On the study of Penrose limits of some supergravity backgrounds and their non-Abelian T-duals

by

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Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgment of collaborative research and discussions.

The work was done under the supervision of Prof. *Prasanta K. Tripathy*, at Chennai Mathematical Institute (CMI), Chennai, India.

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April, 2022

In our capacity as the committee members of the candidate's thesis, we certify that the above statements are true to the best of our knowledge.

Prof. Prasanta K. Tripathy
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Abstract

In the first part of the thesis, we consider various type-IIA supergravity solutions that originated by applying non-Abelian T-duality on well-known type-IIB supergravity backgrounds. Abelian T-duality is an exact symmetry of the string theory of all orders of α'. However, non-Abelian T-duality works as a solution-generating technique at the supergravity level. We present the non-Abelian T-dual supergravity solutions originating from $\mathrm{AdS}_5\times\ \mathrm{S}^5$ geometry, Klebanov-Witten geometry, Klebanov-Tseytlin background and the $D1-D5$ system, where T-duality acts on an appropriate non-Abelian isometry group $SU(2)$.

We study the Penrose limits in these T-dual backgrounds. In most cases, the Penrose limits lead to pp-wave geometries in the neighbourhood of appropriate null geodesics. The quantization of closed string on the resulting pp-wave has also been discussed. We also study the supersymmetry analysis of these type-IIA pp-wave geometries and draw some comments on the dual gauge theory.

In the remaining part of the thesis, we discuss some aspects of Double Field Theory (DFT). Double Field Theory is constructed to incorporate T-duality as a manifested symmetry of string theory. In the thesis, we explore the Heterotic version of it. We introduce the relaxed version of the generalized Kerr-Schild ansatz (GKSA) to study perturbations of the fundamental fields in the theory. We also discuss the classical double copy correspondence in this framework.

List of publications

This thesis is based on the following publications:

1. S. Roychowdhury and P. K. Tripathy, "*The non-Abelian T-dual of Klebanov-Witten Background and its Penrose Limits*," JHEP 11, 125 (2019) doi:10.1007/JHEP11(2019)125 [arXiv:1907.01904 [hep-th]].
2. S. Roychowdhury and P. K. Tripathy, "*Penrose limits in non-Abelian T dual of Klebanov-Tseytlin background*," Phys. Rev. D 104, no.12, 126016 (2021) doi:10.1103/PhysRevD.104.126016 [arXiv:2109.09263 [hep-th]].
3. S. Roychowdhury and P. K. Tripathy, "*Penrose limits in massive type-IIA AdS_3 background*," Phys. Rev. D 105, 106024 (2022) doi:10.1103/PhysRevD.105.106024 [arXiv:2203.14212 [hep-th]].
4. E. Lescano and S. Roychowdhury, "*Heterotic Kerr-Schild Double Field Theory and its double Yang-Mills formulation*," JHEP 04, 090 (2022) doi:10.1007/JHEP04(2022)090 [arXiv:2201.09364 [hep-th]].

Contents

Chapter 1

Introduction

Duality plays a significant role in understanding a large class of Physical systems. The notion of duality in Physics is quite familiar since the 1800's through Maxwell's equations. The Maxwell's equations describe the duality between the electric field and the magnetic field. In mid-seventies 't Hooft proposed a duality between a gauge theory and string theory [1]. The gauge theory consists of $SU(N)$ gauge group with large N. The perturbations of the theory is governed by the expansion parameter $\lambda(= g_{YM}^2 N)$, known as the 't Hooft parameter. The perturbation of the $SU(N)$ gauge theory is a double expansion of the parameters λ and $1/N^2$ [1]. In the proposal of 't Hooft, there exists a precise correspondence between these parameters in gauge theory with the expansion parameter $g_s \sim 1/N$ and the world-sheet expansion parameter $\alpha' \sim 1/\sqrt{\lambda}$ in string theory [1].

The precise duality between gauge theory and string theory came out from the seminal work by Maldacena through his AdS/CFT conjecture [2]. The conjecture states a duality between gravity in Anti de-sitter (AdS) spacetime and operators in a Conformal Field Theory (CFT) that resides at the boundary of the AdS spacetime. Conformal Field Theory is a very special kind of Quantum Field Theory that exhibits conformal symmetry [3]. The original conjecture of AdS/CFT correspondence establishes the relation between type-IIB string theory in $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in four spacetime dimensions [2]. Moreover, this correspondence has been generalized to a very wide range of physical systems, the simplest ones among them consist of type-IIB string theory in other spacetimes in the form $AdS_5 \times X^5$; where X^5 is a five-dimensional internal manifold [4,5,6]. The correspondence in the low energy limit is known as gauge/gravity duality which relates supergravity (the low energy excitation of string theory) with strongly coupled gauge theory. This is one of the most interesting and useful aspects of the correspondence.

Symmetry plays a key role in any physical theory to understand the fundamental nature of the theory. In string theory there exists several dualities: S-duality and T-duality. The former relates the theory at various orders of the string-coupling. T-Duality relates low energy effective actions of various string theories among each other [7,8]. For the closed string case, T-duality exchanges the momentum and winding modes of the closed string compactified in the circles with radius R and α'/R respectively. It also connects the type-IIB and the type-IIA versions of the superstring theories. The familiar description of T-duality concerns $U(1)$-isometry and it is known as the Abelian T-duality. Abelian T-duality is an exact symmetry of the full string theory [9,10]. A non-trivial generalization of this Abelian T-duality exists for the non-Abelian isometric groups [11]. However, this non-Abelian T-duality is not a symmetry of the full string theory. It is an elegant solution-generating technique at the supergravity level. Initially, this duality was formulated in the NS-NS sector of the supergravity theory. For quite some time it was not known how to incorporate the Ramond-Ramond sector in the context of non-Abelian T-Duality. The remarkable works by Sfetsos and Thompson gave insights along this line [12]. For a decade, these techniques have been applied to known supergravity backgrounds to construct several new supergravity backgrounds [13,14,15,16,17,18,19,20]. These new T-dual backgrounds give various insights in the context of AdS/CFT correspondence to construct new CFT duals [21,22,23,24,25,26, 27,28].

For several supergravity backgrounds, the Penrose limits lead to pp-wave geometry. String theory on the pp-wave wave background has been analysed in detail for more than a decade for its interesting features [29,30,31,32,33]. These pp-wave solutions originated by applying Penrose limits in supergravity backgrounds and M-theory backgrounds [34, 35, 36]. The most interesting feature of the pp-wave geometries is that they provide exact string theory backgrounds to all orders in α' as well as g_s [37, 38]. It was a path-breaking work by Berenstein, Maldacena and Nastase (BMN) that shed new insights to do stringy calculations from gauge theory perspective [35]. In this work, BMN showed that pp-waves in the gravity side correspond to operators with large R-charge in $\mathcal{N} = 4$ SYM in four dimensions [35]. Here the pp-wave originated by considering Penrose limits in $AdS_5 \times S^5$ background. In later works, the Penrose limits and pp-wave geometries have been studied for other supergravity backgrounds as well [39,40,41,42]. The corresponding BMN sector has also been discussed. Recently, the Penrose limits in non-Abelian T-dual of $AdS_5 \times S^5$ background have been studied in detail [26]. The Penrose limits along non-singular null geodesic lead to pp-wave geometry in the T-dual background. The closed string quantization and the BMN sector in the dual gauge theory have been discussed in detail for the above case. The work in [26] is the main motivation for this thesis. In the core part of the thesis, we generalised the ideas of [26] in other non-Abelian T-dual supergravity backgrounds. In the thesis first, we review the Penrose limits and pp-waves in the non-Abelian T-dual geometry of $AdS_5 \times S^5$ background and then we consider the non-Abelian T-dual corresponding to the compaction of string theory on Klebanov-Witten background, Klebanov-Tseytlin background and the $D1 - D5$ system. The Klebanov-Witten background arises upon placing a large number of $D3$ branes on a conifold singularity [5]. The corresponding dual gauge theory is $\mathcal{N} = 1$ super Yang-Mills theory in four spacetime dimensions [6]. The Penrose limits in this background and the BMN operators in the dual gauge theory have been studied in detail in [39, 40, 41]. The non-Abelian T-duality along $SU(2)$ isometry of this background has been analysed in [15, 23]. The T-dual geometry is a solution in type-IIA supergravity. In our work [43] and in this thesis we consider this T-dual background and examine the Penrose limits. We also make comments on gauge theory dual of the resulting pp-wave geometries. Adding M

fractional $D3$-branes together with N regular $D3$-branes at conifold singularity leads to $\mathcal{N} = 1$ supersymmetric $SU(N + M) \times SU(N)$ gauge theory in four spacetime dimensions. Due to the presence of fractional branes, the theory is no longer conformal and the gravity dual is a modification of the Klebanov-Witten background and is known Klebanov-Tseytlin background [44]. This background admits an $SU(2)$-isometry which is used for non-Abelian T-dualization. The resulting solution is a massive type-IIA supergravity background with $\mathcal{N} = 1$ supersymmetry [15, 23]. In one of our works as well in this thesis we consider the non-Abelian T-dual background and investigate the Penrose limits. We show that the T-dual geometry admits pp-wave solution around suitable null geodesics [45]. The non-Abelian T-dual solutions that we considered here all are $\mathcal{N} = 1$ supersymmetric. Along this line of investigation, we consider the non-Abelian T-dual of the near horizon geometry $D1 - D5$ system. The near horizon geometry is described by $AdS_3 \times S^3 \times T^4$ background. The Penrose limits in this background as well as its Abelian T-dual background have been studied in [42]. The non-Abelian T-dualization of this background along an $SU(2)$ isometry has been first carried out in [12] and studied in detail in [46]. The non-Abelian T-dual background is a massive type-IIA supergravity solution with $\mathcal{N} = 1$ supersymmetry. In our work [47] we consider this non-Abelian T-dual background and examine variation Penrose limits in it. We show that the T-dual background admits pp-wave solution around some non-singular null geodesic [47]. We make some comments on the probable field theory duals.

In the remaining part of the thesis, we discuss some aspects of Double Field Theory (DFT). Abelian T-duality is an exact symmetry of string theory [9, 10]. In low energy limit, the theory can be written in a duality invariant fashion before compactification with the framework of Double Field Theory (DFT) [48, 49]. It is shown that non-Abelian T-duality can be described in the language of double field theory [50]. Among the different formulations of string theory, heterotic double field theory focuses on the low energy limit of heterotic string and its embedding into the double geometry [51]. The standard construction is built on the D-dimensional metric tensor, two-form and a non-Abelian gauge field. The underline symmetry group of heterotic double field theory is $O(D, D+K)$; where K is the gauge group for the gauge field.

The Kerr-Schild formalism is a powerful tool to construct exact solutions in general relativity [52]. In beginning, it was formulated for the perturbations around the Minkowski space and later it has been generalized to any on-shell spacetime geometry. However, the standard Kerr-Schild (KS) formalism does not describe the entire NS-NS sector of the closed string. Recently the remarkable work by Lee establishes that it is possible to generalized the conventional Kerr-Schild formalism to the double field theory and supergravities by introducing a pair of null vectors [53]. The classical double copy structure of this generalized Kerr-Schild formalism describes the entire NS-NS sector of massless string [53]. But this formalism also does not consider the Ramond-Ramond sector of massless string. Then a subsequent development by Cho and Lee introduced a generalized Kerr-Schild ansatz (GKSA) in the context of heterotic double field theory and supergravity to incorporate the Ramond-Ramond sector [54]. Here, the pair of null vectors are represented by a pair of null $O(d, d+G)$ generalized tangent vectors. In the picture of GKSA, the null condition can be partially relaxed in a consistent way [54]. The classical double copy structure of GKSA describes the heterotic supergravity [54]. However, to study perturbations of the gauge field within this framework was still lacking [55, 56].

In our work [57] and in this thesis we shed some insights into the study of gauge field perturbation in heterotic double field theory. We present a formulation of heterotic double field theory where the fundamental fields are in $O(D, D)$ representations [57]. We obtain the theory by splitting an $O(D, D + K)$ duality invariant DFT. Within this framework, we explore the perturbative properties of heterotic DFT. We use a relaxed version of the generalized Kerr-Schild ansatz which is known as relaxed generalized Kerr-Schild ansatz (R-GKSA). The R-GKSA was first introduced in [58] to study the classical double copy correspondence of point charge. The R-GKSA in our context consists of the perturbation of the generalized background metric up to quadratic order considering a single null vector and the gauge field is linearly perturbed before parametrization [57]. We also study the classical double copy correspondence at the DFT level in our picture.

The thesis is organized as follows.

• In Chapter 2 we discuss the basic notion of T-duality. Starting from the duality of string compactification on circles of radius R and α'/R we discuss the *Buscher formulation* of T-duality for the generic background which admits $U(1)$ isometry together with background fields. This construction is known as the Abelian T-duality. In the same chapter, we discuss the generalisation of Abelian T-duality for non-Abelian isometries, where in most cases the symmetry group is $SU(2)$. We discuss the construction of the NS-NS sector and RR-sector of dualization in detail. To give some examples we consider some supergravity solutions that originate by applying Abelian and non-Abelian T-duality on well-known type-IIB supergravity backgrounds: $AdS_5 \times S^5$ geometry, Klebanov-Witten background, Klebanov-Tseytlin background and the near horizon geometry of $D1 - D5$ system.

• In Chapter 3 we discuss the AdS/CFT correspondence. First, we revisit Maldacena's work for the $AdS_5 \times S^5$ background and its correspondence with $\mathcal{N} = 4$ supersymmetric Yang-Mills theory that resides at the boundary of AdS_5 followed by the work of Klebanov and Witten for the $AdS_5 \times T^{1,1}$ geometry. Together with the above, the Penrose limits of the type-IIB supergravity backgrounds have also been discussed. Starting from the seminal work by BMN for the $AdS_5 \times S^5$ background, the Penrose limits in the Klebanov-Witten background, Klebanov-Tseytlin background and the Penrose limits of near horizon geometry of $D1 - D5$ system have been discussed. For all the above cases, Penrose limits lead to pp-wave geometry. Field theory duals of the corresponding pp-wave geometries have also been discussed.

• In Chapter 4 we review the recent work on Penrose limits in T-dual backgrounds originated from the $AdS_5 \times S^5$ geometry. We discuss the pp-wave geometries both in the Abelian as well as the non-Abelian case with corresponding field theory duals. The supersymmetry discussion of the pp-wave geometry is also provided.

• Chapter 5 is the core part of this thesis. In this chapter, we discussed the Penrose limits in the non-Abelian T-dual of the Klebanov-Witten and Klebanov-Tseytlin background. We discuss various null geodesics in the non-Abelian T-dual geometry, some of them lead to pp-wave geometries. The closed string quantization and the supersymmetry analysis of these pp-wave geometries have been discussed. We briefly mention the dual field theory for these pp-wave geometries.

• Finally in Chapter 6 we consider the T-dual backgrounds originated from the near horizon geometry of $D1 - D5$ system. The near horizon geometry is described by the background $AdS_3 \times S^3 \times T^4$. We discuss the Penrose limits and pp-waves in the T-dual backgrounds. The supersymmetry analysis states that the pp-wave geometry preserves 16 supercharges in the non-Abelian case. The corresponding field theory discussion is also provided.

• In Chapter 7 we first review the basic notation of Double Field Theory and the Kerr-Schild ansatz in this context. We discuss both in metric formalism as well as in frame formulation. We introduce the double Yang-Mills formulation of heterotic DFT and explore the relaxed version of generalized Kerr-Schild ansatz to study perturbations of the fundamental fields in the theory. We also discuss the classical double copy correspondence in this framework.

• Finally we summarise the result and discuss some future directions in Chapter 8.

Chapter 2

Abelian and non-Abelian T-duality

In this chapter we will first introduce the nuts and bolts of the familiar T-duality along with the extended version of it. For the perturbation of closed bosonic strings, T-duality is the simplest version of string duality. The familiar notation of T-duality, it relates the string theories with a large spacetime radius R and with a small spacetime radius α'/R. In the following we will review the basics of this T-duality together with some examples.

2.1 Abelian T-duality

In this section we will consider the closed bosonic string and show how T-duality acts on it. Consider the bosonic string compactified in Minkowski spacetime with 26 spacetime dimensions. Out of 26 spacetime directions, one direction is a circle $\mathbb{S}^1$ with radius R. Here the string propagates in the space defined by $\mathbb{R}^{24,1} \times \mathbb{S}^1$. Let X^{25} be the compactified coordinate which satisfies the periodic boundary condition

$$X^{25}(\sigma + \pi, \tau) = X^{25}(\sigma, \tau) + 2\pi RW \ , \tag{2.1}$$

where (σ, τ) are the worldsheet coordinates and W is known as the winding number that measures the number of times the closed string winds around the compact direction $\mathbb{S}^1$. Considering the familiar left mover and right mover decomposition of the mode expansion of the closed string modes, from (2.1) we have

$$X^{25}(\sigma, \tau) = X^{25}_L(\tau + \sigma) + X^{25}_R(\tau - \sigma) \ , \tag{2.2}$$

with the leading order expansion

$$\begin{aligned} X^{25}_R(\tau - \sigma) = & \ \frac{1}{2}\big(x^{25} - \tilde{x}^{25}\big) + \Big(\alpha' \frac{\kappa}{R} - WR\Big)(\tau - \sigma) + ... \ , \\ X^{25}_L(\tau + \sigma) = & \ \frac{1}{2}\big(x^{25} + \tilde{x}^{25}\big) + \Big(\alpha' \frac{\kappa}{R} + WR\Big)(\tau + \sigma) + ... \ . \end{aligned} \tag{2.3}$$

Here κ is the Kaluza-Klein excitation number and it originates from the quantized momentum associated with direction X^{25}. Because X^{25} is identified with the circle $\mathbb{S}^1$, hence the associated momentum is quantized with the expression $p_{25} = \kappa/R$.

Now the mass spectrum of the closed string is given by

$$M^2 = \left(\frac{\kappa}{R}\right)^2 + \left(\frac{WR}{\alpha'}\right)^2 + \frac{2}{\alpha'}\left[\left(N_L + N_R\right) - 2\right] , \tag{2.4}$$

where $N_{L,R}$ denotes the number of left-moving and right-moving modes respectively and they satisfy the level matching condition $N_R - N_L = W\kappa$. The expression of the mass spectrum (2.4) is invariant under the exchange of $R \leftrightarrow \alpha'/R$ followed by $W \leftrightarrow \kappa$. This duality symmetry is known as T-duality for closed bosonic string. Under T-duality the closed string momentum modes become the winding modes and vice versa. In terms of the mode expansion the T-duality acts as

$$X^{25}_R \to -X^{25}_R \ ; \ X^{25}_L \to X^{25}_L \ . \tag{2.5}$$

In the case of superstring, the T-duality acts in the fermionic sector as well. The action of the T-duality is as follows

$$X^9_R \to -X^9_R \ ; \ X^9_L \to X^9_L \ , \tag{2.6}$$

where X^9 is the compactified coordinate in the ten-dimensional type-II theories. Together with the bosonic modes the fermonic modes exchange as

$$\psi^9_R \to -\psi^9_R \ ; \ \psi^9_L \to \psi^9_L \ . \tag{2.7}$$

It is shown that under this transformation the chirality of the theory changes from type-IIB to type-IIA and vice versa.

Now in the following we will examine T-duality in the presence of background fields in the NS-NS sector as well as in the RR sector. In the NS-NS sector we have the background metric $g_{\mu\nu}$, NS-NS two form $B_{\mu\nu}$ and the background dilaton Φ. The RR sector is contained with p-forms gauge field. Here the background metric $g_{\mu\nu}$ admits at least one $U(1)$ isometric direction along which the T-duality acts. In this case the construction is the following three steps procedure known as the *Buscher Rules* [7, 8]. First consider a sigma model action in this background which supports the $U(1)$-isometry. Then gauge the isometry and impose a constraint by introducing a set of Lagrange multipliers such that the field strength will remain zero. This leads to state that the degrees of freedom of the system will remain unchanged during dualization.

Now the duality picture is the following. If we solve the equation of motions of the Lagrange multipliers and substitute in the original action we recover the original sigma model action. In order to construct the T-dual sigma model, instead of the Lagrange multipliers we integrate out the gauge fields and gauge fix the remaining part. Then the Lagrange multipliers become the coordinates in the dual sigma model.

Consider the worldsheet action of the string together with the background fields as follows

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \left[\sqrt{-h}h^{\alpha\beta}g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu - \epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\right] , \tag{2.8}$$

where, $h_{\alpha\beta}$ is the metric on the worldsheet and consider the background has manifest $U(1)$-isometry along X^9 direction. Following Buscher rules, we introduce a Lagrange multiplier $\tilde{X}^9$ and then the worldsheet action in (2.8) takes the form

$$\begin{aligned} 4\pi\alpha' S = & \int d^2\sigma \Bigg[\sqrt{-h} h^{\alpha\beta} \Big(- g_{99} V_\alpha V_\beta - 2 g_{\mu 9} V_\alpha \partial_\beta X^\mu - g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \Big) \\ & + \epsilon^{\alpha\beta} \Big(B_{9\mu} V_\alpha \partial_\beta X^\mu + B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \Big) + \tilde{X}^9 \epsilon^{\alpha\beta} \partial_\alpha V_\beta \Bigg] \ . \end{aligned} \tag{2.9}$$

Solving the Lagrange multiplier $\tilde{X}^9$ in (2.9) we have $\epsilon^{\alpha\beta}\partial_\alpha V_\beta = 0$. Upon solving this we find $V_\alpha = \partial_\alpha X^9$. Substituting the solution in (2.9) one recovers the original sigma model action (2.8). Instead of that, upon solving V_α to integrate the Lagrange multiplier $\tilde{X}^9$ we find the dual sigma model action as

$$\tilde{S} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \Bigg[\sqrt{-h} h^{\alpha\beta} \tilde{g}_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu - \epsilon^{\alpha\beta} \tilde{B}_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \Bigg] \ , \tag{2.10}$$

where the T-dual fields are given by

$$\begin{aligned} & \tilde{g}_{99} = \frac{1}{g_{99}} \ ; \ \tilde{g}_{9\mu} = \frac{B_{9\mu}}{g_{99}} \ ; \ \tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{B_{9\mu} B_{9\nu} - g_{9\mu} g_{9\nu}}{g_{99}} \ , \\ & \tilde{B}_{9\mu} = \frac{g_{9\mu}}{g_{99}} \ ; \ \tilde{B}_{\mu\nu} = B_{\mu\nu} + \frac{g_{9\mu} B_{9\nu} - B_{9\mu} g_{9\nu}}{g_{99}} \ , \\ & e^{2\left(\Phi - \tilde{\Phi}\right)} = g_{99} \ . \end{aligned} \tag{2.11}$$

To construct the T-dual dilaton $\tilde{\Phi}$, we considered the invariance of the quantity $\sqrt{-g}e^{-2\Phi} \to \sqrt{-\tilde{g}}e^{-2\tilde{\Phi}}$ in T-duality. The transformation rules in (2.11) is known as *Buscher Rules* of T-duality.

Now we shall consider the action of T-duality on the RR p-forms. One can construct the T-dual RR forms by various approaches: spacetime perspective, worldsheet perspective and the bispinor formulation. In the following we will discuss the bispinor formulation introduced in [59]. Consider the bispinor in type-IIB theory as

$$P = \frac{e^\Phi}{2} \sum_{n=0}^{4} \not{F}_{2n+1} \ . \tag{2.12}$$

Similarly in type-IIA theory we have

$$\tilde{P} = \frac{e^\Phi}{2} \sum_{n=0}^{5} \not{F}_{2n} \ , \tag{2.13}$$

where $\not{F} = \frac{1}{p!} \Gamma_{\mu_1 \dots \mu_p} F_p^{\mu_1 \dots \mu_p}$. Now the dual fluxes can be obtained from

$$\tilde{P} = P. \, \Omega^{-1} \ , \tag{2.14}$$

where the Ω is given by

$$\Omega = \frac{1}{\sqrt{g_{99}}}\,\Gamma_{11}\Gamma_9 \ . \tag{2.15}$$

At the end of this chapter we consider some of the well-known supergravity backgrounds and study the application of T-duality on it.

2.2 Non-Abelian T-duality

In the previous section we showed how T-duality works on the background that admits an $U(1)$-isometry. The procedures are described in three steps are known as the *Buscher Rules*. Now Buscher's formulation can be extended to the supergravity background where the corresponding isometric group is non-Abelian [11]. The procedure follows the same steps as in the Abelian case, although the gauge field we introduce here is a non-Abelian group valued quantity. However, unlike its Abelian counterpart, the non-Abelian T-duality is not a symmetry of the full string theory. The non-Abelian T-duality works as a solution generating technique at the supergravity level and maps one supergravity solution to another. Also, in this case, during dualization the isometry of the seed background is partially lost but it can be recovered as a non-local symmetry in the corresponding sigma model. Initially, the non-Abelian T-duality was formulated for purely NS-NS backgrounds. In a remarkable work, the authors in [12] incorporated the RR sector in non-Abelian T-duality. The new supergravity backgrounds correspond to new CFT duals and give new insights into the context of the AdS/CFT correspondence. One interesting example is the dualization of an $SU(2)$ subgroup of the $AdS_5 \times S^5$ geometry. The T-dual background is a solution in type-IIA supergravity and its M-lift is very close to Giaotto-Maldacena geometry that arises from $M5$-branes on Riemann surfaces. Now in the following we will discuss the *Buscher Rules* in non-Abelian T-duality where the corresponding isometry group is $SU(2)$.

First we write the background metric in the form

$$ds^2 = G_{\mu\nu}(x)dx^\mu dx^\nu + 2G_{\mu i}(x)dx^\mu L^i + g_{ij}(x)L^i L^j \ , \tag{2.16}$$

where the index μ runs from 1,2,..7 and describes the spectator part denoted by the coordinate x^μ. In the above, $L^i; i = 1, 2, 3$ are the $SU(2)$ left invariant Maurer-Cartan one forms and are expressed in terms of the $SU(2)$ Euler angles θ, ϕ and ψ. Together with metric, we have the same decomposition of the NS-NS two-form gauge field given by

$$B_2 = B_{\mu\nu}(x)dx^\mu \wedge dx^\nu + B_{\mu i}(x)dx^\mu \wedge L^i + \frac{1}{2}b_{ij}(x)L^i \wedge L^j \ . \tag{2.17}$$

In addition to the metric and NS-NS two-form we have background dilaton Φ depends on the spectator directions, $\Phi = \Phi(x)$.

Now we assign the frames associated with the metric (2.16) as follows

$$\begin{aligned} e^A =& \ \ e^A_\mu dx^\mu \ ; \ A = 1, 2, ..., 7 \\ e^a =& \ \ \kappa^a_j L^j + \lambda^a_\mu dx^\mu \ ; \ a = 1, 2, 3 \ . \end{aligned} \tag{2.18}$$

Imposing the metric in the form

$$ds^2 = \eta_{AB} e^A e^B + e^a e^a \ , \tag{2.19}$$

and then from (2.16)-(2.19) we find

$$G_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu + K_{\mu\nu} \ ; \ \kappa^a_i \lambda^a_\mu = G_{\mu i} \ ; \ \kappa^a_i \kappa^a_j = g_{ij} \ , \tag{2.20}$$

where we define $\lambda^a_\mu \lambda^a_\nu = K_{\mu\nu}$. Now in the following we will consider the construction of the NS-NS sector of the T-dual background. We will follow the worldsheet formulation introduced in [15, 23]. The Lagrangian density for the bosonic sector takes the form

$$\mathcal{L}_0 = Q_{\mu\nu} \partial_+ X^\mu \partial_- X^\nu + Q_{\mu i} \partial_+ X^\mu L^i_- + Q_{i\mu} L^i_+ \partial_- X^\mu + E_{ij} L^i_+ L^j_- \ , \tag{2.21}$$

where $\partial_\pm$ are the conventional worldsheet derivatives and $L^i_\pm = -i \, Tr\big(t^i g^{-1} \partial_\pm g\big)$; t^i are the generators of the isometry group $SU(2)$. The Q's are defined as

$$Q_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu} \ ; \ Q_{\mu i} = G_{\mu i} + B_{\mu i} \ ; \ Q_{i\mu} = G_{i\mu i} + B_{i\mu} \ ; \ E_{ij} = g_{ij} + b_{ij} \ . \tag{2.22}$$

Now we replace the ordinary derivative $\partial_\pm$ by the covariant derivatives in the expression of $L^i_\pm$

$$\partial_\pm g \longrightarrow D_\pm g = \partial_\pm g - A_\pm g \ . \tag{2.23}$$

In addition we add the Lagrangian multiplier term given by

$$i \, Tr\big(v F_\pm\big) \ ; \ F_\pm = \partial_+ A_- - \partial_- A_+ - \big[A_+ \ , A_-\big] \ . \tag{2.24}$$

Equation of motion of the Lagrangian multipliers drives back to the original Lagrangian $\mathcal{L}_0$, whereas integrating out the gauge fields and gauge fixing the group part (here we consider $g = \mathbb{I}$) we obtain the T-dual Lagrangian

$$\hat{\mathcal{L}} = Q_{\mu\nu} \partial_+ X^\mu \partial_- X^\nu + \big(\partial_+ v_i \ + \ \partial_+ X^\mu Q_{\mu i}\big) M^{-1}_{ij} \big(\partial_- v_j - Q_{j\mu} \partial_- X^\mu\big) \ , \tag{2.25}$$

where $M_{ij} = g_{ij} + f_{ijk} v_k$ and f_{ijk} are the structure constants of the corresponding Lie-Algebra.

From (2.25) the dual Q's read as

$$\begin{aligned} \hat{Q}_{\mu\nu} &= Q_{\mu\nu} - Q_{\mu i} M^{-1}_{ij} Q_{j\nu} \ , \ \hat{Q}_{\mu i} = Q_{\mu j} M^{-1}_{ji} \ , \\ \hat{Q}_{i\mu} &= -M^{-1}_{ij} Q_{j\mu} \ , \ \hat{E}_{ij} = M^{-1}_{ij} \ . \end{aligned} \tag{2.26}$$

Now the symmetric combination of Q's generate the components of the T-dual metric and the antisymmetric part leads to NS-NS two-form in the T-dual geometry

$$\begin{aligned} \hat{G}_{\mu\nu} &= G_{\mu\nu} - \frac{1}{2}\big(Q_{\mu i} M^{-1}_{ij} Q_{j\nu} + Q_{\nu i} M^{-1}_{ij} Q_{j\nu}\big) \ , \\ \hat{G}_{\mu i} &= \frac{1}{2}\big(Q_{\mu j} M^{-1}_{ji} - Q_{j\mu} M^{-1}_{ij}\big) \ , \ \hat{g}_{ij} = \frac{1}{2}\big(M^{-1}_{ij} + M^{-1}_{ji}\big) \ , \\ \hat{B}_{\mu\nu} &= B_{\mu\nu} - \frac{1}{2}\big(Q_{\mu i} M^{-1}_{ij} Q_{j\nu} - Q_{\nu i} M^{-1}_{ij} Q_{j\nu}\big) \ , \\ \hat{B}_{\mu i} &= \frac{1}{2}\big(Q_{\mu j} M^{-1}_{ji} + Q_{j\mu} M^{-1}_{ij}\big) \ , \ \hat{b}_{ij} = \frac{1}{2}\big(M^{-1}_{ij} + M^{-1}_{ji}\big) \ . \end{aligned} \tag{2.27}$$

Considering the above metric components the line element of the T-dual geometry is given by

$$d\hat{s}^2 = \hat{G}_{\mu\nu}dx^\mu dx^\nu + \sqrt{2}\hat{G}_{\mu i}dx^\mu dv^i + \frac{1}{2}\hat{g}_{ij}dv^i dv^j \ . \tag{2.28}$$

Similarly for the NS-NS two-form we have

$$\hat{B}_2 = \hat{B}_{\mu\nu}dx^\mu \wedge dx^\nu + \frac{1}{\sqrt{2}}\hat{B}_{\mu i}dx^\mu \wedge dv^i + \frac{1}{4}\hat{b}_{ij}dv^i \wedge dv^j \ . \tag{2.29}$$

In addition the dual dilaton takes the form

$$e^{2(\Phi-\hat{\Phi})} = \det\ M \ . \tag{2.30}$$

This completes the discussion about the NS-NS sector. Now we will turn our focus on the construction of T-duality in the RR sector. It is straightforward to see from (2.21) and (2.25) that the left and right movers transform differently under the T-dual transformations

$$\hat{L}^i_+ = -(M^{-1})_{ji}\big(\partial_+ v_j + Q_{\mu j}\partial_+ X^\mu\big) \ ; \ \hat{L}^i_- = M^{-1}_{ij}\big(\partial_- v_j + Q_{j\mu}\partial_- X^\mu\big) \ . \tag{2.31}$$

These lead to two different frames $\hat{e}^a_+$ and $\hat{e}^a_-$ in the T-dual background. Both the frames describe the same geometry hence they must be related by a Lorentz transformation

$$\hat{e}^a_+ = \Lambda^a_b\ \hat{e}^b_- \ . \tag{2.32}$$

The Lorentz transformation Λ acts on the spinor representation of the Lorentz group as

$$\Omega^{-1}\Gamma^a\Omega = \Lambda^a_b\Gamma^b \ . \tag{2.33}$$

In this case also we consider the bi-spinor formalism (2.12)-(2.14) as we discussed in the Abelian part to construct the RR-form in the T-dual theory. In the original background we write the RR field strengths in basis of e^A and e^a as

$$F_p = G^{(0)}_p + G^a_{p-1}\wedge e^a + \frac{1}{2}G^{ab}_{p-2}\wedge e^a \wedge e^b + G^{(3)}_{p-3}\wedge e^1 \wedge e^2 \wedge e^3 \ . \tag{2.34}$$

Similarly the T-dual RR strength takes the form

$$\hat{F}_p = \hat{G}^{(0)}_p + \hat{G}^a_{p-1}\wedge e'^a + \frac{1}{2}\hat{G}^{ab}_{p-2}\wedge e^a \wedge e'^b + \hat{G}^{(3)}_{p-3}\wedge e'^1 \wedge e'^2 \wedge e'^3 \ , \tag{2.35}$$

where

$$\begin{aligned}
\hat{G}^{(0)}_p &= e^{\Phi-\hat{\Phi}}\Bigg(-A_0 G^{(3)}_p + A_a G^a_p\Bigg) \ , \\
\hat{G}^a_{p-1} &= e^{\Phi-\hat{\Phi}}\Bigg(-\frac{A_0}{2}\epsilon^{abc}G^{bc}_{p-1} + A_b G^{ab}_{p-1} + A_a G^{(0)}_{p-1}\Bigg) \ , \\
\hat{G}^{ab}_{p-2} &= e^{\Phi-\hat{\Phi}}\Bigg[\epsilon^{abc}\Bigg(A_c G^{(3)}_{p-2} + A_0 G^c_{p-2}\Bigg) - \Bigg(A_a G^b_{p-2} - A_b G^a_{p-2}\Bigg)\Bigg] \ , \\
\hat{G}^{(3)}_{p-3} &= e^{\Phi-\hat{\Phi}}\Bigg(\frac{A_a}{2}\epsilon^{abc}G^{bc}_{p-3} + A_0 G^{(0)}_{p-3}\Bigg) \ .
\end{aligned} \tag{2.36}$$

The coefficients in the above transformation are given by

$$A_0 = \frac{\sqrt{\det g}}{\sqrt{\det g + \left(\kappa_i^a y^i\right)^2}} \ ; \ A_a = \frac{\kappa_i^a y^i}{\sqrt{\det g + \left(\kappa_i^a y^i\right)^2}} \ . \tag{2.37}$$

where $y_i = b_i + v_i$ and b_i arises from the b_{ij} as $b_{ij} = \epsilon_{ijk} b_k$.

In the next section we consider some well-known supergravity solutions and study their T-dual backgrounds.

2.3 Abelian and non-Abelian T-dual supergravity backgrounds

Finally, in this section we will consider some of the type-IIB supergravity backgrounds and apply the Abelian as well as non-Abelian T-duality to these backgrounds. The Abelian T-duality acts on a $U(1)$-isometry of the type-IIB backgrounds. The resulting backgrounds are the solutions in type-IIA supergravity. The background we consider in the following also admits non-Abelian isometries. We consider an $SU(2)$ subgroup for the dualization. The T-dual backgrounds solve supergravity equations in type-IIA supergravity. In the following we will review the T-dual background presented in [12, 15, 21, 23, 27, 42, 46].

T-dual backgrounds with AdS$_5$ factor

First we will consider the type-IIB supergravity backgrounds of the form $AdS_5 \times X^5$, where X^5 is some five-dimensional Einstein manifold. We start our discussion with $AdS_5 \times S^5$ geometry. The geometry describes the near horizon geometry of a stack of N $D3$-branes placed on top of each other [2, 4]. The corresponding metric of the geometry is given by

$$ds^2 = 4L^2 ds^2(AdS_5) + 4L^2 d\Omega_2^2(\alpha,\beta) + L^2\cos^2\alpha\Big(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\phi d\psi\Big) \ , \tag{2.38}$$

where

$$\begin{aligned} ds^2(AdS_5) = &\ -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 \ , \\ d\Omega_2^2(\alpha,\beta) = &\ d\alpha^2 + \sin^2\alpha d\beta^2 \ . \end{aligned} \tag{2.39}$$

The RR sector of the background is described by the self dual five-form field strength as

$$F_5 = \frac{2}{g_s L}(1 + \star_{10}) \ \mathrm{Vol}(AdS_5) \ . \tag{2.40}$$

The metric in (2.38) has manifest $U(1)$-isometries along β, ϕ directions as well as in the fibre direction ψ in the transverse S^5 manifold. The Abelian T-duality along the ψ-direction has been studied in detail in [21]. The T-dual background is specified by the metric

$$ds^2_{\mathrm{ATD}} = 4L^2 ds^2(AdS_5) + 4L^2 d\Omega_2^2(\alpha,\beta) + \frac{d\psi^2}{L^2\cos^2\alpha} + L^2\cos^2\alpha \ d\Omega_2^2\big(\chi,\xi\big) \ , \tag{2.41}$$

where we still denote the T-dual coordinate as ψ and the remaining S^2 coordinates as $\big(\chi, \xi\big)$. In the T-dual geometry, the range of the ψ-coordinate becomes $(0, \pi)$. In the above we consider $\alpha' = 1$ for convenience. Now we focus on the remaining background fields of the T-dual geometry.

The background fields in NS-NS sector takes the form

$$\begin{aligned} B_2 &= \quad \psi \sin\chi d\chi \wedge d\xi \ , \\ e^{-2\Phi} &= \quad \frac{L^2 \cos^2\alpha}{g_s} \ . \end{aligned} \tag{2.42}$$

The Ramond-Ramond sector of the T-dual background is supported by the four-form field strength

$$F_4 = \frac{8L^4}{g_s} \cos^3\alpha \sin\alpha \sin\chi d\alpha \wedge d\beta \wedge d\chi \wedge d\xi \ . \tag{2.43}$$

The metric component $g_{\psi\psi}$ in (2.41) is singular at $\alpha = \pi/2$ and also the background dilaton (2.42) blows up. Hence the T-dual background is no longer maximally supersymmetric. It turns out that the above background (2.41)-(2.43) preserves 16 supercharges and solves the equations of motion of type-IIA supergravity [21].

Now we will turn our attention to the non-Abelian T-dual solution originated from (2.38). The non-Abelian T-duality on $SU(2)$ subgroup of the internal manifold S^5 has been carried out first in [12] and then studied in detail in [21, 25]. The T-dual background is described by the metric

$$ds^2_{\text{NATD}} = 4L^2 ds^2(AdS_5) + 4L^2 d\Omega_2^2(\alpha, \beta) + \frac{d\rho^2}{L^2\cos^2\alpha} + \frac{L^2\rho^2\cos^2\alpha}{\rho^2 + L^4\cos^4\alpha} d\Omega_2^2(\chi, \xi) \ . \tag{2.44}$$

In addition, the NS-NS sector of the background is contained by two-form and dilaton as

$$\begin{aligned} B_2 &= \quad \frac{\rho^3\cos^2\alpha}{\rho^2 + L^4\cos^4\alpha} \sin\chi d\chi \wedge d\xi \ , \\ e^{-2\Phi} &= \quad \frac{L^2\cos^2\alpha}{g_s^2}\Big(\rho^2 + L^4\cos^4\alpha\Big) \ . \end{aligned} \tag{2.45}$$

The RR sector of the background is described by two-form and four-form field strengths

$$\begin{aligned} F_2 &= \quad \frac{8L^4}{g_s} \sin\alpha \cos^3\alpha d\alpha \wedge d\beta \ , \\ F_4 &= \quad \frac{8L^4}{g_s} \ \frac{\rho^3\cos^3\alpha}{\rho^2 + L^4\cos^4\alpha} \sin\alpha \sin\chi d\alpha \wedge d\beta \wedge d\chi \wedge d\xi \ . \end{aligned} \tag{2.46}$$

As in the Abelian T-dual background, the metric of the non-Abelian T-dual background (2.44) is singular at $\alpha = \frac{\pi}{2}$ where the $g_{\rho\rho}$-component and dilaton blow up. It is shown in [21] that the background (2.44)-(2.46) is a solution in type-IIA supergravity and preserves 16 supercharges.

