \mathrm{End}(k^n)$. This follows from the isomorphism of k-modules

$$\mathrm{Hom}_{S_d}(V^{\otimes d}, W^{\otimes d}) \simeq ((\mathrm{Hom}_k(V,W))^{\otimes d})^{S_d} \simeq \Gamma^d(\mathrm{Hom}(V,W)). \tag{2.1}$$

by taking $V = W = k^n$, and in this case 2.1 is algebra isomorphism.

Definition 2.15. Let I be the 2-sided ideal in $T(V)$ generated by $(v \otimes w - w \otimes v)$ where $v, w \in V$. The symmetric algebra is the quotient $T(V)/I$ and is denoted by $\mathrm{Sym}(V)$. Since the ideal I is homogeneous, $\mathrm{Sym}(V)$ is a graded algebra. The d^{th} homogeneous piece of $\mathrm{Sym}(V)$ is denoted by $\mathrm{Sym}^d(V)$.

Note that the modules of co-invariants of S_d acting on $V^{\otimes d}$, $(V^{\otimes d})_{S_d}$, equals to $\mathrm{Sym}^d(V)$.

For a representation N of a finite group G, one has the following relation between the invariants N^G and the co-invariants N_G

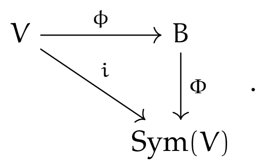
$$(N^*)^G \simeq (N_G)^*. \tag{2.2}$$

Here M^* is the k-linear dual $\mathrm{Hom}_k(M,k)$ of a k-module M.
Applying the isomorphism (2.2) for $G = S_d$ and $N = V^{\otimes d}$, we get $(\mathrm{Sym}^d(V))^* \simeq \Gamma^d(V^*)$.
When V is a finitely generated projective module over k, we have

$$(\mathrm{Sym}^d(V^*))^* \simeq \Gamma^d(V). \tag{2.3}$$

Remark 2.16. The polynomial algebra $k[T_{ij}]$ in n^2 variables can be identified with $\mathrm{Sym}((\mathrm{End}_k(k^n))^*)$. Thus a polynomial representation M of degree d of GL_n is a comodule over the co-algebra $\mathrm{Sym}^d((\mathrm{End}_k(k^n))^*)$. This is equivalent to say that M is a module over the algebra $(\mathrm{Sym}^d((\mathrm{End}_k(k^n))^*))^*$. Since $\mathrm{End}_k(k^n)$ is free over k, by isomorphism (2.3) we have M is a module over the algebra $\Gamma^d(\mathrm{End}_k(k^n))$ which is $S_k(n,d)$.

Universal property of symmetric algebra: Given a morphism $\phi : V \to B$ of k-modules where B is a commutative algebra over k. There exists unique morphism $\Phi : \mathrm{Sym}(V) \to B$ of k-algebras such that following diagram commutes



where i the inclusion morphism. Moreover, if $B = \bigoplus_{i=0} B_i$ is a graded algebra and image of ϕ is inside B_1, Φ is a morphism of graded algebras.

Lemma 2.17. *[Exponential property of the symmetric algebra] For V and W in k-Mod, we have isomorphism of bicommutative graded Hopf algebras*

$$\mathrm{Sym}(V \oplus W) \simeq \mathrm{Sym}(V) \otimes \mathrm{Sym}(W). \tag{2.4}$$

Proof. For A a commutative graded algebra over k, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{gr}_k\text{-alg}}(\mathrm{Sym}(V) \otimes \mathrm{Sym}(W), A) &\simeq \mathrm{Hom}_{\mathrm{gr}_k\text{-alg}}(\mathrm{Sym}(V), A) \oplus \mathrm{Hom}_{\mathrm{gr}_k\text{-alg}}(\mathrm{Sym}(W), A)\\ &\simeq \mathrm{Hom}_{k\text{-Mod}}(V, A) \oplus \mathrm{Hom}_{k\text{-Mod}}(W, A)\\ &\simeq \mathrm{Hom}_{k\text{-Mod}}(V \oplus W, A) \simeq \mathrm{Hom}_{\mathrm{gr}_k\text{-alg}}(\mathrm{Sym}(V \oplus W), A).\end{aligned}$$

Where the first isomorphism is because tensor product of algebras is the coproduct in the category of commutative graded k-algebras gr_k-alg, the second and the last isomorphism follow from the universal property of the symmetric algebra. See also [Kas95, Proposition II.5.2.d]. □

Some consequences of exponential property of the symmetric algebra: Let V and W be finitely generated projective over k.

1. We have an isomorphism of commutative graded algebras

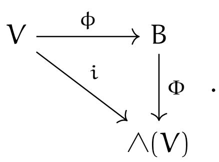
$$\Gamma(V \oplus W) \simeq \Gamma(V) \otimes \Gamma(W). \tag{2.5}$$

   This follows from the isomorphism (2.3) and Lemma 2.17.

2. For $d \geqslant 1$, $\Gamma^d(V)$ and $\mathrm{Sym}^d(V)$ are finitely generated projective over k.

Definition 2.18. Let I be the 2-sided ideal in $T(V)$ generated by $(v \otimes v)$ where $v \in V$. The exterior power algebra is the quotient $T(V)/I$ and is denoted by $\bigwedge(V)$. Since the ideal I is homogeneous, $\bigwedge(V)$ is a graded algebra. The d^{th} homogeneous piece of $\bigwedge(V)$ is denoted by $\bigwedge^d(V)$.

Universal property of the exterior algebra: Given a morphism $\phi : V \to A$ where A is an algebra and $\phi(v)^2 = 0$ for every $v \in V$. Then there exists unique morphism $\Phi : \bigwedge(V) \to A$ such that the following diagram commutes



Moreover, if $A = \bigoplus_{i=0} A_i$ is a graded algebra and image of ϕ is inside A_1, Φ is a morphism of graded commutative algebras.

Lemma 2.19. *[Exponential property of exterior power algebra] Given V and W in k-modules we have isomorphism of graded commutative algebras*

$$\wedge(V \oplus W) \simeq \wedge(V) \otimes \wedge(W). \tag{2.6}$$

For homogeneous elements $a, a' \in \wedge(V)$ and homogeneous elements $b, b' \in \wedge(W)$, the algebra structure on k-module $\wedge(V) \otimes \wedge(W)$ is

$$(a \otimes b)(a' \otimes b') = (-1)^{deg(b) \times deg(a')} aa' \otimes bb'. \tag{2.7}$$

Proof. It follows from the universal property of the exterior algebra and the similar arguments as in the proof of Lemma 2.17. □

Some consequences of exponential property of the exterior algebra:

1. If V is a finitely generated projective then so is $\wedge^d(V)$ for every $d \geqslant 1$.

2. For k-module V, consider the following map

$$\wedge^d(V^*) \xrightarrow{\theta} (\wedge^d(V))^* \quad \text{where} \quad \theta(f_1 \wedge \cdots \wedge f_d)(v_1 \wedge \cdots \wedge v_d) = \det(f_i(v_j)). \tag{2.8}$$

   The map θ is k-linear. If V is finitely generated, $\wedge^d(V)$ is finitely generated and in this case we get θ is surjective. For a free module of finite rank, by computing ranks of both sides of θ, we have that θ is an isomorphism. If V is a finitely generated projective over k, it is a direct summand of a free module F of finite rank. Thus $\wedge^d(V)$ is a submodule of $\wedge^d(F)$. Now the map θ for V is the restriction of θ for F and hence θ is an isomorphism for a finitely generated projective module.

3. The anti-symmetrization map,

$$\wedge^d(V) \quad \to \quad V^{\otimes d} \text{ is given by} \tag{2.9}$$

$$v_{i_1} \wedge \ldots \wedge v_{i_d} \quad \longmapsto \quad \sum_{\sigma \in S_d} sgn(\sigma) v_{i_{\sigma(1)}} \otimes \ldots \otimes v_{i_{\sigma(d)}}. \tag{2.10}$$

   For finitely generated projective V, the anti-symmetrization is injective and therefore $\wedge^d(V)$ is isomorphic to a k-submodule of $V^{\otimes d}$.

Example 2.20. The following are examples of polynomial representations of GL_n.

1. **Natural representation:** GL_n acts on k^n by matrix multiplication. It is a degree 1 polynomial representation of GL_n.

2. **Tensor product:** Let V and W be polynomial representations of degree d_1 and d_2, respectively, of GL_n. Then the diagonal action of GL_n on $V \otimes W$ makes it a polynomial representation of degree $d_1 + d_2$.

3. The d^{th} divided power $\Gamma^d(k^n)$, the d^{th} symmetric power $\mathrm{Sym}^d(k^n)$ and the d^{th} exterior power $\wedge^d(k^n)$ are all polynomial representations of degree d of GL_n.

   Let λ be a partition of d. One associates two important types of examples, namely Weyl and Dual Weyl modules. We illustrate them for $\lambda = (3, 2)$, for the general definition see [ABW82, Definition II.1.4].

   Represent λ as the Young diagram  .

4. **Weyl module:** The following map is the tensor product of the inclusion maps $\Gamma^3(V) \to V \otimes V \otimes V$ and $\Gamma^2(V) \to V \otimes V$

$$\begin{pmatrix} \Gamma^3(V) \\ \otimes \\ \Gamma^2(V) \end{pmatrix} \xrightarrow{\theta_1} \begin{pmatrix} (V \otimes V \otimes V) \\ \otimes \\ (V \otimes V) \end{pmatrix}.$$

   The following map is the tensor products of two copies of the quotient map $\begin{pmatrix} V \\ \otimes \\ V \end{pmatrix} \to \wedge^2(V)$ and the map $V \to \wedge^1(V)$

$$\begin{pmatrix} V \\ \otimes \\ V \end{pmatrix} \otimes \begin{pmatrix} V \\ \otimes \\ V \end{pmatrix} \otimes V \xrightarrow{\theta_2} \wedge^2(V) \otimes \wedge^2(V) \otimes \wedge^1(V)$$

   The Weyl module: $\Delta(\lambda)(V)$:= image of the composite map $\theta_2 \circ \theta_1$.

5. **Dual Weyl module:** The following map is the tensor products of two copies of the anti-symmetrization map $\wedge^2(V) \to \begin{pmatrix} V \\ \otimes \\ V \end{pmatrix}$ and the map $\wedge^1(V) \to V$

$$\wedge^2(V) \otimes \wedge^2(V) \otimes \wedge^1(V) \xrightarrow{\psi_1} \begin{pmatrix} V \\ \otimes \\ V \end{pmatrix} \otimes \begin{pmatrix} V \\ \otimes \\ V \end{pmatrix} \otimes V.$$

   The following map is the tensor products of the quotient maps $(V \otimes V \otimes V) \to \mathrm{Sym}^3(V)$ and $(V \otimes V) \to \mathrm{Sym}^2(V)$

$$\begin{pmatrix} (V\otimes V\otimes V) \\ \otimes \\ (V\otimes V) \end{pmatrix} \overset{\psi_2}{\to} \begin{pmatrix} \mathrm{Sym}^3(V) \\ \otimes \\ \mathrm{Sym}^2 V \end{pmatrix}.$$

Dual Weyl module: $\nabla(\lambda)(V)$:= image of the composite map $\psi_2 \circ \psi_1$.

For $\lambda \in \Lambda^+(n,d)$, there are canonical decompositions

$$\Gamma^{\lambda_1}(V)\otimes\cdots\otimes\Gamma^{\lambda_m}(V) \overset{\pi_1}{\twoheadrightarrow} \Delta(\lambda)(V) \overset{i_1}{\hookrightarrow} \wedge^{\lambda'_1}(V)\otimes\cdots\otimes\wedge^{\lambda'_n}(V) \tag{2.11}$$
$$\wedge^{\lambda'_1}(V)\otimes\cdots\otimes\wedge^{\lambda'_n}(V) \overset{\pi_2}{\twoheadrightarrow} \nabla(\lambda)(V) \overset{i_2}{\hookrightarrow} \mathrm{Sym}^{\lambda_1}(V)\otimes\cdots\otimes\mathrm{Sym}^{\lambda_m}(V) \tag{2.12}$$

where λ' is the transpose of λ. When V is a free k-module of finite rank, it follows from the universal freeness of $\Delta(\lambda)(V)$ and $\nabla(\lambda)(V)$ that the maps π_j and i_j split for $1 \leqslant j \leqslant 2$ (see [ABW82, Theorem II.3.16 and Theorem II.2.16]).

We recall now a basic vanishing result related to the Weyl and dual Weyl modules which we will need later in Chapter 4.

Theorem 2.21. *[AB88, Theorem 5.3] Let R be a commutative hereditary ring, $R \to \bar{R}$ a homomorphism of commutative rings and A an R-algebra. Let M and N be left A-modules, which are free R-modules. Furthermore, assume that M has a projective resolution over A by finitely generated projective A-modules. Then there is a short exact sequence of $\bar{R}$-modules*

$$0 \to \bar{R}\otimes_R \mathrm{Ext}^i_A(M,N) \to \mathrm{Ext}^i_{\bar{A}}(\bar{M},\bar{N}) \to \mathrm{Tor}^R_1(\bar{R},\mathrm{Ext}^{i+1}_A(M,N)) \to 0, \tag{2.13}$$

for each $i \geqslant 0$ where $\bar{A} = \bar{R}\otimes A$, $\bar{M} = \bar{R}\otimes M$ and $\bar{N} = \bar{R}\otimes N$.

Proposition 2.22. *Let k be a commutative ring with unity. Then $\mathrm{Ext}^i_{S_k(n,d)}(\Delta(\lambda)(k^n),\nabla(\mu)(k^n)) = 0$ for all partitions λ, μ and all $i > 0$.*

Proof. From Remark 2.16, $\mathrm{Ext}^i_{S_k(n,d)}(\Delta(\lambda)(k^n),\nabla(\mu)(k^n)) \simeq \mathrm{Ext}^i_{\mathrm{Pol}_n(d)}(\Delta(\lambda)(k^n),\nabla(\mu)(k^n))$.

It is is well known when k is a field, $\mathrm{Ext}^i_{\mathrm{Pol}_n(d)}(\Delta(\lambda)(k^n),\nabla(\mu)(k^n)) = 0$ for $i \geqslant 1$ (first use [Jan03, A.10] or [FS97, Corollary 3.12.] and then [Jan03, Propositions II.4.13]).

The Schur algebra $S_k(n,d)$ is isomorphic to $S_{\mathbb{Z}}(n,d)\otimes k$ for any commutative ring k. Then using the exact sequence (2.13) for the ring homomorphism $\mathbb{Z} \to k$ where k is a field we get

$$k\otimes_{\mathbb{Z}} \mathrm{Ext}^i_{S_{\mathbb{Z}}(n,d)}(\Delta(\lambda)(k^n),\nabla(\mu)(k^n)) = 0 \text{ for } i \geqslant 1. \tag{2.14}$$

We first prove the proposition when $k = \mathbb{Z}$ adapting the method more-less from [AB88, Proposition 7.4].

Notice that for $i \geqslant 1$, the abelian group $\mathrm{Ext}^i_{S_{\mathbb{Z}}(n,d)}(\Delta(\lambda)(k^n), \nabla(\mu)(k^n))$ is finitely generated. When $k = \mathbb{Q}$, we get from equation (2.14) that $\mathrm{Ext}^i_{S_{\mathbb{Z}}(n,d)}(\Delta(\lambda)(k^n), \nabla(\mu)(k^n))$ is a finitely generated torsion $\mathbb{Z}$-module. Next take $k = \mathbb{Z}/p\mathbb{Z}$ where p is a prime number. Again using equation (2.14) we get $\mathrm{Ext}^i_{S_{\mathbb{Z}}(n,d)}(\Delta(\lambda)(k^n), \nabla(\mu)(k^n))$ has no p-torsion part also. Thus $\mathrm{Ext}^i_{S_{\mathbb{Z}}(n,d)}(\Delta(\lambda)(k^n), \nabla(\mu)(k^n)) = 0$.

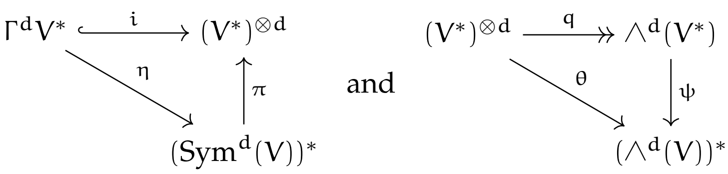
For an arbitrary commutative ring k we use once more the exact sequence (2.13) for the ring homomorphism $\mathbb{Z} \to k$ and the vanishing over $\mathbb{Z}$ to conclude $\mathrm{Ext}^i_{S_k(n,d)}(\Delta(\lambda)(k^n), \nabla(\mu)(k^n)) = 0$ for $i \geqslant 1$. □

2.2.5 Duality

The contravariant dual M° of a polynomial representation M is M^*, made into a left GL_n-module via matrix transpose. For $f \in M^*$, $g \in GL_n$ and $m \in M$

$$g.f(m) = f(g^{tr}.m) \tag{2.15}$$

where g^{tr} denotes the transpose of g, see [Gre80, Section 2.7]. For a finitely generated projective module V recall from isomorphisms (2.2) and (2.8) that $\Gamma^d(V^*) \simeq (\mathrm{Sym}^d(V))^*$ and $\wedge^d(V^*) \simeq (\wedge^d(V))^*$. Using these, splitting of maps in the decompositions (2.11) and commutativity of the following diagrams, we get more generally that $\Delta(\lambda)(V^*) \simeq (\nabla(\lambda)(V))^*$ for a free k-module of finite rank V and a partition λ of d (see [ABW82, Proposition II.4.1]).



Here (a) i is the inclusion morphism, π is the dual of the quotient morphism $V^{\otimes d} \to \mathrm{Sym}^d(V)$ and η is the isomorphism (2.2); (b) q is the quotient morphism, θ is the dual of anti-symmetrization morphism (2.9) and ψ is the morphism (2.8).

Chapter 3

Double centralizer properties of Schur algebras of Weyl groups of types A and B

The Weyl group of types **A** and **B** are the symmetric group S_d and the hyperoctahedral group $B_d := (\mathbb{Z}/2\mathbb{Z})^d \rtimes S_d$ respectively. The Schur algebra of Weyl group of type **A** is the centralizer algebra $\mathrm{End}_{S_d}((k^n)^{\otimes d})$. In [Gre97], R. M. Green defined an action of B_d on $(k^{2n})^{\otimes d}$. The centralizer algebra $H(2n, d) := \mathrm{End}_{B_d}((k^{2n})^{\otimes d})$ is called the Schur algebra of Weyl group of type **B** or the hyperoctahedral Schur algebra.

It is very well known that the double centralizer property between $S_k(n, d)$ and kS_d can be established by using an idempotent in $S_k(n, d)$ ([Gre80, 6.4f], [Mar09, Theorem 4.1]). We give a simple proof of these facts which does not require any explicit calculations in $S_k(n, d)$. Our method also works for proving the double centralizer property (in non-quantum case) between $H(2n, d)$ and kB_d ([Gre97, Proposition 4.5.4]).

3.1 The hom functor

Let A be an associative algebra with unity over k. Let $\mathbf{e}$ be a non-trivial idempotent in A. The k-module $A\mathbf{e}$ (resp. $\mathbf{e}A$) is $(A, \mathbf{e}A\mathbf{e})$-bimodule (resp. $(\mathbf{e}A\mathbf{e}, A)$-bimodule). We have a functor

$$\mathrm{Hom}_A(A\mathbf{e}, -) : A\text{-Mod} \to \text{Mod-}\mathrm{End}_A(A\mathbf{e}). \tag{3.1}$$

One has an algebra anti-homomorphism

$$\mathbf{e}A\mathbf{e} \xrightarrow{f} \mathrm{Hom}_A(A\mathbf{e}, A\mathbf{e}) \text{ given by } \mathbf{e}a\mathbf{e} \longmapsto f_{\mathbf{e}a\mathbf{e}}(t\mathbf{e}) = t\mathbf{e}.\mathbf{e}a\mathbf{e}. \tag{3.2}$$

Note that an element of $\mathrm{Hom}_A(A\mathbf{e}, A\mathbf{e})$ is determined by its value at the idempotent $\mathbf{e}$. So the homomorphism (3.2) is an isomorphism. Thus

$$\mathrm{End}_A(A\mathbf{e}) \simeq (\mathbf{e}A\mathbf{e})^{\mathrm{op}}. \tag{3.3}$$

Now the hom functor (3.1) takes the form

$$\mathrm{Hom}_A(A\mathbf{e}, -) : A\text{-Mod} \rightarrow \mathbf{e}A\mathbf{e}\text{-Mod and} \tag{3.4}$$

$$\mathrm{Hom}_A(A\mathbf{e}, M) \simeq \mathbf{e}M \text{ as left } \mathbf{e}A\mathbf{e}\text{-modules.} \tag{3.5}$$

For functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ we write $F \dashv G$ if F is left adjoint to G.

It is easy to check $\mathrm{Hom}_A(A\mathbf{e}, M) \simeq \mathbf{e}M \simeq eA \otimes_A M$ as left $\mathbf{e}A\mathbf{e}$-module. Consider the functors

$$\mathcal{L} : \mathbf{e}A\mathbf{e}\text{-Mod} \rightarrow A\text{-Mod} \quad , \quad \mathcal{R} : \mathbf{e}A\mathbf{e}\text{-Mod} \rightarrow A\text{-Mod} \quad \text{given by}$$
$$\mathcal{L}(N) = A\mathbf{e} \otimes_{\mathbf{e}A\mathbf{e}} N \quad , \quad \mathcal{R}(N) = \mathrm{Hom}_{\mathbf{e}A\mathbf{e}}(\mathbf{e}A, N).$$

Then $\mathcal{L} \dashv \mathrm{Hom}_A(A\mathbf{e}, -)$ and $\mathrm{Hom}_A(A\mathbf{e}, -) \dashv \mathcal{R}$. Moreover, we have

$$\mathrm{Hom}_A(A\mathbf{e}, \mathcal{L}(N)) \simeq N \text{ and } \mathrm{Hom}_A(A\mathbf{e}, \mathcal{R}(N)) \simeq N. \tag{3.6}$$

3.2 Double centralizer property

Theorem 3.1. *Let G be a finite group acting on the right of a finite set X. Suppose there exists $x_0 \in X$ such that $|O_{x_0}| = |G|$ (i.e., the stabilizer of x_0 is trivial) where O_{x_0} is the orbit of x_0. Then there exists an idempotent $\xi \in A := \mathrm{End}_{kG}(kX)$ such that*

1. $A\xi \simeq kX$ *as* (A, kG)*-bimodules.*
2. $\xi A \xi \simeq kG$.
3. $\mathrm{End}_A(kX) \simeq (kG)^{\mathrm{op}}$.

Proof. Let $X = \coprod_{x \in \mathcal{J}} O_x$ be the orbit space decomposition. Then kO_x is a right kG-module and

$$kX = \bigoplus_{x \in \mathcal{J}} kO_x. \tag{3.7}$$

We have $kO_{x_0} \simeq kG$ as right kG-module because $|O_{x_0}| = |G|$. Let ξ be the projection of kX, under the isomorphism (3.7), onto the summand kO_{x_0}, i.e., $\xi : kX \rightarrow kX$ such that $\xi^2 = \xi$ and the image of ξ is kO_{x_0}. Notice that ξ is kG-linear and so $\xi \in A$.

The left (resp. right) A-module action on A is given by post-composition (resp. pre-composition). The k-module kX is (A, kG)-bimodule. The following are isomorphism of left A-modules

$$A\xi = \mathrm{Hom}_{kG}(kX, kX) \circ \xi = \{\phi \circ \xi \,|\, \phi \in A\} \simeq \mathrm{Hom}_{kG}(kO_{x_o}, kX) \quad (3.8)$$
$$\simeq \mathrm{Hom}_{kG}(kG, kX) \simeq kX \quad (3.9)$$

We have an algebra isomorphism $\xi A\xi = \{\xi \circ \phi \circ \xi \,|\, \phi \in A\} \simeq \mathrm{Hom}_{kG}(kG, kG) \simeq kG$ (recall here kG is right kG-module). Under this identification $A\xi$ is right kG-module and the isomorphism in (3.8) is (A, kG)-linear.

Finally the isomorphism $\mathrm{End}_A(kX) \simeq (kG)^{op}$ of algebras follows from the parts *1*, *2* and the isomorphism (3.3). □

Similar idea as in the proof of the above theorem was used in establishing the double centralizer property between Hecke algebra and quantum Schur algebra in [DJ91, Theorem 6.6]. If $G = S_d$ acting on $(k^n)^{\otimes d}$ by permuting the tensor factors, then Proposition 3.1 is well known for example see [Mar09, Section 4.1].

Remark 3.2. Let A and ξ be as in Proposition 3.1. Here we identify $\xi A\xi$ with kG. Let M be a right G-module. Then $\mathrm{Hom}_{kG}(M, kX)$ is a left A-module. Moreover, if M is free of finite rank over k then we have isomorphisms of left $\xi A\xi$-modules

$$\xi\mathrm{Hom}_{kG}(M, kX) = \xi\mathrm{Hom}_{kG}(M, kX) = \{\xi \circ f \,|\, f \in \mathrm{Hom}_{kG}(M, kX)\}$$
$$\simeq \mathrm{Hom}_{kG}(M, kG) \simeq M^*.$$

Using Theorem 3.1, we will see in the following examples that double centralizer properties in the cases of Weyl groups of types **A** and **B** can be easily derived.

Example 3.3. $(k^n)^{\otimes d}$ has a basis indexed by the set $I(n, d)$ where $I(n, d) := \{f : [d] \to [n]\}$ and $[m] := \{1, 2, \ldots, m\}$. The symmetric group S_d acts on $(k^n)^{\otimes d}$ by permuting the tensor factors. This action is equivalent to S_d acting on the right of $I(n, d)$. More precisely, for $\sigma \in S_d$ and $f \in I(n, d)$, $(\sigma.f)(i) = f(\sigma(i))$ where $i \in [d]$. The centralizer algebra $\mathrm{End}_{S_d}((k^n)^{\otimes d})$ is the Schur algebra $S_k(n, d)$.

Assume $n \geqslant d$. Let $f : [d] \to [n]$ such that $f(i) = i$ for all $i \in [d]$. Notice that the cardinality of the S_d-orbit of f is $d!$. Therefore from Theorem 3.1, $\mathrm{End}_{S_k(n,d)}((k^n)^{\otimes d}) \simeq (kS_d)^{op}$ which is the classical Schur-Weyl duality between the Schur algebra and the symmetric group. Now if we restrict the action of S_d to a subgroup G then also the cardinality of the G-orbit of f is $|G|$. Thus by Theorem 3.1 the classical Schur-Weyl duality extends to any subgroup G of S_d that is $\mathrm{End}_A((k^n)^{\otimes d}) \simeq (kG)^{op}$ where $A = \mathrm{End}_G((k^n)^{\otimes d})$.

Example 3.4. We recover the double centralizer property between the hyperoctahedral group $B_d := (\mathbb{Z}/2\mathbb{Z})^d \rtimes S_d$ and the hyperoctahedral Schur algebra of R. M. Green [Gre97, Proposition 4.5.4]. By identifying an element $\sigma \in S_d$ with the d-tuple $(\sigma(1), \cdots, \sigma(d))$ one can see B_d is $\{((-1)^{l_1}a_1, \cdots, (-1)^{l_d}a_d) \mid (a_1, \cdots, a_d) \in S_d$ and $l_j \in \{0,1\} \forall 1 \leqslant j \leqslant d\}$ (which is the signed symmetric group). The elements $s_0 := (-1, 2, \cdots, d)$ and $s_t := (1, \cdots, t+1, t, \cdots, d)$ generate B_d where $1 \leqslant t \leqslant (d-1)$. From ([HL06, Page 604] and [Gre97, Section 1.1]) the relations satisfied by these generators are

$$\begin{aligned} s_t^2 &= 1, \text{ for } 0 \leqslant t \leqslant (d-1), \\ (s_0 s_1)^2 &= (s_1 s_0)^2, \\ s_t s_{t+1} s_t &= s_{t+1} s_t s_{t+1}, \text{ for } 1 \leqslant t \leqslant (d-2), \\ s_i s_j &= s_j s_i, \text{ for } 0 \leqslant i < (j-1) \leqslant (d-2). \end{aligned}$$

The action of generators of B_d on the right of the set $\Omega(2n, d) = \{\underline{i} = (i_1, \cdots, i_d) \mid i_j \in [2n]\}$, where $[2n] = \{-n, \cdots, -1, 1, \cdots, n\}$, is given as follows ([HL06, Lemma 3.1]). For $\underline{i} \in \Omega(2n, d)$

1. $(\underline{i} * s_0) = (-i_1, \cdots, i_d)$,
2. for $t > 0$, $(\underline{i} * s_t) = (i_1, \cdots, i_{t+1}, i_t, \cdots, i_d)$.

See also [Gre97, Lemma 2.1.1]. Note that $k\Omega(2n, d)$ is isomorphic to the tensor space $(k^{2n})^{\otimes d}$. The centralizer algebra $H(2n, d) := \mathrm{End}_{B_d}((k^{2n})^{\otimes d})$ is called the Schur algebra of Weyl group of type **B** or the hyperoctahedral Schur algebra (see [Gre97, Definition 2.2.2]). Assume $n \geqslant d$. Let $\underline{i} = (1, 2, \cdots, d) \in \Omega(2n, d)$. Notice that the subset $\{((-1)^{r_1}b_1, \cdots, (-1)^{r_d}b_d) \mid b_i \neq b_j$ for $i \neq j$ and $r_i \in \{0,1\}\}$ of $\Omega(2n, d)$ is the subset of the orbit of $\underline{i}$. The cardinality of this subset is $|B_d|$ and hence the cardinality of the orbit of $\underline{i}$ is $|B_d|$. Thus by Theorem 3.1 $\mathrm{End}_{H(2n,d)}((k^{2n})^{\otimes d})$ is $(kB_d)^{op}$ for $n \geqslant d$.

We gather some informations of orbits of actions of S_d and B_d as in Examples 3.3 and 3.4 in the following remarks which will be required in Chapter 6.

Remark 3.5. The orbits of S_d acting on $I(n, d)$ are indexed by the set $\Lambda(n, d)$ where $\Lambda(n, d) = \{\lambda = (\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{N} \cup \{0\}$ and $\sum_{i=1}^{n} \lambda_i = d\}$. The orbit corresponding to $\lambda \in \Lambda(n, d)$ is the set $I_\lambda = \{f \in I(n, d) \mid \ |f^{-1}(j)| = \lambda_j \ \forall\, j \in [n]\}$. The orbit space decomposition $I(n, d) = \coprod_{\lambda \in \Lambda(n,d)} I_\lambda$ induces an isomorphism of right S_d-modules

$$(k^n)^{\otimes d} \simeq kI(n, d) = \bigoplus_{\lambda \in \Lambda(n,d)} kI_\lambda. \tag{3.10}$$

We write $\lambda \sim \mu$ for $\lambda \in \Lambda(n,d)$ and $\mu \in \Lambda(m,d)$ if they are same after ignoring the zeros and the ordering of their elements. We call λ is associated to μ if $\lambda \sim \mu$. Then we have,

$$I_\lambda \simeq I_\mu \qquad \text{as } S_d\text{-sets.} \tag{3.11}$$

Let G be a subgroup of S_d. Then the S_d-orbits are G-sets. Under the isomorphism (3.11), $\coprod_{s \in \mathcal{J}(\lambda)} I_\lambda^s \simeq \coprod_{s \in \mathcal{J}(\mu)} I_\mu^t$ where $\mathcal{J}(\lambda)$ and $\mathcal{J}(\mu)$ are the sets of G-orbits of I_λ and I_μ respectively. For every $s \in \mathcal{J}(\lambda)$, under the isomorphism (3.11) there exists unique $t \in \mathcal{J}(\mu)$ such that $I_\lambda^s \simeq I_\mu^t$ as G-sets. Thus kI_λ^s is isomorphic to kI_μ^t as G-modules.

