E_0-SEMIGROUPS AND PRODUCT SYSTEMS

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A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

to

Chennai Mathematical Institute

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DECLARATION

I hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

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CERTIFICATE

I certify that the thesis entitled **"E_0-semigroups and product systems"** submitted for the degree of **Doctor of Philosophy in Mathematics** by S.P.Murugan is the record of research work carried out by him under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

Chennai Mathematical Institute
Date: March, 2019.

Professor R.Srinivasan
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LIST OF PUBLICATIONS ARISING FROM THE THESIS

Journal

1. S.P.Murugan, S. Sundar. *An essential representation for a product system over a finitely generated subsemigroup of* $\mathbb{Z}^d$. the Proceedings of Math- ematical Sciences.
2. S.P.Murugan, S. Sundar. *On the existence of* E_0 *-semigroups - the multiparameter case*. Infinite Dimensional Analysis, Quantum Probability and Related Topics.

Acknowledgements

First and foremost, I would like to express my deep sincere love and thanks to my father Mr. S. Paramasivam and my mother Mrs. P. Malaiyammal as they brought me up to this stage with their love and care. I thank my brothers S.P.Ganesan and S.P. Shiva Sankar for their support and encouragement.

I express my heartfelt gratitude to my advisor Prof. R.Srinivasan. He unconditionally took me as his student and gave me constant support throughout my Ph.D. life with his immense patience and excellent knowledge. I thank him for introducing me to the theory of E_0-semigroups and Quantum Stochastic Calculus and also for having many discussions on these topics which helped me in entering this branch of Mathematics. I learned many things from him and he clarified even my simple doubts with a smiling face. I am inspired by the way he deals with various situations of life in a very cool manner and having observed this from him helps me a lot in leading a pleasant life. I thank him once again for being my all-time inspiration.

I take this opportunity to thank S.Sundar (IMSc) for his consistent support and encouragement throughout my stay in Chennai Mathematical Institute. He treated me as a brother. I learned from him how to do research in mathematics.

I am indebted to Dr. M. Saravanan and Mr. G. Arun Kumar, my seniors, in Ayya nadar Janakiammal College of Arts and Science, Sivakasi. I fondly recall the joyful Mathematical discussions we had.

I thank all the academic and the administrative members of the Chennai Mathematical Institute for giving a conducive environment to pursue research.

I take this opportunity to thank Ayyanar, Hari Krishnan, Srinivasan(Tech.,IMSC), Raja(IIT) and Anand Karunanithi for their support and encouragement throughout my stay in Chennai.

This thesis is dedicated to all of my friends.

S.P.Murugan

Contents

Introduction

The theory of E_0-semigroups over $\mathbb{R}_+$, initiated by Powers [20] has been an active area of research in mathematics for the past 30 years. Arveson extensively analyzed E_0-semigroups in a sequence of papers (see [3],[4],[5],[6]) and, he introduced the theory of product systems as a tool to study E_0-semigroups. Arveson divided E_0-semigroup naturally into three types: type I, II and III. Arveson completely classified all type I E_0-semigroups up to cocycle conjugacy. It is also known that type II and type III E_0-semigroups exist. In fact, there are uncountably many type II and type III E_0-semigroups. Masaki Izumi [14] has written a survey about the developments of the one-parameter theory up to 2012, to which we refer.

In this thesis, we attempt to generalize this theory to E_0-semigroups over the closed convex cone in $\mathbb{R}^d$. Some notable references where E_0-semigroups over semigroups other than $\mathbb{N}$ and $\mathbb{R}_+$ occure, are [25, 27, 22, 23, 24, 17, 16, 18, 1, 13]. Let P be a closed convex cone in $\mathbb{R}^d$. We assume that P is spanning i.e. $P-P=\mathbb{R}^d$ and pointed i.e.$P\cap -P=\{0\}$. Denote the interior of P by Ω. It is a fact that Ω is dense in P. Also Ω is an ideal in P in the sense that $P+\Omega \subseteq \Omega$. For $x,y \in \mathbb{R}^d$, we write $x \leq_P y$ and $x <_\Omega y$ if $y-x \in P$ and $y-x \in \Omega$ respectively. Throughout, the letter M stands for a von Neumann algebra with separable predual. Denote the predual of M by M_*. For $T \in M_*$ and $A \in M$, we write $T(A)$ by $\langle T, A\rangle$.

We define E_0-semigroups over P by simply replacing $[0,\infty)$ by P as follows. By an E_0-semigroup over P on M, we mean a family $\alpha=\{\alpha_x\}_{x\in P}$ of normal $*$-endomorphisms of M such that the following conditions hold:

1) For $x,y \in P$, $\alpha_x \circ \alpha_y = \alpha_{x+y}$,

2) For $T \in M_*$ and $A \in M$, the map $P \ni x \to \langle T, \alpha_x(A))\rangle \in \mathbb{C}$ is continuous,

3) For $x \in P$, $\alpha_x(1)=1$, and

4) the endomorphism α_0 is identity.

A measurable E_0-semigroup over P on M is a family $\alpha = \{\alpha_x\}_{x\in P}$ of normal $*$-endomorphisms of M which satisfies (1),(3),(4) and the following measurability condition: 2') for $T \in M_*$ and $A \in M$, the map $P \ni x \to \langle T, \alpha_x(A)\rangle \in \mathbb{C}$ is measurable.

When P is clear, we will simply call an E_0-semigroup over P by E_0-semigroup. The first theorem of this thesis is as follows.

Theorem A. *Let $\alpha = \{\alpha_x\}_{x\in P}$ be a measurable E_0-semigroup on a von Neumann algebra M. Assume that α is faithful in the sense that $\bigcap_{x\in\Omega} ker(\alpha_x) = \{0\}$. Then $\alpha = \{\alpha_x\}_{x\in P}$ is an E_0-semigroup on M.*

The following example is an illustration of the Theorem A.

Let $\Gamma_s(\mathcal{H}) = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\text{Ⓢ}^n}$ be the symmetric Fock space over $\mathcal{H}$. For any $\xi \in \mathcal{H}$, let $W(\xi)$ be the unitary operator on $\Gamma_s(\mathcal{H})$ defined on the exponential vectors $\{e(\xi) : \xi \in \mathcal{H}\}$ by the relation $W(\xi)e(\eta) = e^{-1/2||\xi||^2 - \langle \xi, \eta\rangle} e(\xi + \eta)$ for all $\eta \in \mathcal{H}$. Note that the family $\{W(\xi) : \xi \in \mathcal{H}\}$ is irreducible in $\mathcal{B}(\Gamma_s(\mathcal{H}))$ (refer [19]). Let us recall an isometric representation of P.