Now we shall consider the Klebanov-Witten background and the geometries that originated from it by applying Abelian and non-Abelian T-duality. This type-IIB background corresponds to the near horizon limit of parallel $D3$ branes at conical singularities [5,6]. The metric of the background has the following form

$$ds^2 = L^2 ds^2_{AdS_5} + L^2 ds^2_{T^{1,1}} \ , \tag{2.47}$$

where the AdS_5 metric is given in (2.39) and metric of the $T^{1,1}$ is given by

$$ds^2_{T^{1,1}} = \ \lambda_1^2 \ d\Omega_2^2\big(\theta_1, \phi_1\big) + \lambda_2^2 \ d\Omega_2^2\big(\theta_2, \phi_2\big) + \lambda^2\big(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\big)^2 \ . \tag{2.48}$$

In (2.47), L is the size of AdS_5 and the λ_i's are constant parameters with the numerical values $\lambda_1 = \lambda_2 = \frac{1}{\sqrt{6}}$ and $\lambda = \frac{1}{3}$. The RR sector of the background is described by the self dual five-form field strengths

$$F_5 = \frac{4}{g_s L}\Bigg[\text{Vol}(AdS_5) - L^5 \text{Vol}(T^{1,1})\Bigg]. \tag{2.49}$$

The background (2.47)-(2.49) has $U(1)$ isometries along ϕ_1, ϕ_2 and ψ directions in the transverse $T^{1,1}$ part. Also, the metric is symmetric under the exchange of the two azimuthal coordinates (ϕ_1, ϕ_2) followed by (θ_1, θ_2). First we will consider Abelian T-duality along the ϕ_2-direction. The geometry of the dual background is given by the metric [27]

$$ds^2_{\text{ATD}} = L^2 ds^2_{AdS_5} + L^2\lambda_1^2\Bigg[d\Omega_2^2\big(\theta_1, \phi_1\big) + d\theta_2^2 + \frac{\lambda^2 \sin^2\theta_2}{P(\theta_2)}\big(d\psi + \cos\theta_1 d\phi_1\big)^2 + \frac{d\phi_2^2}{\lambda_1^2 P(\theta_2)}\Bigg]. \tag{2.50}$$

where $P(\theta_2) = \lambda^2 \cos^2\theta_2 + \lambda_2^2 \sin^2\theta_2$. The background dilaton and the NS-NS two-form admits by the T-dual background are given by

$$e^{-2\hat{\Phi}} = \frac{L^2}{g_s^2} P(\theta_2) \ ; \ \hat{B}_2 = -\frac{L^2\lambda^2\cos\theta_2}{P(\theta_2)} \ \Big(d\phi_2 \wedge d\psi + \cos\theta_1 d\phi_2 \wedge d\phi_1\Big) \ . \tag{2.51}$$

The RR four form F_4 for the T-dual background has the following expression

$$\hat{F}_4 = \frac{4L^4\lambda\lambda_1^4}{g_s} \sin\theta_1 \sin\theta_2 \ d\theta_1 \wedge d\phi_1 \wedge d\theta_2 \wedge d\psi \ . \tag{2.52}$$

Now we will focus on the other isometry direction ψ of the geometry given in (2.48). By applying standard rules of T-duality we have the following dual background

$$ds^2_{\text{ATD}} = L^2 ds^2_{AdS_5} + L^2\Bigg[\lambda_1^2 d\Omega_2^2\big(\theta_1, \phi_1\big) + \lambda_2^2 d\Omega_2^2\big(\theta_2, \phi_2\big) + \frac{1}{\lambda^2} d\psi^2\Bigg] \ . \tag{2.53}$$

In the metric (2.53) we rescaled the ψ-coordinate as $\psi \to \frac{L^2}{\alpha'}\psi$ to get L^2 as a common factor in the metric and as like in the previous case here also we set $\alpha' = 1$ for convenience. In the dual solution the NS-NS sector contains constant dilaton together with the two-form field

$$e^{-2\hat{\Phi}} = \frac{\lambda^2 L^2}{g_s^2} \ ; \ \hat{B}_2 = -L^2\Bigg[\cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\Bigg] \wedge d\psi \ , \tag{2.54}$$

The NS-NS three-form flux corresponding to the two-form $\hat{B}_2$ takes the form

$$\hat{H}_3 = L^2\Bigg[\sin\theta_1 d\theta_1\wedge d\phi_1 + \sin\theta_2 d\theta_2\wedge d\phi_2\Bigg]\wedge d\psi\ . \tag{2.55}$$

The RR sector of the dual background described by a non-vanishing four-form flux as

$$\hat{F}_4 = \frac{4L^4\lambda\lambda_1^2\lambda_2^2}{g_s}\sin\theta_1\sin\theta_2\ d\phi_1\wedge d\theta_1\wedge d\phi_2\wedge d\theta_2\ . \tag{2.56}$$

Now in the following we will consider the non-Abelian T-dual background originating from the geometry (2.47)-(2.49). The non-Abelian T-duality on an $SU(2)$ subgroup of the symmetry group of the internal manifold $T^{1,1}$ was first carried out in [15] and studied extensively in [23,27]. The geometry of the T-dual solution is specified by the metric [15,16,17,18,23,24]

$$d\hat{s}^2 = L^2 ds^2_{AdS_5} + L^2 d\hat{s}^2_{T^{1,1}}\ , \tag{2.57}$$

where

$$d\hat{s}^2_{T^{1,1}} = \lambda_1^2 d\Omega_2^2(\theta_1,\phi_1)+\frac{\lambda_2^2\lambda^2}{\Delta}x_1^2\sigma_{\hat{3}}^2+\frac{1}{\Delta}\Bigg[(x_1^2+\lambda^2\lambda_2^2)dx_1^2+(x_2^2+\lambda_2^4)dx_2^2+2x_1x_2dx_1dx_2\Bigg]\ , \tag{2.58}$$

and

$$\Delta = \lambda_2^2x_1^2+\lambda^2(x_2^2+\lambda_2^4)\ ,\ \sigma_{\hat{3}} = d\psi+\cos\theta_1 d\phi_1\ . \tag{2.59}$$

In the dual metric we rescaled the coordinates x_1 and x_2 to get an overall L^2 factor. The dual background is supported by a NS-NS two-form along with the dilaton as follows

$$\hat{B}_2 = -\frac{\lambda^2L^2}{\Delta}\Big[x_1x_2dx_1+\Big(x_2^2+\lambda_2^4\Big)dx_2\Big]\wedge\sigma_{\hat{3}}\ ;\ e^{-2\hat{\Phi}} = \frac{8L^6}{g_s^2}\Delta\ . \tag{2.60}$$

The NS-NS three form flux for the $\hat{B}_2$ is given by

$$\begin{aligned}\hat{H}_3 &= \frac{\lambda^2L^2}{\Delta^2}\Bigg[\lambda_2^2x_1^3+\lambda^2x_1\Big(x_2^2+\lambda_2^4\Big)-2\lambda^2x_1x_2^2+2\lambda_2^2x_1\Big(x_2^2+\lambda_2^4\Big)\Bigg]dx_1\wedge dx_2\wedge\sigma_{\hat{3}}\\ &- \frac{\lambda^2L^2}{\Delta}\Bigg[x_1x_2dx_1+\Big(x_2^2+\lambda_2^4\Big)dx_2\Bigg]\sin\theta_1 d\theta_1\wedge d\phi_1\ .\end{aligned} \tag{2.61}$$

The RR sector of the background is described by the two-form and four-form field strengths

$$\begin{aligned}\hat{F}_2 &= \frac{8\sqrt{2}}{g_s}\lambda\lambda_1^4L^4\sin\theta_1 d\phi_1\wedge d\theta_1\ ,\\ \hat{F}_4 &= -\frac{8\sqrt{2}}{g_s}L^6\lambda\lambda_1^4\frac{x_1}{\Delta}\sin\theta_1 d\phi_1\wedge d\theta_1\wedge\sigma_{\hat{3}}\wedge\Big(\lambda_1^2x_1dx_2-\lambda^2x_2dx_1\Big)\ .\end{aligned} \tag{2.62}$$

Unlike the $AdS_5 \times S^5$ case, it is shown that this T-dual background (2.57)-(2.62) preserves $\mathcal{N}=1$ supersymmetry and solves the equations of motion in type-IIA supergravity [15, 23].

Now we consider the Klebanov-Tseytlin background. This background arises upon placing a stack of N regular and M fractional branes at a conifold singularity. The corresponding gauge theory is governed by $\mathcal{N}=1$ supersymmetric $SU(N+M)\times SU(N)$ gauge theory introduced in [44]. The geometry of the background is given by the metric

$$ds^2 = H(r)^{-\frac{1}{2}}\ \eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{\frac{1}{2}}\Big(dr^2 + r^2 ds^2_{T^{1,1}}\Big)\ , \tag{2.63}$$

where $\eta_{\mu\nu}$ is the metric denoting the stranded $(3+1)$-dimensional Minkowski space and $T^{1,1}$ metric is given in (2.48). The expression of the warp factor $H(r)$ is given by [60]

$$H(r) = \frac{1}{r^4}\left[R^4 + 2L^4\left(\ln\left(\frac{r}{r_0}\right) + \frac{1}{4}\right)\right]\ . \tag{2.64}$$

In addition to the metric, the background NS-NS two-form field B_2 has the expression

$$B_2 = \frac{T(r)}{6\sqrt{2}}\Big(\sin\theta_1 d\theta_1\wedge d\phi_1 - \sin\theta_2 d\theta_2\wedge d\phi_2\Big)\ . \tag{2.65}$$

The corresponding NS-NS three form takes the form

$$H_3 = \frac{L^2}{3r}\Big(\sin\theta_1 dr\wedge d\theta_1\wedge d\phi_1 - \sin\theta_2 dr\wedge d\theta_2\wedge d\phi_2\Big)\ . \tag{2.66}$$

Due to the presence of fractional branes, in the RR sector we have three-form field strengths F_3 along with the five form field strengths F_5 as

$$\begin{aligned} F_3 =&\ \frac{P}{18\sqrt{2}}\Big(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\Big)\wedge\Big(\sin\theta_1 d\theta_1\wedge d\phi_1 - \sin\theta_2 d\theta_2\wedge d\phi_2\Big)\ , \\ F_5 =&\ \big(1+\star_{10}\big)K(r)\mathrm{Vol}\big(T^{1,1}\big)\ . \end{aligned} \tag{2.67}$$

Here, $\star_{10}$ denotes the Hodge dual with respect to the background metric (2.63). In the above we follow the notations [44]

$$\begin{aligned} P =&\ \frac{L^2}{g_s}\ 2\sqrt{2}\ , \\ T(r) =&\ 2\sqrt{2}L^2\ln\left(\frac{r}{r_0}\right)\ , \\ K(r) =&\ \frac{r^4}{30}\ H(r)\left[1 - \frac{L^4}{2r^4H(r)}\right]\ . \end{aligned} \tag{2.68}$$

It turns out that the constant P is proportional to the number M of fractional $D3$ branes present in the type-IIB background.

Now in the following we consider the non-Abelian T-dual of the background (2.63)-(2.68). The non-Abelian T-duality acts on an $SU(2)$ isometry of the background metric and has been studied in [15, 23]. The metric of the T-dual geometry is given by

$$\begin{aligned} d\hat{s}^2_{\text{NATD}} = \; & H(r)^{-\frac{1}{2}}\; \eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{\frac{1}{2}}\Big(dr^2 + \frac{1}{6}r^2 d\Omega_2^2(\theta_1,\phi_1)\Big) \\ & + \frac{1}{2r^2\Delta\; H(r)^{\frac{1}{2}}}\Bigg[12r^4H(r)v_2^2\sigma_{\hat{3}}^2 + 12\Big(r^4H(r)+27v_2^2\Big)dv_2^2 \\ & + 9\Big(2r^4H(r)+\mathcal{V}^2\Big)dv_3^2 + 108\mathcal{V}v_2\; dv_2dv_3\Bigg]\;, \end{aligned} \tag{2.69}$$

where, $\sigma_{\hat{3}}$ has the expression

$$\sigma_{\hat{3}} = d\psi + \cos\theta_1 d\phi_1\;, \tag{2.70}$$

together with the functions Δ and $\mathcal{V}$ takes the form

$$\Delta = 2r^4H(r) + \mathcal{V}^2 + 54v_2^2\;,\; \mathcal{V} = 6v_3 + 2\sqrt{2}L^2\ln\left(\frac{r}{r_0}\right)\;. \tag{2.71}$$

The NS-NS sector of the background is described by the following NS-NS two-form $\hat{B}_2$ and dilaton $\hat{\Phi}$

$$\begin{aligned} \hat{B}_2 \; &= \frac{L^2}{3}\ln\left(\frac{r}{r_0}\right)\sin\theta_1 d\theta_1\wedge d\phi_1 + \frac{3\sqrt{2}}{\Delta}\;\mathcal{V}v_2\sigma_{\hat{3}}\wedge dv_2 + \frac{1}{\sqrt{2}\Delta}\Big(2r^4H(r)+\mathcal{V}^2\Big)\sigma_{\hat{3}}\wedge dv_3\;, \\ e^{-2\hat{\Phi}} \; &= \frac{1}{81g_s^2}\; r^2H(r)^{\frac{1}{2}}\;\Delta\;. \end{aligned} \tag{2.72}$$

The field strengths corresponding to the RR sector are given by

$$\begin{aligned} \hat{F}_0 = \; & -L^2\;\frac{2\sqrt{2}}{9g_s}\;, \\ \hat{F}_2 = \; & \frac{1}{162\sqrt{2}\;g_s}\left[\frac{r^4}{5}\;H(r)\Big(1-\frac{L^4}{2r^4H(r)}\Big) + L^2\;6\sqrt{2}\Big(6v_3+2\sqrt{2}L^2\ln\left(\frac{r}{r_0}\right)\Big)\right]\sin\theta_1 d\theta_1\wedge d\phi_1 \\ & -\;L^2\frac{4}{3g_s}\;\frac{6v_3+2\sqrt{2}L^2\ln\left(\frac{r}{r_0}\right)}{\Delta}\;v_2\;\sigma_{\hat{3}}\wedge dv_2 + L^2\;\frac{12}{g_s}\frac{v_2^2}{\Delta}\;\sigma_{\hat{3}}\wedge dv_3\;, \\ \hat{F}_4 = \; & \frac{v_2}{18\Delta\;g_s}\sin\theta_1 d\theta_1\wedge d\phi_1\wedge d\psi\wedge\Bigg[\Bigg(-18\sqrt{2}L^2\Big(6v_3+2\sqrt{2}L^2\ln\left(\frac{r}{r_0}\right)\Big) \\ & -\frac{3r^4}{5}\;H(r)\Big(1-\frac{L^4}{2r^4H(r)}\Big)\Bigg)v_2dv_3 + \;2\Bigg(-2\sqrt{2}L^2r^4H(r) + \frac{r^4}{30}\Big(6v_3 \\ & +2\sqrt{2}L^2\ln\left(\frac{r}{r_0}\right)\Big)H(r)\Big(1-\frac{L^4}{2r^4H(r)}\Big) - 54\sqrt{2}L^2v_2^2\Bigg)dv_2\Bigg]\;. \end{aligned} \tag{2.73}$$

The presence of $\hat{F}_0$ in the RR sector indicates that the T-dual background is a solution (2.69)-(2.73) in massive type-IIA supergravity. As like the Klebanov-Witten case, here also the non-Abelian T-dual solution preserves $\mathcal{N}=1$ supersymmetry [15, 23].

non-Abelian T-dual backgrounds with AdS$_3$ factor

Finally we turn on our interest in a background that contains an AdS$_3$ factor. We consider the type-IIB supergravity on the background with $AdS_3 \times CY_2 \times S^3$ geometry. The geometry describes the near horizon limit of a stake intersecting $D1-D5$ branes configuration. The metric of the corresponding background is given by

$$ds^2 = 4L^2 ds^2\big(AdS_3\big) + L^2 ds^2\big(CY_2\big) + 4L^2 ds^2\big(S^3\big) \ . \tag{2.74}$$

where

$$\begin{aligned} ds^2\big(AdS_3\big) &= \ -\cosh^2 r dt^2 + dr^2 + \sinh^2 r \ d\chi^2 \ , \\ ds^2\big(S^3\big) &= \ d\alpha^2 + d\beta^2 + d\gamma^2 + 2\cos\alpha d\beta d\gamma \ . \end{aligned} \tag{2.75}$$

Together with the above $ds^2(CY_2)$ represents the metric of a Calabi-Yau twofold, *i.e.* it can be a $K3$ or a T^4.

In the above, the parameter L describes the size of the internal manifold CY_2. For the case of T^4, the metric is given by

$$ds^2\big(T^4\big) = \sum_{i=1}^{4} dz_i^2 \ . \ i = 1,2,3,4 \ . \tag{2.76}$$

In addition to the background metric, the above type-IIB background is supported by the dilaton field $e^{2\Phi}=1$ in the NS-NS sector and the RR sector is described by three-form field strengths as

$$F_3 = 8L^2 \ \mathrm{Vol}\big(S^3\big) \ . \tag{2.77}$$

The presence of the S^3 factor in the metric (2.74) indicates that the background admits an $SO(4)$ R-symmetry. The non-Abelian T-duality along an $SU(2)$ subgroup of it is studied in [12], where $CY_2 = T^4$. The T-dual metric reads

$$d\hat{s}^2_{\mathrm{NATD}} = 4L^2 ds^2\big(AdS_3\big) + L^2 ds^2\big(T^4\big) + \frac{d\rho^2}{4L^2} + \frac{L^2\rho^2}{4L^4+\rho^2} \ d\Omega_2^2\big(\theta,\phi\big) \ , \tag{2.78}$$

Together with the metric the NS-NS sector of the background is contained with a dilaton and two-form gauge field

$$e^{-2\hat{\Phi}} = \frac{1}{4g_s^2}\big(4L^6 + L^2\rho^2\big) \ , \ \hat{B}_2 = -\frac{\rho^3}{2\big(4L^4+\rho^2\big)} \sin\theta d\theta\wedge d\phi \ . \tag{2.79}$$

The RR sector is described by the following zero form, two-form and four-field strengths

$$\hat{F}_0 = \frac{L^2}{g_s} \ , \ \hat{F}_2 = -\frac{L^2 \rho^3}{2 g_s \big(4L^4 + \rho^2\big)} \sin\theta d\theta \wedge d\phi \ , \ \hat{F}_4 = -\frac{L^6}{g_s} \mathrm{Vol}\big(T^4\big) \ . \tag{2.80}$$

The T-dual geometry describes a massive type-IIA supergravity on $AdS_3 \times T^4 \times M_3$. It turns out that the background (2.78)-(2.80) preserves only half the supersymmetry of the original background [12, 46].

Chapter 3

Gauge-Gravity duality, Penrose limits, PP-wave and BMN correspondence

In this chapter first we will review the revolutionary AdS/CFT correspondence and then we will discuss some aspects of this correspondence in the Penrose limits. The AdS/CFT correspondence draws a clear picture between a gravitational theory with certain gauge theory. In the following we will discuss some aspects of this correspondence.

3.1 Gauge-Gravity duality

The AdS/CFT correspondence was introduced by Maldacena in his groundbreaking work in [2]. It provides a correspondence between a certain Quantum Field Theory and a theory of gravity. The original correspondence states that type-IIB string theory on $AdS_5 \times S^5$ is equivalent to $SU(N)$ super Yang-Mills theory in four spacetime dimensions that resides at the boundary of AdS_5. Here the Yang-Mills theory is $\mathcal{N} = 4$ supersymmetric with large colour group N. In the correspondence the 't Hooft coupling holds fixed; $\lambda = g_S N =$ fixed, where g_s is the type-IIB string coupling. The correspondence relates a strongly coupled gauge theory with a weakly coupled gravity theory in AdS spacetime. On the string theory side, we have two expansion parameters α and g_s. On the other hand, the gauge theory contains two parameters N and the t'Hooft expansion parameter λ. They are related among each other as

$$g_s \leftrightarrow \frac{1}{N} \ ; \ \alpha' \leftrightarrow \lambda \ . \tag{3.1}$$

There are various description exists to study the AdS/CFT correspondence. In the following we will discuss the correspondence in the Dp-brane scenario.

In the string theory Dp-branes are defined as hypersurfaces where the open string end. Dp-branes are extended objects in $p+1$ spacetime dimensions. Due to the existence of the Dp-branes, string theory inhibits a non-perturbative expansion. The motion and the deformation of the Dp-brane is parametrized by the scalar fields ϕ^i ; $i = 1, .., 9 - p$, which reside on the world-volume of the Dp-brane. The associate's action is known as the Born-Infeld action and is given by the expression

$$S_{BI} = -T_{Dp} \int d^{p+1}x - det\sqrt{\Big(g_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\Big)} \tag{3.2}$$

where $F_{\mu\nu}$ is the field strength of the gauge field A_μ, and $g_{\mu\nu}$ is the induced metric on the worldvolume geometry. Considering the flat space background the induced metric takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \big(2\pi\alpha'\big)^2 \partial_\mu\phi^i\partial_\nu\phi^i \ . \tag{3.3}$$

Now one can expand the action in powers of $F_{\mu\nu}$ and $\partial_\mu\phi^i$. In the leading order we have

$$S = -T_{Dp}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi^i\partial_\nu\phi^i + ...\right) \ . \tag{3.4}$$

Now by rewriting the T_{Dp} as $1/g^2_{YM}$, it turns out that the first term in the above equation describes an $U(N)$ gauge theory. The above analysis states that the N Dp-branes have a $SU(N)$ gauge theory description in $p+1$ dimensions. The closed strings in the bulk give us supergravity.

Now we consider another set-up governed by this Dp-brane. Consider a stack of N $D3$-branes take place on top of each other. The classical ten-dimensional supergravity solution obtained from this system is given by

$$ds^2 = H(r)^{-\frac{1}{2}}\Big(-dt^2 + dx^2_{1,2,3}\Big) + H(r)^{\frac{1}{2}}\Big(dr^2 + r^2 d\Omega_5^2\Big) \ , \tag{3.5}$$

where

$$H(r) = 1 + \frac{L^4}{r^4} \ ; \ L^4 \sim N \ , \tag{3.6}$$

and $d\Omega_5^2$ is the metric of the five-dimensional round sphere.

Together with the metric, the background contains a five-form field strengths given by

$$F_5 = \Big(1 + \star\Big) \ dH(r)^{-1} \wedge dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \ . \tag{3.7}$$

Considering the near horizon limit *i.e.* $r << L$, the $H(r)$ becomes $H(r) \sim L^4/r^4$. Then the metric in (3.5) takes the form

$$ds^2 = \frac{r^2}{L^2}\Big(-dt^2 + dx^2_{1,2,3}\Big) + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2 \ . \tag{3.8}$$

The above geometry describes the $AdS_5 \times S^5$ background in Poincare coordinates.

Considering the asymptotic limit, $r >> L$ of the metric (3.5) we have

$$ds^2 = \Big(- dt^2 + dx_{1,2,3}^2 \Big) + dr^2 + r^2 d\Omega_5^2 \ , \tag{3.9}$$

The above is just the metric of flat Minkowski space in ten spacetime dimensions.

Hence at the near horizon limit the geometry (3.5) describes the $AdS_5 \times S^5$ background and at asymptotic limit it is just the geometry of the flat ten-dimensional space. Now we have two description of the Dp-brane picture. First one corresponds to the system that is decoupled into supergravity of the bulk and $\mathcal{N} = 4$ super Yang-Mills. The second one corresponds to a system in bulk supergravity together with the near horizon geometry that describes type-IIB string theory in $AdS_5 \times S^5$. In both the descriptions the bulk part is common; this lead to state that the $\mathcal{N} = 4$ super Yang-Mills theory is equivalent to the gravity theory on $AdS_5 \times S^5$ with the following identification of the couplings of the two theories

$$g_s = g_{YM}^2 \ ; \frac{L^4}{\alpha'^2} = g_{YM}^2 N \equiv \lambda \ . \tag{3.10}$$

Moreover, there is some consistency present in the symmetry description in both sides of the correspondence. Considering the hyperboloid coordinates the global AdS can be written

$$-(X^{-1})^2 - (X^0)^2 + (X^1)^2 + ... + (X^{d-1})^2 = -L^2 \ , \tag{3.11}$$

where L is the size of the AdS space. It is straightforward to see that the isometry of the AdS_5 geometry is $SO(2,4)$. In addition to AdS_5, we have the S^5 part that admits an $SO(6)$ isometry. Hence the background exhibits a global $SO(2,4) \times SO(6)$ symmetry. Now the field theory shows the same global isometry as follows. In four-dimensional the conformal group is $SO(2,4)$. Together with in $\mathcal{N} = 4$ super Yang-Mills theory we have six scalars $\phi^i \ ; i = 1, ..., 6$ that admit an $SO(6)$ isometry. This $SO(6)$ isometry rotates the ϕ^is. Hence the field theory also exhibits a global $SO(2,4) \times SO(6)$ symmetry.

Now in the following we will discuss the mapping of the fundamental observables on both sides of the correspondence. Let $\mathcal{O}$ be an operator in the boundary super Yang-Mills theory and the field ϕ is in the bulk of AdS. The field ϕ takes the value ϕ_0 at the boundary of AdS. Then the correspondence states that

$$\begin{aligned} Z_{\mathcal{O}}\Big[\phi_0\Big]_{CFT} &= \int \mathcal{D}\Big[\text{super Yang-Mills fields}\Big] e^{-S_{\text{super Yang-Mills fields}} + \int d^4x \mathcal{O}(x)\phi_0(x)} \\ &= Z_{classical}\Big[\phi_0\Big]_{AdS} = e^{S_{SUGRA}\left[\phi[\phi_0]\right]} \end{aligned} \tag{3.12}$$

where large-N and large-λ limit are taken into account. From (3.12) one can calculate the correlation functions for the conformal operators in boundary gauge theory [4]

$$\langle \mathcal{O}(x_1) .... \mathcal{O}(x_n) \rangle = \frac{\delta^n}{\delta\phi_0(x_1) ... \delta\phi_0(x_n)} Z_{\mathcal{O}} \Big[\phi_0\Big]_{\phi_0 = 0} \ . \tag{3.13}$$

This correspondence has been generalized for the various AdS spaces with a compact manifold. In general the correspondence states that string theory in the bulk of any AdS_{d+1} spacetime is equivalent to a conformal gauge theory with d dimension which resides at the boundary of the AdS_{d+1} spacetime.

$$\text{String theory on } AdS_{d+1} \longleftrightarrow \text{Conformal gauge theory } CFT_d \text{ on } \partial AdS_{d+1}\,. \tag{3.14}$$

The correspondence is also generalized for the cases where the compact manifold is some X^5 rather than S^5. Namely in those cases the correspondence is known as gauge / gravity duality. One simplest example of such geometry is $AdS_5 \times T^{1,1}$. The background arises upon placing N $D3$ branes at conifold singularity [6]. The corresponding gauge theory was introduced by Klebanov and Witten in [6]. The metric of $T^{1,1}$ admits an $SU(2) \times SU(2) \times U(1)$ isometry that is also present in the dual $\mathcal{N} = 1$ SYM theory.

In the various literature the correspondence has been investigated for the AdS_d backgrounds where $d = 2, 3, 4, 5, 6, 7$. However, the direct proof of this correspondence is still lacking. For the T-dual backgrounds we discussed in the last chapter, the corresponding gauge theories have been discussed in [21, 22, 23, 24, 25, 26, 27, 28].

3.2 Penrose limits, PP-wave and BMN correspondence

Penrose limits have been studied in great detail for a large class of supergravity backgrounds for more than a decade [32, 34]. Supergravity backgrounds for which the Penrose limits lead to pp-wave geometries play a significant role to construct interacting string states through AdS/CFT correspondence [35, 36]. The most interesting feature of the PP-wave geometries is that they provide exact backgrounds to all orders in α' and g_s in string theory including the Ramond-Ramond sector of the closed string [37, 38]. PP-wave geometries are obtained by considering the Penrose limit along appropriate null geodesic [33]. Those null geodesic are singularity free. The Penrose limits, PP-wave together with BMN sector in the dual gauge theory have been studied in various type-IIB supergravity solutions [39, 40, 41, 42]. In the following sections we will review some of them.

We will start our discussion by considering the pioneering work by *Berenstein, Maldacena* and *Nastase* in the maximally supersymmetric background $AdS_5 \times S^5$. For this background, it the work by BMN has opened up new insights in stringy calculations in gauge theories [35]. We have presented the geometry of the $AdS_5 \times S^5$ background together with the background fields in (2.38)-(2.40). The gauge theory dual of the type-IIB string modes in the above background has one-to-one correspondence with a set of operators in $\mathcal{N} = 4$ super Yang-Mills theory that resides at the boundary of the AdS_5-spacetime [2, 4]. Now in the following we will review the work by BMN [35]. Consider the motion of a particle with very high speed along ψ-direction and sitting at the equator of S^5 manifold: $\alpha = \frac{\pi}{2}, \theta = 0$. In addition, the null geodesic resides at the origin of the AdS_5 background *i.e.* $\rho = 0$. We expand the background metric (2.38) in the vicinity of this null geodesic with the following form

$$\rho = \frac{r}{L}\ ;\ \alpha = \frac{\pi}{2} + \frac{y}{L}\ ;\ x^+ = \frac{1}{2}\big(t+\beta\big)\ ;\ x^- = \frac{L^2}{2}\big(t-\beta\big)\ , \tag{3.15}$$

and subsequently take the $L \to \infty$ limit. In this limit, the leading order terms provide

$$ds^2_{pp} = 4dx^+dx^- - \mu^2\big(\bar{r}_4^2 + \bar{y}_4^2\big)dx^{+2} + d\bar{r}_4^2 + d\bar{y}_4^2\ . \tag{3.16}$$

In this limit the self-dual five-form field strength in (2.40) takes the form

$$F_{+1234} = F_{+5678} = \mu\ . \tag{3.17}$$

Here the $\bar{r}_4$ and $\bar{y}_4$ parametrizes the transverse $\mathbb{R}^8$ geometry in the metric (3.16) and the parameter μ is introduced by rescaling the $x^\pm$ coordinates: $x^+ \to \mu x^+$ and $x^- \to \frac{x^-}{\mu}$.

The BMN proposal [35] states that the Penrose limit on the gravity side restricts the dual gauge theory to a certain sector that carries large $U(1)$ R-symmetry charge. The dual gauge theory corresponding to $AdS_5 \times S^5$ background is governed by $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four spacetime dimensions on $\mathbb{R} \times S^3$. The gauge theory dual for the pp-wave geometry (3.16)-(3.17) is the set of operators with large $U(1)_R$-symmetry charge J in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory together with large-N limit [35]. This large R sector in the dual gauge theory is known as the 'BMN' sector of the super-Yang-Mills theory:

$$\boxed{\text{pp-wave in gravity side}} \longleftrightarrow \boxed{\text{BMN operators in the dual gauge theory}}\ . \tag{3.18}$$

In the BMN proposal $J\big(=i\partial_\psi\big)$ is the angular momentum operator that rotates the (1-2) plane of transverse $\mathbb{R}^6$. In addition to the angular momentum operator J, we have energy operator E in the theory which is the same as the conformal dimension Δ of the operators in $\mathbb{R}^4$ and it is given by $E = i\partial_t = \Delta$. In the Penrose limits, the operators Δ and J take the form

$$\begin{aligned} &\frac{p^-}{\mu} = \Delta - J = \text{fixed}\ ,\ \alpha'\mu p^+ = \frac{\Delta + J}{\sqrt{4g^2_{YM}N}} = \text{fixed}\ , \\ &g^2_{YM} = \text{fixed}\ ,\ \frac{J^2}{N} = \text{fixed}\ ,\ N \to \infty\ ,\ J \to \infty\ . \end{aligned} \tag{3.19}$$

Here p^- is the light-cone Hamiltonian in string theory and the BMN proposal it measures the difference between conformal dimension (Δ) and the R-charge (J) of the operators in dual $\mathcal{N} = 4$ SYM gauge theory.

The BMN correspondence states that the interacting string states give rise to the Hilbert space of $\mathcal{N} = 4$ SYM generated by the BMN operators acting on the CFT vacuum state. The string theory vacuum state has correspondence with the BMN operator acting on the CFT vacuum state

$$|0, p^+\rangle_{\text{string vacuum state}} \longleftrightarrow \mathcal{N}\ Tr\big(Z^J\big)|0\rangle_{\text{CFT vacuum state}}\ , \tag{3.20}$$

where $\mathcal{N}$ is the normalization factor and Z is a chiral multiplet in $\mathcal{N}=4$ SYM with R-charge $J[Z]=1$. For the excited string states, BMN proposed that the corresponding field theory operators are nearly BPS: $\Delta - J \sim \mathcal{O}(1)$. The state described in (3.20) has $\Delta - J = 0$. The first excited string state is defined by $\Delta - J = 1$. The corresponding states in the gauge theory are obtained by acting the BMN operators on the CFT vacuum state. On the field theory side there are eight bosonic modes together with eight fermionic modes corresponding to the first excited string state $\Delta - J = 1$. These bosonic and fermionic modes in the dual $\mathcal{N}=4$ SYM are obtained as follows. In the gauge theory, there are four scalars (ϕ^i ; $i=1,2,3,4$) in the directions not rotated by J. We also have derivatives of the field Z as $D_iZ = \partial_i Z + [A_i, Z]$, where $i=1,2,3,4$; orientated along $\mathbb{R}^4$. Together with the above bosonic operators, there are eight fermionic operators $\chi^a_{J=\frac{1}{2}}$ with $J=\frac{1}{2}$ and other eight with $J=-\frac{1}{2}$ of the sixteen component gaugino χ in the super Yang-Mills theory. The eight components of the gaugino transform in the spinor representation of $SO(4)\times SO(4)$, which is also the symmetry of the background metric (3.16).

Now in the following we will discuss the Penrose limits and PP-waves in Klebanov-Witten background. The background provides one of the earliest and interesting examples of AdS/CFT correspondence. The dual gauge theory of the type-IIB string in this background is $\mathcal{N}=1$ SYM in four spacetime dimensions [5,6]. The background metric and the background gauge fields are given in (2.47)-(2.49). The Penrose limits of this background have been studied in detail in [39,40,41]. The corresponding BMN operators have also been presented in the literature. In the following we will review the work in [40].

Consider the motion of a particle with a very high speed along the direction given by $\big(\psi+\phi_1+\phi_2\big)$ in the $T^{1,1}$ geometry. The geodesic resides at the origin of AdS_5 i.e. $\rho=0$ together with $\theta_i=0$. Before considering the Penrose limits we rewrite the coordinates of $T^{1,1}$ as

$$\begin{aligned}
\tilde{x}^+ &= \frac{1}{2}\Big[t+\frac{1}{3}\big(\psi+\phi_1+\phi_2\big)\Big]\ ,\\
\tilde{x}^- &= \frac{1}{2}\Big[t-\frac{1}{3}\big(\psi+\phi_1+\phi_2\big)\Big]\ ,\\
\Phi_1 &= \phi_1-\frac{1}{2}\Big[t+\frac{1}{3}\big(\psi+\phi_1+\phi_2\big)\Big]\ ,\\
\Phi_2 &= \phi_2-\frac{1}{2}\Big[t+\frac{1}{3}\big(\psi+\phi_1+\phi_2\big)\Big]
\end{aligned} \tag{3.21}$$

and

$$x^+=\tilde{x}^+\ ,\ x^-=L^2\tilde{x}^-\ ,\ \rho=\frac{r}{L}\ \text{and}\ \theta_i=\sqrt{6}\ \frac{r_i}{L}\ . \tag{3.22}$$

In the large L limit, the leading order of the expansion provides

$$\begin{aligned}
ds^2 = &-4dx^+dx^- - \mu^2\big(r^2+r_1^2+r_2^2\big)dx^{+2}+dr^2+r^2d\Omega_3^2\\
&+dr_1^2+r_1^2d\Phi_1^2+dr_2^2+r_2^2d\Phi_2^2\ .
\end{aligned} \tag{3.23}$$

Here we introduce the parameter μ via rescaling the $x^{\pm}$ coordinates as $x^+ \to \mu x^+$ and $x^- \to \frac{x^-}{\mu}$. In the above metric the $\left(r_i, \Phi_i\right);\ i = 1, 2$ part parametrizes an $\mathbb{R}^2$ space. The transverse part of (3.23) defines an $\mathbb{R}^8$ space. Parametrized the transverse $\mathbb{R}^8$ space by the coordinates $\bar{z}_i;\ i = 1, 2, ..., 8$ the metric in (3.23) takes the form

$$ds^2 = -4dx^+dx^- - \mu^2 \bar{z}^2 dx^{+2} + d\bar{z}_i^2 \ , \tag{3.24}$$

where $\bar{z}^2 = \sum_{i=1}^{8} \bar{z}_i^2$.

The expression of five-form field strength in this limit is given as

$$F_{+1234} = F_{+5678} \sim \mu \ . \tag{3.25}$$

The above pp-wave background (3.24)-(3.25) provides an exactly solvable string theory including fields in RR sector [61, 62].

The gravity dual for the above PP-wave background is described in [40]. The conifold is described by a quadric in $\mathbb{C}^4$ as $\sum_{i=1}^{4} w_i^2 = 0$. It can also be written as

$$\begin{aligned} &det\ \mathcal{W} = 0\ ;\ Z_1 Z_2 - Z_3 Z_4 = 0\ , \\ &\mathcal{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} w_3 + i w_4 & w_1 - i w_2 \\ w_1 + i w_2 & -w_3 + i w_4 \end{pmatrix} \equiv \begin{pmatrix} Z_1 & Z_3 \\ Z_4 & Z_2 \end{pmatrix} \ . \end{aligned} \tag{3.26}$$

The above quadric equation exhibits an $SO(4)$ isometry that can be written as $SU(2)_A \times SU(2)_B$ together with an $U(1)$ symmetry: $w_i \to e^{i\alpha} w_i$. This $U(1)$ symmetry is identified with $U(1)$ R-symmetry in the dual gauge theory.

Now we will discuss the BMN sector in the gauge theory corresponding to the pp-wave geometry studied in (3.23)-(3.25). In the Klebanov-Witten theory the lowest components of chiral operators are identified with the conifold coordinates as

$$Z_1 = A_+ B_+\ ;\ Z_2 = A_- B_-\ ;\ Z_3 = A_+ B_-\ ;\ Z_4 = A_- B_+\ . \tag{3.27}$$

Here the fields $A_{\pm}$ and $B_{\pm}$ form doublets under $SU(2)_A$ and $SU(2)_B$ respectively and carry $U(1)$ R-charge of $1/2$. In the BMN correspondence, the light-cone Hamiltonian of the string is identified with $\Delta - J$ of the operators in the field theory, where $J = -i\partial_\psi$ is the angular momentum operator [35]. For the Klebanov-Witten background, the angular momentum operator J takes the form

$$J = -i \left[\frac{1}{2} \partial_\psi + \partial_{\phi_1} + \partial_{\phi_2} \right] . \tag{3.28}$$

In the following table we represent the fields in the dual $SU(N) \times SU(N)$ gauge theory together with their gauge transformation. In the table, $U(1)_A$ and $U(1)_B$ are identified with the T_3 generators of $SU(2)_A$ and $SU(2)_B$ respectively and J is given by $J = \frac{1}{2} U(1)_R + U(1)_A + U(1)_B$.

	$SU_L(N)$	$SU_R(N)$	$U_A(1)$	$U_B(1)$	$U_R(1)$	J	Δ	$\Delta - J$
$(A_+\psi_+^A)$	N	$\bar{N}$	1/2	0	(1/2, -1/2)	(3/4, 1/4)	(3/4, 5/4)	(0, 1)
$(A_-\psi_-^A)$	N	$\bar{N}$	-1/2	0	(1/2, -1/2)	(-1/4, -3/4)	(3/4, 5/4)	(1, 2)
$(B_+\psi_+^B)$	$\bar{N}$	N	0	1/2	(1/2, -1/2)	(3/4, 1/4)	(3/4, 5/4)	(0, 1)
$(B_-\psi_-^B)$	$\bar{N}$	N	0	-1/2	(1/2, -1/2)	(-1/4, -3/4)	(3/4, 5/4)	(1, 2)
$Z \equiv A_+B_+$	$adj \bigoplus 1$	$adj \bigoplus 1$	1/2	1/2	1	3/2	3/2	0
$\phi_1 \equiv A_+B_-$	$adj \bigoplus 1$	$adj \bigoplus 1$	1/2	- 1/2	1	1/2	3/2	1
$\phi_2 \equiv A_-B_+$	$adj \bigoplus 1$	$adj \bigoplus 1$	-1/2	1/2	1	1/2	3/2	1
$\psi_1 \equiv A_+\psi_+^B$	$adj \bigoplus 1$	$adj \bigoplus 1$	1/2	- 1/2	0	1	2	1
$\psi_2 \equiv B_+\psi_+^A$	$adj \bigoplus 1$	$adj \bigoplus 1$	-1/2	1/2	0	1	2	1

From the above table, it is straightforward to see that the BMN operator corresponds to the ground state of the string is $Z\big(\equiv A_+B_+\big)$ with proper normalization factor as

$$|0,p^+\rangle_{\text{string vacuum state}} \longleftrightarrow \frac{1}{\sqrt{J}N^{\frac{J}{2}}}Tr\Big[\big(A_+B_+\big)^J\Big] \,. \tag{3.29}$$

The operator Z carries $\Delta - J = 0$. The bosonic operators correspond to $\Delta - J = 1$ are given by D_iZ together with ϕ_1 and ϕ_2. In the fermionic sector we have ψ_1 and ψ_2 associated with $\Delta - J = 1$. These states correspond to higher energy string states.

The BMN correspondence in the Klebanov-Witten geometry is:

$$\begin{aligned} &\Delta - J = 0 \longrightarrow Z \longleftrightarrow \text{ ground state of string },\\ &\Delta - J = 1 \longrightarrow \{D_iZ \;,\; \phi_1 \;,\; \phi_2 \;,\; \psi_1 \;,\; \psi_2\} \longleftrightarrow \text{ first excited state of string }. \end{aligned} \tag{3.30}$$

There are two operators which are missing in the bosonic sector. They are associated with non-chiral operators build out by basic bosonic fields as described in [63].

Now we will turn our attention to the non-conformal Klebanov-Tseytlin background [44]. The type-IIB metric together with the background fields are given in (2.63)-(2.68). The Penrose limits in the background have been studied in detail in [40]. However, unlike the $AdS_5 \times S^5$ case and the Klebanov-Witten background here the Penrose limits do not lead to the PP-wave background. In the following we will discuss the work of [40].

To inspect the Penrose limits, we consider the motion of a particle along the geodesic given by $\big(\psi + \phi_1 + \phi_2\big)$ in the transverse space $T^{1,1}$ and we concentrate the region around $\theta_i = 0$. Consider the motion of a massless particle in the (r, ψ) plane that resides at $\theta_1 = 0 = \theta_2$. The Lagrangian that describes the system is given by

$$\mathcal{L} = -\frac{r^2}{\sqrt{1 + P\ln\left(\frac{r}{r_0}\right)}}\,\dot{t}^2 + \frac{1}{r^2}\sqrt{1 + P\ln\Big(\frac{r}{r_0}\Big)}\;\dot{r}^2 + \sqrt{1 + P\ln\Big(\frac{r}{r_0}\Big)}\;\dot{\psi}^2 \,. \tag{3.31}$$

The Lagrangian (3.31) does not depend on the generalized coordinates t and ψ explicitly. Hence, the conjugate momenta associated with t and ψ coordinates will be conserved. Denoting them as E and μ respectively we get

$$\dot{t} = \frac{E}{r^2}\sqrt{1 + P\ln\big(\frac{r}{r_0}\big)} \text{ and } \dot{\psi} = \frac{\mu}{\sqrt{1 + P\ln\big(\frac{r}{r_0}\big)}} \ . \tag{3.32}$$

The null geodesic condition $i.e.$ $\mathcal{L} = 0$ leads to

$$\dot{r}^2 + \frac{\mu^2 r^2}{1 + P\ln\big(\frac{r}{r_0}\big)} = E^2 \ . \tag{3.33}$$

Following [40], consider the following expansion

$$\partial_u = \dot{r}\partial_r + \dot{t}\partial_t + \dot{\psi}\partial_\psi \ , \ \partial_v = -\frac{1}{E}\partial_t \ , \ \partial_x = \mu\partial_t + E\partial_\psi \tag{3.34}$$

and set $E = 1$ for convenience.

Now rescale the coordinates of the background geometry as

$$u \to u \ ; \ v \to \frac{v}{L^2} \ ; \ \theta_i = \sqrt{6}\,\frac{r_i}{L} \ ; \ x \to \frac{x}{L} \tag{3.35}$$

and subsequently take $L \to \infty$ limit. In the expansion, leading order of the metric provides

$$\begin{aligned} ds^2 = {} & 2dudv + \frac{r^2}{\sqrt{1 + P\ln\big(\frac{r}{r_0}\big)}}\, dx_3^2 + \sqrt{1 + P\ln\big(\frac{r}{r_0}\big)}\ \left[1 - \frac{\mu^2 r^2}{1 + P\ln\big(\frac{r}{r_0}\big)}\right] dx^2 \\ & + \sqrt{1 + P\ln\big(\frac{r}{r_0}\big)}\left[dr_1^2 + r_1^2 d\phi_1^2 + dr_2^2 + r_2^2 d\phi_2^2\right] - \frac{\mu^2}{\sqrt{1 + P\ln\big(\frac{r}{r_0}\big)}}\Big(r_1^2 + r_2^2\Big) du^2. \end{aligned} \tag{3.36}$$

Together with the above metric the NS-NS sector and the RR sector fields in this limit take the form

$$\begin{aligned} B_2 \sim {} & P\ln\big(\frac{r}{r_0}\big)\Big(dr_1 \wedge r_1 d\phi_1 - dr_2 \wedge r_2 d\phi_2\Big) \ , \\ F_3 \sim {} & P\dot{\psi}\ du \wedge \Big(dr_1 \wedge r_1 d\phi_1 - dr_2 \wedge r_2 d\phi_2\Big) \ , \\ F_5 \sim {} & \Big(1 + \star\Big)\Big(1 + P\ln\big(\frac{r}{r_0}\big)\Big)\dot{\psi}\ du \wedge dr_1 \wedge r_1 d\phi_1 \wedge dr_2 \wedge r_2 d\phi_2 \ . \end{aligned} \tag{3.37}$$

It is shown that the above background leads to pp-wave geometry by setting $P = 0$ [40]. As mentioned earlier, P measures the number of fractional $D3$-brane. Hence $P = 0$ leads to the removal of fractional $D3$-brane and restoring the undeformed Klebanov-Witten background. This analysis showed that the Klebanov-Tsyetlin background does not provide PP-wave geometry [40].

Finally we will discuss the Penrose limits in the AdS_3 background. Due to the lack of an explicit expression for the metric of $K3$, we consider the geometry to be $AdS_3 \times S^3 \times T^4$ with the metric

$$ds^2_{AdS_3\times S^3\times T^4} = R^2 ds^2\big(AdS_3\big) + R^2 ds^2\big(S^3\big) + R^2 ds^2\big(T^4\big) \ , \tag{3.38}$$

where

$$\begin{aligned} ds^2\big(AdS_3\big) =& \ -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho \, d\varphi^2 \ , \\ ds^2\big(S^3\big) =& \ d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\chi^2 \ , \\ ds^2\big(T^4\big) =& \ \frac{\varsigma}{R^2}\Big(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2\Big) \ . \end{aligned} \tag{3.39}$$

Here R is the size of the manifold S^3 and ς is the size of the four-dimensional torus T^4. In addition, the RR sector of the type-IIB background is described by the following three-form field strength

$$F_3 = \mathrm{Vol}\big(S^3\big) \ . \tag{3.40}$$

The Penrose limits of the background have been studied in [42]. In the following we review the work of [42]. First we redefine background coordinates in the following way

$$\begin{pmatrix} \tilde{\psi} \\ \tilde{x}_1/R \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \psi \\ x_1/R \end{pmatrix} \ . \tag{3.41}$$

Now we consider the motion of a particle with a very high speed along the direction $\tilde{\psi}$. The null geodesic resides at the origin of AdS_3 *i.e.* $\rho = 0$ and at $\theta = 0$. We expand the background by defining the coordinates

$$t = x^+ + \frac{x^-}{R^2} \ ; \ \tilde{\psi} = x^+ - \frac{x^-}{R^2} \ ; \ \rho = \frac{z}{R} \ ; \ \theta = \frac{y}{R} \tag{3.42}$$

and subsequently take $R \to \infty$ limit while keeping $l^2 \equiv R^2\big(\varsigma - 1\big)$ finite. The leading order of the expansion of background metric provides

$$\begin{aligned} ds^2 =& \ -4dx^+dx^- - \Big(z^2 + y^2\cos^2\alpha + l^2\sin^2\alpha\Big)dx^{+2} + dz^2 + dy^2 + z^2\varphi^2 + y^2 d\chi^2 \\ & + \Big(\sin^2\alpha + \varsigma\cos^2\alpha\Big)d\tilde{x}_1^2 + \varsigma\Big(dx_2^2 + dx_3^2 + dx_4^2\Big) \ , \end{aligned} \tag{3.43}$$

In the limit the RR three-form takes the form

$$F_3 = 2N_5 \ dx^+ \wedge \Big(\frac{z}{R^2}dz\wedge d\varphi + \cos\alpha\frac{y}{R^2}dy\wedge d\chi\Big) \ , \tag{3.44}$$

where N_5 is the $D5$-brane charge and we consider $\alpha' = g_s = 1$ for convenience.