Remark 3.6. For $\underline{i} \in \Omega(2n,d)$, define $\lambda_j =$ number of times j and $-j$ occur in $\underline{i}$ where $1 \leqslant j \leqslant n$ (see [Gre97, Definition 4.1.2]). Then $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Lambda(n,d)$ and it is called the type of $\underline{i}$. Notice that the types of two elements in $\Omega(2n,d)$ are the same if and only if they are in the same orbit. Thus an indexing set of orbits of B_d-action on $\Omega(2n,d)$ is $\Lambda(n,d)$. Using the orbit space decomposition $\Omega(2n,d) = \coprod_{\lambda \in \Lambda(n,d)} O_\lambda$ where O_λ is the orbit corresponding to $\lambda \in \Lambda(n,d)$, we have

$$(k^{2n})^{\otimes d} = \coprod_{\lambda \in \Lambda(n,d)} kO_\lambda. \tag{3.12}$$

Moreover, if $\lambda \in \Lambda(n,d)$ and $\mu \in \Lambda(m,d)$ such that $\lambda \sim \mu$ then

$$O_\lambda \simeq O_\mu \text{ as } B_d\text{-sets.} \tag{3.13}$$

So in particular $kO_\lambda \simeq kO_\mu$ as kB_d-modules.

The Schur functors: Let n and d be positive integers such that $n \geqslant d$.

Case 1: (type A) From Example 3.3 and the part 1 of Theorem 3.1, there exists an idempotent $\mathbf{e} \in S_k(n,d)$ such that $S_k(n,d)\mathbf{e} \simeq (k^n)^{\otimes d}$ as $(S_k(n,d), kS_d)$-bimodules. (In addition, using Remark 3.2 we also have $\mathbf{e}S_k(n,d) \simeq ((k^n)^{\otimes d})^*$ as $(kS_d, S_k(n,d))$-bimodules.) These allow us to use the theory of hom functors from Section 3.1. So using the part 3 of Theorem 3.1, define Sch:$= \mathrm{Hom}_{S_k(n,d)}((k^n)^{\otimes d}, -) : S_k(n,d)$-Mod$\to kS_d$-Mod and this is called the Schur functor (see [Gre80, Chapter 6] and [Mar09, Chapter 4]). Let M be an object of kS_d-Mod. From Section 3.1, the left adjoint $\mathcal{L}$ and the right adjoint $\mathcal{R}$ of Sch are respectively given by $\mathcal{L}(M) = ((k^n)^{\otimes d} \otimes_{kS_d} M)$ and $\mathcal{R}(M) = \mathrm{Hom}_{kS_d}(((k^n)^{\otimes d})^*, M)$.

Case 2: (type B) From Example 3.4 and the part 1 of Theorem 3.1, there exists an idempotent $\mathbf{e} \in H(2n,d)$ such that $H(2n,d)\mathbf{e} \simeq (k^{2n})^{\otimes d}$ as $(H(2n,d), kB_d)$-bimodules. (In addition, using Remark 3.2 $\mathbf{e}H(2n,d) \simeq ((k^{2n})^{\otimes d})^*$ as $(kB_d, H(2n,d))$-bimodules.) As in Case 1, we define a functor HSch:$= \mathrm{Hom}_{H(2n,d)}((k^{2n})^{\otimes d}, -) : H(2n,d)$-Mod$\to kB_d$-Mod and this is called the hyperoctahedral Schur functor. Let M be an object of kB_d-Mod. The left adjoint $\mathcal{L}$

and the right adjoint $\mathcal{R}$ of HSch are respectively given by $\mathcal{L}(M) = ((k^{2n})^{\otimes d} \otimes_{kB_d} M)$ and $\mathcal{R}(M) = \mathrm{Hom}_{kB_d}(((k^{2n})^{\otimes d})^*, M)$.

Chapter 4

Strict polynomial functors and internal tensor product

The strict polynomial functors of degree d provide a unified way of studying the polynomial representations of GL_n for all $n \geqslant d$. The polynomial representations of degree d of GL_n are equivalently $S_k(n,d)$-modules by Remark 2.16. From [ABW82], the constructions of some polynomial representations of degree d of GL_n namely Weyl and dual Weyl modules are functorial in k^n. The Schur algebra $S_k(n,d)$ is isomorphic to the d^{th} divided power of the endomorphism algebra $\text{End}_k(k^n)$. Taking all of this into consideration, a polynomial representation of fixed degree of GL_n could be the evaluation at k^n of a functor from an appropriate category to k-Mod. This appropriate category is the *divided power category* and a k-linear covariant functor from this category to k-Mod is called a strict polynomial functor. The strict polynomial functors were originally defined by Friedlander and Suslin, in a slightly different but equivalent fashion, in [FS97]. On evaluating a strict polynomial functor at k^n becomes a left $S_k(n,d)$-module. In fact, the category of strict polynomial functors of degree d is equivalent to $S_k(n,d)$-Mod for $n \geqslant d$ (see [FS97, Kra13]).

An internal Hom of strict polynomial functors was introduced by Chałupnik in [Cha05] and by Touzé in [Tou13]. The monoidal structure/internal tensor product was discovered by Krause in [Kra13], as a consequence of Day convolution. This is the topic of study of this chapter.

Several of the important and fundamental calculations of this internal tensor product were carried out in [Kra13] (see Proposition 4.11 and Proposition 4.15). We discuss them in Section 4.2 and also give our computation of internal tensor product involving Sym^d, $\wedge^d$, Γ^λ, $\Delta(\lambda)$, and $\nabla(\lambda)$ (see Lemma 4.13, Proposition 4.14, Proposition 4.16 and Corollary 4.19).

The left (resp. right) derived functors of the internal tensor products (resp. the internal homs) exist from the unbounded derived category of $\text{Rep}\Gamma_k^d$, see [Kra13]. The left derived internal tensor product is the left adjoint of the right derived internal hom by [Kra13]. We show (Proposition 4.21) that this adjunction together with the Yoneda lemma supplies some interesting natural isomorphisms involving the higher derived internal tensors and the higher derived internal Hom.

It was shown that the higher derived internal tensors of the exterior power functor and a Weyl functor vanish in [Kra13]. We generalize this by showing that the higher derived internal tensor products of any two Weyl functors vanish in Proposition 4.28. We consider the higher derived internal tensor product with base change and prove a form of the universal coefficient theorem (Proposition 4.31). As a consequence, we obtain that over the ring of integers the values of the higher derived internal tensor products of certain strict polynomial functors are finite abelian groups (Corollary 4.33). Similar results for the higher derived internal Hom were proved in [BMT16, Section 7].

4.1 Strict polynomial functor

We recall some definitions from [Kra13].

Definition 4.1. The objects of d^{th} divided power category are same as the objects of P_k. The morphism space from V to W is $\Gamma^d \mathrm{Hom}_k(V, W) \simeq \mathrm{Hom}_{S_d}(V^{\otimes d}, W^{\otimes d})$. Here the compositions of morphisms is given by compositions of maps. We denote this category by $\Gamma^d P_k$.

The category $\Gamma^d P_k$ is k-linear (see Definition 8.3). Since the tensor product over k ($\otimes$) of any finitely projective modules over k is again finitely generated projective, the category $\Gamma^d P_k$ is monoidal (or has an internal tensor product, see Definition 8.11). Also, the k-linear homomorphism (Hom) of two finitely generated projective modules is again finitely generated projective so the category $\Gamma^d P_k$ is closed (or has an internal Hom, see 8.11).

Definition 4.2. A strict polynomial functor of degree d is a k-linear covariant functor from $\Gamma^d P_k$ to k-Mod. The category of strict polynomial functor of degree d is denoted as $\mathrm{Rep}\Gamma^d_k$.

Remark 4.3. Notice that $\mathrm{Rep}\Gamma^d_k$ is the functor category $\mathrm{Fct}(\Gamma^d P_k, k\text{-Mod})$. Therefore the category $\mathrm{Rep}\Gamma^d_k$ is k-linear, abelian, complete and co-complete. We refer to Proposition 8.8 for the details.

Let $\mathrm{rep}\Gamma^d_k$ be the functor category $\mathrm{Fct}(\Gamma^d P_k, P_k)$. This is a full subcategory of $\mathrm{Rep}\Gamma^d_k$.

Example 4.4. The following are examples of the strict polynomial functors of degree d. We use $\mathrm{Hom}_{\Gamma^d P_k}(V, W) = \Gamma^d(\mathrm{Hom}_k(V, W)) \simeq \mathrm{Hom}_{S_d}(V^{\otimes d}, W^{\otimes d})$ in order to produce maps at the level of morphisms from V to W, where V and W are objects in $\Gamma^d P_k$

1. The d^{th} tensor power functor, $\otimes^d : \Gamma^d P_k \to k\text{-Mod}$ is given by $\otimes^d(V) = V^{\otimes d}$ and

$$\mathrm{Hom}_{\Gamma^d P_k}(V, W) \to \mathrm{Hom}_k(V^{\otimes d}, W^{\otimes d}) \tag{4.1}$$

is the inclusion map.

2. The d^{th} divided power functor, $\Gamma^d : \Gamma^d P_k \to k\text{-Mod}$ is given by $\Gamma^d(V) = (V^{\otimes d})^{S_d}$ and

$$\mathrm{Hom}_{\Gamma^d P_k}(V, W) \to \mathrm{Hom}_k((V^{\otimes d})^{S_d}, (W^{\otimes d})^{S_d}) \tag{4.2}$$

is obtained by the fact that an S_d-linear map from $V^{\otimes d}$ to $W^{\otimes d}$ induces a k-linear map on the invariants.

3. The d^{th} symmetric power functor, $\mathrm{Sym}^d : \Gamma^d P_k \to k\text{-Mod}$ is given by $\mathrm{Sym}^d(V) = (V^{\otimes d})_{S_d}$ and $\mathrm{Hom}_{\Gamma^d P_k}(V, W) \to \mathrm{Hom}_k(\mathrm{Sym}^d(V), \mathrm{Sym}^d(W))$ is obtained by the fact that an S_d-linear map from $V^{\otimes d}$ to $W^{\otimes d}$ induces a k-linear map on the coinvariants.

4. The d^{th} exterior power functor, $\wedge^d : \Gamma^d P_k \to k\text{-Mod}$ is given by $\wedge^d(V) = V^{\otimes d} / < v \otimes v > \cap V^{\otimes d}$ and $\mathrm{Hom}_{\Gamma^d P_k}(V, W) \to \mathrm{Hom}_k(\wedge^d(V), \wedge^d(W))$ is obtained by the easy observation that the image of subspace $< v \otimes v > \cap V^{\otimes d}$ under an S_d-linear map is inside the subspace $< w \otimes w > \cap W^{\otimes d}$.

5. For λ a partition of d, the Weyl functor $\Delta(\lambda)$ and the dual Weyl functor $\nabla(\lambda)$ are also objects of $\mathrm{Rep}\Gamma^d_k$. See [ABW82] for their definition. When $\lambda = (d)$ then $\Delta(\lambda) = \Gamma^d$ and $\nabla(\lambda) = \mathrm{Sym}^d$. When $\lambda = (1, \cdots, 1) = (1^d)$ then $\Delta(\lambda) = \wedge^d = \nabla(\lambda)$.

It is clear that $\otimes^d$ is an object of $\mathrm{rep}\Gamma^d_k$. Using item 2 (in some consequences of exponential property of symmetric algebra) at page 16, we get Γ^d and Sym^d are also objects of $\mathrm{rep}\Gamma^d_k$. Similarly, from item 1 (in some consequences of exponential property of exterior algebra) at page 17. Later we will see that more generally $\Delta(\lambda)$ and $\nabla(\lambda)$ are also objects of $\mathrm{rep}\Gamma^d_k$ (see item 2 in applications of equivalence at page 33).

Duality: Let X be an object of $\mathrm{Rep}\Gamma^d_k$. Following [Kou91, Page 105] and [Kuh94, 3.4], we define the contravariant dual (also known as Kuhn dual) X° of X by $X^\circ(V) := X(V^*)^*$. Evaluation on k^n takes this duality to the usual contravariant duality as given in Section 2.2.5 for $S_k(n, d)$-modules. E.g. $(\mathrm{Sym}^d)^\circ = \Gamma^d, (\wedge^d)^\circ = \wedge^d$ and more generally we will see that $(\Delta(\lambda))^\circ = \nabla(\lambda)$ (item 2 in applications of equivalence at page 33). This duality when restricted to $\mathrm{rep}\Gamma^d_k$ is exact and its square is naturally isomorphic to the identity functor.

We recall an important construction of strict polynomial functors from [Tou14] and also a class of projective generators in $\mathrm{Rep}\Gamma^d_k$ from [Kra13].

Parametrization: For an object V in $\Gamma^d P_k$ and a strict polynomial functor X, the lower and upper parametrization of X were defined in [Tou14, Section 4.1] as follows:

$$X_V(W) := X(V \otimes W) \qquad \text{and} \qquad X^V(W) := X(\mathrm{Hom}_k(V, W)).$$

The functors X_V and X^V are objects of $\mathrm{Rep}\Gamma^d_k$. The following lemma gives that the upper parametrization is the left adjoint of the lower parametrization.

Lemma 4.5. *[Tou14, Lemma 4.1] For strict polynomial functors X, Y and a finitely generated projective module V, we have the following natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Rep}\Gamma^d_k}(X^V, Y) \simeq \mathrm{Hom}_{\mathrm{Rep}\Gamma^d_k}(X, Y_V).$$

Let us elaborate the upper parametrization of Γ^d. This is a functor $\Gamma^{d,V} : \Gamma^d P_k \to k\text{-Mod}$ where $\Gamma^{d,V}(W) = \Gamma^d(\mathrm{Hom}_k(V,W)) = \mathrm{Hom}_{\Gamma^d P_k}(V,W) = \mathrm{Hom}_{\Gamma^d P_k}(V,-)(W)$ which is precisely a representable functor in the functor category $\mathrm{Rep}\Gamma^d_k$.

Projective objects: The functor $\Gamma^{d,V}$, being a representable functor, is a projective object in $\mathrm{Rep}\Gamma^d_k$. Moreover, by Proposition 8.8 the class $\{\Gamma^{d,V}\}$ as V runs over the objects of $\Gamma^d P_k$ form a projective generator (see Definition 8.7) in $\mathrm{Rep}\Gamma^d_k$. As we notice that the class $\{\Gamma^{d,V}\}$ as V runs over the objects of $\Gamma^d P_k$ is quite big. Later we will see, whenever $n \geqslant d$, in fact Γ^{d,k^n} is enough to generate the category $\mathrm{Rep}\Gamma^d_k$ (see application of exponential property at page 31).

External tensor product: Let X and Y be two strict polynomial functors of degree d_1 and d_2 respectively. Then one defines a strict polynomial functor of degree $d_1 + d_2$ from [Kra13, Page 1002] as follows

$$(X \otimes Y)(V) := X(V) \otimes Y(V).$$

For $\lambda \in \Lambda(n,d)$, define

$$\Gamma^\lambda = \Gamma^{\lambda_1} \otimes \ldots \otimes \Gamma^{\lambda_n}, \qquad \mathrm{Sym}^\lambda = \mathrm{Sym}^{\lambda_1} \otimes \ldots \otimes \mathrm{Sym}^{\lambda_n} \quad \text{and} \quad \wedge^\lambda = \wedge^{\lambda_1} \otimes \ldots \otimes \wedge^{\lambda_n}.$$

If the sequences $\lambda \in \Lambda(n,d)$ and $\mu \in \Lambda(m,d)$ are the same after ignoring the zeros and the ordering then we have isomorphisms of strict polynomial functors

$$\Gamma^\lambda \simeq \Gamma^\mu \quad , \quad \mathrm{Sym}^\lambda \simeq \mathrm{Sym}^\mu \quad \text{and} \quad \wedge^\lambda \simeq \wedge^\mu. \tag{4.3}$$

Exponential property: Recall the exponential properties of the symmetric algebra , the divided power algebra and the exterior algebra from Chapter 2. Taking the d^{th} homogeneous piece from equations (2.4), (2.5) and (2.6) we get

$$\begin{aligned} \mathrm{Sym}^d(V \oplus W) &\simeq \bigoplus_{i=0}^{d} \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^{d-i}(W), \\ \Gamma^d(V \oplus W) &\simeq \bigoplus_{i=0}^{d} \Gamma^i(V) \otimes \Gamma^{d-i}(W), \\ \wedge^d(V \oplus W) &\simeq \bigoplus_{i=0}^{d} \wedge^i(V) \otimes \wedge^{d-i}(W). \end{aligned}$$

Now proceeding inductively we get

$$\begin{aligned} \mathrm{Sym}^{d,k^n}(W) &\simeq \mathrm{Sym}^d(W\oplus\cdots\oplus W) \simeq \bigoplus_{\lambda\in\Lambda(n,d)} \mathrm{Sym}^\lambda(W), \\ \Gamma^{d,k^n}(W) &\simeq \Gamma^d(W\oplus\cdots\oplus W) \simeq \bigoplus_{\lambda\in\Lambda(n,d)} \Gamma^\lambda(W), \\ \wedge^{d,k^n}(W) &\simeq \wedge^d(W\oplus\cdots\oplus W) \simeq \bigoplus_{\lambda\in\Lambda(n,d)} \wedge^\lambda(W). \end{aligned}$$

We have the canonical decomposition of functors Sym^{d,k^n}, Γ^{d,k^n} and $\wedge^{d,k^n}$ as follows:

$$\Gamma^{d,k^n} \simeq \bigoplus_{\lambda\in\Lambda(n,d)} \Gamma^\lambda \quad , \quad \mathrm{Sym}^{d,k^n} \simeq \bigoplus_{\lambda\in\Lambda(n,d)} \mathrm{Sym}^\lambda \quad \text{and} \quad \wedge^{d,k^n} \simeq \bigoplus_{\lambda\in\Lambda(n,d)} \wedge^\lambda . \tag{4.4}$$

Application of exponential property: Let V be a finitely generated projective module over k. Then we can write $k^s = V\oplus V'$ for some $s\in\mathbb{N}$ and $V'\in P_k$. We have $\Gamma^{d,k^s} = \bigoplus_{i=0}^{d} \Gamma^{i,V}\otimes\Gamma^{d-i,V'}$ and $d = i$ gives that functor $\Gamma^{d,V}$ is a direct summand of Γ^{d,k^s}. Therefore in the class of projective generator $\{\Gamma^{d,V}\}$, we could take V to be a free module of finite rank.

Now assume $n \geqslant d$. Then for every $m\in\mathbb{N}$ and $\mu\in\Lambda(m,d)$ there exists $\lambda\in\Lambda(m,d)$ such that $\lambda\sim\mu$. So by using the isomorphism (4.3), we get that every direct summand of Γ^{d,k^m} in the decomposition (4.4) is already a direct summand of Γ^{d,k^n}. Thus the direct sum of a finite number of copies of Γ^{d,k^n} surjects onto Γ^{d,k^m}. Finally, we conclude that Γ^{d,k^n} is a small projective generator (see Definition 8.9) in $\mathrm{Rep}\Gamma^d_k$.

The evaluation functor: Let X be an object of $\mathrm{Rep}\Gamma^d_k$. Then $X(k^n)$ is a left module over $\mathrm{End}_{\Gamma^d P_k}(k^n) = \mathrm{End}_{S_d}((k^n)^{\otimes d})$. From Definition 2.14 recall that $\mathrm{End}_{S_d}((k^n)^{\otimes d})$ is the Schur algebra $S_k(n,d)$. Thus evaluation at k^n gives a functor $\mathrm{Rep}\Gamma^d_k \xrightarrow{ev_{k^n}} S_k(n,d)$-Mod. First we note a useful lemma.

Lemma 4.6. *[FS97, Corollary 2.12] For λ in $\Lambda(n,d)$ and an object X in $\mathrm{Rep}\Gamma^d_k$, we have*

$$\mathrm{Hom}_{\mathrm{Rep}\Gamma^d_k}(\Gamma^\lambda, X) \simeq \lambda\text{-weight space of } X(k^n). \tag{4.5}$$

Moreover, the above isomorphism is functorial in X.

Proof. Since Γ^{d,k^n} is a representable functor therefore from the Yoneda lemma 8.2, we have

$$\mathrm{Hom}_{\mathrm{Rep}\Gamma^d_k}(\Gamma^{d,k^n}, X) \simeq X(k^n). \tag{4.6}$$

Note that $\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(\Gamma^{d,k^n}, X)$ is a right $\mathrm{End}_{\mathrm{Rep}\Gamma_k^d}(\Gamma^{d,k^n})$-module. From Proposition 8.4, $\mathrm{End}_{\mathrm{Rep}\Gamma_k^d}(\Gamma^{d,k^n}) = (\Gamma^{d,k^n}(k^n))^{op}$ and so it is isomorphic to $(S_k(n,d))^{op}$. So $\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(\Gamma^{d,k^n}, X)$ is a left $S_k(n,d)$-module.

Functionality in the first slot of the Yoneda lemma 8.2 gives that the isomorphism (4.6) is $S_k(n,d)$-linear. So by taking the λ-weight space under the isomorphism (4.6) we get the desired result. Functionality in the variable X follows immediately again from the Yoneda lemma. □

We have the following commutative diagram,

$$\begin{array}{ccc} \mathrm{Rep}\Gamma_k^d & \xrightarrow{ev_{k^n}} & S_k(n,d)\text{-Mod} \\ \downarrow{\scriptstyle \mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(\Gamma^{d,k^n},-)} & & \downarrow \wr \\ \text{Mod-}\mathrm{End}_{\mathrm{Rep}\Gamma_k^d}(\Gamma^{d,k^n}) & \xrightarrow{\sim} & \text{Mod-}(\Gamma^{d,k^n}(k^n))^{op}. \end{array}$$

Here the bottom arrow is given by the Yoneda lemma (see Proposition 8.4). The rightmost arrow is given by the isomorphism $\Gamma^{d,k^n}(k^n) \simeq S_k(n,d)$. Thus the functor ev_{k^n} is naturally isomorphic to the hom functor $\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(\Gamma^{d,k^n}, -)$. One of the advantage of this isomorphism is that it gives insight for the existence of the left adjoint of ev_{k^n}. For an object M in $S_k(n,d)$-Mod, the left and right adjoints of ev_{k^n}, respectively, are given by

$$\theta_!(M) = \Gamma^{d,k^n} \otimes_{S_k(n,d)} M \qquad , \qquad \theta_*(M) = \mathrm{Hom}_{S_k(n,d)}(\Gamma^{d,-}(k^n), M).$$

(The functor $\theta_!$ first appeared in [FS97].) Moreover, we have $ev_{k^n} \circ \theta_! \simeq 1_{S_k(n,d)}$ and $ev_{k^n} \circ \theta^* \simeq 1_{S_k(n,d)}$ where $1_{S_k(n,d)}$ denotes the identity functor on $S_k(n,d)$-Mod. For $n \geqslant d$, ev_{k^n} was shown to be an equivalence of categories in [FS97] when k is a field and in [Kra13] when k is a commutative ring with unity.

Theorem 4.7. *[FS97, Theorem 3.2] [Kra13, Theorem 2.10] Let n and d be positive integers such that $n \geqslant d$. Then ev_{k^n} is an equivalence of categories.*

Proof. We have noticed that ev_{k^n} is naturally isomorphic to the functor $\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(\Gamma^{d,k^n}, -)$. As $n \geqslant d$, the functor Γ^{d,k^n} is a small projective generator (see in application of exponential property at page 31). In addition, we have $\mathrm{Rep}\Gamma_k^d$ is abelian, complete and cocomplete. Thus from Theorem 8.10 we get that ev_{k^n} is an equivalence. □

Applications of the equivalence (4.7):

1. Let X be an object of $\mathrm{Rep}\Gamma^d_k$. Then X is in $\mathrm{rep}\Gamma^d_k$ if and only $X(k^d) \in P_k$, see [Kra13, Remark 2.11]. We amplify this slightly. First, if $X(k^d)$ is a projective k-module, then so is $X(V)$ for all V in P_k, because

$$X(V) \simeq \hom_{\mathrm{Rep}\Gamma^d_k}(\Gamma^{d,V}, X) \overset{\oplus}{\hookrightarrow} \hom_{\mathrm{Rep}\Gamma^d_k}(\Gamma^{d,k^m}, X) \overset{\oplus}{\hookrightarrow} \hom_{\mathrm{Rep}\Gamma^d_k}(\bigoplus_{\text{finite}} \Gamma^{d,k^d}, X) \simeq \bigoplus_{\text{finite}} X(k^d).$$

For finite generation, we again have equivalent characterizations given by the following circular implications: X in $\mathrm{Rep}\Gamma^d_k$ is finitely generated $\overset{\text{def}}{\Leftrightarrow}$ X is a quotient of a finite direct sum of representable functors $\Gamma^{d,W}$ $\Rightarrow$ each $X(V)$ is a finitely generated k-module $\Rightarrow$ $X(k^d)$ is a finitely generated k-module $\Rightarrow$ $X(k^d)$ is a finitely generated $S_k(d,d)$-module $\Rightarrow$ (by the equivalence in (4.7)) X is a quotient of a finite direct sum of Γ^{d,k^d}, i.e. X is finitely generated.

2. If $X(k^d)$ is a torsion k-module then $X(V)$ is a torsion k-module for every object V in P_k.

3. For $\lambda \in \Lambda^+(n,d)$, from [ABW82] $\Delta(\lambda)(k^d)$ and $\nabla(\lambda)(k^d)$ are free k-modules of finite rank. So from item 1 we get that $\Delta(\lambda)$ and $\nabla(\lambda)$ are objects of $\mathrm{rep}\Gamma^d_k$. Using $\Delta(\lambda)(k^d)^* \simeq (\nabla(\lambda)(k^d))^*$ (from Section 2.2.5) and item 1 we deduce that $(\Delta(\lambda))^\circ \simeq \nabla(\lambda)$.

4.2 Closed monoidal structure of strict polynomial functors

One has the contravariant Yoneda embedding of $\Gamma^d P_k$ inside $\mathrm{Rep}\Gamma^d_k$,

$$\begin{aligned} \Gamma^d P_k &\hookrightarrow \mathrm{Rep}\Gamma^d_k \\ V &\mapsto \mathrm{Hom}_{\Gamma^d P_k}(V,-) = \Gamma^{d,V}. \end{aligned}$$

Recall that $\Gamma^d P_k$ is a closed monoidal category. Therefore the Day convolution induces a closed, monoidal structure on $\mathrm{Rep}\Gamma^d_k$ such that the Yoneda embedding preserves these structures (for example see [IK86]). More precisely, it defines the following bifunctors

$$(-\underline{\otimes}-) : \mathrm{Rep}\Gamma^d_k \times \mathrm{Rep}\Gamma^d_k \to \mathrm{Rep}\Gamma^d_k \text{ and } \mathbb{H}(-,-) : (\mathrm{Rep}\Gamma^d_k)^{op} \times \mathrm{Rep}\Gamma^d_k \to \mathrm{Rep}\Gamma^d_k,$$

which on representable functors become,

$$\Gamma^{d,V} \underline{\otimes} \Gamma^{d,W} = \Gamma^{d,V\otimes W} \quad \text{and} \quad \mathbb{H}(\Gamma^{d,V}, \Gamma^{d,W}) = \Gamma^{d,\mathrm{Hom}_k(V,W)}.$$

Since the category k-Mod is cocomplete, using Proposition 8.6, we have for an object X in $\mathrm{Rep}\Gamma^d_k$, the canonical isomorphism

$$X \simeq \operatorname*{colim}_{\Gamma^{d,V} \to X} \Gamma^{d,V}. \tag{4.7}$$

Following [Kra13] one uses the isomorphism (4.7) to define the bifunctors $(-\underline{\otimes}-)$ and $\mathbb{H}(-,-)$ for arbitrary objects X and Y. More explicitly, we write $X = \operatorname{colim}_{\Gamma^{d,V}\to X} \Gamma^{d,V}$ and $Y = \operatorname{colim}_{\Gamma^{d,W}\to Y} \Gamma^{d,W}$. Then,

$$X\underline{\otimes}Y := \operatorname{colim}_{\Gamma^{d,V}\to X}\big(\operatorname{colim}_{\Gamma^{d,W}\to Y}(\Gamma^{d,V}\underline{\otimes}\Gamma^{d,W})\big) \qquad \text{and} \qquad \mathbb{H}(X,Y) := \lim_{\Gamma^{d,V}\to X}\big(\operatorname{colim}_{\Gamma^{d,W}\to Y}(\mathbb{H}(\Gamma^{d,V},\Gamma^{d,W}))\big).$$

The bi-functors $(-\underline{\otimes}-)$ and $\mathbb{H}(-,-)$ are as called the *internal tensor* and the *internal hom* of the strict polynomial functors respectively. There is another expression for the bi-functor $\mathbb{H}$ which does not involve limits/colimits and is as follows:

$$V \mapsto \operatorname{Hom}_{\operatorname{Rep}\Gamma^d_k}(X, Y_V).$$

This was introduced by Chałupnik when the first slot is $\wedge^d$ in [Cha05] and by Touzé in general in [Tou13]. We show that

$$\mathbb{H}(X,Y)(V) \simeq \operatorname{Hom}_{\operatorname{Rep}\Gamma^d_k}(X, Y_V). \tag{4.8}$$

Write $X = \operatorname{colim}_{\alpha} \Gamma^{d,W}$ and $Y = \operatorname{colim}_{\beta} \Gamma^{d,U}$. Then $\mathbb{H}(X,Y)(V) = \lim_{\alpha}(\operatorname{colim}_{\beta}(\Gamma^{d,\operatorname{Hom}(W,U)}(V)))$. The parametrization commutes with the colimit, we have $Y_V = \operatorname{colim}_{\beta} \Gamma^{d,U}_V$. The morphism space $\operatorname{Hom}_{\operatorname{Rep}\Gamma^d_k}(X, Y_V) = \lim_{\alpha}(\operatorname{colim}_{\beta}(\operatorname{Hom}_{\operatorname{Rep}\Gamma^d_k}(\Gamma^{d,W}, \Gamma^{d,U}_V))$ where

$$\begin{aligned}\operatorname{Hom}_{\operatorname{Rep}\Gamma^d_k}(\Gamma^{d,W}, \Gamma^{d,U}_V) &\simeq \Gamma^{d,U}_V(W) = \Gamma^d(\operatorname{Hom}(U, V\otimes W)) \\ &= \Gamma^d(\operatorname{Hom}(\operatorname{Hom}(U,W),V)) = \Gamma^{d,\operatorname{Hom}(U,W)}(V).\end{aligned}$$

It is interesting to note that the internal tensor and internal hom form the usual hom-tensor adjunction.

Proposition 4.8. *[Kra13, Proposition 2.4] For objects X, Y and Z in $\operatorname{Rep}\Gamma^d_k$, there is an isomorphism of abelian groups*

$$\operatorname{Hom}_{\operatorname{Rep}\Gamma^d_k}(X\underline{\otimes}Y, Z) \simeq \operatorname{Hom}_{\operatorname{Rep}\Gamma^d_k}(X, \mathbb{H}(Y,Z)) \tag{4.9}$$

which is natural in X, Y and Z.

In the next proposition we show that the adjunction (4.9) remains true if we replace ordinary hom by the internal Hom.