An isometric representation of P on a separable Hilbert space $\mathcal{H}$ is a map $V : P \to \mathcal{B}(\mathcal{H})$ such that

1) For $x \in P$, V_x is an isometry;

2) For $x, y \in P$, $V_{x+y} = V_x V_y$; and

3) For $\xi \in \mathcal{H}$, the map $P \ni x \to V_x(\xi) \in \mathcal{H}$ is continuous.

Given an isometric representation $\{V_x\}_{x\in P}$ of P, there exists an E_0-semigroup on $\mathcal{B}(\Gamma_s(\mathcal{H}))$, say $\alpha = \{\alpha_x\}_{x\in P}$ satisfying

$$\alpha_x(W(\xi)) = W(V_x\xi) \text{ for all } \xi \in \mathcal{H}.$$

The E_0-semigroup $\alpha = \{\alpha_x\}_{x\in P}$ is called the CCR flow associated to V (see [1]).

Let $\alpha = \{\alpha_x\}_{x\in P}$ be an E_0-semigroup on M. A σ-weakly continuous family $\{U_x\}_{x\in P}$ of unitaries in M is called an α-cocycle if $U_{x+y} = U_x\alpha_x(U_y)$ for all x , $y \in P$. Suppose that $\{U_x\}_{x\in P}$ is an α-cocycle. For $x \in P$, define $\beta_x = Ad(U_x) \circ \alpha_x$. Then $\beta = \{\beta_x\}_{x\in P}$ is an E_0-semigroup over P on M. We call such an E_0-semigroup a cocycle perturbation of α.

We prove the following proposition in this thesis.

Proposition. *Let $\alpha = \{\alpha_x\}_{x\in P}$ be an E_0-semigroup on M. Assume that α is faithful. Suppose $\{U_x : x \in \Omega\}$ is a σ-weakly measurable family of isometries in M such that $U_{x+y} = U_x\alpha_x(U_y)$ for all $x,\ y \in \Omega$. Then there exists a unique σ-weakly continuous family of isometries $\{\widetilde{U}_x\}_{x\in P}$ in M such that $\widetilde{U}_x = U_x$ for $x \in \Omega$. Moreover, if U_x, for $x \in \Omega$, is unitary in M, then $\widetilde{U}_x$, $x \in P$, is unitary in M.*

Arveson introduced the notion of a product system over $[0, \infty)$. He proved that two E_0-semigroups are cocycle conjugate if and only if their product systems are isomorphic. Conversely, Arveson proved every abstract product system can be realized as a product system associated with an E_0-semigroup.

We treat $\mathcal{B}(\mathcal{H})$ as a measurable space where the measurable structure given by σ-weak topology of $\mathcal{B}(\mathcal{H})$. Then $\mathcal{B}(\mathcal{H})$ is a standard Borel space with this measurable structure. Let $p : \Omega \times \mathcal{B}(\mathcal{H}) \to \Omega$ be the projection $p(x, T) = x$.

We define the notion of a concrete product system as in (page-35 [7]) with $(0, \infty)$ replaced by Ω. Let $\mathcal{E}$ be a standard Borel subset of $\Omega \times \mathcal{B}(\mathcal{H})$. For $x \in \Omega$, let

$$\mathcal{E}(x) = \{T \in \mathcal{B}(\mathcal{H}) : (x, T) \in \mathcal{E}\}.$$

We say that $\mathcal{E}$ is a concrete product system over Ω if the following conditions hold:

1) The map $p : \mathcal{E} \to \Omega$ is onto.

2) For $x \in \Omega$, the set of operators $\mathcal{E}(x)$ is a linear subspace of $\mathcal{B}(\mathcal{H})$. For $T, S \in \mathcal{E}(x)$, T^*S is a scalar which we denote by $\langle T, S\rangle$. With respect to this inner product, $\mathcal{E}(x)$ is a separable Hilbert space.

3) For $x, y \in \Omega$, the linear span of $\{TS : T \in \mathcal{E}(x), S \in \mathcal{E}(y)\}$ is dense in $\mathcal{E}(x + y)$.

4) There exists a sequence $\{V_n\}_{n\in\mathbb{N}}$ of measurable maps from Ω to $\mathcal{B}(\mathcal{H})$ such that for every $x \in \Omega$, the set $\{V_n(x) : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{E}(x)$.

Let $\alpha = \{\alpha_x\}_{x\in P}$ be a fixed E_0-semigroup. For $x \in \Omega$, the intertwining space for α_x is the set

$$\mathcal{E}(x) = \{T \in \mathcal{B}(\mathcal{H}) : \alpha_x(A)T = TA \ \forall\ A \in \mathcal{B}(\mathcal{H})\}.$$

Then, $\mathcal{E}(x)$ is a separable Hilbert space with the inner product is given by $\langle T, S\rangle = T^*S$. Let $\mathcal{E}_\alpha = \coprod_{x\in\Omega} \mathcal{E}(x)$ be the disjoint union of family $\{\mathcal{E}(x)\}_{x\in\Omega}$ of Hilbert spaces associated with an E_0-semigroup α.

Theorem B. *Let $\alpha = \{\alpha_x\}_{x\in P}$ and $\beta = \{\beta_x\}_{x\in P}$ be E_0 semigroups on $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ respectively. Then α and β are cocycle conjugate if and only if $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are isomorphic.*

Fix $d \geq 1$. Let $S \subset \mathbb{Z}^d$ be a non-zero finitely generated subsemigroup. Assume that $S - S = \mathbb{Z}^d$.

By a discrete product system of Hilbert spaces over S, we mean a disjoint union of non-zero separable Hilbert spaces over S, say $E := \coprod_{t\in P} E(t)$, such that

(1) the Hilbert space $E(0) = \mathbb{C}$,

(2) for $s, t \in S$, there exists a unitary $u_{s,t} : E(s) \otimes E(t) \to E(s + t)$ such that for $r, s, t \in S$, $u_{r+s,t}(u_{r,s} \otimes id) = u_{r,s+t}(id \otimes u_{s,t})$, and

(3) for $s \in S$, $u_{s,0}(x \otimes \lambda) = \lambda x$ and $u_{0,s}(\lambda \otimes x) = \lambda x$ for $\lambda \in E(0)$ and $x \in E(s)$.

It is known that a product system over S is isomorphic to a product system associated to an E-semigroup over S. This is due to the fact that any product system has representation on a separable Hilbert space. (See Lemma 1.10 of [12]).

We state the Theorem which is proved in this thesis.

Theorem C. *Let S be a finitely generated sub-semigroup of $\mathbb{Z}^d$. Let $E := \coprod_{t\in S} E(t)$ be a discrete product system over S. Then the discrete product system E admits an essential representation. Consequently, there exists an E_0-semigroup over S whose product system is isomorphic to E.*

By an abstract product system over Ω, we mean a standard Borel space E together with a measurable surjection $p : E \to \Omega$ such that the following holds.