The supersymmetry analysis for this PP-wave geometry has also been worked out in [42]. It is shown that the background preserves 16 supercharges.

Chapter 4

Penrose limits in non-Abelian T-dual of AdS$_5\times$ S^5 background

This chapter is one of the most important chapters in this thesis. Here we will describe some of the recent works on pp-wave geometries in the T-dual backgrounds. The Penrose limits in Abelian and non-Abelian T-duals of $AdS_5 \times S^5$ geometries have been studied in detail in [26]. The Penrose limits around the smooth null geodesic lead to pp-wave geometries both in Abelian as well as the non-Abelian case. The authors in [26] considered the closed string quantization in these PP-wave backgrounds and the corresponding BMN sector has been constructed. The supersymmetry discussion of these pp-wave geometries has also been carried out in the literature. In the following two sections we will review the work of [26] in detail. First we will discuss the Penrose limits in the Abelian T-dual background followed by the non-Abelian T-dual geometry.

4.1 Penrose limits in Abelian T-dual background

In this section we will discuss the Penrose limits in the Abelian T-dual geometry of string theory on $AdS_5 \times S^5$ background. The Abelian T-duality acts on the $U(1)$-fibre direction of the transverse S^5 manifold. Before considering the Penrose limits, we first recall the T-dual background as follows.

The background metric of the T-dual geometry is given by

$$ds^2_{\rm ATD} = 4L^2 ds^2(AdS_5) + 4L^2 d\Omega_2^2(\alpha,\beta) + \frac{L^2}{\cos^2\alpha}\, d\psi^2 + L^2\cos^2\alpha\, d\Omega_2^2\big(\chi,\xi\big) \ . \tag{4.1}$$

Here we rescaled the T-dual coordinate ψ as $\psi \to L^2\psi$ to get L^2 common factor in the metric and we consider $\alpha' = 1$ for convenience. After rescaling the ψ-coordinate the background fields in the NS-NS sector take the form

$$B_2 = L^2\psi\sin\chi d\chi\wedge d\xi \ ; \ e^{-2\Phi} = \frac{L^2\cos^2\alpha}{g_s} \ . \tag{4.2}$$

Similarly, the RR field strength becomes

$$F_4 = \frac{8L^4}{g_s}\cos^3\alpha\sin\alpha\sin\chi d\alpha\wedge d\beta\wedge d\chi\wedge d\xi \ . \tag{4.3}$$

Now we consider the Penrose limits in the T-dual solution mentioned above. Denoting the spacetime coordinates as $\{x^\mu\}$, the geodesic equation is expressed as

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{du}\frac{dx^\rho}{du} = 0 \ . \tag{4.4}$$

Here u is the affine parameter along the geodesic. We will consider the motion of a particle along various $U(1)$-isometry directions in the T-dual background. Denote x^λ one such isometry direction and the velocity and acceleration along any direction $x^\mu, \mu\neq\lambda$ zero, we have

$$\frac{dx^\mu}{du} = 0 = \frac{d^2x^\mu}{du^2} \ , \ \text{for } \mu\neq\lambda \ . \tag{4.5}$$

Substituting the above in (4.4), the geodesic equation for motion along an isometry direction provides

$$\partial^\mu g_{\lambda\lambda} = 0 \ . \tag{4.6}$$

In additon to the above condition, we need to impose $ds^2 = 0$ to obtain null geodesics for our purpose.

The T-dual background (4.1) has isometry along ξ ,β and ψ directions. For motion along ξ direction, the geodesic condition (4.6) leads to $\alpha = \{0, \frac{\pi}{2}, \pi\} = \chi$. For the values $\chi = \{0,\pi\}$ the $g_{\xi\xi}$-component of the metric vanishes so we ruled out these values. The same is true for $\alpha = \frac{\pi}{2}$. So we have two smooth null geodesic: one resides at $\{\alpha = 0, \ \chi = \frac{\pi}{2}\}$ and the second one at $\{\alpha = \pi, \ \chi = \frac{\pi}{2}\}$. Both null geodesics lead to the same pp-wave geometry as discussed in [26]. We will expand the background metric around $\{\alpha = 0, \ \chi = \frac{\pi}{2}\}$ with the following expansion

$$r = \frac{\bar{r}}{2L} \ , \ \alpha = \frac{x}{2L} \ , \ \psi = \frac{y}{L} \ , \ \chi = \frac{\pi}{2} + \frac{y}{L} \ , \ t = x^+ \ , \ \xi = 2x^+ + \frac{x^-}{L^2} \ . \tag{4.7}$$

We keep the β coordinate unchanged and rescale the string coupling g_s as $g_s = L\tilde{g}_s$ in order to get finite dilaton at Penrose limits. Substituting the expansion into the background metric (4.1), the leading order geometry provides

$$ds^2 = 4dx^+dx^- + d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 + dx^2 + x^2 d\beta^2 + dz^2 + dy^2 - \Big(\bar{r}^2 + x^2 + 4z^2\Big)dx^{+2} \ . \tag{4.8}$$

In the Penrose limits, the background fields in the NS-NS sector and the RR sector take the form

$$B_2 = 2y \ dz\wedge dx^+ \ ; \ e^{-2\Phi} = \frac{1}{\tilde{g}_s} \ ,$$
$$\text{and } F_4 = \frac{4x}{\tilde{g}_s} \ dx\wedge d\beta\wedge dz\wedge dx^+ \ . \tag{4.9}$$

The metric in (4.8) is in Brinkmann form with $R = 0$ and satisfies the type-IIA equation of motions as described in Appendix (A). So the geometry is indeed a pp-wave solution in type-IIA supergravity.

Now we will consider the other isometric directions of the T-dual geometry. First consider the direction β. By imposing the geodesic condition (4.6) we have $\alpha = \{0, \frac{\pi}{2}, \pi\}$. However, all of them give singular values for the T-dual geometry (4.1). Hence, motion along the β-direction does not lead to PP-wave geometries.

In the Abelian T-dual geometry finally we consider the null geodesic that carries non-zero angular momentum. For this we consider motion along ψ and ξ directions. The geodesic equation now implies that $\alpha = 0, \chi = \pi/2$. Consider the Lagrangian for a massless particle moving along this geodesic

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \ . \tag{4.10}$$

Here we choose u to be the affine parameter and the dots denote derivative with respect to it. Substituting the explicit expression in the T-dual metric (4.1) in the above Lagrangian we have

$$\mathcal{L} = \frac{L^2}{2} \Big(-4\dot{t}^2 + \dot{\psi}^2 + \dot{\xi}^2 \Big) \ . \tag{4.11}$$

The Lagrangian does depend on the generalized coordinates t, ξ and ψ conjugate momenta corresponding to the generalized coordinates t, ϕ_1 and ψ. The corresponding momentum will be conserved. By choosing the affine parameter u suitably we have

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -4L^2 \dot{t} = -L^2 \ , \ \frac{\partial \mathcal{L}}{\partial \dot{\xi}} = L^2 \dot{\xi} = -JL^2 \ . \tag{4.12}$$

Here we fixed the energy $L^2 p_t$ appropriately and J is the conserved angular momentum along the ξ direction. The conserved angular momentum along the generalized coordinate ψ is fixed by considering the geodesic to be null, *i.e.*, we set $\mathcal{L} = 0$. We have

$$\dot{\psi}^2 = \frac{1}{4}\big(1 - 4J^2\big) \longrightarrow \psi = \frac{1}{2}\sqrt{1 - 4J^2} \ u \ . \tag{4.13}$$

Here we set the additive constant to zero. From the above expression we find that J must be bounded by

$$0 \leq J \leq \frac{1}{2} \ . \tag{4.14}$$

Now we expand the background metric around the null geodesic $r = \alpha = 0, \chi = \pi/2$ and carries angular momentum J. First we redefine the coordinates

$$r = \frac{\bar{r}}{2L} \ , \ \alpha = \frac{x}{2L} \ , \ \chi = \frac{\pi}{2} + \frac{z}{L} \ , \tag{4.15}$$

together with

$$dt = c_1 du,\ d\xi = c_2 du + c_3 \frac{dw}{L},\ d\psi = c_4 du + c_5 \frac{dw}{L} + c_6 \frac{dv}{L^2}\ , \tag{4.16}$$

and subsequently take $L \to \infty$ limit.

The null geodesic condition sets the constant coefficients c_1, c_2 and c_4 as

$$c_1 = \frac{1}{4}\ ,\ c_2 = -J\ ,\ c_4 = \frac{1}{2}\sqrt{1-4J^2}\ . \tag{4.17}$$

In the expansion of the metric there will be some $\mathcal{O}(L)$ terms that can be removed by setting $c_2 c_3 + c_4 c_5 = 0$. Normalizing the coefficient of dw^2 to unity provides $c_3^2 + c_5^2 = 1$. Imposing the coefficients of light-cone terms in the metric as $2dudv$ we obtain $c_4 c_6 = 1$. These conditions solve to obtain the remaining coefficients c_3, c_5 and c_6 as

$$c_3 = \sqrt{1-4J^2}\ ,\ c_5 = 2J\ ,\ c_6 = \frac{2}{\sqrt{1-4J^2}}\ . \tag{4.18}$$

The resulting PP-wave metric in this case has the expression

$$\begin{aligned} ds^2_{pp} = \ & 2dudv + d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 + dz^2 + dx^2 + x^2 d\beta^2 \\ & + dw^2 - \left[\frac{\bar{r}^2}{16} + \frac{8J^2-1}{16} x^2 + J^2 z^2\right] du^2. \end{aligned} \tag{4.19}$$

In the Penrose limits, the background fields in the NS-NS sector are found to be

$$e^{-2\Phi} = \frac{1}{\tilde{g}_s^2}\ ,\ B_2 = \frac{u}{2}\ dz \wedge dw\ . \tag{4.20}$$

The RR sector field strength at Penrose limits becomes

$$F_4 = \frac{2Jx}{\tilde{g}_s}\ du \wedge dz \wedge dx \wedge d\beta\ . \tag{4.21}$$

Now we shall consider the closed string quantization moving in the above pp-wave background (4.19)-(4.21). Consider the world-sheet action

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \left[\sqrt{g} g^{\alpha\beta} G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu + \epsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' \sqrt{g}\ \mathcal{R}^{(2)} \Phi\right]\ , \tag{4.22}$$

In the action, $\{\alpha, \beta\}$ denote the worldsheet coordinates (τ, σ) and $\{\mu, \nu\}$ denote the spacetime coordinates. $G_{\mu\nu}$ is the metric of the background, $B_{\mu\nu}$ is the NS-NS two-form and Φ is the dilaton. We fixed the convention as $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1$ and gauge fix the worldsheet metric $g_{\alpha\beta}$ such that $\sqrt{g} g^{\alpha\beta} = \eta^{\alpha\beta}$, with $-\eta_{\tau\tau} = \eta_{\sigma\sigma} = 1$. We assigned the string coordinates as

$$U = u,\ V = v,\ \left(X^1, X^2, X^3, X^4\right) \in \bar{r}, \Omega_3,\ \left(X^5, X^6\right) \in x, \beta,\ \text{and}\ \left(X^7, X^8\right) \in z, w\ . \tag{4.23}$$

In quantization, we consider the light cone gauge $U=\tau$ with $p^+=1$. Then worldsheet action for the pp wave background (4.19) takes the form

$$S=-\frac{1}{4\pi\alpha'}\int d\tau d\sigma\,\Bigg[\sum_{i=1}^{8}\partial X^i.\,\partial X^i+\frac{1}{16}\sum_{i=1}^{4}(X^i)^2+\frac{8J^2-1}{16}\sum_{i=5}^{6}(X^i)^2 \\ +J^2(X^7)^2-\Big(\tilde{k}_1X^7\partial_\sigma X^8-\tilde{k}_2X^8\partial_\sigma X^7\Big)\Bigg]\ , \tag{4.24}$$

where in the action we choose the gauge fixed of NS-NS two-form as

$$B_2=\frac{1}{2}\Big(\tilde{k}_2w\ du\wedge dz-\tilde{k}_1z\ du\wedge dw\Big)\ \text{with}\ \Big(\tilde{k}_1+\tilde{k}_2\Big)=1\ . \tag{4.25}$$

The equations of motion the transverse coordinates X^i ; $i=1,...,8$ generated from the action (4.24) are given by

$$\begin{aligned}
&\Box X^i-\frac{1}{16}X^i=0\ ;\ i=1,2,3,4\ ,\\
&\Box X^i-\frac{8J^2-1}{16}X^i=0\ ;\ i=5,6\ ,\\
&\Box X^7-J^2X^7+\frac{1}{2}\partial_\sigma X^8=0\ ,\\
&\Box X^8-\frac{1}{2}\partial_\sigma X^7=0\ .
\end{aligned} \tag{4.26}$$

Consider an ansatz of the form $X^i\sim e^{-i\omega t+in\sigma}$, the frequencies of the transverse modes are given by

$$\begin{aligned}
\omega_{n,i}^2=&\ \ n^2+\frac{1}{16}\ ;\ i=1,2,3,4\ ,\\
\omega_{n,i}^2=&\ \ n^2+\frac{8J^2-1}{16}\ ;\ i=5,6\ ,\\
\omega_{n,\pm}^2=&\ \ n^2+\frac{J^2}{2}\pm\frac{1}{2}\sqrt{n^2+J^4}\ .
\end{aligned} \tag{4.27}$$

These modes have a correspondence with the set of operators (BMN operators) in the dual gauge theory. In the dual field theory section we shall discuss the corresponding BMN operators.

4.2 Penrose limits in non-Abelian T-dual background

In this section we turn on our interest in the Penrose limits in non-Abelian T-dual of $AdS_5\times S^5$ geometry. The non-Abelian T-duality acts on an $SU(2)$ subgroup of the S^5 manifold. Before discussing the Penrose limits we first recall the T-dual solution as presented in [21].

The metric is given by

$$ds^2_{\text{NATD}} = 4L^2 ds^2(AdS_5) + 4L^2 d\Omega_2^2(\alpha,\beta) + \frac{L^2 d\rho^2}{\cos^2\alpha} + \frac{L^2\rho^2\cos^2\alpha}{\rho^2+\cos^4\alpha}\, d\Omega_2^2(\chi,\xi) \ , \tag{4.28}$$

here we rescale the T-dual coordinate ρ as $\rho \to L^2\rho$ to get L^2 common factor in the metric and we set $\alpha' = 1$ for convenience.

The background is supported by a two-form and a non-zero dilaton in the NS-NS sector

$$\begin{aligned} B_2 = &\ \frac{L^2\rho^3\cos^2\alpha}{\rho^2+\cos^4\alpha}\sin\chi d\chi\wedge d\xi \ , \\ e^{-2\Phi} = &\ \frac{L^6\cos^2\alpha}{g_s^2}\big(\rho^2+\cos^4\alpha\big) \ . \end{aligned} \tag{4.29}$$

The RR sector of the background consists of a two-form and a four-form field strengths

$$\begin{aligned} F_2 = &\ \frac{8L^4}{g_s}\sin\alpha\cos^3\alpha d\alpha\wedge d\beta \ , \\ F_4 = &\ \frac{8L^6}{g_s}\ \frac{\rho^3\cos^3\alpha}{\rho^2+\cos^4\alpha}\sin\alpha\sin\chi d\alpha\wedge d\beta\wedge d\chi\wedge d\xi \ . \end{aligned} \tag{4.30}$$

Now we discuss the Penrose limits in T-dual geometry (4.28)-(4.30). The background has an $U(1)$-isometry along β as well as ξ directions. For motion along β-direction the geodesic condition (4.6) leads to $\alpha = \pi/2$, however, for $\alpha = \pi/2$ the background metric (4.28) is singular and does not lead to pp-wave geometry [26]. Considering the other $U(1)$-isometric direction ξ, the geodesic condition provides

$$\begin{aligned} &\rho = 0 \ , \ \alpha = \frac{\pi}{2} \ , \\ &\chi = \{0, \frac{\pi}{2}, \pi\} \ , \\ &\alpha = \{0, \frac{\pi}{2}, \pi\} \ , \ \chi = \{0,\pi\} \text{ or } \cos^4\alpha = \rho^2 \ . \end{aligned} \tag{4.31}$$

From the above values, $\alpha = \frac{\pi}{2}$ together $\chi = \{0,\pi\}$ are ruled out as the first one corresponds to a singular point in the T-dual geometry and for $\chi = \{0,\pi\}$ the metric component $g_{\xi\xi}$ vanishes. Hence, motion along the ξ-direction does not provide any smooth null geodesic to examine the Penrose limits.

As in the Abelian case, we finally consider the geodesic that carries angular momentum. Such a geodesic exists for motion along ρ and ξ directions. The geodesic equation now implies that $\alpha = 0, \chi = \pi/2$. Now consider the Lagrangian of a massless particle moving along this geodesic

$$\mathcal{L} = \frac{L^2}{2}\Big(-4\dot{t}^2 + \dot{\rho}^2 + \frac{\rho^2}{\rho^2+1}\dot{\xi}^2\Big) \ , \tag{4.32}$$

as also mentioned earlier the dot is referred to derivative with respect to the affine parameter u. In the above Lagrangian t and ξ are cyclic coordinates. The corresponding momenta will be conserved. By choosing the affine parameter u suitably we have

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = -4L^2\dot{t} = -L^2 \; ; \; \frac{\partial\mathcal{L}}{\partial\dot{\xi}} = L^2\frac{\rho^2}{\rho^2+1}\dot{\xi} = -JL^2 \; , \tag{4.33}$$

where J is the conserved angular momentum along the ξ direction. The null geodesic condition $i.e.$ $\mathcal{L}=0$ gives the solution of $\dot{\rho}^2$

$$\dot{\rho}^2 = \frac{1}{4} - \frac{\rho^2}{\rho^2+1}J^2 \; . \tag{4.34}$$

Now we expand the background metric (4.28) around the null geodesic $r=\alpha=0, \; \chi=\pi/2$ and carries angular momentum J by proposing the following expansion

$$r = \frac{\bar{r}}{2L} \; , \; \alpha = \frac{x}{2L} \; , \; \chi = \frac{\pi}{2} + \frac{z}{2L} \; , \tag{4.35}$$

together with

$$\begin{aligned} dt = & \; c_1 du \; , \\ d\xi = & \; c_2 du + \Big(1+4Jc_2\Big)\frac{dw}{L} + \Big(c_2 - \frac{3}{4J}\Big)\frac{dv}{L^2} \; , \\ d\rho = & \; c_3\Big(du + \frac{4J}{L}dw + \frac{dv}{L^2}\Big) \; . \end{aligned} \tag{4.36}$$

and subsequently take $L\to\infty$ limit. We redefine the string coupling g_s as $g_s = L^3\tilde{g}_s$ in order to get finite dilaton at Penrose limits.

The null geodesic condition determines the coefficients c_i's of the above expansion as

$$c_1 = \frac{1}{4} \; , \; c_2 = -\frac{\rho^2+1}{\rho^2}J \; , \; c_3 = \sqrt{\frac{1}{4} - \frac{\rho^2+1}{\rho^2}J^2} \; . \tag{4.37}$$

The leading order expansion of the background provides

$$\begin{aligned} ds^2 = & \; 2dudv + d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 + dx^2 + x^2 d\beta^2 + \frac{\rho^2}{\rho^2+1}dz^2 + \left[\frac{\rho^2}{\rho^2+1} - 4J^2\right]dw^2 \\ & - \left[\frac{\bar{r}^2}{16} + \frac{8J^2-1}{16}x^2 + \frac{\rho^2+1}{\rho^2}J^2 z^2\right]du^2. \end{aligned} \tag{4.38}$$

The metric is not yet in the Brinkmann form, we shall transform it into Brinkmann form later. The NS-NS two-form and the dilaton for the background are given by

$$B_2 = \frac{\rho^3}{\rho^2+1} \; dz\wedge dw \; ; \; e^{-2\Phi} = \frac{\rho^2+1}{\tilde{g}_s^2} \; . \tag{4.39}$$

In Penrose limits only RR four-form field strength survives

$$F_4 = \frac{2Jx\rho}{\tilde{g}_s} \; du\wedge dz\wedge dx\wedge d\beta \; . \tag{4.40}$$

In order to bring the above pp-wave background (4.38)-(4.40) in Brinkmann form, we follow the formalism presented in [26]. Following [26], consider a line element of the form

$$ds^2 = 2dudv \ + \sum_i A_i(u) \ dx_i^2 \ . \tag{4.41}$$

Now, replace the transverse coordinates x_i and one light-cone coordinate v as

$$x_i \to \frac{x_i}{\sqrt{A_i}} \ , \ v \to v + \frac{1}{4} \sum_i \frac{\dot{A}_i}{A_i} \ x_i^2 \ . \tag{4.42}$$

The line element in (4.41) now takes the Brinkmann form

$$ds^2 = 2dudv \ + \sum_i dx_i^2 + \Big(\sum_i F_i(u) \ x_i^2 \Big) du^2 \ , \tag{4.43}$$

where the functions F_i are given by

$$F_i = \frac{1}{4} \frac{\dot{A}_i^2}{A_i^2} + \frac{1}{2} \frac{d}{du} \left(\frac{\dot{A}_i}{A_i} \right) \ . \tag{4.44}$$

For the pp-wave metric (4.38) we have

$$A_z = \frac{\rho^2}{1+\rho^2} \ , \ A_w = \frac{\rho^2}{1+\rho^2} - 4J^2 \ . \tag{4.45}$$

After making the replacement

$$\begin{aligned} & z \to \frac{z}{\sqrt{A_z}} \ , \ w \to \frac{w}{\sqrt{A_w}} \ \ \text{and} \\ & v \to v + \frac{1}{4} \left[\frac{\dot{A}_z}{A_z} \ z^2 + \frac{\dot{A}_w}{A_w} \ w^2 \right] \ , \end{aligned} \tag{4.46}$$

we find

$$\begin{aligned} ds^2 = \ & 2dudv + d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 + dx^2 + x^2 d\beta^2 + dz^2 + dw^2 \\ & - \left[\frac{\bar{r}^2}{16} + \frac{x^2}{16} \Big(8J^2 - 1 \Big) + \frac{\big(\rho^2+1\big)^2}{\rho^4} J^2 z^2 - F_z z^2 - F_w w^2 \right] du^2 \ . \end{aligned} \tag{4.47}$$

The functions F_i's can be read from the expression (4.44) as:

$$F_z = \frac{4J^2\Big(4\rho^2+1\Big) + 3\Big(4J^2-1\Big)\rho^4}{4\rho^4\Big(\rho^2+1\Big)^2} \ ; \ F_w = -\frac{3}{4\Big(\rho^2+1\Big)^2} \ . \tag{4.48}$$

Together with the metric, the background fields in the NS-NS sector take the following forms in the Brinkmann representation

$$e^{-2\Phi}=\frac{\rho^2+1}{\tilde{g}_s^2}\ ;\ B_2=-\frac{1}{2}\frac{\rho^2+3}{\rho^2+1}\Big(k_1z\ du\wedge dw-k_2w\ du\wedge dz\Big)\ , \tag{4.49}$$

where $k_1+k_2=1$. The RR four-form field strengths in Brinkmann representation takes the form

$$F_4=\frac{2Jx\sqrt{\rho^2+1}}{\tilde{g}_s}\ du\wedge dx\wedge dz\wedge d\beta\ . \tag{4.50}$$

Before going to the dual field theory description, we consider closed string eigenmodes in the pp-wave background (4.47)-(4.50). We use the same gauge fixing and same notation for the string coordinates as we discussed in the Abelian T-dual case (4.23).

In this case worldsheet action for the pp wave background (4.47)-(4.50) takes the form

$$\begin{aligned}S=-\frac{1}{4\pi\alpha'}\int d\tau d\sigma\ \Bigg[\sum_{i=1}^{8}\partial X^i.\ \partial X^i+\frac{1}{16}\sum_{i=1}^{4}(X^i)^2+\frac{8J^2-1}{16}\sum_{i=5}^{6}(X^i)^2+\frac{\big(\rho^2+1\big)^2}{\rho^4}J^2(X^7)^2\\ -F_z(X^7)^2-F_w(X^8)^2-\frac{\rho^2+3}{\rho^2+1}\Big(k_1X^7\partial_\sigma X^8-k_2X^8\partial_\sigma X^7\Big)\Bigg]\end{aligned} \tag{4.51}$$

The equations of motion of the transverse scalars $X^i\ ;\ i=1,...,8$ by varying the action (4.51) are given by

$$\begin{aligned}&\Box X^i-\frac{1}{16}X^i=0\ ;\ i=1,2,3,4\ ,\\ &\Box X^i-\frac{8J^2-1}{16}X^i=0\ ;\ i=5,6\ ,\\ &\Box X^7-\left[\frac{\big(\rho^2+1\big)^2}{\rho^4}J^2-F_z\right]X^7+\frac{1}{2}\frac{\rho^3+3}{\rho^2+1}\ \partial_\sigma X^8=0\ ,\\ &\Box X^8+F_wX^8-\frac{1}{2}\frac{\rho^3+3}{\rho^2+1}\ \partial_\sigma X^7=0\ .\end{aligned} \tag{4.52}$$

Considering the solution of the form $X^i\sim e^{-i\omega t+in\sigma}$, the frequencies of some of the modes are given by

$$\begin{aligned}\omega_{n,i}^2=&\ \ n^2+\frac{1}{16}\ ;\ i=1,2,3,4\ ,\\ \omega_{n,i}^2=&\ \ n^2+\frac{8J^2-1}{16}\ ;\ i=5,6\ .\end{aligned} \tag{4.53}$$

Finally, we will discuss the supersymmetry preserved by the pp-wave solution in the non-Abelian T-dual geometry. The non-Abelian T-dual of $AdS_5\times S^5$ background preserves the half of the supersymmetries of the original seed background due to the Killing spinors of the $AdS_5 \times S^5$ background are not fully invariant under the $SU(2)_L$ symmetry that is considered for the dualization [12, 21]. In the following we will discuss the supersymmetry the PP-wave background originated from the non-Abelian T-dual solution.

First we assign the Brinkmann coordinates y^i of the PP-wave geometry as

$$\begin{aligned}&d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 = \left(dy^i\right)^2 \ ; \ i = 1,2,3,4 \ , \ dx^2 + x^2 d\beta^2 = \left(dy^5\right)^2 + \left(dy^6\right)^2 \ , \\ &z = y^7 \ , \ w = y^8 \ .\end{aligned} \tag{4.54}$$

In this representation the pp-wave background (4.47)-(4.50) takes the form

$$\begin{aligned} ds^2 =& \ 2dudv + \sum_{i=1}^{8} dy_i^2 + \mathcal{H}\ du^2 \ , \\ \Phi =& \ \Phi(u) \ , \\ H_3 =& \ f_1(u)\ du \wedge dy^7 \wedge dy^8 \ , \\ F_4 =& \ f_2(u)\ du \wedge dy^5 \wedge dy^6 \wedge dy^7 \ , \end{aligned} \tag{4.55}$$

where we introduced the notation

$$\begin{aligned} \mathcal{H} =& \ \sum_{i,j=1}^{8} F_{ij} y^i y^j \ , \ F_{ij} = F_{ji} \ , \\ \Phi(u) =& \ -\frac{1}{2} \ln\left[\frac{\rho^2+1}{\tilde{g}_s^2}\right] \ , \\ f_1(u) =& \ \frac{1}{2}\frac{\rho^2+3}{\rho^2+1} \ , \\ f_2(u) =& \ -\frac{2Jx\sqrt{\rho^2+1}}{\tilde{g}_s} \ , \end{aligned} \tag{4.56}$$

and F_{ij} are defined by

$$\begin{aligned} &F_{11} = F_{22} = F_{33} = F_{44} = -\frac{1}{16} \ , \\ &F_{55} = F_{66} = \frac{1-8J^2}{16} \ , \\ &F_{77} = F_z - \frac{(1+\rho^2)^2}{\rho^4} J^2 \ , \ F_{88} = F_w \ , \end{aligned} \tag{4.57}$$

all other F_{ij}'s are zero. Now we introduce the frame $\{e^a\}$ as

$$e^- = du \ , \ e^+ = dv + \frac{1}{2}\mathcal{H} du \ , \ e^i = dy^i \ , \tag{4.58}$$

then the PP-wave metric (4.55) can be written as

$$ds^2 = 2e^+e^- + \sum_{i=1}^{8}\left(e^i\right)^2 = \eta_{ab}e^a e^b \ , \tag{4.59}$$

with $\eta_{+-} = \eta_{-+} = 1$ and $\eta_{ij} = \delta_{ij}$. The non-zero components of spin-connections are given by

$$\omega_{-i} = -\omega_{i-} = \omega^{+i} = -\omega^{i+} = \frac{1}{2}\partial_i\mathcal{H}\ du \ . \tag{4.60}$$

In terms of the frame (4.58), the background fields (4.55) take the form

$$\begin{aligned} \Phi =&\quad \Phi(u)\ , \\ H_3 =&\quad f_1(u)\ e^-\wedge e^7\wedge e^8\ , \\ F_4 =&\quad f_2(u)\ e^-\wedge e^5\wedge e^6\wedge e^7\ . \end{aligned} \tag{4.61}$$

The supersymmetric variations of the dilatino and gravitino in type-IIA supergravity are given by

$$\begin{aligned} \delta\lambda &= \frac{1}{2}\partial\!\!\!/\Phi_\epsilon - \frac{1}{24}H\!\!\!/\sigma_3\epsilon \ + \frac{e^\Phi}{8\times 24}F\!\!\!/_4\sigma_1\epsilon\ , \\ \delta\psi_\mu &= D_\mu\epsilon - \frac{1}{8}H_{\mu\nu\rho}\Gamma^{\nu\rho}\sigma_3\epsilon + \frac{e^\Phi}{8\times 24}F\!\!\!/_4\sigma_1\Gamma_\mu\epsilon\ . \end{aligned} \tag{4.62}$$

The conventions we used here presented in Appendix (A). The Killing spinor ϵ consists of Majorana-Weyl spinors $\epsilon_\pm$, such that

$$\epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} \ . \tag{4.63}$$

In type-IIA supergravity, the chirality of the spinor ϵ satisfies $\Gamma_{11}\epsilon = -\sigma_3\epsilon$. Together with above we define $\Gamma^\pm$ as

$$\Gamma^\pm = \frac{1}{\sqrt{2}}\Big(\Gamma^9 \pm \Gamma^0\Big)\ . \tag{4.64}$$

Now we substitute the background fields in (4.62), and setting the dilatino variation to zero, we have

$$\Gamma^-\left[\dot{\Phi} - \frac{1}{2}f_1(u)\Gamma^{78}\sigma_3 + \frac{e^{\hat{\Phi}}}{4}\ f_2(u)\Gamma^{567}\sigma_1\right]\epsilon = 0\ . \tag{4.65}$$

The above holds only $\Gamma^-\epsilon = 0$, which states the PP-wave solution we studied in (4.47)-(4.50) preserves 16 supercharges.

Now we consider the spinor condition arising from the variation of the gravitino. First consider the $\delta\psi_+$ variation. The NS-NS three-form H_3 does not have any leg along e^+ and

$\Gamma_+\epsilon = \Gamma^-\epsilon = 0$. Hence $\delta\psi_+ = 0$ leads to $\partial_+\epsilon = 0$. It provides that the Killing spinor ϵ is independent of v, *i.e.* $\epsilon = \epsilon(u, y^i)$.

Now we consider the variation of the transverse components of gravitino $\delta\psi_i$, $i = 1, ..., 8$. Setting it to zero we have

$$\partial_i\epsilon = \Gamma^-\mathcal{R}\ \epsilon \ , \tag{4.66}$$

where

$$\mathcal{R} = \frac{1}{4}f_1(u)\Big(\delta_{i8}\Gamma^7 - \delta_{i7}\Gamma^8\Big)\sigma_3 - \frac{e^\Phi}{8}f_2(u)\Gamma^{567}\Gamma^i\sigma_1 \ . \tag{4.67}$$

As Γ^- anticommutes with $\mathcal{R}$ and from the dilatino variation we have $\Gamma^-\epsilon = 0$. Hence, $\partial_i\epsilon = 0$ leads to $\epsilon = \chi(u)$ with $\Gamma^-\chi(u) = 0$.

Finally, consider the variation $\delta\psi_- = 0$, in this case the covariant derivative D_- becomes

$$D_- = \partial_- + \frac{1}{2}F_{ij}y^j\Gamma^{-i} \ . \tag{4.68}$$

The above gives rise to

$$\partial_u\chi(u) - \frac{1}{4}f_1(u)\Gamma^{78}\sigma_3\chi(u) - \frac{e^\Phi}{8}f_2(u)\Gamma^-\Gamma^+\Gamma^{567}\sigma_1\chi(u) = 0 \ . \tag{4.69}$$

The above equation is the form $\partial_u\chi(u) - \mathcal{A}(u)\chi(u) = 0$, which can be integrated as follows

$$\chi(u) = e^{\int du\mathcal{A}(u)}\chi_0 \ ,$$

with $\Gamma^-\chi_0 = 0$. This concludes that the PP-wave geometry we studied in non-Abelian T-dual solution preserves 16 supercharges.

4.3 Dual Field theory

We will now discuss the corresponding field theory dual for the pp-wave geometries studied both in Abelian and non-Abelian T-dual solutions. The gauge theory dual corresponding to the T-dual background has been discussed in [21]. The dual quiver theory with k nodes contained with $\mathcal{N} = 2$ vector multiplet and $\mathcal{N} = 2$ bifundamental hypermultiplet. It turns out that the holographic central charge matches with the field-theoretic central charge [21]. In this section we will discuss the BMN sector of the quiver gauge theory. First we consider the PP-wave geometries originated from the Abelian T-dual background (4.1)-(4.3).

In the quiver [21], there is an adjoint complex scalar presents at each node i together with two bifundamental complex scalar fields residing in the next node *i.e.* i to $i+1$. We denote these scalars as: X_i and the remaining two as V_i and W_i. Among V_i and W_i, one of them is in the representation of $(i, \overline{i+1})$ another one in the complex conjugate $(\bar{i}, i+1)$ in the corresponding gauge group. Now in the following we shall consider the global symmetries of the T-dual background in (4.1) and match them with the corresponding symmetries in the dual gauge theory. In the gauge theory an $SU(2)_R$ rotates the V_i and $\overline{W}_i$. The fields $(V_i, \overline{W}_i)$ and $(\overline{V}_i, W_i)$ forms a doublet under this $SU(2)_R$. We denote the Cartan generator J_1 for the $U(1)$-symmetry resides inside the $SU(2)_R$. Together with the above, there is an $U(1)_R$ symmetry that acts only the chiral field X_i and the $d^2\theta$ as

$$X_i \to e^{i\alpha} X_i \ ; \ d^2\theta \to e^{-i\alpha} d^2\theta \ . \tag{4.70}$$

In addition, there is an extra $U(1)$ symmetry that rotates the V_i and W_i in opposite directions one of other by the same amount of phase

$$V_i \to e^{i\alpha} V_i \ ; \ W_i \to e^{-i\alpha} W_i \ . \tag{4.71}$$

This is not an R-symmetry and we denote the corresponding Cartan generator in this case is J_2.

Now we will identify the above symmetries in the gravity background studied in (4.1)-(4.3). It is straightforward to see that the T-dual background metric (4.1) shows the global $SU(2) \times U(1) \times U(1)$ isometry. The $SU(2)$ symmetry corresponds to the round 2-sphere parametrized by $\big(\chi, \xi\big)$ in (4.1). Together with the $U(1)_R$ symmetry is identified with the isometry along the β-direction in the T-dual geometry. The extra $U(1)$-symmetry in the dual field theory is associated with the shift symmetry along the compact direction ψ in the T-dual metric (4.1). Following [26], the dual Gravity coordinates can be expressed as field theory scalars by parametrize the six-dimension space with three complex scalars as

$$Z_1 = L \sin\alpha \ e^{i\beta} \ ; \ Z_2 = L \cos\alpha \cos\chi \ e^{i(\xi+\psi)} \ ; \ Z_3 = L \cos\alpha \sin\chi \ e^{i(\xi-\psi)} \ . \tag{4.72}$$

Here $L^2 = \sum_i |Z_i|^2$ is the size of the space $\mathbb{C}^3$. In this notation, we identify Z_1 as the multiplet X along with Z_2 and Z_3 as V and W respectively. The $SU(2)_R$ acts on the scalars Z_2 and Z_3 and the corresponding Cartan $U(1)$-symmetry charge is J_1. Similarly, the $U(1)_R$ acts on the scalar Z_1 giving a phase $e^{i\alpha}$ and the corresponding Cartan generator denoted by J_2. In addition, the extra $U(1)$ rotates Z_2 and Z_3 as $Z_2 \to e^{i\alpha} Z_2$ along with $Z_3 \to e^{-i\alpha} Z_3$. In the following table we represent the multiplets and the corresponding charge under the symmetry transformation. Here the total angular momentum is given by $J = J_1 + kJ_2$ and the light cone Hamiltonian is identified with the BMN operators given by the expression $H = \Delta - \big(J_1 + kJ_2\big)$

	X	V	W	$\overline{X}$	$\overline{V}$	$\overline{W}$
Δ	1	1	1	1	1	1
J_1	0	1/2	1/2	0	-1/2	-1/2
kJ_2	0	1/2	-1/2	0	-1/2	1/2
$H = \Delta - kJ_2 - J_1$	1	0	1	1	2	1

From the above table it is straightforward to see that $H = 0$ for V_i, defines the ground state and X_i together with W_i corresponds to the oscillator modes of the string. Hence, the vacuum is contained with only V_i fields and wraps the circular quiver. The vacuum state has zero winding mode($m = 0$) and one unit of momentum ($p = 1$). The vacuum is given by the operator

$$|m = 0\ ;\ p = 1\rangle = \mathcal{O}_k = \frac{1}{\sqrt{\mathcal{N}}}\, Tr\Big[V_1 V_2 ... V_k\Big]\ . \tag{4.73}$$

The higher energy states are constructed by the action of the derivative operator $D_a\ ; a = 0, 1, 2, 3.$ on vacuum along with the fields X_i and W_i and their conjugates [26]. Now at $n = 0$ the corresponding frequencies of the closed string (4.27) are given by

$$\begin{aligned}
\omega_{n=0,\ a} = &\ \ \frac{1}{4}\ ;\ a = 1, 2, 3, 4\ ,\\
\omega_{n=0,\ i} = &\ \ \frac{\sqrt{8J^2 - 1}}{4}\ ;\ i = 5, 6\ ,\\
\omega^2_{n=0,\ +} = &\ \ J\ ,\\
\omega^2_{n=0,\ +} = &\ \ 0\ .
\end{aligned} \tag{4.74}$$

The $H = 1$ corresponds $\omega_{n=0,\ a}$ but it does not match with the light cone energy. Same phenomenon is present in the insertions of $X, \bar{X}, W, \bar{W}$ the corresponding energy does not match with $H = 1$. This is because the term $g^2_{YM} N/k$ is divergent in this case and the same term is also present in the superpotential as $\mathcal{W} \sim g_{YM}\ Tr_{i+1}\Big[V_i X_i W_i\Big]$. Hence, the effective coupling is large so interaction effects are no longer negligible and change the eigenenergies of the corresponding states.

In the non-Abelian T-dual case, the frequencies of corresponding pp-wave geometry depend on the radial coordinate (4.52). Interestingly, in the limit of large radial coordinate ρ the PP-wave geometry in non-Abelian T-dual solution takes the form of PP-wave in the Abelian T-dual geometry, apart from the expression of dilaton Φ . It turns out that the eigenenergies for the closed string modes flow in the lightcone time $u = 0$ to $u = \infty$ [26]. These flows can be identified with the RG flow between UV to IR of the string modes frequencies. Where the UV corresponds to $\rho, u \to \infty$ and the IR limit is identified with $u \to 0$. However, in this case, further investigation is needed to precisely find the BMN operators.

Chapter 5

Penrose limits in non-Abelian T-dual of Klebanov-Witten and Klebanov-Tseytlin background

This chapter is the core part of the thesis. In this chapter we will present a series of works where we studied the Penrose limits in non-Abelian T-dual supergravity backgrounds [43,45]. In our work we first considered the Klebanov-Witten background followed by the Klebanov-Tseytlin background. In the following we will present the Penrose limits in these T-dual geometries.

5.1 Penrose limits in non-Abelian T-dual of Klebanov-Witten background

In this section we will discuss the Penrose limits in T-dual geometries originating from Klebanov-Witten background. First we consider the Abelian T-dual solution and subsequently examine the Penrose limits in it. We have presented the T-dual backgrounds in Chapter 2, where the T-duality acts along ϕ_2-direction as well as ψ-direction in the $T^{1,1}$ geometry. For convenience, we reproduce the geometry here.

First we consider the isometric ϕ_2, by applying Buscher rules the T-dual background reads

$$ds^2_{\rm ATD} = L^2 ds^2_{AdS_5} + L^2\lambda_1^2\left[d\Omega_2^2(\theta_1,\phi_1) + d\theta_2^2 + \frac{\lambda^2\sin^2\theta_2}{P(\theta_2)}\big(d\psi + \cos\theta_1 d\phi_1\big)^2 + \frac{d\phi_2^2}{\lambda_1^2 P(\theta_2)}\right]. \quad (5.1)$$

where $P(\theta_2) = \lambda^2\cos^2\theta_2 + \lambda_2^2\sin^2\theta_2$.

The NS-NS sector of the T-dual background is contained with a background dilaton the two-form gauge field given by

$$e^{-2\hat{\Phi}} = \frac{L^2}{g_s^2} P(\theta_2) \; ; \; \hat{B}_2 = -\frac{L^2 \lambda^2 \cos\theta_2}{P(\theta_2)} \Big(d\phi_2 \wedge d\psi + \cos\theta_1 d\phi_2 \wedge d\phi_1 \Big) \; . \tag{5.2}$$

The RR sector is described by the four-form field strength

$$\hat{F}_4 = \frac{4L^4 \lambda \lambda_1^4}{g_s} \sin\theta_1 \sin\theta_2 \; d\theta_1 \wedge d\phi_1 \wedge d\theta_2 \wedge d\psi \; . \tag{5.3}$$

Now we will turn our attention to the isometry direction ψ of the geometry given in (2.48). In this case the T-dual background described by the metric

$$ds^2_{\text{ATD}} = L^2 ds^2_{AdS_5} + L^2 \left[\lambda_1^2 d\Omega_2^2(\theta_1, \phi_1) + \lambda_2^2 d\Omega_2^2(\theta_2, \phi_2) + \frac{1}{\lambda^2} d\psi^2 \right] \; . \tag{5.4}$$

In the above metric we rescaled the ψ-coordinate as $\psi \to \frac{L^2}{\alpha'}\psi$ to get L^2 as a common factor in the metric. Now recall the background gauge fields for the T-dual background as follows. The NS-NS sector of the background is supported by a non-vanishing dilaton and a NS-NS two-form given by

$$e^{-2\hat{\Phi}} = \frac{\lambda^2 L^2}{g_s^2} \; ; \; \hat{B}_2 = -L^2 \left[\cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 \right] \wedge d\psi \; , \tag{5.5}$$

The NS-NS three-form flux arises from the two-form $\hat{B}_2$ is given by

$$\hat{H}_3 = L^2 \left[\sin\theta_1 d\theta_1 \wedge d\phi_1 + \sin\theta_2 d\theta_2 \wedge d\phi_2 \right] \wedge d\psi \; . \tag{5.6}$$

In the RR sector we have a non-vanishing four-form flux

$$\hat{F}_4 = \frac{4L^4 \lambda \lambda_1^2 \lambda_2^2}{g_s} \sin\theta_1 \sin\theta_2 \; d\phi_1 \wedge d\theta_1 \wedge d\phi_2 \wedge d\theta_2 \; . \tag{5.7}$$

We consider the motion of a particle along various $U(1)$-isometry directions of the T-dual geometry (5.1) and obtain the corresponding Penrose limits. Consider first the ϕ_1 isometry. The relevant metric component is

$$g_{\phi_1\phi_1} = L^2 \left[\lambda_1^2 \sin^2\theta_1 + \frac{\lambda_1^2 \lambda^2 \sin^2\theta_2}{\lambda^2 \cos^2\theta_2 + \lambda_2^2 \sin^2\theta_2} \cos^2\theta_1 \right] \; . \tag{5.8}$$

Applying the geodesic condition (4.6) we have $\theta_1 = (0, \frac{\pi}{2}, \pi)$ together with $\theta_2 = (0, \frac{\pi}{2}, \pi)$. This leads to four geodesics: $\{\theta_1 = 0, \theta_2 = \pi/2\}$, $\{\theta_1 = \pi, \theta_2 = \pi/2\}$, $\{\theta_1 = \pi/2, \theta_2 = 0\}$

and $\{\theta_1 = \pi/2, \theta_2 = \pi\}$. We first consider the geodesic $\{\theta_1 = 0, \theta_2 = \frac{\pi}{2}\}$ and propose a large L expansion around that:

$$r = \frac{\bar{r}}{L}\ ,\ \theta_1 = \frac{z}{L}\ ,\ \theta_2 = \frac{\pi}{2} + \frac{x}{L}\ ,\ t = ax^+\ ,\ \phi_1 = bx^+ + \frac{x^-}{L^2}\ ,\ \phi_2 = \frac{\phi_2}{L}\ , \tag{5.9}$$

and keeping the ψ-coordinate unchanged. In the expansion a and b are unknown parameters. Ignoring the subleading terms in $L \to \infty$ limit, the expansion of the T-dual metric provides

$$\begin{aligned} ds^2 &= d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 + \lambda_1^2 dz^2 + \lambda_1^2 dx^2 - \Big[\bar{r}^2 a^2 + b^2 z^2 \big(\lambda^2 - \lambda_1^2\big)\Big](dx^+)^2 \\ &- \lambda^2 \Big[bz^2 d\psi dx^+ - 2d\psi dx^- - 2bdx^+ dx^-\Big] - \frac{\lambda^4}{\lambda_2^2} x^2 \big(d\psi + bdx^+\big)^2 + \frac{1}{\lambda_2^2} d\phi_2^2 \\ &- L^2 \Big[a^2 (dx^+)^2 - \lambda^2 \big(d\psi + bdx^+\big)^2\Big]\ . \end{aligned} \tag{5.10}$$

Due to the presence of $\mathcal{O}(L^2)$ terms the metric diverges in the $L \to \infty$ limit . This divergence occurs because, in this case we have not been able to impose the geodesic to be null. This amounts to setting

$$a^2(dx^+)^2 - \lambda^2\big(d\psi + bdx^+\big)^2 = 0\ .$$

For the geodesic $\{\theta_1 = \pi, \theta_2 = \pi/2\}$, leads to a divergent metric in the large L expansion. So it also doesn't provide any PP-wave geometry.