Proposition 4.9. *For objects X, Y and Z in $\operatorname{Rep}\Gamma^d_k$, we have*

$$\mathbb{H}(X\underline{\otimes}Y, Z) \simeq \mathbb{H}(X, \mathbb{H}(Y,Z)). \tag{4.10}$$

Proof. For an object W in $\mathrm{Rep}\Gamma_k^d$, we have

$$\begin{aligned}\mathrm{Hom}(W, \mathbb{H}(X \underline{\otimes} Y, Z)) &\simeq \mathrm{Hom}(W \underline{\otimes}(X \underline{\otimes} Y), Z) \simeq \mathrm{Hom}((W \underline{\otimes} X) \underline{\otimes} Y, Z) \\ &\simeq \mathrm{Hom}(W \underline{\otimes} X, \mathbb{H}(Y, Z)) \simeq \mathrm{Hom}(W, \mathbb{H}(X, \mathbb{H}(Y, Z))).\end{aligned}$$

All homomorphism spaces in the series of the above isomorphisms are in the category $\mathrm{Rep}\Gamma_k^d$ Thus using the Yoneda lemma we get $\mathbb{H}(X \underline{\otimes} Y, Z) \simeq \mathbb{H}(X, \mathbb{H}(Y, Z))$. □

Note that by evaluating at k of both sides of the isomorphism (4.10) and using the isomorphism (4.8), we get $\mathrm{Hom}(X \underline{\otimes} Y, Z_k) \simeq \mathrm{Hom}(X, \mathbb{H}(Y, Z)_k)$. Parametrization of an object of $\mathrm{Rep}\Gamma_k^d$ at k is isomorphic to the same object. So in particular we obtain the adjunction (4.9).

Remark 4.10. We use the isomorphism (4.8) to show $X^\circ = \mathbb{H}(X, \mathrm{Sym}^d)$, see also [Kra13, Section 4]. Write $X = \underset{\alpha}{\mathrm{colim}} \Gamma^{d,V}$, where α is a morphism $\Gamma^{d,V} \to X$.

$$\begin{aligned}\mathbb{H}(\underset{\alpha}{\mathrm{colim}} \Gamma^{d,V}, \mathrm{Sym}^d)(W) &\simeq \lim_{\alpha} \mathbb{H}(\Gamma^{d,V}, \mathrm{Sym}^d)(W) \\ &\simeq \lim_{\alpha} \mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(\Gamma^{d,V}, \mathrm{Sym}_W^d) \\ &\simeq \lim_{\alpha} \mathrm{Sym}_W^d(V) \\ &\simeq \lim_{\alpha} \mathrm{Hom}_k(\Gamma^{d,V}(W^*), k) \\ &\simeq \mathrm{Hom}_k(\underset{\alpha}{\mathrm{colim}}\, \Gamma^{d,V}(W^*), k) \\ &\simeq \mathrm{Hom}_k(X(W^*), k) = X^\circ(W).\end{aligned}$$

Monoidal dual or the internal dual: Recall that Γ^d is the identity of the bifunctor $\underline{\otimes}$. In [Rei16], monoidal dual or the internal dual of an object X in $\mathrm{Rep}\Gamma_k^d$ is defined to be $X^\vee := \mathbb{H}(X, \Gamma^d)$.

4.2.1 Computations of internal tensors and internal homs

We have seen that in the process of defining $\underline{\otimes}$ and $\mathbb{H}$ the functor $\Gamma^{d,V}$ play a crucial role. It would be a first step to compute $\underline{\otimes}$ and $\mathbb{H}$ when one of the variable is $\Gamma^{d,V}$ and the other one is an arbitrary strict polynomial functor.

Proposition 4.11. *[Kra13, Lemma 2.5] For an object X in $\mathrm{Rep}\Gamma_k^d$, we have $\Gamma^{d,V} \underline{\otimes} X \simeq X^V$ and $\mathbb{H}(\Gamma^{d,V}, X) \simeq X_V$.*

Several of the important functors, e.g. Weyl functors, are resolved by the functors Γ^λ. We show in Proposition 4.13 that $(\Gamma^\lambda \underline{\otimes} -)$ takes X to its λ-weight space X^λ, which we define in Definition 4.12. For a strict polynomial functor X, the k-module $X_V(W) := X(V \otimes W)$ has $GL(V) \times GL(W)$-action. When $V = k^n$, any weight space of $X(k^n \otimes W)$ with respect to the action of $GL(k^n)$ is still functorial in W and thus yields a strict polynomial functor of degree d.

Definition 4.12. [KSS16, Definition 3.2.1] Let X be an object of $\mathrm{Rep}\Gamma^d_k$ and $\lambda \in \Lambda(n, d)$. Then we define a strict polynomial functor X^λ by $X^\lambda(V) = X(k^n \otimes V)_\lambda$, i.e., the λ-weight space of the $GL(k^n)$-module $X(k^n \otimes V)$ (see Definition 2.11).

From the decompositions (4.4), we have $(\Gamma^{d,k^n})^\lambda = \Gamma^\lambda$, $(\mathrm{Sym}^{d,k^n})^\lambda = \mathrm{Sym}^\lambda$ and $(\wedge^{d,k^n})^\lambda = \wedge^\lambda$.

Lemma 4.13. *[KSS16, Lemma 3.2.2] For $\lambda \in \Lambda(n, d)$ and $X \in \mathrm{Rep}\Gamma^d_k$, we have $X \underline{\otimes} \Gamma^\lambda \simeq X^\lambda$.*

Proof. First let $X = \Gamma^{d,V}$. We have $(\Gamma^{d,V})^\lambda(U) =$ the λ-weight space of the $GL(k^n)$-module $\Gamma^{d,V}(k^n \otimes U) = \Gamma^d(V^* \otimes k^n \otimes U)_\lambda$. Using Proposition 4.11, $(\Gamma^{d,V} \underline{\otimes} \Gamma^\lambda)(U) \simeq \Gamma^\lambda(\mathrm{Hom}(V, U)) = \Gamma^\lambda(V^* \otimes U) = \Gamma^d(k^n \otimes V^* \otimes U)_\lambda$, where the λ-weight space is again taken for the $GL(k^n)$-action. In general we again use $X \simeq \operatorname{colim}_{\Gamma^{d,V} \to X} \Gamma^{d,V}$ and check compatibility with colimits. We have

$$\left(\operatorname*{colim}_{\Gamma^{d,V} \to X} \Gamma^{d,V}\right) \underline{\otimes}\, \Gamma^\lambda \quad \simeq \quad \operatorname*{colim}_{\Gamma^{d,V} \to X} (\Gamma^{d,V} \underline{\otimes} \Gamma^\lambda) \quad \simeq \quad \operatorname*{colim}_{\Gamma^{d,V} \to X} (\Gamma^{d,V})^\lambda \quad \simeq \quad \left(\operatorname*{colim}_{\Gamma^{d,V} \to X} \Gamma^{d,V}\right)^\lambda.$$

The first isomorphism is true since $\underline{\otimes}$ commutes with colimits. The second follows from the special case proved above. Finally, the functor $Y \to Y^\lambda$ preserves colimits, since it is exact and preserves arbitrary direct sums in the abelian category $\mathrm{Rep}\Gamma^d_k$. □

Note that from Lemma 4.13 we have recovered [Kra13, Proposition 3.4], which is $\wedge^d \underline{\otimes} \Gamma^\lambda \simeq \wedge^\lambda$.

When $X = \Gamma^\mu$, we are able to compute $X \underline{\otimes} \Gamma^\lambda$ explicitly and thus answering a question posed by Krause in [Kra13]. Let S be the set of all $m \times n$ matrices whose row sums is μ and column sums is λ, and such matrices can naturally be identified with an element of $\Lambda(mn, d)$.

Proposition 4.14. *[KSS16, Proposition 3.2.3] For $\mu \in \Lambda(n, d)$ and $\lambda \in \Lambda(m, d)$, we have $\Gamma^\mu \underline{\otimes} \Gamma^\lambda \simeq \bigoplus_{\nu \in S} \Gamma^\nu$.*

Proof. We have $\Gamma^\mu \underline{\otimes} \Gamma^\lambda \simeq (\Gamma^\mu)^\lambda$. To unwind this, let $U \in \Gamma^d P_k$. Then using Definition 4.12 we get

$$(\Gamma^\mu \underline{\otimes} \Gamma^\lambda)(U) \quad \simeq \quad \Gamma^\mu(k^m \otimes U)_\lambda \quad \simeq \quad (\Gamma^d(k^n \otimes k^m \otimes U)_\mu)_\lambda$$

where the λ-weight space is taken with respect to $GL(k^m)$ and the μ-weight space is taken with respect to $GL(k^n)$. Thus, $(\Gamma^\mu \underline{\otimes} \Gamma^\lambda)(U)$ is the (μ, λ)-weight space of $\Gamma^d(k^n \otimes k^m \otimes U)$ with respect to the action of $GL(k^n) \times GL(k^m)$.

On the other hand $\Gamma^d(k^n \otimes k^m \otimes U)$ is a polynomial representation of $GL(k^n \otimes k^m)$ whose weights are given by $n \times m$ non-negative integer matrices (ν_{ij}) with the entries adding up to d. We can pull back the action of $GL(k^n \otimes k^m)$ to $GL(k^n) \times GL(k^m)$ via the morphism $A \times B \mapsto A \otimes B$. This induces another weight space decomposition of $\Gamma^d(k^n \otimes k^m \otimes U)$, in which elements of the (ν_{ij})-weight space under the $GL(k^n \otimes k^m)$-action now have weight $(\mu_1, \dots, \mu_n, \lambda_1, \dots, \lambda_m)$ for the $GL(k^n) \times GL(k^m)$-action, where $\mu_i = \sum_{j=1}^m \nu_{ij}$, and $\beta_j = \sum_{i=1}^n \nu_{ij}$. □

We compute $\underline{\otimes}$ involving the exponential strict polynomial functors Sym^d and $\wedge^d$.

Proposition 4.15. *[Kra13, Lemma 4.3]* $\wedge^d \underline{\otimes} \wedge^d \simeq \mathrm{Sym}^d$.

Proof. We discuss the same proof as in [Kra13] but in more detail.

We use the following presentation from [ABW82, Page 216] of $\wedge^d$,

$$\bigoplus_{i=0}^{d} \otimes^i \otimes \Gamma^2 \otimes \otimes^{d-i-2} \xrightarrow{\bigoplus_{i=0}^{d} id^{\otimes i} \otimes \Delta_1 \otimes id^{\otimes (d-i-2)}} \otimes^d \longrightarrow \wedge^d \longrightarrow 0 \tag{4.11}$$

where $\Delta_1 \colon \Gamma^2 \to \otimes^2$ is the co-multiplication map (or the inclusion map). We show that on applying $(\wedge^d \underline{\otimes} -)$ to the presentation (4.11), we get the following presentation (4.12) from [ABW82, Page 214] of Sym^d,

$$\bigoplus_{i=0}^{d} \otimes^i \otimes \wedge^2 \otimes \otimes^{d-i-2} \xrightarrow{\bigoplus_{i=0}^{d} id^{\otimes i} \otimes \Delta_2 \otimes id^{\otimes (d-i-2)}} \otimes^d \longrightarrow \mathrm{Sym}^d \longrightarrow 0 \tag{4.12}$$

where $\Delta_2 \colon \wedge^2 \to \otimes^2$ is the co-multiplication map (or the quotient map).

It follows from Lemma 4.13 that $\wedge^d \underline{\otimes} (\otimes^i \otimes \Gamma^2 \otimes \otimes^{d-i-2}) \simeq \otimes^i \otimes \wedge^2 \otimes \otimes^{d-i-2}$ and $\wedge^d \underline{\otimes} \otimes^d \simeq \otimes^d$. In the following we show that $(\wedge^2 \underline{\otimes}\, \Delta_1) = \Delta_2$.

Let e_1, e_2 be the standard basis of k^2. Take $E_{ij} = e_i^* \otimes e_j$. The $(2,0)$-weight space of Γ^{2,k^2} in its decomposition is isomorphic to Γ^2. In fact, the span of $E_{11} \otimes E_{11}, E_{12} \otimes E_{12}$ and $E_{11} \otimes E_{12} + E_{12} \otimes E_{11}$ inside $\Gamma^{2,k^2}(k^2)$ is isomorphic to $\Gamma^2(k^2)$. Similarly, the functor $\wedge^2$ is isomorphic to the $(2,0)$-weight space of $\wedge^{2,k^2}$ and in this case $\wedge^2(k^2)$ is isomorphic to the span of $E_{11} \wedge E_{12}$ inside $\wedge^{2,k^2}(k^2)$. (One could also use that Γ^2 (resp. $\wedge^2$) as $(0,2)$-weight space of Γ^{2,k^2} (resp. $\wedge^{2,k^2}$).) The copy of $\otimes^2$ occurs as $(1,1)$-weight space in the weight space decomposition of both the functors Γ^{2,k^2} and $\wedge^{2,k^2}$. The span of $E_{11} \otimes E_{21} + E_{21} \otimes E_{11}, E_{11} \otimes E_{22} + E_{22} \otimes E_{11}, E_{12} \otimes E_{21} + E_{21} \otimes E_{12}, E_{12} \otimes E_{22} + E_{22} \otimes E_{12}$ inside $\Gamma^{2,k^2}(k^2)$ is isomorphic to $\otimes^2(k^2)$. Likewise, $\otimes^2(k^2)$ is also isomorphic to the span of $(E_{11} \wedge E_{21})$, $(E_{11} \wedge E_{22})$, $(E_{12} \wedge E_{21})$, $(E_{12} \wedge E_{22})$ inside $\wedge^{2,k^2}(k^2)$.

The morphism $\triangle_1$ (resp. $\triangle_2$) therefore corresponds to an element of $\mathrm{End}_{\mathrm{Rep}\Gamma^2_k}(\Gamma^{2,k^2})$ (resp. $\mathrm{End}_{\mathrm{Rep}\Gamma^2_k}(\wedge^{2,k^2})$). By the Yoneda lemma $\mathrm{End}_{\mathrm{Rep}\Gamma^2_k}(\Gamma^{2,k^2}) \simeq \Gamma^{2,k^2}(k^2)$. Let $a = E_{11} \otimes E_{21} + E_{21} \otimes E_{11}$. The element a of $\Gamma^{2,k^2}(k^2)$ gives a morphism α from Γ^{2,k^2} to itself. Note that $\alpha(k^2)(id) = a$. Consider the following diagrams:

$$\begin{array}{ccc} \Gamma^2(k^2) & \xrightarrow{\theta_1} & \Gamma^{2,k^2}(k^2) \\ \triangle_1 (k^2)\big\downarrow & & \alpha(k^2)\big\downarrow \\ \otimes^2(k^2) & \xrightarrow{\theta_2} & \Gamma^{2,k^2}(k^2) \end{array} \quad (D_1) \qquad ; \qquad \begin{array}{ccc} \wedge^2(k^2) & \xrightarrow{\theta_1'} & \wedge^{2,k^2}(k^2) \\ \triangle_2 (k^2)\big\downarrow & & \beta\big\downarrow \\ \otimes^2(k^2) & \xrightarrow{\theta_2'} & \wedge^{2,k^2}(k^2) \end{array} \quad (D_2)$$

The map θ_1 is as follows:

$$\begin{aligned} e_1 \otimes e_1 &\mapsto E_{11} \otimes E_{11} \\ e_1 \otimes e_2 + e_2 \otimes e_1 &\mapsto E_{11} \otimes E_{12} + E_{12} \otimes E_{11} \\ e_2 \otimes e_2 &\mapsto E_{12} \otimes E_{12}. \end{aligned}$$

The map θ_2 is as follows:

$$\begin{aligned} e_1 \otimes e_1 &\mapsto E_{11} \otimes E_{21} + E_{21} \otimes E_{11} \\ e_1 \otimes e_2 &\mapsto E_{11} \otimes E_{22} + E_{22} \otimes E_{11} \\ e_2 \otimes e_1 &\mapsto E_{12} \otimes E_{21} + E_{21} \otimes E_{12} \\ e_2 \otimes e_2 &\mapsto E_{12} \otimes E_{22} + E_{21} \otimes E_{12}. \end{aligned}$$

The commutativity of the diagram (D_1) gives that restriction of α to Γ^2 is $\triangle_1$.

Now we proceed to show the commutativity of the diagram (D_2) to assert that $\wedge^2 \underline{\otimes}\ \triangle_1 = \triangle_2$. Since a is a morphism from k^2 to k^2 in $\Gamma^2 P_k$, it will induce a morphism $\beta := \wedge^{2,k^2}(a)$ from $\wedge^{2,k^2}(k^2)$ to $\wedge^{2,k^2}(k^2)$. The maps θ_1' and θ_2', respectively, are given by

$$e_1 \wedge e_2 \mapsto E_{11} \wedge E_{12} \text{ and}$$

$$\begin{aligned} e_1 \otimes e_1 &\mapsto E_{11} \wedge E_{21} \\ e_1 \otimes e_2 &\mapsto E_{11} \wedge E_{22} \\ e_2 \otimes e_1 &\mapsto E_{12} \wedge E_{21} \\ e_2 \otimes e_2 &\mapsto E_{12} \wedge E_{22}. \end{aligned}$$

The morphism β takes $E_{11} \wedge E_{12}$ to $E_{11} \wedge E_{22} + E_{12} \wedge E_{21} = E_{11} \wedge E_{12} - E_{21} \wedge E_{12}$. Now $\theta_2'(\triangle_2 (k^2))(e_1 \wedge e_2) = \theta_2'(e_1 \otimes e_2 - e_2 \otimes e_1) = E_{11} \wedge E_{12} - E_{12} \wedge E_{21}$.

Finally, in addition using that $(\wedge^d \underline{\otimes} -)$ is a right exact functor we get the required presentation (4.12). □

Proposition 4.16. *[KSS16, Proposition 3.2.5]* $\mathrm{Sym}^d \underline{\otimes} \mathrm{Sym}^d \simeq \mathrm{Sym}^d$. *If 2 is unit,* $\wedge^d \underline{\otimes} \mathrm{Sym}^d \simeq \wedge^d$. *If* $2 = 0$, $\wedge^d \underline{\otimes} \mathrm{Sym}^d \simeq \mathrm{Sym}^d$.

Proof. The strategy here is to use some appropriate presentations of objects and use the right exactness of $\underline{\otimes}$.
From the definition of Sym^d, we have a presentation

$$\bigoplus_{i=0}^{d-1} \otimes^d \xrightarrow{\oplus(1-\sigma_i)} \otimes^d \longrightarrow \mathrm{Sym}^d \longrightarrow 0,$$

where σ_i switches the i and $i+1$ tensor factors. To apply $\wedge^d \underline{\otimes} -$ we realize $\otimes^d$ as a summand of Γ^{d,k^d} via the exponential property (or (1^d)-weight space of Γ^{d,k^d}). Then σ_i can be realized as the restriction of the morphism τ_i in $\mathrm{End}_{\mathrm{Rep}\Gamma^d_k}(\Gamma^{d,k^d})$ that corresponds to $f_i^{\otimes d} \in \Gamma^d \mathrm{End}(k^d)$ where $f_i \in \mathrm{End}(k^d)$ switches the standard basis vectors e_i and e_{i+1} in k^d. Now $\wedge^d \underline{\otimes} \otimes^d \simeq \otimes^d$ by Lemma 4.13 and the exponential property of $\wedge^d$. The map $\wedge^d \underline{\otimes} \tau_i(U) \in \mathrm{End}\wedge^{d,k^d}(U)$ switches the occurrences of e_i^* and e_{i+1}^* in d-fold wedge products of vectors $e_j^* \otimes u \in (k^d)^* \otimes U$. Identifying $U^{\otimes d}$ inside $\wedge^d((k^d)^* \otimes U)$ gives $\wedge^d \underline{\otimes} \sigma_i = -\sigma_i$. The claims about $\wedge^d \underline{\otimes} \mathrm{Sym}^d$ follow from the resulting presentation

$$\bigoplus_{i=0}^{d-1} \otimes^d \xrightarrow{\oplus(1+\sigma_i)} \otimes^d \longrightarrow \wedge^d \underline{\otimes} \mathrm{Sym}^d \longrightarrow 0.$$

Applying $\wedge^d \underline{\otimes} -$ again gives a presentation of $\mathrm{Sym}^d \underline{\otimes} \mathrm{Sym}^d$, as $\wedge^d \underline{\otimes} \wedge^d \simeq \mathrm{Sym}^d$. But this is also the original presentation as the sign of σ_i switches once again. □

Remark 4.17. The internal Hom involving Γ^d, Sym^d and $\wedge^d$ were computed in [Tou13].

Remark 4.18. [KSS16, Remark 3.2.6] If 2 is a nonzero nonunit, then $\wedge^d \underline{\otimes} \mathrm{Sym}^d$ can be more complicated as can be seen when $k = \mathbb{Z}$ and $d = 2$. In this case we have the exact sequence $0 \to \bar{I} \to \wedge^2 \underline{\otimes} \mathrm{Sym}^2 \to \wedge^2 \to 0$, where $\bar{I}$ is the 2-torsion functor defined by $\bar{I}(V) = \mathbb{Z}$-span of $\{v \otimes v\}/\mathbb{Z}$-span of $\{v_1 \otimes v_2 + v_2 \otimes v_1\}$. (Let $f : V^{\otimes 2} \to W^{\otimes 2}$ be S_2-linear then $f(\mathbb{Z}$-span of $\{v_1 \otimes v_2 + v_2 \otimes v_1\})$ is contained in $\mathbb{Z}$-span of $\{w_1 \otimes w_2 + w_2 \otimes w_1\}$. So it is clear that $\bar{I}$ is a strict polynomial functor.) The functor $\bar{I} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ (see Definition 4.20 of the base change of strict polynomial functor) can be identified with the Frobenius twist, $I^{(1)}$, of the identity functor I over $k = \mathbb{Z}/2\mathbb{Z}$. In contrast, over any hereditary k, if functors X and Y take values in P_k then so does $\mathbb{H}(X, Y)$.

From Propositions 4.14 and 4.16, we have the following calculations where all direct sums are over the same set S as in Proposition 4.14.

Corollary 4.19. *[KSS16, Corollary 3.2.7] For* $\lambda \in \Lambda(m, d)$ *and* $\mu \in \Lambda(n, d)$, *we have*

$$\Gamma^\lambda \underline{\otimes} \wedge^\mu \simeq \oplus \wedge^\nu \quad , \quad \Gamma^\lambda \underline{\otimes} \mathrm{Sym}^\mu \simeq \wedge^\lambda \underline{\otimes} \wedge^\mu \simeq \mathrm{Sym}^\lambda \underline{\otimes} \mathrm{Sym}^\mu \simeq \oplus \mathrm{Sym}^\nu,$$

$$\mathrm{Sym}^{\lambda}\underline{\otimes}\wedge^{\mu} \simeq \oplus\wedge^{\nu} \text{ if 2 is a unit and } \mathrm{Sym}^{\lambda}\underline{\otimes}\wedge^{\mu} \simeq \oplus\mathrm{Sym}^{\nu} \text{ if } 2 = 0.$$

4.3 Higher derived internal structures

Let $D(\mathrm{Rep}\Gamma_k^d)$ be the unbounded derived category of $\mathrm{Rep}\Gamma_k^d$. Following [Kra13, Section 4], the bifunctor $(-\underline{\otimes}-)$ has the left derived functor which can be computed using K-projective resolution from [Spa88] in either slot. Similarly, the bi-functor $\mathbb{H}(-,-)$ has the right derived functor and can be computed using K-injective resolution from [Spa88] (resp. K-projective resolution) in the second slot (resp. first slot). It was proved that the left derived internal tensors is left adjoint to the right derived internal hom.

Proposition 4.20. *[Kra13, Proposition 4.1] The bifunctors $(-\underline{\otimes}-)$ and $\mathbb{H}(-,-)$ have derived functors*

$$-\overset{\mathbf{L}}{\underline{\otimes}}- : D(\mathrm{Rep}\Gamma_k^d) \times D(\mathrm{Rep}\Gamma_k^d) \to D(\mathrm{Rep}\Gamma_k^d),$$

$$\mathbf{R}\mathbb{H}(-,-) : D(\mathrm{Rep}\Gamma_k^d)^{op} \times D(\mathrm{Rep}\Gamma_k^d) \to D(\mathrm{Rep}\Gamma_k^d).$$

Moreover, for objects X, Y and Z in $D(\mathrm{Rep}\Gamma_k^d)$ there is a natural isomorphism

$$\mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}(X\overset{\mathbf{L}}{\underline{\otimes}}Y, Z) \simeq \mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}(X, \mathbf{R}\mathbb{H}(Y,Z)). \tag{4.13}$$

The adjunction (4.13) gives two other types of adjunctions involving $\overset{\mathrm{L}}{\underline{\otimes}}$ and $\mathrm{R}\mathbb{H}$. This is our next proposition.

Proposition 4.21. *Let X, Y and $Z \in D(\mathrm{Rep}\Gamma_k^d)$. Then the following are equivalent,*

1. $\mathbf{R}\mathbb{H}(X\overset{\mathbf{L}}{\underline{\otimes}}Y, Z) \simeq \mathbf{R}\mathbb{H}(X, \mathbf{R}\mathbb{H}(Y,Z))$,
2. $\mathbf{R}\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(X\overset{\mathbf{L}}{\underline{\otimes}}Y, Z) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(X, \mathbf{R}\mathbb{H}(Y,Z))$,
3. $\mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}(X\overset{\mathbf{L}}{\underline{\otimes}}Y, Z) \simeq \mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}(X, \mathbf{R}\mathbb{H}(Y,Z))$.

Proof. Since parametrization at an object V of $\Gamma^d P_k$ is an exact functor of $\mathrm{Rep}\Gamma_k^d$, it extends to a functor of $D(\mathrm{Rep}\Gamma_k^d)$. Any object of $D(\mathrm{Rep}\Gamma_k^d)$ can be evaluated at V. Notice that

$$\mathbf{R}\mathbb{H}(X,Y)(V) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(X, Y_V). \tag{4.14}$$

Assume that $\mathbf{R}\mathbb{H}(X\overset{\mathbf{L}}{\underline{\otimes}}Y, Z) \simeq \mathbf{R}\mathbb{H}(X, \mathbf{R}\mathbb{H}(Y,Z))$. By evaluating at k and using the isomorphism (4.14), we get $\mathbf{R}\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(X\overset{\mathbf{L}}{\underline{\otimes}}Y, Z) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(X, \mathbf{R}\mathbb{H}(Y,Z))$. This proves *1* implies 2.

Next we prove 2 implies 3. By applying the functor H^0 in the isomorphism of complexes $\mathbf{R}\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(X\overset{\mathbf{L}}{\underline{\otimes}}Y,Z) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(X,\mathbf{R}\mathbb{H}(Y,Z))$, we get

$$\mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}(X\overset{\mathbf{L}}{\underline{\otimes}}Y,Z) \simeq \mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}(X,\mathbf{R}\mathbb{H}(Y,Z)).$$

Finally, to prove 3 implies 1 we use the Yoneda lemma and the associativity of $(-\overset{\mathbf{L}}{\underline{\otimes}}-)$. Let $W \in D(\mathrm{Rep}\Gamma_k^d)$. Then

$$\begin{aligned}\mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}(W,\mathbf{R}\mathbb{H}(X\overset{\mathbf{L}}{\underline{\otimes}}Y,Z)) &\simeq \mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}(W\overset{\mathbf{L}}{\underline{\otimes}}(X\overset{\mathbf{L}}{\underline{\otimes}}Y)),Z)\\ &\simeq \mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}((W\overset{\mathbf{L}}{\underline{\otimes}}X)\overset{\mathbf{L}}{\underline{\otimes}}Y,Z)\\ &\simeq \mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}(W\overset{\mathbf{L}}{\underline{\otimes}}X,\mathbf{R}\mathbb{H}(Y,Z))\\ &\simeq \mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}(W,\mathbf{R}\mathbb{H}(X,\mathbf{R}\mathbb{H}(Y,Z))).\end{aligned}$$

□

The derived Koszul duality and the derived Ringel duality:

The derived Ringel dual functor $\mathbf{R}\mathbb{H}(\wedge^d,-)$ of strict polynomial functors was introduced by Chałupnik in [Cha05] and also studied by Touzé in [Tou13]. The derived Koszul dual functor $(\wedge^d\overset{\mathbf{L}}{\underline{\otimes}}-)$ was introduced by Krause. It turned out that the derived Koszul dual is quasi-inverse of the derived Ringel dual.

Theorem 4.22. *[Kra13, Theorem 4.9] The functor $(\wedge^d\overset{\mathbf{L}}{\underline{\otimes}}-) : D(\mathrm{Rep}\Gamma_k^d) \to D(\mathrm{Rep}\Gamma_k^d)$ is an equivalence and its quasi-inverse is the functor $\mathbf{R}\mathbb{H}(\wedge^d,-)$.*

In [Kra13, Proposition 4.16] the Koszul dual of a Weyl functor was given.

Proposition 4.23. *For λ a partition of d, we have $\wedge^d\overset{\mathbf{L}}{\underline{\otimes}}\Delta(\lambda) \simeq \nabla(\lambda')$ where λ' is the conjugate partition of λ. In particular, $\wedge^d\overset{\mathbf{L}}{\underline{\otimes}}\wedge^d \simeq \mathrm{Sym}^d$.*

Derived duals: The derived dual of the Kuhn dual was defined in [Kra13, Section 4]. Recall from Remark 4.10 the Kuhn dual is the internal hom functor $\mathbb{H}(-,\mathrm{Sym}^d)$. Therefore the derived Kuhn dual of a complex X of strict polynomial functors is $X^\diamond := \mathbf{R}\mathbb{H}(X,\mathrm{Sym}^d)$. For $X \in D(\mathrm{rep}\Gamma_k^d)$, we have $X^\diamond = X^\circ$.

Following arguments of Krause but slightly more general, we relate duality with $\mathbb{H}$ and $\underline{\otimes}$, compare [Kra13, Lemmas 2.7 and 2.8]. We also include a part of [Kra13, Lemma 4.5], which we need in Proposition 4.28. Note that the hypothesis on X in the last sentence of the following lemma is valid when X is a Weyl functor, see [AB88, Section 4].

Lemma 4.24. *[KSS16, Lemma 2.8.1] In* $\mathrm{Rep}\Gamma^d_k$, $\mathbb{H}(X, Y^\circ) \simeq (X\underline{\otimes}Y)^\circ \simeq \mathbb{H}(Y, X^\circ)$. *Then* $Y = \Gamma^d$ *gives* $X^\circ \simeq \mathbb{H}(X, \mathrm{Sym}^d)$. *In* $D(\mathrm{Rep}\Gamma^d_k)$, *we have* $X\overset{\mathbf{L}}{\underline{\otimes}}Y^\diamond \simeq \mathbf{R}\mathbb{H}(X, Y)^\diamond$ *when* $X \simeq$ *a bounded complex of finite direct sums of representable functors.*

Proof. By symmetry of $\underline{\otimes}$, it suffices to show $\mathbb{H}(X, Y^\circ) \simeq (X\underline{\otimes}Y)^\circ$. As both expressions turn colimits in the X variable into limits, it suffices to check this for $X = \Gamma^{d,W}$ which is immediate. The last statement follows from a direct calculation by using a projective resolution of Y. □

In general, one cannot move all occurrences of duality in Lemma 4.24 to one side due to a potentially bad behavior of duality outside $\mathrm{rep}\Gamma^d_k$. If Y is in $\mathrm{rep}\Gamma^d_k$, then $\mathbb{H}(X, Y) \simeq \mathbb{H}(X, Y^{\circ\circ}) \simeq (X\underline{\otimes}Y^\circ)^\circ$. If $X\underline{\otimes}Y$ is in $\mathrm{rep}\Gamma^d_k$, then $X\underline{\otimes}Y \simeq \mathbb{H}(X, Y^\circ)^\circ$ but for general k this can fail even if X and Y are in $\mathrm{rep}\Gamma^d_k$, see Remark 4.18.