(1) For $x \in \Omega$, $E(x) := p^{-1}(x)$ is a non-zero separable Hilbert space.

(2) There exists an associative multiplication $E \times E \ni (u, v) \to uv \in E$ such that $p(uv) = p(u) + p(v)$ for $u, v \in E$. Also the multiplication $E \times E \ni (u, v) \to uv \in E$ is measurable.

(3) Let $x, y \in \Omega$ be given. Then there exists a unitary $u_{x,y} : E(x) \otimes E(y) \to E(x + y)$ such that $u_{x,y}(u \otimes v) = uv$ for $(u, v) \in E(x) \times E(y)$.

(4) Let $\Delta := \{(u, v) \in E \times E : p(u) = p(v)\}$. The maps $\Delta \ni (u, v) \to u + v \in E$ and $\Delta \ni (u, v) \to \langle u|v\rangle \in \mathbb{C}$ are measurable.

(5) The map $\mathbb{C} \times E \ni (\lambda, u) \to \lambda u \in E$ is measurable.

(6) As a measurable field of Hilbert spaces, E is trivial by which we mean the following: There exists a separable Hilbert space $\mathcal{H}_0$ and a Borel isomorphism $\theta : E \to \Omega \times \mathcal{H}_0$ such that $\pi_1 \circ \theta = p$ and for every $x \in \Omega$, the map $\pi_2 \circ \theta : E(x) \to \mathcal{H}_0$ is unitary. Hereby π_1 and π_2, we mean the first and second projections from $\Omega \times \mathcal{H}_0$ onto Ω and $\mathcal{H}_0$ respectively. We shall abbreviate the above by saying that as a measurable field of Hilbert spaces, E is isomorphic to $\Omega \times \mathcal{H}_0$ and write $E \simeq \Omega \times \mathcal{H}_0$.

The following question arises naturally: Is every product system isomorphic to a concrete one that arises from E_0-semigroup? Arveson settled the question in affirmative when $P = [0, \infty)$, using the machinery of the spectral C^* -algebra of a product system. The proof is technical and long. Later Skeide in [26] found a simpler proof. Consequently, Arveson himself found a simpler proof in [8]. We should also mention that Skeide proved in [26] that the E_0 -semigroups arising out of his construction and Arvesons construction are in fact conjugate and not just cocycle conjugate.

Let $E := \coprod_{x \in \Omega} E(x)$ and $F := \coprod_{x \in \Omega} F(x)$ be abstract product systems over Ω. We say that E is isomorphic to F if for every $x \in \Omega$, there exists a unitary map $\theta_x : E(x) \to F(x)$ such that

(1) for $x, y \in \Omega$ and $(u, v) \in E(x) \times E(y)$, $\theta_{x+y}(uv) = \theta_x(u)\theta_y(v)$, and

(2) the map $\theta := \coprod_{x \in \Omega} \theta_x : E \to F$ is a Borel isomorphism.

The main theorem of this thesis is as follows.

Theorem D. *Let $E := \coprod_{x \in \Omega} E(x)$ be a product system over Ω. Then there exists an E_0-semigroup $\alpha := \{\alpha_x\}_{x \in P}$ such that E is isomorphic to the product system associated to α.*

The organization of this thesis is as follows:

1) In the first chapter, we deal with the preliminaries.

2) In Chapter Two, we prove Theorem A, Proposition and Theorem B.

3) In Chapter Three, we prove Theorem C.

4) In Chapter Four, we prove Theorem D.

Chapter 1

Preliminaries

In this chapter, we collect ingredients which are necessary for understanding this thesis. The material in this section is well known and borrowed from Arveson [2]. We have included the details to make the thesis self-contained.

1.1 Normal $*$-endomorphism and intertwining space

For a fixed complex separable Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ denote the algebra of all bounded operators on $\mathcal{H}$ and the algebra of all compact operators on $\mathcal{H}$ respectively. Let $\mathcal{L}^1(\mathcal{H})$ be the set of trace class operators on $\mathcal{H}$. Note that $\mathcal{L}^1(\mathcal{H})$ is the predual of $\mathcal{B}(\mathcal{H})$. Denote the identity operator on $\mathcal{H}$ by 1. Put $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. For $A \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{L}^1(\mathcal{H})$, we shall write $Tr(AT)$ by $\langle A, T\rangle$.

Definition 1.1.1. *By a normal $*$-endomorphism, we mean a map $\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ which satisfies the following conditions:*

1) The map α is a $*$-homomorphism, and

2) For $A \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{L}^1(\mathcal{H})$, the map $\mathcal{B}(\mathcal{H}) \ni A \to \langle \alpha(A), T\rangle \in \mathbb{C}$ is continuous.

We say that α is unital if $\alpha(1) = 1$.

Let $End(\mathcal{B}(\mathcal{H}))$ be the set of all normal $*$-endomorphism of $\mathcal{B}(\mathcal{H})$. Let $\mathcal{E}$ be the set of all separable norm closed subspace E of $\mathcal{B}(\mathcal{H})$ such that for $S, T \in E$, S^*T is a scalar.

For a normal $*$-endomorphism $\alpha \in End(\mathcal{B}(\mathcal{H}))$, let

$$E(\alpha) := \{T \in \mathcal{B}(\mathcal{H}) : \alpha(A)T = TA \ \forall \ A \in \mathcal{B}(\mathcal{H})\}.$$

Then $E(\alpha)$ is called the intertwining space of α. Note that:

1) the set $E(\alpha)$ is a norm closed subspace of $\mathcal{B}(\mathcal{H})$, and

2) for S, $T \in E(\alpha)$, S^*T is a scalar and denote the scalar by $\langle S, T\rangle$

The fact (2) allows us to define an inner product on $E(\alpha)$ as follows, for S, $T \in E(\alpha)$ let $\langle S, T\rangle = S^*T$. Note that the operator norm on $E(\alpha)$ coincides with the norm defined by this inner product on E. Therefore $E(\alpha)$ is a Hilbert space with respect to this inner product.