The remaining two geodesics give rise to PP-wave geometry as follows. Consider the geodesic $\{\theta_1 = \pi/2, \theta_2 = 0\}$ with the expansion

$$r = \frac{\bar{r}}{L}\ ,\ \theta_1 = \frac{\pi}{2} + \frac{z}{L}\ ,\ \theta_2 = \frac{x}{L}\ ,\ t = ax^+,\ \phi_1 = bx^+ + \frac{x^-}{L^2}\ ,\ \phi_2 = \frac{\phi_2}{L}\ , \tag{5.11}$$

while keeping the ψ-coordinate unchanged. Like the earlier case here also a and b are unknown parameters to be chosen suitable[1] in order to obtain

$$ds_{pp}^2 = 2dx^+dx^- + d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 + dz^2 + dx^2 + x^2 d\psi^2 + d\phi_2^2 - 6\big(\bar{r}^2 + 6z^2 - 6x^2\big)(dx^+)^2\ . \tag{5.12}$$

The metric is indeed a pp-wave solution in the standard Brinkmann form. In the limit, the dilaton has the expression

$$e^{-2\hat{\Phi}} = \frac{1}{\tilde{g}_s^2}\lambda^2\ , \tag{5.13}$$

together with NS-NS three-form field strength

$$\hat{H}_3 = 2\sqrt{6}\ dz \wedge d\phi_2 \wedge dx^+\ . \tag{5.14}$$

The RR sector of the background is described by two-form and four-form field strengths

$$\hat{F}_2 = 0\ ,\ \hat{F}_4 = \frac{4\sqrt{6}}{3\tilde{g}_s}\ x\ dz \wedge dx^+ \wedge dx \wedge d\psi\ . \tag{5.15}$$

[1] To get the metric in standard form, we set $a = 1/\lambda_1, b = 1/\lambda_1^2$ and, in addition, we rescale some of the coordinates as $x \to \sqrt{6}x,\ z \to \sqrt{6}z,\ \phi_2 \to \frac{1}{3}\phi_2$.

For the remaining geodesic $\{\theta_1 = \pi/2, \theta_2 = \pi\}$ also leads to a pp wave geometry which is identical to (5.12) with the background fields.

The T-dual background (5.1) has also isometry along the ϕ_2-direction. In the following we discuss the motion along ϕ_2-direction and examine Penrose limits. First consider the metric component $g_{\phi_2\phi_2}$:

$$g_{\phi_2\phi_2} = \frac{L^2}{\lambda^2\cos^2\theta_2 + \lambda_2^2\sin^2\theta_2} \ . \tag{5.16}$$

The geodesic condition (4.6) leads to $\theta_2 = \big(0, \pi/2, \pi\big)$. Consider the following expansion around the geodesic $\{\theta_1 = 0, \theta_2 = 0\}$ with large L limit:

$$r = \frac{\bar{r}}{L},\ \theta_1 = \frac{z}{L},\ \theta_2 = \frac{x}{L},\ t = ax^+,\ \phi_2 = bx^+ + \frac{x^-}{L^2}, \tag{5.17}$$

keeping ϕ_1 and ψ unchanged. At prior, there is an $\mathcal{O}(L^2)$ divergent term present in the metric. In order to remove we need to choose $a = \lambda, b = \lambda^2$. This leads to a null geodesic. By redefining the x and z coordinates, we find

$$ds^2 = 2dx^+dx^- + d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 + dz^2 + z^2 d\phi_1^2 + dx^2 + x^2(d\psi + d\phi_1)^2 - \frac{1}{9}\big(\bar{r}^2 + 3x^2\big)(dx^+)^2. \tag{5.18}$$

Although there are no more diverging terms present in the metric, but the scalar curvature R for the metric is non-vanishing and hence it does not correspond to a pp-wave geometry. This is because the metric component $g_{\phi_1\phi_1}$ vanishes for the values $\{\theta_1 = 0, \theta_2 = 0\}$.

Finally we consider the isometric direction ψ in the T-dual metric. The corresponding metric component is given by

$$g_{\psi\psi} = L^2\ \frac{\lambda_1^2\lambda^2\sin^2\theta_2}{\lambda^2\cos^2\theta_2 + \lambda_2^2\sin^2\theta_2} \ . \tag{5.19}$$

From the geodesic condition (4.6) we have $\theta_2 = \big(0, \pi/2, \pi\big)$. We will not consider the values $\theta_2 = (0, \pi)$ as for those values the $g_{\psi\psi}$-component vanishes. Consider the geodesic $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$ with the following expansion:

$$r = \frac{\bar{r}}{L},\ \theta_1 = \frac{x}{L},\ \theta_2 = \frac{\pi}{2} + \frac{z}{L},\ t = ax^+,\ \psi = bx^+ + \frac{x^-}{L^2},\ \phi_2 = \frac{\phi_2}{L}, \tag{5.20}$$

and keeping the ϕ_1-coordinate unchanged.

In the limit $L \to \infty$ limit the leading order terms provide

$$\begin{aligned} ds^2 &= d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 + \lambda_1^2 dx^2 + \lambda_1^2 dz^2 + \Big[\Big(\lambda_1^2 - \lambda^2\Big)x^2 - \frac{\lambda^4}{\lambda_2^2}z^2\Big]d\phi_1^2 + \frac{1}{\lambda_2^2}d\phi_2^2 \\ &- \Big[\bar{r}^2a^2 + \frac{\lambda^4}{\lambda_2^2}b^2z^2\Big](dx^+)^2 + \lambda^2\Big[2bdx^+dx^- - b\Big(x^2 + \frac{\lambda^2}{\lambda_2^2}2z^2\Big)dx^+d\phi_1 + 2dx^-d\phi_1\Big] \\ &- L^2\Big[a^2(dx^+)^2 - \lambda^2\big(bdx^+ + d\phi_1\big)^2\Big] \ . \end{aligned} \tag{5.21}$$

This metric contains a divergent term of $\mathcal{O}(L^2)$ that cannot be removed by fixing the parameters a and b. This is because, in this case also, there is no null geodesic for any choice of the parameters a and b. Hence in this case the background does not give PP-wave geometry.

Now we consider the second T-dual background, where T-duality acts along ψ-direction. The geometry of the background is given by the metric (5.4). The background has $U(1)$-isometries along ϕ_1, ϕ_2 and ψ directions. The Motion along the ψ-direction does not give any non-trivial constraint because the $g_{\psi\psi}$-component of the metric is constant. As the metric is symmetric under the exchange of ϕ_1 and ϕ_2 followed by θ_1 and θ_2, it will be sufficient to consider geodesics along one of these directions. For the ϕ_1 isometry direction the geodesic condition leads to $\theta_1 = (0, \frac{\pi}{2}, \pi)$. However, for the values $\theta_1 = 0$ and π the $g_{\phi_1\phi_1}$-component of the metric vanishes. Hence, we have only $\theta_1 = \pi/2$. In order to examine the Penrose limit, we consider the large L expansion of the T-dual metric (5.4) and retain the leading terms as:

$$r = \frac{\bar{r}}{L},\ \theta_1 = \frac{\pi}{2} + \frac{z}{L},\ \theta_2 = \frac{x}{L},\ t = ax^+,\ \phi_1 = bx^+ + \frac{x^-}{L^2},\ \psi = \frac{y}{L},\ \phi_2 = \beta, \tag{5.22}$$

and redefine the string coupling as $g_s = L\ \tilde{g}_s$ to make dilaton finite in Penrose limits. To get a null geodesic we must impose the condition $a = \lambda_1 b$ together with $\lambda_1^2 b = 1$ and make the co-ordinate redefinitions $x^+ = u,\ x^- = v,\ z \to \sqrt{6}z,\ x \to \sqrt{6}x,\ y \to \frac{1}{3}y$ to bring the PP-wave metric into the standard form:

$$ds^2 = 2dudv + d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 + dz^2 + dx^2 + x^2 d\beta^2 + dy^2 - 6(\bar{r}^2 + 6z^2)du^2. \tag{5.23}$$

The expressions for the background fields in the NS-NS sector in this limit are given by:

$$e^{-2\hat{\Phi}} = \frac{\lambda^2}{\tilde{g}_s^2}\ , \tag{5.24}$$

and

$$\hat{H}_3 = 2\sqrt{6}\ dz \wedge du \wedge dy\ . \tag{5.25}$$

In the Penrose limits the field strengths for the RR fluxes have the expression:

$$\hat{F}_2 = 0,\ \hat{F}_4 = \frac{4\sqrt{6}}{3\tilde{g}_s} x\ du \wedge dz \wedge d\beta \wedge dx\ . \tag{5.26}$$

As in the $AdS_5 \times S^5$ case, now we will consider the null geodesic that carries angular momentum. To find such a geodesic, we consider motion along ϕ_1 and ψ directions. From the geodesic equation we have $\theta_1 = \pi/2, \theta_2 = 0$. Considering the Lagrangian of a massless particle as (4.10) and substituting the explicit expression for the background metric (5.4) we have

$$\mathcal{L} = \frac{L^2}{2}\Big(- \dot{t}^2 + \frac{1}{6}\dot{\phi}_1^2 + 9\dot{\psi}^2\Big)\ . \tag{5.27}$$

The Lagrangian does not depend on the coordinates t, ϕ_1 and ψ. Hence, the conjugate momenta corresponding to these generalized coordinates are conserved. We fixed the affine parameter u as

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -L^2 \dot{t} = -L^2 \ .$$

Assigning J be the conserved angular momentum along the ϕ_1-direction, we have

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = \frac{1}{6} \dot{\phi}_1 L^2 = -JL^2 \ .$$

However, the conserved angular momentum associated with the variable ψ can no longer be arbitrary. It is determined by imposing that the geodesic is null, *i.e.*, we set $\mathcal{L} = 0$ and it leads to

$$\dot{\psi}^2 = \frac{1}{9}\left(1 - 6J^2\right) \ , \tag{5.28}$$

can be integrated as

$$\psi = \frac{1}{3}\sqrt{1 - 6J^2} \ u \ .$$

The above expression gives a bound to the angular momentum J:

$$0 \leq J \leq \frac{1}{\sqrt{6}} \ . \tag{5.29}$$

To obtain the Penrose limit for the null geodesic carrying angular momentum J around $r = 0,\ \theta_1 = \frac{\pi}{2},\ \theta_2 = 0$, first we redefine the background coordinates:

$$r = \frac{\bar{r}}{L},\ \theta_1 = \frac{\pi}{2} + \frac{z}{L},\ \theta_2 = \frac{x}{L} \ . \tag{5.30}$$

and consider the following expansion with $L \to \infty$ limit:

$$dt = c_1 du,\ d\phi_1 = c_2 du + c_3 \frac{dw}{L},\ d\psi = c_4 du + c_5 \frac{dw}{L} + c_6 \frac{dv}{L^2} \ , \tag{5.31}$$

The null geodesic condition sets the constant coefficients c_1, c_2 and c_4

$$c_1 = 1,\ c_2 = -6J,\ c_4 = \frac{1}{3}\sqrt{1 - 6J^2} \ . \tag{5.32}$$

In the expansion there is a diverging term $\mathcal{O}(L)$ which can be removed upon setting $\lambda_1^2 c_2 c_3 + \frac{1}{\lambda^2} c_4 c_5 = 0$. Normalizing the coefficient of dw^2 to unity gives the condition $\lambda_1^2 c_3^2 + \frac{1}{\lambda^2} c_5^2 = 1$. By choosing the normalization factor of the light cone term $2dudv$ to 2, we have $\frac{c_4 c_6}{\lambda^2} = 1$. Solving these conditions provides the remaining coefficients c_3, c_5 and c_6 as

$$c_3 = \sqrt{6(1 - 6J^2)} \ ,\ c_5 = J\sqrt{\frac{2}{3}} \ ,\ c_6 = \frac{1}{3}\frac{1}{\sqrt{1 - 6J^2}} \ . \tag{5.33}$$

Then the resulting PP-wave metric after a rescaling the coordinates: $x \to \sqrt{6}x, z \to \sqrt{6}z$ has the expression

$$ds_{pp}^2 = 2dudv + d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 + dz^2 + dx^2 + x^2 d\beta^2 + dw^2 - \left(\bar{r}^2 + 36J^2 z^2\right) du^2. \tag{5.34}$$

The PP-wave background is supported by the following dilaton and B_2 field

$$e^{-2\hat{\Phi}} = \frac{\lambda^2}{\tilde{g}_s^2} \ , \ \hat{B}_2 = 2z \ dw \wedge du + x^2\sqrt{1-6J^2} \ d\beta \wedge du \ , \tag{5.35}$$

with the corresponding NS-NS three-form flux

$$\hat{H}_3 = 2 \ dz \wedge dw \wedge du + 2x\sqrt{1-6J^2} \ dx \wedge d\beta \wedge du \ . \tag{5.36}$$

In the Penrose limit we have only four-form field strengths

$$\hat{F}_4 = \frac{4\sqrt{6}}{3\tilde{g}_s} Jx \ du \wedge dz \wedge dx \wedge d\beta \ . \tag{5.37}$$

Now we show the PP-wave backgrounds we have obtained satisfy the type-IIA supergravity equations. In the following, we consider the pp-wave background specified by eqs.(5.34)-(5.37). For type-IIA supergravity, the Bianchi identity and gauge field equation are given in Appendix (A). In this PP-wave background the Bianchi identities are satisfied trivially. The equation of motion for B_2 is satisfied for our background, because the dilaton is constant, $F_2 = 0$ and $F_4 \wedge F_4 = 0$.

Now the Hodge dual of the three-form NS-NS flux H_3 is given by

$$\star H_3 = 2\big(dx \wedge d\beta + \sqrt{1-6J^2} \ dz \wedge d\omega\big) \wedge du \wedge d\Omega_4$$

, which is a closed differential and the Hodge dual of F_4 is given by

$$\star F_4 = \frac{4\sqrt{6}}{3\tilde{g}_s} J \ du \wedge d\Omega_4 \ ,$$

which leads to $H_3 \wedge \star F_4 = 0$. Also, $\star F_4$ is closed and $H_3 \wedge F_4 = 0$.

In order to verify the Einstein's equations, first note that $\hat{\Phi} = \text{const}$, together with $H^2 = F_4^2 = 0 = R$. Computation of the Ricci tensor for the background shows that only R_{uu} is non-vanishing. Similarly the only H_{uu}^2 and $(F_4^2)_{uu}$ are non-zero. The expression of these are given by

$$H_{uu}^2 = 16 - 48J^2 \ , \ (F_4^2)_{uu} = 64J^2/\tilde{g}_s^2 \ , \ \text{and} \ R_{uu} = 4 + 36J^2 \ . \tag{5.38}$$

The above analysis shows that the equation of motion for R_{uu} component is indeed satisfied.

Now we shall discuss the closed string modes propagating in the PP-wave background. We will consider the pp-wave solution (5.34) that carries angular momentum J. The PP-wave solution has been obtained from the geometry by performing an Abelian T-duality along ψ-isometry of the Klebanov-Witten background. The string world sheet action is given by (4.22) and we consider the same conventions as described in (4.22).

We assign the string coordinates in the following manner:

$$U = u, \ V = v, \ \big(X^1, X^2, X^3, X^4\big) \in \bar{r}, \Omega_3, \ \big(X^5, X^6\big) \in x, \beta, \ \big(X^7, X^8\big) \in z, w \ , \tag{5.39}$$

Then the worldsheet action (4.22) for the PP-wave background (5.34) takes the form

$$\begin{aligned} S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \Big[\sum_{i=1}^{8}\partial X^i.\,\partial X^i + \sum_{i=1}^{4}(X^i)^2 + X^8\partial_\sigma X^7 - X^7\partial_\sigma X^8 \\ -(X^5)^2\,\sqrt{1-6J^2}\,\partial_\sigma X^6 + 36J^2(X^7)^2\Big]\,, \end{aligned} \tag{5.40}$$

The equations of motions for the transverse modes originating from the above action are given by

$$\begin{aligned} &\Box X^i - X^i = 0\ ;\ i = 1,2,3,4\ , \\ &\Box X^5 + \sqrt{1-6J^2}\ X^5\ \partial_\sigma X^6 = 0\ , \\ &\Box X^6 - \sqrt{1-6J^2}\ X^5\ \partial_\sigma X^5 = 0\ , \\ &\Box X^7 - 36J^2X^7 + \frac{1}{2}\partial_\sigma X^8 = 0\ , \\ &\Box X^8 - \frac{1}{2}\partial_\sigma X^7 = 0\ . \end{aligned} \tag{5.41}$$

Considering the ansatz $X^i \sim e^{-i\omega t + in\sigma}$, the frequencies of the uncoupled modes are given by

$$\omega_{n,i}^2 = n^2 + 1\ ;\ i = 1,2,3,4\ . \tag{5.42}$$

The modes X^7 and X^8 can be decoupled leads to two fourth-order linear partial differential equations with corresponding frequencies

$$\omega_{n,i}^2 = n^2 + \frac{1}{2}\Big[36J^2 \pm \sqrt{(36J^2)^2 + n^2}\Big], i = 7, 8. \tag{5.43}$$

Defining a complex mode Z as $Z = X^6 + iX^5$, the equations involving X^5 and X^6 provides

$$\Box Z + \frac{1}{2}\ \sqrt{1-6J^2}\ \left(Z - \bar{Z}\right)\ \partial_\sigma Z = 0 \tag{5.44}$$

This corresponds to a non-linear complex harmonic oscillator and the exact analytic solutions cannot be obtained. Although for small value of $\sqrt{1-6J^2}$ one can use perturbation theory to obtain the frequencies of the corresponding modes.

Now we turn our interest to examining Penrose limits around various null geodesics in the non-Abelian T-dual geometry. The metric and the background fields of the T-dual background are given in (2.57)-(2.62). In the T-dual metric we have along $U(1)$-isometry along

ϕ_1 and ψ directions. In the following we consider motion along these isometric directions and examine Penrose limits. First consider motion along the ϕ_1-direction. The relevant metric component to examine Penrose limits is

$$g_{\phi_1\phi_1} = L^2 \left[\lambda_1^2 \sin^2\theta_1 + \frac{\lambda_2^2\lambda^2}{\Delta} x_1^2 \cos^2\theta_1 \right] . \tag{5.45}$$

This metric component depends on x_1, x_2 and θ_1 directions. From the geodesic condition, $\partial_\mu g_{\phi_1\phi_1} = 0$ we have $x_1 = 0, \theta_1 = \pi/2$ for $\mu = x_1$, and $x_1 = 0 = x_2, \theta_1 = \pi/2$ for $\mu = x_2$. Among these values $\mu = \theta_1$ this gives rise to the values $\theta_1 = (0, \pi/2, \pi)$. However the only non-singular choice for a geodesic is $x_1 = 0,\ x_2 = 0$ and $\theta_1 = \pi/2$. Now we propose the following large L expansion around this geodesic:

$$r = \frac{\bar{r}}{L},\ x_1 = \frac{y_1}{L},\ x_2 = \frac{y_2}{L},\ \theta_1 = \frac{\pi}{2} + \frac{z}{L},\ t = ax^+,\ \phi_1 = bx^+ + \frac{x^-}{L^2}\ , \tag{5.46}$$

and keeping the ψ-coordinate unchanged. In the expansion, the parameters a and b are set to be $1/\lambda_1$ and $1/\lambda_1^2$ respectively. We also redefine the light cone coordinates $x^+ = u,\ x^- = v$ and rescale $z \to \sqrt{6}z,\ y_1 \to y_1/\sqrt{6},\ y_2 \to y_2/3$. The leading order terms in the limit $L \to \infty$ provides

$$ds^2 = 2dudv + d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 + dz^2 + dy_1^2 + y_1^2 d\psi^2 + dy_2^2 - 6\big(\bar{r}^2 + 6z^2\big)du^2. \tag{5.47}$$

This is a PP-wave solution in the standard Brinkmann form.

In order to keep the dilaton finite in the Penrose limit, we redefine the string coupling as

$$g_s = L^3 \tilde{g}_s\ , \tag{5.48}$$

then for the above PP-wave background the dilaton takes the form

$$e^{-2\hat{\Phi}} = \frac{8}{\tilde{g}_s^2}\lambda^2\lambda_2^4\ , \tag{5.49}$$

and the NS-NS two-form becomes

$$\hat{B}_2 = 2\sqrt{6}z\ dy_2 \wedge du\ , \tag{5.50}$$

which leads to the following NS-NS three-form flux

$$\hat{H}_3 = 2\sqrt{6}\ du \wedge dz \wedge dy_2\ . \tag{5.51}$$

At Penrose limit we have only two-form field strength in the RR sector

$$\hat{F}_2 = \frac{8}{3\sqrt{3}\tilde{g}_s}\ du \wedge dz\ . \tag{5.52}$$

Now we will consider the other isometric direction ψ in the T-dual metric. In the following we will show that motion along ψ-direction does not give pp-wave geometry. In this case the relevant component of the metric is

$$g_{\psi\psi} = L^2\ \frac{\lambda_2^2\lambda^2}{\Delta} x_1^2\ . \tag{5.53}$$

From the geodesic condition we obtain $x_2 = 0, \theta_1 = 0$. Proposing the following expansion around the geodesic:

$$r = \frac{\bar{r}}{L},\ x_2 = \frac{y_2}{L},\ \theta_1 = \frac{z}{L},\ t = ax^+,\ \psi = bx^+ + \frac{x^-}{L^2}\ , \tag{5.54}$$

and keeping x_1 and ϕ_1 coordinates unchanged. In the $L \to \infty$ limit the dual metric gives

$$\begin{aligned} ds^2 &= -\bar{r}^2 a^2 (dx^+)^2 + d\bar{r}^2 + \bar{r}^2 d\Omega_3^2 + \lambda_1^2 dz^2 + \lambda_1^2 z^2 d\phi_1^2 \\ &+ \frac{\lambda_2^2\lambda^2}{\sum}\Big[2bx_1^2\, dx^+ dx^- + 2x_1^2 dx^- d\phi_1 - bx_1^2 z^2 dx^+ d\phi_1 - z^2 x_1^2 d\phi_1^2 \\ &- \frac{\lambda^2 y_2^2 x_1^2}{\sum}\big(bdx^+ + d\phi_1\big)^2\Big] - \frac{1}{\sum}\Big[\frac{1}{\sum}\lambda^2 y_2^2\big(x_1^2 + \lambda_2^2\lambda^2\big)dx_1^2 - \lambda_2^4 dy_2^2 - 2x_1 y_2 dx_1 dy_2\Big] \\ &- L^2 a^2 (dx^+)^2 + \frac{L^2}{\sum}\Big[\lambda_2^2\lambda^2 x_1^2\Big\{\big(bdx^+ + d\phi_1\big)^2 + 2bdx^+ d\phi_1\Big\} + \big(x_1^2 + \lambda_2^2\lambda^2\big)dx_1^2\Big] \end{aligned} \tag{5.55}$$

where,

$$\sum = \lambda_2^2 x_1^2 + \lambda^2 \lambda_2^4\ . \tag{5.56}$$

Here also we have diverging terms and that cannot be removed by any choice of the parameters a and b. Hence the motion along ψ-isometry does not lead to PP-wave background.

Now we consider the closed string modes propagating in the PP-wave background (5.47), Here also we use the same worldsheet action and gauge fixing defined in (4.22). For the PP-wave background (5.47) we assign the string coordinates X^i as

$$U = u,\ V = v,\ \Big(X^1, X^2, X^3, X^4\Big) \in \bar{r}, \Omega_3,\ \Big(X^5, X^6\Big) \in y_1, \psi,\ \Big(X^7, X^8\Big) \in z, y_2\ . \tag{5.57}$$

Then the worldsheet action for the PP-wave background (5.47) takes the form

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\ \Big[\partial X^i.\, \partial X^i + 6\Big(\sum_{i=1}^{4}(X^i)^2 + 6(X^7)^2\Big) - \sqrt{6}X^7\partial_\sigma X^8 + \sqrt{6}X^8\partial_\sigma X^7\Big]\ . \tag{5.58}$$

The equations of motion for the transverse modes X^i varying from the action are given by

$$\begin{aligned} &\Box X^i - 6X^i = 0\ ;\ i = 1,2,3,4\ , \\ &\Box X^i = 0\ ;\ i = 5,6\ , \\ &\Box X^7 - 36X^7 + \frac{\sqrt{6}}{2}\,\partial_\sigma X^8 = 0\ , \\ &\Box X^8 - \frac{\sqrt{6}}{2}\,\partial_\sigma X^7 = 0\ . \end{aligned} \tag{5.59}$$

Considering the ansatz of the form $X^i \sim e^{-i\omega t + in\sigma}$. We have

$$
\begin{aligned}
\omega_{n,i}^2 = &\ \ n^2 + 6\ ;\ i = 1,2,3,4\ , \\
\omega_{n,i}^2 = &\ \ n^2\ ;\ i = 5,6\ , \\
\omega_{n,i}^2 = &\ \ n^2 + \frac{1}{2}\left[36 \pm \sqrt{(36)^2 + 6n^2}\right]\ ;\ i = 7,8\ .
\end{aligned} \tag{5.60}
$$

5.1.1 Dual Gauge theory

In this section we shall make comments on possible gauge theory dual for the pp-wave geometries that we have studied in the previous section. The gauge theory dual has been presented in [23,27]. The construction of dual gauge theory is based on intersecting $D4 - NS5 - NS5'$ branes.

In the Abelian T-dual geometry (2.53), there is an underlying $SU(2)_A \times SU(2)_B \times U(1)_R$ global symmetry present in the background metric. The chiral fields (A_1, A_2) form a doublet under the $SU(2)_A$ subgroup of the global symmetry and similarly (B_1, B_2) form a doublet corresponds to $SU(2)_B$. $U(1)_R$ is the R-symmetry group corresponds to a shift along the circle coordinate ψ, and all the fields A_1, A_2, B_1, B_2 get the same charge under this symmetry. Denoting J_1 and J_2 to be the Cartan generators of $SU(2)_A$ and $SU(2)_B$ respectively and let J_3 be the generator of $U(1)_R$. The Cartan generators J_1 and J_2 correspond to the shift in the azimuthal coordinates ϕ_1 and ϕ_2 respectively and J_3 corresponds to shift in ψ.

Now we will try to identify the large-R sector of the dual gauge theory. The BMN sector has been constructed and studied in detail in [40]. First we define the conifold coordinates in the theory as: $Z_1 = A_1B_1, Z_2 = A_2B_2, Z_3 = A_1B_2, Z_4 = A_2B_1$. The light cone Hamiltonian is given by $H = \Delta - (J_1 + J_2 + J_3)$. From the BMN proposal it can be shown that the ground state of the string corresponds to the operator Z_1 and the first excited state: $H = 1$, is described by the operators Z_3, Z_4 and the covariant derivatives D_i acting on Z_1.

In closed string quantization of the PP-wave geometry (5.34) corresponding to the Abelian T-dual background we find the frequencies (5.42)-(5.43) corresponding to $n = 0$ modes as [2]

$$
\begin{aligned}
&\omega_{0,i} = 1,\ i = 1,2,3,4, \\
&\omega_{0,7+,8+} = 6J\ , \\
&\omega_{0,7-,8-} = 0\ .
\end{aligned} \tag{5.61}
$$

This mismatch of the frequencies is due to large effective interaction which causes the energies of the states to change. The same phenomenon is also present in the PP-wave background corresponding to the Abelian T-dual of $AdS_5 \times S^5$ geometry [26].

[2] For small value of $\sqrt{1 - 6J^2}$ it can be shown using perturbation theory that the lowest mode will correspond to $n = 1$ and will have a higher frequency than the above modes.

For the PP-wave geometry originated from non-Abelian T-dual background, the BMN operators correspond to a class of operators in the quiver theory that we discussed in Chapter 3. The construction of dual field theory is based on the calculations of the brane charge and the central charge [27]. The authors in [27] show that the holographic entanglement entropy agrees with the corresponding supergravity dual. However, for the PP-wave geometry, care must be taken as it originated by zooming a particular region and hence it is globally *not complete* [27]. In this case, the holographic entanglement entropy can be obtained by imposing a hard cutoff on the non-compact directions. But its field theory interpretation is not clear. It might have correspondence to the entanglement entropy in some excited state in the dual field theory.

5.2 Penrose limits in non-Abelian T-dual of Klebanov-Tseytlin background

In this section we will discuss the Penrose limits in non-Abelian T-dual of Klebanov-Tseytlin background. The T-dual background was presented in Chapter 2. The metric together with the background fields are given by the expression (2.69)-(2.73). Before going to examine Penrose limits first we rescale some of the coordinates in T-dual geometry and introduce the parameter $\tilde{r}$ as

$$\ln \tilde{r} = \ln r_0 - \frac{1}{4} - \frac{R^4}{2L^4} \,,$$

Then the wrap factor $H(r)$ takes the form

$$H(r) = \frac{2L^4}{r^4} \ln\left(\frac{r}{\tilde{r}}\right) \,. \tag{5.62}$$

Now we consider the Minkowski coordinates x^μ and rescale as $x^\mu \to L^2 x^\mu$. We also rescale T-dual coordinates $v_{2,3}$ as $v_{2,3} \to L^2 v_{2,3}$. Then T-dual metric (2.69) becomes

$$\begin{aligned} d\hat{s}^2_{\text{NATD}} = \; & L^2\Bigg[\frac{1}{\sqrt{2}}\frac{r^2}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\;\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{2}\;\frac{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}{r^2}\Big(dr^2 + \frac{1}{6}r^2 d\Omega_2^2(\theta_1,\phi_1)\Big)\Bigg] \\ & + \frac{L^2}{A}\Bigg[6\sqrt{2}\,\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\;v_2^2\sigma_{\hat{3}}^2 + \Bigg(6\sqrt{2}\,\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)} + \frac{81\sqrt{2}}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\;v_2^2\Bigg)dv_2^2 \\ & + \Bigg(9\sqrt{2}\,\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)} + \frac{9}{2\sqrt{2}}\;\frac{1}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\;\left(6v_3 + 2\sqrt{2}\ln\left(\frac{r}{r_0}\right)\right)^2\Bigg)dv_3^2 \\ & + \frac{27\sqrt{2}}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\;\left(6v_3 + 2\sqrt{2}\ln\left(\frac{r}{r_0}\right)\right)\;v_2 dv_2 dv_3\Bigg]\;. \end{aligned} \tag{5.63}$$

Where A stands for

$$A = 4\ln\left(\frac{r}{\tilde{r}}\right) + \left(6v_3 + 2\sqrt{2}\ln\left(\frac{r}{r_0}\right)\right)^2 + 54v_2^2\;.$$

After the rescaling, the NS-NS sector of the T-dual background (2.72) takes the form

$$\begin{aligned} \hat{B}_2 = \; & \frac{L^2}{3}\ln\left(\frac{r}{r_0}\right)\sin\theta_1 d\theta_1\wedge d\phi_1 + \frac{L^2}{A}\Bigg[3\sqrt{2}\Big(6v_3 + 2\sqrt{2}\ln\left(\frac{r}{r_0}\right)\Big)\;v_2\;\sigma_{\hat{3}}\wedge dv_2 \\ & + \Bigg(2\sqrt{2}\;\ln\left(\frac{r}{\tilde{r}}\right) + \frac{1}{\sqrt{2}}\Big(6v_3 + 2\sqrt{2}\ln\left(\frac{r}{r_0}\right)\Big)^2\Bigg)\;\sigma_{\hat{3}}\wedge dv_3\Bigg]\;, \\ e^{-2\hat{\Phi}} = \; & \frac{L^6}{81g_s^2}\;\sqrt{2}\;\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\;A\;. \end{aligned} \tag{5.64}$$

Considering the rescaling, the field strengths in the RR sectors (2.73) become

$$\begin{aligned}
\hat{F}_0 = & -L^2\,\frac{2\sqrt{2}}{9g_s}\ ,\\
\hat{F}_2 = & \ \frac{L^4}{162\sqrt{2}\ g_s}\left[\frac{2}{5}\ln\left(\frac{r}{\tilde{r}}\right)-\frac{1}{10}+6\sqrt{2}\left(6v_3+2\sqrt{2}\ln\left(\frac{r}{r_0}\right)\right)\right]\sin\theta_1 d\theta_1\wedge d\phi_1\\
& +\frac{L^4}{g_s A}\frac{4}{3}\left[-\left(6v_3+2\sqrt{2}\ln\left(\frac{r}{r_0}\right)\right)v_2\ \sigma_{\hat{3}}\wedge dv_2+9v_2^2\ \sigma_{\hat{3}}\wedge dv_3\right]\ ,\\
\hat{F}_4 = & \ \frac{L^6}{g_s A}\ v_2\ \sin\theta_1 d\theta_1\wedge d\phi_1\wedge d\psi\wedge\Bigg[\Bigg(-\sqrt{2}\left(6v_3+2\sqrt{2}\ln\left(\frac{r}{r_0}\right)\right)-\frac{1}{15}\ln\left(\frac{r}{\tilde{r}}\right)\\
& +\frac{1}{60}\Bigg)v_2 dv_3+\Bigg(-\frac{4\sqrt{2}}{9}\ln\left(\frac{r}{\tilde{r}}\right)+\frac{1}{9}\left(\frac{1}{15}\ln\left(\frac{r}{\tilde{r}}\right)-\frac{1}{60}\right)\left(6v_3+2\sqrt{2}\ln\left(\frac{r}{r_0}\right)\right)\\
& -6\sqrt{2}v_2^2\Bigg)dv_2\Bigg]\ .
\end{aligned}\tag{5.65}$$

Now we consider various isometric directions in the T-dual geometry with rescaled coordinates (5.63). The background admits $U(1)$-isometry along ψ and ϕ_1 directions. First we turn our interest to the motion along ψ direction. The geodesics equation for this case is

$$\partial_\mu g_{\psi\psi} = 0\ . \tag{5.66}$$

The relevant component of the metric for the discussion is:

$$g_{\psi\psi} = \frac{L^2}{A}\ 6\sqrt{2}\ \sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\ v_2^2\ . \tag{5.67}$$

Now the component $g_{\psi\psi}$ depends on r together with the T-dual coordinates v_2 and v_3. The geodesic condition (5.66), for $\mu = r$, leads to $v_2 = 0$ and for $\mu = v_2, v_3$ we have $\{r = \tilde{r}, v_2 = 0\}$. However, for all these values the metric component $g_{\psi\psi}$ vanishes and leads to singular geometries. Hence, we will not examine Penrose limits for these singular geometries.

We will now consider the motion along another symmetric direction ϕ_1. Consider the relevant metric component as:

$$g_{\phi_1\phi_1} = L^2\ \sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\ \left[\frac{1}{3\sqrt{2}}\sin^2\theta_1+\frac{6\sqrt{2}}{A}\ v_2^2\cos^2\theta_1\right]\ . \tag{5.68}$$

In this case the geodesic condition becomes:

$$\partial_\mu g_{\phi_1\phi_1} = 0\ .$$

For $\mu = r$, the geodesic condition provides $\{\theta_1 = (0,\pi), v_2 = 0\}$. For $\mu = \theta_1$, we have $\{r = \tilde{r}, \theta_1 = (0, \frac{\pi}{2}, \pi)\}$. Similarly, for $\mu = v_2, v_3$ the geodesic condition leads to $\{r = \tilde{r}, \theta_1 =$

$\frac{\pi}{2}, v_2 = 0\}$. However, for the values $r = \tilde{r}$, $\{\theta_1 = (0,\pi), v_2 = 0\}$, the metric component $g_{\phi_1\phi_1}$ vanishes and give singular geometries.

In the following first we will consider Penrose limit around the geodesic $\theta_1 = \frac{\pi}{2}, v_2 = 0 = v_3$ and keeping the r-coordinate unchanged, *i.e.*, $r = c$; for some constant $c \neq \tilde{r} \neq 0$. Consider the following expansion around the above-mentioned geodesic with large L limit:

$$\begin{aligned} & x_i = \frac{y_i}{L}\ ;\ i = 1,2,3,\ r = c + \frac{x}{L}\ ,\ \theta_1 = \frac{\pi}{2} + \frac{z}{L}\ ,\ t = ax^+\ , \\ & \phi_1 = bx^+ + \frac{x^-}{L^2}\ . \end{aligned} \tag{5.69}$$

In addition, we also rescale the T-dual coordinates v_2 and v_3 as $v_2 \to \frac{v_2}{L}$, $v_3 \to \frac{v_3}{L}$ while keeping the ψ-coordinate unchanged. In the expansion a and b are some unknown constant parameters. By imposing the null geodesic condition, we have the relation between the parameters a, b with c as:

$$a^2 = \frac{b^2}{3c^2}\ln\left(\frac{c}{\tilde{r}}\right)\ . \tag{5.70}$$

Substituting the above expansion in the T-dual metric (5.63) and keeping the leading order terms in $L \to \infty$ we have

$$\begin{aligned} ds^2 = \ & \frac{1}{3\sqrt{2}}\sqrt{\ln\left(\frac{c}{\tilde{r}}\right)}\ 2bdx^+dx^- + \frac{1}{\sqrt{2}}\ \frac{c^2}{\sqrt{\ln\left(\frac{c}{\tilde{r}}\right)}}\Big(dy_1^2 + dy_2^2 + dy_3^2\Big) + \frac{\sqrt{2}}{c^2}\sqrt{\ln\left(\frac{c}{\tilde{r}}\right)}\ dx^2 \\ & + \frac{1}{3\sqrt{2}}\sqrt{\ln\left(\frac{c}{\tilde{r}}\right)}\ dz^2 + \frac{3}{\sqrt{2}}\ \frac{\sqrt{\ln\left(\frac{c}{\tilde{r}}\right)}}{\ln\left(\frac{c}{\tilde{r}}\right) + 2\Big(\ln\left(\frac{c}{r_0}\right)\Big)^2}\Big(dv_2^2 + v_2^2 d\psi^2\Big) + \frac{9}{2\sqrt{2}}\frac{1}{\sqrt{\ln\left(\frac{c}{\tilde{r}}\right)}}\ dv_3^2 \\ & - \frac{b^2}{3\sqrt{2}c^2}\sqrt{\ln\left(\frac{c}{\tilde{r}}\right)}\left[\frac{x^2}{\ln\left(\frac{c}{\tilde{r}}\right)} + x^2 + c^2z^2\right](dx^+)^2 - L\ \frac{2b^2x}{3c}\ln\left(\frac{c}{\tilde{r}}\right)\ (dx^+)^2\ . \end{aligned} \tag{5.71}$$

The metric contains a divergent term with $\mathcal{O}(L)$ and that cannot be removed for any choice of the parameters present in the metric. Also the null geodesic condition (5.70) inhibits to set $b = 0$. Hence, motion along the isometric direction ϕ_1 does not lead to any smooth geometry.

As in the case of $AdS_5 \times S^5$ and Klebanov-Witten backgrounds, we consider the null geodesic that carries nonzero angular momentum. In order to find such a geodesic, we consider motion in the (r, ϕ_1) plane. We will confine our analysis in the neighbourhood of $\theta_1 = \frac{\pi}{2}$ and $v_2 = v_3 = 0$. Consider the Lagrangian for a massless particle as in (4.10) and substitute the background metric (5.63) we find

$$\mathcal{L} = \frac{L^2}{2}\left(-\frac{1}{\sqrt{2}}\frac{r^2}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ \dot{t}^2 + \frac{\sqrt{2}}{r^2}\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\ \dot{r}^2 + \frac{1}{3\sqrt{2}}\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\ \dot{\phi}_1^2 \right)\ . \tag{5.72}$$

Now, in the Lagrangian (5.72) t and ϕ_1 are cyclic coordinates and their corresponding momenta will be conserved. Let $-EL^2$ be the conserved momentum associated with t. Then,

$$E = -\frac{1}{L^2}\frac{\partial\mathcal{L}}{\partial\dot{t}} = \frac{r^2}{\sqrt{2\ln\left(\frac{r}{\tilde{r}}\right)}}\,\dot{t}\ . \tag{5.73}$$

Denoting $-JL^2$ be the conserved momentum associated with the cyclic coordinate ϕ_1 we have

$$J = -\frac{1}{L^2}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = -\frac{1}{3\sqrt{2}}\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\,\dot{\phi}_1\ . \tag{5.74}$$

The null geodesic condition *i.e.* $\mathcal{L}=0$ leads to:

$$\dot{r}^2 + \frac{3r^2}{\ln\left(\frac{r}{\tilde{r}}\right)}J^2 = E^2\ . \tag{5.75}$$

Now we will examine the Penrose limit for the above null geodesic carrying angular momentum J. We expand the T-dual metric around $x_i = 0, i = 1,2,3$, $\theta_1 = \frac{\pi}{2}$ and $v_2 = v_3 = 0$. First we redefine the coordinates in the T-dual geometry as

$$x_i = \frac{y_i}{L}\ ;\ i=1,2,3,\ \ \theta_1 = \frac{\pi}{2} + \frac{z}{L}\ ,\ v_2 \to \frac{v_2}{L}\ ,\ v_3 \to \frac{v_3}{L}\ , \tag{5.76}$$

while keeping the ψ-coordinate unchanged. We redefine the string coupling g_s as $g_s = L^3\,\tilde{g}_s$, in order to keep the dilaton finite at the Penrose limit. Finally, we will consider the following expansion :

$$dt = c_1 du,\ \ dr = c_2 du + c_3\frac{dw}{L}\ ,\ d\phi_1 = c_4 du + c_5\frac{dw}{L} + c_6\frac{dv}{L^2}\ , \tag{5.77}$$

and subsequently take $L\to\infty$ limit. We need to fix the unknown coefficients c_i. Imposing the null geodesic condition determines the coefficients c_1, c_2 and c_4 as follows:

$$\begin{aligned} c_1 &= \frac{E\sqrt{2}}{r^2}\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\ , \\ c_2 &= \left[E^2 - \frac{3r^2}{\ln\left(\frac{r}{\tilde{r}}\right)}J^2\right]^{\frac{1}{2}}, \\ c_4 &= -\frac{3\sqrt{2}}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\,J\ . \end{aligned} \tag{5.78}$$

Now we substitute the expansion (5.77) in the T-dual metric (5.63) and consider the leading terms. At first, the expansion will contain divergent terms of $\mathcal{O}(L)$ as well as $\mathcal{O}(L^2)$.