We proceed to compute some of the higher derived internal tensor products. For the higher derived internal homs we refer to Chałupnik [Cha05] and Touzé [Tou13, Tou14].

Internal tensor product and Weyl functors:

The functor Γ^ν is a projective object in $\mathrm{Rep}\Gamma^d_k$ therefore $(-\overset{\mathbf{L}}{\underline{\otimes}}\Gamma^\nu) = (-\underline{\otimes}\Gamma^\nu)$. We show that $\Delta(\lambda)\underline{\otimes}\Gamma^\nu$ has an explicit Weyl filtration. We obtain parallel results for dual Weyl functors using Koszul duality. By contrast, the internal tensor product of two Weyl functors need not have a Weyl filtration e.g., $\wedge^d\underline{\otimes}\wedge^d \simeq \mathrm{Sym}^d$ by Proposition 4.15. However, we show that their higher derived internal tensor products do vanish which is not true for two dual Weyl functors.

Proposition 4.25. *[KSS16, Proposition 3.3.1]* $\Delta(\lambda)\underline{\otimes}\Gamma^\nu$ *has a Weyl filtration that is independent of the ground ring* k *and in which the multiplicity of any Weyl functor can be calculated as a sum of products of Littlewood-Richardson coefficients.*

Proof. More generally, we will give a precise description of $\Delta(\lambda/\mu)\underline{\otimes}\Gamma^\nu$ where $\Delta(\lambda/\mu)$ is the skew Weyl functor [ABW82] corresponding to partitions λ and μ with $\mu \subset \lambda$, i.e., the Young diagram of μ is contained in that of λ. Let $\nu = (\nu_1, \ldots, \nu_n) \in \Lambda(n, d)$. By Lemma 4.13,

$$(\Delta(\lambda/\mu)\underline{\otimes}\Gamma^\nu)(V) \simeq \Delta(\lambda/\mu)^\nu(V) \simeq \Delta(\lambda/\mu)(\mathrm{hom}(k^n, V))_\nu \simeq \Delta(\lambda/\mu)(V \oplus V \oplus \cdots \oplus V)_\nu. \quad (4.15)$$

Let U and W be free k-modules of finite rank. In [AB85], Akin and Buchsbaum give an explicit construction of a filtration of the skew Weyl module $\Delta(\lambda/\mu)(U \oplus W)$ that is universal (i.e., independent of the ground ring k) and functorial in U and W and whose associated graded object is

$$\bigoplus_{\substack{\text{partitions } \alpha \\ \mu\subset\alpha\subset\lambda}} \Delta(\alpha/\mu)(U) \otimes \Delta(\lambda/\alpha)(W). \quad (4.16)$$

We use (4.16) to first calculate $\Delta(\lambda/\mu)\underline{\otimes}\Gamma^{d,k^n} \simeq \Delta(\lambda/\mu)^{k^n}$. Taking V to be free of finite rank and using (4.16) repeatedly gives a filtration of $\Delta(\lambda/\mu)\underline{\otimes}\Gamma^{d,k^n}(V) \simeq \Delta(\lambda/\mu)(V \oplus V \oplus \cdots \oplus V)$. As this description is true for any free module V of finite rank, the same description is valid in $\mathrm{Rep}\Gamma^d_k$ by the equivalence (4.7). Altogether, we get a filtration of $\Delta(\lambda/\mu)\underline{\otimes}\Gamma^{d,k^n}$ whose associated graded object is

$$\bigoplus_{\substack{\text{partitions } \alpha^1,\alpha^2,\ldots,\alpha^{n-1} \\ \mu\subset\alpha^1\subset\alpha^2\subset\cdots\subset\alpha^{n-1}\subset\lambda}} \Delta(\alpha^1/\mu) \otimes \Delta(\alpha^2/\alpha^1) \otimes \cdots \otimes \Delta(\lambda/\alpha^{n-1}). \tag{4.17}$$

Taking the ν-weight space in (4.15) is equivalent to requiring $|\alpha^i/\alpha^{i-1}| = \nu_i$ for all $i = 1, \ldots, n$ in (4.17), with the understanding that $\alpha^0 = \mu$ and $\alpha^n = \lambda$. Therefore we get a filtration of $\Delta(\lambda/\mu)\underline{\otimes}\Gamma^\nu$ whose associated graded object is

$$\bigoplus_{\substack{\text{partitions } \alpha^0\subset\alpha^1\subset\alpha^2\subset\cdots\subset\alpha^{n-1}\subset\alpha^n \\ \alpha^0=\mu,\alpha^n=\lambda,|\alpha^i/\alpha^{i-1}|=\nu_i}} \Delta(\alpha^1/\mu) \otimes \Delta(\alpha^2/\alpha^1) \otimes \cdots \otimes \Delta(\lambda/\alpha^{n-1}). \tag{4.18}$$

Finally note that each tensor product in (4.18) itself has a Weyl filtration in which the multiplicity of any $\Delta(\beta)$ can be calculated as a sum of products of Littlewood-Richardson coefficients. This is because $\Delta(\alpha^i/\alpha^{i-1})$ has a Weyl filtration in which the multiplicity of $\Delta(\beta)$ equals the Littlewood-Richardson coefficient $c^{\alpha^i}_{\alpha^{i-1}\beta}$, by ([Kou91, Theorem 2.6] or contravariant dual of [Bof89, Theorem 1.3]). Applying this in all tensor slots leads to a filtration whose successive quotients are n-fold tensor products of Weyl functors. These in turn have Weyl filtrations with multiplicities given by products of Littlewood-Richardson coefficients. □

Corollary 4.26. *[KSS16, Corollary 3.3.2] $\Delta(\lambda)\underline{\otimes}\wedge^\nu \simeq \nabla(\lambda')\underline{\otimes}\Gamma^\nu$ has a dual Weyl filtration that is independent of the ground ring k and in which the multiplicity of any dual Weyl functor can be calculated as a sum of products of Littlewood-Richardson coefficients.*

Proof. Apply $-\overset{\mathbf{L}}{\underline{\otimes}}\wedge^d$ to Proposition 4.25. By [Kra13, Propositions 3.4, 4.16] we have, respectively, $\wedge^d\overset{\mathbf{L}}{\underline{\otimes}}\Gamma^\nu \simeq \wedge^\nu$ (or see Lemma 4.13) and $\wedge^d\overset{\mathbf{L}}{\underline{\otimes}}\Delta(\lambda) \simeq \nabla(\lambda')$. We calculate

$$\Delta(\lambda)\overset{\mathbf{L}}{\underline{\otimes}}\wedge^\nu \simeq \Delta(\lambda)\overset{\mathbf{L}}{\underline{\otimes}}\wedge^d\overset{\mathbf{L}}{\underline{\otimes}}\Gamma^\nu \simeq \nabla(\lambda')\overset{\mathbf{L}}{\underline{\otimes}}\Gamma^\nu$$

and remark that applying $\wedge^d\underline{\otimes}-$ turns a Weyl filtration into a dual Weyl filtration, because a functor with a Weyl filtration is acyclic for the left exact functor $\wedge^d\underline{\otimes}-$. □

When k is a PID, the property of having a finite (dual) Weyl filtration passes to finite direct sums and to summands via the Ext^1 vanishing criterion [Jan03, Lemmas II.B.9,10]. As any projective object in the $\mathrm{Rep}\Gamma_k^d$ is a summand of a direct sum of Γ^{d,k^n}, we get from Proposition 4.25 and Corollary 4.26.

Corollary 4.27. *[KSS16, Corollary 3.3.3] When k is a PID, the internal tensor product of a (dual) Weyl functor and a finitely generated projective object in $\mathrm{Rep}\Gamma_k^d$ has a (dual) Weyl filtration.*

Recall from Proposition 4.23 that $\wedge^d \underline{\otimes}^{\mathbf{L}} \Delta(\lambda) \simeq \nabla(\lambda')$ this implies in particular that higher derived internal tensor products of $\wedge^d$ and a Weyl functor vanish. More generally we show that even though $\Delta(\lambda)\underline{\otimes}\Delta(\mu)$ is hard to compute, it is in $\mathrm{rep}\Gamma_k^d$ and the corresponding higher derived internal tensor products always vanish.

Proposition 4.28. *[KSS16, Proposition 3.3.5] Let $\Delta(\lambda)$ and $\Delta(\mu)$ be Weyl functors corresponding to partitions λ and μ of d. Then $H^i(\Delta(\lambda)\underline{\otimes}^{\mathbf{L}}\Delta(\mu)) = 0$ for $i \neq 0$ and $\Delta(\lambda)\underline{\otimes}\Delta(\mu)$ is in $\mathrm{rep}\Gamma_k^d$.*

Proof. In the context of derived functors, single objects will be considered as complexes concentrated in degree 0. We will use the derived duality $(-)^\diamond$ from page no. 41. We have $(\nabla(\mu))^\diamond \simeq (\nabla(\mu))^\circ \simeq \Delta(\mu)$ where the first isomorphism is due to $\nabla(\mu) \in D(\mathrm{Rep}\Gamma_k^d)$. As $\Delta(\lambda)$ has a finite projective resolution by finite direct sums of various Γ^ν by [AB88, Section 4], we may use [Kra13, Lemma 4.5] (see Lemma 4.24) to get

$$\Delta(\lambda)\underline{\otimes}^{\mathbf{L}}\Delta(\mu) \simeq \mathbf{R}\mathbb{H}(\Delta(\lambda), \nabla(\mu))^\diamond.$$

It is enough to prove two claims.

1. $H^i(\mathbf{R}\mathbb{H}(\Delta(\lambda), \nabla(\mu))) = 0$ for $i \neq 0$ which would give $\mathbf{R}\mathbb{H}(\Delta(\lambda), \nabla(\mu)) \simeq \mathbb{H}(\Delta(\lambda), \nabla(\mu))$.
2. $\mathbb{H}(\Delta(\lambda), \nabla(\mu))$ is in $\mathrm{rep}\Gamma_k^d$ which would give $\mathbb{H}(\Delta(\lambda), \nabla(\mu))^\diamond \simeq \mathbb{H}(\Delta(\lambda), \nabla(\mu))^\circ$.

To prove the first claim we use the equivalence in Theorem 4.7 and the isomorphism (4.14) to calculate

$$H^i(\mathbf{R}\mathbb{H}(\Delta(\lambda), \nabla(\mu))(k^d)) \simeq \mathrm{Ext}^i_{\mathrm{Rep}\Gamma_k^d}(\Delta(\lambda)^{k^d}, \nabla(\mu)).$$

These Ext^i vanish for $i \neq 0$ by Proposition 2.22 because, by the discussion in Proposition 4.25, $\Delta(\lambda)^{k^d}$ has a Weyl filtration. For the second claim, by item 1 (in the application of equivalence 4.7) at page no. 33 it is enough to prove that $\mathbb{H}(\Delta(\lambda), \nabla(\mu))(k^d)$ is in P_k. We have

$$\mathbb{H}(\Delta(\lambda), \nabla(\mu))(k^d) \simeq \mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(\Delta(\lambda)^{k^d}, \nabla(\mu)) \simeq \mathrm{Hom}_{S_k(d,d)}(\Delta(\lambda)^{k^d}(k^d), \nabla(\mu)(k^d))$$

which is seen to be free of finite rank over k as follows. When $k = \mathbb{Z}$ it is a subgroup of $\mathrm{Hom}_{\mathbb{Z}}(\Delta(\lambda)^{\mathbb{Z}^d}(\mathbb{Z}^d), \nabla(\mu)(\mathbb{Z}^d))$ which is a free abelian group of finite rank. Now change base to k [AB88, Theorem 5.3] and use vanishing of Ext^1 resulting from Proposition 2.22. The two claims together prove the result. □

Remark 4.29. [KSS16, Remark 3.3.6] Proposition 4.28 is not true for dual Weyl functors. For example, let k be of characteristic 2. Applying $(-\underline{\otimes}\mathrm{Sym}^2)$ to the projective resolution $0 \to \Gamma^2 \to \otimes^2 \to \wedge^2 \to 0$ of $\wedge^2$, we get from Proposition 4.16 the sequence $0 \to \mathrm{Sym}^2 \to \otimes^2 \to \mathrm{Sym}^2 \to 0$. It follows that $H^{-1}(\wedge^2 \overset{\mathbf{L}}{\underline{\otimes}} \mathrm{Sym}^2) \simeq I^{(1)}$, the Frobenius twist of the identity functor.

4.3.1 Base change and higher derived internal tensors

In this section, we obtain a form of the universal coefficient theorem for strict polynomial functors of degree d. The strict polynomial functors $\Gamma^d_{\mathbb{Z}}$ and Γ^d_k are the divided power functors over $\mathbb{Z}$ and k respectively (not to be confused with *parametrization* of strict polynomial functors). Let $\mathbb{Z} \to k$ be the canonical ring homomorphism and V be an object in $P_{\mathbb{Z}}$. We have the following base change for the divided powers

$$\Gamma^d_k(V) \underset{\mathbb{Z}}{\otimes} k \simeq \Gamma^d_k(V \underset{\mathbb{Z}}{\otimes} k) \tag{4.19}$$

When $V = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^n)$, the isomorphism (4.19) in particular gives $S_{\mathbb{Z}}(n,d) \underset{\mathbb{Z}}{\otimes} k \simeq S_k(n,d)$. Using the equivalence (4.7) between the category of strict polynomial functors of degree d and the category of modules over the Schur algebra one defines a base change for strict polynomial functors of degree d.

Using the base change isomorphism for Schur algebra, we can define a *base change functor*

$$(-\underset{\mathbb{Z}}{\otimes} k) : \mathrm{Rep}\Gamma^d_{\mathbb{Z}} \to \mathrm{Rep}\Gamma^d_k \quad \text{which is given by} \tag{4.20}$$

$$X \underset{\mathbb{Z}}{\otimes} k = \Gamma^{d,k^d} \otimes_{S_k(d,d)} (X(\mathbb{Z}^d) \underset{\mathbb{Z}}{\otimes} k).$$

Notice that

$$\Gamma^{d,V}_{\mathbb{Z}} \underset{\mathbb{Z}}{\otimes} k \simeq \Gamma_k^{d,V \underset{\mathbb{Z}}{\otimes} k}. \tag{4.21}$$

Lemma 4.30. *For objects X and Y in $\mathrm{Rep}\Gamma^d_{\mathbb{Z}}$, we have*

$$(X \underline{\otimes} Y) \underset{\mathbb{Z}}{\otimes} k \simeq (X \underset{\mathbb{Z}}{\otimes} k) \underline{\otimes} (Y \underset{\mathbb{Z}}{\otimes} k). \tag{4.22}$$

Proof. First take $X = \Gamma^{d,\mathbb{Z}^m}_{\mathbb{Z}}$ and $Y = \Gamma^{d,\mathbb{Z}^n}_{\mathbb{Z}}$. The left hand side of (4.22) is

$$\begin{aligned}(\Gamma^{d,\mathbb{Z}^m}_{\mathbb{Z}} \underline{\otimes} \Gamma^{d,\mathbb{Z}^n}_{\mathbb{Z}}) \underset{\mathbb{Z}}{\otimes} k &= \Gamma_{\mathbb{Z}}^{d,\mathbb{Z}^m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}^n} \underset{\mathbb{Z}}{\otimes} k \\ &= \Gamma_k^{d,(\mathbb{Z}^m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}^n) \underset{\mathbb{Z}}{\otimes} k}.\end{aligned}$$

The right hand side of (4.22) is

$$\begin{aligned}(\Gamma^{d,\mathbb{Z}^m}_{\mathbb{Z}} \underset{\mathbb{Z}}{\otimes} k)\underline{\otimes}(\Gamma^{d,\mathbb{Z}^n}_{\mathbb{Z}} \underset{\mathbb{Z}}{\otimes} k) &= \Gamma_k^{d,\mathbb{Z}^m \underset{\mathbb{Z}}{\otimes} k} \underline{\otimes} \Gamma_k^{d,\mathbb{Z}^n \underset{\mathbb{Z}}{\otimes} k} \\ &= \Gamma_k^{d,k^m} \underline{\otimes} \Gamma_k^{d,k^n} \\ &= \Gamma_k^{d,k^m \underset{k}{\otimes} k^n}.\end{aligned}$$

For arbitrary X and Y we express them as colimits of $\Gamma^{d,\mathbb{Z}^m}$ and use that the functors $\underline{\otimes}$ and the base change functor preserves colimit.

□

The following proposition tells us the behavior of the higher derived internal tensors with the base change functor (4.20).

Proposition 4.31. *For* X *and* Y *in* $\mathrm{rep}\Gamma^d_{\mathbb{Z}}$*. Then we have following short exact sequence of strict polynomial functors over* k :

$$0 \to H_n(X\overset{\mathbf{L}}{\underline{\otimes}} Y) \underset{\mathbb{Z}}{\otimes} k \to H_n((X \underset{\mathbb{Z}}{\otimes} k)\overset{\mathbf{L}}{\underline{\otimes}}(Y \underset{\mathbb{Z}}{\otimes} k)) \to \Gamma_k^{d,k^d} \otimes_{S_k(d,d)} \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^d), k) \to 0$$

(To see how $\mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^d), k)$ is a $S_k(d,d)$-module we refer Remark 4.32.)

Proof. Let $P_\bullet$ be a projective resolution of X over $\mathbb{Z}$. Note that terms of $P_\bullet$ are finite direct sums of $\Gamma^{d,\mathbb{Z}^d}_{\mathbb{Z}}$. Let $C_\bullet$ be the complex $P_\bullet \underline{\otimes} Y$ and we denote the boundary maps by $\{\mathbf{d}_i\}$.

For every $n \geqslant 1$, we have the short exact sequence of strict polynomial functors over $\mathbb{Z}$

$$0 \to \mathrm{Ker}\ \mathbf{d}_n \to C_n \to \mathrm{Im}\ \mathbf{d}_n \to 0.$$

Evaluating at $\mathbb{Z}^d$ gives $0 \to \mathrm{Ker}\ \mathbf{d}_n(\mathbb{Z}^d) \to C_n(\mathbb{Z}^d) \to \mathrm{Im}\ \mathbf{d}_n(\mathbb{Z}^d) \to 0$ is exact. On applying $(- \underset{\mathbb{Z}}{\otimes} k)$, we get the complex

$$0 \to \mathrm{Ker}\ \mathbf{d}_n(\mathbb{Z}^d) \underset{\mathbb{Z}}{\otimes} k \to C_n(\mathbb{Z}^d) \underset{\mathbb{Z}}{\otimes} k \to \mathrm{Im}\ \mathbf{d}_n(\mathbb{Z}^d) \underset{\mathbb{Z}}{\otimes} k \to 0. \tag{4.23}$$

This complex is also exact as $\mathrm{Im}\ \mathbf{d}_n(\mathbb{Z}^d)$ is projective over $\mathbb{Z}$ so $\mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{Im}\ \mathbf{d}_n(\mathbb{Z}^d), k)$ is 0. The short exact sequence (4.23) and the definition of base change of strict polynomial functor together with the equivalence (4.7) gives the complex

$$0 \to \mathrm{Ker}\ \mathbf{d}_n \underset{\mathbb{Z}}{\otimes} k \to C_n \underset{\mathbb{Z}}{\otimes} k \to \mathrm{Im}\ \mathbf{d}_n \underset{\mathbb{Z}}{\otimes} k \to 0, \text{ for every } n \geqslant 1, \tag{4.24}$$

is exact. Let Ker $\mathbf{d}$ and Im $\mathbf{d}$ be the chain sub-complexes of $C_\bullet$ with the zero boundary maps. The short exact sequences in (4.24) assemble to give short exact of chain complexes.

$$0 \to \text{Ker } \mathbf{d} \underset{\mathbb{Z}}{\otimes} k \to C_\bullet \underset{\mathbb{Z}}{\otimes} k \to \text{Im } \mathbf{d} \underset{\mathbb{Z}}{\otimes} k \to 0 \tag{4.25}$$

So for every $n \geqslant 1$, we get an exact sequence of homologies

$$H_{n+1}(\text{Im } \mathbf{d} \underset{\mathbb{Z}}{\otimes} k) \xrightarrow{\delta_n} H_n(\text{Ker } \mathbf{d} \underset{\mathbb{Z}}{\otimes} k) \to H_n(C_\bullet \underset{\mathbb{Z}}{\otimes} k) \to H_n(\text{Im } \mathbf{d} \underset{\mathbb{Z}}{\otimes} k) \xrightarrow{\delta_{n-1}} H_{n-1}(\text{Ker } \mathbf{d} \underset{\mathbb{Z}}{\otimes} k) \tag{4.26}$$

Since the boundary maps of both the chain complexes Ker $\mathbf{d}$ and Im $\mathbf{d}$ are zero, the exact sequence (4.26) becomes

$$\text{Im } \mathbf{d}_{n+1} \underset{\mathbb{Z}}{\otimes} k \xrightarrow{\delta_n} \text{Ker } \mathbf{d}_n \underset{\mathbb{Z}}{\otimes} k \xrightarrow{\psi} H_n(C_\bullet \underset{\mathbb{Z}}{\otimes} k) \xrightarrow{\theta} \text{Im } \mathbf{d}_n \underset{\mathbb{Z}}{\otimes} k \xrightarrow{\delta_{n-1}} \text{Ker } \mathbf{d}_{n-1} \underset{\mathbb{Z}}{\otimes} k \tag{4.27}$$

The exact sequence (4.27) give rise to the short exact sequence

$$0 \to \text{Ker } \theta \to H_n(C_\bullet \underset{\mathbb{Z}}{\otimes} k) \to \text{Im } \theta \to 0 \tag{4.28}$$

In the next discussion, we will simplify each term of the sequence (4.28). By isomorphism (4.21), each term of $P_\bullet \underset{\mathbb{Z}}{\otimes} k$ is a projective object of $\text{rep}\Gamma^d_k$. Evaluating at k^d we can check that $P_\bullet \underset{\mathbb{Z}}{\otimes} k$ is exact thus $P_\bullet \underset{\mathbb{Z}}{\otimes} k$ is a projective resolution of $X \underset{\mathbb{Z}}{\otimes} k$. By Lemma 4.30 the complex $(P_\bullet \underset{\mathbb{Z}}{\otimes} k)\underline{\otimes}(Y \underset{\mathbb{Z}}{\otimes} k)$ is isomorphic to the complex $C_\bullet \underset{\mathbb{Z}}{\otimes} k$. We have

$$H_n(C_\bullet \underset{\mathbb{Z}}{\otimes} k) \simeq H_n((X \underset{\mathbb{Z}}{\otimes} k)\overset{\mathbf{L}}{\underline{\otimes}}(Y \underset{\mathbb{Z}}{\otimes} k)) \tag{4.29}$$

Now $\text{Im } \theta = \text{Ker } \delta_{n-1}$ and $\text{Ker } \theta = \text{Im } \psi \simeq \text{Ker } \mathbf{d}_n \underset{\mathbb{Z}}{\otimes} k / \text{Ker } \psi = \text{Ker } \mathbf{d}_n \underset{\mathbb{Z}}{\otimes} k / \text{Im } \delta_n$. By definition of connecting homomorphism $\delta_j = i_j \underset{\mathbb{Z}}{\otimes} k$, where i_j is the inclusion of Im $\mathbf{d}_{j+1}$ inside Ker $\mathbf{d}_j$. Since the j^{th} homologies of the complex $C_\bullet$ computes $H_j(X\overset{\mathbf{L}}{\underline{\otimes}}Y)$, we exact sequence

$$0 \to \text{Im } \mathbf{d}_{j+1} \xrightarrow{i_j} \text{Ker } \mathbf{d}_j \to H_j(X\overset{\mathbf{L}}{\underline{\otimes}}Y) \to 0. \tag{4.30}$$

The objects Im $\mathbf{d}_{j+1}$ and Ker $\mathbf{d}_j$ are acyclic with respect to the base change functor $(- \underset{\mathbb{Z}}{\otimes} k)$, for every $j \geqslant 1$. Thus the homologies of the complex $0 \to \text{Im } \mathbf{d}_j \underset{\mathbb{Z}}{\otimes} k \xrightarrow{\delta_j} \text{Ker } \mathbf{d}_j \underset{\mathbb{Z}}{\otimes} k \to 0$ computes $\Gamma^{d,k^d} \otimes_{S_k(d,d)} \text{Tor}^{\mathbb{Z}}_l(H_j(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^d), k)$ for $l \geqslant 0$.

Remark 4.32. Notice that $\text{Tor}^{\mathbb{Z}}_l(H_j(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^d), k)$ being quotient of $S_k(d,d)$-modules is again a $S_k(d,d)$-module.

When $j = (n-1)$, Ker δ_{n-1} is $\Gamma^{d,k^d} \otimes_{S_k(d,d)} \text{Tor}^{\mathbb{Z}}_1(H_{n-1}(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^d), k)$. When $j = n$,

$\operatorname{Ker}\theta = \operatorname{Im}\psi$ is $\Gamma^{d,k^d} \otimes_{S_k(d,d)} \operatorname{Tor}_0^{\mathbb{Z}}(H_n(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^d), k)$, which is precisely $H_n(X\overset{\mathbf{L}}{\underline{\otimes}}Y) \underset{\mathbb{Z}}{\otimes} k$. Finally, we substitute the simplified terms in (4.28) to get the short exact sequence

$$0 \to H_n(X\overset{\mathbf{L}}{\underline{\otimes}}\, Y) \underset{\mathbb{Z}}{\otimes} k \to H_n((X \underset{\mathbb{Z}}{\otimes} k)\overset{\mathbf{L}}{\underline{\otimes}}(Y \underset{\mathbb{Z}}{\otimes} k)) \to \Gamma_k^{d,k^d} \otimes_{S_k(d,d)} \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^d), k) \to 0.$$

□

Alternate proof: Let X and Y be objects in $\operatorname{rep}\Gamma_{\mathbb{Z}}^d$. Let $P_X^\bullet$ and $P_Y^\bullet$ be the projective resolutions of X and Y respectively. Then

$$(P_X^\bullet \underline{\otimes} P_Y^\bullet) \underset{\mathbb{Z}}{\otimes} k \simeq ((P_X^\bullet \underset{\mathbb{Z}}{\otimes} k)\underline{\otimes}(P_Y^\bullet \underset{\mathbb{Z}}{\otimes} k)) \simeq P^\bullet_{X\underset{\mathbb{Z}}{\otimes}k} \underline{\otimes} P^\bullet_{Y\underset{\mathbb{Z}}{\otimes}k}.$$

Here the first isomorphism follows from Lemma 4.30 and the second isomorphism follows from the equation (4.21). Then applying the Künneth formula, for eg. see [Wei94, Theorem 3.6.1] we get the desired exact sequence in Proposition 4.31.

Corollary 4.33. *When X and Y are in $\operatorname{rep}\Gamma_{\mathbb{Z}}^d$, the abelian groups $H_n(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^m)$ are finite for every m and $n \in \mathbb{N}$.*

Proof. By taking $k = \mathbb{Q}$ in Proposition 4.31 and using that the higher derived internal tensor vanish over $\mathbb{Q}$, for $n \geqslant 1$, we get

$$H_n(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^d) \otimes \mathbb{Q} = 0. \tag{4.31}$$

From the definition of higher derived internal tensor functors of two finitely generated strict polynomial functors is again finitely generated. In particular, $H_n(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^d)$ is finitely generated $\mathbb{Z}$-module. Using this and the equation (4.31), we get $H_n(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^d)$ is a torsion $\mathbb{Z}$-module. Since a finitely generated torsion $\mathbb{Z}$-module is a finite abelian group therefore $H_n(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^d)$ is a finite abelian group. For arbitrary $m \geqslant 1$, using item 2 (in applications to equivalence) at page number 33, we deduce that $H_n(X\overset{\mathbf{L}}{\underline{\otimes}}Y)(\mathbb{Z}^m)$ is a finite abelian group. □

4.4 The Schur functor

For $n \geqslant d$, we recall from Section 3.2 the Schur functor and its adjoints

1. $\operatorname{Sch} : S_k(n,d)\text{-Mod} \to kS_d\text{-Mod}$ is $\operatorname{Hom}_{S_k(n,d)}((k^n)^{\otimes d}, -)$.
2. $\mathcal{L} : kS_d\text{-Mod} \to S_k(n,d)\text{-Mod}$ and $\mathcal{R} : kS_d\text{-Mod} \to S_k(n,d)\text{-Mod}$, for M in $kS_d\text{-Mod}$, are given by $\mathcal{L}(M) = ((k^n)^{\otimes d} \otimes_{kS_d} -)$ and $\mathcal{R} = \operatorname{Hom}_{kS_d}(((k^n)^{\otimes d})^*, -)$.

(Recall that $(k^n)^{\otimes d}$ is a right kS_d-module. So $((k^n)^{\otimes d})^*$ is a left kS_d-module.) Using the equivalence (4.7), the Schur functor becomes a functor from $\mathrm{Rep}\Gamma^d_k$ to kS_d-Mod and its adjoints become functors from kS_d-Mod $\to \mathrm{Rep}\Gamma^d_k$ (by abuse of notations we denote these functors also by Sch, $\mathcal{L}$ and $\mathcal{R}$). Explicitly,

1. $\mathrm{Sch} : \mathrm{Rep}\Gamma^d_k \to kS_d\text{-Mod}$ is $\mathrm{Hom}_{\mathrm{Rep}\Gamma^d_k}(\otimes^d, -)$.
2. $\mathcal{L} : kS_d\text{-Mod} \to \mathrm{Rep}\Gamma^d_k$ and $\mathcal{R} : kS_d\text{-Mod} \to \mathrm{Rep}\Gamma^d_k$ respectively are given by

$$\mathcal{L}(M) = (\otimes^d \otimes_{kS_d} M) \text{ and } \mathcal{R}(M) = \mathrm{Hom}_{kS_d}((\otimes^d)^*, M).$$

   where M is in kS_d-Mod.

Using the isomorphism (3.6), for N in kS_d-Mod, we have

$$\mathrm{Sch}(\mathcal{L}(N)) \simeq N \text{ and } \mathrm{Sch}(\mathcal{R}(N)) \simeq N. \tag{4.32}$$

Recall from the decomposition (4.4), the functor $\otimes^d$ is a direct summand of the representable functor Γ^{d,k^d}. Thus $\otimes^d$ is a projective object in $\mathrm{Rep}\Gamma^d_k$ and so Sch is an exact functor.

Remark 4.34. When $d!$ is a unit in k (e.g., when k is a field of characteristic 0 or $p > d$), Sch is well known to be an equivalence of categories. This is because, for any λ in $\Lambda(n, d)$, the appropriate symmetrization map $\otimes^d \to \Gamma^\lambda$ is surjective. Thus $\otimes^d$ itself is a projective generator for $\mathrm{Rep}\Gamma^d_k$.

Chapter 5

Compatibility of the internal tensor product with the Kronecker product

It is well known that the external tensor product of objects $X_1 \in \mathrm{Rep}\Gamma_k^d$ and $X_2 \in \mathrm{Rep}\Gamma_k^e$ corresponds via a suitable Schur functor to inducing the representation $\mathrm{Sch}(X_1) \otimes \mathrm{Sch}(X_2)$ of $S_d \times S_e$ to S_{d+e}. Recall from Chapter 4 that there is an internal tensor product and an internal Hom on $\mathrm{Rep}\Gamma_k^d$. In this chapter, we show that the internal tensor product (resp. internal Hom) via the Schur functor corresponds to the Kronecker product (resp. Kronecker hom) in kS_d-Mod (Theorem 5.3). We also show that the left adjoint (resp. right adjoint) of the Schur functor takes the Kronecker product (resp. Kronecker Hom) to the internal tensor product (resp. internal Hom) (Corollary 5.5). In addition, we also observe that the right adjoint functor is a lax monoidal functor (Corollary 5.6).