Lemma 1.1.2. *For $d \in \mathbb{N}$, let $\{V_i\}_{i=1}^d$ be isometries on $\mathcal{H}$ such that $V_i^*V_j = 0$ for $i \neq j$ and $\sum_{i=1}^{d} V_iV_i^*$ is a projection. Then for $A \in \mathcal{B}(\mathcal{H})$ and $\xi \in \mathcal{H}$, we have $||\sum_{i=1}^{d} V_iAV_i^*\xi||^2 \leq ||A||^2||\xi||\ ||\sum_{i=1}^{d} V_iV_i^*\xi||$*

Proof: Let $A \in \mathcal{B}(\mathcal{H})$ and $\xi \in \mathcal{H}$ be given. Note that for $i \neq j$, $V_i^*V_j = 0$. Now calculate as follows to observe that

$$\begin{aligned}
||\sum_{i=1}^{d} V_iAV_i^*\xi||^2 &= \sum_{i=1}^{d} ||V_iAV_i^*\xi||^2 \\
&\leq \sum_{i=1}^{d} ||A||^2||V_i^*\xi||^2 \\
&\leq ||A||^2 \sum_{i=1}^{d} \langle V_iV_i^*\xi, \xi\rangle \\
&\leq ||A||^2||\xi|| \sum_{i=1}^{d} ||V_iV_i^*\xi||.
\end{aligned}$$

This completes the proof. □

Remark 1.1.3. *Suppose α is a $*$-endomorphism of $\mathcal{B}(\mathcal{H})$. Assume that α is continuous with respect to the weak operator toplogy. Then α is normal (Thm.3.12 [28]).*

Proposition 1.1.4. *Let (V_n) be a sequence of isometries on $\mathcal{H}$ such that $V_i^*V_j = 0$. Let $Q = \sum_{i=1}^{\infty} V_iV_i^*$. Then*

1) For $A \in \mathcal{B}(\mathcal{H})$, $\sum_{i=1}^{\infty} V_iAV_i^*$ converges strongly.

2) If $\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is defined by the relation $\alpha(A) = \sum_{i=1}^{\infty} V_i A V_i^*$, for all $A \in \mathcal{B}(\mathcal{H})$, then α is a normal $*$-endomorphism.

Proof : Suppose $A \in \mathcal{B}(\mathcal{H})$ and $\xi \in \mathcal{H}$ are given. Let For every $k \geq 1$, put $S_k = \sum_{i=1}^{k} V_i A V_i^* \xi$. We claim that (S_k) is a Cauchy sequence in $\mathcal{H}$. Fix m and n with $m \geq n$. By Lemma 1.2., we have $||S_m - S_n||^2 = \sum_{i=n+1}^{m} ||V_i A V_i^* \xi||^2 \leq ||A||^2 ||\xi|| \sum_{i=n+1}^{m} ||V_i V_i^* \xi||$.

Since $\sum_{i=1}^{n} V_i V_i^*$ converges strongly to the projection Q, (S_k) is a Cauchy sequence in $\mathcal{H}$. This proves (1).

For $k \geq 1$, let $\alpha_k : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be the normal $*$-endomorphism of $\mathcal{B}(\mathcal{H})$ defined by the formula $\alpha_k(A) = \sum_{i=1}^{k} V_i A V_i^*$ for all $A \in \mathcal{B}(\mathcal{H})$. Define a map $\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by the relation $\alpha(A) = \sum_{i=1}^{\infty} V_i A V_i^*$ for all $A \in \mathcal{B}(\mathcal{H})$. By (1), α is well-defined. We claim that α is a normal $*$-endomorphism of $\mathcal{B}(\mathcal{H})$. Since $\alpha_k(A) \to \alpha(A)$ strongly for any A and α_k is an endomorphism, α is a $*$-endomorphism of $\mathcal{B}(\mathcal{H})$.

Suppose $A \in \mathcal{B}(\mathcal{H})$ is positive. Let (A_λ) be a bounded increasing net of positive elements in $\mathcal{B}(\mathcal{H})$ such that (A_λ) converges weakly to A. We claim that $(\alpha(A_\lambda))$ converges weakly to $\alpha(A)$. Let $\xi \in \mathcal{H}$. Calculate as follows to observe that

$$
\begin{aligned}
|\langle \alpha(A_\lambda)\xi, \xi\rangle - \langle \alpha(A)\xi, \xi\rangle| &= |\langle \alpha(A_\lambda)\xi - \alpha(A)\xi, \xi\rangle| \\
&\leq |\langle \alpha(A_\lambda)\xi - \alpha_k(A_\lambda)\xi, \xi\rangle| + |\langle \alpha_k(A_\lambda)\xi - \alpha_k(A)\xi, \xi\rangle| \\
&+ |\langle \alpha_k(A)\xi - \alpha(A)\xi, \xi\rangle| \\
&\leq ||A_\lambda||^2 ||\xi|| \sum_{i=k+1}^{\infty} ||V_i V_i^* \xi|| + |\langle \alpha_k(A_\lambda)\xi - \alpha_k(A)\xi, \xi\rangle| \\
&+ |\langle \alpha_k(A)\xi - \alpha(A)\xi, \xi\rangle|
\end{aligned}
$$

The above estimate yields our claim. □

We need the following standard result about the representation theory of the algebra of compact operators.

Theorem 1.1.5. *Let α be a representation of $\mathcal{K}(\mathcal{H})$ on a separable Hilbert space $\widetilde{\mathcal{H}}$. Then there exists a separable Hilbert space L and an isometry $U : \mathcal{H} \otimes L \to \widetilde{\mathcal{H}}$ such that $\alpha(A) = U(A \otimes 1)U^*$ for all $A \in \mathcal{K}(\mathcal{H})$. (page 19 [2])*

Theorem 1.1.6. *Let α be a normal $*$-endomorphism of $\mathcal{B}(\mathcal{H})$. Then there exists $d \in \mathbb{N}_\infty$ and isometries $\{V_i\}_{i=1}^d$ on $\mathcal{H}$ such that $V_i^* V_j = 0$ for $i \neq j$. Then $\alpha(A) = \sum_{i=1}^{d} V_i A V_i^*$ for all $A \in \mathcal{B}(\mathcal{H})$.*

Proof: Consider α restricted to $\mathcal{K}(\mathcal{H})$. Suppose α restricted to $\mathcal{K}(\mathcal{H})$ is non-degenerate representation. By Theorem 1.1.5, there exists a separable Hilbert space L and an unitary $U : \mathcal{H} \otimes L \to \mathcal{H}$ such that $\alpha(A) = U(A \otimes 1)U^*$ for all $A \in \mathcal{K}(\mathcal{H})$. The Hilbert spaces $\mathcal{H} \otimes L$ and $\bigoplus_i^d H$ can be identified through the unitary $W : \mathcal{H} \otimes L \to \bigoplus_i^d H$ which is given by the equation $W(\xi \otimes \eta) = \bigoplus_i^d \langle \xi_i, \eta \rangle \xi$ where $\{\xi_i\}$ is a fixed ortho-normal basis for L. For any i, let $W_i : \mathcal{H} \to \bigoplus_i^d \mathcal{H}$ be the isometry defined by the formula $W_i(\xi) = (0, 0, 0..., \xi, 0, 0, ...)$ for all $\xi \in \mathcal{H}$. Set $V_i = UW_i$. Then V_i is an isometry on $\mathcal{H}$. Note that for $i \neq j$, $V_i^* V_j = 0$. It is now an easy verification that $\alpha(A) = \sum_{i=1}^{d} V_i A V_i^*$ for all $A \in \mathcal{K}(\mathcal{H})$. Since $\mathcal{K}(\mathcal{H})$ is strongly dense in $\mathcal{B}(\mathcal{H})$, $\alpha(A) = \sum_{i=1}^{d} V_i A V_i^*$ for all $A \in \mathcal{B}(\mathcal{H})$. □