Imposing the null geodesic condition automatically cancels the $\mathcal{O}(L^2)$ terms each other. The $\mathcal{O}(L)$ term can be removed upon setting the coefficients as:

$$c_2c_3 + \frac{r^2}{6}c_4c_5 = 0 \ . \tag{5.79}$$

Considering the value of c_2 and c_4 from (5.78) and substituting in the above equation one can express the coefficient c_3 in terms of c_5 as

$$c_3 = \left[E^2 - \frac{3r^2}{\ln\left(\frac{r}{\tilde{r}}\right)}J^2\right]^{-\frac{1}{2}} \frac{r^2}{\sqrt{2\ln\left(\frac{r}{\tilde{r}}\right)}}\ J\ c_5 \ . \tag{5.80}$$

The coefficient c_5 can be determined by requiring the background fields to satisfy the Einstein's equations that we will present later. Finally, we fixed the only remaining coefficient c_6. This is obtained by choosing the normalization factor of cross-term $dudv$ in the metric appropriately. Then we have

$$c_6 = -\frac{1}{J} \ . \tag{5.81}$$

Substituting the above expansion in T-dual metric (5.63), and subsequently taking the limit $L \to \infty$, we find

$$\begin{aligned} ds^2 = \ & 2dudv + \frac{1}{\sqrt{2}}\ \frac{r^2}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\left(dy_1^2 + dy_2^2 + dy_3^2\right) + \sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\ \left(\frac{c_3^2\sqrt{2}}{r^2} + \frac{c_5^2}{3\sqrt{2}}\right)dw^2 \\ & + \frac{1}{3\sqrt{2}}\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\ dz^2 + \frac{3}{\sqrt{2}}\ \frac{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}{\ln\left(\frac{r}{\tilde{r}}\right) + 2\Big(\ln\left(\frac{r}{r_0}\right)\Big)^2}\left(dv_2^2 + v_2^2 d\psi^2\right) \\ & + \frac{9}{2\sqrt{2}}\frac{1}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ dv_3^2 - \frac{3\sqrt{2}}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}J^2z^2\ du^2 \ . \end{aligned} \tag{5.82}$$

In the following we will show that the above metric is a PP-wave geometry by rewriting it in the standard Brinkmann form. Now in this limit, the NS-NS sector takes the form

$$\begin{aligned} \hat{B}_2 = \ & \frac{c_5}{3}\ln\left(\frac{r}{r_0}\right)\ dz\wedge dw - 3\ \frac{\ln\left(\frac{r}{r_0}\right)}{\ln\left(\frac{r}{\tilde{r}}\right) + 2\Big(\ln\left(\frac{r}{r_0}\right)\Big)^2}\ v_2 dv_2\wedge d\psi + \frac{3Jz}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ du\wedge dv_3 \ , \\ e^{-2\hat{\Phi}} = \ & \frac{4\sqrt{2}}{81\ \tilde{g}_s^2}\ \sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\ \left[\ln\left(\frac{r}{\tilde{r}}\right) + 2\Big(\ln\left(\frac{r}{r_0}\right)\Big)^2\right] \ . \end{aligned} \tag{5.83}$$

Similarly the field strengths corresponding to the RR sector become

$$
\begin{aligned}
\hat{F}_0 = &\ 0 \ , \\
\hat{F}_2 = &\ \frac{J}{54\tilde{g}_s}\frac{1}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\left[\frac{2}{5}\ln\left(\frac{r}{\tilde{r}}\right)+24\ln\left(\frac{r}{r_0}\right)-\frac{1}{10}\right]du\wedge dz \ , \\
\hat{F}_4 = &\ \frac{2J}{3\tilde{g}_s\Big(\ln\left(\frac{r}{\tilde{r}}\right)+2\Big(\ln\left(\frac{r}{r_0}\right)\Big)^2\Big)}\frac{1}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\, v_2\Bigg[-\ln\left(\frac{r}{\tilde{r}}\right)+\frac{1}{2}\ln\left(\frac{r}{r_0}\right)\Big(\frac{1}{15}\ln\left(\frac{r}{\tilde{r}}\right) \\
&-\frac{1}{60}\Big)\Bigg]du\wedge dz\wedge d\psi\wedge dv_2 \ .
\end{aligned}
\tag{5.84}
$$

At Penrose limits, the NS-NS three-form field strength $\hat{H}_3$ has the expression

$$
\begin{aligned}
\hat{H}_3 = &\ \frac{1}{3}\left[E^2-\frac{3r^2}{\ln\left(\frac{r}{\tilde{r}}\right)}J^2\right]^{\frac{1}{2}}\left[c_5'\ln\left(\frac{r}{r_0}\right)+\frac{c_5}{r}\right]du\wedge dz\wedge dw-3v_2\left[E^2-\frac{3r^2}{\ln\left(\frac{r}{\tilde{r}}\right)}J^2\right]^{\frac{1}{2}} \\
&\frac{\ln\left(\frac{r}{\tilde{r}}\right)-\ln\left(\frac{r}{r_0}\right)-2\Big(\ln\left(\frac{r}{r_0}\right)\Big)^2}{r\Big(\ln\left(\frac{r}{\tilde{r}}\right)+2\Big(\ln\left(\frac{r}{r_0}\right)\Big)^2\Big)^2}\, du\wedge dv_2\wedge d\psi+\frac{3J}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\, du\wedge dv_3\wedge dz \ .
\end{aligned}
\tag{5.85}
$$

In order to get the above we used $dr = c_2 du$ and the expression for the coefficient c_2 is given by (5.78).

Now in the following we will represent the metric and the background fields in the standard Brinkmann form [33]. In order to do that we will follow the same formalism developed in [26] and presented in (4.41)-(4.44). For the PP-wave metric studied in (5.82) we have

$$
\begin{aligned}
&A_{y_1} = A_{y_2} = A_{y_3} = \frac{1}{\sqrt{2}}\ \frac{r^2}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}} \ , \ A_w = \sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\ \left(\frac{c_3^2\sqrt{2}}{r^2}+\frac{c_5^2}{3\sqrt{2}}\right) \ , \\
&A_z = \frac{1}{3\sqrt{2}}\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)} \ , \ A_{v_2} = \frac{3}{\sqrt{2}}\ \frac{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}{\ln\left(\frac{r}{\tilde{r}}\right)+2\Big(\ln\left(\frac{r}{r_0}\right)\Big)^2} \ , \ A_{v_3} = \frac{9}{2\sqrt{2}}\frac{1}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}} \ .
\end{aligned}
\tag{5.86}
$$

Then after making the following replacement

$$
\begin{aligned}
& y_1 \to \frac{y_1}{\sqrt{A_{y_1}}} \ , \ y_2 \to \frac{y_2}{\sqrt{A_{y_2}}} \ , \ y_3 \to \frac{y_3}{\sqrt{A_{y_3}}} \ , \ w \to \frac{w}{\sqrt{A_w}} \ , \ z \to \frac{z}{\sqrt{A_z}} \ , \\
& v_2 \to \frac{v_2}{\sqrt{A_{v_2}}} \ , \ v_3 \to \frac{v_3}{\sqrt{A_{v_3}}} \quad \text{and} \\
& v \to v + \frac{1}{4}\left[\frac{\dot{A}_{y_1}}{A_{y_1}}\ y_1^2 + \frac{\dot{A}_{y_2}}{A_{y_2}}\ y_2^2 + \frac{\dot{A}_{y_3}}{A_{y_3}}\ y_3^2 + \frac{\dot{A}_w}{A_w}\ w^2 + \frac{\dot{A}_z}{A_z}\ z^2 + \frac{\dot{A}_{v_2}}{A_{v_2}}\ v_2^2 + \frac{\dot{A}_{v_3}}{A_{v_3}}\ v_3^2\right] \ ,
\end{aligned}
\tag{5.87}
$$

we have

$$
\begin{aligned}
ds^2 = \ & 2dudv + dy_1^2 + dy_2^2 + dy_3^2 + dw^2 + dz^2 + dv_2^2 + v_2^2\ d\psi^2 + dv_3^2 \\
& + \left[F_{y_1} y_1^2 + F_{y_2} y_2^2 + F_{y_3} y_3^2 + F_w w^2 + F_z z^2 + F_{v_2} v_2^2 + F_{v_3} v_3^2 - \frac{3\sqrt{2}}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ J^2 z^2\right] du^2 \ ,
\end{aligned}
\tag{5.88}
$$

where the functions F_i can be read from the expression (4.44).

Now we will consider the background fields in the Brinkmann form. The fields in the NS-NS sector are given as

$$
\begin{aligned}
e^{-2\hat{\Phi}} = \ & \frac{4\sqrt{2}}{81\ \tilde{g}_s^2}\ \sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}\ \left[\ln\left(\frac{r}{\tilde{r}}\right) + 2\Big(\ln\left(\frac{r}{r_0}\right)\Big)^2\right] \ , \\
\hat{H}_3 = \ & \frac{2^{\frac{1}{4}}}{\sqrt{3}}\left[E^2 - \frac{3r^2}{\ln\left(\frac{r}{\tilde{r}}\right)} J^2\right]^{\frac{1}{2}}\left[c_5' \ln\left(\frac{r}{r_0}\right) + \frac{c_5}{r}\right]\left[\frac{c_3^2\sqrt{2}}{r^2} + \frac{c_5^2}{3\sqrt{2}}\right]^{-\frac{1}{2}} \frac{1}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}} \\
& du \wedge dz \wedge dw - \ \sqrt{2} v_2 \left[E^2 - \frac{3r^2}{\ln\left(\frac{r}{\tilde{r}}\right)} J^2\right]^{\frac{1}{2}} \frac{\ln\left(\frac{r}{\tilde{r}}\right) - \ln\left(\frac{r}{r_0}\right) - 2\Big(\ln\left(\frac{r}{r_0}\right)\Big)^2}{r\Big(\ln\left(\frac{r}{\tilde{r}}\right) + 2\Big(\ln\left(\frac{r}{r_0}\right)\Big)^2\Big)} \\
& \frac{1}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ du \wedge dv_2 \wedge d\psi + \frac{2\sqrt{3}\ J}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ du \wedge dv_3 \wedge dz \ .
\end{aligned}
\tag{5.89}
$$

Similarly, the RR field strengths become

$$\begin{aligned}
\hat{F}_0 =& \ 0 \ , \\
\hat{F}_2 =& \ \frac{2^{(-\frac{3}{4})}}{9\sqrt{3}\ \tilde{g}_s}\frac{J}{\left(\ln\left(\frac{r}{\tilde{r}}\right)\right)^{\frac{3}{4}}}\left[\frac{2}{5}\ln\left(\frac{r}{\tilde{r}}\right)+24\ln\left(\frac{r}{r_0}\right)-\frac{1}{10}\right]du\wedge dz \ , \\
\hat{F}_4 =& \ \frac{2^{\frac{7}{4}}}{3\sqrt{3}\ \tilde{g}_s}\frac{J}{\left(\ln\left(\frac{r}{\tilde{r}}\right)\right)^{\frac{5}{4}}}\ v_2\Bigg[-\ln\left(\frac{r}{\tilde{r}}\right)+\frac{1}{2}\ln\left(\frac{r}{r_0}\right)\left(\frac{1}{15}\ln\left(\frac{r}{\tilde{r}}\right)\right. \\
& \left.\left.-\frac{1}{60}\right)\right]du\wedge dz\wedge d\psi\wedge dv_2 \ .
\end{aligned} \tag{5.90}$$

We will now show that these fields satisfy the Bianchi identities and the gauge field equation of motion in type-IIA supergravity (A). For the background fields in (5.88)-(5.90), the field strengths $\hat{H}_3, \hat{F}_2$ and $\hat{F}_4$ are all closed together with $\hat{F}_0$ as well as $\hat{H}_3\wedge\hat{F}_2$ are indeed zero. Hence, the Bianchi identities (A) are indeed satisfied.

The expression of Hodge duals for the above background fields are given by

$$\begin{aligned}
\star\ \hat{H}_3 =& \ \frac{1}{7!}\ du\wedge dy_1\wedge dy_2\wedge dy_3\Bigg[\frac{2^{\frac{1}{4}}}{\sqrt{3}}\ v_2\left(E^2-\frac{3r^2}{\ln\left(\frac{r}{\tilde{r}}\right)}J^2\right)^{\frac{1}{2}}\left(c_5'\ln\left(\frac{r}{r_0}\right)+\frac{c_5}{r}\right) \\
& \left(\frac{c_3^2\sqrt{2}}{r^2}+\frac{c_5^2}{3\sqrt{2}}\right)^{-\frac{1}{2}}\frac{1}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ dv_2\wedge d\psi\wedge dv_3 \\
& -\sqrt{2}\left(E^2-\frac{3r^2}{\ln\left(\frac{r}{\tilde{r}}\right)}J^2\right)^{\frac{1}{2}}\frac{\ln\left(\frac{r}{\tilde{r}}\right)-\ln\left(\frac{r}{r_0}\right)-2\left(\ln\left(\frac{r}{r_0}\right)\right)^2}{r\left(\ln\left(\frac{r}{\tilde{r}}\right)+2\left(\ln\left(\frac{r}{r_0}\right)\right)^2\right)^2} \\
& \frac{1}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ dz\wedge dw\wedge dv_3+\ v_2\ \frac{2\sqrt{3}\ J}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ dw\wedge dv_2\wedge d\psi\Bigg] \ , \\
\star\ \hat{F}_2 =& \ \frac{1}{8!}\ \frac{2^{(-\frac{3}{4})}}{9\sqrt{3}\ \tilde{g}_s}\frac{J}{\left(\ln\left(\frac{r}{\tilde{r}}\right)\right)^{\frac{3}{4}}}\ v_2\left[\frac{2}{5}\ln\left(\frac{r}{\tilde{r}}\right)+24\ln\left(\frac{r}{r_0}\right)-\frac{1}{10}\right]du\wedge dy_1\wedge dy_2\wedge dy_3 \\
& \wedge\ dw\wedge d\psi\wedge dv_2\wedge dv_3 \ , \\
\star\ \hat{F}_4 =& \ \frac{1}{6!}\ \frac{2^{\frac{7}{4}}}{3\sqrt{3}\ \tilde{g}_s}\frac{J}{\left(\ln\left(\frac{r}{\tilde{r}}\right)\right)^{\frac{5}{4}}}\ \left[-\ln\left(\frac{r}{\tilde{r}}\right)+\frac{1}{2}\ln\left(\frac{r}{r_0}\right)\left(\frac{1}{15}\ln\left(\frac{r}{\tilde{r}}\right)-\frac{1}{60}\right)\right]du\wedge dy_1 \\
& \wedge dy_2\wedge dy_3\wedge\ dw\wedge dv_3 \ .
\end{aligned} \tag{5.91}$$

The above expressions provide that $\star\hat{H}_3$ and $e^{-2\hat{\Phi}}\star\hat{H}_3$ are closed. Also, $\hat{F}_2\wedge\star\hat{F}_4$ and

$\hat{F}_4 \wedge \hat{F}_4$ vanishes. Similarly, both $\star\hat{F}_2$ as well as $\star\hat{F}_4$ are exact in forms. In addition, $\hat{H}_3 \wedge \hat{F}_4$ and $\hat{H}_3 \wedge \star\hat{F}_4$ also vanishes. Hence the gauge field equations (A) in type-IIA supergravity are also satisfied.

Now to find the undetermined coefficient c_5, we consider the Einstein's equations in type-IIA supergravity:

$$\hat{R}_{\mu\nu} + 2D_\mu D_\nu \hat{\Phi} = \frac{1}{4}\hat{H}^2_{\mu\nu} + e^{2\hat{\Phi}}\left[\frac{1}{2}(\hat{F}^2_2)_{\mu\nu} + \frac{1}{12}(\hat{F}^2_4)_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\Big(\hat{F}^2_0 + \frac{1}{2}\hat{F}^2_2 + \frac{1}{4!}\hat{F}^2_4\Big)\right] , \tag{5.92}$$

together with the dilation equation

$$\hat{R} + 4D^2\hat{\Phi} - 4(\partial\hat{\Phi})^2 - \frac{1}{12}\hat{H}^2 = 0 \ . \tag{5.93}$$

For the pp-wave geometry (in the Brinkmann form) the only non vanishing component of Ricci tensor is $\hat{R}_{uu}$ with the equation

$$\hat{R}_{uu} + 2D_u D_u \hat{\Phi} = \frac{1}{4}\hat{H}^2_{uu} + e^{2\hat{\Phi}}\left[\frac{1}{2}(\hat{F}^2_2)_{uu} + \frac{1}{12}(\hat{F}^2_4)_{uu}\right] . \tag{5.94}$$

This equation involves the undetermined coefficient c_5 in the expression of $\hat{H}^2_{uu}$. One can solve this equation to determine the expression for the coefficient c_5.

Before going to discuss the field theory dual, in the following we will discuss the supersymmetry preserved by the pp-wave background studied in (5.88)-(5.90). As like for the $AdS_5 \times S^5$ case, here also we introduce the Brinkmann coordinates X^i as

$$\begin{gathered} dy_i^2 = \big(dX^i\big)^2 \ ; \ i = 1,2,3 \ , \ w = X^4 \ , \ z = X^5 \ , \\ dv_2^2 + v_2^2 \, d\psi^2 = \big(dX^6\big)^2 + \big(dX^7\big)^2 \ , \ v_3 = X^8 \ . \end{gathered} \tag{5.95}$$

Then, in these X^i-coordinates the PP-wave background (5.88)-(5.90) takes the form

$$\begin{aligned} ds^2 =& \ 2dudv + \sum_{i=1}^{8} dX_i^2 + \mathcal{H} \, du^2 \ , \\ \hat{\Phi} =& \ \Phi(u) \ , \\ \hat{H}_3 =& \ f_1(u) \, du \wedge dX^5 \wedge dX^4 - f_2(u) \, du \wedge dX^6 \wedge dX^7 + f_3(u) \, du \wedge dX^8 \wedge dX^5 \ , \\ \hat{F}_2 =& \ f_4(u) \, du \wedge dX^5 \ , \\ \hat{F}_4 =& \ f_5(u) \, du \wedge dX^5 \wedge dX^7 \wedge dX^6 \ , \end{aligned} \tag{5.96}$$

where for easy reading we introduce the notation

$$\mathcal{H} = \quad F_{ij}X^iX^j = \Big[F_{X^1}(X^1)^2 + F_{X^2}(X^2)^2 + F_{X^3}(X^3)^2 + F_{X^4}(X^4)^2$$

$$
\begin{aligned}
+ \quad & \Big(F_{X^5} - \frac{3\sqrt{2}}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ J^2\Big)(X^5)^2 + F_{X^6}(X^6)^2 + F_{X^7}(X^7)^2 + F_{X^8}(X^8)^2\Bigg]\ , \\
f_1(u) = \quad & \frac{2^{\frac{1}{4}}}{\sqrt{3}}\left[E^2 - \frac{3r^2}{\ln\left(\frac{r}{\tilde{r}}\right)}J^2\right]^{\frac{1}{2}}\left[c_5'\ln\left(\frac{r}{r_0}\right) + \frac{c_5}{r}\right]\left[\frac{c_3^2\sqrt{2}}{r^2} + \frac{c_5^2}{3\sqrt{2}}\right]^{-\frac{1}{2}}\frac{1}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ , \\
f_2(u) = \quad & \sqrt{2}\left[E^2 - \frac{3r^2}{\ln\left(\frac{r}{\tilde{r}}\right)}J^2\right]^{\frac{1}{2}}\frac{\ln\left(\frac{r}{\tilde{r}}\right) - \ln\left(\frac{r}{r_0}\right) - 2\Big(\ln\left(\frac{r}{r_0}\right)\Big)^2}{r\Big(\ln\left(\frac{r}{\tilde{r}}\right) + 2\Big(\ln\left(\frac{r}{r_0}\right)\Big)^2\Big)}\frac{1}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ , \\
f_3(u) = \quad & \frac{2\sqrt{3}\ J}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ , \\
f_4(u) = \quad & \frac{2^{(-\frac{3}{4})}}{9\sqrt{3}\ \tilde{g}_s}\frac{J}{\Big(\ln\left(\frac{r}{\tilde{r}}\right)\Big)^{\frac{3}{4}}}\left[\frac{2}{5}\ln\left(\frac{r}{\tilde{r}}\right) + 24\ln\left(\frac{r}{r_0}\right) - \frac{1}{10}\right]\ , \\
f_5(u) = \quad & \frac{2^{\frac{7}{4}}}{3\sqrt{3}\ \tilde{g}_s}\frac{J}{\Big(\ln\left(\frac{r}{\tilde{r}}\right)\Big)^{\frac{5}{4}}}\left[-\ln\left(\frac{r}{\tilde{r}}\right) + \frac{1}{2}\ln\left(\frac{r}{r_0}\right)\left(\frac{1}{15}\ln\left(\frac{r}{\tilde{r}}\right) - \frac{1}{60}\right)\right]\ ,
\end{aligned} \tag{5.97}
$$

where the functions F_{ij} are defined by

$$
\begin{aligned}
& F_{11} = F_{22} = F_{33} = F_{X^i}\ ;\ i = 1,2,3\ , \\
& F_{44} = F_{X^4}\ ,\ F_{55} = F_{X^5} - \frac{3\sqrt{2}}{\sqrt{\ln\left(\frac{r}{\tilde{r}}\right)}}\ J^2\ ,\ F_{66} = F_{X^6}\ ,\ F_{77} = F_{X^7}\ ,\ F_{88} = F_{X^8}
\end{aligned} \tag{5.98}
$$

By considering the same frame notation as in (4.58), in this case we have

$$
\begin{aligned}
\hat{\Phi} = \quad & \Phi(u)\ , \\
\hat{H}_3 = \quad & f_1(u)\ e^- \wedge e^5 \wedge e^4 - f_2(u)\ e^- \wedge e^6 \wedge e^7 + f_3(u)\ e^- \wedge e^8 \wedge e^5\ , \\
\hat{F}_2 = \quad & f_4(u)\ e^- \wedge e^5\ , \\
\hat{F}_4 = \quad & f_5(u)\ e^- \wedge e^5 \wedge e^7 \wedge e^6\ .
\end{aligned} \tag{5.99}
$$

Here the supersymmetric variations of the dilatino and gravitino are given by

$$
\begin{aligned}
\delta\hat{\lambda} &= \frac{1}{2}\not{\partial}\hat{\Phi}\hat{\epsilon} - \frac{1}{24}\hat{\not{H}}\sigma_3\hat{\epsilon}\ + \frac{1}{8}e^{\hat{\Phi}}\left[\frac{3}{2}\hat{\not{F}}_2\,(i\sigma_2) + \frac{1}{24}\hat{\not{F}}_4\sigma_1\right]\hat{\epsilon}\ , \\
\delta\hat{\psi}_\mu &= D_\mu\hat{\epsilon} - \frac{1}{8}\hat{H}_{\mu\nu\rho}\Gamma^{\nu\rho}\sigma_3\hat{\epsilon} + \frac{1}{8}e^{\hat{\Phi}}\left[\frac{1}{2}\hat{\not{F}}_2\,(i\sigma_2) + \frac{1}{24}\hat{\not{F}}_4\sigma_1\right]\Gamma_\mu\hat{\epsilon}\ ,
\end{aligned} \tag{5.100}
$$

Here we use the same convention as in $AdS_5 \times S^5$ case.

First considering the dilatino variation to zero we obtain

$$\Gamma^{-}\left[\dot{\hat{\Phi}} - \frac{1}{2}\Big(f_1(u)\Gamma^{54} - f_2(u)\Gamma^{67} + f_3(u)\Gamma^{85}\Big)\sigma_3 + \frac{e^{\hat{\Phi}}}{4}\Big(3f_4(u)\Gamma^{5}\left(i\sigma_2\right) + f_5(u)\Gamma^{576}\sigma_1\Big)\right]\hat{\epsilon} = 0\ . \tag{5.101}$$

The above condition gives us $\Gamma^{-}\hat{\epsilon} = 0$. This states that the pp-wave background (5.96) preserves 16 supercharges subject to gravitino variation. We now consider the supersymmetric variation of gravitino. As the NS-NS three-form $\hat{H}_3$ does not have any leg along e^{+} and $\Gamma_{+}\hat{\epsilon} = \Gamma^{-}\hat{\epsilon} = 0$, the variation $\delta\hat{\psi}_{+} = 0$ leads to $\partial_{+}\hat{\epsilon} = 0$. Hence we find that the Killing spinor $\hat{\epsilon}$ is independent of v, *i.e.* $\hat{\epsilon} = \hat{\epsilon}(u, X^{i})$.

Finally we consider the variation $\delta\hat{\psi}_i$; $i = 1, ..., 8$. Set it to 0, we obtain

$$\partial_i\hat{\epsilon} = \Gamma^{-}\mathcal{R}\ \hat{\epsilon}\ , \tag{5.102}$$

where $\mathcal{R}$ is given by

$$\begin{aligned}\mathcal{R} = \ & \frac{1}{4}\bigg(f_1(u)\Big(\delta_{i4}\Gamma^{5} - \delta_{i5}\Gamma^{4}\Big) - f_2(u)\Big(\delta_{i7}\Gamma^{6} - \delta_{i6}\Gamma^{7}\Big) + f_3(u)\Big(\delta_{i5}\Gamma^{8} - \delta_{i8}\Gamma^{5}\Big)\bigg)\sigma_3 \\ & -\frac{e^{\hat{\Phi}}}{8}\bigg(f_4(u)\Gamma^{5}\big(i\sigma_2\big) + f_5(u)\Gamma^{576}\sigma_1\bigg)\Gamma^{i}\ .\end{aligned} \tag{5.103}$$

Now, Γ^{-} anticommutes with $\mathcal{R}$ and we also have $\Gamma^{-}\hat{\epsilon} = 0$. Then $\partial_i\hat{\epsilon} = 0$ leads to $\hat{\epsilon} = \chi(u)$ with $\Gamma^{-}\chi(u) = 0$. Finally, the condition $\delta\hat{\psi}_{-} = 0$ gives rise to

$$\partial_u\chi(u) - \frac{1}{4}\Big(f_1(u)\Gamma^{54} - f_2(u)\Gamma^{67} + f_3(u)\Gamma^{85}\Big)\sigma_3\chi(u) - \frac{e^{\hat{\Phi}}}{4}\bigg(f_4(u)\Gamma^{5}\big(i\sigma_2\big) + f_5(u)\Gamma^{576}\sigma_1\bigg)\chi(u) = 0\ . \tag{5.104}$$

The above can be written in the simple form $\partial_u\chi(u) - \mathcal{M}(u)\chi(u) = 0$, which can be integrated to give rise

$$\chi(u) = e^{\int du\mathcal{M}(u)}\chi_0\ .$$

Hence, the supersymmetry discussion shows that the PP-wave background (5.88)-(5.90) preserves 16 supercharges.

5.2.1 Gauge theory duals

In this section we will discuss the dual gauge theory for the pp-wave background studied in (5.88)-(5.90). The dual gauge theory for the non-Abelian T-dual of Klebanov-Tseytlin background has been discussed in [23]. The construction of the gauge theory is based on the D-brane charges present in the dual background. It turns out that in the dual theory the Seiberg duality is present much like as its seed background [23]. Here we will consider the Maxwell and Page charges for PP-wave background. We have shown that the RR field strengths for the PP-wave background in Brinkmann coordinates are given by

$$\begin{aligned}
\hat{F}_0 =& \; 0 \; , \\
\hat{F}_2 =& \; \frac{2^{(-\frac{3}{4})}}{9\sqrt{3}\;\tilde{g}_s} \frac{J}{\left(\ln\left(\frac{r}{\tilde{r}}\right)\right)^{\frac{3}{4}}} \left[\frac{2}{5}\ln\left(\frac{r}{\tilde{r}}\right) + 24\ln\left(\frac{r}{r_0}\right) - \frac{1}{10}\right] du\wedge dz \; , \\
\hat{F}_4 =& \; \frac{2^{\frac{7}{4}}}{3\sqrt{3}\;\tilde{g}_s} \frac{J}{\left(\ln\left(\frac{r}{\tilde{r}}\right)\right)^{\frac{5}{4}}} \; v_2 \Bigg[-\ln\left(\frac{r}{\tilde{r}}\right) \\
& + \frac{1}{2}\ln\left(\frac{r}{r_0}\right)\left(\frac{1}{15}\ln\left(\frac{r}{\tilde{r}}\right) - \frac{1}{60}\right)\Bigg] du\wedge dz\wedge d\psi\wedge dv_2 \; . \qquad (5.105)
\end{aligned}$$

In type-IIA theory, the Maxwell and page charge for various brane is given by

$$\begin{aligned}
\hat{Q}_{\text{Max, D6}} =& \; \frac{1}{\sqrt{2}\pi^2}\int \hat{F}_2 \; , \\
\hat{Q}_{\text{Max, D8}} =& \; \sqrt{2}\int \hat{F}_0 \; , \qquad (5.106)
\end{aligned}$$

and

$$\begin{aligned}
\hat{Q}_{\text{Page, D6}} =& \; \frac{1}{\sqrt{2}\pi^2}\int \hat{F}_2 - \hat{B}_2\hat{F}_0 \; , \\
\hat{Q}_{\text{Page, D8}} =& \; \sqrt{2}\int \hat{F}_0 \; . \qquad (5.107)
\end{aligned}$$

Now from (5.105) it is very straightforward to see the $D8$ charges are all zero. Also, the Maxwell for $D6$-branes is the same as the Page charge of it. The Maxwell and Page charges for $D2$-branes also vanish. We have

$$\begin{aligned}
\hat{Q}_{\text{Max, D2}} &= \frac{1}{2k_{10}^2 T_{D2}}\int_{c6} \hat{F}_6 \; , \\
\hat{Q}_{\text{Page, D2}} &= \frac{1}{2k_{10}^2 T_{D2}}\int_{c6} \Bigg[\hat{F}_6 - \hat{B}_2\wedge\hat{F}_4 + \frac{1}{2}\hat{B}_2\wedge\hat{B}_2\wedge\hat{F}_2 \\
& - \frac{1}{6}\,\hat{F}_0\hat{B}_2\wedge\hat{B}_2\wedge\hat{B}_2\Bigg] \; . \qquad (5.108)
\end{aligned}$$

Keeping v_2 as fixed, $\hat{F}_4$ and $\hat{F}_6$ are zero together with $\hat{F}_0$ as well as $\hat{B}_2\wedge\hat{F}_2$ vanish for the PP-wave background. Hence, from (5.107), we find that there is no longer any cascading due to the large gauge transformation of $\hat{B}_2$. This leads that the quiver theory dual to the PP-wave geometry (5.88)-(5.90) correspond to the end point of the cascade.

Chapter 6

Penrose limits in non-Abelian T-dual of AdS$_3$ background

In this chapter we will discuss the Penrose limits in T-dual of AdS$_3$ backgrounds. We will consider both Abelian as well as non-Abelian T-dual of $AdS_3 \times S^3 \times T^4$ geometry and examine the Penrose limits. The Penrose limits in Abelian T-dual geometry have been discussed in [42]. The supersymmetry discussion of the PP-wave geometry is also provided and it states that the resulting background preserves 16 supercharges. In our work [47], we consider the non-Abelian T-dual background described in [46] as well as in Chapter 2 in the thesis and inspect Penrose limits in the dualized geometry. In the following first we will revisit the work in [42] subsequently discuss our work [47] for the non-Abelian T-dual solution.

6.1 Penrose limits in Abelian T-dual background

In this section we will discuss the Penrose limits in the Abelian T-dual of $AdS_3 \times S^3 \times T^4$ background. The T-dual geometry together with the background fields are presented in Chapter 2 in the thesis. The Abelian T-duality acts along the fibre direction in S^3 as discussed in [42]. In the following we will examine the Penrose limits in the resulting geometry.

First recall the T-dual geometry as presented in Chapter 2 in the thesis. The metric of the geometry is given by

$$\begin{aligned} ds^2 = \;& R^2\Big(-\cosh^2\rho\; dt^2 + d\rho^2 + \sinh^2\rho\; d\varphi^2\Big) + \frac{R^2}{4}\Big(\cos^2\bar{\theta} d\bar{\psi}^2 + d\bar{\theta}^2\Big) \\ & + \frac{4\alpha'^2}{R^2} d\bar{\chi}^2 + \varsigma\Big(dx_6^2 + dx_7^2 + dx_8^2 + dx_9^2\Big) \end{aligned} \tag{6.1}$$

The NS-NS sector of the background is described by a non-zero dilaton and NS-NS three-form flux as

$$\begin{aligned} e^{\Phi'} = \;& \frac{2R}{g l_s N_5}\,, \\ H_3 = \;& -\alpha' \cos\bar{\theta}\; d\bar{\psi} \wedge d\bar{\theta} \wedge d\bar{\chi}\,. \end{aligned} \tag{6.2}$$

The RR sector of the background is described by a two-form field strengths together with a four-form field strengths

$$\begin{aligned} F_2 = & \ -\frac{1}{4} g l_s N_5 \cos\bar\theta \ d\bar\psi \wedge d\bar\theta \ , \\ F_4 = & \ -2 g l_s \alpha' N_5 \cosh\rho \sinh\rho \ dt \wedge d\rho \wedge d\varphi \wedge d\bar\chi \ . \end{aligned} \tag{6.3}$$

Now we inspect the Penrose limits in the background (6.1)-(6.3). To find the null geodesic consider the motion of a particle with very speed along $\bar\psi$-direction in the dual background. The null geodesic resides at $\bar\theta = 0$ and at the origin of AdS$_3$ geometry, $\rho = 0$. Consider the following expansion around the null geodesic $\rho = \bar\theta = 0$:

$$t = x^+ + \frac{x^-}{R^2} \ ; \ \frac{\bar\psi}{2} = x^+ - \frac{x^-}{R^2} \ ; \ \rho = \frac{z}{R} \ ; \ \frac{\bar\theta}{2} = \frac{\bar x_4}{2} \ ; \ \bar\chi = \frac{R \bar x_5}{2\alpha'} \ ; \ \varsigma^{\frac{1}{2}} x_i = \bar x_i \ , \tag{6.4}$$

and subsequently take $R \to \infty$ limit. In the leading order of the expansion the background metric provides

$$ds^2 = -4dx^+dx^- - \Big(\bar x_2^2 + \bar x_3^2 + \bar x_2^2 + 4\bar x_4^2\Big) dx^{+2} + \sum_{i=2}^{9} d\bar x_i^2 \ . \tag{6.5}$$

In the limit, the background dilaton vanishes and the NS-NS three-form flux takes the form

$$H_3 = 2dx^+ \wedge d\bar x_5 \wedge d\bar x_4 \ . \tag{6.6}$$

The supersymmetry analysis for the above background is carried out in [42]. The vanishing of transverse components of the gravitino projection states that the Killing Spinor ϵ is linear in both along one light-cone direction x^+ and the transverse coordinates $\bar x_i$. Subject to dilatino variation and the periodicity condition of the light-cone coordinate x^+, it is shown that the PP-wave geometry (6.5)-(6.6) preserves 16 supercharges [42].

6.2 Penrose limits in non-Abelian T-dual background

We will now turn our attention to the Penrose limits for the non-Abelian T-dual background. First we recall the T-dual geometry along with the background fields as

$$d\hat s^2_{\mathrm{NATD}} = 4L^2 ds^2\big(AdS_3\big) + L^2 ds^2\big(T^4\big) + \frac{L^2}{4} d\rho^2 + \frac{L^2\rho^2}{4+\rho^2} \ d\Omega_2^2\big(\theta, \phi\big) \ , \tag{6.7}$$

In the above metric we rescaled the ρ-coordinate as $\rho \to L^2\rho$ to get L^2 common factor in the metric.

Due to the rescaling of the radial coordinate in the T- dual metric, the NS-NS sector of the background takes form

$$e^{-2\hat\Phi} = \frac{L^6}{4g_s^2}\big(4+\rho^2\big) \ , \ \hat B_2 = -\frac{L^2\rho^3}{2\big(4+\rho^2\big)} \sin\theta d\theta \wedge d\phi \ , \tag{6.8}$$

similarly the field strengths in the RR sector become

$$\hat{F}_0 = \frac{L^2}{g_s} \ , \ \hat{F}_2 = -\frac{L^4\rho^3}{2g_s\big(4+\rho^2\big)} \sin\theta d\theta \wedge d\phi \ , \ \hat{F}_4 = -\frac{L^6}{g_s}\mathrm{Vol}\big(T^4\big) \ . \tag{6.9}$$

In the following we will consider the motion of a particle of various isometric directions in the T-dual geometry (6.7) and examine Penrose limits. It is straightforward to see that z_i's along the T^4 manifold are the symmetric directions. However, the geodesic condition (4.6) for them is trivially satisfied. Hence we will not consider motion along the T^4 directions.

Now we turn our focus to the other symmetric direction in the T-dual geometry. The background has $U(1)$-isometry along ϕ-direction. The relevant component of the metric is given by

$$g_{\phi\phi} = \frac{L^2\rho^2}{4+\rho^2}\sin^2\theta \ . \tag{6.10}$$

The metric component has a nontrivial dependence on ρ and θ coordinates. The geodesic equation (4.6) for $\mu = \rho$ gives $\rho = 0$ and $\theta = \{0, \pi\}$. Similarly for $\mu = \theta$, we get $\rho = 0$ and $\theta = \{0, \frac{\pi}{2}, \pi\}$. Among these values, we will not consider $\rho = 0$ and $\theta = \{0, \pi\}$ as for these values the $g_{\phi\phi}$ component vanishes. This leads us to the only possibility of considering the motion of a particle carrying nonzero angular momentum in the (ρ, ϕ) plane. To find such a geodesic we confine our discussion to the neighbourhood of $r = 0 = z_i$ and $\theta = \frac{\pi}{2}$. Considering the same Lagrangian for a massless particle as in (4.10) and substituting the background metric (6.7) we obtained

$$\mathcal{L} = \frac{L^2}{2}\bigg(-4\dot{t}^2 + \frac{1}{4}\dot{\rho}^2 + \frac{\rho^2}{4+\rho^2}\dot{\phi}^2\bigg) \ . \tag{6.11}$$

The above Lagrangian does not depend on the coordinates t and ϕ. Hence the corresponding generalized momenta will be conserved. First consider the momentum conjugate to t as

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = -4L^2\dot{t} \ . \tag{6.12}$$

Thus, we have $\dot{t} = \mathrm{const}$. Set the affine parameter u appropriately we have $\dot{t} = 1$. Now consider the equation of motion of the generalized coordinate ϕ

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = L^2\frac{\rho^2}{4+\rho^2}\dot{\phi} = \mathrm{const} \ . \tag{6.13}$$

Let J be the angular momentum associated with the motion along ϕ-direction we have

$$J = -\frac{\rho^2}{4+\rho^2}\dot{\phi} \ . \tag{6.14}$$

Imposing the geodesics to be null *i.e.* $\mathcal{L} = 0$, we obtain the equation for ρ-coordinate as

$$\dot{\rho}^2 = 4\bigg(4 - \frac{4+\rho^2}{\rho^2}J^2\bigg) \ . \tag{6.15}$$

This equation can be solved exactly and we obtain in a simple form

$$\rho^2 = \frac{4J^2 + 4\big(J^2-4\big)^2\big(u+c_\rho\big)^2}{4-J^2} \ , \tag{6.16}$$

here c_ρ is the integration constant. One can set it to zero by redefining the parameter u.

Now in the following we obtain the Penrose limit for the above null geodesic around $r = 0 = z_i$ and $\theta = \frac{\pi}{2}$ with carrying angular momentum J. Before going to Penrose expansion, we redefine the background coordinates as

$$r = \frac{\bar{r}}{L} \ , \ z_i = \frac{y_i}{L} \ ; \ i = 1,2,3,4 \ , \quad \theta = \frac{\pi}{2} + \frac{x}{L} \ . \tag{6.17}$$

We also rescale the string coupling as $g_s = L^3\tilde{g}_s$, in order to keep the dilaton finite in this limit. Finally, we consider the following expansion

$$dt = c_1 du \ , \ d\phi = c_2 du + c_3\frac{dw}{L} + c_4\frac{dv}{L^2} \ , \ d\rho = c_5 du + c_6\frac{dw}{L} \ , \tag{6.18}$$

and subsequently take $L \to \infty$ limit.

Now we will solve the coefficients c_i in the above expansion. The null geodesic condition determines three of the coefficients as follows

$$c_1 = 1 \ , \ c_2 = -\frac{4+\rho^2}{\rho^2}J \ , \ c_5 = 2\left[4 - \frac{\big(4+\rho^2\big)}{\rho^2}J^2\right]^{\frac{1}{2}} \ . \tag{6.19}$$

Now we substitute the expansion (6.18) in the T-dual metric (6.7) and take the $L \to \infty$ limit. In the expansion, at first it contains diverging terms of order $\mathcal{O}(L^2)$. By imposing the null geodesic condition, they cancel each other. The expansion also contains the order of $\mathcal{O}(L)$ terms. Those can be ruled away upon setting the coefficients c_i as

$$c_5 c_6 + \frac{4\rho^2}{4+\rho^2}\ c_2 c_3 = 0 \ . \tag{6.20}$$

Now substitute the values of c_2 and c_5 from (6.19) in above we find

$$\frac{c_3}{c_6} = \frac{1}{2J}\left[4 - \frac{\big(4+\rho^2\big)}{\rho^2}J^2\right]^{\frac{1}{2}} \ . \tag{6.21}$$

Later we will show that the coefficient c_3 can be determined from Einstein's equations in type-IIA supergravity. The only leftover coefficient c_4 is fixed by considering the appropriate normalization for the cross-term $dudv$ and it leads to $c_4 = -\frac{1}{J}$.

Now substitute the above results in the background metric (6.7) the leading terms provide

$$\begin{aligned} ds^2 = \ & 2dudv + 4d\bar{r}^2 + 4\bar{r}^2 d\chi^2 + dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2 + \left(\frac{c_6^2}{4} + \frac{\rho^2}{4+\rho^2}c_3^2\right)dw^2 \\ & + \frac{\rho^2}{4+\rho^2}dx^2 - \left(4\bar{r}^2 + \frac{4+\rho^2}{\rho^2}J^2x^2\right)du^2 \ . \end{aligned} \tag{6.22}$$

Now we will consider the Penrose limit for the remaining background fields in the NS-NS sector as well as in the RR sector. In this limit the NS-NS two-form $\hat{B}_2$ and the dilaton takes the form

$$\begin{aligned}\hat{B}_2 =& \quad \frac{c_3\rho^3}{2\big(4+\rho^2\big)}\ dw\wedge dx\ ,\\ e^{-2\hat{\Phi}} =& \quad \frac{1}{4\tilde{g}_s^2}\big(4+\rho^2\big)\ .\end{aligned} \tag{6.23}$$

The NS-NS three-form field strength corresponding to the two-form $\hat{B}_2$ in this limit becomes

$$\hat{H}_3 = \rho\ \frac{\sqrt{4\rho^2-\big(4+\rho^2\big)J^2}}{\big(4+\rho^2\big)^2}\ \Big[\big(12+\rho^2\big)c_3+\rho\big(4+\rho^2\big)c_3'\Big]\ du\wedge dw\wedge dx\ , \tag{6.24}$$

In the above we have used $d\rho = c_5 du$ and c_5 is given by (6.19). In the Penrose limits, the only RR two-form field strength is non-vanishing and given by the expression

$$\hat{F}_2 = \frac{J\rho}{2\tilde{g}_s}\ dx\wedge du\ . \tag{6.25}$$

The metric in (6.22) is not in the standard Brinkmann form [33]. In the following we will transform it into the Brinkmann form by following [26] and also presented in (4.41)-(4.44). For the pp-wave geometry in (6.22) we have the following A_i's

$$A_{\bar{r}} = 4\ ,\ \ A_w = \frac{c_6^2}{4}+\frac{\rho^2}{4+\rho^2}c_3^2\ ,\ \ A_x = \frac{\rho^2}{4+\rho^2}\ . \tag{6.26}$$

Now we redefine the coordinates in (6.22) as

$$\begin{aligned}&\bar{r}\to\frac{\bar{r}}{\sqrt{A_{\bar{r}}}}\ ,\ \ w\to\frac{w}{\sqrt{A_w}}\ ,\ \ x\to\frac{x}{\sqrt{A_x}}\quad \text{and}\\ &v\to v+\frac{1}{4}\left[\frac{\dot{A}_w}{A_w}\ w^2+\frac{\dot{A}_x}{A_x}\ x^2\right]\ ,\end{aligned} \tag{6.27}$$

then we get

$$\begin{aligned}ds^2 =& \quad 2dudv+d\bar{r}^2+\bar{r}^2d\chi^2+dy_1^2+dy_2^2+dy_3^2+dy_4^2+dw^2+dx^2\\ &-\left[\bar{r}^2+\left(\frac{(4+\rho^2)^2}{\rho^4}J^2-F_x\right)x^2-F_w w^2\right]du^2\ ,\end{aligned} \tag{6.28}$$

where functions F_i are determined using the expression (4.44).

Now we will express the background fields in the Brinkmann coordinates. The NS-NS fields take the form

$$\begin{aligned}e^{-2\hat{\Phi}} =& \quad \frac{1}{4\tilde{g}_s^2}\big(4+\rho^2\big)\ ,\\ \hat{H}_3 =& \quad \frac{\sqrt{4\rho^2-\big(4+\rho^2\big)J^2}}{\big(4+\rho^2\big)^{\frac{3}{2}}}\ \Big[\big(12+\rho^2\big)c_3+\rho\big(4+\rho^2\big)c_3'\Big]\left[\frac{c_6^2}{4}+\frac{\rho^2}{4+\rho^2}c_3^2\right]^{-\frac{1}{2}}\\ &\quad du\wedge dw\wedge dx\ .\end{aligned} \tag{6.29}$$

Similarly, the RR two-form RR field strength in Brinkmann coordinates becomes

$$\hat{F}_2 = \frac{J}{2\tilde{g}_s}\sqrt{4+\rho^2}\ dx\wedge du\ . \tag{6.30}$$

Now in the following we show that the above background indeed satisfies the equation of motions and Bianchi identities in type-IIA supergravity. The background fields (6.29)-(6.30) for the PP-wave geometry show that the NS-NS three form field strength $\hat{H}_3$ together with $\hat{F}_2$ are closed. Also we have $\hat{F}_0 = 0 = \hat{F}_4$. Thus the Bianchi identities (A) are trivially satisfied.