Moreover, we show that the results about the preservation of these internal structures under the Schur functor and its adjoints remain true when we consider their appropriate derived functors on the unbounded derived categories (Theorem 5.9 and Corollary 5.11).

In characteristic 0, we give an application to the known Kronecker multiplicities of simple kS_d-modules when one of the partitions is a two row partition or a hook (Section 5.4).

5.1 The Schur functor is closed monoidal

We first compute Sch of some important strict polynomial functors.

Proposition 5.1. *[KSS16, Proposition 3.1.1] We have* $\mathrm{Sch}(\Gamma^{d,V}) \simeq (V^{\otimes d})^*$ *as left* kS_d*-modules. Here* S_d *acts on* $V^{\otimes d}$ *by permuting the tensor factors. Moreover, this isomorphism is functorial in the variable* V *in* $\Gamma^d P_k$.

Proof. Let $e_1, e_2, \dots, e_d$ be a basis of the vector space k^d. We identify $V^* \otimes k^d$ with $V_1 \oplus V_2 \oplus \cdots \oplus V_d$ where $V_i = V^* \otimes ke_i \simeq V^*$. As $\otimes^d = \Gamma^{1^d}$, from the isomorphism (4.5) $\mathrm{Sch}(X) =$

$\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(\Gamma^{1^d}, X) \simeq$ the 1^d-weight space of $X(k^d)$. Using these identifications we get

$$\begin{aligned} \mathrm{Sch}(\Gamma^{d,V}) &= \mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(\otimes^d, \Gamma^{d,V}) \\ &\simeq 1^d\text{-weight space of the } GL(k^d)\text{-module } \Gamma^d(V^* \otimes k^d) \\ &\simeq \Gamma^d(V_1 \oplus V_2 \oplus \cdots \oplus V_d)_{1^d}. \end{aligned}$$

By the exponential property $\Gamma(U \oplus W) \simeq \Gamma(U) \otimes \Gamma(W)$, we have

$$\Gamma^d(V_1 \oplus V_2 \oplus \cdots \oplus V_d) \simeq \bigoplus_{\lambda \in \Lambda(d,d)} \Gamma^{\lambda_1}(V_1) \otimes \cdots \otimes \Gamma^{\lambda_d}(V_d).$$

In this decomposition, $\Gamma^{\lambda_1}(V_1) \otimes \Gamma^{\lambda_2}(V_2) \otimes \cdots \otimes \Gamma^{\lambda_d}(V_d)$ is the $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$-weight space of the left $GL(k^d)$-module $\Gamma^d(V^* \otimes k^d)$. So it follows from a fact that the $1^d = (1, 1, \ldots, 1)$-weight space is a left kS_d-module. As we require the $1^d = (1, 1, \ldots, 1)$-weight space, we have $\mathrm{Sch}(\Gamma^{d,V}) \simeq V_1 \otimes \cdots \otimes V_d \simeq (V^*)^{\otimes d}$. Here the left action of kS_d on $(V^*)^{\otimes d}$ is given as follows, for $\sigma \in S_d$ and $f_1 \otimes \cdots \otimes f_d \in (V^*)^{\otimes d}$, $\sigma.(f_1 \otimes \cdots \otimes f_d) = (f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(d)})$.

The action of kS_d on $V^{\otimes d}$ given by permuting the tensor factors is a right action and so $(V^{\otimes d})^*$ is a left kS_d-module. Moreover, notice that the canonical k-linear isomorphism between left kS_d-modules $(V^*)^{\otimes d}$ and $(V^{\otimes d})^*$ is kS_d-linear.

One checks that the right action of $\mathrm{End}_{\mathrm{Rep}\Gamma_k^d}(\otimes^d)$ by precomposition on $\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(\otimes^d, \Gamma^{d,V})$, after applying the isomorphisms with $(kS_d)^{\mathrm{op}}$ and $(V^*)^{\otimes d} \simeq (V^{\otimes d})^*$ respectively, translates into the claimed left kS_d-action.

It is easy to observe that a morphism from W to V in $\Gamma^d P_k$ induces a morphism from $\Gamma^{d,V}$ to $\Gamma^{d,W}$ in $\mathrm{Rep}\Gamma_k^d$ and a morphism from $(V^{\otimes d})^*$ to $(W^{\otimes d})^*$ in kS_d-Mod. Recall that $\mathrm{Sch} = \mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(\otimes^d, -)$. Therefore from Lemma 4.6 and the fact that the process of taking λ-weight space is functorial, we get the isomorphism $\mathrm{Sch}(\Gamma^{d,V}) \simeq (V^{\otimes d})^*$ is functorial in the variable V in $\Gamma^d P_k$.

□

Remark 5.2. Using the exponential property for the symmetric algebra in the place of divided power algebra in the proof above proposition, we easily conclude that $\mathrm{Sch}(\mathrm{Sym}^{d,V}) \simeq (V^{\otimes d})^*$. However, the exterior algebra being graded commutative, we have $\mathrm{Sch}(\wedge^{d,V}) \simeq \mathrm{sgn} \otimes (V^{\otimes d})^*$.

Theorem 5.3. *[KSS16, Theorem 3.1.3] For* X *and* $Y \in \mathrm{Rep}\Gamma_k^d$*, we have* $\mathrm{Sch}(X \underline{\otimes} Y) \simeq \mathrm{Sch}(X) \otimes \mathrm{Sch}(Y)$ *and* $\mathrm{Sch}(\mathbb{H}(X, Y)) \simeq \mathrm{Hom}(\mathrm{Sch}(X), \mathrm{Sch}(Y))$. *In fact,* Sch *is a closed monoidal functor.*

Proof. First consider the case when $X = \Gamma^{d,V}$ and $Y = \Gamma^{d,W}$. From Proposition 5.1 we have

$$\mathrm{Sch}(\Gamma^{d,V} \underline{\otimes} \Gamma^{d,W}) \simeq \mathrm{Sch}(\Gamma^{d,V\otimes W}) \simeq ((V \otimes W)^*)^{\otimes d} \simeq (V^*)^{\otimes d} \otimes (W^*)^{\otimes d} \simeq \mathrm{Sch}(\Gamma^{d,V}) \otimes \mathrm{Sch}(\Gamma^{d,W}).$$

Similarly we have $\mathrm{Sch}(\mathbb{H}(\Gamma^{d,V}, \Gamma^{d,W})) \simeq \mathrm{Hom}(\mathrm{Sch}(\Gamma^{d,V}), \mathrm{Sch}(\Gamma^{d,W}))$.

For general X and Y we express them as colimits of representable functors and use the fact that Sch, $\underline{\otimes}$ and the Kronecker $\otimes$ product preserve colimits, since they are left adjoints.

Recall that the isomorphism $\mathrm{Sch}(\Gamma^{d,V}) \simeq (V^{\otimes d})^*$ is functorial in the variable V. Using this and the definition of $\underline{\otimes}$ for representable functors, we readily see the commutativity of the diagrams in Definition 8.12 when F is Sch and all the objects involved are of the form $\Gamma^{d,V}$. We then use colimits to obtain the commutativity of the diagrams in Definition 8.12 for arbitrary strict polynomial functors. Thus Sch is a monoidal functor.

The proof for $\mathbb{H}$ proceeds the same way. For the second isomorphism, one observes additionally that Sch preserves limits and for the fourth isomorphism that $\mathrm{Hom}(\mathrm{Sch}(\Gamma^{d,V}), -)$ preserves colimits because $\mathrm{Sch}(\Gamma^{d,V}) \simeq (V^{\otimes d})^*$ is a finitely generated projective k-module. So we get Sch is a closed functor. □

5.2 Internal structures under the adjoints

The left and right adjoints of Sch were expressed as the internal tensor and the internal Hom respectively in [Rei16, Theorems 4.3 and 5.4]. We give a simple proof of these as an application of the fact that a left adjoint functor preserves arbitrary colimits and the Yoneda lemma. Also, she showed the left adjoint takes the Kronecker product to the internal tensor product in [Rei16, Corollary 4.4], the right adjoint takes the Kronecker hom to the internal Hom in [Rei16, Corollary 5.5] up to duality. But we show that the right adjoint takes the Kronecker hom to the internal Hom in general.

Lemma 5.4. *For an object X of $\mathrm{Rep}\Gamma^d_k$, we have $\mathcal{L}(\mathrm{Sch}(X)) \simeq X \underline{\otimes} \mathrm{Sym}^d$ and $\mathcal{R}(\mathrm{Sch}(X)) \simeq \mathbb{H}(\mathrm{Sym}^d, X)$.*

Proof. For the left adjoint, we have

$$\mathcal{L}(\mathrm{Sch}(\Gamma^{d,V})) \simeq \mathcal{L}((V^{\otimes d})^*) \simeq \mathrm{Sym}^{d,V} \simeq \Gamma^{d,V} \underline{\otimes} \mathrm{Sym}^d.$$

(By the definition, $\mathcal{L}((V^{\otimes d})^*) = \otimes^d \otimes_{kS_d} (V^{\otimes d})^* \simeq \mathrm{Sym}^{d,V}$.) For arbitrary X, we express it as colimit of $\Gamma^{d,V}$ and use that all three functors Sch, $\mathcal{L}$ and $(-\underline{\otimes}\mathrm{Sym}^d)$ preserve colimit to get $\mathcal{L}(\mathrm{Sch}(X)) \simeq X \underline{\otimes} \mathrm{Sym}^d$.

For every object Y in $\mathrm{Rep}\Gamma_k^d$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(Y, \mathcal{R}(\mathrm{Sch}(X))) &\simeq \mathrm{Hom}_{kS_d}(\mathrm{Sch}(Y), \mathrm{Sch}(X)) \simeq \mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(\mathcal{L}(\mathrm{Sch})(Y), X) \\ &\simeq \mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(Y \underline{\otimes} \mathrm{Sym}^d, X) \simeq \mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(Y, \mathbb{H}(\mathrm{Sym}^d, X)).\end{aligned}$$

Now it follows from the Yoneda lemma that $\mathcal{R}(\mathrm{Sch}(X)) \simeq \mathbb{H}(\mathrm{Sym}^d, X)$. □

Corollary 5.5. *[KSS16, Corollary 3.1.6] For M and N in kS_d-Mod and X in $\mathrm{Rep}\Gamma_k^d$ such that $\mathrm{Sch}(X) \simeq M$, we have*

$$\mathcal{L}(M \otimes N) \simeq X \underline{\otimes} N \text{ and } \mathcal{R}\mathrm{Hom}(M, N) \simeq \mathcal{R}(X, N).$$

The isomorphisms are functorial in N, and functorial in M as well if $X = \mathcal{L}(M)$ or if $X = \mathcal{R}(M)$.

Proof. The second result follows from the Yoneda lemma by the following sequence of functorial isomorphisms.

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(Z, \mathcal{R}(\mathrm{Hom}(M, N))) &\simeq \mathrm{Hom}_{kS_d}(\mathrm{Sch}(Z), \mathrm{Hom}(M, N)) && \text{by } \mathrm{Sch} \dashv \mathcal{R} \\ &\simeq \mathrm{Hom}_{kS_d}(\mathrm{Sch}(Z) \otimes M, N) && \text{by } -\otimes M \dashv \mathrm{Hom}(M, -) \\ &\simeq \mathrm{Hom}_{kS_d}(\mathrm{Sch}(Z) \otimes \mathrm{Sch}(X), N) && \text{by choice of } X \\ &\simeq \mathrm{Hom}_{kS_d}(\mathrm{Sch}(Z \underline{\otimes} X), N) && \text{by Theorem 5.3} \\ &\simeq \mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(Z \underline{\otimes} X, \mathcal{R}N) && \text{by } \mathrm{Sch} \dashv \mathcal{R} \\ &\simeq \mathrm{Hom}_{\mathrm{Rep}\Gamma_k^d}(Z, \mathbb{H}(X, \mathcal{R}N)) && \text{by } -\underline{\otimes} X \dashv \mathbb{H}(X, -).\end{aligned}$$

The proof of the first result is entirely parallel by using a test object in the second slot, see the next proof. The last sentence follows from the isomorphisms $\mathrm{Sch} \circ \mathcal{L} \simeq \mathrm{id} \simeq \mathrm{Sch} \circ \mathcal{R}$. □

Both the left and right adjoints are not monoidal. The left adjoint does not preserve the monoidal identity. The right adjoint does not preserve the monoidal structure in general but it turns out to be the lax monoidal from Proposition 8.13. See Definition 8.12 for the definition of lax monoidal functor.

Proposition 5.6. *The functor $\mathcal{R}$ is lax monoidal.*

5.2.1 Duals under the adjoints

In this section we study the k-linear dual of an object of kS_d-Mod under the adjoints of Sch.

Proposition 5.7. *For an object X in $\mathrm{Rep}\Gamma_k^d$, we have $(\mathcal{L}(\mathrm{Sch}(X)))^\circ \simeq X^\vee$ and $\mathcal{R}(\mathrm{Sch}(X)^*) \simeq X^\vee$.*

Proof. Write $X = \operatorname{colim}_{\alpha} \Gamma^{d,V}$ where α is a morphism from $\Gamma^{d,V}$ to X. Then

1. $$\begin{aligned} \mathcal{L}(\mathrm{Sch}(X))^{\circ} &= \mathcal{L}(\mathrm{Sch}(\operatorname{colim}_{\alpha}\Gamma^{d,V}))^{\circ} \simeq (\operatorname{colim}_{\alpha} \mathcal{L}(\mathrm{Sch}(\Gamma^{d,V}))^{\circ} \simeq \lim_{\alpha} (\mathcal{L}((V^{\otimes d})^{*})^{\circ} \\ &\simeq \lim_{\alpha} (\mathrm{Sym}^{d,V})^{\circ} \simeq \lim_{\alpha} \Gamma^{d}_{V} \simeq \lim_{\alpha} \mathbb{H}(\Gamma^{d,V}, \Gamma^{d}) = \mathbb{H}(\operatorname{colim}_{\alpha} \Gamma^{d,V}, \Gamma^{d}) = X^{\vee}. \end{aligned}$$

2. $$\begin{aligned} \mathcal{R}(\mathrm{Sch}(X)^{*}) &= \mathcal{R}(\mathrm{Sch}(\operatorname{colim}_{\alpha} \Gamma^{d,V})^{*}) \simeq \mathcal{R}(\lim_{\alpha} (((V^{\otimes d})^{*})^{*}) \simeq \mathcal{R}(\lim_{\alpha} V^{\otimes d}) \simeq \lim_{\alpha} \mathcal{R}(V^{\otimes d}) \\ &\simeq \lim_{\alpha} \mathbb{H}(\Gamma^{d,V}, \Gamma^{d}) \simeq \mathbb{H}(\operatorname{colim}_{\alpha} \Gamma^{d,V}, \Gamma^{d}) = \mathbb{H}(X, \Gamma^{d}) = X^{\vee}. \end{aligned}$$

□

Remark 5.8. Relations of duals with adjoints were first studied in [Rei16, Corollary 4.5] but those hold when k is a field. The isomorphism, $\mathrm{Sym}^{d} \underline{\otimes} X \simeq \mathbb{H}(X, \Gamma^{d})^{\circ}$, used in the proof of [Rei16, Corollary 4.5] may not be true if k is not a field. For $d = 1$, $\mathrm{Rep}\Gamma^{d}_{k}$ is equivalent to k-Mod. Under this equivalence $\underline{\otimes}$ (resp. $\mathbb{H}$) reduces to $\otimes$ (resp. Hom), the Kuhn dual is the k-linear dual and the functors $\mathrm{Sym}^{d}, \Gamma^{d}$ both are the module k. Take $k = \mathbb{Z}$ and X to be a torsion $\mathbb{Z}$-module. Then $\mathrm{Sym}^{d} \underline{\otimes} X = X$ but $\mathbb{H}(X, \Gamma^{d})^{\circ} = 0$.

5.3 Results on the unbounded derived categories

We show that Theorem 5.3 and Corollary 5.5 are true even at the level of (unbounded) derived categories. Let $D(\mathrm{Rep}\Gamma^{d}_{k})$ and $D(kS_{d}\text{-Mod})$ denote the unbounded derived categories of $\mathrm{Rep}\Gamma^{d}_{k}$ and kS_{d}-Mod respectively.

Theorem 5.9. *[KSS16, Theorem 3.1.4] For $X, Y \in D(\mathrm{Rep}\Gamma^{d}_{k})$ there are functorial isomorphisms*

$$\mathrm{Sch}(X \overset{L}{\underline{\otimes}} Y) \simeq \mathrm{Sch}(X) \overset{L}{\otimes} \mathrm{Sch}(Y) \textit{ and } \mathrm{Sch}(R\mathbb{H}(X, Y)) \simeq R\mathrm{Hom}(\mathrm{Sch}(X), \mathrm{Sch}(Y)).$$

Proof. We first recall some generalities, see [Spa88, BN93] for these matters. $\mathrm{Rep}\Gamma^{d}_{k}$ and kS_{d}-Mod are Grothendieck categories. There are the usual quotient functors $K(\mathrm{Rep}\Gamma^{d}_{k}) \xrightarrow{Q_1} D(\mathrm{Rep}\Gamma^{d}_{k})$ and $K(kS_{d}\text{-Mod}) \xrightarrow{Q_2} D(kS_{d}\text{-Mod})$. Each Q_j has a left adjoint $\mathbf{p}_j$ ("K-projective resolution" – used to calculate left derived functors) and a right adjoint $\mathbf{i}_j$ ("K-injective resolution" – used to calculate right derived functors). Arbitrary products and coproducts exist in $K(\mathrm{Rep}\Gamma^{d}_{k})$ and $K(kS_{d}\text{-Mod})$, which pass to the respective derived categories by the quotient functors. Coproducts are preserved by $\mathbf{p}_j$ and products by $\mathbf{i}_j$.

The functors Sch, $\underline{\otimes}, \mathbb{H}, \otimes$, and Hom pass to respective Homotopy categories (where one uses the same notation for them) and have derived functors. As Sch is exact, it extends to the derived category by term-wise application, the extension still denoted by Sch. It preserves (co)products. Derived bifunctors of $\underline{\otimes}$ and $\otimes$ (respectively, $\mathbb{H}$ and Hom) are calculated by taking double complexes obtained from appropriate resolutions and forming the corresponding total complexes via coproduct (respectively, product). $\underline{\otimes}$ and $\otimes$ preserve coproducts in each variable. $\mathbb{H}$ and Hom preserve products in the second variable and turn coproducts in the first variable into products. These properties pass to the respective derived functors because, e.g., $\mathbf{R}\mathbb{H}(X,-) \simeq Q_1 \circ \mathbb{H}(X,-) \circ \mathbf{i}_1$. (To avoid clutter we will continue to denote $Q_1 X$ in $D(\mathrm{Rep}\Gamma^d_k)$ by X.)

First, we prove the result about tensor products. By Theorem 5.3 we have the following isomorphism of bifunctors from $\mathrm{Rep}\Gamma^d_k \times \mathrm{Rep}\Gamma^d_k$ to kS_d-Mod.

$$F := \mathrm{Sch} \circ (-\underline{\otimes}-) \simeq G \circ (\mathrm{Sch} \times \mathrm{Sch}) \quad \text{where} \quad G := - \otimes -$$

As Sch is exact, $\mathbf{L}F \simeq \mathrm{Sch} \circ (-\overset{\mathbf{L}}{\underline{\otimes}}-)$. By the universal property of the derived bifunctor $\mathbf{L}F$, there is a natural transformation of triangulated bifunctors

$$\eta : \mathbf{L}G \circ (\mathrm{Sch} \times \mathrm{Sch}) = (- \overset{\mathbf{L}}{\otimes} -) \circ (\mathrm{Sch} \times \mathrm{Sch}) \to \mathbf{L}F$$

We will prove that η is an isomorphism in two steps.

Step 1. We claim that the restriction of η to $D^-(\mathrm{Rep}\Gamma^d_k) \times D^-(\mathrm{Rep}\Gamma^d_k)$ is an isomorphism. Consider the full subcategory $\mathcal{P}$ of $\mathrm{Rep}\Gamma^d_k$ consisting of direct sums of representable objects $\Gamma^{d,V}$. We observe below that $K^-(\mathrm{Sch})(\mathcal{P}) \times K^-\mathrm{Sch}(\mathrm{Rep}\Gamma^d_k)$ is $(- \otimes -)$-projective in the terminology of [KS06, Definitions 10.3.9, 13.4.2] or see Definition 8.18. This would prove the claim by a suitable analogue of Grothendieck spectral sequence as formulated in [KS06, Proposition 13.3.13(ii)] and also see Proposition 8.20.

To see the $(- \otimes -)$-projectivity of $K^-\mathrm{Sch}(\mathcal{P}) \times K^-\mathrm{Sch}(\mathrm{Rep}\Gamma^d_k)$, first note that direct sums of objects of the form $\mathrm{Sch}(\Gamma^{d,V}) \simeq (V^*)^{\otimes d}$ form a generating subcategory of kS_d-Mod because, e.g., the projective generator $kS_d \simeq \mathrm{Sch}(\Gamma^{1^d}) \overset{\oplus}{\hookrightarrow} \mathrm{Sch}(\Gamma^{d,k^d})$. So one needs to check that for X and Y in $K^-(kS_d\text{-Mod})$, the total complex $X \otimes Y$ is exact whenever one of the following two conditions is met. (i) all nonzero terms of X are of the form $\oplus_\alpha (V^*_\alpha)^{\otimes d}$ and Y is exact; or (ii) X is an exact complex all of whose nonzero terms are of the form $\oplus_\alpha (V^*_\alpha)^{\otimes d}$ and Y is arbitrary. As $(V^*_\alpha)^{\otimes d}$ is a projective k-module, this amounts to the exactness of the total complex of a fourth quadrant double complex with exact columns/rows.

Step 2. Fix an X in $D^-(\mathrm{Rep}\Gamma^d_k)$. Let $\mathcal{C}$ be the full subcategory of $D(\mathrm{Rep}\Gamma^d_k)$ whose objects are those Y for which $\eta(X, Y)$ is an isomorphism. Clearly $\mathcal{C}$ is a triangulated subcategory and by Step 1 it contains $D^-(\mathrm{Rep}\Gamma^d_k)$. As $\mathbf{L}F$ and $\mathbf{L}G \circ (\mathrm{Sch} \times \mathrm{Sch})$ preserve coproducts in each slot, $\mathcal{C}$ is closed under formation of arbitrary coproducts. It follows that $\mathcal{C}$ is all of $D(\mathrm{Rep}\Gamma^d_k)$ because any object $Y \simeq$ the cone of $(1-\text{shift})$ endomorphism of the coproduct $\oplus_{n \geqslant 0} Y^{\leqslant n}$ of bounded above truncations of Y [Spa88], [BN93, Section 2] or Corollary 8.26. Repeating the argument by fixing an arbitrary Y gives the result for arbitrary X and Y.

The result about $\mathbb{H}$ is obtained similarly. We indicate only the changes. By Theorem 5.3 we have the isomorphism of bifunctors from $(\mathrm{Rep}\Gamma_k^d)^{op} \times \mathrm{Rep}\Gamma_k^d$ to kS_d-Mod

$$F' := \mathrm{Sch} \circ \mathbb{H}(-,-) \simeq G' \circ (\mathrm{Sch} \times \mathrm{Sch}) \quad \text{where} \quad G' := \mathrm{Hom}(-,-)\ .$$

leading to a natural transformation of triangulated bifunctors

$$\eta' : \mathbf{R}F' \to \mathbf{R}G' \circ (\mathrm{Sch} \times \mathrm{Sch}) = \mathbf{R}\mathrm{Hom}(-,-) \circ (\mathrm{Sch} \times \mathrm{Sch}).$$

As before, the restriction of η' to $D^-(\mathrm{Rep}\Gamma_k^d) \times D^+(\mathrm{Rep}\Gamma_k^d)$ is an isomorphism. One sees this by using $\mathrm{Hom}(-,-)$-injectivity of $\mathrm{Sch}(\mathcal{P}) \times \mathrm{Sch}(\mathrm{Rep}\Gamma_k^d)$, which follows by exactness of the total complex of a first quadrant double complex with exact columns/rows.

The triangulated bifunctors $\mathbf{R}F'$ and $\mathbf{R}G' \circ (\mathrm{Sch} \times \mathrm{Sch})$ preserve products in the second slot and turn coproducts in the first slot into products. By using homotopy limits (see Definition 8.27) in the second slot and homotopy colimits (see Definition 8.25) in the first slot, one gets as before that η' is an isomorphism on all of $D((\mathrm{Rep}\Gamma_k^d)^{op}) \times D(\mathrm{Rep}\Gamma_k^d)$. □

Let k be a field of positive characteristic p. By a theorem of Clausen, James [Gre80, Theorem 6.4b], simple modules of kS_d are indexed by p-restricted partitions. If Sch of a finitely generated strict polynomial functor X vanishes then all the composition factors of X are non p-restricted. Sch being an exact functor it commutes with taking (co)homologies. The Kronecker product is exact but $\underline{\otimes}$ is not exact functor in general. From Theorem 5.9 we get that Sch vanishes on higher derived internal tensor and hom. So we obtain

Corollary 5.10. *[KSS16, Corollary 3.1.5] Let k be a field of positive characteristic p. Let $X, Y \in \mathrm{Rep}\Gamma_k^d$ be finitely generated. Then $H_i(X \overset{\mathbf{L}}{\underline{\otimes}} Y)$ and $H^i(\mathbf{R}\mathbb{H}(X,Y))$ have non p-restricted composition factors for all $i \geqslant 1$.*

Corollary 5.11. *[KSS16, Corollary 3.1.6] For $M^\bullet$ and $N^\bullet$ in $D(kS_d\text{-Mod})$ and $X^\bullet$ in $D(\mathrm{Rep}\Gamma_k^d)$ such that $\mathrm{Sch}(X^\bullet) \simeq M^\bullet$, we have $\mathcal{L}'(M^\bullet \overset{\mathbf{L}}{\otimes} N^\bullet) \simeq X \overset{\mathbf{L}}{\underline{\otimes}} N^\bullet$ and $\mathcal{R}'\mathbf{R}\mathrm{Hom}(M^\bullet, N^\bullet) \simeq \mathbf{R}\mathbb{H}(X^\bullet, N^\bullet)$ where $\mathcal{L}' = \mathbf{L}\mathcal{L}$ and $\mathcal{R}' = \mathbf{R}\mathcal{R}$. The isomorphisms are functorial in $N^\bullet$, and functorial in $M^\bullet$ as well if $X^\bullet = \mathcal{L}'(M^\bullet)$ or if $X = \mathcal{R}'(M^\bullet)$.*

Proof. In general, an adjunction $F \dashv G$ of additive functors between Grothendieck categories leads to an adjunction $\mathbf{L}F \dashv \mathbf{R}G$ between the corresponding derived categories. (The adjunction first passes to homotopy categories and then one combines with the adjunctions $\mathbf{p} \dashv Q \dashv \mathbf{i}$, e.g. see [Kra13, Proposition 4.1] for the case of $(X^\bullet \underline{\otimes} -) \dashv \mathbb{H}(X^\bullet, -)$. This case also involves appropriate total complexes of bicomplexes as $X^\bullet$ itself is a complex.) Thus we may repeat the argument in the proof of Corollary 5.5 by using derived versions of all adjunctions and Theorem 5.9. We sketch this for the first isomorphism.

$$\begin{aligned}\mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}(\mathcal{L}'(M^\bullet \overset{\mathbf{L}}{\otimes} N^\bullet), Z^\bullet) &\simeq \mathrm{Hom}_{D(kS_d\text{-Mod})}(N^\bullet, \mathbf{R}\mathrm{Hom}(M^\bullet, \mathrm{Sch}(Z^\bullet)))\\ &\simeq \mathrm{Hom}_{D(kS_d\text{-Mod})}(N^\bullet, \mathrm{Sch}(\mathbf{R}\mathbb{H}(X^\bullet, Z^\bullet)))\\ &\simeq \mathrm{Hom}_{D(\mathrm{Rep}\Gamma_k^d)}(X^\bullet \overset{\mathbf{L}}{\underline{\otimes}} \mathcal{L}'(N^\bullet), Z^\bullet)\end{aligned}$$

Here for first and third isomorphisms we use ($\mathcal{L}' \dashv \mathrm{Sch}$, $\overset{\mathbf{L}}{\otimes} \dashv \mathbf{R}\mathrm{Hom}$). Second isomorphism follows from $\mathrm{Sch}(X^\bullet) \simeq M^\bullet$ and Theorem 5.9. Finally, as Sch is exact, deriving $\mathrm{Sch} \circ \mathcal{L} \simeq \mathrm{id} \simeq \mathrm{Sch} \circ \mathcal{R}$ gives $\mathrm{Sch} \circ \mathcal{L}' \simeq \mathrm{id} \simeq \mathrm{Sch} \circ \mathcal{R}'$. This gives the last assertion. □

5.4 Application to the Kronecker multiplicities

For this section, let k be a field of characteristic 0. Now Sch is an equivalence of semisimple monoidal categories. The Kronecker problem for the symmetric group asks for a good description of multiplicities of Specht modules in the tensor product of two Specht modules. Via Sch, this is equivalent to decomposing $\Delta(\lambda)\underline{\otimes}\Delta(\mu)$, where λ and μ are partitions of d. One can do this, e.g., by combining the Jacobi-Trudi formula to express $\Delta(\mu)$ as an alternating sum of various Γ^ν and then using Proposition 4.25 to calculate each $\Delta(\lambda)\underline{\otimes}\Gamma^\nu$. Such an algorithm involves cancellations and its ingredients translate into standard facts about the internal product of symmetric functions. Even so, in two special cases (namely when μ is either a two-row partition or a hook) we show below that one can devise a reasonably simple procedure in terms of $\underline{\otimes}$. In the case of a hook, this uses a signed version of polynomial functors defined by Axtell [Axt13].

Example 5.12. [KSS16, Example 3.4.1] We calculate $\Delta(\lambda)\underline{\otimes}\Delta((a,b))$, where $a \geqslant b$ are positive integers with $a+b=d$.

$$\begin{aligned}\Delta(\lambda)\underline{\otimes}\Gamma^{(a,b)} &\simeq \Delta(\lambda)^{k^2}_{(a,b)} \simeq \bigoplus_{\mu\subset\lambda,|\mu|=a} \Delta(\mu)\otimes\Delta(\lambda/\mu)\\ &\simeq \bigoplus_{\mu\subset\lambda,|\mu|=a} \Delta(\mu)\otimes\Big(\bigoplus_{\nu\subset\lambda,|\nu|=b} c^\lambda_{\mu,\nu}\Delta(\nu)\Big)\\ &\simeq \bigoplus_{|\mu|=a,|\nu|=b,|\alpha|=d} c^\lambda_{\mu,\nu}\, c^\alpha_{\mu,\nu}\Delta(\alpha).\end{aligned}$$

We also have $\Gamma^{(a,b)} \simeq \Gamma^{(a+1,b-1)} \oplus \Delta((a,b))$ by, e.g., Pieri's formula. Apply $-\underline{\otimes}\Delta(\lambda)$ and use the above calculation to get

$$\Delta(\lambda)\underline{\otimes}\Delta((a,b)) \simeq \bigoplus_\alpha \Big(\sum_{|\mu|=a,|\nu|=b} c^\lambda_{\mu,\nu}\, c^\alpha_{\mu,\nu} - \sum_{|\bar\mu|=a+1,|\bar\nu|=b-1} c^\lambda_{\bar\mu,\bar\nu}\, c^\alpha_{\bar\mu,\bar\nu}\Big)\Delta(\alpha).$$

As a special case consider $b = 1$ and let $c =$ the number of outer corners of λ. Now $\Gamma^{(a+1,b-1)} = \Gamma^d$, which is the identity for $\underline{\otimes}$. We get

$$\Delta(\lambda)\underline{\otimes}\Delta((a,1)) \simeq (c-1)\Delta(\lambda) \bigoplus \bigoplus_{\alpha}\Delta(\alpha),$$

where α ranges over partitions obtained by moving exactly one box in the Young diagram of λ elsewhere. Note that if we apply the Schur functor to this, we get the well known formula for the Kronecker product of a Specht module with the standard module.