Proposition 1.1.7. *Let α be a normal $*$-endomorphism of $\mathcal{B}(\mathcal{H})$. Then*

1) *The intertwining space of α, $E(\alpha)$ is a non-zero separable Hilbert space, and*

2) *The projection $\alpha(1) = [E(\alpha)\mathcal{H}]$ onto the closed vector subspace of $\mathcal{H}$ generated by $E(\alpha)\mathcal{H} = \{T\xi : T \in E(\alpha), \xi \in \mathcal{H}\}$.*

Proof: By Theorem 1.1.6, there exists $d \in \mathbb{N}_\infty$ and isometries $V_1, V_2, ...V_d$ on $\mathcal{H}$ such that $V_i^* V_j = 0$ for $i \neq j$ and $\alpha(A) = \sum_{i=1}^{d} V_i A V_i^*$ for all $A \in \mathcal{B}(\mathcal{H})$. Note that $\{V_i\}$ in $E(\alpha)$ forms an orthonormal basis. Hence, $E(\alpha)$ is an non-zero separable Hilbert space. This proves (1).

Let $[E(\alpha)\mathcal{H}]$ be a closed vector subspace of $\mathcal{H}$ generated by $E(\alpha)\mathcal{H}$. We shall denote by the projection $[E(\alpha)\mathcal{H}]$ corresponds to $[E(\alpha)\mathcal{H}]$ closed vector subspace of $\mathcal{H}$. Note that $\alpha(1) \geq [E(\alpha)\mathcal{H}]$. Since $V_iV_i^* \leq [E(\alpha)\mathcal{H}]$, $\alpha(1) = [E(\alpha)\mathcal{H}]$. This proves (2) □

Let $\mathcal{E}$ be the set of all separable norm closed subspace E of $\mathcal{B}(\mathcal{H})$ such that for S, $T \in E$, S^*T is a scalar.

Theorem 1.1.8. *The map $End(\mathcal{B}(\mathcal{H})) \ni \alpha \to E(\alpha) \in \mathcal{E}$ is a bijection.*

Proof: Let $\pi : End(\mathcal{B}(\mathcal{H})) \to \mathcal{E}$ be map defined by $\pi(\alpha) = E(\alpha)$ for all $\alpha \in End(\mathcal{B}(\mathcal{H}))$.

Claim: π is one-one. Let α, $\beta \in End(\mathcal{B}(\mathcal{H}))$. Suppose $E(\alpha) = E(\beta)$. By theorem 1.1.6, there exists $d \in \mathbb{N}_\infty$ and isometries $V_1, V_2, ...V_d$ on $\mathcal{H}$ such that $V_i^*V_j = 0$ for $i \neq j$ and $\alpha(A) = \sum_{i=1}^{d} V_iAV_i^*$ for all $A \in \mathcal{B}(\mathcal{H})$. Since $E(\alpha) = E(\beta)$, we have $\beta(A)V_i = V_iA$ for all $1 \leq i \leq d$. Note that $\sum_{i}^{n} V_iV_i^*$ converges strongly to the projection $\beta(1)$. Since $\beta(A)V_iV_i^* = V_iAV_i^*$, $\alpha = \beta$. □

Note that E is Hilbert space with respect to the inner-product defined by the relation $\langle S, T\rangle = S^*T$ for all S, $T \in E$. Let $V_1, V_2, ..., V_d$, where $d = dim(E)$ be an orthonormal basis for E such that $V_i^*V_j = 0$ for $i \neq j$. By Theorem 1.1.6, normal $*$-endomorphism defined by the relation $\alpha(A) = \sum_{i=1}^{d} V_iAV_i^*$ for all $A \in \mathcal{B}(\mathcal{H})$. Therefore $E(\alpha) = E$. This proves π is onto. □

Proposition 1.1.9. *Let α and β be two normal $*$-endomorphism of $\mathcal{B}(\mathcal{H})$ Suppose $E(\alpha)$ and $E(\beta)$ are intertwining spaces for α and β respectively. Then we have*

(i) For $S \in E(\alpha)$ and $T \in E(\beta)$, $ST \in E(\alpha \circ \beta)$.

(ii) For $S, S' \in E(\alpha)$ and $TT' \in E(\beta)$, $\langle ST, S'T'\rangle = \langle S, S'\rangle\langle T, T'\rangle$

(iii) the linear span of $\{TS : T \in E(\alpha), S \in E(\beta)\}$ is dense in $E(\alpha \circ \beta)$.

(Prop.2.4.1 [2])

1.2 Tensor Product of Normal $*$-Homomorphism

Let α and β be normal $*$-endomorphism of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{L})$ respectively. Let $E(\alpha)$ and $E(\beta)$ be the intertwining spaces for α and β respectively. Put $E(\alpha) \otimes E(\beta)$ = norm closure of linear span of $\{S \otimes T : S \in E(\alpha), T \in E(\beta)\}$ in $\mathcal{B}(\mathcal{H} \otimes \mathcal{L})$. Then $E(\alpha) \otimes E(\beta)$ is norm closed and separable subspace of $\mathcal{B}(\mathcal{H} \otimes \mathcal{L})$ such that $S, T \in E(\alpha) \otimes E(\beta)$, S^*T is scalar operator.

Lemma 1.2.1. *Let α and β be normal $*$-endomorphism of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{L})$ respectively. Then there exists unique normal $*$-homomorphism on $\mathcal{B}(\mathcal{H} \otimes \mathcal{L})$ say $\alpha \otimes \beta$, such that $\alpha \otimes \beta(A \otimes B) = \alpha(A) \otimes \beta(B)$ for all $A \otimes B \in \mathcal{B}(\mathcal{H} \otimes \mathcal{L})$.*

Proof: By Theorem 1.1.6, there exist unique normal $*$-endomorphism γ on $\mathcal{B}(\mathcal{H} \otimes \mathcal{L})$ such that $E(\gamma) = E(\alpha) \otimes E(\beta)$. First, we claim that the projection $\gamma(1)$ and $\alpha(1) \otimes \beta(1)$ are equal. Note that $\gamma(1)$ and $\alpha(1) \otimes \beta(1)$ are projections onto $E(\gamma)\mathcal{H}$ and $[E(\alpha)\mathcal{H}] \otimes [E(\beta)\mathcal{H}]$ respectively. Now observe that $\{S \otimes T(\xi \otimes \eta) : S \in E(\alpha), T \in E(\beta) \text{ and } \xi, \eta \in \mathcal{H}\}$ is a total set of both $[E(\gamma)\mathcal{H}]$ and $[E(\alpha)\mathcal{H}] \otimes [E(\beta)\mathcal{H}]$. Therefore, $\gamma(1) = \alpha(1) \otimes \beta(1)$.

Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{L})$ be given. Note that the range of $\gamma(A \otimes B)$ is contained in $\gamma(1)$. Hence it suffices to show that $\gamma(A \otimes B) = \alpha(A) \otimes \beta(B)$ on $\{S \otimes T(\xi \otimes \eta) : S \in E(\alpha), T \in E(\beta) \text{and} \xi, \eta \in \mathcal{H}\}$. Let $S \in E(\alpha), T \in E(\beta)$ and ξ,$\eta \in \mathcal{H}$ be given. Now calculate as follows to observe that

$$\begin{aligned}\gamma(A \otimes B)(S\xi \otimes T\eta) &= \gamma(A \otimes B)S \otimes T(\xi \otimes \eta)\\ &= (S \otimes T)(A \otimes B)(\xi \otimes \eta)\\ &= (SA \otimes TB)(\xi \otimes \eta)\\ &= \alpha(A) \otimes \beta(B)(S\xi \otimes T\eta)\end{aligned}$$

Therefore, $\gamma(A \otimes B) = \alpha(A) \otimes \beta(B)$. This proves our claim.

1.3 Measurable Field of Hilbert spaces

Let X be any non-empty set. We denote a σ-algebra of subsets of X by $\mathcal{B}_X$. The letter $\mathcal{H}$ stands for a complex Hilbert space.

A family $\{\mathcal{H}_x\}_{x \in X}$ of Hilbert spaces indexed by a measurable set X is called a field of Hilbert spaces. Let $\prod_{x \in X} \mathcal{H}_x$ be the Cartesian product of the family $\{\mathcal{H}_x\}_{x \in X}$. We denote the vector subspace of $\prod_{x \in X} \mathcal{H}_x$ by $\mathcal{V}$. An element f of $\mathcal{V}$ is a map $X \ni x \to f(x) \in \bigcup_{x \in X} \mathcal{H}_x$ such that $f(x) \in \mathcal{H}_x$ for all $x \in X$. In this case, we call f is a vector field.

Definition 1.3.1. *By a measurable field of Hilbert spaces, we mean a field* $\{\mathcal{H}_x\}_{x\in X}$ *of Hilbert spaces together with a vector subspace* $\mathcal{V}$ *of* $\prod_{x\in X}\mathcal{H}_x$ *which satisfies the following conditions:*

1) *For* $x \in X$, $\mathcal{H}_x$ *is non-zero;*

2) *For* $x \in X$, *the map* $X \ni x \to ||f(x)|| \in \mathbb{C}$ *is measurable;*

3) *If* $g \in \prod_{x\in X}\mathcal{H}_x$ *is such that the map* $X \ni x \to \langle f(x), g(x)\rangle \in \mathbb{C}$ *is measurable for* $f \in \mathcal{V}$, *then* $g \in \mathcal{V}$;

4) *There exists a sequence* (f_n) *in* $\mathcal{V}$ *such that for* $x \in X$, *the set* $\{f_n(x)\}$ *forms total set in* $\mathcal{H}_x$.

We denote the measurable field of Hilbert spaces by $(\{\mathcal{H}_x\}, \mathcal{V})$. For each $x \in X$, we denote the dimension of $\mathcal{H}_x$ by $d(x)$. Condition (3) ensures that each $\mathcal{H}_x$ is a separable Hilbert space.

Now, we state the basic properties of a measurable field of Hilbert spaces which will be useful in the following chapters. The next Theorem gives some information about the basic properties of a measureble field of Hilbert spaces. The proof of the following theorem contained in (page 221 [11]).

Theorem 1.3.2. *Let* $(\{H_x\}, \mathcal{V})$ *be a measurable field of Hilbert spaces. Suppose the sequence* (f_n) *in* $\mathcal{V}$ *such that for* $x \in X$, *the set* $\{f_n(x)\}$ *forms total set in* $\mathcal{H}_x$. *Then for* $1 \leq m \leq \infty$, *the set* $\{x \in X : d(x) = m\}$ *is measurable. Moreover, there is a sequence* (g_n) *of vector fields with the following properties:*

i. *for each* $x \in X$, $\{g_n(x) : 1 \leq n \leq d(x)\}$ *is an orthonormal basis for* $\mathcal{H}_x$.

ii. *for each* n, *there is a measurable partition* $\{X_k^n\}_{k=1}^{\infty}$ *of* X *such that on each* X_k^n, $g_n(x)$ *is a finite linear combination of the* $f_n(x)$*'s with coefficients depending measurably on* x.

1.4 Standard Borel space

Let X be a topological space. We say X is a Polish space if the space X is homeomorphic to a separable complete metric space. We mention some of the facts about Polish space.

- An open subspace of Polish space is also Polish.
- A closed subspace of a Polish space is also Polish.
- Every compact metric space is a Polish space.

We shall denote the Borel σ-algebra on X generated by open subsets of X by $\mathcal{B}_X$. An element belongs to $\mathcal{B}_X$ is called a measurable subset of X. The pair $(X, \mathcal{B}_X)$ is called measurable space.

Let $(Y, \mathcal{B}_Y)$ be another measurable space. A map $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is called a measurable isomorphism if it is a bijection and both f and f^{-1} is measurable functions.

Definition 1.4.1. *A Borel space $(X, \mathcal{B}_X)$ is called standard if X is measurable isomorphic to a Borel subset of a Polish space.*

Let $\mathcal{H}$ be a separable Hilbert space. Let $B_1 = \{T \in \mathcal{B}(\mathcal{H}) : ||T|| \leq 1\}$ be the closed unit ball in $\mathcal{B}(\mathcal{H})$. Then B_1 is a compact with respect to the weak operator topology. Let $\{\xi_n \in \mathcal{H} : n \in \mathbb{N}\}$ be countable dense subset of unit vectors in $\mathcal{H}$. Let d be a metric defined on B_1 by the following equation

$$d(S, T) = \sum_{i,j=1}^{\infty} 2^{-i-j} |\langle S\xi_i - T\xi_i, \xi_j \rangle|$$

for all S ,$T \in B_1$. It is now an easy verification that the metric gives rise to the weak operator topology on B_1. Therefore, B_1 is a Polish space. The set $\{T \in \mathcal{B}(\mathcal{H}) : ||T|| < 1\}$ is Borel subset of B_1 becasue $\{T \in \mathcal{B}(\mathcal{H}) : ||T|| < 1\} = \bigcup_n \{T \in \mathcal{B}(\mathcal{H}) : ||T|| \leq 1 - 1/n\}$.