Now consider the Hodge duals of the field strengths $\hat{H}_3$ and $\hat{F}_2$ as follows:

$$\begin{aligned}\star\ \hat{H}_3 =& \ \frac{\sqrt{4\rho^2-\left(4+\rho^2\right)J^2}}{\left(4+\rho^2\right)^{\frac{3}{2}}}\left[\left(12+\rho^2\right)c_3+\rho\left(4+\rho^2\right)c_3'\right]\left[\frac{c_6^2}{4}+\frac{\rho^2}{4+\rho^2}c_3^2\right]^{-\frac{1}{2}}\\ & du\wedge d\Omega_2\wedge dy_1\wedge dy_2\wedge dy_3\wedge dy_4\ ,\\ \star\ \hat{F}_2 =& \ \frac{J}{2\tilde{g}_s}\sqrt{4+\rho^2}\ du\wedge d\Omega_2\wedge dy_1\wedge dy_2\wedge dy_3\wedge dy_4\wedge dw\ .\end{aligned} \tag{6.31}$$

Using the above expression it can be shown $\star\hat{H}_3$, $\left(e^{-2\hat{\Phi}}\star\hat{H}_3\right)$ as well as $\star\ \hat{F}_2$ are all closed. Hence, the gauge field equations (A) are satisfied indeed. In order to fixed the undetermined coefficient c_3 we consider the Einstein's equations for type IIA supergravity:

$$\hat{R}_{\mu\nu}+2D_\mu D_\nu\hat{\Phi} = \frac{1}{4}\hat{H}^2_{\mu\nu}+e^{2\hat{\Phi}}\left[\frac{1}{2}(\hat{F}_2^2)_{\mu\nu}+\frac{1}{12}(\hat{F}_4^2)_{\mu\nu}-\frac{1}{4}g_{\mu\nu}\left(\hat{F}_0^2+\frac{1}{2}\hat{F}_2^2+\frac{1}{4!}\hat{F}_4^2\right)\right]\ . \tag{6.32}$$

together with the dilaton equation

$$\hat{R}+4D^2\hat{\Phi}-4(\partial\hat{\Phi})^2-\frac{1}{12}\hat{H}^2 = 0\ . \tag{6.33}$$

In the Brinkmann form the only non-vanishing component of the Ricci tensor of PP-wave geometry is $\hat{R}_{uu}$ and it follows the equation

$$\hat{R}_{uu}+2D_uD_u\hat{\Phi} = \frac{1}{4}\hat{H}^2_{uu}+\frac{1}{2}e^{2\hat{\Phi}}\left(\hat{F}_2^2\right)_{uu}\ . \tag{6.34}$$

This indeed provides a nontrivial constraint involving the coefficient c_3 due to the presence of $\hat{H}^2_{uu}$ and the coefficient c_3 can be determined by solving this equation.

Now we will discuss the number of supersymmetries preserved by the pp-wave background studied in (6.28)-(6.30). As like the earlier case, first we introduce the Brinkmann coordinates X^i as follows

$$\begin{aligned}& d\bar{r}^2+\bar{r}^2d\chi^2 = \left(dX^i\right)^2\ ;\ i=1,2\ ,\ y_i = X^i\ ;\ i=3,4,5,6\ ,\\ & w = X^7\ ,\ x = X^8\ .\end{aligned} \tag{6.35}$$

In these X^i coordinates, the PP-wave background in (6.28)-(6.30) becomes

$$\begin{aligned}
ds^2 =& \quad 2dudv + \sum_{i=1}^{8} dX_i^2 + \mathcal{H}\ du^2 \ , \\
\hat{\Phi} =& \quad \Phi(u) \ , \\
\hat{H}_3 =& \quad f_1(u)\ du \wedge dX^7 \wedge dX^8 \ , \\
\hat{F}_2 =& \quad f_2(u)\ dX^8 \wedge du \ .
\end{aligned} \tag{6.36}$$

where we have introduced the notations

$$\begin{aligned}
\Phi(u) =& \quad \frac{1}{2} \ln \left[\frac{4\tilde{g}_s^2}{4+\rho^2} \right] \ , \\
\mathcal{H} =& \quad \sum_{i,j=1}^{8} F_{ij} X^i X^j \ ; \ F_{ij} = F_{ji} \ , \\
f_1(u) =& \quad \frac{\sqrt{4\rho^2 - \left(4+\rho^2\right)J^2}}{\left(4+\rho^2\right)^{\frac{3}{2}}} \left[\left(12+\rho^2\right)c_3 + \rho\left(4+\rho^2\right)c_3' \right] \left[\frac{c_6^2}{4} + \frac{\rho^2}{4+\rho^2} c_3^2 \right]^{-\frac{1}{2}} \ , \\
f_2(u) =& \quad \frac{J}{2\tilde{g}_s} \sqrt{4+\rho^2} \ .
\end{aligned} \tag{6.37}$$

The functions F_{ij} are given by the expressions

$$\begin{aligned}
& F_{11} = F_{22} = -1 \ , \\
& F_{77} = F_w \ , \ F_{88} = F_x - \frac{(4+\rho^2)^2}{\rho^4} J^2 \ .
\end{aligned} \tag{6.38}$$

Assigning the same frame notation as in (4.58), here we obtained

$$\begin{aligned}
\hat{\Phi} =& \quad \Phi(u) \ , \\
\hat{H}_3 =& \quad f_1(u)\ e^- \wedge e^7 \wedge e^8 \ , \\
\hat{F}_2 =& \quad f_2(u)\ e^8 \wedge e^- \ .
\end{aligned} \tag{6.39}$$

Let us now focus on the supersymmetric variations for the dilatino and gravitino as

$$\begin{aligned}
\delta\hat{\lambda} =& \frac{1}{2}\not{\partial}\hat{\Phi}\hat{\epsilon} - \frac{1}{24}\not{\hat{H}}\sigma_3\hat{\epsilon} \ + \frac{3}{16} e^{\hat{\Phi}} \not{\hat{F}}_2 \left(i\sigma_2\right) \hat{\epsilon} \ , \\
\delta\hat{\psi}_\mu =& D_\mu\hat{\epsilon} - \frac{1}{8}\hat{H}_{\mu\nu\rho}\Gamma^{\nu\rho}\sigma_3\hat{\epsilon} + \frac{1}{16} e^{\hat{\Phi}} \not{\hat{F}}_2 \left(i\sigma_2\right) \Gamma_\mu\hat{\epsilon} \ .
\end{aligned} \tag{6.40}$$

Considering the same convention as like in $AdS_5 \times S^5$ case and setting the dialton variation to zero we find

$$\Gamma^- \left[\dot{\hat{\Phi}} - \frac{1}{2} f_1(u)\Gamma^{78}\sigma_3 - \frac{3e^{\hat{\Phi}}}{4}\ f_2(u)\Gamma^8 \left(i\sigma_2\right) \right] \hat{\epsilon} = 0 \ . \tag{6.41}$$

The above leads to the solution $\Gamma^{-}\hat{\epsilon} = 0$. This states that the pp-wave solution (6.36) we studied here preserves 16 supercharges. We will now proceed to solve the spinor conditions for the gravitino variation. We need to solve $\delta\hat{\psi}_{\mu} = 0$ for $\mu = +, -, i$. Now from the NS-NS three form flux, we have $\hat{H}_{+\mu\nu} = 0$ for all μ, ν. Considering this and together with the condition $\Gamma_{+}\hat{\epsilon} = \Gamma^{-}\hat{\epsilon} = 0$, the variation $\delta\hat{\psi}_{+} = 0$ provides that the Killing spinor $\hat{\epsilon}$ is independent of the light cone coordinate v, *i.e.*, $\partial_{+}\hat{\epsilon} = 0$. Thus, we have $\hat{\epsilon} = \hat{\epsilon}(u, X^{i})$.

Now we consider the variations $\delta\hat{\psi}_{i}$, $i = 1, ..., 8$. Set it to zero, we find

$$\partial_{i}\hat{\epsilon} = \Gamma^{-}\mathcal{R}\ \hat{\epsilon}\ , \tag{6.42}$$

where the notation $\mathcal{R}$ is given by

$$\mathcal{R} = \frac{1}{4}f_{1}(u)\Big(\delta_{i8}\Gamma^{7} - \delta_{i7}\Gamma^{8}\Big)\sigma_{3} + \frac{e^{\hat{\Phi}}}{8}f_{2}(u)\Gamma^{8}\big(i\sigma_{2}\big)\Gamma^{i}\ . \tag{6.43}$$

Now it can be shown that Γ^{-} anticommutes with $\mathcal{R}$ and we also have $\Gamma^{-}\hat{\epsilon} = 0$. Then the above provides $\hat{\epsilon} = \chi(u)$ for some $\chi(u)$ such that $\Gamma^{-}\chi(u) = 0$.

Finally solving the condition $\delta\hat{\psi}_{-} = 0$ we find

$$\partial_{u}\chi(u) - \frac{1}{4}f_{1}(u)\Gamma^{78}\sigma_{3}\chi(u) + \frac{e^{\hat{\Phi}}}{4}f_{2}(u)\Gamma^{8}\big(i\sigma_{2}\big)\chi(u) = 0\ . \tag{6.44}$$

Introducing the matrix $\mathcal{M}(u)$ as

$$\mathcal{M}(u) = \frac{1}{4}\Big(f_{1}(u)\Gamma^{78}\sigma_{3} - e^{\hat{\Phi}}f_{2}(u)\Gamma^{8}\big(i\sigma_{2}\big)\Big) \tag{6.45}$$

the above will take simplified form

$$\partial_{u}\chi(u) - \mathcal{M}(u)\chi(u) = 0\ . \tag{6.46}$$

This equation can be integrated and lead to $\chi(u) = e^{\int du\mathcal{M}(u)}\chi_{0}$. Hence, the supersymmetry analysis shows that the PP-wave background (6.36)-(6.38) preserves 16 supercharges.

6.3 Field theory dual

The dual field theory corresponds to the non-Abelian T-dual background has been studied in [46]. The dual field theory is based on the intersecting branes and the holographic central charge matches with the field-theoretic central charge discussed in [46]. The BMN operators corresponding to PP-wave background belong to a class of operator resides in the quiver gauge theory. In the following we will consider the brane charges for the PP-wave background discussed in (6.28)-(6.30). In type IIA supergravity the expressions of Page charges of various

D-branes are given by [25]

$$\begin{aligned}
\hat{Q}_{\text{page},D6} &= \frac{1}{2\kappa_{10}^2 T_{D6}} \int_{C_2} \hat{F}_2 - \hat{F}_0 \hat{B}_2 \ , \\
\hat{Q}_{\text{page},D4} &= \frac{1}{2\kappa_{10}^2 T_{D4}} \int_{C_4} \hat{F}_4 - \hat{B}_2 \wedge \hat{F}_2 + \frac{1}{2} \hat{F}_0 \hat{B}_2 \wedge \hat{B}_2 \ , \\
\hat{Q}_{\text{page},D2} &= \frac{1}{2\kappa_{10}^2 T_{D2}} \int_{C_6} \hat{F}_6 - \hat{B}_2 \wedge \hat{F}_4 + \frac{1}{2} \hat{B}_2 \wedge \hat{B}_2 \wedge \hat{F}_2 - \frac{1}{6} \hat{F}_0 \hat{B}_2 \wedge \hat{B}_2 \wedge \hat{B}_2 \ , \\
\hat{Q}_{\text{page},D8} &= \frac{1}{2\kappa_{10}^2 T_{D8}} \int \hat{F}_0 \ ,
\end{aligned} \tag{6.47}$$

and the Maxwell charges are given by [23]

$$\begin{aligned}
\hat{Q}_{\text{Max},D6} &= \frac{1}{\sqrt{2}\pi^2} \int_{C_2} \hat{F}_2 \ , \\
\hat{Q}_{\text{Max},D4} &= \frac{1}{\sqrt{2}\pi^2} \int_{C_4} \hat{F}_4 \ , \\
\hat{Q}_{\text{Max},D2} &= \frac{1}{\sqrt{2}\pi^2} \int_{C_6} \hat{F}_6 \ , \\
\hat{Q}_{\text{Max},D8} &= \sqrt{2} \int \hat{F}_0 \ .
\end{aligned} \tag{6.48}$$

Here C_n is the n-cycle admits by the corresponding geometry. The background also carries $NS5$ branes with charge

$$\hat{Q}_{NS5} = \frac{1}{4\pi\alpha'} \int_{C_3} \hat{H}_3 \ . \tag{6.49}$$

Substituting the values of the field strengths for the PP-wave solution in (6.28)-(6.30), we find the non-vanishing charges

$$\begin{aligned}
\hat{Q}_{\text{page},D6} = \ & \hat{Q}_{\text{Max},D6} = \frac{1}{2\kappa_{10}^2 T_{D6}} \int_{C_2} \hat{F}_2 \ , \\
\hat{Q}_{NS5} = \ & \frac{1}{4\pi\alpha'} \int_{C_3} \hat{H}_3 \ ,
\end{aligned} \tag{6.50}$$

This indicates that at the Penrose limit we have only $D6$ and $NS5$ branes. Hence, The BMN operators corresponding to the holographic dual of the pp-wave geometry will reside in this quiver theory governed by intersecting configuration of $D6$ and $NS5$ branes.

Chapter 7

Heterotic Double Field Theory, Generalized Kerr-Schild and its double Yang-Mills formulation

In this Chapter we will discuss our work [57] on the Heterotic version of the Double Field Theory (DFT). First we will review the basic construction of Heterotic Double Field Theory and then we will study the Kerr-Schild ansatz in this context. It has been quite well known for a long time that the Abelian T-duality is an exact symmetry of string theory of all order of the expansion parameter α' [9]. However, this symmetry does not manifest from the spacetime action of any versions of the superstring theories. In the framework of Double Field Theory, the low energy version of string theories namely known as supergravity limit can be written in a duality invariant fashion [48,49,64,65,66,67]. In our work and in this thesis we consider the low energy limit of the heterotic string and its formulation in the framework of double field theory. The canonical approach to study heterotic DFT was discussed in detail in [51]. It is shown that the construction is based on the D-dimensional metric tensor g, NS-NS two form field B_2 field and a non-Abelian gauge field A *i.e.* $\mathcal{H} = \mathcal{H}(g, B_2, A)$ [68]. Here $\mathcal{H}$ is known as generalized metric which is a multiplet of the corresponding symmetry group in the theory. In addition to the $\mathcal{H}$, the theory is contained with the generalized dilaton d and these are the fundamental fields of heterotic DFT. The frame formalism of heterotic DFT was introduced in [69], where the associated generalized frame has field dependence $\mathcal{E} = \mathcal{H}(g, B_2, A)$

In DFT, the generalized metric formulation and generalized frame formulation are not always equivalent [70]. As like in general relativity, the generalized frame contains extra degrees of freedom leads to a gauge fixing procedure that some components of this frame field are fixed and extra conditions appear if/when, for instance, one desires to work considering perturbations around a background. One simple and powerful proposal to explore this idea is the generalized Kerr-Schild ansatz (GKSA). For the case of ordinary DFT, GKSA was introduced in [53] and its heterotic version was discussed in [54]. In both cases, the ansatz describes a linear perturbation around $\mathcal{H}$ or $\mathcal{E}$ by considering a pair of null vectors together with an arbitrary perturbation for the generalized dilaton d. Upon parametrization at the supergravity level, the ansatz gives rise to the perturbations around the background g_o along with the NS-NS two-form gauge field B_o and the background gauge field, A_o as follows

$$
\begin{aligned}
g_{\mu\nu} &= g_{o\mu\nu} - \frac{\kappa}{1+\frac{1}{2}\kappa l.\bar{l}} l_{(\mu}\bar{l}_{\nu)} &\quad (7.1)\\
b_{\mu\nu} &= b_{o\mu\nu} - \frac{\kappa}{1+\frac{1}{2}\kappa l.\bar{l}} l_{[\mu}(\bar{l}_{\nu]} - \frac{1}{\sqrt{2}} j_i A_{\nu]}{}^i) &\quad (7.2)\\
A_{\mu i} &= A_{o\mu i} + \frac{1}{\sqrt{2}} \frac{\kappa}{1+\frac{1}{2}\kappa l.\bar{l}} l_\mu j_i\,, &\quad (7.3)
\end{aligned}
$$

where κ is the order parameter of the expansion, l is a null vector and $\bar{l}$ and j satisfy the condition

$$
\bar{l}^2 + j^2 = 0\,. \quad (7.4)
$$

Both l and $\bar{l}$ are pair of null geodesics and together with j satisfy geodesic equations. The above relations simplify the dynamics partially and can be easily imposed at the DFT level [54]. Interestingly, the perturbations in (7.1)-(7.3), along with the perturbation of the background dilaton, integrate a family of theories that can be described using the GKSA. Such like the charged black string discussed in [71] and the charged dilaton black hole geometry discussed in [72], both of them studied in detail by considering the generalized metric formalism [54]. The case $j = 0$ and $l = \bar{l}$ is described by the ordinary Kerr-Schild ansatz [73].

However, the family of theories that can be studied using the GKSA in the generalized frame formalism of DFT is slightly smaller than the theories that can be described in the generalized metric formulation, since the gauge fixing condition forces $j = 0$, and then both l and $\bar{l}$ are null vectors. In the following first we rewrite the heterotic DFT in a double Yang-Mills forms by following [74]. We write the $O(D, D+K)$ multiplets in heterotic DFT in terms of $O(D, D)$ multiplets. The advantage of this formulation is that it in a generalized gauge field at the DFT level and one can study the perturbations of this field before parametrization, irrespective of the fundamental fields of the theory. In order to study perturbations we consider the relaxed version of GKSA introduced in [58]. At the very end, we discuss the double copy correspondence in this formulation.

7.1 Review of Heterotic Double Field Theory and Generalized Kerr-Schild Ansatz

In this section we discuss the basic notions of Heterotic Double Field Theory and its double Yang-Mills formulation. The Heterotic DFT is defined on double spacetime coordinates: $X^{\mathcal{M}}$; where $\mathcal{M} = 0, \ldots, D-1+K$, and K is the dimension of the corresponding heterotic gauge group. The $X^{\mathcal{M}}$ transforms under the fundamental representation of the underlying symmetry group $G = O(D, D+K)$. The theory is invariant under a global G symmetry which reads

$$\delta_G V_{\mathcal{M}} = V_{\mathcal{N}} h^{\mathcal{N}}{}_{\mathcal{M}} \,, \tag{7.5}$$

$V_{\mathcal{M}}$ is a generic G-multiplet and $h \in O(D, D+K)$ is the G-parameter. The invariant metric of G is $\eta_{\mathcal{M}\mathcal{N}} \in G$. The G-invariance constraint the $h_{\mathcal{M}\mathcal{N}}$ as

$$h_{\mathcal{M}\mathcal{N}} = -h_{\mathcal{N}\mathcal{M}} \,. \tag{7.6}$$

In the theory the Infinitesimal generalized diffeomorphisms are governed by the generalized version of the Lie derivative and the transformation takes the form

$$\hat{\mathcal{L}}_\xi V_{\mathcal{M}} = \xi^{\mathcal{N}} \partial_{\mathcal{N}} V_{\mathcal{M}} + (\partial_{\mathcal{M}} \xi^{\mathcal{N}} - \partial^{\mathcal{N}} \xi_{\mathcal{M}}) V_{\mathcal{N}} + f_{\mathcal{M}\mathcal{N}\mathcal{P}} \xi^{\mathcal{N}} V^{\mathcal{P}} + t \partial_{\mathcal{M}} \xi^{\mathcal{M}} \,, \tag{7.7}$$

here $V_{\mathcal{M}}$ is an arbitrary generalized tensor in the theory, t is a weight constant of the tensor and $f_{\mathcal{M}\mathcal{N}\mathcal{P}}$ is the generalized structure constants that satisfy the anti-symmetric conditions

$$f_{\mathcal{M}\mathcal{N}\mathcal{P}} = f_{[\mathcal{M}\mathcal{N}\mathcal{P}]} \,, \qquad f_{[\mathcal{M}\mathcal{N}}{}^{\mathcal{R}} f_{\mathcal{P}]R}{}^{\mathcal{Q}} = 0 \,. \tag{7.8}$$

The strong constraint is given by

$$\begin{aligned} \partial_{\mathcal{M}} A \, \partial^{\mathcal{M}} B &= 0 \,, && (7.9) \\ \partial_{\mathcal{M}} (\partial^{\mathcal{M}} A) &= 0 \,, && (7.10) \\ f^{\mathcal{M}\mathcal{N}\mathcal{P}} \partial_{\mathcal{M}} B &= 0 \,. && (7.11) \end{aligned}$$

These conditions indeed to ensure the closure of the generalized diffeomorphisms. Here A and B are generic fields or parameters that appear in the heterotic DFT.

For a vector field $V_{\mathcal{N}}$, the covariant derivative takes the form

$$\nabla_{\mathcal{M}} V_{\mathcal{N}} = \partial_{\mathcal{M}} V_{\mathcal{N}} - \Gamma_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} V_{\mathcal{P}} \,, \tag{7.12}$$

where $\Gamma_{\mathcal{M}\mathcal{N}\mathcal{P}}$ is a generalized affine connection. The metric compatibility in this framework is given by the expression

$$\nabla_{\mathcal{M}} \mathcal{H}_{\mathcal{N}\mathcal{P}} \;=\; 0 \,, \quad \nabla_{\mathcal{M}} \eta_{\mathcal{N}\mathcal{P}} = 0 \,, \tag{7.13}$$

which gives rise

$$\Gamma_{[\mathcal{M}\mathcal{N}\mathcal{P}]} = 0 \,, \tag{7.14}$$

the above determines some projections of the generalized affine connection [75].

The theory is also invariant under the local double Lorentz $\mathcal{H} = O(D-1,1)_L \times O(1, D-1+K)_R$ symmetry governed by the generalized parameter $\Gamma_{\mathcal{A}\mathcal{B}}$ where $\mathcal{A} = (\underline{\mathcal{A}}, \overline{\mathcal{A}})$ splitting into $O(D-1,1)_L$ and $O(1, D-1+K)_R$ vector indices, $\underline{\mathcal{A}} = \underline{A} = 0, \ldots, D-1$ and $\overline{A} = (\overline{A}, \bar{i}) = 0, \ldots, D-1+K$, *i*.e.,

$$\delta_{\mathcal{H}} V_{\mathcal{A}} = V_{\mathcal{B}} \Gamma^{\mathcal{B}}{}_{\mathcal{A}} \,, \tag{7.15}$$

where $\mathcal{H}$ is a generic vector. The $\mathcal{H}$-invariance of $\eta_{\mathcal{AB}}$ leads to $\Gamma_{\mathcal{AB}} = -\Gamma_{\mathcal{BA}}$.

In terms of generalized frame picture, the fundamental fields of the theory contained a generalized frame $\mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}}$ and a generalized background dilaton d. The frame-formulation of DFT leads to the existence of two constant, symmetric and invertible $\mathcal{H}$-invariant metrics $\eta_{\mathcal{AB}}$ and $\mathcal{H}_{\mathcal{AB}}$. The first one is used to raise and lower the indices that are rotated by $\mathcal{H}$ and the latter one is constrained according to

$$\mathcal{H}_{\mathcal{A}}{}^{\mathcal{C}}\mathcal{H}_{\mathcal{C}}{}^{\mathcal{B}} = \delta^{\mathcal{B}}_{\mathcal{A}} . \tag{7.16}$$

In addition, the generalized frame is constrained to relate the metrics $\eta_{\mathcal{AB}}$ and $\eta_{\mathcal{MN}}$ and defines a generalized metric $\mathcal{H}_{\mathcal{MN}}$ from $\mathcal{H}_{\mathcal{AB}}$

$$\eta_{\mathcal{AB}} = \mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}\eta_{\mathcal{MN}}\mathcal{E}^{\mathcal{N}}{}_{\mathcal{B}} , \quad \mathcal{H}_{\mathcal{MN}} = \mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}}\mathcal{H}_{\mathcal{AB}}\mathcal{E}_{\mathcal{N}}{}^{\mathcal{B}} . \tag{7.17}$$

As the generalized metric $\mathcal{H}_{\mathcal{MN}}$ is also an element of $O(D, D + K)$, it satisfies

$$\mathcal{H}_{\mathcal{MP}}\eta^{\mathcal{PQ}}\mathcal{H}_{\mathcal{QN}} = \eta_{\mathcal{MN}} . \tag{7.18}$$

The projector operators in DFT take the form

$$\mathcal{P}_{\mathcal{MN}} = \frac{1}{2}\left(\eta_{\mathcal{MN}} - \mathcal{H}_{\mathcal{MN}}\right) \quad \text{and} \quad \overline{\mathcal{P}}_{\mathcal{MN}} = \frac{1}{2}\left(\eta_{\mathcal{MN}} + \mathcal{H}_{\mathcal{MN}}\right) . \tag{7.19}$$

The action of a flat covariant derivative acting on a generic vector $V_{\mathcal{B}}$ is given by

$$\mathcal{D}_{\mathcal{A}}V_{\mathcal{B}} = \mathcal{E}_{\mathcal{A}}V_{\mathcal{B}} + \omega_{\mathcal{AB}}{}^{\mathcal{C}}V_{\mathcal{C}} , \tag{7.20}$$

where $\mathcal{E}_{\mathcal{A}} = \sqrt{2}\mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}\partial_{\mathcal{M}}$ and $\omega_{\mathcal{AB}}{}^{\mathcal{C}}$ is the generalized spin connection that follows

$$\omega_{\mathcal{ABC}} = -\omega_{\mathcal{ACB}} \qquad \text{and} \qquad \omega_{\mathcal{A}\overline{\mathcal{B}}\underline{\mathcal{C}}} = \omega_{\mathcal{A}\underline{\mathcal{B}}\overline{\mathcal{C}}} = 0 . \tag{7.21}$$

However unlike general relativity, in DFT we cannot fully determine a generalized spin connection, $\omega_{\mathcal{ABC}}$. Only the totally antisymmetric and trace parts of $\omega_{\mathcal{ABC}}$ can be represented in terms of generalized frame and the generalized dilaton, i.e.

$$\omega_{[\mathcal{ABC}]} = -\mathcal{E}_{[\mathcal{A}}\mathcal{E}^{\mathcal{N}}{}_{\mathcal{B}}\mathcal{E}_{\mathcal{N}\mathcal{C}]} - \frac{\sqrt{2}}{3}f_{\mathcal{MNP}}\mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}\mathcal{E}^{\mathcal{N}}{}_{\mathcal{B}}\mathcal{E}^{\mathcal{P}}{}_{\mathcal{C}} \equiv -\frac{1}{3}\mathcal{F}_{\mathcal{ABC}} , \tag{7.22}$$

$$\omega_{\mathcal{BA}}{}^{\mathcal{B}} = -\sqrt{2}e^{2d}\partial_{\mathcal{M}}(\mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}e^{-2d}) = -\mathcal{F}_{\mathcal{A}} . \tag{7.23}$$

The generalized metric formulation, the $O(D, D + K)$ invariant DFT action takes the form

$$S = \int d^{2D+K}Xe^{-2d}\mathcal{L} , \tag{7.24}$$

where,

$$\begin{aligned}\mathcal{L} \;=\;& \frac{1}{8}\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{KL}}\partial_{\mathcal{N}}\mathcal{H}_{\mathcal{KL}} - \frac{1}{2}\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{N}}\mathcal{H}^{\mathcal{KL}}\partial_{\mathcal{L}}\mathcal{H}_{\mathcal{MK}} \\ & +4\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{M}}d\partial_{\mathcal{N}}d - 2\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{N}}d - \frac{1}{2}f^{\mathcal{M}}{}_{\mathcal{NK}}\mathcal{H}^{\mathcal{NP}}\mathcal{H}^{\mathcal{KQ}}\partial_{\mathcal{P}}\mathcal{H}_{\mathcal{MQ}} \\ & -\frac{1}{12}f^{\mathcal{M}}{}_{\mathcal{KP}}f^{\mathcal{N}}{}_{\mathcal{LQ}}\mathcal{H}_{\mathcal{MN}}\mathcal{H}^{\mathcal{KL}}\mathcal{H}^{\mathcal{PQ}} \\ & -\frac{1}{4}f^{\mathcal{M}}{}_{\mathcal{NK}}f^{\mathcal{N}}{}_{\mathcal{ML}}\mathcal{H}^{\mathcal{KL}} - \frac{1}{6}f^{\mathcal{MNK}}f_{\mathcal{MNK}}\,. \end{aligned} \tag{7.25}$$

Up to some total derivative, the same Lagrangian (7.25) can be written in terms of generalized fluxes as

$$2\mathcal{E}_{\underline{\mathcal{A}}}\mathcal{F}^{\underline{\mathcal{A}}} + \mathcal{F}_{\underline{\mathcal{A}}}\mathcal{F}^{\underline{\mathcal{A}}} - \frac{1}{6}\mathcal{F}_{\underline{\mathcal{ABC}}}\mathcal{F}^{\underline{\mathcal{ABC}}} - \frac{1}{2}\mathcal{F}_{\overline{\mathcal{A}}\underline{\mathcal{BC}}}\mathcal{F}^{\overline{\mathcal{A}}\underline{\mathcal{BC}}}\,. \tag{7.26}$$

The equations of motion from the varying the action we obtained

$$\begin{aligned}\mathcal{R} =\;& \frac{1}{8}\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{KL}}\partial_{\mathcal{N}}\mathcal{H}_{\mathcal{KL}} - \frac{1}{2}\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{N}}\mathcal{H}^{\mathcal{KL}}\partial_{\mathcal{L}}\mathcal{H}_{\mathcal{MK}} + 4\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{M}}\partial_{\mathcal{N}}d \\ & +4\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{N}}d - 4\mathcal{H}^{\mathcal{MN}}\partial_{\mathcal{M}}d\partial_{\mathcal{N}}d - \partial_{\mathcal{M}}\partial_{\mathcal{N}}\mathcal{H}^{\mathcal{MN}} - \frac{1}{2}f^{\mathcal{M}}{}_{\mathcal{NK}}\mathcal{H}^{\mathcal{NP}}\mathcal{H}^{\mathcal{KQ}}\partial_{\mathcal{P}}\mathcal{H}_{\mathcal{MQ}} \\ & -\frac{1}{12}f^{\mathcal{M}}{}_{\mathcal{KP}}f^{\mathcal{N}}{}_{\mathcal{LQ}}\mathcal{H}_{\mathcal{MN}}\mathcal{H}^{\mathcal{KL}}\mathcal{H}^{\mathcal{PQ}} - \frac{1}{4}f^{\mathcal{M}}{}_{\mathcal{NK}}f^{\mathcal{N}}{}_{\mathcal{ML}}\mathcal{H}^{\mathcal{KL}} \\ & -\frac{1}{6}f^{\mathcal{MNK}}f_{\mathcal{MNK}} = 0\,, \end{aligned} \tag{7.27}$$

along with generalized Ricci tensor

$$\mathcal{R}_{\mathcal{MN}} = P_{\mathcal{M}}{}^{\mathcal{P}}\mathcal{K}_{\mathcal{PQ}}\overline{P}^{\mathcal{Q}}{}_{\mathcal{N}} + \overline{P}_{\mathcal{M}}{}^{\mathcal{P}}\mathcal{K}_{\mathcal{PQ}}P^{\mathcal{Q}}{}_{\mathcal{N}} = 0\,, \tag{7.28}$$

where

$$\begin{aligned}\mathcal{K}_{\mathcal{MN}} \;=\;& \frac{1}{8}\partial_{\mathcal{M}}\mathcal{H}^{\mathcal{KL}}\partial_{\mathcal{N}}\mathcal{H}_{\mathcal{KL}} - \frac{1}{4}\left(\partial_{\mathcal{L}} - 2\partial_{\mathcal{L}}d\right)\left(\mathcal{H}^{\mathcal{LK}}\partial_{\mathcal{K}}\mathcal{H}_{\mathcal{MN}}\right) + 2\partial_{\mathcal{M}}\partial_{\mathcal{N}}d \\ & -\frac{1}{2}\partial_{(\mathcal{M}}\mathcal{H}^{\mathcal{KL}}\partial_{\mathcal{L}}\mathcal{H}_{\mathcal{N})\mathcal{K}} + \frac{1}{2}\left(\partial_{\mathcal{L}} - 2\partial_{\mathcal{L}}d\right)\left(\mathcal{H}^{\mathcal{KL}}\partial_{(\mathcal{M}}\mathcal{H}_{\mathcal{N})\mathcal{K}} + \mathcal{H}^{\mathcal{K}}{}_{(\mathcal{M}}\partial_{\mathcal{K}}\mathcal{H}^{\mathcal{L}}{}_{\mathcal{N})}\right) \\ & +\frac{1}{2}f_{(\mathcal{ML}}{}^{\mathcal{K}}\mathcal{H}^{\mathcal{LP}}\partial_{\mathcal{P}}\mathcal{H}_{\mathcal{N})\mathcal{K}} - \frac{1}{2}f_{(\mathcal{ML}}{}^{\mathcal{K}}\mathcal{H}^{\mathcal{LP}}\partial_{\mathcal{N})}\mathcal{H}_{\mathcal{PK}} - \frac{1}{4}f_{\mathcal{ML}}{}^{\mathcal{K}}f_{\mathcal{NK}}{}^{\mathcal{L}} \\ & +\frac{1}{2}e^{2d}\partial_{\mathcal{P}}(e^{-2d}\mathcal{H}^{\mathcal{LP}}\mathcal{H}_{\mathcal{Q}(\mathcal{M}})f_{\mathcal{N})\mathcal{L}}{}^{\mathcal{Q}} - \frac{1}{4}f_{\mathcal{MK}}{}^{\mathcal{P}}f_{\mathcal{NL}}{}^{\mathcal{Q}}\mathcal{H}^{\mathcal{KL}}\mathcal{H}_{\mathcal{PQ}}\,. \end{aligned} \tag{7.29}$$

This completes the discussion on the basic construction of heterotic DFT. In this formulation there is no signature of the gauge field as it is encoded in the generalized metric (or in frame) and the gauge symmetry is embedded in the generalized diffeomorphisms.

7.2 Heterotic Kerr-Schild ansatz in metric formulation

Now we introduce Kerr-Schild ansatz in the context of Heterotic DFT. First we will discuss the metric formulation of it. The formulation of the GKSA for heterotic DFT describes exact and linear perturbation of the generalized background metric [54]

$$\mathcal{H}_{\mathcal{M}\mathcal{N}} = \mathcal{H}_{o\mathcal{M}\mathcal{N}} + \kappa(\overline{\mathcal{K}}_{\mathcal{M}}\mathcal{K}_{\mathcal{N}} + \mathcal{K}_{\mathcal{M}}\overline{\mathcal{K}}_{\mathcal{N}})\,, \tag{7.30}$$

where $\overline{\mathcal{K}}_{\mathcal{M}} = \overline{\mathcal{P}}_{\mathcal{M}}{}^{\mathcal{N}}\overline{\mathcal{K}}_{\mathcal{N}}$ and $\mathcal{K}_{\mathcal{M}} = \mathcal{P}_{\mathcal{M}}{}^{\mathcal{N}}\mathcal{K}_{\mathcal{N}}$ are a pair of generalized null vectors satisfy

$$\eta^{\mathcal{M}\mathcal{N}}\overline{\mathcal{K}}_{\mathcal{M}}\overline{\mathcal{K}}_{\mathcal{N}} = 0\,, \tag{7.31}$$

$$\eta^{\mathcal{M}\mathcal{N}}\mathcal{K}_{\mathcal{M}}\mathcal{K}_{\mathcal{N}} = 0\,. \tag{7.32}$$

According to (7.30), the corresponding DFT projectors are perturbed as follows,

$$\begin{aligned} \mathcal{P}_{\mathcal{M}\mathcal{N}} &= \mathcal{P}_{o\mathcal{M}\mathcal{N}} - \frac{1}{2}\kappa(\overline{K}_{\mathcal{M}}\mathcal{K}_{\mathcal{N}} + \mathcal{K}_{\mathcal{M}}\overline{\mathcal{K}}_{\mathcal{N}}) \\ \overline{\mathcal{P}}_{\mathcal{M}\mathcal{N}} &= \overline{\mathcal{P}}_{\mathcal{M}\mathcal{N}} + \frac{1}{2}\kappa(\overline{\mathcal{K}}_{\mathcal{M}}\mathcal{K}_{\mathcal{N}} + \mathcal{K}_{\mathcal{M}}\overline{\mathcal{K}}_{\mathcal{N}})\,. \end{aligned} \tag{7.33}$$

In addition, the generalized background dilaton can be perturbed with a generic κ expansion given by

$$d = d_o + \kappa f\,, \qquad f = \sum_{n=0}^{\infty}\kappa^n f_n\,. \tag{7.34}$$

Following the ordinary Kerr-Schild ansatz [73], the generalized vectors $\mathcal{K}_{\mathcal{M}}$, $\overline{\mathcal{K}}_{\mathcal{M}}$ and f obey some conditions to produce finite deformations in the DFT action and EOM's. Following the original construction of the GKSA [53] we impose the following,

$$\begin{aligned} \overline{\mathcal{K}}^{\mathcal{P}}\partial_{\mathcal{P}}\mathcal{K}^{\mathcal{M}} + \mathcal{K}_{\mathcal{P}}\partial^{\mathcal{M}}\overline{K}^{\mathcal{P}} - \mathcal{K}^{\mathcal{P}}\partial_{\mathcal{P}}\overline{\mathcal{K}}^{\mathcal{M}} &= 0\,, \\ \mathcal{K}^{\mathcal{P}}\partial_{\mathcal{P}}\overline{\mathcal{K}}^{\mathcal{M}} + \overline{\mathcal{K}}_{\mathcal{P}}\partial^{\mathcal{M}}\mathcal{K}^{\mathcal{P}} - \overline{\mathcal{K}}^{\mathcal{P}}\partial_{\mathcal{P}}\mathcal{K}^{\mathcal{M}} &= 0\,, \end{aligned} \tag{7.35}$$

and

$$\mathcal{K}^{\mathcal{M}}\partial_{\mathcal{M}}f = \overline{\mathcal{K}}^{\mathcal{M}}\partial_{\mathcal{M}}f = 0\,. \tag{7.36}$$

Using (7.14), and change $\partial \to \nabla$ in (7.35) we have ,

$$\begin{aligned} \overline{\mathcal{K}}^{\mathcal{P}}\nabla_{\mathcal{P}}\mathcal{K}^{\mathcal{M}} + \mathcal{K}_{\mathcal{P}}\nabla^{\mathcal{M}}\overline{\mathcal{K}}^{\mathcal{P}} - \mathcal{K}^{\mathcal{P}}\nabla_{\mathcal{P}}\overline{\mathcal{K}}^{\mathcal{M}} &= 0\,, \\ \mathcal{K}^{\mathcal{P}}\nabla_{\mathcal{P}}\overline{\mathcal{K}}^{\mathcal{M}} + \overline{\mathcal{K}}_{\mathcal{P}}\nabla^{\mathcal{M}}\mathcal{K}^{\mathcal{P}} - \overline{\mathcal{K}}^{\mathcal{P}}\nabla_{\mathcal{P}}\mathcal{K}^{\mathcal{M}} &= 0\,. \end{aligned} \tag{7.37}$$

In order to read heterotic supergravity, we parametrized the generalized metric as

$$\mathcal{H}_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} g_o^{\mu\nu} & -g_o^{\mu\rho}C_{o\rho\nu} & -g_o^{\mu\rho}A_{o\rho i} \\ -g_o^{\nu\rho}C_{o\rho\mu} & g_{o\mu\nu} + C_{o\rho\mu}C_{o\sigma\nu}g_o^{\rho\sigma} + A_{o\mu}{}^{i}\kappa_{ij}A_{o\nu}{}^{j} & C_{o\rho\mu}g_o^{\rho\sigma}A_{o\sigma i} + A_{o\mu}{}^{j}\kappa_{ji} \\ -g_o^{\nu\rho}A_{o\rho i} & C_{o\rho\nu}g_o^{\rho\sigma}A_{o\sigma i} + A_{o\nu}{}^{j}\kappa_{ij} & \kappa_{ij} + A_{o\rho i}g_o^{\rho\sigma}A_{o\sigma j} \end{pmatrix}\,, \tag{7.38}$$

where $C_{o\mu\nu} = b_{o\mu\nu} + \frac{1}{2} A_{o\mu}{}^{i} A_{o\nu i}$ and $\mu = 0, \ldots, D-1$, $i = 1, \ldots, K$, while the parametrization of the generalized background dilaton is given by

$$e^{-2d_o} = \sqrt{-g_o} e^{-2\phi_o} . \tag{7.39}$$

In addition, the generalized vectors $\mathcal{K}_{\mathcal{M}}$ and $\overline{\mathcal{K}}_{\mathcal{M}}$ can be parametrized as

$$K_M = \frac{1}{\sqrt{2}} \begin{pmatrix} l^{\mu} \\ -l_{\mu} - C_{o\rho\mu} l^{\rho} \\ -A_{o\rho i} l^{\rho} \end{pmatrix} , \quad \bar{K}_M = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{l}^{\mu} \\ \bar{l}_{\mu} - C_{o\rho\mu} \bar{l}^{\rho} - \sqrt{2} A_{o\mu i} j^{i} \\ -A_{o\rho i} \bar{l}^{\rho} + \sqrt{2} j_i \end{pmatrix} , \tag{7.40}$$

with l and $\bar{l}$ satisfy,

$$l_{\mu} l^{\mu} = 0 , \tag{7.41}$$
$$\bar{l}_{\mu} \bar{l}^{\mu} + j^{i} j_{i} = 0 . \tag{7.42}$$

These satisfy geodesic conditions, which are inherited from generalized conditions at the DFT level.

Finally, the perturbations of the heterotic supergravity field are given by

$$g_{\mu\nu} = g_{o\mu\nu} - \frac{\kappa}{1 + \frac{1}{2}\kappa l.\bar{l}} l_{(\mu} \bar{l}_{\nu)} \tag{7.43}$$
$$b_{\mu\nu} = b_{o\mu\nu} - \frac{\kappa}{1 + \frac{1}{2}\kappa l.\bar{l}} l_{[\mu} (\bar{l}_{\nu]} - \frac{1}{\sqrt{2}} j_i A_{\nu]}{}^{i}) \tag{7.44}$$
$$A_{\mu i} = A_{o\mu i} + \frac{1}{\sqrt{2}} \frac{\kappa}{1 + \frac{1}{2}\kappa l.\bar{l}} l_{\mu} j_i \tag{7.45}$$
$$\phi = \phi_o + \kappa f , \tag{7.46}$$

where we keep the same notation for the perturbation of the standard dilaton. While the ordinary Kerr-Schild ansatz is based on linear perturbations on the metric tensor, the generalized Kerr-Schild ansatz contains a tower of perturbations due to $l.\bar{l} \neq 0$. Moreover, only l is a null vector when the gauge sector is taken into account.

In the next part we discuss about an obstruction when the generalized frame formulation is used. Here, the background gauge field $A_{o\mu i}$ cannot be perturbed when one considers heterotic DFT written in terms of a fundamental $O(D, D+K)$ covariant frame.

7.2.1 Heterotic Kerr-Schild ansatz in frame formulation

The generalized Kerr-Schild ansatz for the DFT frame is given by,

$$\begin{aligned} \mathcal{E}_{\mathcal{M}}{}^{\overline{\mathcal{A}}} &= \mathcal{E}_{o\mathcal{M}}{}^{\overline{\mathcal{A}}} + \frac{1}{2}\kappa \mathcal{E}_{o\mathcal{N}}{}^{\overline{\mathcal{A}}} \mathcal{K}_{\mathcal{M}} \overline{\mathcal{K}}^{\mathcal{N}} , \\ \mathcal{E}_{\mathcal{M}}{}^{\underline{\mathcal{A}}} &= \mathcal{E}_{oM}{}^{\underline{\mathcal{A}}} - \frac{1}{2}\kappa \mathcal{E}_{o\mathcal{N}}{}^{\underline{\mathcal{A}}} \overline{\mathcal{K}}_{\mathcal{M}} \mathcal{K}^{\mathcal{N}} . \end{aligned} \tag{7.47}$$

In the frame formulation of DFT, by imposing conditions (7.35) and (7.36) we have [56]

$$\begin{aligned} \mathcal{K}^{\underline{\mathcal{A}}}\mathcal{E}_{\underline{\mathcal{A}}}\overline{\mathcal{K}}^{\overline{\mathcal{C}}} + \mathcal{K}^{\underline{\mathcal{A}}}\overline{\mathcal{K}}^{\overline{\mathcal{B}}}\mathcal{F}_{\underline{\mathcal{A}}\overline{\mathcal{B}}}{}^{\overline{\mathcal{C}}} &= 0\,, \\ \overline{\mathcal{K}}^{\overline{\mathcal{A}}}\mathcal{E}_{\overline{\mathcal{A}}}\mathcal{K}^{\underline{\mathcal{C}}} + \overline{\mathcal{K}}^{\overline{\mathcal{A}}}\mathcal{K}^{\underline{\mathcal{B}}}\mathcal{F}_{\overline{\mathcal{A}}\underline{\mathcal{B}}}{}^{\underline{\mathcal{C}}} &= 0\,, \end{aligned} \tag{7.48}$$

and

$$\mathcal{K}^{\underline{\mathcal{A}}}\mathcal{E}_{\underline{\mathcal{A}}}f = \overline{\mathcal{K}}^{\mathcal{A}}\mathcal{E}_{\overline{\mathcal{A}}}f = 0\,, \tag{7.49}$$

where $\mathcal{K}_{\underline{\mathcal{A}}} = \mathcal{E}^{\mathcal{M}}{}_{\underline{\mathcal{A}}}\mathcal{K}_{\mathcal{M}}$ and $\overline{\mathcal{K}}_{\overline{\mathcal{A}}} = \mathcal{E}^{\mathcal{M}}{}_{\overline{\mathcal{A}}}\overline{\mathcal{K}}_{\mathcal{M}}$.