Example 5.13. [KSS16, Example 3.4.2] We calculate $\Delta(\lambda)\underline{\otimes}\Delta((p,1^q))$ where p, q are positive integers with $d = p+q$. Straightforward imitation of earlier procedure would require us to involve Γ^ν where ν has several parts. Instead, we use polynomial functors whose arguments are super-vector spaces (see [Axt13]). In this language $\Gamma^p(V) \otimes \wedge^q(V) \simeq \Gamma^{d,k^+\oplus k^-}_{(p,q)}(V)$. On the right hand side, the parametrization by $k^+ \oplus k^-$ and taking the (p,q) weight space amounts to requiring p letters from the argument V to commute and remaining q letters to anticommute. Similarly, using obvious terminology, $\Delta_+(\mu)(V) = \Delta(\mu)(V_+) = \Delta(\mu)(V)$ whereas $\Delta_-(\mu)(V) = \Delta(\mu)(V_-) \simeq \nabla(\mu')(V)$, which is $\Delta(\mu')(V)$ for k a field of characteristic 0. Proceeding as before via a super-analogue of the relevant filtration, we have

$$\begin{aligned}
\Delta(\lambda)\underline{\otimes}\left(\Gamma^p \otimes \wedge^q\right) &\simeq \Delta(\lambda)^{k^+\oplus k^-}_{(p,q)} \simeq \bigoplus_{\mu\subset\lambda,|\mu|=p} \Delta_+(\mu)\otimes\Delta_-(\lambda/\mu) \\
&\simeq \bigoplus_{\mu\subset\lambda,|\mu|=p} \Delta_+(\mu)\otimes\Delta_+(\lambda'/\mu') \\
&\simeq \bigoplus_{\mu\subset\lambda,|\mu|=p} \Delta(\mu)\otimes\left(\bigoplus_{\nu\subset\lambda,|\nu|=q} c^{\lambda'}_{\mu',\nu}\Delta(\nu)\right) \\
&\simeq \bigoplus_{|\mu|=p,|\nu|=q,|\alpha|=d} c^{\lambda'}_{\mu',\nu}\, c^{\alpha}_{\mu,\nu}\Delta(\alpha).
\end{aligned}$$

Now again by Pieri's rule, $\Gamma^p \otimes \wedge^q \simeq \Delta((p,1^q)) \oplus \Delta((p+1,1^{q-1}))$. This allows one to calculate $\Delta(\lambda)\underline{\otimes}\Delta(p,1^q)$ as the alternating sum of $\Delta(\lambda)\underline{\otimes}(\Gamma^{p+i}\otimes\wedge^{q-i})$ with $i = 0,\dots,q$.

Chapter 6

Sesquialgebra structures of Schur algebras of Weyl groups of types A and B

A sesquialgebra structure was defined by Tang, Weinstein and Zhu in [TWZ07] as a variant of a Hopf algebra structure. In [TWZ07], it was shown that a sesquialgebra structure on an unital associative algebra A gives an internal tensor product of A-modules. Converse was also shown to be true by Hovey in [Hov11], i.e., an internal tensor product of A-modules which preserves colimit in each slot must arise from a sesquialgebra structure of A.

Recall from Chapter 3 that the centralizer algebras $S_k(n,d) = \mathrm{End}_{S_d}((k^n)^{\otimes d})$ and $H(2n,d) = \mathrm{End}_{B_d}((k^{2n})^{\otimes d})$ are the Schur algebras of Weyl groups of types **A** and **B** respectively.

From Chapter 4 there is an internal tensor product of $S_k(n,d)$-modules by passing to the category of strict polynomial functors. In this chapter, we give analogous set-up in two cases: (1) for the new centralizer algebra $\mathrm{End}_G((k^n)^{\otimes d})$ (see Example 3.3) where G is a subgroup of S_d acting on $(k^n)^{\otimes d}$ by restriction, (2) Schur algebra of type **B**, $H(2n,d)$. If $n \geqslant d$, then we show that $(1)\,\mathrm{End}_G((k^n)^{\otimes d})$-Mod is equivalent to the functor category of certain monoidal category (Theorem 6.1), (2) (in addition assume that 2 is unit in k) the same is true for $H(2n,d)$-Mod (Theorem 6.4). So in particular, there exists an internal tensor product on $\mathrm{End}_G((k^n)^{\otimes d})$-Mod and also on $H(2n,d)$-Mod.

Similar to the functor Sch, here also we have a functor Sch_G (resp. HSch) relating the corresponding functor category with kG-Mod (resp. kB_d-Mod), and it has both the adjoints.

As in the type **A** case, we give analogs of results (Theorem 5.3 and Corollary 5.5) in these cases. The internal tensor product on the functor category is compatible with the Kronecker product on kG-Mod via Sch_G (Proposition 6.3). Also, the same is true in type **B** case (or simply kB_d-Mod) via HSch (Theorem 6.5).

For $n < d$, we give an example that the bifunctor on $S_k(n,d)$-Mod induced by the internal tensor product of strict polynomial functors is not associative (Example 6.13).

The results of this chapter are a part of the preprint [Sri].

6.1 The categories of functors

Case 1: Schur algebra of Weyl group of type A

Any subgroup G of S_d acts on $(k^n)^{\otimes d}$ by restriction and there is the corresponding centralizer algebra $\mathrm{End}_G((k^n)^{\otimes d})$. Recall $S_k(n,d)$ is isomorphic to $\mathrm{End}_{S_d}((k^n)^{\otimes d})$ and $S_k(n,d)$-Mod is equivalent to the functor category $\mathrm{Rep}\Gamma_k^d$ (when $n \geqslant d$). We produce analogous set-up for these new centralizer algebras $\mathrm{End}_G((k^n)^{\otimes d})$.

Define a category $\mathcal{G}^d$ whose objects are the free modules of finite rank over k. The morphism space from V to W is $\mathrm{Hom}_G(V^{\otimes d}, W^{\otimes d})$. This is a k-linear category. For objects V and W in $\mathcal{G}^d$, the tensor product over k, $V \otimes W$ and k-linear homomorphism $\mathrm{Hom}(V,W)$ define an internal tensor product and an internal Hom in $\mathcal{G}^d$ respectively.

Let $\mathrm{Rep}\,\mathcal{G}^d$ be the functor category $\mathrm{Fct}(\mathcal{G}^d, k\text{-Mod})$. Thus $\mathrm{Rep}\,\mathcal{G}^d$ is abelian, complete and co-complete.

The evaluation functor: For an object X in $\mathrm{Rep}\,\mathcal{G}^d$, $X(k^n)$ is a left $\mathrm{End}_G((k^n)^{\otimes d})$-module. Thus we have a functor $ev_{k^n} : \mathrm{Rep}\,\mathcal{G}^d \to \mathrm{End}_G((k^n)^{\otimes d})$-Mod. Define $h_V(W)$ to be $\mathrm{Hom}_{\mathcal{G}^d}(V,W) = \mathrm{Hom}_G(V^{\otimes d}, W^{\otimes d})$. For $G = S_d$, note that h_V is the strict polynomial functor $\Gamma^{d,V}$. For $G = S_d$, ev_{k^n} was shown to be an equivalence of categories when $n \geqslant d$ in ([FS97, Theorem 3.2] and [Kra13, Theorem 2.10]). We generalize this for any subgroup G of S_d.

Theorem 6.1. *The functor* $ev_{k^n} : \mathrm{Rep}\,\mathcal{G}^d \to \mathrm{End}_G((k^n)^{\otimes d})$-Mod *is an equivalence of categories when* $n \geqslant d$.

Proof. From Theorem 8.10 it is enough to find a small projective generator P in $\mathrm{Rep}\,\mathcal{G}^d$ such that $\mathrm{End}_{\mathrm{Rep}\,\mathcal{G}^d}(P) \simeq (\mathrm{End}_G((k^n)^{\otimes d}))^{op}$ (this idea was also used in [Kra13, Theorem 2.10]).

From Proposition 8.8 the class of representable functors $\mathrm{Hom}_{\mathcal{G}^d}(k^m, -)$ forms a projective generator in $\mathrm{Rep}\,\mathcal{G}^d$. We have the decomposition

$$\begin{aligned} \mathrm{Hom}_{\mathcal{G}^d}(k^m, -) &= \mathrm{Hom}_G((k^m)^{\otimes d}, -) && (6.1)\\ &= \mathrm{Hom}_G(\bigoplus_{\lambda\in\Lambda(m,d)} kI_\lambda, -) \text{ using isomorphism (3.10)} && (6.2)\\ &= \bigoplus_{\lambda\in\Lambda(m,d)} \mathrm{Hom}_G(kI_\lambda, -). && (6.3) \end{aligned}$$

If $\lambda \sim \mu$ then by Remark 3.5, $kI_\lambda \simeq kI_\mu$ as S_d-modules so in particular $kI_\lambda \simeq kI_\mu$ as G-modules. Therefore $\mathrm{Hom}_G(kI_\lambda, -) \simeq \mathrm{Hom}_G(kI_\mu, -)$ in $\mathrm{Rep}\,\mathcal{G}^d$.

Suppose $n \geqslant d$, then every direct summand of $\mathrm{Hom}_{\mathcal{G}^d}(k^m, -)$ in the decomposition (6.1) has already occurred in the decomposition of $\mathrm{Hom}_{\mathcal{G}^d}(k^n, -)$. Therefore $\mathrm{Hom}_{\mathcal{G}^d}(k^n, -)$ is a small projective generator in $\mathrm{Rep}\,\mathcal{G}^d$. For $P = \mathrm{Hom}_{\mathcal{G}^d}(k^n, -)$, $\mathrm{End}_{\mathrm{Rep}\,\mathcal{G}^d}(P)$ is isomorphic to $(\mathrm{End}_G((k^n)^{\otimes d}))^{op}$ and the functor ev_{k^n} is naturally isomorphic to $\mathrm{Hom}_{\mathrm{Rep}\,\mathcal{G}^d}(P, -)$ due to the Yoneda lemma. □

Remark 6.2. The condition $n \geqslant d$ is not always necessary. For example, when G is the trivial subgroup we could take $n = 1$ for any d. But when $G = S_d$ it is necessary also (for $n < d$, ev_{k^n} is not faithful for example $\wedge^d(k^n) = 0$). One possibility is to investigate for a necessary condition could be to determine explicitly the orbit space decomposition of G-action on $I(n, d)$.

For $n < d$, the functor ev_{k^n} need not be an equivalence but it has both the left and right adjoints which are given respectively as follows:

$$\mathrm{End}_G((k^n)^{\otimes d}) - \mathrm{Mod} \xrightarrow{\theta_!} \mathrm{Rep}\,\mathcal{G}^d \quad \text{and} \quad \mathrm{End}_G((k^n)^{\otimes d}) - \mathrm{Mod} \xrightarrow{\theta_*} \mathrm{Rep}\,\mathcal{G}^d \text{ where} \qquad (6.4)$$

$\theta_!(M) = \mathrm{Hom}_G((k^n)^{\otimes d}, \otimes^d) \otimes_{\mathrm{End}_G((k^n)^{\otimes d})} M$ and
$\theta_*(M) = \mathrm{Hom}_{\mathrm{End}_G((k^n)^{\otimes d})}(\mathrm{Hom}_G(\otimes^d, (k^n)^{\otimes d}), M)$.

We also have $ev_{k^n} \circ \theta_! \simeq \mathrm{id}$ and $ev_{k^n} \circ \theta_* \simeq \mathrm{id}$.

Internal structures on Rep $\mathcal{G}^d$:

Recall from Chapter 4 that $\mathrm{Rep}\Gamma_k^d$ has an internal tensor product and an internal Hom. We generalize these constructions for objects of $\mathrm{Rep}\,\mathcal{G}^d$ following the technique used in [Kra13]. Also we show compatibility of this internal tensor product with the internal tensor product of kG-modules. (This was proved for $G = S_d$ in [AR15, Theorem 4.4] and in [KSS16, Theorem 3.1.3] but we present an elementary proof which works for any subgroup G of S_d.) Note that $\otimes^d$ is an object of $\mathrm{Rep}\,\mathcal{G}^d$.

Proposition 6.3. *We have*

1. *There are bi-functors*

$$\begin{aligned}(-\underline{\otimes}-) &: \operatorname{Rep}\mathcal{G}^d \times \operatorname{Rep}\mathcal{G}^d \to \operatorname{Rep}\mathcal{G}^d \quad \text{and}\\ \mathbb{H}(-,-) &: (\operatorname{Rep}\mathcal{G}^d)^{op} \times \operatorname{Rep}\mathcal{G}^d \to \operatorname{Rep}\mathcal{G}^d,\end{aligned}$$

which are given on representable functors. $h_V \underline{\otimes} h_W = h_{(V\otimes W)}$, $\mathbb{H}(h_V, h_W) = h_{(\operatorname{Hom}_k(V,W))}$. *Moreover, there is a natural isomorphism,* $\operatorname{Hom}_{\operatorname{Rep}\mathcal{G}^d}(X\underline{\otimes}Y, Z) \simeq \operatorname{Hom}_{\operatorname{Rep}\mathcal{G}^d}(X, \mathbb{H}(Y,Z))$.

2. *There exist an internal tensor product and an internal Hom on* $\operatorname{End}_G((k^n)^{\otimes d})$-Mod *if* $n \geqslant d$.

3. $\operatorname{End}_{\operatorname{Rep}\mathcal{G}^d}(\otimes^d) \simeq (kG)^{op}$.

4. *There exists a functor* $\operatorname{Sch}_G := \operatorname{Hom}_{\operatorname{Rep}\mathcal{G}^d}(\otimes^d, -) : \operatorname{Rep}\mathcal{G}^d \to kG\text{-Mod}$ *such that* $\operatorname{Sch}_G(h_{k^m}) = ((k^m)^{\otimes d})^*$.

5. *For objects* X *and* Y *in* $\operatorname{Rep}\mathcal{G}^d$, *there are natural isomorphisms* $\operatorname{Sch}_G(X\underline{\otimes}Y) \simeq \operatorname{Sch}_G(X) \otimes \operatorname{Sch}_G(Y)$ *and* $\operatorname{Sch}_G(\mathbb{H}(X,Y)) \simeq \operatorname{Hom}(\operatorname{Sch}_G(X), \operatorname{Sch}_G(Y))$.

6. *The functors* $\mathcal{L} : kG\text{-Mod} \to \operatorname{Rep}\mathcal{G}^d$ *and* $\mathcal{R} : kG\text{-Mod} \to \operatorname{Rep}\mathcal{G}^d$ *given on abject* M *in* kG-Mod *as follows:* $\mathcal{L}(M) = (\otimes^d \otimes_{kG} M)$ *and* $\mathcal{R}(M) = \operatorname{Hom}_{kG}((\otimes^d)^*, M)$ *are the left and right adjoints of* Sch_G *respectively. For an object* X *in* $\operatorname{Rep}\mathcal{G}^d$ *such that* $\operatorname{Sch}_G(X) \simeq M$, *we have* $\mathcal{L}(M \otimes N) \simeq X\underline{\otimes}\mathcal{L}(N)$ *and* $\mathcal{R}(\operatorname{Hom}(M,N)) \simeq \mathbb{H}(X, \mathcal{R}(N))$.

Proof. 1. This is merely [Kra13, Proposition 2.4] but we spell out the important ingredient used there. We have the contra-variant Yoneda embedding,

$$\mathcal{G}^d \hookrightarrow \operatorname{Rep}\mathcal{G}^d \tag{6.5}$$

$$V \longmapsto h_V \tag{6.6}$$

As $\mathcal{G}^d$ has an internal tensor product and an internal Hom, Day convolution induces an internal tensor product and an internal Hom on $\operatorname{Rep}\mathcal{G}^d$. Moreover, these internal structures are preserved by the Yoneda embedding. Now k-Mod is a cocomplete category so from [ML98, III.7, Theorem 1] for an object X in $\operatorname{Rep}\mathcal{G}^d$ we have $X = \operatorname*{colim}_{h_V \to X} h_V$.

2. Follows from Theorem 6.1 and part 1. More precisely, for $\operatorname{End}_G((k^n)^{\otimes d})$-modules M and N, define an internal tensor product as follows

$$M\underline{\otimes}N := (X\underline{\otimes}Y)(k^n) \tag{6.7}$$

where X and Y are such that $X(k^n) \simeq M$ and $Y(k^n) \simeq N$.

3. Let $A = \mathrm{End}_G((k^n)^{\otimes d})$. Then,

$$\begin{aligned}\mathrm{End}_{\mathrm{Rep}\,\mathcal{G}^d}(\otimes^d) &\simeq \mathrm{End}_A((k^d)^{\otimes d}) \text{ by Theorem 6.1}\\ &\simeq (kG)^{op} \text{ by part 3 of Theorem 3.1.}\end{aligned}$$

4. For the action of G on $(k^n)^{\otimes d}$, let ξ be the idempotent in A as in Theorem 3.1. Recall that $h_{k^m} = \mathrm{Hom}_G((k^m)^{\otimes d}, -)$. We have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Rep}\,\mathcal{G}^d}(\otimes^d, \mathrm{Hom}_G((k^m)^{\otimes d}, -)) &\simeq \mathrm{Hom}_A((k^d)^{\otimes d}, \mathrm{Hom}_G((k^m)^{\otimes d}, (k^d)^{\otimes d})) \text{ by Theorem 6.1}\\ &\simeq \mathrm{Hom}_A(A\xi, \mathrm{Hom}_G((k^m)^{\otimes d}, (k^d)^{\otimes d})) \text{ by Theorem 3.1}\\ &\simeq \xi\mathrm{Hom}_G((k^m)^{\otimes d}, (k^d)^{\otimes d}) \text{ from equation (3.4)}\\ &\simeq ((k^m)^{\otimes d})^* \text{ by Remark 3.2.}\end{aligned}$$

The action of kG on $(k^m)^{\otimes d}$ is a right action so $((k^m)^{\otimes d})^*$ is a left kG-module. We also have that $((k^m)^{\otimes d})^* \simeq ((k^m)^*)^{\otimes d}$ where action of kG on $((k^m)^*)^{\otimes d}$ made into a left action by sending g to g^{-1}.

5. For $G = S_d$, the key computation was Proposition 5.1, $\mathrm{Sch}(\Gamma^{d,V}) = (V^{\otimes d})^*$. In our case, it is the computation from part 4. Then the proof follows from the exact arguments as in Theorem 5.3.

6. It follows from parallel ideas as in Corollary 5.5.

□

Case 2: Schur algebra of Weyl group of type B

Define a category $\mathcal{C}$ whose objects are k^{2n} for $n \in \mathbb{N}$ and a distinguished object k (we will see relevance of this object in the next section). For objects V and W in $\mathcal{C}$ the morphism space $\mathrm{Hom}_{\mathcal{C}}(V, W) = \mathrm{Hom}_{B_d}(V^{\otimes d}, W^{\otimes d})$ where action of B_d on $(k^{2n})^{\otimes d}$ as given in Example 3.4 and $(k)^{\otimes d} = k$ is the trivial B_d-module. Let $\mathrm{Rep}\mathcal{C}$ denote the functor category $\mathrm{Fct}(\mathcal{C}, k\text{-Mod})$. As usual $\mathrm{Rep}\mathcal{C}$ is abelian, complete and co-complete.

Theorem 6.4. *If 2 is a unit in k and $n \geqslant d$, $ev_{k^{2n}} : \mathrm{Rep}\mathcal{C} \to H(2n, d)\text{-Mod}$ is an equivalence.*

Proof. From Theorem 8.10 it is enough to find a small projective generator P in $\mathrm{Rep}\mathcal{C}$ such that $\mathrm{End}_{\mathrm{Rep}\mathcal{C}}(P) \simeq (H(2n, d))^{op}$ and $\mathrm{Hom}_{\mathrm{Rep}\mathcal{C}}(P, -) \simeq ev_{k^{2n}}$.
By Remark 3.6, $\mathrm{Hom}_{B_d}((k^{2n})^{\otimes d}, -) = \bigoplus_{\lambda \in \Lambda(n,d)} \mathrm{Hom}_{B_d}(kO_\lambda, -)$. If $\lambda \in \Lambda(n, d)$ and $\mu \in \Lambda(m, d)$ such that $\lambda \sim \mu$ then by isomorphism (3.13) $\mathrm{Hom}_{B_d}(kO_\lambda, -) \simeq \mathrm{Hom}_{B_d}(kO_\mu, -)$ in $\mathrm{Rep}\mathcal{C}$. So we conclude that if $n \geqslant d$, representable functor $\mathrm{Hom}_{B_d}((k^{2m})^{\otimes d}, -)$ is a direct summand of direct sum of finitely many copies of $\mathrm{Hom}_{B_d}((k^{2n})^{\otimes d}, -)$ for $m \geqslant 1$.

We show that $\mathrm{Hom}_{B_d}(k,-)$ is a direct summand of $\mathrm{Hom}_{B_d}((k^2)^{\otimes d},-)$ (in particular a direct summand of $\mathrm{Hom}_{B_d}((k^{2n})^{\otimes d},-)$) if 2 is a unit in k.

Let e_1 and e_{-1} be the standard basis for k^2. Define a map $\phi : (k^2)^{\otimes d} \to (k^2)^{\otimes d}$ which on the basis element $e_{\underline{i}}$ is given by $\phi(e_{\underline{i}}) = \sum\limits_{\underline{j}\in\Omega(2,d)} e_{\underline{j}}$ where $e_{\underline{l}} = e_{l_1} \otimes e_{l_2} \otimes \cdots \otimes e_{l_d}$ for $\underline{l} = (l_1, \cdots, l_d) \in \Omega(2, d)$ (see Example 3.4). Note that B_d acts trivially on $\sum\limits_{\underline{j}\in\Omega(2,d)} e_{\underline{j}}$ (enough to check this for generators of B_d). So ϕ is B_d-linear and image of ϕ is the trivial B_d-module k. Notice that $\phi \circ \phi = 2^d\phi$. Since 2 is a unit in k, the map $\frac{1}{2^d}\phi$ is the projection onto the trivial B_d-module k. Thus k is a direct summand of $(k^2)^{\otimes d}$ as B_d-module. This gives $\mathrm{Hom}_{B_d}(k,-)$ is a direct summand of $\mathrm{Hom}_{B_d}((k^2)^{\otimes d},-)$.

So we conclude that $P = \mathrm{Hom}_{B_d}((k^{2n})^{\otimes d},-)$ is a small projective generator in $\mathrm{Rep}\mathcal{C}$. The Yoneda lemma gives $\mathrm{Hom}_{\mathrm{Rep}\mathcal{C}}(P,-) \simeq ev_{k^{2n}}$ and $\mathrm{End}_{\mathrm{Rep}\mathcal{C}}(P) \simeq (H(2n, d))^{op}$. □

Internal tensor product on Rep$\mathcal{C}$:

We claim that tensor product of modules over k induces an internal tensor product on $\mathcal{C}$. At the level of object this is $k^{2n} \otimes k^{2m} = k^{2.2nm}$ and $k \otimes k^{2n} = k^{2n} \otimes k = k^{2n}$. (Distinguished object k in $\mathcal{C}$ plays the role of tensor identity.) We show in the following how this is a tensor product of morphisms also.

We provide an isomorphism $\Omega(2n, d) \times \Omega(2m, d) \to \Omega(2.2nm, d)$ of B_d-sets where B_d acts on $\Omega(2n, d) \times \Omega(2m, d)$ diagonally. For this we first need the following set-up.
For $d = 1$, $\Omega(2n, 1) = [2n]$ and so there is an action of $\mathbb{Z}/2\mathbb{Z}$ on $[2n]$ via $s_0 \in B_d$. Consider the map $[2n] \times [2m] \xrightarrow{\phi} [2.2nm]$ defined as follows. For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$,

1. $\phi(-i, -j) = -2im + (m - j)$,
2. $\phi(-i, j) = -2im + (m + (j - 1))$,
3. $\phi(i, -j) = 2im - (m + (j - 1))$,
4. $\phi(i, j) = 2im - (m - j)$.

For example, when $n = m = 2$ then $\phi(-1, -1) = -2$, $\phi(-1, 1) = -1$, $\phi(1, -1) = 1$ and $\phi(1, 1) = 2$. (We remark that our choice of ϕ is not canonical but it is enough for our purpose to produce a such map.)

One can easily check that ϕ is a bijective map which is also $\mathbb{Z}/2\mathbb{Z}$-linear. Define a map $\Omega(2n, d) \times \Omega(2m, d) \xrightarrow{\psi} \Omega(2.2nm, d)$ by $\psi((i_1, \cdots, i_d), (j_1, \cdots, j_d)) = (\phi(i_1, j_1), \cdots, \phi(i_d, j_d))$. We can check that ψ is an isomorphism which respects the action of B_d (enough to do this for generators of B_d).

Finally we conclude that ψ induces an isomorphism $(k^{2n})^{\otimes d} \otimes (k^{2m})^{\otimes d} \to (k^{2.2nm})^{\otimes d}$ of B_d-modules. With the help of this isomorphism it is clear that tensor product of modules over k induces an internal tensor product of morphisms in $\mathcal{C}$.

The internal tensor product on $\mathcal{C}$ via Day convolution induces an internal tensor product on $\mathrm{Rep}\mathcal{C}$. Note that $\otimes^d$ is an object of $\mathrm{Rep}\mathcal{C}$. The next theorem and corollary follow from analogous arguments as in the proof of Proposition 6.3 where the role of Theorem 6.1 and Example 3.3 will be replaced by now Theorem 6.4 and Example 3.4 respectively.

Theorem 6.5. *The following hold,*

1. *There exists an internal product on* $\mathrm{Rep}\mathcal{C}$.
2. $\mathrm{End}_{\mathrm{Rep}\mathcal{C}}(\otimes^d) \simeq (kB_d)^{op}$.
3. $\mathrm{HSch} := \mathrm{Hom}_{\mathrm{Rep}\mathcal{C}}(\otimes^d, -) : \mathrm{Rep}\mathcal{C} \to kB_d\text{-Mod}$ *preserves internal tensor product.*
4. *If* 2 *is a unit in* k *and* $n \geqslant d$, $H(2n, d)\text{-Mod}$ *has an internal tensor product.*

6.2 Sesquialgebras

Notions of sesquialgebra structures and hopfish algebra structures were introduced by Tang, Weinstein and Zhu [TWZ07] as variants of a Hopf algebra structure. Their motivation arises from Poisson geometry. The basic idea is to replace the comultiplication map and the counit map in the definition of a coalgebra A by an $(A, A \otimes A)$-bimodule and an (A, k)-bimodule respectively. One of the interesting properties is unlike Hopf algebra structures both of these structures can be transported via a Morita equivalence [TWZ07].

Definition 6.6. Let A be an unital associative algebra over k. We say that A is a sesquialgebra if there exist a $(A, A \otimes A)$-bimodule $\triangle$ and a (A, k)-bimodule ϵ such that the following properties hold:

1. (Coassociativity) $\triangle \otimes_{A\otimes A} (\triangle \otimes A) \simeq \triangle \otimes_{A\otimes A} (A \otimes \triangle)$ as $(A, A \otimes A \otimes A)$-bimodules
2. (Counit) $\triangle \otimes_{A\otimes A} (\epsilon \otimes A) \simeq A \simeq \triangle \otimes_{A\otimes A} (A \otimes \epsilon)$ as (A, A)-bimodules

The following theorem gives a relevance of a sesquialgebra structure. We only give a sketch of a proof and more details can be found in [Hov11].

Theorem 6.7. *[TWZ07][Hov11, Theorem 2.3] If* A *is a sesquialgebra,* $A\text{-Mod}$ *has an internal tensor product that preserve colimits in each slot. Moreover, the left and right internal Hom also exist.*

Proof. For left A-modules M and N, $M \otimes N$ is a left $A \otimes A$-module. Suppose A is a sesquialgebra. Then the following define an internal tensor product on A-Mod,

$$M \underline{\otimes} N := \triangle \otimes_{A \otimes A} (M \otimes N). \tag{6.8}$$

Since the functors $(M \underline{\otimes} -)$ and $(- \underline{\otimes} N)$ preserve colimits, by [ML98, Theorem 2, Chapter X] their right adjoints exist and these define left and right internal Hom. More precisely, left and right internal Hom are respectively given as follows: $\mathbb{L}\mathcal{H}om(M, -)(T) = \mathrm{Hom}_A(M \underline{\otimes} A, T)$ and $\mathbb{R}\mathcal{H}om(-, N)(T) = \mathrm{Hom}_A(A \underline{\otimes} N, T)$. □

In fact, the converse is also true.

Theorem 6.8. *[Hov11, Theorem 2.3] If* A-Mod *has an internal tensor product such that it preserves colimits in each slot then* A *is a sesquialgebra.*

Proof. Suppose there exists an internal tensor product $\underline{\otimes}$ on A-Mod. Then $\triangle := A \underline{\otimes} A$ and ϵ the unit of $\underline{\otimes}$ makes A into a sesquialgebra. □

6.2.1 Hopfish algebra

Definition 6.9. For a sesquialgebra A, a right $A \otimes A$-module S is called a pre-antipode if $\mathrm{Hom}_k(S, k) \simeq \mathrm{Hom}_A(\triangle, \epsilon)$ as left $A \otimes A$-modules. Moreover, if S considered as (A^{op}, A)-bimodule, is free right A-module of rank 1, we say that S is an antipode. The algebra A along with the antipode S is called a hopfish algebra.

Convention: In [TWZ07], $\mathrm{Hom}(B, C)$ means maps from C to B. But for us $\mathrm{Hom}(B, C)$ means maps from B to C.

6.2.2 Examples: certain centralizer algebras

Corollary 6.10. *For* $n \geqslant d$, *the centralizer algebra* $\mathrm{End}_G((k^n)^{\otimes d})$ *is a sesquialgebra with a preantipode. Moreover, if char* k *is* 0 *then* $\mathrm{End}_G((k^n)^{\otimes d})$ *is a Hopfish algebra .*

Proof. By Theorem 6.8 and Proposition 6.3, we can take $\triangle = \mathrm{End}_G((k^n)^{\otimes d}) \underline{\otimes} \mathrm{End}_G((k^n)^{\otimes d}) = h_{(k^n \otimes k^n)}(k^n) = \mathrm{Hom}_G((k^n \otimes k^n)^{\otimes d}, (k^n)^{\otimes d})$ and $\epsilon = h_k(k^n) = \mathrm{Hom}_G(k^{\otimes d}, (k^n)^{\otimes d}) = \mathrm{Hom}_G(k, (k^n)^{\otimes d}) = ((k^n)^{\otimes d})^G$.