We treat $\mathcal{B}(\mathcal{H})$ as a measurable space where the σ-algebra of subsets of $\mathcal{B}(\mathcal{H})$ is generated by the weak operator topology.

Proposition 1.4.2. *$\mathcal{B}(\mathcal{H})$ is a standard Borel space.*

Proof: Let $\{\xi_n \in \mathcal{H} : n \in \mathbb{N}\}$ be countable dense subset of unit vectors in $\mathcal{H}$. Note that the map $\mathcal{B}(\mathcal{H}) \ni T \to ||T|| \in \mathbb{R}$ is measurable because $||T|| = \sup_{i,j} |\langle T\xi_i, \xi_j \rangle|$. Let $f : \mathcal{B}(\mathcal{H}) \to \{T \in \mathcal{B}(\mathcal{H}) : ||T|| < 1\}$ be a bijective map defined by the formula $f(T) = T/(1 + ||T||)$ for all $T \in \mathcal{B}(\mathcal{H})$. Observe that f is a measurable function. The inverse of f is given by the equation $f^{-1}(S) = S/(1 - ||S||)$ for all $S \in \{T \in \mathcal{B}(\mathcal{H}) : ||T|| < 1\}$. Also note that the inverse of f is a measurable function. This proves our claim. □

Chapter 2

Concrete Product Systems

Let P be a closed convex cone in $\mathbb{R}^d$. We assume that P is spanning i.e. $P - P = \mathbb{R}^d$ and pointed i.e.$P \cap -P = \{0\}$. Denote the interior of P by Ω. Note that Ω is an ideal in P in the sense that $P + \Omega \subseteq \Omega$. For $x, y \in \mathbb{R}^d$, we write $x \leq_P y$ and $x <_\Omega y$ if $y - x \in P$ and $y - x \in \Omega$ respectively. For a Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ denotes the algebra of bounded operators on $\mathcal{H}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $U : \mathcal{H}_1 \to \mathcal{H}_2$ be unitary. We denote the map $B(\mathcal{H}_1) \ni T \to UTU^* \in B(\mathcal{H}_2)$ by $Ad(U)$.

For the rest of this chapter, the letter M stands for a von Neumann algebra with separable predual. Denote the predual of M by M_*. For $T \in M_*$ and $A \in M$, we write $T(A)$ by $\langle T, A \rangle$.

2.1 Definition of E_0 semigroup

The following definition of E_0-semigroup is inspired by Defn.2.0.4 of Arveson [7]. We simply replace $[0, \infty)$ by the closed convex cone P in $\mathbb{R}^d$. Let us recall that definition.

Definition 2.1.1. *By an E_0-semigroup over P on M, we mean a family $\alpha = \{\alpha_x\}_{x \in P}$ of normal $*$-endomorphisms of M such that the following conditions hold:*

(1) For $x, y \in P$, $\alpha_x \circ \alpha_y = \alpha_{x+y}$,

(2) For $T \in M_*$and $A \in M$, the map $P \ni x \to \langle T, \alpha_x(A) \rangle \in \mathbb{C}$ is continuous,

(3) For $x \in P$, $\alpha_x(1) = 1$, and

(4) the endomorphism α_0 is identity.

An E_0-semigroup over Ω of M is a family $\{\alpha_x\}_{x \in \Omega}$ of normal $*$-endomorphisms on M which satisfies the first three conditions of Definition 2.1.1.

Let $\alpha = \{\alpha_x\}_{x \in P}$ be an E_0-semigroup over P on M. We say that α is faithful if $\bigcap_{x \in \Omega} ker(\alpha_x) = \{0\}$.

2.2 Convex Analysis

In this section, we collect the necessary technical tools from the convex analysis that we need in the following section. We refer the reader to Chapter 1 of [10] for proofs. Let us first recall the notion of a dual cone. The dual cone of P, denoted P^*, is defined as

$$P^* = \{a \in \mathbb{R}^d : \langle a, x \rangle \geq 0, x \in P\}.$$

Then P^* is also a closed, convex, spanning and a pointed cone. It is well known that the second dual of P, i.e. $(P^*)^*$ is P itself. We denote the interior of P^* by Ω^* . The interior of P^* is given by

$$\Omega^* = \{a \in \mathbb{R}^d : \langle a, x \rangle > 0, x \in P \setminus \{0\}\}.$$

The facts that we need about closed convex cones are summarised below.

Lemma 2.2.1. *Let P be a closed convex cone in $\mathbb{R}^d$. Assume that P is spanning. Then we have the following:*

1) The interior of P, Ω is dense in P.

2) The boundary of P, i.e. $P \setminus \Omega$ has Lebesgue measure zero.

3) Let (a_k) be a sequence in $\mathbb{R}^d$ such that $(a_k) \to a$. Then $1_{P+a_k} \to 1_{P+a}$ a.e. Also $1_{\Omega+a_k} \to 1_{\Omega+a}$ a.e.

Proof: We leave it to the reader to convince himself that to prove (1), it is enough to show that is non-empty. The fact that Ω is non-empty is a direct consequence of Prop. I.1.4 of [10]. For proof of (2), we refer the reader to Lemma 4.1. of [21]. Note that if $(a_k) \to a$, then 1_{P+a_k} converges pointwise to 1_{P+a} on the complement of the boundary of $P + a$. The almost convergence of $1_{\Omega+a_k} \to 1_{\Omega+a}$ follows from the fact that the boundary of P has Lebesgue measure zero. This completes the proof. □

Definition 2.2.2. *Let $f : \Omega \to \mathbb{C}$ be a bounded and measurable function. The Laplace transform of f, denoted $L(f)$, is a function on Ω^* defined by the following relation: for $a \in \Omega^*$*

$$L(f)(a) = \int_\Omega e^{-\langle a,x\rangle} f(x)d(x)$$

Remark 2.2.3. *The Laplace transform of the constant function 1 is usually called the characteristic function of the cone Ω [10].*

Note that the Laplace transform well defined. To see this use lemma I.1.5 of [10] to observe that $a \in \Omega^*$, there exist $M > 0$ and k large such that for $x \in P$

$$e^{-\langle a,x\rangle} \leq \frac{M}{(1+||x||^2)^k}$$

Another application of Lemma I.1.5 of [10] implies that the Laplace transform is continuous.

Proposition 2.2.4. *Let $f : \Omega \to \mathbb{C}$ be a bounded and measurable function. If $L(f)(a) = 0$ for all $a \in \Omega^*$, then $f = 0$ a.e.*

Proof: First assume that $f \in L^1(\Omega)$. We consider f as a function on $L^1(P)$ by declaring its value outside Ω to be zero. A simple application of Lemma I.1.5. of [[10]] and Stone-Weierstrass theorem implies that the linear span of $\{e^{-\langle a,x\rangle} : a \in \Omega^*\}$ is dense in $C_0(P)$. As a consequence, it follows that $\int_P \phi(x)f(x)d(x) = 0$ for all $\phi \in C_0(P)$. But $f \in L^1(P)$. Consequently $f = 0$ a.e.