Using these identifications, conditions (7.48) and (7.49) can be written as

$$\begin{aligned} \mathcal{K}_{\underline{\mathcal{A}}}\mathcal{D}^{\underline{\mathcal{A}}}\overline{\mathcal{K}}^{\overline{\mathcal{B}}} &= \overline{\mathcal{K}}_{\overline{\mathcal{A}}}\mathcal{D}^{\overline{\mathcal{A}}}\mathcal{K}^{\underline{\mathcal{B}}} = 0\,, \\ \mathcal{K}_{\underline{\mathcal{A}}}\mathcal{D}^{\underline{\mathcal{A}}}f &= \mathcal{K}_{\overline{\mathcal{A}}}\mathcal{D}^{\overline{\mathcal{A}}}f = 0\,, \end{aligned} \tag{7.50}$$

where $\mathcal{D}_{\mathcal{A}}$ is a background Lorentz covariant derivative.

The parametrization of the generalized background frame takes the form

$$\mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -e_{o\mu a} - C_{o\rho\mu}e^{\rho}_{oa} & e^{\mu}_{o\,a} & -A_{o\rho}{}^{i}e^{\rho}_{oa}\,, \\ \overline{e}_{o\mu a} - C_{o\rho\mu}\overline{e}^{\rho}_{oa} & \overline{e}^{\mu}_{o\,a} & -A_{o\rho}{}^{i}\overline{e}^{\rho}_{oa} \\ \sqrt{2}A_{o\mu i}e^{i}{}_{\overline{i}} & 0 & \sqrt{2}e^{i}{}_{\overline{i}} \end{pmatrix}\,, \tag{7.51}$$

and we impose the standard gauge fixing for the double Lorentz group,

$$e_{o\mu a}\eta^{ab}e_{o\nu b} = \overline{e}_{o\mu a}\eta^{ab}\overline{e}_{o\nu b} = g_{o\mu\nu}\,, \tag{7.52}$$

with η_{ab} the ten dimensional flat metric, $a, b = 0, \ldots, D-1$.

Since $\mathcal{E}^{\mu}{}_{\overline{i}}$ cannot be perturbed, we are forced to impose $j_i = 0$ from the very beginning. Then, the parametrization of the generalized vector fields is

$$\mathcal{K}_{\mathcal{M}} = \frac{1}{\sqrt{2}}\begin{pmatrix} l^{\mu} \\ -l_{\mu} - C_{o\rho\mu}l^{\rho} \\ -A_{oi\rho}l^{\rho} \end{pmatrix}\,, \quad \overline{\mathcal{K}}_{\mathcal{M}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{l}^{\mu} \\ \bar{l}_{\mu} - C_{o\rho\mu}\bar{l}^{\rho} \\ -A_{oi\rho}\bar{l}^{\rho} \end{pmatrix} \tag{7.53}$$

In this framework, both l and $\bar{l}$ are null vectors in order to preserve the condition

$$\mathcal{E}^{\mu}{}_{\overline{i}} = 0\,. \tag{7.54}$$

Then, it is possible to recover perturbations for the supergravity fields but part of the perturbations of the b-field and the full perturbation for the gauge field are missed.

In the next section we rewrite heterotic DFT in terms of $O(D,D)$ fundamental fields in order to present a double Yang-Mills formulation of this theory. The leading order terms of the construction are closely related to the construction given in [74].

7.3 Double Yang-Mills formulation

We start by considering the splitting of heterotic DFT. We parametrize the $O(D, D + K)$ generalized frame in terms of $O(D, D)$ multiplets, i.e.,

$$\mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}} = \begin{pmatrix} \mathcal{E}_M{}^A & \mathcal{E}_M{}^{\bar{\alpha}} \\ \mathcal{E}_\alpha{}^A & \mathcal{E}_\alpha{}^{\bar{\alpha}} \end{pmatrix} = \begin{pmatrix} (\chi^{\frac{1}{2}})_M{}^N E_N{}^A & -A_M{}^\beta e_\beta{}^{\overline{\alpha}} \\ A^M{}_\alpha E_M{}^A & (\Box^{\frac{1}{2}})_\alpha{}^\beta e_\beta{}^{\overline{\alpha}} \end{pmatrix} , \tag{7.55}$$

where $e_\alpha{}^{\overline{\alpha}}$ satisfies

$$e_\alpha{}^{\overline{\alpha}} \kappa_{\overline{\alpha}\overline{\beta}} e_\beta{}^{\overline{\beta}} = \kappa_{\alpha\beta} \, , \tag{7.56}$$

$$e^\alpha{}_{\overline{\alpha}} \kappa_{\alpha\beta} e^\beta{}_{\overline{\beta}} = \kappa_{\overline{\alpha}\overline{\beta}} \, . \tag{7.57}$$

Where χ_{MN} and $\Box_{\alpha\beta}$ are defined as follows

$$\chi_{MN} = \eta_{MN} - A_M{}^\alpha A_{N\alpha} \, , \quad \Box_{\alpha\beta} = \kappa_{\alpha\beta} - A_{M\alpha} A^M{}_\beta \, , \tag{7.58}$$

and we also impose

$$E^{M\overline{A}} A_{M\alpha} = 0 \tag{7.59}$$

which requires a gauge fixing. Similarly, the projectors $\mathcal{P}_{\mathcal{M}\mathcal{N}}$, $\overline{\mathcal{P}}_{\mathcal{M}\mathcal{N}}$ and $\mathcal{P}_{\mathcal{A}\mathcal{B}}$, $\overline{\mathcal{P}}_{\mathcal{A}\mathcal{B}}$ are decomposed as

$$\mathcal{P}_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} P_{MN} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \overline{\mathcal{P}}_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{P}_{MN} & 0 \\ 0 & 0 & \kappa_{\alpha\beta} \end{pmatrix} , \tag{7.60}$$

$$\mathcal{P}_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} P_{AB} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \overline{\mathcal{P}}_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{P}_{AB} & 0 \\ 0 & 0 & \kappa_{\overline{\alpha}\overline{\beta}} \end{pmatrix} . \tag{7.61}$$

In [76], the ansatz (7.55) was used to obtain α'-corrections. There, the authors considered that the $A_{M\alpha}$ is not a fundamental field and, consequently, they identified it with some projections of the DFT fluxes. Here we follow a different philosophy. We consider $A_{M\alpha}$ as a fundamental field and we construct a double Yang-Mills action. Moreover, the parametrization of this field gives rise to the ordinary Yang-Mills connection $A_{\mu i}$. As we previously mentioned, $\Big\{E_{MA}, A_{M\alpha}, d\Big\}$ are now the fundamental fields of this alternative formulation of heterotic DFT. Similarly, the symmetry rules given by $\hat{\xi}_{\mathcal{M}}$ and $\Gamma_{\mathcal{A}\mathcal{B}}$ must be decomposed in a consistent way.

The symmetry transformations of the $O(D, D + K)$ fields are given by

$$\delta\mathcal{E}_{\mathcal{M}\mathcal{A}} = \hat{\xi}^{\mathcal{N}} \partial_{\mathcal{N}} \mathcal{E}_{\mathcal{M}\mathcal{A}} + (\partial_{\mathcal{M}} \hat{\xi}^{\mathcal{P}} - \partial^{\mathcal{P}} \hat{\xi}_{\mathcal{M}}) \mathcal{E}_{\mathcal{P}\mathcal{A}} + f_{\mathcal{M}\mathcal{N}\mathcal{P}} \hat{\xi}^{\mathcal{N}} \mathcal{E}^{\mathcal{P}}{}_{\mathcal{A}} + \mathcal{E}_{\mathcal{M}\mathcal{B}} \Gamma^{\mathcal{B}}{}_{\mathcal{A}} \, , \tag{7.62}$$

$$\delta d = \hat{\xi}^{\mathcal{N}} \partial_{\mathcal{N}} d - \frac{1}{2} \partial_{\mathcal{M}} \hat{\xi}^{\mathcal{M}} \, , \tag{7.63}$$

where $f_{\mathcal{MNP}}$ only takes values when $\mathcal{MNP} = \alpha\beta\gamma$. The $\partial_{\mathcal{M}}$ derivative is split according to $\partial_{\mathcal{M}} = (\partial_M, 0)$ and the $O(D,D)$ strong constraint is

$$\begin{aligned} \partial_M A \partial^M B &= 0\,, & (7.64)\\ \partial_M(\partial^M B) &= 0\,. & (7.65) \end{aligned}$$

The generalized diffeomorphisms parameter is split accordingly

$$\hat{\xi}^{\mathcal{M}} = (\hat{\xi}^M, \lambda^\alpha)\,, \tag{7.66}$$

with λ_α a gauge parameter.

Some of the components of the double Lorentz parameter, $\Gamma_{\mathcal{AB}}$, require a gauge fixing to ensure $\delta\mathcal{E}_\alpha{}^{\overline{A}} = 0$ and $\delta e_i{}^{\bar{i}} = 0$. From the former we find,

$$\Gamma_{\overline{\beta}}{}^{\overline{A}} = (\Box^{-1/2})^\alpha{}_\beta e^\beta{}_{\overline{\beta}} (\partial^P \lambda_\alpha) E_P{}^{\overline{A}}\,. \tag{7.67}$$

Similarly, we can now demand $\delta\mathcal{E}_\alpha{}^{\overline{\alpha}} = \delta(\Box^{\frac{1}{2}})_\alpha{}^\beta e_\beta{}^{\overline{\alpha}}$, and then $\delta e_\alpha{}^{\overline{\alpha}} = 0$. The previous condition is satisfied if

$$\Gamma_{\overline{\beta}\overline{\alpha}} = (\Box^{-1/2})^\alpha{}_\beta e^\beta{}_{[\overline{\beta}} \Big(- \hat{\xi}_P \partial^P \mathcal{E}_{\alpha\overline{\alpha}]} + \partial^P \lambda_\alpha \mathcal{E}_{P\overline{\alpha}]} - f_{\alpha\delta\gamma} \lambda^\delta \mathcal{E}^\gamma{}_{\overline{\alpha}]} \Big)\,. \tag{7.68}$$

Equations (7.67) and (7.68) are the gauge fixing conditions that we need to write heterotic DFT in terms of fields which are in representations of $O(D,D)$.

7.3.1 Transformation rules

First we define the field

$$C_{M\alpha} = -A_M{}^\beta (\Box^{-\frac{1}{2}})_{\beta\alpha} \tag{7.69}$$

which is constrained by $E_M{}^{\overline{A}} C_{M\alpha} = 0$. In addition we also define

$$\begin{aligned} \Delta_{\alpha\beta} &= \kappa_{\alpha\beta} + C_{M\alpha} C^M{}_\beta & (7.70)\\ \Theta_{MN} &= \eta_{MN} + C_M{}^\alpha C_{N\alpha}\,. & (7.71) \end{aligned}$$

with satisfy

$$\Delta_{\alpha\beta} = (\Box^{-1})_{\alpha\beta}\ , \qquad \Theta_{MN} = (\chi^{-1})_{MN}\ . \tag{7.72}$$

There relations are useful to write the action principle in terms of $A_{M\alpha}$ or in terms of $C_{M\alpha}$.

Considering the transformation of $\delta\mathcal{E}_{M\overline{A}} = \delta E_{M\overline{A}}$ and rewriting in terms of the $C_{M\alpha}$ field we obtain,

$$\delta E_{M\overline{A}} = \mathcal{L}_{\hat{\xi}} E_{M\overline{A}} + E_M{}^{\overline{B}} \Lambda_{\overline{BA}} + C_M{}^\gamma \partial^P \lambda_\gamma E_{P\overline{A}}\,, \tag{7.73}$$

where we have identified as

$$\Gamma_{\overline{AB}} = \Lambda_{\overline{AB}}\,. \tag{7.74}$$

Now we turn on the frame projection $\delta E_{M\underline{A}}$,

$$\delta \mathcal{E}_{M\underline{A}} = \delta(\chi^{\frac{1}{2}})_M{}^N E_N{}^{\underline{A}} + (\chi^{\frac{1}{2}})_M{}^N \delta E_N{}^{\underline{A}}\,. \tag{7.75}$$

By using the previous results we find,

$$\delta E_{N\underline{A}} = \mathcal{L}_{\hat{\xi}} E_{N\underline{A}} + E_N{}^{\underline{B}}\Gamma_{\underline{BA}} - (\Theta^{\frac{1}{2}})^M{}_N\Big(\partial_M\lambda^\alpha C^R{}_\beta(\Delta^{\frac{1}{2}})^\beta{}_\alpha + \delta_\Lambda(\Theta^{-\frac{1}{2}})_M{}^R\Big)E_{R\underline{A}}\,. \tag{7.76}$$

The above expressed needs a parameter redefinition to express it in a generalized Green-Schwarz form. We define

$$Q_N{}^R = (\Theta^{\frac{1}{2}})^M{}_N\Big(\partial_M\lambda^\alpha C^R{}_\beta(\Delta^{\frac{1}{2}})^\beta{}_\alpha + \delta_\Lambda(\Theta^{-\frac{1}{2}})_M{}^R\Big)E_{R\underline{A}}\,, \tag{7.77}$$

and $S_{MN} = \overline{P}_M{}^P\partial_P\lambda^\alpha C_{N\alpha} - Q_{MN}$, in order to identify the following

$$\Gamma_{\underline{AB}} = \Lambda_{\underline{AB}} - E^M{}_{\underline{A}} S_{MN} E^N{}_{\underline{B}}\,. \tag{7.78}$$

Then in terms of the previous parameter, the transformation of this component takes the form,

$$\delta E_{N\underline{A}} = \mathcal{L}_{\hat{\xi}} E_{N\underline{A}} + E_N{}^{\underline{B}}\Lambda_{\underline{BA}} - \partial_{\overline{N}}\lambda^\alpha C^R{}_\alpha E_{R\underline{A}}\,. \tag{7.79}$$

Together with the transformation of $C_{M\alpha}$ reads

$$\delta C_M{}^\gamma = \mathcal{L}_{\hat{\xi}} C_M{}^\gamma + \partial_M\lambda^\gamma - (\Box^{-1})^{\gamma\alpha}\partial_{\overline{M}}\lambda_\alpha + C_M{}^\alpha\partial^P\lambda_\alpha C_P{}^\gamma - C_M{}^\alpha f_{\alpha\beta}{}^\gamma\lambda^\beta\,. \tag{7.80}$$

7.3.2 Flux formulation

Now in this section we will write the Fluxes in the terms of $O(D,D)$ multiplets. The $O(D,D+K)$ invariant fluxes [77] can be written in terms of $O(D,D)$ multiplets by following (7.55) as

$$\begin{aligned}\mathcal{F}_{\underline{ABC}} \;=\;& 3\sqrt{2}(\chi^{\frac{1}{2}})^M{}_P E^P{}_{[\underline{A}}\Big(\partial_M((\chi^{\frac{1}{2}})^{NQ}E_{Q\underline{B}})(\chi^{\frac{1}{2}})_{NR}E^R{}_{\underline{C}]} + \partial_M A_{\underline{B}}{}^\alpha A_{\underline{C}]\alpha}\Big)\\ &+\sqrt{2}f^{\alpha\beta\gamma}A_{\underline{A}\alpha}A_{\underline{B}\beta}A_{\underline{C}\gamma}\,,\end{aligned} \tag{7.81}$$

$$\begin{aligned}\mathcal{F}_{\overline{A}\underline{BC}} \;=\;& \sqrt{2}E^M{}_{\overline{A}}\partial_M((\chi^{\frac{1}{2}})^{NQ}E_{Q\underline{B}})(\chi^{\frac{1}{2}})_{NR}E^R{}_{\underline{C}} + 2\sqrt{2}(\chi^{\frac{1}{2}})^M{}_P E^P{}_{[\underline{C}}\partial_M(E^N{}_{\overline{A}})(\chi^{\frac{1}{2}})_{NR}E^R{}_{\underline{B}]}\\ &+2\sqrt{2}E^M{}_{\overline{A}}\partial_M(A_{[\underline{B}}{}^\alpha)A_{\underline{C}]\alpha}\,,\end{aligned} \tag{7.82}$$

$$\begin{aligned}\mathcal{F}_{\overline{\alpha}\underline{BC}} \;=\;& -\sqrt{2}A^M{}_{[\alpha}e^\alpha{}_{\overline{\alpha}}\partial_M((\chi^{\frac{1}{2}})^{NQ}E_{Q\underline{B}})(\chi^{\frac{1}{2}})_{NR}E^R{}_{\underline{C}} + 2\sqrt{2}(\chi^{\frac{1}{2}})^M{}_P E^P{}_{[\underline{C}}\partial_M((\Box^{\frac{1}{2}})_{\alpha\beta}e^\beta{}_{\overline{\alpha}})A_{\underline{B}]}{}^\alpha\\ &+\sqrt{2}A^M{}_\beta e^\beta{}_{\overline{\alpha}}\partial_M A_{\underline{B}}{}^\alpha A_{\underline{C}\alpha} + \sqrt{2}f^{\alpha\beta\gamma}(\Box^{\frac{1}{2}})^{\alpha\delta}e_{\delta\overline{\alpha}}A_{\underline{B}\beta}A_{\underline{C}\gamma}\,,\end{aligned} \tag{7.83}$$

$$\mathcal{F}_{\underline{A}} \;=\; \sqrt{2}\partial_M((\chi^{\frac{1}{2}})^{MP}E_{P\underline{A}}) - 2\sqrt{2}(\chi^{\frac{1}{2}})^{MP}E_{P\underline{A}}\partial_M d\,. \tag{7.84}$$

where $A_{\underline{B}\alpha} = A_{M\alpha}E^M{}_{\underline{B}}$. The remaining projections can be decomposed in the same way.

7.3.3 Action principle

Now we will write the components of the $O(D,D+K)$ generalized metric in terms of the following $O(D,D)$ multiples $\{H_{MN}, A_{M\alpha}\}$ (similarly to the decomposition given in [78]). In our conventions,

$$\begin{aligned}
\mathcal{H}_{MN} &= H_{MN} + 2A_{M\alpha}A_N{}^{\alpha} = H_{MN} + 2\eta_{MN} - 2(\Theta^{-1})_{MN}\,, && (7.85)\\
\mathcal{H}_{M\beta} &= -2(\chi^{\frac{1}{2}})_M{}^P A_{P\beta} = 2C_{M\beta}\,, && (7.86)\\
\mathcal{H}_{\alpha N} &= -2(\chi^{\frac{1}{2}})_N{}^Q A_{Q\alpha} = 2C_{N\alpha}\,, && (7.87)\\
\mathcal{H}_{\alpha\beta} &= \kappa_{\alpha\beta} - 2A^M{}_{\alpha}A_{M\beta} = -\kappa_{\alpha\beta} + 2(\Delta^{-1})_{\alpha\beta}\,. && (7.88)
\end{aligned}$$

Then the decomposed DFT Lagrangian takes the form

$$\begin{aligned}
\mathcal{L} = \; & \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} + \frac{1}{4}\mathcal{H}^{MN}\partial_M\mathcal{H}^{K\alpha}\partial_N\mathcal{H}_{K\alpha} + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{\alpha\beta}\partial_N\mathcal{H}_{\alpha\beta}\\
& -\frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\partial_L\mathcal{H}_{MK} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{\beta K}\partial_K\mathcal{H}_{M\beta} - \frac{1}{2}\mathcal{H}^{\alpha K}\partial_K\mathcal{H}^{MN}\partial_N\mathcal{H}_{\alpha M}\\
& -\frac{1}{2}\mathcal{H}^{\alpha K}\partial_K\mathcal{H}^{\beta N}\partial_N\mathcal{H}_{\alpha\beta} + 4\mathcal{H}^{MN}\partial_M d\partial_N d - 2\partial_M\mathcal{H}^{MN}\partial_N d - \frac{1}{2}f^{\alpha}_{\beta\gamma}\mathcal{H}^{\beta M}\mathcal{H}^{\gamma N}\partial_M\mathcal{H}_{\alpha N}\\
& -\frac{1}{2}f^{\alpha}_{\beta\gamma}\mathcal{H}^{\beta M}\mathcal{H}^{\gamma\lambda}\partial_M\mathcal{H}_{\alpha\lambda} - \frac{1}{12}f^{\alpha}_{\beta\gamma}f^{\lambda}_{\mu\rho}\mathcal{H}_{\alpha\lambda}\mathcal{H}^{\beta\mu}\mathcal{H}^{\gamma\rho} - \frac{1}{4}f^{\alpha}_{\beta\gamma}f^{\beta}_{\alpha\lambda}\mathcal{H}^{\gamma\lambda} - \frac{1}{6}f^{\alpha\beta\gamma}f_{\alpha\beta\gamma}\,,
\end{aligned} \tag{7.89}$$

and the double Yang-Mills action principle leads the following form,

$$\begin{aligned}
S \;=\; & \int d^{2D}X e^{-2d}\Big(\frac{1}{8}(H^{MN} - 2(\Theta^{-1})^{MN})\partial_M(H^{KL} - 2(\Theta^{-1})^{KL})\partial_N(H_{KL} - 2(\Theta^{-1})_{KL})\\
& +(H^{MN} - 2(\Theta^{-1})^{MN})\partial_M C^{K\alpha}\partial_N C_{K\alpha} + \frac{1}{2}(H^{MN} - 2(\Theta^{-1})^{MN})\partial_M(\Delta^{-1})^{\alpha\beta}\partial_N(\Delta^{-1})_{\alpha\beta}\\
& -\frac{1}{2}(H^{MN} + 2\eta^{MN} - 2(\Theta^{-1})^{MN})\partial_N(H^{KL} - 2(\Theta^{-1})^{KL})\partial_L(H_{MK} - 2(\Theta^{-1})_{MK})\\
& -2(H^{MN} + 2\eta^{MN} - 2(\Theta^{-1})^{MN})\partial_N C^{K\beta}\partial_K C_{M\beta} - 2C^{K\alpha}\partial_K(H^{MN} - 2(\Theta^{-1})^{MN})\partial_N C_{M\alpha}\\
& -4C^{K\alpha}\partial_K C^{N\beta}\partial_N(\Delta^{-1})_{\alpha\beta} + 4(H^{MN} - 2(\Theta^{-1})^{MN})\partial_M d\partial_N d - 2\partial_M(H^{MN} - 2(\Theta^{-1})^{MN})\partial_N d\\
& -4f^{\alpha}_{\beta\gamma}C^{M\beta}C^{N\gamma}\partial_M C_{N\alpha} - 2f^{\alpha}_{\beta\gamma}C^{M\beta}(-\kappa^{\gamma\lambda} + 2(\Delta^{-1})^{\gamma\lambda})\partial_M(\Delta^{-1})_{\alpha\lambda}\\
& -\frac{1}{12}f^{\alpha}_{\beta\gamma}f^{\lambda}_{\mu\rho}(-\kappa_{\alpha\lambda} + 2(\Delta^{-1})_{\alpha\lambda})(-\kappa^{\beta\mu} + 2(\Delta^{-1})^{\beta\mu})(-\kappa^{\gamma\rho} + 2(\Delta^{-1})^{\gamma\rho})\\
& -\frac{1}{4}f^{\alpha}_{\beta\gamma}f^{\beta}_{\alpha\lambda}(-\kappa^{\gamma\lambda} + 2(\Delta^{-1})^{\gamma\lambda}) - \frac{1}{6}f^{\alpha\beta\gamma}f_{\alpha\beta\gamma}\Big)\,.
\end{aligned} \tag{7.90}$$

The dynamics of the above system can be figure out by varying the previous Lagrangian or splitting the equations of motion from the $O(D,D+K)$ perspective. Following the second one the equation of motion for the $O(D,D)$ dilaton reads

$$\mathcal{R} = 0\,, \tag{7.91}$$

which is equivalent to $\mathcal{L}=0$. The equation of motion for the $O(D,D)$ metric is given by

$$\begin{aligned}
&\mathcal{K}_{MN} - (H_M{}^P + 2\delta_M{}^P - 2(\Theta^{-1})_M{}^P)\mathcal{K}_{QP}(H^Q{}_N + 2\delta^Q{}_N - 2(\Theta^{-1})^Q{}_N)\\
&-4C_M{}^{\alpha}\mathcal{K}_{\alpha Q}(H^Q{}_N + 2\delta^Q{}_N - 2(\Theta^{-1})^Q{}_N) - 4C_M{}^{\alpha}\mathcal{K}_{\alpha\beta}C_N{}^{\beta} = 0\,,
\end{aligned} \tag{7.92}$$

in addition, the equation of motion for the gauge field is given by

$$\begin{aligned}&\mathcal{K}_{M\alpha}-2(H_M{}^P+2\delta_M{}^P-2(\Theta^{-1})_M{}^P)\mathcal{K}_{PQ}C^Q{}_\alpha\\&-(H_M{}^P+2\delta_M{}^P-2(\Theta^{-1})_M{}^P)\mathcal{K}_{P\beta}(-\kappa^\beta{}_\alpha+2(\Delta^{-1})^\beta{}_\alpha)\\&-2C_M{}^\gamma\mathcal{K}_{\gamma\beta}(-\kappa^\beta{}_\alpha+2(\Delta^{-1})^\beta{}_\alpha)-4C_M{}^\gamma\mathcal{K}_{\gamma Q}C^Q{}_\alpha=0\,,\end{aligned}\tag{7.93}$$

where

$$\begin{aligned}\mathcal{K}_{MN}=\;&\frac{1}{8}\partial_M(H^{KL}-2(\Theta^{-1})^{KL})\partial_N(H_{KL}-2(\Theta^{-1})_{KL})+\partial_M C^{K\alpha}\partial_N C_{K\alpha}\\&-\frac{1}{4}\big(\partial_L-2\partial_L d\big)\big((H^{KL}-2(\Theta^{-1})^{KL})\partial_K(H_{MN}-2(\Theta^{-1})_{MN})\big)+2\partial_M\partial_N d\\&-\frac{1}{2}\partial_{(M}(H^{KL}-2(\Theta^{-1})^{KL})\partial_L(H_{N)K}-2(\Theta^{-1})_{N)K})-2\partial_{(M}C^{L\alpha}\partial_L C_{N)\alpha}\\&+\frac{1}{2}\big(\partial_L-2\partial_L d\big)\big((H^{KL}+2\eta^{KL}-2(\Theta^{-1})^{KL})\partial_{(M}(H_{N)K}-2(\Theta^{-1})_{N)K})\\&+4C^{L\alpha}\partial_{(M}C_{N)\alpha}+(H_{(MK}+2\eta_{(MK}-2(\Theta^{-1})_{(MK})\partial^K(H^L{}_{N)}-2(\Theta^{-1})^L{}_{N)})\big)\\&+\frac{1}{2}\partial_M(\Delta^{-1})^{\alpha\beta}\partial_N(\Delta^{-1})_{\alpha\beta}\,,\end{aligned}\tag{7.94}$$

$$\begin{aligned}\mathcal{K}_{M\alpha}=\;&-\frac{1}{2}\big(\partial_L-2\partial_L d\big)\big((H^{LK}-2(\Theta^{-1})^{LK})\partial_K C_{M\alpha}\big)-\frac{1}{2}\partial_M(H^{LK}-2(\Theta^{-1})^{LK})\partial_L C_{K\alpha}\\&-\partial_M C^{L\beta}\partial_L(\Delta^{-1})_{\alpha\beta}+\frac{1}{2}\big(\partial_L-2\partial_L d\big)\big((H^{LK}+2\eta^{LK}-2(\Theta^{-1})^{LK})\partial_M C_{K\alpha}\\&+2C^{L\beta}\partial_M(\Delta^{-1})_{\alpha\beta}+2(H_M{}^K+2\delta_M{}^K-2(\Theta^{-1})_M{}^K)\partial_K C^L{}_{\alpha)}\big)\\&+e^{2d}\partial_P\Big(e^{-2d}C^{P\beta}C_{M\gamma}\Big)f^\gamma_{\alpha\beta}\,,\end{aligned}\tag{7.95}$$

and

$$\begin{aligned}\mathcal{K}_{\alpha\beta}=\;&-\frac{1}{4}\big(\partial_L-2\partial_L d\big)\big((H^{LK}-2(\Theta^{-1})^{LK})\partial_K(\Delta^{-1})_{\alpha\beta}\big)+2\big(\partial_L-2\partial_L d\big)\big(C^K{}_{(\alpha}\partial_K C^L{}_{\beta)}\big)\\&+f^\lambda_{(\alpha\gamma}C^{P\gamma}\partial_P(\Delta^{-1})_{\beta)\lambda}-\frac{1}{4}f^\lambda_{\alpha\gamma}f^\gamma_{\beta\lambda}+e^{2d}\partial_P\Big(e^{-2d}C^{P\gamma}(\Delta^{-1})_{\lambda(\alpha}\Big)f^\lambda_{\beta)\gamma}\\&-\frac{1}{4}f^\lambda_{\alpha\gamma}f^\sigma_{\beta\rho}(-\kappa^{\gamma\rho}+2(\Delta^{-1})^{\gamma\rho})(-\kappa_{\lambda\sigma}+2(\Delta^{-1})_{\lambda\sigma})\,.\end{aligned}\tag{7.96}$$

7.3.4 Leading order terms

In this section we consider the leading order terms of the double Yang-Mills formulation that we have presented. It turns out that the formulation of heterotic DFT leads to an infinite expansion of gauge fields in the action principle. In the leading order terms the non-covariant transformation of the gauged vector is given by [74],

$$\delta_{\text{non-cov}}A_M{}^\gamma=-\partial_{\underline{M}}\lambda^\gamma+\mathcal{O}(A^2)\,.\tag{7.97}$$

The transformation of a generic DFT gauged vector is given by

$$\delta_\lambda V^\gamma = -A_M{}^\alpha f_{\alpha\beta}{}^\gamma \lambda^\beta \,, \tag{7.98}$$

hence a projected covariant derivative can be defined as

$$\nabla_{\underline{M}} V^\gamma = \partial_{\underline{M}} V^\gamma - V^\alpha f_{\alpha\beta}{}^\gamma A_{\underline{M}}{}^\beta \,. \tag{7.99}$$

However, the combination $\partial_{\overline{M}} V^\gamma$ cannot be put in a covariant form using the gauge connection. When deriving a more general object one has to consider the full DFT covariant derivative as follows

$$\nabla_{\underline{M}} V_N{}^\gamma = \partial_{\underline{M}} V_N{}^\gamma - \Gamma_{\underline{M}N}{}^P V_P{}^\gamma - V_N{}^\alpha f_{\alpha\beta}{}^\gamma A_{\underline{M}}{}^\beta \,, \tag{7.100}$$

where Γ_{MNP} is the generalized affine connection in DFT, which is not fully determined [75].

The curvature for this generalized gauge connection takes the form

$$F_{MN}{}^\gamma = 2\partial_{[M} A_{N]}{}^\gamma + f^\gamma{}_{\alpha\beta} A_M{}^\alpha A_N{}^\beta \,. \tag{7.101}$$

The previous one transforms covariantly with respect to (7.98) by considering the generalized Jacobi identities for the structure constant, in addition it agrees with the constraints of the heterotic DFT.

With respect to gauge transformations, the transformation rule of the generalized metric reads

$$\delta_\lambda H_{MN} = 4 C_{(\underline{M}}{}^\alpha \partial_{\overline{N})} \lambda_\alpha \; = -4 A_{(\underline{M}}{}^\alpha \partial_{\overline{N})} \lambda_\alpha + \mathcal{O}(A^3). \tag{7.102}$$

Interestingly, the previous transformation is non-covariant, F^2 terms withη contractions are only fully covariant. It turns out that the leading order term in (7.102) is a generalized Green-Schwarz transformation with considering $A_{M\alpha}$ as a fundamental field. Upon parametrization, the B_2 shows this kind of transformation, as well as the metric tensor needs to be redefined.

Now we will turn on our interest to construct the leading order Lagrangian, we consider the Abelian terms such that in variation they are linear in the gauge field. The leading order terms of the double Yang-Mills action provide

$$\begin{aligned} S \;\; = \;\; & \int d^{2D} X e^{-2d} \Big(\frac{1}{8} H^{MN} \partial_M H^{KL} \partial_N H_{KL} - \frac{1}{2} H^{MN} \partial_N H^{KL} \partial_L H_{MK} \\ & + 4 H^{MN} \partial_M d \partial_N d - 2\partial_M H^{MN} \partial_N d + H^{MN} \eta^{KL} F^\alpha_{LM} F_{KN\alpha} \\ & - 2 A^{N\alpha} (\partial_N H^{KL})(\partial_L A_{K\alpha}) - 2 H^{MN} \partial_N H^{KL} (\partial_L A_{(K}{}^\alpha) A_{M)\alpha} \\ & + \frac{1}{4} A_{M\alpha} A_N{}^\alpha \partial^M H^{KL} \partial^N H_{KL} - A^M{}_\alpha A^{N\alpha} \partial_N H^{KL} \partial_L H_{MK} \\ & + H_{MN} \partial^M A^K{}_\alpha A^{L\alpha} \partial^N H_{KL} - 2 H^{MN} \partial_N A^{(K\alpha} A^{L)}{}_\alpha \partial_L H_{MK} \\ & + 8 A^{M\alpha} A^N{}_\alpha (\partial_M d)(\partial_N d) - 8 (\partial_M A^{(M\alpha}) A^{N)}{}_\alpha (\partial_N d) \; + \ldots \Big) \,. \end{aligned} \tag{7.103}$$

In addition $A_{M\alpha} = -C_{M\alpha}$ in this limit.

However, the combination $H^{MN}\eta^{KL}F^{\alpha}_{LM}F_{KN\alpha}$ is not fully covariant, so extra pieces are needed to formulate the invariant action. Now, the ungauged part of the Lagrangian is not gauge invariant as the generalized Green-Schwarz mechanism. Also, the double gauge field, $A_{M\alpha}$ transforms as a gauge connection to the leading order in gauge fields. Then, some part of the leading order Lagrangian can be written in terms of its curvature but still the non-covariant contributions appear.

7.3.5 Parametrization

In this part we parametrized the theory and show that it leads to standard heterotic supergravity. The parametrization of the fundamental fields in the theory take the form

$$\begin{aligned} H_{MN} &= \begin{pmatrix} \bar{g}^{\mu\nu} & -\bar{g}^{\mu\rho}b_{\rho\nu} \\ b_{\mu\rho}\bar{g}^{\rho\nu} & \bar{g}_{\mu\nu} - b_{\mu\rho}\bar{g}^{\rho\sigma}b_{\sigma\nu} \end{pmatrix} , && (7.104) \\ C_M{}^{\alpha} &= \frac{1}{2}\begin{pmatrix} -\bar{g}^{\mu\rho}A_\rho{}^i \\ -b_{\mu\rho}\bar{g}^{\rho\sigma}A_\sigma{}^i + A_\mu{}^i \end{pmatrix} . && (7.105) \end{aligned}$$

The metric tensor $\bar{g}_{\mu\nu}$ receives an anomalous gauge transformation from (7.102) and the following field redefinition is needed,

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \frac{1}{2}A_\mu{}^i A_{\nu i} \ . \tag{7.106}$$

The above procedure is same the as the gravitational Green-Schwarz mechanism where the metric tensor is redefined by taking the terms proportional to the spin connection [79].

The generalized dilaton can be parametrized in the following way,

$$e^{-2d} = \sqrt{-\tilde{g}}e^{-2\tilde{\phi}} = \sqrt{-g}e^{-2\phi} \ , \tag{7.107}$$

because the metric redefinition (7.106) leads a dilaton redefinition to obtain the standard integral measure in the corresponding theory. Now, the transformation rules for the supergravity fields are given by

$$\begin{aligned} \delta g_{\mu\nu} &= L_\xi g_{\mu\nu} \ , && (7.108) \\ \delta b_{\mu\nu} &= L_\xi b_{\mu\nu} + 2\partial_{[\mu}\zeta_{\nu]} - \partial_{[\mu}\lambda^i A_{\nu]i} \ , && (7.109) \\ \delta A_\mu{}^i &= L_\xi A_\mu{}^i + \partial_\mu \lambda^i - f_{jk}{}^i \lambda^j A_\mu{}^k \ , && (7.110) \\ \delta\phi &= L_\xi \phi \ , && (7.111) \end{aligned}$$

where L_ξ denotes the ordinary Lie derivative and $\hat{\xi}_\mu = \zeta_\mu$ for the abelian gauge transformation of the NS-NS two form field. Finally, we obtain the action as

$$S = \int d^D x\sqrt{-g}e^{-2\phi}\Big(R - 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}\hat{H}_{\mu\nu\rho}\hat{H}^{\mu\nu\rho} - \frac{1}{4}F_{\mu\nu i}F^{\mu\nu i}\Big) , \tag{7.112}$$

where

$$\begin{aligned} \hat{H}_{\mu\nu\rho} &= 3\left[\partial_{[\mu}b_{\nu\rho]} - \left(A^i_{[\mu}\partial_\nu A_{\rho]i} - \frac{1}{3}f_{ijk}A^i_\mu A^j_\nu A^k_\rho\right)\right] && (7.113) \\ F^i_{\mu\nu} &= 2\partial_{[\mu}A^i_{\nu]} - f^i{}_{jk}A^j_\mu A^k_\nu \, . && (7.114) \end{aligned}$$

7.4 Double Yang-Mills and the relaxed GKSA

Now we introduce the relaxed form of the GKSA in order to study perturbations in the theory. The GKSA was originally introduced in [53] and has been studied in heterotic DFT framework [54] [55] [56], Kaluza-Klein DFT [80], Exceptional Field Theory [81], together with [58] [82]. In the following we extend the ansatz in the context of double Yang-Mills formulation of heterotic field theory.

7.4.1 Generalized metric formulation

First we consider a generalized metric H_{MN}, together with a generalized dilaton d and a generalized gauged field $A_{M\alpha}$. The generalized metric H_{MN} satisfies a relaxed Kerr-Schild ansatz [58] while the gauge field is linearly perturbed and the generalized dilaton can take an arbitrary perturbation, i.e.,

$$\begin{aligned} H_{MN} &= H_{oMN} + \kappa(\overline{K}_M K_N + K_M \overline{K}_N) + \frac{\kappa^2}{2}\overline{K}^2 K_M K_N\,, && (7.115)\\ C_{M\alpha} &= C_{oM\alpha} - \kappa K_M J_\alpha\,, && (7.116)\\ d &= d_o + \kappa f\,, && (7.117)\end{aligned}$$

where H_{oMN}, C_{oMN} and d_o are generalized fields in the background, $f = \sum_{n=0}^{\infty} f^{(n)}\kappa^n$, K_M and $\bar{K}_M$ are the projected vectors,

$$\begin{aligned} \bar{K}_M &= \frac{1}{2}(\eta_{MN} + H_{MN})\bar{K}^N = \bar{P}_{MN}\bar{K}^N\,, && (7.118)\\ K_M &= \frac{1}{2}(\eta_{MN} - H_{MN})K^N = P_{MN}K^N\,, && (7.119)\end{aligned}$$

and κ is the order parameter of the expansion. The null condition acts only on K_M as following

$$\eta^{MN}K_M K_N = 0\,, \tag{7.120}$$

and $\overline{K}_M$ is related to J_α by the expression

$$\eta^{MN}\overline{K}_M\overline{K}_N = -J_\alpha J_\beta \kappa^{\alpha\beta}\,. \tag{7.121}$$

The parametrization of the ansatz takes the same form as (7.43)-(7.46) but the inverse metric inherits a second-order perturbation,

$$g^{\mu\nu} = \tilde{g}_o^{\mu\nu} + \kappa l^{(\mu}\bar{l}^{\nu)} + \frac{\kappa^2}{4}\bar{l}^2 l^\mu l^\nu\,. \tag{7.122}$$

Now we consider the pure Abelian Yang-Mills theory at the DFT level. First we start as

$$\begin{aligned} S &= \int d^{2D}X e^{-2d} H^{MN} F^K{}_{M\alpha} F_{KN}{}^\alpha && (7.123)\\ &= \int d^{2D}X e^{-2d}(H^{MN}\partial_M A^{K\alpha}\partial_N A_{K\alpha} - 2H^{MN}\partial_M A^{K\alpha}\partial_K A_{N\alpha})\,, && (7.124)\end{aligned}$$

where d and H^{MN} are constants and $A_{M\alpha}$ is the dynamical field with the equation of motion

$$H^{MN}\partial_M F_{KN} + 2H^M{}_K\partial^N F_{MN} = 0\,, \tag{7.125}$$

the above can be understood as a duality invariant generalization of the Maxwell equation. Now we apply the generalized Kerr-Schild ansatz on $A_{M\alpha}$ as follows

$$A_{M\alpha} = A_{oM\alpha} + \kappa K_M J_\alpha\,, \tag{7.126}$$

which agrees with (7.117). The dynamics of the system is governed by the equation

$$\begin{aligned}&H^{MN}\partial_M(\partial_K(A_{oN}{}^\alpha + \kappa K_N J^\alpha) - \partial_N(A_{oK}{}^\alpha + \kappa K_K J^\alpha))\\&+H^M{}_K\partial_M\partial_N(A_o^{N\alpha} + \kappa K^N J^\alpha) = 0\,.\end{aligned} \tag{7.127}$$

In order to express in terms of the supergravity fields we parametrize the background fields as

$$H_{oMN} = \begin{pmatrix}\eta^{\mu\nu} & 0\\ 0 & \eta_{\mu\nu}\end{pmatrix}\,, \tag{7.128}$$

$$C_M{}^\alpha = \frac{1}{2}\begin{pmatrix}-\eta^{\mu\rho}A_{o\rho}{}^i\\ A_{o\mu}{}^i\end{pmatrix}\,, \tag{7.129}$$

together the perturbations take the form

$$K_M = \frac{1}{\sqrt{2}}\begin{pmatrix}l^\mu\\ -l_\mu\end{pmatrix}\,,\quad \overline{K}_M = \frac{1}{\sqrt{2}}\begin{pmatrix}\bar{l}^\mu\\ \bar{l}_\mu\end{pmatrix}\,,\quad J_\alpha = \frac{1}{\sqrt{2}}j_i\,. \tag{7.130}$$

Upon parametrization of the pure Abelian Yang-mills dynamics leads to the following contributions

$$-\frac{\kappa}{2}\eta^{\mu\nu}\partial_{\mu\rho}(l_\nu j^i) + \frac{\kappa}{2}\eta^{\mu\nu}\partial_{\mu\nu}(l_\rho j^i) = 0\,, \tag{7.131}$$

which is first order in κ.

7.4.2 Generalized frame formulation

Now in the following we will discuss the relaxed generalized Kerr-Schild ansatz in the generalized frame formulation. The relaxed generalized Kerr-Schild ansatz for the double Yang-Mills fields in the generalized frame formulation takes the form

$$E_M{}^{\overline{A}} = E_{oM}{}^{\overline{A}} + \frac{1}{2}\kappa E_{oN}{}^{\overline{A}}K_M\overline{K}^N\,, \tag{7.132}$$

$$E_M{}^{\underline{A}} = E_{oM}{}^{\underline{A}} - \frac{1}{2}\kappa E_{oN}{}^{\underline{A}}\overline{K}_M K^N - \frac{1}{8}\kappa^2\overline{K}^2 K_M K_N E_o^N{}_{\underline{A}}\,, \tag{7.133}$$

in addition, the generalized gauge field and the generalized dilaton are perturbed as in the metric formalism. Here, the second order perturbation of $E_M{}^{\underline{A}}$ has its coefficient fixed by

$$E_{M\underline{A}}E^M{}_{\underline{B}} + E_{M\overline{A}}E^M{}_{\overline{B}} = \eta_{AB}\,, \tag{7.134}$$

$$-E_{M\underline{A}}E^M{}_{\underline{B}} + E_{M\overline{A}}E^M{}_{\overline{B}} = H_{AB} \tag{7.135}$$

as the right-hand side of the previous expressions must remain invariant. In this picture, the generalized metric belongs to $O(D,D)$ only if $\overline{K}^2 = 0$. That leads to state the perturbation (7.115) is not compatible with (7.133), which explains why the gauge field cannot be perturbed in the ordinary $O(D, D+K)$ framework. However, this formulation still has an obstruction, but it arises before parametrization. This is very straightforward, because the double Yang-Mills formulation of heterotic DFT written in terms of $O(D,D)$ multiplets is equivalent to the standard familiar formulation written in terms of $O(D, D+K)$ multiplets.

7.4.3 Field redefinitions

The different formulations of DFT is the need of field redefinitions to match with standard supergravity framework. This situation arises when the generalized metric is contained with a non-trivial symmetry invariance as it encodes the B_2. The most straightforward example is the Green-Schwarz mechanism. Analyzing the equation (7.106), it turns out that the redefinition is mandatory as the metric $\tilde{g}$ transforms non-covariantly under gauge transformations according to (7.102). Considering at the perturbative level, background objects could encode perturbations.