Pre-antipode: Let $\mathcal{S} = \mathrm{Hom}_k(((k^n \otimes k^n)^{\otimes d})^G, k)$. Then the left action of $A \otimes A$ on $((k^n \otimes k^n)^{\otimes d})^G$ makes $\mathcal{S}$ into a right $A \otimes A$-module. We have,

$$\begin{aligned}\mathrm{Hom}_A(\triangle, \epsilon) &= \mathrm{Hom}_A(h_{k^n \otimes k^n}(k^n), h_k(k^n)) \simeq \mathrm{Hom}_{\mathrm{Rep}\,\mathcal{G}^d}(h_{k^n \otimes k^n}, h_k) \\ &\simeq h_k(k^n \otimes k^n) = \mathrm{Hom}_G(k, (k^n \otimes k^n)^{\otimes d}) \simeq ((k^n \otimes k^n)^{\otimes d})^G \simeq \mathrm{Hom}_k(\mathcal{S}, k).\end{aligned}$$

The second and third isomorphisms are due to Theorem 6.1 and the Yoneda lemma respectively. Since $((k^n \otimes k^n)^{\otimes d})^G$ is a free k-module of finite rank, so is $\mathcal{S}$ and this justifies the last isomorphism. Using the duality between invariants and coinvariants we have $\mathcal{S} = (((k^n \otimes k^n)^{\otimes d})^*)_G$. If char k =0, invariants and coinvariants are isomorphic. So $\mathcal{S} = (((k^n \otimes k^n)^{\otimes d})^*)^G$ which is $\mathrm{Hom}_G(k, ((k^n \otimes k^n)^{\otimes d})^*) \simeq \mathrm{Hom}_G((k^n)^{\otimes d}, (k^n)^{\otimes d})^*)$. The module $\mathrm{Hom}_G((k^n)^{\otimes d}, (k^n)^{\otimes d})^*)$ is a left $\mathrm{End}_G((k^n)^{\otimes d})^*)$-module of rank 1 thus $\mathcal{S}$ is a right $\mathrm{End}_G((k^n)^{\otimes d})$-module of rank 1. ☐

Thus for $A = \mathrm{End}_G((k^n)^{\otimes d})$, an internal tensor product of A-modules, which was induced from Day convolution, is simply given by

$$M \underline{\otimes} N \simeq \mathrm{Hom}_G((k^n \otimes k^n)^{\otimes d}, (k^n)^{\otimes d}) \otimes_{A \otimes A} (M \otimes N). \tag{6.9}$$

Remark 6.11. Note that when G is the trivial group and $d = 1$, invariants and coinvariants are always isomorphic. So from the proof of Corollary 6.10, we conclude that over any commutative ring k with unity $\mathrm{End}_k(k^n)$ is a Hopfish algebra. It is interesting to observe that a Hopfish algebra structure on $\mathrm{End}_k(k^n)$ given as in [TWZ07, Example 5.6] and the one discussed in the proof of Corollary 6.10 are the same.

Corollary 6.12. *If 2 is a unit k and $n \geqslant d$, the hyperoctahedral Schur algebra $H(2n, d)$ is a sesquialgebra with a preantipode. Moreover, if char $k = 0$ then $H(2n, d)$ is a Hopfish algebra.*

Proof. By using part 4 of Theorem 6.5 and Theorem 6.8, we take $\triangle = \mathrm{Hom}_{B_d}((k^{2n} \otimes k^{2n})^{\otimes d}, (k^{2n})^{\otimes d})$ and $\epsilon = ((k^{2n})^{\otimes d})^{B_d}$ to define a sesquialgebra structure on $H(2n, d)$. The module $\mathcal{S} = \mathrm{Hom}_k(((k^{2n} \otimes k^{2n})^{\otimes d})^{B_d}, k)$ is the pre-antipode. ☐

We give an example in the case $n < d$, that the bifunctor (6.9) may not be associative for $S_k(n, d)$-modules even over a field of characteristic 0. Let us assume henceforth that k is a field of characteristic 0. From [Gre80, Theorem (6.5e)], if a partition λ of d has more than n parts then $ev_{k^n}(L_\lambda) = 0$ and otherwise (i.e. $\lambda \in \Lambda^+(n, d)$) $ev_{k^n}(L_\lambda)) \neq 0$ where L_λ denotes the simple object corresponding to λ in $\mathrm{Rep}\Gamma_k^d$. Recall from the equation (6.4) the definition of $\theta_!$ and its properties. In the case $\lambda \in \Lambda^+(n, d)$, by using [Gre80, Theorem (6.5f)] $ev_{k^n}(L_\lambda)$ is a simple $S_k(n, d)$-module, and in addition using [Mar09, Theorem 4.1.7(ii)], we get $\theta_!(ev_{k^n}(L_\lambda)) \simeq L_\lambda$.

Since $\theta_!$ is the left adjoint of ev_{k^n}, using colimits we realized that the bifunctor (6.9) is isomorphic to $ev_{k^n}(\theta_!(M)\underline{\otimes}\theta_!(N))$.

If $n = 1$ then for any d, $S_k(n,d)$ is k and the bifunctor (6.9) reduces to ordinary tensor product. So the bifunctor (6.9) is associative. If $n = 2$ and $d = 3$ then there are only two simple $S_k(n,d)$-modules, one for each partitions $(2,1), (3)$. In this case also the bifunctor (6.9) turns out to be associative. So the next case is to be analyzed when $n = 2$ and $d = 4$.

Example 6.13. Let $M_1 = ev_{k^2}(L_{(2,2)})$ and $M_2 = M_3 = ev_{k^2}(L_{(3,1)})$ be left $S_k(2,4)$-modules. Note that $\theta_!(M_1) = L_{(2,2)}$ and $\theta_!(M_2) = \theta_!(M_3) = L_{(3,1)}$. We show that $(M_1\underline{\otimes}M_2)\underline{\otimes}M_3$ is not isomorphic to $M_1\underline{\otimes}(M_2\underline{\otimes}M_3)$. Since the compatibility of internal tensor product of objects in $\mathrm{Rep}\Gamma_k^d$ with Kronecker product of kS_d-modules, for simple functors it could be computed from [BDVO15, Corollary 4.2.5] or also from Section 5.4.

Computation of $(M_1\underline{\otimes}M_2)\underline{\otimes}M_3$:

- $(M_1\underline{\otimes}M_2) = ev_{k^2}((\theta_!(M_1)\underline{\otimes}\theta_!(M_2)) = ev_{k^2}(L_{(2,2)}\underline{\otimes}L_{(3,1)}) = ev_{k^2}(L_{(3,1)} \oplus L_{(2,1,1)}) = M_3$
- $(M_1\underline{\otimes}M_2)\underline{\otimes}M_3 = ev_{k^2}(L_{(3,1)}\underline{\otimes}L_{(3,1)}) = ev_{k^2}(L_{(3,1)} \oplus L_{(2,2)} \oplus L_{(2,1,1)} \oplus L_{(4)})$
$= ev_{k^2}(L_{(3,1)}) \oplus ev_{k^2}(L_{(2,2)}) \oplus ev_{k^2}(L_{(4)})$.

Computation of $M_1\underline{\otimes}(M_2\underline{\otimes}M_3)$:

- $M_2\underline{\otimes}M_3 = ev_{k^2}(L_{(3,1)} \oplus L_{(2,2)} \oplus L_{(2,1,1)} \oplus L_{(4)}) = ev_{k^2}(L_{(3,1)}) \oplus ev_{k^2}(L_{(2,2)}) \oplus ev_{k^2}(L_{(4)})$.
- $M_1\underline{\otimes}(M_2\underline{\otimes}M_3) = ev_{k^2}(L_{(2,2)}\underline{\otimes}(L_{(3,1)} \oplus L_{(2,2)} \oplus L_{(4)})) = ev_{k^2}(L_{(3,1)}) \oplus ev_{k^2}(L_{(2,2)}) \oplus ev_{k^2}(L_{(2,2)}) \oplus ev_{k^2}(L_{(4)})$. (Every computation in this case has already appeared before in the course of this example except $L_{(2,2)}\underline{\otimes}L_{(2,2)}$ which is $L_{(2,2)} \oplus L_{(4)} \oplus L_{(1,1,1,1)}$).

6.3 Concluding remarks

From [TWZ07], an algebra which is Morita equivalent to a quasi-Hopf algebra becomes a sesquialgebra. As specified in the introduction of [TWZ07] that it would be nice to have a new class of sesquialgebras which are not Morita equivalent to a quasi-Hopf algebra. Now, these new centralizer algebras $\mathrm{End}_G((k^n)^{\otimes d})$ and $H(2n,d)$ certainly give a potentially large class of sesquialgebras with pre-antipode. It seems hard but still interesting to explore whether these new centralizer algebras are Morita equivalent to quasi-Hopf algebras.

In a joint work with A. Prasad and G. Thanganvelu, we are exploring the representation theory of centralizer algebra $AS_k(n,d) := \mathrm{End}_G((k^n)^{\otimes d})$ when G is the alternating group $A_d \subset S_d$ [PST]. (Over the field of complex numbers $\mathbb{C}$, this algebra was studied in a work of Regev [Reg02].) Interestingly, representations of $AS_k(n, d)$ is closely related to the representations of $S_k(n, d)$ and the Koszul duality on $S_k(n, d)$-Mod.

Chapter 7

Coherent functors and strict polynomial functors

Franjou and Pirashivli [FP08] studied a connection between the coherent functors in the sense of Auslander [Aus66a] on the category of finite dimensional kS_d-modules with the strict polynomial functors of degree d. More precisely, there is a quotient functor j^* from the category of coherent functors $\mathcal{C}(S_d)$ to $\mathrm{rep}\Gamma^d_k$. The external tensor product of representations of S_{d_1} and S_{d_2} induces an external tensor product on the category $\mathcal{C}(S_{d_1+d_2})$ [FP08]. It was shown in [FP08] that this external tensor product of coherent functors via j^* corresponds to the external tensor product of strict polynomial functors at page no. (30). In this chapter we observe that the category of coherent functors has an internal tensor product (Proposition 7.3). The aim of this chapter is to relate via j^* this internal tensor product of coherent functors with the internal tensor product of strict polynomial functors (Proposition 7.5).

7.1 Internal tensor product of coherent functors

Throughout this chapter we assume k is a field.

Let kG-mod be the category of finite dimensional representations of a finite group G. Let $A(G)$ be the category of representations of kG-mod, i.e., k-linear covariant functors from kG-mod to k-Mod.

We recall projective and injective objects of $A(G)$ from [FP08].

Projective objects of $A(G)$: From Proposition 8.8, the representable functor $h_M : kG\text{-mod} \to k$-Mod given by $h_M(N) = \mathrm{Hom}_G(M, N)$ is a projective object. Notice that $h_M(kG) = M^*$, k-linear dual of M.

Injective Objects of $A(G)$: For $M \in kG$-mod, the functor $t_M : kG\text{-mod} \to k\text{-Mod}$ given by $t_M(N) := (M \otimes N)_G$, where G acts on $M \otimes N$ diagonally, is an injective object of $A(G)$.

Hopf algebra structure of kG induces a closed, symmetric monoidal structure on $A(G)$ via Day convolution [IK86]. From Proposition 8.6, for every $F \in A(G)$, $F = \underset{h_M \to F}{\operatorname{colim}} h_M$. We denote this monoidal structure on $A(G)$ by $(-\underline{\odot}-)$. More explicitly, $(-\underline{\odot}-)$ is given as follows:

$$h_M \underline{\odot} h_N := h_{M \otimes N}, \qquad F \underline{\odot} h_N := \underset{h_M \to F}{\operatorname{colim}} h_M \underline{\odot} h_N, \qquad F \underline{\odot} G := \underset{h_N \to G}{\operatorname{colim}} F \underline{\odot} h_N.$$

We also have an internal Hom (closed structure) on $A(G)$ which is given as follows:

$$\mathcal{H}(h_M, h_N) := h_{\mathrm{Hom}(M,N)}, \quad \mathcal{H}(h_M, G) := \underset{h_N \to G}{\operatorname{colim}} \mathcal{H}(h_M, h_N), \quad \mathcal{H}(F, G) := \underset{h_M \to F}{\lim} \mathcal{H}(h_M, G)$$

Proposition 7.1. *The bifunctors* $(-\underline{\odot}-) : A(G) \times A(G) \to A(G)$ *and* $\mathcal{H}(-,-) : A(G)^{op} \times A(G) \to A(G)$ *form an adjunction pair, i.e., we have a natural isomorphism,*

$$\mathrm{Hom}_{A(G)}(F_1 \underline{\odot} F_2, F_3) \simeq \mathrm{Hom}_{A(G)}(F_1, \mathcal{H}(F_2, F_3)),$$

where F_1, F_2 *and* F_3 *are objects of* $A(G)$.

Definition 7.2. An object F in $A(G)$ is said to be coherent if there is an exact sequence $h_N \to h_M \to F \to 0$ where M and N are objects in kG-mod. See [FP08, Definition 2.2.1].

The full subcategory of $A(G)$ consisting of coherent functors is denoted by $\mathcal{C}(G)$. The kernel and co-kernel of morphism between coherent functors are again coherent functors for example see [Har98, Theorem 1.1]. In fact, $\mathcal{C}(G)$ is an abelian category from [Aus66b].

Proposition 7.3. *The closed, symmetric monoidal structure of* $A(G)$ *restricts to the category of coherent functors* $\mathcal{C}(G)$.

Proof. We need to show that the bifunctors $(-\underline{\odot}-)$ and $\mathcal{H}(-,-)$ restrict to the full subcategory $\mathcal{C}(G)$ of $A(G)$. We will show it only for the bifunctor $(-\underline{\odot}-)$ and the arguments for $\mathcal{H}(-,-)$ are similar. By definition, $h_M \underline{\odot} h_N = h_{M \otimes N}$ for M, N in kG-mod. Let F be a coherent functor and M_1 be an object in kG-mod. We first show that $F \underline{\odot} h_{M_1}$ is coherent. We have an exact sequence $h_{M_2} \to h_{M_3} \to F \to 0$ where M_1 and M_2 are objects in kG-mod. We apply the functor $(h_{M_1} \underline{\odot} -)$ to the above the exact sequence. Since $(-\underline{\odot}-)$ is right exact in both the slots, we get $F \underline{\odot} h_{M_1}$ is the quotient of the morphism of the coherent functors, $h_{M_2} \underline{\odot} h_{M_1} \to h_{M_3} \underline{\odot} h_{M_1}$. Since $\mathcal{C}(G)$ is abelian, we get $F \underline{\odot} h_{M_1}$ is a coherent functor. Now repeating the exact arguments in 2^{nd} slot of $(-\underline{\odot}-)$ we conclude if $F, G \in \mathcal{C}(G)$ then $F \underline{\odot} G \in \mathcal{C}(G)$. □

We recall the functors from [FP08, Section 2.4] which relate $\mathcal{C}(G)$ with kG-mod. The functors $t^* : \mathcal{C}(G) \to kG\text{-Mod}$, $t_* : kG\text{-Mod} \to \mathcal{C}(G)$ and $t_! : kG\text{-Mod} \to \mathcal{C}(G)$ are respectively given by $t^*(f) = f(k[G])$, $t_*(M) = h_{M^*}$ and $t_!(M) = t_M$. Moreover, $t_!$ (resp. t_*) is the left (resp. right) adjoint of t^* and $t^* \circ t_! \simeq \mathrm{id}$ (resp. $t^* \circ t_* \simeq \mathrm{id}$). In the next proposition, we observe the compatibility of internal structures under these functors.

Proposition 7.4. *The following hold*

1. *The functor* t^* *is a closed monoidal functor.*

2. *The functor* $t_!$ *preserves internal tensor product.*

3. *The functor* t_* *is closed, lax monoidal functor.*

Proof. We have $t^*(h_M) = h_M(kG) = M^*$. By definition of $(- \odot -)$ and the fact that the Kronecker product commutes with the k-linear duality $*$ we get $t^*(h_M \underline{\odot} h_N) \simeq t^*(h_M) \otimes t^*(h_N)$.

For arbitrary F_1 and F_2 in $\mathcal{C}(G)$, we write them as colimits of representable functors. Now use that the functor t^* preserves colimits and also the bifunctors $(-\underline{\odot}-)$ and $(- \otimes -)$ preserve colimits in each slot. And the fact that t^* preserves internal homs follows from parallel arguments in Theorem 5.3. Since $t_!$ (resp. t_*) is being left (resp. right) adjoint of t^*, the part 2 (resp. part 3) follows by parallel arguments in the proof of Corollary 5.5. □

We recall the functors from [FP08, Section 3] relating coherent functors and strict polynomial functors. The functors $j^* : \mathcal{C}(S_d) \to \mathrm{rep}\Gamma^d_k$, $j_* : \mathrm{rep}\Gamma^d_k \to \mathcal{C}(S_d)$ and $j_! : \mathrm{rep}\Gamma^d_k \to \mathcal{C}(S_d)$ are respectively given by $j^*(F)(V) = F(V^{\otimes d})$, $j_*(X)(M) = \mathrm{Hom}_{\mathrm{rep}\Gamma^d_k}(\mathcal{R}(M^*), X)$ and $j_!(X) = \mathbb{D}((j^*(X^\circ)))$ where $\mathbb{D}(F)(M) = (F(M^*))^*$. Moreover, $j_!$ (resp. j_*) is the left (resp. right) adjoint of j^* and $j^* \circ j_! \simeq \mathrm{id}$ (resp. $j^* \circ j_* \simeq \mathrm{id}$). The functor j^* being a quotient functor, we get every strict polynomial functor must arise from a coherent functor. The following proposition compares $(-\underline{\odot}-)$ with $(-\underline{\otimes}-)$.

Proposition 7.5. *The functor* j^* *is a lax monoidal functor.*

Proof. We prove that the left adjoint $j_!$ of j^* is a monoidal functor. Then it will follow from Proposition 8.13 that j^* is a lax monoidal functor. From [FP08, Lemma 3.2.5] $j_!(\Gamma^{d,V}) = h_{V^{\otimes d}}$. Now by definitions of $(-\underline{\odot}-)$ and $(-\underline{\otimes}-)$, we get that $j_!$ preserves internal tensor product on representable functors. Now it is easy to derive that $j_!$ is a monoidal functor as the bifunctors $(-\underline{\odot}-)$ and $(-\underline{\otimes}-)$ preserve colimits in each slot and also $j_!$ being a left adjoint preserves colimits. □

Example 7.6. If characteristic of k is 0 then j^* is an equivalence and $j_!$ is a quasi-inverse of j^*. Thus in this case j^* is a monoidal functor. But j^* is not a monoidal functor in general. Let k be a field of odd characteristic. Let ϵ be the sign representation of S_d. Then $j^*(h_\epsilon) = \wedge^d$ and $j^*(h_k) = \Gamma^d$. Now $j^*(h_\epsilon \underline{\odot} h_\epsilon) = j^*(h_k) = \Gamma^d$. But $j^*(h_\epsilon) \underline{\otimes} j^*(h_\epsilon) = \wedge^d \underline{\otimes} \wedge^d \simeq \mathrm{Sym}^d$.

From [FP08, Proposition 3.2.9], the functor j^* and its adjoints $j_!$, j_* are part of a recollement of abelian categories. See [FP08, A.1] for the definition of recollement of abelian categories. In the following remark we give some information about the kernel of j^*.

Remark 7.7. Let F be a coherent functor such that $j^*(F) = 0$. We claim that there exists a kS_d-module M such that $\mathrm{Tor}_1^{kS_d}(M, -) \twoheadrightarrow F$. By an equivalent definition of a coherent functor (see [FP08, Equation 3]) there exist kS_d-modules N_1 and N_2 with $N_1 \xrightarrow{f} N_2$ such that $0 \to F \to t_{N_1} \xrightarrow{t_f} t_{N_2}$ is exact. By evaluating this exact sequence at kS_d and using $F(kS_d) = 0$ (from [FP08, Proposition 3.29]), we get $N_1 \xrightarrow{f} N_2$ is injective. So we have a short exact sequence $0 \to N_1 \to N_2 \to M \to 0$ where $M = \mathrm{coker}\, f$. This gives rise the following long exact sequence

$$\to \mathrm{Tor}_1^{kS_d}(N_1, -) \to \mathrm{Tor}_1^{kS_d}(N_2, -) \to \mathrm{Tor}_1^{kS_d}(M, -) \to t_{N_1} \xrightarrow{t_f} t_{N_2} \to t_M \to 0.$$

But F is the kernel of t_f, so $\mathrm{Tor}_1^{kS_d}(M, -) \twoheadrightarrow F$.

Chapter 8

Appendix

8.1 Functor categories

The standard references for this section are [ML98] and [Fre64].

Definition 8.1. A functor category from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a category whose objects are the covariant functors from $\mathcal{C}$ to $\mathcal{D}$. A morphism between two objects is given by a natural transformation. We denote the functor category from $\mathcal{C}$ to $\mathcal{D}$ by $\mathrm{Fct}(\mathcal{C}, \mathcal{D})$. See [ML98, Section II.4] and [Fre64, Chapter 5].

8.1.1 The Yoneda lemma

Let $\mathcal{C}$ be a category whose morphism spaces are sets. For an object A in $\mathcal{C}$, we denote the representable functor $\mathrm{Hom}_{\mathcal{C}}(A, -)$ by h_A. Let $\mathcal{S}ets$ be the category of sets.

Lemma 8.2. *[ML98, Lemma (Yoneda)] For a functor $F : \mathcal{C} \to \mathcal{S}ets$, there is an isomorphism of sets $\mathrm{Hom}_{\mathrm{Fct}(\mathcal{C},\mathcal{S}ets)}(h_A, F) \simeq F(A)$. Moreover, this isomorphism is natural both in A and F.*

Proof. Consider the map $\mathrm{Hom}_{\mathrm{Fct}(\mathcal{C},\mathcal{S}ets)}(h_A, F) \xrightarrow{\varphi} F(A)$ given by $\varphi(\eta) = \eta_A(\mathrm{id}_A)$.

Moreover, a natural transformation η from h_A to F is completely determined by $\eta_A(\mathrm{id}_A)$. For any object A' in $\mathcal{C}$ and $f \in h_A(A') = \mathrm{Hom}_{\mathcal{C}}(A, A')$, then $\eta_{A'}(f) = F(f)(\eta_A(\mathrm{id}_A))$. We write $\eta_{A'}(-) = F(-)(\eta_A(\mathrm{id}_A))$. For more detail see [ML98, Section III.2]. □

For objects A, B and C in $\mathcal{C}$, consider the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Fct}(\mathcal{C},\mathcal{S}ets)}(h_A, h_B) \times \mathrm{Hom}_{\mathrm{Fct}(\mathcal{C},\mathcal{S}ets)}(h_B, h_C) & \longrightarrow & \mathrm{Hom}_{\mathrm{Fct}(\mathcal{C},\mathcal{S}ets)}(h_A, h_C) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}}(B, A) \times \mathrm{Hom}_{\mathcal{C}}(C, B) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(C, A), \end{array} \qquad (\dagger)$$

where the vertical maps are given as in Yoneda lemma 8.2 and the lower horizontal map is the composition of morphisms in $\mathcal{C}$. Now we describe the upper horizontal map in (†). Let $\eta \in \mathrm{Hom}_{\mathrm{Fct}(\mathcal{C},\mathcal{S}ets)}(h_A, h_B)$ and $\beta \in \mathrm{Hom}_{\mathrm{Fct}(\mathcal{C},\mathcal{S}ets)}(h_B, h_C)$. Define $\beta\eta$, an element of $\mathrm{Hom}_{\mathrm{Fct}(\mathcal{C},\mathcal{S}ets)}(h_A, h_C)$, as follows: given $T \in \mathcal{C}$, $(\beta\eta)_T = \beta_T \circ \eta_T$. This composition of natural transformations is called the vertical composition.

From the Yoneda lemma, we have $\varphi(\eta) = \eta_A(\mathrm{id}_A)$ and $\varphi(\beta) = \beta_B(\mathrm{id}_B)$. Then,

$$\begin{aligned}\varphi(\beta\eta) = (\beta\eta)_A(\mathrm{id}_A) &= (\beta_A \circ \eta_A)(\mathrm{id}_A) = \beta_A(\eta_A(\mathrm{id}_A)) = h_C(\eta_A(\mathrm{id}_A))(\beta_B(\mathrm{id}_B)) \\ &= (\eta_A(\mathrm{id}_A) \circ -)(\beta_B(\mathrm{id}_B)) = \eta_A(\mathrm{id}_A) \circ \beta_B(\mathrm{id}_B).\end{aligned}$$

This shows that the diagram (†) is commutative.

Definition 8.3. A category $\mathcal{C}$ is said to be a k-linear category if the morphism space of any two objects in $\mathcal{C}$ is a k-module and the composition of morphisms is bilinear.

The endomorphism space of an object in a k-linear category $\mathcal{C}$ is a k-algebra, where the algebra structure is given by compositions.

Proposition 8.4. *Let $\mathcal{C}$ be a k-linear category and A be an object of $\mathcal{C}$. Then k-linear isomorphism in the Yoneda lemma* $\mathrm{End}_{Fct(\mathcal{C},\mathcal{S}ets)}(h_A, h_A) \simeq \mathrm{End}_{\mathcal{C}}(A)$ *is an algebra anti-isomorphism.*

Proof. Take all objects of $\mathcal{C}$ in the diagram (†) to be A. Then the proof follows from the commutativity of (†). □

Let $(\mathrm{End}_{\mathcal{C}}(A))^{\mathrm{op}}$ be the opposite algebra of $\mathrm{End}_{\mathcal{C}}(A)$. Then the k-linear isomorphism, $\mathrm{End}_{\mathrm{Fct}(\mathcal{C},\mathcal{S}ets)}(h_A) \simeq (\mathrm{End}_{\mathcal{C}}(A))^{\mathrm{op}}$, in the Yoneda lemma, is an algebra isomorphism.

Remark 8.5. Throughout this thesis we deal with functor categories when $\mathcal{C}$ is a k-linear category and $\mathcal{D}$ is the category k-Mod. The objects of $\mathrm{Fct}(\mathcal{C}, k\text{-Mod})$ are k-linear covariant functors. In this case, $\mathrm{Fct}(\mathcal{C}, k\text{-Mod})$ is abelian, complete and co-complete category. The exactness of a chain complex of objects in $\mathrm{Fct}(\mathcal{C}, k\text{-Mod})$ can be checked pointwise, i.e., by by evaluating at each object c in $\mathcal{C}$.

Proposition 8.6. *[ML98, III.7, Theorem 1] Let $\mathcal{C}$ be a category whose morphism spaces are sets. Let $F : \mathcal{C} \to \mathcal{S}ets$ be a covariant functor. The F is the colimit of representable functors.*

Proof. Consider a category $\mathcal{J}$ whose objects are the natural transformations $\phi_c : \mathrm{Hom}_{\mathcal{C}}(c, -) \to F$ for every object c in $\mathcal{C}$. A morphism from ϕ_c and ϕ_d is a morphism $\mathrm{Hom}_{\mathcal{C}}(c, -) \to \mathrm{Hom}_{\mathcal{C}}(d, -)$ such that the following diagram of natural transformations commute

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(c,-) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(d,-) \\ & \searrow^{\phi_c} & \downarrow{\phi_d} \\ & & F \end{array} .$$

Define a functor $\theta : \mathcal{J} \to \mathrm{Fct}(\mathcal{C}, \mathcal{S}ets)$ such that $\theta(\phi_c) = \mathrm{Hom}_{\mathcal{C}}(c,-)$. Then F together with family of morphisms $\{\phi_c\}_{c \in \mathcal{C}}$ is the universal co-cone of θ or simply it is the colimit of θ. We write $F = \operatorname{colim}_{\phi_c} \mathrm{Hom}_{\mathcal{C}}(c,-)$. For explicit details we refer [ML98, III.7, Theorem 1]. □

For the result in Proposition 8.6, one also uses the terminology that the functor category is *locally presentable* by representable functors. Proposition 8.6 remains true if we replace the category of $\mathcal{S}ets$ by a co-complete category $\mathcal{D}$ such that $\mathcal{C}$ is enriched over $\mathcal{D}$.

Projective generator :
In an abelian category $\mathcal{C}$ an object c is projective if $\mathrm{Hom}_{\mathcal{C}}(c,-)$ is an exact functor.

Definition 8.7. A class of projective objects in an abelian category $\mathcal{C}$ is said to be a projective generator if every object in $\mathcal{C}$ is a quotient of direct sum of some of objects in the given class.

Proposition 8.8. *Let $\mathcal{C}$ be a k-linear category. Then the representable functor $h_c = \mathrm{Hom}_{\mathcal{C}}(c,-)$ is a projective object in $\mathrm{Fct}(\mathcal{C}, k\text{-Mod})$. Moreover, the class $\{h_c\}$ as c varies over the class of objects of $\mathcal{C}$ is a projective generator in $\mathrm{Fct}(\mathcal{C}, k\text{-Mod})$.*

Proof. Follows from the Yoneda lemma and Remark 8.5 or see for example also [Fre64, Exercise 5.G]. □

Definition 8.9. An object c in an abelian category $\mathcal{C}$ is called a small projective generator if $\{c\}$ is a projective generator and preserves set-indexed direct sums.

Theorem 8.10. *[Fre64, Page 106] An abelian category $\mathcal{C}$ which is complete and has a small projective generator P if and only if $\mathcal{C}$ is equivalent to $\text{Mod-}R$ where $R \simeq \mathrm{End}_{\mathcal{C}}(P)$.*

8.2 Monoidal categories

We recall the definitions of monoidal categories and monoidal functors from [ML98, Chapter VII].

Definition 8.11. A data $(\mathcal{C}, \underline{\otimes}, e, \alpha, \lambda, \rho)$, where $\mathcal{C}$ is a category and $\underline{\otimes}$ is a bifunctor on $\mathcal{C}$, is called a monoidal category if (a) for objects X, Y and $Z \in \mathcal{C}$, $\alpha_{X,Y,Z} : X\underline{\otimes}(Y\underline{\otimes}Z) \to (X\underline{\otimes}Y)\underline{\otimes}Z$ is an isomorphism such that the following pentagon diagram commutes, (b) for object $X \in \mathcal{C}$, $\lambda_X : e\underline{\otimes}X \to X$ and $\rho_X : X\underline{\otimes}e \to X$ are the isomorphisms such that the following triangular diagram commutes.

$$
\begin{array}{ccccc}
 & & X\underline{\otimes}(Y\underline{\otimes}(Z\underline{\otimes}T)) & & \\
 & \swarrow^{id_X\underline{\otimes}\alpha_{Y,Z,T}} & & \searrow^{\alpha_{X,Y\underline{\otimes}Z,T}} & \\
X\underline{\otimes}((Y\underline{\otimes}Z)\underline{\otimes}T)) & & & & (X\underline{\otimes}Y)\underline{\otimes}(Z\underline{\otimes}T) \\
\downarrow{\scriptstyle \alpha_{X,Y\underline{\otimes}Z,T}} & & & & \downarrow{\scriptstyle \alpha_{X\underline{\otimes}Y,Z,T}} \\
(X\underline{\otimes}(Y\underline{\otimes}Z))\underline{\otimes}T & & \xrightarrow{\alpha_{X,Y,Z}\underline{\otimes}id_T} & & ((X\underline{\otimes}Y)\underline{\otimes}Z)\underline{\otimes}T
\end{array}
\quad , \quad
\begin{array}{ccc}
X\underline{\otimes}(e\underline{\otimes}Y) & \xrightarrow{\alpha_{X,e,Y}} & (X\underline{\otimes}e)\underline{\otimes}Y \\
\downarrow{\scriptstyle id_X\underline{\otimes}\lambda_Y} & \swarrow{\scriptstyle \rho_Y\underline{\otimes}id_Y} & \\
X\underline{\otimes}Y & &
\end{array}
$$

The bifunctor $\underline{\otimes}$ in the data of a monoidal category is called the internal tensor product. A symmetric monoidal category is said to be closed if there exists a bifunctor $\mathbb{H}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ such that for objects X, Y and Z in $\mathcal{C}$ one has the following natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X\underline{\otimes}Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X, \mathbb{H}(Y, Z)).$$

The bifunctor $\mathbb{H}$ is called an internal Hom on $\mathcal{C}$.