For a generic $\tilde{g}_{\mu\nu}$ the familiar Kerr-Schild ansatz [73] leads to ,

$$\tilde{g}_{\mu\nu} = \tilde{g}_{o\mu\nu} + \kappa \tilde{l}_\mu \tilde{l}_\nu \,. \tag{7.136}$$

In order to make compatible at the perturbative level, we are forced to impose a vector redefinition given by

$$\tilde{l}_\mu = l_\mu + A_{o\mu}{}^i j_i \,, \tag{7.137}$$

together with a field redefinition for the backgrounds

$$\tilde{g}_{o\mu\nu} = g_{o\mu\nu} + \frac{\kappa^2}{2} l_\mu l_\nu j^2 \,. \tag{7.138}$$

Now it is easy to see that the background $\tilde{g}$ admits a second-order perturbation in terms of the ordinary l vector. In addition, the null and geodesic equations for the tilde fields lead to impose furthermore conditions. These were discussed in [56] for Lorentz Green-Schwarz contributions in the generalized Kerr-Schild picture, and it turns out that they are equivalent to the gauge Green-Schwarz contributions when the Lorentz connection is replaced by the gauge connection.

7.5 Application

Now in the following we will discuss some of the applications of our framework.

7.5.1 Classical double copy at the DFT level

The main outcomes of this framework lead us inspect to the single and zeroth copy at the DFT level. The foundations of this relation and DFT are introduced in [83].

Consider the ungauged part of the equation of motion of the generalized metric with zero dilaton,

$$\mathcal{R}_{MN} = P_M{}^P \mathcal{K}_{PQ} \bar{P}^Q{}_N + \bar{P}_M{}^P \mathcal{K}_{PQ} P^Q{}_N = 0\,, \tag{7.139}$$

where

$$\begin{aligned}\mathcal{K}_{MN} \;=\;& \frac{1}{8}\partial_M H^{KL}\partial_N H_{KL} - \frac{1}{4}\partial_L H^{LK}\partial_K H_{MN} - \frac{1}{4}H^{LK}\partial_K\partial_L H_{MN}\\ &-\frac{1}{2}\partial_{(M}H^{KL}\partial_L H_{N)K} + \frac{1}{2}\partial_L H^{KL}\partial_{(M}H_{N)K} + \frac{1}{2}\partial_L H^K{}_{(M}\partial_K H^L{}_{N)}\\ &+\frac{1}{2}H^{KL}\partial_{L(M}H_{N)K} + \frac{1}{2}H^K{}_{(M}\partial_K\partial_L H^L{}_{N)}\,.\end{aligned} \tag{7.140}$$

By imposing the ordinary generalized Kerr-Schild ansatz on a flat H_{oMN} background,

$$H_{MN} \;=\; H_{oMN} + \kappa(\overline{K}_M K_N + K_M \overline{K}_N)\,. \tag{7.141}$$

In this case it is needed to invoke the generalized version of the geodesic condition [53] as follows

$$\overline{K}^P \partial_P K^M = K^P \partial_P \overline{K}^M \;=\; 0\,. \tag{7.142}$$

After imposing the conditions, the contributions in linear reads

$$\begin{aligned}&-\frac{1}{2}H_o^{KL}\partial_K(\partial_L K_M \overline{K}_N + K_M \partial_L \overline{K}_N)\\ &+\overline{P}_{oM}{}^K\partial_K(\partial_L K^L \overline{K}_N) - P_{oN}{}^K \partial_K(K_M \partial_L \overline{K}^L) = 0\,,\end{aligned} \tag{7.143}$$

and the quadratic contributions are null because they depend on the perturbation of the generalized dilaton. Defining a generalized killing vector, ξ_M, in the following way

$$\mathcal{L}_\xi H_{MN} = 0\,, \tag{7.144}$$

where $\mathcal{L}_\xi$ is the generalized Lie derivative. In addition, we consider a set of double coordinates such that $\xi = const.$ with

$$\xi^M K_M = \xi^M \overline{K}_M = 1\,. \tag{7.145}$$

Upon contracting $\xi^M R_{MN}$ we get,

$$-\frac{1}{2}H_o^{KL}\partial_K\partial_L \overline{K}_N + \xi^M \overline{P}_{oM}{}^K \partial_K(\partial_L K^L \overline{K}_N) - P_{oN}{}^K \partial_K \partial_L \overline{K}^L = 0\,, \tag{7.146}$$

and contracting $\xi^N R_{MN}$ we have

$$-\frac{1}{2}H_o^{KL}\partial_K\partial_L K_M - \xi^N P_{oN}{}^K\partial_K(\partial_L\overline{K}^L K_M) + \overline{P}_{oM}{}^K\partial_K\partial_L K^L = 0\,. \tag{7.147}$$

Now identify each null vector as a generalization of a $U(1)$ vector field, $A_M = A_{\underline{M}}$ and $\overline{A}_M = \overline{A}_{\overline{M}}$, we obtain the generalization of the single copy expression at the DFT level as the following

$$\begin{aligned}
-\frac{1}{2}H_o^{KL}\partial_K\partial_L\overline{A}_N + \xi^M\overline{P}_{oM}{}^K\partial_K(\partial_L A^L\overline{A}_N) - P_{oN}{}^K\partial_K\partial_L\overline{A}^L &= 0\,,\\
-\frac{1}{2}H_o^{KL}\partial_K\partial_L A_M - \xi^N P_{oN}{}^K\partial_K(\partial_L\overline{A}^L A_M) + \overline{P}_{oM}{}^K\partial_K\partial_L A^L &= 0\,.
\end{aligned} \tag{7.148}$$

Although, the previous expressions do not match with the generalization of the Maxwell equation at the DFT level, still it is straightforward to show that they encoded a pair of Maxwell equations at the supergravity level. Consider a null B_2 gauge field at the supergravity level, then we have $\xi^M = (0, \xi^\mu)$. The parametrized generalized metric takes the form

$$H_{MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}\,, \tag{7.149}$$

together, the pair of generalized gauge vectors inherit the ordinary Abelian fields $A_M \to A_\mu$ and $\overline{A}_M \to \overline{A}_\mu$. The geodesic condition leads to

$$\begin{aligned}
A_\mu\partial^\mu\overline{A}_\nu &= 0\,,\\
\overline{A}_\mu\partial^\mu A_\nu &= 0\,.
\end{aligned} \tag{7.150}$$

and the ordinary killing vector satisfies

$$\xi^\mu\partial_\mu(A_\nu\overline{A}_\rho) = 0\,. \tag{7.151}$$

The parametrization of (7.148) is given by the expression

$$\Box A_\nu - \partial_\nu(\partial^\rho A_\rho) = 0 \leftrightarrow \partial_\mu F^{\mu\nu} = 0 \tag{7.152}$$

$$\Box A_\nu - \partial_\nu(\partial^\rho \overline{A}_\rho) = 0 \leftrightarrow \partial^\mu\overline{F}_{\mu\nu} = 0\,, \tag{7.153}$$

where we have introduced the curvatures of the Abelian gauge fields in the following way

$$\begin{aligned}
F_{\mu\nu} &= 2\partial_{[\mu}A_{\nu]}\,,\\
\overline{F}_{\mu\nu} &= 2\partial_{[\mu}\overline{A}_{\nu]}\,.
\end{aligned} \tag{7.154}$$

The zeroth copy relation can be obtained by contracting the extra generalized Killing vector ξ in (7.146) or (7.147), with introducing the scalar function φ and considering the redefinition $\kappa_{\text{DFT}} \to \kappa\varphi$, we obtained

$$-\frac{1}{2}H_o^{KL}\partial_K\partial_L\varphi + \xi^M\overline{P}_{oM}{}^K\partial_K\partial_L(\varphi K^L) - \xi^N P_{oN}{}^K\partial_{KL}(\varphi\overline{K}^L) = 0\,, \tag{7.155}$$

which leads to the standard supergravity zeroth copy relation

$$\Box\varphi = 0\,. \tag{7.156}$$

7.6 Relaxed GKSA and the DFT Lagrangian

Considering the ungauged part of the double Yang-Mills Lagrangian with assuming constant backgrounds and the zero background dilaton we have

$$\frac{1}{8}H^{MN}\partial_M H^{KL}\partial_N H_{KL} - \frac{1}{2}H^{MN}\partial_N H^{KL}\partial_L H_{MK}\,, \tag{7.157}$$

it turns out that the sixth and fifth-order contributions vanish by considering K_M is a null vector.

The contributions come from the κ^4 as

$$\begin{aligned}
&-\frac{1}{8}K^M K^N \overline{K}^4 \partial_M K_Q \partial_N K^Q - \frac{1}{8}K^M K^N \overline{K}_P \overline{K}^Q \overline{K}^2 \partial_M K_Q \partial_N K^P \\
&+\frac{1}{8}K^M K^N \overline{K}^P \overline{K}^Q \overline{K}^2 \partial_M K_P \partial_N K_Q - \frac{1}{8}H_o^{MN} K_M K^P \overline{K}^4 \partial_P K_Q \partial_N K^Q\,,
\end{aligned} \tag{7.158}$$

and the κ^3 contributions are

$$\begin{aligned}
&-\frac{1}{4}K^M \overline{K}^2 \overline{K}^P \partial_P K_Q \partial_M K^Q - \frac{1}{4}K^M \overline{K}_N \overline{K}^P \overline{K}^Q \partial_P K_Q \partial_M K^N \\
&+\frac{1}{4}K^M \overline{K}^2 \overline{K}^P \partial_M K_Q \partial_P K^Q + \frac{1}{4}K^M \overline{K}_N \overline{K}^P \overline{K}^Q \partial_M K_P \partial_Q K^N \\
&-\frac{1}{4}H_o^{MN} K_M \overline{K}^P \overline{K}^2 \partial_P K_Q \partial_N K^Q + \frac{1}{4}H_o^{MN} K_M K^P \overline{K}^Q \partial_N K_Q \partial_P \overline{K}^2 \\
&+\frac{1}{4}H_o^{MN} K^P \overline{K}^Q \overline{K}^2 \partial_M K_Q \partial_P K_N - \frac{1}{4}H_o^{MN} K_M K^P \overline{K}^2 \partial_P K_Q \partial_N \overline{K}^Q \\
&+\frac{1}{4}H_o^{MN} K_M K^P \overline{K}^Q \partial_P K_Q \partial_N \overline{K}^2 - \frac{1}{4}H_o^{MN} K^P \overline{K}_Q \overline{K}^2 \partial_P K_M \partial_N K^Q \\
&-\frac{1}{4}H_o^{MN} K_M K^P \overline{K}^2 \partial_N K^Q \partial_P \overline{K}_Q - \frac{1}{4}H_o^{MN} K^P \overline{K}_M \overline{K}^2 \partial_P K_Q \partial_N K^Q \\
&-K^M \overline{K}^2 \overline{K}^P \partial_Q K_P \partial_M K^Q - \frac{1}{2}K^M K^N \overline{K}^2 \partial_M K_Q \partial_N \overline{K}^Q\,,
\end{aligned} \tag{7.159}$$

similarly the κ^2 contributions are

$$\begin{aligned}
&\frac{1}{4}H_o^{MN} \overline{K}^2 \partial_M K_Q \partial_N K^Q + \frac{1}{4}H_o^{MN} \overline{K}_P \overline{K}^Q \partial_M K_Q \partial_N K^P \\
&+\frac{1}{4}H_o^{MN} \overline{K}^P \overline{K}^Q \partial_M K_P \partial_N K_Q - \frac{1}{2}H_o^{MN} \overline{K}_P \overline{K}^Q \partial_Q K_M \partial_N K^P \\
&-\frac{1}{2}H_o^{MN} K_M \overline{K}^P \partial_N K^Q \partial_P \overline{K}_Q + \frac{1}{2}H_o^{MN} K_M \overline{K}^P \partial_Q K_P \partial_N \overline{K}^Q \\
&-\frac{1}{2}H_o^{MN} \overline{K}_M \overline{K}^P \partial_P K_Q \partial_N K^Q - \frac{1}{2}H_o^{MN} K^P \overline{K}^Q \partial_P K_M \partial_N \overline{K}_Q \\
&-\frac{1}{2}H_o^{MN} K_M K^P \partial_P \overline{K}_Q \partial_N \overline{K}^Q - \frac{1}{2}H_o^{MN} \overline{K}^2 \partial_Q K_M \partial_N K^Q \\
&-\frac{1}{2}H_o^{MN} K_M \overline{K}^P \partial_N K^Q \partial_Q \overline{K}_P + \frac{1}{2}H_o^{MN} K^P \overline{K}^Q \partial_M K_Q \partial_P \overline{K}_N \\
&-\frac{1}{2}H_o^{MN} K^P \overline{K}_M \partial_P K_Q \partial_N \overline{K}^Q - \frac{1}{2}H_o^{MN} \overline{K}_M \overline{K}^P \partial_Q K_P \partial_N K^Q\,.
\end{aligned} \tag{7.160}$$

Now the generalized dilaton contributions are given by

$$\begin{aligned}
\mathcal{L}_d &= 4H^{MN}\partial_M d\partial_N d - 2\partial_M H^{MN}\partial_N d \\
&= 4\kappa^2\left[H_o^{MN} + \kappa\Big(\bar{K}^M K^N + K^M\bar{K}^N\Big)\right]\partial_M f\partial_N f \\
&\quad -2\kappa^2\partial_M\left[\Big(\bar{K}^M K^N + K^M\bar{K}^N\Big)\right]\partial_N f \ . \qquad (7.161)
\end{aligned}$$

The gauged part can be traced out easily by considering the linear ansatz for the C-field as follows

$$\Delta_{\alpha\beta} - \Delta_{o\alpha\beta} = -2\kappa K^M J_{(\alpha}C_{oM\beta)} \qquad (7.162)$$

$$\Theta_{MN} - \Theta_{oMN} = -2\kappa K_{(M}J^\alpha C_{oN)\alpha} + \kappa^2 K_M K_N J^2 \, . \qquad (7.163)$$

The first-order contributions can be computed using (7.90) with $\Delta^{-1} = \Delta_o^{-1} + \mathcal{O}(\kappa^2)$. Then, the κ contribution for the gauge sector will be

$$4f^\alpha_{\beta\gamma}C^{M\beta}C^{N\gamma}\partial_M(K_N J_\alpha) \ . \qquad (7.164)$$

Chapter 8

Conclusions

In the first part of the thesis, we reviewed the non-Abelian T-duality and its connection with AdS/CFT correspondence. Although non-Abelian T-duality is not an exact symmetry of string theory, it is an elegant solution-generating technique at the supergravity level. It provides new supergravity backgrounds. In this thesis we consider the T-dual backgrounds of $\mathrm{AdS}_5\times\ \mathrm{S}^5$, Klebanov-Witten geometry, Klebanov-Tseytlin background as well background with AdS_3 factor.

We considered various null geodesic in the T-dual backgrounds and examined the Penrose limits. Apart from singular geodesic, Penrose limits lead to PP-wave geometries. We considered the closed string modes propagating in these new PP-wave geometries and briefly mentioned the corresponding field theory duals. We also discussed the supersymmetry analysis for these PP-wave geometries. Although further investigation is needed to identify BMN sector precisely in the dual gauge theory. It would be interesting to explore PP-wave geometry in the non-Abelian T-dual of Klebanov-Strassler background as well λ-deformed supergravity backgrounds. Together with the above, it is shown in the various literature by C. Nunez *et.al.* that the holographic central charge analysis plays a crucial role to constructing CFT duals for non-Abelian T-dual backgrounds. It is interesting to examine the same for the PP-wave geometries by imposing some hard cutoffs along the non-compact directions and exploring the significance of those hard cutoffs in the dual gauge theory.

In the second part of the thesis, we discussed the double Yang-Mills formulation of heterotic DFT. In our picture, the fundamental fields are in $O(D,D)$-framework that has appeared by rewriting the $O(D,D+K)$ fields in terms of $O(D,D)$ multiplets. In this framework, we explored the relaxed version of Generalized Kerr-Schild (GKSA) ansatz to study perturbations of these fields. In relaxed GKSA ansatz, the generalized background metric is perturbed up to quadratic order but the gauge field is linearly perturbed. As an application of it, we explored the classical double copy correspondence at the DFT level. It would be interesting to explore the relaxed GKSA in the context of α'-corrections, non-Abelian T-duality, Fractons as well as in the non-Riemannian geometries.

Appendix A

type-IIB and type-IIA supergravity

In this appendix we discuss basic notions of type-IIB and type-IIA supergravity. We will discuss the action, equations of motion, Bianchi identities together with supersymmetry analysis of type-IIB and type-IIA supergravity. In the following we will start discussions by considering type-IIB supergravity.

A.1 type-IIB supergravity

The action of type-IIB supergravity is given by

$$\begin{aligned} S_{IIB} &= \frac{1}{2k^2}\int_{M_{10}}\sqrt{-g}\left[e^{-2\Phi}\Big(R+4(\partial\Phi)^2-\frac{H^2}{12}\Big)-\frac{1}{2}\Big(F_1^2+\frac{F_3^2}{3!}+\frac{1}{2}\frac{F_5^2}{5!}\Big)\right] \\ &-\frac{1}{2}C_4\wedge H\wedge dC_2\ , \end{aligned} \tag{A.1}$$

where the field strengths in terms of the potentials are

$$H = dB_2\ ,\ F_1 = dC_0\ ,\ F_3 = dC_2 - C_0 H\ ,\ F_5 = dC_4 - H\wedge C_2\ . \tag{A.2}$$

In addition, F_5 has to be self dual. The Bianchi identities are

$$dH = 0\ ,\ dF_1 = 0\ ,\ dF_3 = H\wedge F_1\ ,\ dF_5 = H\wedge F_3\ . \tag{A.3}$$

Einstein's equations that follow varying the metric are

$$R_{\mu\nu}+2D_\mu D_\nu\Phi = \frac{1}{4}H^2_{\mu\nu}+e^{2\Phi}\left[\frac{1}{2}(F_1^2)_{\mu\nu}+\frac{1}{4}(F_3^2)_{\mu\nu}+\frac{1}{96}(F_5^2)_{\mu\nu}-\frac{1}{4}g_{\mu\nu}\Big(F_1^2+\frac{1}{6}F_3^2\Big)\right]\ . \tag{A.4}$$

The equation comes from varying the dilaton is

$$R+4D^2\Phi-4(\partial\Phi)^2-\frac{1}{12}H^2 = 0\ . \tag{A.5}$$

Finally, from the variation of the various fluxes we obtain

$$\begin{aligned}
&d\left(e^{-2\Phi} \star H\right) - F_1 \wedge \star F_3 - F_3 \wedge F_5 = 0 \ , \\
&d \star F_1 + H \wedge \star F_3 = 0 \ , \\
&d \star F_3 + H \wedge F_5 = 0 \ , \\
&d \star F_5 + H \wedge F_3 = 0 \ .
\end{aligned} \tag{A.6}$$

A.2 Massive type-IIA supergravity

For massive IIA, the field strengths are given by

$$H = dB \ , \ F_2 = dC_1 + mB \ , \ F_4 = dC_3 - H \wedge C_1 + \frac{m}{2} B \wedge B \ . \tag{A.7}$$

The field strenghts are invariant under the gauge transformations

$$\delta B = d\Lambda \ , \ \delta C_1 = -m\Lambda \ , \ \delta C_3 = -m\Lambda \wedge B \ , \tag{A.8}$$

where Λ is a one-form. The Bianchi identities become

$$dH = 0 \ , \ dF_2 = mH \ , \ dF_4 = H \wedge F_2 \ . \tag{A.9}$$

The action of the massive IIA supergravity is

$$\begin{aligned}
S_{\text{Massive } IIA} &= \frac{1}{2k^2} \int_{M_{10}} \sqrt{-g} \left[e^{-2\Phi} \left(R + 4(\partial\Phi)^2 - \frac{H^2}{12} \right) - \frac{1}{2} \left(m^2 + \frac{F_2^2}{2} + \frac{F_4^2}{4!} \right) \right] \\
&- \frac{1}{2} \left(dC_3 \wedge dC_3 \wedge B + \frac{m}{3} dC_3 \wedge B^3 + \frac{m^2}{20} B^5 \right) \ .
\end{aligned} \tag{A.10}$$

Einstein's equations are

$$R_{\mu\nu} + 2D_\mu D_\nu \Phi = \frac{1}{4} H^2_{\mu\nu} + e^{2\Phi} \left[\frac{1}{2} (F_2^2)_{\mu\nu} + \frac{1}{12} (F_4^2)_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \left(F_0^2 + \frac{1}{2} F_2^2 + \frac{1}{4!} F_4^2 \right) \right] . \tag{A.11}$$

The equation coming from varying the dilaton is

$$R + 4D^2\Phi - 4(\partial\Phi)^2 - \frac{1}{12} H^2 = 0 \ . \tag{A.12}$$

The equation of motion for the gauge fields in type-IIA case are given by

$$\begin{aligned}
&d\left(e^{-2\Phi} \star H\right) - F_2 \wedge \star F_4 - \frac{1}{2} F_4 \wedge F_4 = m \star F_2 \ , \\
&d \star F_2 + H \wedge \star F_4 = 0 \ , \\
&d \star F_4 + H \wedge F_4 = 0 \ .
\end{aligned} \tag{A.13}$$

A.3 Supersymmetry in type-IIB and type-IIA supergravity

The Killing spinor ϵ consists of real Majorana-Weyl spinors $\epsilon_\pm$, such that

$$\epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} \ . \tag{A.14}$$

In type-IIA supergravity, the Killing spinor ϵ follows the condition $\Gamma_{11}\epsilon = -\sigma_3\epsilon$. Here σ_i's denotes the Pauli matrices with $i = 1, 2, 3$. For the type-IIB supergravity, ϵ satifies $\Gamma_{11}\epsilon = \mathbb{I}_2\epsilon$. The type-$IIA$ Killing spinor equations are

$$\begin{aligned}
\delta\lambda &= \frac{1}{2}\not{\partial}\Phi_\epsilon - \frac{1}{24}\not{H}\sigma_3\epsilon \ + \frac{1}{8}e^\Phi\left[5m\sigma_1 + \frac{3}{2}\not{F}_2\,(i\sigma_2) + \frac{1}{24}\not{F}_4\sigma_1\right]\epsilon \ , \\
\delta\psi_\mu &= D_\mu\epsilon - \frac{1}{8}H_{\mu\nu\rho}\Gamma^{\nu\rho}\sigma_3\epsilon + \frac{1}{8}e^\Phi\left[m\sigma_1 + \frac{1}{2}\not{F}_2\,(i\sigma_2) + \frac{1}{24}\not{F}_4\sigma_1\right]\Gamma_\mu\epsilon \ ,
\end{aligned} \tag{A.15}$$

where $D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\epsilon$.

The Killing spinor equations of type-IIB are

$$\begin{aligned}
\delta\lambda &= \frac{1}{2}\not{\partial}\Phi_\epsilon - \frac{1}{24}\not{H}\sigma_3\epsilon \ + \frac{1}{2}e^\Phi\left[\not{F}_1\,(i\sigma_2) + \frac{1}{12}\not{F}_3\sigma_1\right]\epsilon \ , \\
\delta\psi_\mu &= D_\mu\epsilon - \frac{1}{8}H_{\mu\nu\rho}\Gamma^{\nu\rho}\sigma_3\epsilon - \frac{1}{8}e^\Phi\left[\not{F}_1\,(i\sigma_2) + \frac{1}{6}\not{F}_3\sigma_1 + \frac{1}{240}\not{F}_5\,(i\sigma_2)\right]\Gamma_\mu\epsilon \ ,
\end{aligned} \tag{A.16}$$

where we use the notation $\not{F}_n \equiv F_{i_1\ldots i_n}\Gamma^{i_1\ldots i_n}$.

Bibliography

[1] G. 't Hooft, "*A Planar Diagram Theory for Strong Interactions*," Nucl. Phys. B **72**, 461 (1974) doi:10.1016/0550-3213(74)90154-0

[2] J. M. Maldacena, "*The Large N limit of superconformal field theories and supergravity*," Adv. Theor. Math. Phys. **2**, 231-252 (1998) doi:10.1023/A:1026654312961 [arXiv:hep-th/9711200 [hep-th]].

[3] P. H. Ginsparg, "*APPLIED CONFORMAL FIELD THEORY*," [arXiv:hep-th/9108028 [hep-th]].

[4] E. Witten, "*Anti-de Sitter space and holography*," Adv. Theor. Math. Phys. **2**, 253-291 (1998) doi:10.4310/ATMP.1998.v2.n2.a2 [arXiv:hep-th/9802150 [hep-th]].

[5] I. R. Klebanov and E. Witten, "*Superconformal field theory on three-branes at a Calabi-Yau singularity*," Nucl. Phys. B **536**, 199 (1998) doi:10.1016/S0550-3213(98)00654-3 [hep-th/9807080].

[6] I. R. Klebanov and E. Witten, "*AdS / CFT correspondence and symmetry breaking*," Nucl. Phys. B **556**, 89 (1999) doi:10.1016/S0550-3213(99)00387-9 [hep-th/9905104].

[7] T. H. Buscher, "*A Symmetry of the String Background Field Equations*," Phys. Lett. B **194** (1987) 59. doi:10.1016/0370-2693(87)90769-6

[8] T. H. Buscher, "*Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models*," Phys. Lett. B **201** (1988) 466. doi:10.1016/0370-2693(88)90602-8

[9] A. Sen, "*$O(d) \times O(d)$ symmetry of the space of cosmological solutions in string theory, scale factor duality and two-dimensional black holes*," Phys. Lett. B **271**, 295-300 (1991) doi:10.1016/0370-2693(91)90090-D

[10] M. Rocek and E. P. Verlinde, "*Duality, quotients, and currents*," Nucl. Phys. B **373**, 630 (1992) doi:10.1016/0550-3213(92)90269-H [hep-th/9110053].

[11] X. C. de la Ossa and F. Quevedo, "*Duality symmetries from nonAbelian isometries in string theory*," Nucl. Phys. B **403**, 377-394 (1993) doi:10.1016/0550-3213(93)90041-M [arXiv:hep-th/9210021 [hep-th]].

[12] K. Sfetsos and D. C. Thompson, "*On non-abelian T-dual geometries with Ramond fluxes*," Nucl. Phys. B **846**, 21-42 (2011) doi:10.1016/j.nuclphysb.2010.12.013 [arXiv:1012.1320 [hep-th]].

[13] Y. Lozano, E. O Colgain, K. Sfetsos and D. C. Thompson, "*Non-abelian T-duality, Ramond Fields and Coset Geometries*," JHEP **1106**, 106 (2011) doi:10.1007/JHEP06(2011)106 [arXiv:1104.5196 [hep-th]].

[14] G. Itsios, Y. Lozano, E. O Colgain and K. Sfetsos, "*Non-Abelian T-duality and consistent truncations in type-II supergravity*," JHEP **1208**, 132 (2012) doi:10.1007/JHEP08(2012)132 [arXiv:1205.2274 [hep-th]].

[15] G. Itsios, C. Nunez, K. Sfetsos and D. C. Thompson, "*On Non-Abelian T-Duality and new $\mathcal{N}=1$ backgrounds*," Phys. Lett. B **721**, 342 (2013) doi:10.1016/j.physletb.2013.03.033 [arXiv:1212.4840 [hep-th]].

[16] A. Barranco, J. Gaillard, N. T. Macpherson, C. Nunez and D. C. Thompson, "*G-structures and Flavouring non-Abelian T-duality*," JHEP **1308**, 018 (2013) doi:10.1007/JHEP08(2013)018 [arXiv:1305.7229 [hep-th]].

[17] N. T. Macpherson, C. Núñez, L. A. Pando Zayas, V. G. J. Rodgers and C. A. Whiting, "*Type IIB supergravity solutions with AdS_5 from Abelian and non-Abelian T dualities*," JHEP **02**, 040 (2015) doi:10.1007/JHEP02(2015)040 [arXiv:1410.2650 [hep-th]].

[18] K. S. Kooner and S. Zacarias, "*Non-Abelian T-Dualizing the Resolved Conifold with Regular and Fractional D3-Branes*," JHEP **1508**, 143 (2015) doi:10.1007/JHEP08(2015)143 [arXiv:1411.7433 [hep-th]].

[19] H. Dimov, S. Mladenov, R. C. Rashkov and T. Vetsov, "*Non-abelian T-duality of Pilch-Warner background*," Fortsch. Phys. **64**, 657 (2016) doi:10.1002/prop.201600032 [arXiv:1511.00269 [hep-th]].

[20] H. Dimov, R. C. Rashkov, S. Mladenov and T. Vetsov, "*Non-Abelian T-Duality from Penrose Limit of the Pilch-Warner Solution*," Bulg. J. Phys. **43**, no. 4, 251 (2016).

[21] Y. Lozano and C. Nuenz, "*Field theory aspects of non-Abelian T-duality and $\mathcal{N}=2$ linear quivers*," JHEP **1605**, 107 (2016) doi:10.1007/JHEP05(2016)107 [arXiv:1603.04440 [hep-th]].

[22] Y. Lozano, N. T. Macpherson, J. Montero and C. Nunez, "*Three-dimensional $\mathcal{N}=4$ linear quivers and non-Abelian T-duals*," JHEP **1611**, 133 (2016) doi:10.1007/JHEP11(2016)133 [arXiv:1609.09061 [hep-th]].

[23] G. Itsios, C. Nunez, K. Sfetsos and D. C. Thompson, "*Non-Abelian T-duality and the AdS/CFT correspondence:new $\mathcal{N}=1$ backgrounds*," Nucl. Phys. B **873**, 1 (2013) doi:10.1016/j.nuclphysb.2013.04.004 [arXiv:1301.6755 [hep-th]].

[24] T. R. Araujo and H. Nastase, "*$\mathcal{N} = 1$ SUSY backgrounds with an AdS factor from non-Abelian T duality*," Phys. Rev. D **91**, no. 12, 126015 (2015) doi:10.1103/PhysRevD.91.126015 [arXiv:1503.00553 [hep-th]].

[25] C. A. Whiting, "*Duality symmetries in string-inspired supergravity: T-dualities and the gauge/gravity correspondence*", Ph.D. Thesis, University of Iowa, 2015. https://doi.org/10.17077/etd.qc6n8jma

[26] G. Itsios, H. Nastase, C. Núñez, K. Sfetsos and S. Zacarías, "*Penrose limits of Abelian and non-Abelian T-duals of $AdS_5 \times S^5$ and their field theory duals*," JHEP **1801**, 071 (2018) doi:10.1007/JHEP01(2018)071 [arXiv:1711.09911 [hep-th]].

[27] G. Itsios, Y. Lozano, J. Montero and C. Nunez, "*The AdS_5 non-Abelian T-dual of Klebanov-Witten as a $\mathcal{N} = 1$ linear quiver from M5-branes*," JHEP **1709**, 038 (2017) doi:10.1007/JHEP09(2017)038 [arXiv:1705.09661 [hep-th]].

[28] J. Gaillard, N. T. Macpherson, C. Núñez and D. C. Thompson, "*Dualising the Baryonic Branch: Dynamic SU(2) and confining backgrounds in IIA*," Nucl. Phys. B **884**, 696-740 (2014) doi:10.1016/j.nuclphysb.2014.05.004 [arXiv:1312.4945 [hep-th]].

[29] J. Kowalski-Glikman, "*Vacuum States in Supersymmetric Kaluza-Klein Theory*," Phys. Lett. B **134**, 194-196 (1984) doi:10.1016/0370-2693(84)90669-5

[30] P. T. Chrusciel and J. Kowalski-Glikman, "*The Isometry group and Killing spinors for the p p wave space-time in D = 11 supergravity*," Phys. Lett. B **149**, 107-110 (1984) doi:10.1016/0370-2693(84)91563-6

[31] J. M. Figueroa-O'Farrill and G. Papadopoulos, "*Homogeneous fluxes, branes and a maximally supersymmetric solution of M theory*," JHEP **08**, 036 (2001) doi:10.1088/1126-6708/2001/08/036 [arXiv:hep-th/0105308 [hep-th]].

[32] M. Blau, J. M. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "*A New maximally supersymmetric background of IIB superstring theory*," JHEP **01**, 047 (2002) doi:10.1088/1126-6708/2002/01/047 [arXiv:hep-th/0110242 [hep-th]].

[33] M. Blau, "*Plane Waves and Penrose Limits* " . http://www.blau.itp.unibe.ch/lecturesPP.pdf

[34] M. Blau, J. M. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "*Penrose limits and maximal supersymmetry*," Class. Quant. Grav. **19**, L87-L95 (2002) doi:10.1088/0264-9381/19/10/101 [arXiv:hep-th/0201081 [hep-th]].

[35] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, "*Strings in flat space and pp waves from $\mathcal{N}$=4 superYang-Mills*," JHEP **04**, 013 (2002) doi:10.1088/1126-6708/2002/04/013 [arXiv:hep-th/0202021 [hep-th]].

[36] D. Sadri and M. M. Sheikh-Jabbari, "*The Plane wave / superYang-Mills duality*," Rev. Mod. Phys. **76**, 853 (2004) doi:10.1103/RevModPhys.76.853 [hep-th/0310119].

[37] D. Amati and C. Klimcik, "*Nonperturbative Computation of the Weyl Anomaly for a Class of Nontrivial Backgrounds*," Phys. Lett. B **219**, 443 (1989). doi:10.1016/0370-2693(89)91092-7

[38] G. T. Horowitz and A. R. Steif, "*Space-Time Singularities in String Theory*," Phys. Rev. Lett. **64**, 260 (1990). doi:10.1103/PhysRevLett.64.260

[39] J. Gomis and H. Ooguri, "*Penrose limit of $\mathcal{N}=1$ gauge theories*," Nucl. Phys. B **635**, 106-126 (2002) doi:10.1016/S0550-3213(02)00396-6 [arXiv:hep-th/0202157 [hep-th]].

[40] L. A. Pando Zayas and J. Sonnenschein, "*On Penrose limits and gauge theories*," JHEP **05**, 010 (2002) doi:10.1088/1126-6708/2002/05/010 [arXiv:hep-th/0202186 [hep-th]].

[41] N. Itzhaki, I. R. Klebanov and S. Mukhi, "*PP wave limit and enhanced supersymmetry in gauge theories*," JHEP **03**, 048 (2002) doi:10.1088/1126-6708/2002/03/048 [arXiv:hep-th/0202153 [hep-th]].

[42] J. H. Cho, T. j. Lee and S. Nam, "*Supersymmetries and Hopf duality in the Penrose limit of $AdS_3 \times S^3 \times T^4$*," Nucl. Phys. B **650**, 3-23 (2003) doi:10.1016/S0550-3213(02)01069-6 [arXiv:hep-th/0207252 [hep-th]].

[43] S. Roychowdhury and P. K. Tripathy, "*The non-Abelian T-dual of Klebanov-Witten Background and its Penrose Limits*," JHEP **11**, 125 (2019) doi:10.1007/JHEP11(2019)125 [arXiv:1907.01904 [hep-th]].

[44] I. R. Klebanov and A. A. Tseytlin, "*Gravity duals of supersymmetric SU(N) $\times$ SU(N+M) gauge theories*," Nucl. Phys. B **578**, 123-138 (2000) doi:10.1016/S0550-3213(00)00206-6 [arXiv:hep-th/0002159 [hep-th]].

[45] S. Roychowdhury and P. K. Tripathy, "*Penrose limits in non-Abelian T dual of Klebanov-Tseytlin background*," Phys. Rev. D **104**, no.12, 126016 (2021) doi:10.1103/PhysRevD.104.126016 [arXiv:2109.09263 [hep-th]].

[46] Y. Lozano, N. T. Macpherson, C. Nunez and A. Ramirez, "*AdS_3 solutions in massive IIA, defect CFTs and T-duality*," JHEP **12**, 013 (2019) doi:10.1007/JHEP12(2019)013 [arXiv:1909.11669 [hep-th]].

[47] S. Roychowdhury and P. K. Tripathy, "*Penrose limits in massive type-IIA AdS_3 background*," Phys. Rev. D **105**, 106024 (2022) doi:10.1103/PhysRevD.105.106024 [arXiv:2203.14212 [hep-th]].

[48] C. Hull and B. Zwiebach, "*Double Field Theory*," JHEP **09**, 099 (2009) doi:10.1088/1126-6708/2009/09/099 [arXiv:0904.4664 [hep-th]].

[49] G. Aldazabal, D. Marques and C. Nunez, "*Double Field Theory: A Pedagogical Review*," Class. Quant. Grav. **30**, 163001 (2013) doi:10.1088/0264-9381/30/16/163001 [arXiv:1305.1907 [hep-th]].

[50] A. Catal-Ozer, "*Non-Abelian T-duality as a Transformation in Double Field Theory*," JHEP **08**, 115 (2019) doi:10.1007/JHEP08(2019)115 [arXiv:1904.00362 [hep-th]].

[51] O. Hohm and S. K. Kwak, "*Double Field Theory Formulation of Heterotic Strings*," JHEP **06**, 096 (2011) doi:10.1007/JHEP06(2011)096 [arXiv:1103.2136 [hep-th]].

[52] R.P. Kerr and A. Schild, "*A new class of vacuum solutions of the Einstein field equations*," Proc. Symp. Appl. Math. **17**, (1965) 199.

[53] K. Lee, "*Kerr-Schild Double Field Theory and Classical Double Copy*," JHEP **10**, 027 (2018) doi:10.1007/JHEP10(2018)027 [arXiv:1807.08443 [hep-th]].

[54] W. Cho and K. Lee, "*Heterotic Kerr-Schild Double Field Theory and Classical Double Copy*," JHEP **07**, 030 (2019) doi:10.1007/JHEP07(2019)030 [arXiv:1904.11650 [hep-th]].

[55] E. Lescano and J. A. Rodríguez, "*$\mathcal{N}$ = 1 supersymmetric Double Field Theory and the generalized Kerr-Schild ansatz*," JHEP **10**, 148 (2020) doi:10.1007/JHEP10(2020)148 [arXiv:2002.07751 [hep-th]].

[56] E. Lescano and J. A. Rodríguez, "*Higher-derivative heterotic Double Field Theory and classical double copy*," JHEP **07**, 072 (2021) doi:10.1007/JHEP07(2021)072 [arXiv:2101.03376 [hep-th]].

[57] E. Lescano and S. Roychowdhury, "*Heterotic Kerr-Schild Double Field Theory and its double Yang-Mills formulation*," JHEP **04**, 090 (2022) doi:10.1007/JHEP04(2022)090 [arXiv:2201.09364 [hep-th]].

[58] K. Kim, K. Lee, R. Monteiro, I. Nicholson and D. Peinador Veiga, "*The Classical Double Copy of a Point Charge*," JHEP **02**, 046 (2020) doi:10.1007/JHEP02(2020)046 [arXiv:1912.02177 [hep-th]].

[59] S. F. Hassan, "*T duality, space-time spinors and RR fields in curved backgrounds*," Nucl. Phys. B **568**, 145-161 (2000) doi:10.1016/S0550-3213(99)00684-7 [arXiv:hep-th/9907152 [hep-th]].

[60] M. Kruczenski, "*Wilson loops and anomalous dimensions in cascading theories*," Phys. Rev. D **69**, 106002 (2004) doi:10.1103/PhysRevD.69.106002 [arXiv:hep-th/0310030 [hep-th]].

[61] R. R. Metsaev, "*Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*," Nucl. Phys. B **625**, 70-96 (2002) doi:10.1016/S0550-3213(02)00003-2 [arXiv:hep-th/0112044 [hep-th]].

[62] R. R. Metsaev and A. A. Tseytlin, "*Exactly solvable model of superstring in Ramond-Ramond plane wave background*," Phys. Rev. D **65**, 126004 (2002) doi:10.1103/PhysRevD.65.126004 [arXiv:hep-th/0202109 [hep-th]].

[63] S. S. Gubser and I. Mitra, "*Some interesting violations of the Breitenlohner-Freedman bound*," JHEP **07**, 044 (2002) doi:10.1088/1126-6708/2002/07/044 [arXiv:hep-th/0108239 [hep-th]].

[64] W. Siegel, "*Two vierbein formalism for string inspired axionic gravity*", Phys. Rev. D **47** (1993) 5453 [hep-th/9302036].

W. Siegel, "*Superspace duality in low-energy superstrings*", Phys. Rev. D **48** (1993) 2826 [hep-th/9305073].

W. Siegel, "*Manifest duality in low-energy superstrings*", In *Berkeley 1993, Proceedings, Strings '93* 353-363, and State U. New York Stony Brook - ITP-SB-93-050 (93,rec.Sep.) 11 p. (315661) [hep-th/9308133].

[65] O. Hohm, C. Hull and B. Zwiebach, "*Generalized metric formulation of Double Field Theory*", JHEP **1008** (2010) 008 [hep-th/1006.4823].

I. Jeon, K. Lee and J. H. Park, "*Stringy differential geometry, beyond Riemann*", Phys. Rev. D **84** (2011) 044022 [hep-th/1105.6294].

I. Jeon, K. Lee and J. H. Park, "*Differential geometry with a projection: Application to Double Field Theory*", JHEP **1104** (2011) 014 [hep-th/1011.1324].

[66] O. Hohm, D. Lust and B. Zwiebach, "*The Spacetime of Double Field Theory: Review, Remarks, and Outlook*", Fortsch. Phys. **61** (2013) 926, [hep-th/1309.2977].

D. S. Berman and D. C. Thompson, "*Duality Symmetric String and M-Theory*", Phys. Rept. **566** (2014) 1, [hep-th/1306.2643].

[67] E. Lescano, "*α'-corrections and their double formulation*", J.Phys.A **55** (2022) 5, 053002, [hep-th/2108.12246]

[68] J. Maharana and J. H. Schwarz, "*Noncompact symmetries in string theory*", Nucl. Phys. B 390 (1993) 3 [hep-th/9207016].

[69] O. Hohm and S. K. Kwak, "*Frame-like Geometry of Double Field Theory*", J. Phys. A **44** (2011) 085404 [hep-th/1011.4101].

[70] O. Hohm, "*Background Independent Double Field Theory at Order α': Metric vs. Frame-like Geometry*", Phys.Rev.D 95 (2017) 6, 066018, [hep-th/1612.06453].

[71] A. Sen, "*Macroscopic charged heterotic string*", Nucl. Phys. B 388 (1992) 457 [hep-th/9206016].

[72] D. Garfinkle, G. T. Horowitz and A. Strominger, "*Charged black holes in string theory*", Phys. Rev. D 43 (1991) 3140 Erratum: [Phys. Rev. D 45 (1992) 3888].

G. W. Gibbons and K. i. Maeda, "*Black Holes and Membranes in Higher Dimensional Theories with Dilaton Fields*", Nucl. Phys. B 298 (1988) 741.

[73] R. P. Kerr, "*Gravitational field of a spinning mass as an example of algebraically special metrics*", Phys. Rev. Lett.**11** (1963) 237.

R. P. Kerr and A. Schild, "*A new class of vacuum solutions of the Einstein field equations*", Proc. Symp. Appl. Math.**17**(1965), 199.

G. C. Debney, R. P. Kerr and A. Schild, "*Solutions of the Einstein and Einstein-Maxwell Equations*", J. Math. Phys.**10** , 1842 (1969).

[74] I.Jeon, K. Lee, J-H Park, "*Double field formulation of Yang-Mills theory*", Phys.Lett.B 701 (2011) 260-264, [hep-th/1102.0419].

[75] O. Hohm and B. Zwiebach, '*On the Riemann Tensor in Double Field Theory*,' JHEP **05** (2012) 126 [hep-th/1112.5296].

[76] W. H. Baron, E. Lescano and D. Marques, "*The generalized Bergshoeff-de Roo identification*", JHEP **1811**, 160 (2018), [hep-th/1810.01427].

[77] D. Geissbuhler, D. Marques, C. Nunez and V. Penas, "*Exploring Double Field Theory*", JHEP **1306** (2013) 101, [hep-th/1304.1472].

[78] O. Hohm, A. Sen and B. Zwiebach, "*Heterotic Effective Action and Duality Symmetries Revisited*," JHEP **1502** (2015) 079 doi:10.1007/JHEP02(2015)079 [arXiv:1411.5696 [hep-th]].

[79] "*T-duality and α'-corrections*," JHEP **1510**, 084 (2015), [hep-th/1507.00652].

[80] S. Angus, K. Cho, K. Lee, "*The Classical Double Copy for Half-Maximal Supergravities and T-duality*", JHEP **10** (2021) 211 [hep-th/2105.12857]

[81] D. S. Berman, K. Kim, K. Lee, "*The Classical Double Copy for M-theory from a Kerr-Schild Ansatz for Exceptional Field Theory*", JHEP **04** (2021) 071, [hep-th/2010.08255].

[82] Kyoungho Cho, Kwangeon Kim, Kanghoon Lee, "*The Off-Shell Recursion for Gravity and the Classical Double Copy for currents*", [hep-th/2109.06392].

E. Lescano and M. Mayo, "*Gauged double field theory as an L_∞ algebra*", JHEP 06 (2021) 058, [hep-th/2103.07361].

[83] F. Diaz-Jaramillo, O. Hohm, J. Plefka, "*Double Field Theory as the Double Copy of Yang-Mills*", [hep-th/2109.01153]