Definition 8.12. A functor $F : (\mathcal{C}, \underline{\otimes}, e, \lambda, \rho) \to (\mathcal{C}', \underline{\otimes}', e', \lambda', \rho')$ is said to be monoidal if there is a natural isomorphism $\phi_{X,Y} : F(X)\underline{\otimes}'F(Y) \to F(X\underline{\otimes}Y)$ and an isomorphism $\phi : e' \to F(e)$. Moreover, if categories $\mathcal{C}$ and $\mathcal{C}'$ are closed and F preserves internal Hom as well then we call F is closed monoidal.

$$
\begin{array}{ccc}
(F(X)\underline{\otimes}'F(Y))\underline{\otimes}'F(Z) & \xrightarrow{\alpha'_{F(X),F(Y),F(Z)}} & F(X)\underline{\otimes}'(F(Y)\underline{\otimes}'F(Z)) \\
\downarrow{\scriptstyle \phi_{X,Y}\underline{\otimes}'id_{F(Z)}} & & \downarrow{\scriptstyle id_{F(X)}\underline{\otimes}'\phi_{Y,Z}} \\
F(X\underline{\otimes}Y)\underline{\otimes}'F(Z) & & F(X)\underline{\otimes}'(F(Y)\underline{\otimes}F(Z)) \\
\downarrow{\scriptstyle \phi_{X\underline{\otimes}Y,Z}} & & \downarrow{\scriptstyle \phi_{X,Y\underline{\otimes}Z}} \\
F((X\underline{\otimes}Y)\underline{\otimes}Z) & \xrightarrow[F(\alpha_{X,Y,Z})]{} & F(X\underline{\otimes}(Y\underline{\otimes}Z))
\end{array}
$$

$$
\begin{array}{ccc}
F(X)\underline{\otimes}'e' & \xrightarrow{id_{F(X)}\underline{\otimes}'\phi} & F(X)\underline{\otimes}'F(e) \\
\downarrow{\scriptstyle \rho'_{F(X)}} & & \downarrow{\scriptstyle \phi_{X,e}} \\
F(X) & \xleftarrow[F(\rho_X)]{} & F(X\underline{\otimes}e)
\end{array}
\qquad
\begin{array}{ccc}
e'\underline{\otimes}'F(X) & \xrightarrow{\phi\underline{\otimes}'id_{F(X)}} & F(e)\underline{\otimes}'F(X) \\
\downarrow{\scriptstyle \lambda'_{F(X)}} & & \downarrow{\scriptstyle \phi_{e,X}} \\
F(X) & \xleftarrow[F(\lambda_X)]{} & F(e\underline{\otimes}X)
\end{array}
$$

If there is natural morphism $\phi_{X,Y} : F(X)\underline{\otimes}'F(Y) \to F(X\underline{\otimes}Y)$ (need not be an isomorphism) and a morphism $\phi : e' \to F(e)$ such that the above diagrams commute then we call F is a lax monoidal functor.

Proposition 8.13. *[AM10, Proposition 3.84] Let $(\mathcal{C}, \underline{\otimes})$ and $(\mathcal{D}, \underline{\otimes}')$ be two monoidal categories. Then the right adjoint $\mathcal{R} : \mathcal{C} \to \mathcal{D}$ of a monoidal functor $\mathcal{L} : \mathcal{D} \to \mathcal{C}$ is a lax monoidal functor.*

Proof. We only provide morphism $\phi_{X,Y} : \mathcal{R}(X)\underline{\otimes}'\mathcal{R}(Y) \to \mathcal{R}(X\underline{\otimes}Y)$ for objects X, Y in $\mathcal{C}$. For the details of commutativity of the diagrams in the definition of a lax monoidal functor we refer [AM10, Proposition 3.84]. We have $\mathrm{Hom}_{\mathcal{D}}(\mathcal{R}(X)\underline{\otimes}'\mathcal{R}(Y), \mathcal{R}(X\underline{\otimes}Y)) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}(\mathcal{R}(X)\underline{\otimes}'\mathcal{R}(Y)), X\underline{\otimes}Y) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathcal{L} \circ \mathcal{R}(X)\underline{\otimes}\mathcal{L} \circ \mathcal{R}(Y), X\underline{\otimes}Y)$, where the first isomorphism is due to $\mathcal{L} \dashv \mathcal{R}$ and the second isomorphism is because of the hypothesis that $\mathcal{L}$ is monoidal. Then using the unit morphism $\mathcal{L} \circ \mathcal{R} \to \mathrm{id}_{\mathcal{D}}$ of the adjunction $\mathcal{L} \dashv \mathcal{R}$ we get a morphism $\mathrm{Hom}_{\mathcal{C}}(X\underline{\otimes}Y, X\underline{\otimes}Y) \xrightarrow{\psi} \mathrm{Hom}_{\mathcal{D}}(\mathcal{R}(X)\underline{\otimes}'\mathcal{R}(Y), \mathcal{R}(X\underline{\otimes}Y))$. The required morphism $\phi_{X,Y}$ is the image of identity morphism $\mathrm{id}_{X\underline{\otimes}Y}$ under ψ. □

The following theorem first appeared in the thesis of Brian Day.

Theorem 8.14. *[IK86] If $\mathcal{C}$ is a symmetric closed monoidal category then* $\mathrm{Fct}(\mathcal{C}, k\text{-Mod})$ *is also a symmetric closed monoidal category.*

8.3 The derived functors on the unbounded derived categories

The results and terminologies in this section are from [KS06]. We also have one small new result (Proposition 8.20) which is analogue of [KS06, Proposition 13.3.13].

For an abelian category $\mathcal{A}$, $\mathrm{Ch}(\mathcal{A})$ and $K(\mathcal{A})$ be the category of the unbounded chain complexes and the unbounded homotopy category respectively. We denote $K^*(\mathcal{A})$ ($* = ub, +, -$) for the full subcategory of $K(\mathcal{A})$ consisting of unbounded complexes (resp. bounded below, resp. bounded above). An additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories induces a triangulated functor $K^*(F) : K^*(\mathcal{A}) \to K^*(\mathcal{B})$ where $* = ub, +, -$. We denote the left (resp. right) derived functor of F by $\mathbf{L}^*F$, if it exists (resp. $\mathbf{R}^*F$) where $* = ub, -$ (resp. $* = ub, +$).

Definition 8.15. Let $\mathcal{J}$ be a full triangulated subcategory of $K^*(A)$. Consider the conditions (1), (2) and (3) below.

1. For every $X \in K^*(\mathcal{A})$, there exists a quasi-isomorphism $Y \to X$ with $Y \in \mathcal{J}$.
2. For every $X \in K^*(\mathcal{A})$, there exists a quasi-isomorphism $X \to Y$ with $Y \in \mathcal{J}$.
3. If $Y \in \mathcal{J}$ is quasi-isomorphic to 0, $F(Y)$ is quasi-isomorphic to 0.

Then

a. if conditions (1) and (3) are satisfied, we say that $\mathcal{I}$ is F-projective.

b. if conditions (2) and (3) are satisfied, we say that $\mathcal{I}$ is F-injective.

Proposition 8.16. *The following hold,*

1. *If $\mathcal{I}$ is F-projective, then the left derived functor $\mathbf{L}F$ exists and $\mathbf{L}F(X) \simeq K^*(F)(Y) = F(Y)$ for $Y \to X$ a quasi-isomorphism with $Y \in \mathcal{I}$.*

2. *If $\mathcal{I}$ is F-injective, then the right derived functor $\mathbf{R}F$ exists and $\mathbf{R}F(X) \simeq K^*(F)(Y) = F(Y)$ for $X \to Y$ a quasi-isomorphism with $Y \in \mathcal{I}$.*

The derived functors of compositions:
Let $F : \mathcal{C} \times \mathcal{D} \to \mathcal{C}' \times \mathcal{D}'$ and $F' : \mathcal{C}' \times \mathcal{D}' \to \mathcal{D}''$ be an additive functor and an additive bifunctor respectively (all the categories here are abelian).

Remark 8.17. We encounter F when it is product two additive functors which is be by definition a bifunctor but not an additive bifunctor in general.

Definition 8.18. A pair of full triangulated subcategories $(\mathcal{I}'_1, \mathcal{I}'_2)$ of $K^*(\mathcal{C}') \times K^*(\mathcal{D}')$ is called F'-projective (resp. injective) if for every $Y_1 \in \mathcal{I}'_1$, $\mathcal{I}_2$ is $F'(Y_1, -)$-projective (resp. injective) and for every $Y_2 \in \mathcal{I}'_2$, $\mathcal{I}'_1$ is $F'(-, Y_2)$-projective (resp. injective).

Proposition 8.19. *The following hold,*

1. *If $(\mathcal{I}'_1, \mathcal{I}'_2)$ is a F'-projective, the left derived bifunctor $\mathbf{L}^*F'$ exists and moreover $\mathbf{L}^*F'(X', Y') \simeq K^*(F')(X'', Y'') = F'(X'', Y'')$ for quasi-isomorphisms $X'' \to X'$ and $Y'' \to Y'$ with $X'' \in \mathcal{I}'_1$ and $Y'' \in \mathcal{I}'_2$.*

2. *If $(\mathcal{I}'_1, \mathcal{I}'_2)$ is a F'-injective, the right derived bifunctor $\mathbf{R}^*F'$ exists and moreover $\mathbf{R}^*F'(X', Y') \simeq K^*(F')(X'', Y'') = F'(X'', Y'')$ for quasi-isomorphisms $X' \to X''$ and $Y' \to Y''$ with $X'' \in \mathcal{I}'_1$ and $Y'' \in \mathcal{I}'_2$.*

The following proposition is the analogue of Grothendieck spectral sequence for bi-functors or simply derived functor of composition of a bi-functor with a functor (compare with [KS06, Proposition 13.3.13]).

Proposition 8.20. *The following hold,*

1. *Let $* = ub, -$. Assume that the left derived functor $\mathbf{L}^*F$ and the left derived bifunctors $\mathbf{L}^*F'$, $\mathbf{L}^*(F' \circ F)$ exist. Then there is a canonical morphism*

$$\mathbf{L}^*F' \circ \mathbf{L}^*F \to \mathbf{L}^*(F' \circ F). \tag{8.1}$$

2. *Assume that there exist full additive subcategories* $(\mathcal{I}_1, \mathcal{I}_2) \subset K^*(\mathcal{C}) \times K^*(\mathcal{D})$ *and* $(\mathcal{I'}_1, \mathcal{I'}_2) \subset K^*(\mathcal{C'}) \times K^*(\mathcal{D'})$ *such that* $(\mathcal{I}_1, \mathcal{I}_2)$ *is* F*-projective,* $(\mathcal{I'}_1, \mathcal{I'}_2)$ *is* F'*-projective and* $K^*(F)((\mathcal{I}_1, \mathcal{I}_2)) \subset (\mathcal{I'}_1, \mathcal{I'}_2)$*. Then* $(\mathcal{I}_1, \mathcal{I}_2)$ *is* $(F' \circ F)$*-projective and (8.1) induces an isomorphism*

$$\mathbf{L}^- F' \circ \mathbf{L}^- F \to \mathbf{L}^-(F' \circ F). \tag{8.2}$$

Proof. 1. From the universal properties of both the left derived functor and bifunctor, there are a bijection

$$\mathrm{Hom}(L^* F' \circ L^* F, L^*(F' \circ F)) \simeq \mathrm{Hom}(L^* F' \circ L^* F \circ Q, Q'' \circ F' \circ F) \tag{8.3}$$

and natural morphisms of functors

$$L^* F' \circ Q' \to Q'' \circ F' \quad \text{and} \quad L^* F \circ Q \to Q' \circ F.$$

We have the canonical morphisms

$$L^* F' \circ L^* F \circ Q \to L^* F' \circ Q' \circ F \to Q'' \circ F' \circ F$$

and the image of this composition under the bijection (8.3) gives the required morphism (8.1).

2. This follows from the part *1* of Proposition 8.19.

□

Similar equations as (8.1) and (8.2) also hold for the right derived functor. This is the next proposition.

Proposition 8.21. *The following hold,*

1. *Let* $* = \mathrm{ub}, +$*. Assume that the right derived functor* $\mathbf{R}^* F$ *and the right derived bifunctors* $\mathbf{R}^* F'$, $\mathbf{R}^*(F' \circ F)$ *exist. Then there is a canonical morphism*

$$\mathbf{R}^*(F' \circ F) \to \mathbf{R}^* F' \circ \mathbf{R}^* F. \tag{8.4}$$

2. *Assume that there exist full additive subcategories* $(\mathcal{I}_1, \mathcal{I}_2) \subset K^*(\mathcal{C}) \times K^*(\mathcal{D})$ *and* $(\mathcal{I'}_1, \mathcal{I'}_2) \subset K^*(\mathcal{C'}) \times K^*(\mathcal{D'})$ *such that* $(\mathcal{I}_1, \mathcal{I}_2)$ *is* F*-injective,* $(\mathcal{I'}_1, \mathcal{I'}_2)$ *is* F'*-injective and* $K^*(F)(\mathcal{I}_1, \mathcal{I}_2) \subset (\mathcal{I'}_1, \mathcal{I'}_2)$*. Then* $(\mathcal{I}_1, \mathcal{I}_2)$ *is* $(F' \circ F)$*-injective and (8.4) induces an isomorphism*

$$\mathbf{R}^+(F' \circ F) \to \mathbf{R}^+ F' \circ \mathbf{R}^+ F. \tag{8.5}$$

8.3.1 Homotopy limits and homotopy colimits

Homotopy limits and colimits were defined by Bökstedt and Neeman in [BN93]. A nice notes on these is given by Murfet [Mur]. The results and their proofs in this section are learnt from [Mur].

Homotopy limits/colimits are often useful to provide isomorphisms between right/left derived functors on unbounded derived category of an abelian category.

Throughout this section X and Y are two objects in $\mathrm{Ch}(\mathcal{A})$ and $u : X \to Y$ is a morphism in $\mathrm{Ch}(\mathcal{A})$.

Let $m \in \mathbb{Z}$. Then one can define shift functors $[m] : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ and $[-m] : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ as follows:

$$(X[m])^n = X^{n+m},\ \partial^n_{X[m]} = (-1)^m \partial^n_X \text{ and } u[m]^n = u^{n+m}.$$

We recall two very important and standard constructions from homological algebra associated to a morphism u.

Mapping cone: Let C_u be the complex $X[1] \oplus Y$. For $n \in \mathbb{Z}$, the n^{th} term of C_u is $X^{n+1} \oplus Y^n$ and the n^{th} differential, $\partial^n_{C_u} : C^n_u \to C^{n+1}_u$, is given by

$$\partial^n_{C_u} = \begin{pmatrix} -\partial^{n+1}_X & 0 \\ u^{n+1} & \partial^n_Y \end{pmatrix}.$$

Proposition 8.22. *Let $u : X \to Y$ be a morphism in an abelian category. Let $v : Y \to C_u$ be the inclusion morphism and $\pi : C_u \to X[1]$ be the projection morphism. Then*

1. *A morphism $f : Y \to Q$ factors through v if and only if $f \circ u$ is homotopic to 0.*
2. *A morphism $f : Q \to X$ factors through $\pi[-1]$ if and only if $u \circ f$ is homotopic to 0.*

Mapping cylinder: The mapping cylinder $\widetilde{C}_u$ is the mapping cone of $\pi[-1] : C_u[-1] \to X$. The n^{th} term of $\widetilde{C}_u$ is $X^{n+1} \oplus Y^n \oplus X^n$ and the n^{th} differential is given by

$$\partial^n_{\widetilde{C}_u} = \begin{pmatrix} -\partial^{n+1}_X & 0 & 0 \\ u^{n+1} & \partial^n_Y & 0 \\ 1 & 0 & \partial^n_X \end{pmatrix}.$$

We have a morphism $\phi : \widetilde{C_u} \to Y$ given by $\phi^n = (0 \ \ -1 \ \ u^n)$. The morphism ϕ is a homotopy equivalence.

Proposition 8.23. *Suppose $0 \to X \xrightarrow{u} Y \xrightarrow{q} Q \to 0$ is an exact sequence of complexes of objects in an abelian category $\mathcal{A}$. Then there is a canonical commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & \widetilde{C}_u & \xrightarrow{\pi} & C_u & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{q} & Q & \longrightarrow & 0, \end{array} \tag{8.6}$$

where the morphism f is given by $f^n = (0, q^n)$. Moreover, the morphism f is a quasi-isomorphism.

Proof. Checking the commutativity of the diagram (8.6) is straightforward. Since the rows of the commutative diagram (8.6) are exact therefore we have a commutative diagram of long exact sequence on homologies

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^n(X) & \longrightarrow & H^n(\widetilde{C}_u) & \longrightarrow & H^n(C_u) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle H^n(\phi)} & & \downarrow{\scriptstyle H^n(f)} & & \\ \cdots & \longrightarrow & H^n(X) & \longrightarrow & H^n(Y) & \longrightarrow & H^n(Q) & \longrightarrow & \cdots \end{array} \tag{8.7}$$

Since ϕ is a homotopy equivalence, $H^n(\phi)$ is an isomorphism. Thus using five-lemma we conclude that $H^n(f)$ is an isomorphism. So the morphism f is a quasi-isomorphism. □

Lemma 8.24. *[KS06, Exercise 8.37] Let $\mathcal{A}$ be a Grothendieck category. Let $\{X_n\}_{n\geqslant 0}$ be an inductive system in $\mathcal{A}$ indexed by $\mathbb{N} \cup \{0\}$. Let $sh : \bigoplus_{n\geqslant 0} X_n \to \bigoplus_{n\geqslant 0} X_n$ be a morphism in $\mathcal{A}$ associated with $X_n \xrightarrow{\phi_n} X_{n+1}$. Then we have an exact sequence*

$$0 \to \bigoplus_{n\geqslant 0} X_n \xrightarrow{id-sh} \bigoplus_{n\geqslant 0} X_n \to \operatorname{colim} X_n \to 0. \tag{8.8}$$

Proof. For every $m \in \mathbb{N}$, we have a short exact sequence

$$0 \to \bigoplus_{n=0}^{m} X_n \xrightarrow{id-sh} \bigoplus_{n=0}^{m+1} X_n \xrightarrow{\theta} X_{m+1} \to 0 \tag{8.9}$$

For example, let $m = 2$ then morphisms $id - sh = \begin{pmatrix} id_{X_1} & 0 \\ -\phi_1 & id_{X_2} \\ 0 & -\phi_2 \end{pmatrix}$ and $\theta = \begin{pmatrix} \phi_2 \circ \phi_1 & \phi_2 & id_{X_3} \end{pmatrix}$. The composition $\theta \circ (id - sh) = 0$. The morphisms $(id - sh)$ and θ are monomorphism and epimorphism respectively. Thus exactness of the sequence (8.9) follows in this case.

Since $\mathcal{A}$ is Grothendieck category, in particular, colimits of exact sequences is exact, we get the exactness of the sequence (8.8) by applying the colimit to the short exact sequence (8.9). $\square$

Let the pair $(\mathcal{K}, T)$ denote a triangulated category.

Definition 8.25. Suppose $(\mathcal{K}, T)$ satisfies AB3 condition (i.e. arbitrary coproducts exist). Let $X_0 \overset{\phi_0}{\to} X_1 \overset{\phi_1}{\to} X_2 \overset{\phi_2}{\to} \cdots$ be a sequence of objects and morphisms in $\mathcal{K}$. Let $sh : \bigoplus_{i \geqslant 0} X_i \to \bigoplus_{i \geqslant 0} X_i$ be the morphism $\bigoplus_{i \geqslant 0} \phi_i$. We say that X is a homotopy colimit of the above sequence if

$$\bigoplus_{i \geqslant 0} X_i \xrightarrow{id - sh} \bigoplus_{i \geqslant 0} X_i \longrightarrow X \longrightarrow T(\bigoplus X_i)$$

is a distinguished triangle. We write $X = \operatorname{hocolim} X_i$.

Corollary 8.26. *Let $\mathcal{A}$ be a Grothendieck category. Let $X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} \cdots$ be a sequence of objects and morphisms in $K(\mathcal{A})$. Then there is a quasi-isomorphism $\operatorname{hocolim} X_i \to \operatorname{colim} X_i$. In particular, we have $H^j(\operatorname{hocolim} X_i) \simeq \operatorname{colim} H^j(X_i)$ for every $j \in \mathbb{Z}$.*

Proof. Let ϕ_i' be a chain complex map representing the class ϕ_i in the homotopy category $K(\mathcal{A})$. We have an inductive system, $X_0 \xrightarrow{\phi_0'} X_1 \xrightarrow{\phi_1'} \cdots$, in the category $Ch(\mathcal{A})$. Since $\mathcal{A}$ is a Grothendieck category, $Ch(\mathcal{A})$ is a Grothendieck category. Thus $\operatorname{colim} X_i$ exists in $Ch(\mathcal{A})$. From Lemma 8.24 we have a short exact sequence of complexes,

$$0 \to \bigoplus_{i \geqslant 0} X_i \xrightarrow{id - sh} \bigoplus_{i \geqslant 0} X_i \to \operatorname{colim} X_i \to 0$$

By Proposition 8.23 we have a quasi-isomorphism from $C_{id-sh} \to \operatorname{colim} X_i$. But by the definition of a homotopy colimit, $\operatorname{hocolim} X_i$ is quasi-isomorphic to C_{id-sh}. $\square$

The following definition is the dual notion of homotopy colimit.

Definition 8.27. Suppose $(\mathcal{K}, T)$ satisfies AB3* condition (i.e. arbitrary products exist). Let $\cdots \to X_3 \xrightarrow{\phi_2} X_2 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_0} X_0$ be a sequence of objects and morphisms. We say that X is a homotopy limit of the above sequence if $T^{-1}(\prod_{i \geqslant 0} X_i) \longrightarrow X \longrightarrow \prod_{i \geqslant 0} X_i \xrightarrow{id - sh} \prod_{i \geqslant 0} X_i$ is a distinguished triangle, where $sh : \prod_{i \geqslant 0} X_i \to \prod_{i \geqslant 0} X_i$ is the morphism $\prod_{i \geqslant 0} \phi_i$.

Lemma 8.28. *Let $\mathcal{A}$ be an abelian category satisfying AB4* condition (i.e. AB3* and product of epimorphisms is epimorphism). Assume that $\mathcal{A}$ has projective generators. Let $\cdots \to X_3 \xrightarrow{\phi_2} X_2 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_0} X_0$ be a sequence of objects in $K(\mathcal{A})$. Let X be an object of $K(\mathcal{A})$ together with morphisms $X \to X_i$ which are compatible with the sequence morphisms ϕ_i. If for every n, the map $H^n(X) \to H^n(X_i)$ is eventually an isomorphism, then we have a quasi-isomorphism $X \to \operatorname{holim}(X_i)$.*

Proof. We only sketch a proof and explicit details can be found for example in [Mur, Lemma 77]. The object X in $K(\mathcal{A})$ is given with the collection of morphisms $X \to X_i$ therefore by the universal property of direct product we have a morphism $X \xrightarrow{\theta} \prod_{i \geqslant 0} X_i$. By hypothesis the morphisms $X \to X_i$ are compatible with the sequence morphisms $X_{i+1} \xrightarrow{\phi_i} X_i$, the composition

$$X \xrightarrow{\theta} \prod_{i \geqslant 0} X_i \xrightarrow{id-sh} \prod_{i \geqslant 0} X_i \tag{8.10}$$

is zero. Using Proposition 8.22 we have commutative diagram of sequences in $K(\mathcal{A})$.

$$\begin{array}{ccccc} X & \xrightarrow{\quad\theta\quad} & \prod_{i \geqslant 0} X_i & \xrightarrow{id-sh} & \prod_{i \geqslant 0} X_i \\ \downarrow f & & \downarrow id & & \downarrow id \\ C_{id-sh}[-1] & \longrightarrow & \prod_{i \geqslant 0} X_i & \xrightarrow{id-sh} & \prod_{i \geqslant 0} X_i \end{array} \tag{8.11}$$

We have commutative diagram of n^{th} homology

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^n(X) & \longrightarrow & H^n(\prod_{i \geqslant 0} X_i) & \xrightarrow{H^n(id-sh)} & H^n(\prod_{i \geqslant 0} X_i) & \longrightarrow & 0 \\ & & \downarrow H^n(f) & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \longrightarrow & H^n(C_{id-sh}[-1]) & \longrightarrow & H^n(\prod_{i \geqslant 0} X_i) & \xrightarrow{H^n(id-sh)} & H^n(\prod_{i \geqslant 0} X_i) & \longrightarrow & 0 \end{array} \tag{8.12}$$

The top row of the diagram (8.12) is exact because of the hypothesis, eventually $H^n(X) \to H^n(X_i)$ is an isomorphism. The bottom row of the diagram (8.12) is exact because

$$C_{id-sh}[-1] \to \prod_{i \geqslant 0} X_i \xrightarrow{id-sh} \prod_{i \geqslant 0} X_i$$

is a distinguished triangle in $K(\mathcal{A})$ and the morphism $H^n(id - sh)$ is an epimorphism. Thus $H^n(f)$ is isomorphism for every $n \in \mathbb{Z}$. So f is a quasi-isomorphism. By definition of homotopy limit, holim X_i is a quasi-isomorphism to $C_{id-sh}[-1]$. □

Index of Symbols

Bibliography

[AB85] Kaan Akin and David A. Buchsbaum. Characteristic-free representation theory of the general linear group. *Adv. in Math.*, 58(2):149–200, 1985.

[AB88] Kaan Akin and David A. Buchsbaum. Characteristic-free representation theory of the general linear group. II. Homological considerations. *Adv. in Math.*, 72(2):171–210, 1988.

[ABW82] Kaan Akin, David A. Buchsbaum, and Jerzy Weyman. Schur functors and Schur complexes. *Adv. in Math.*, 44(3):207–278, 1982.

[AM10] Marcelo Aguiar and Swapneel Mahajan. *Monoidal functors, species and Hopf algebras*, volume 29 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2010. With forewords by Kenneth Brown and Stephen Chase and André Joyal.

[AR15] C. Aquilano and R. Reischuk. The monoidal structure on strict polynomial functors. *arXiv:1503.05108*, 2015.

[Aus66a] Maurice Auslander. *Coherent Functors*, pages 189–231. Springer Berlin Heidelberg, Berlin, Heidelberg, 1966.

[Aus66b] Maurice Auslander. Coherent functors. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 189–231. Springer, New York, 1966.

[Axt13] J. Axtell. Spin polynomial functors and representations of schur superalgebras. *arxiv:1302.0042*, 2013.

[BDVO15] C. Bowman, M. De Visscher, and R. Orellana. The partition algebra and the Kronecker coefficients. *Trans. Amer. Math. Soc.*, 367(5):3647–3667, 2015.

[BMT16] Lawrence Breen, Roman Mikhailov, and Antoine Touzé. Derived functors of the divided power functors. *Geom. Topol.*, 20(1):257–352, 2016.

[BN93] Marcel Bökstedt and Amnon Neeman. Homotopy limits in triangulated categories. *Compositio Math.*, 86(2):209–234, 1993.

[Bof89] Giandomenico Boffi. Characteristic-free decomposition of skew Schur functors. *J. Algebra*, 125(2):288–297, 1989.

[Bou81] Nicolas Bourbaki. *Éléments de mathématique*. Masson, Paris, 1981. Algèbre. Chapitres 4 à 7. [Algebra. Chapters 4–7].

[Cha05] Marcin Chałupnik. Extensions of strict polynomial functors. *Ann. Sci. École Norm. Sup. (4)*, 38(5):773–792, 2005.

[Cha09] Marcin Chałupnik. Extensions of Weyl and Schur functors. *Homology Homotopy Appl.*, 11(2):27–48, 2009.

[DEN04] Stephen R. Doty, Karin Erdmann, and Daniel K. Nakano. Extensions of modules over Schur algebras, symmetric groups and Hecke algebras. *Algebr. Represent. Theory*, 7(1):67–100, 2004.

[DJ91] Richard Dipper and Gordon James. q-tensor space and q-Weyl modules. *Trans. Amer. Math. Soc.*, 327(1):251–282, 1991.

[FP08] Vincent Franjou and Teimuraz Pirashvili. Strict polynomial functors and coherent functors. *Manuscripta Math.*, 127(1):23–53, 2008.

[Fre64] Peter Freyd. *Abelian categories. An introduction to the theory of functors*. Harper's Series in Modern Mathematics. Harper & Row, Publishers, New York, 1964.

[FS97] Eric M. Friedlander and Andrei Suslin. Cohomology of finite group schemes over a field. *Invent. Math.*, 127(2):209–270, 1997.

[Gre80] James A. Green. *Polynomial representations of* GL_n, volume 830 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1980.

[Gre97] R. M. Green. Hyperoctahedral Schur algebras. *J. Algebra*, 192(1):418–438, 1997.

[Har98] Robin Hartshorne. Coherent functors. *Adv. Math.*, 140(1):44–94, 1998.

[HL06] Jun Hu and Zhiqiang Li. On tensor spaces over Hecke algebras of type B_n. *J. Algebra*, 304(1):602–611, 2006.

[Hov11] Mark Hovey. Additive closed symmetric monoidal structures on R-modules. *J. Pure Appl. Algebra*, 215(5):789–805, 2011.

[IK86] Geun Bin Im and G. M. Kelly. A universal property of the convolution monoidal structure. *J. Pure Appl. Algebra*, 43(1):75–88, 1986.

[Jam78] D.G James. *The representation theory of symmetric groups*. Number 682 in Lecture Notes in Mathematics. Springer-Verlag, 1978.

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[Kas95] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Kou91] Frank M. Kouwenhoven. Schur and Weyl functors. *Adv. Math.*, 90(1):77–113, 1991.

[Kra13] Henning Krause. Koszul, Ringel and Serre duality for strict polynomial functors. *Compos. Math.*, 149(6):996–1018, 2013.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[KSS16] U. Kulkarni, S. Srivastava, and K. V. Subrahmanyam. Relating tensor structures on representations of general linear and symmetric groups. *arxiv:1503.09152v2*, 2016. To appear in Transformation group.

[Kuh94] Nicholas J. Kuhn. Generic representations of the finite general linear groups and the Steenrod algebra. I. *Amer. J. Math.*, 116(2):327–360, 1994.

[Mar09] Stuart Martin. *Schur algebras and representation theory*. Cambridge tracts in mathematics. Cambridge University Press, 2009.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[Mur] Daniel Murfet. Derived category part I. http://therisingsea.org/ notes/Derived-Categories.pdf.

[PST] A. Prasad, S. Srivastava, and G. Thanganvelu. Alternating Schur algebra and Koszul duality. In preparation.

[Reg02] Amitai Regev. Double centralizing theorems for the alternating groups. *J. Algebra*, 250(1):335–352, 2002.

[Rei16] Rebecca Reischuk. The adjoints of the schur functor. *arXiv:1601.03513*, 2016.

[Spa88] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.*, 65(2):121–154, 1988.

[Sri] Shraddha Srivastava. Sesquialgebra structures of Schur algebras of Weyl groups of types A and B. Preprint.

[Tou10] Antoine Touzé. Cohomology of classical algebraic groups from the functorial viewpoint. *Adv. Math.*, 225(1):33–68, 2010.

[Tou13] Antoine Touzé. Ringel duality and derivatives of non-additive functors. *J. Pure Appl. Algebra*, 217(9):1642–1673, 2013.

[Tou14] Antoine Touzé. Bar complexes and extensions of classical exponential functors. *Ann. Inst. Fourier (Grenoble)*, 64(6):2563–2637, 2014.

[TvdK10] Antoine Touzé and Wilberd van der Kallen. Bifunctor cohomology and cohomological finite generation for reductive groups. *Duke Math. J.*, 151(2):251–278, 2010.

[TWZ07] Xiang Tang, Alan Weinstein, and Chenchang Zhu. Hopfish algebras. *Pacific J. Math.*, 231(1):193–216, 2007.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.