Now fix $a_0 \in \Omega$ be given. Define $g : \Omega \to \mathbb{C}$ by $g(x) := e^{-\langle a_0,x\rangle} f(x)$ for all $x \in \Omega$. Clearly $g \in L^1(P)$. The given hypothesis implies that $L(g) = 0$. By what we have proved so far, it follows that $g = 0$ a.e. which is equivalent to saying that $f = 0$ a.e. This completes the proof. □

Remark 2.2.5. *We should remark that P is pointed is needed to show that the exponentials $\{e^{-\langle a,x\rangle} : a \in \Omega^*\}$ separate points of P.*

2.3 Measurable Implies Continuous

Let M be a von Neumann algebra with separable predual M_*. For $T \in M_*$ and $A \in M$ we write $T(A)$ by $\langle T, A\rangle$. Suppose α is a normal $*$-endomorphism of M. Then by duality, there exist a linear map β: $M_* \to M_*$ such that $\langle \beta(T), A\rangle = \langle T, \alpha(A)\rangle$ for all $T \in M_*$ and $A \in M$. In this case, we say that α and β are dual to each other.

Let $\alpha = \{\alpha_x\}_{x\in P}$ be a semigroup of normal $*$-endomorphisms of M. Fix $x \in P$ let $\beta_x : M_* \to M_*$ be a linear map on M_* such that $\langle \beta_x(T), A\rangle = \langle T, \alpha_x(A)\rangle$ for all $T \in M_*$

and $A \in M$. Since $||\alpha_x|| = 1$, $||\beta_x|| = 1$. Hence, the family $\beta = \{\beta_x\}_{x \in P}$ is norm bounded.

Proposition 2.3.1. *Let $\alpha = \{\alpha_x\}_{x \in P}$ be a semigroup of unital normal $*$-endomorphism on M. Assume that α restricted to Ω is an E_0-semigroup over Ω on M. If α is faithful, then α is an E_0-semigroup over P on M.*

Proof: It is enough to prove that

(1) the endomorphism α_0 is identity, and

(2) for $T \in M_*$ and $A \in M$, the map $P \ni x \to \langle T, \alpha_x(A) \rangle \in \mathbb{C}$ is continuous.

Let $A \in M$ be given. Let $x \in \Omega$ be given. Now $\alpha_x(A - \alpha_0(A)) = \alpha_x(A) - \alpha_x(A) = 0$. Since α is faithful, $\alpha_0(A) = A$. This proves (1). Let $x \in P$ be given. Let $T \in M_*$ and $A \in M$ be given. Suppose that (x_k) is a sequence in P such that $x_k \to x$. Note that $\alpha_{x_k}(A) \in \{T \in M : ||T|| \leq ||A||\}$. Since $\{T \in M : ||T|| \leq ||A||\}$ is σ-weakly compact, it is enough to prove that any convergent subsequence $(\alpha_{x_{k_i}}(A))$ of $(\alpha_{x_k}(A))$ converges to $\alpha_x(A)$. Let $(\alpha_{x_{k_i}}(A))$ be a subsequence of $(\alpha_{x_k}(A))$ such that $\alpha_{x_{k_i}}(A) \to B$ for some $B \in M$. We claim that $B = \alpha_x(A)$. Let $s \in \Omega$ be given. Note that $(s + x_{k_i})$ is a sequence in Ω and $s + x_{k_i} \to s + x \in \Omega$. Hence, $\alpha_{s+x_{k_i}}(A) \to \alpha_{s+x}(A)$. On the other hand, $\alpha_{s+x_{k_i}}(A) = \alpha_s(\alpha_{x_{k_i}}(A)) \to \alpha_s(B)$. Thus $\alpha_s(B) = \alpha_s(\alpha_x(A))$ for all $s \in \Omega$. Since α is faithful, we have $B = \alpha_x(A)$. This proves (2). Hence α is an E_0-semigroup over P on M. □

Proposition 2.3.2. *Let $\alpha = \{\alpha_x\}_{x \in \Omega}$ be a semigroup over Ω on M such that for $x \in \Omega$, α_x an unital normal $*$-endomorphism on M. Assume that for $T \in M_*$ and $A \in M$, the map $\Omega \ni x \to T(\alpha_x(A)) \in \mathbb{C}$ is measurable. Suppose that α is faithful. Then α is an E_0-semigroup over Ω on M.*

Proof: Let $\{\beta_x\}_{x \in \Omega}$ be the family of linear maps on M_* such that $\langle \beta_x(T), A \rangle = \langle T, \alpha_x(A) \rangle$ for all $x \in \Omega$, $T \in M_*$ and $A \in M$ and $\beta_{x+y} = \beta_x \beta_y$ for all $x, y \in \Omega$. Let $T \in M_*$ and $A \in M$ be given. Since the map $\Omega \ni x \to \langle T, \alpha_x(A) \rangle \in \mathbb{C}$ is measurable, the map $\Omega \ni x \to \langle \beta_x(T), A \rangle \in \mathbb{C}$ is measurable. We claim that α is an E_0-semigroup over Ω on M. It is enough to prove that

1) the endomorphism α_0 is identity, and

2) for $T \in M_*$ and $A \in M$, the map $\Omega \ni x \to \langle \beta_x(T), A \rangle \in \mathbb{C}$ is continuous.

The faithfulness of α implies that α_0 is the identity map on M. This proves (1). Observe that $\beta = \{\beta_x\}_{x\in P}$ is norm bounded. To prove (2), it is enough to check the weak continuity of β on some norm dense set in M_*.

For $a \in \Omega^*$ and $T \in M_*$, let

$$T(a) := \int_\Omega e^{-\langle a,x\rangle}\beta_x(T)d(x).$$

We claim that the linear span of $\{T(a) : T \in M_*, a \in \Omega^*\}$ is norm dense in M_*.

Suppose the linear span of $\{T(a) : T \in M_*, a \in \Omega^*\}$ is not norm dense in M_*. Then by Hahn-Banach Theorem there exists a non-zero $A \in M$ such that

$$\langle T(a), A\rangle = \int_\Omega e^{-\langle a,x\rangle}\langle \beta_x(T), A\rangle d(x) = 0 \text{ for all } a \in \Omega^*.$$

Now Prop.2.2.4 and separability of M_*, it follows that there exists a subset $E \subseteq \Omega$ of measure zero such that $0 = \langle \beta_x(T), A\rangle = \langle T, \alpha_x(A)\rangle$ for all $x \in \Omega \setminus E$, $T \in M_*$. Thus, for $x \in \Omega \setminus E$, $\alpha_x(A) = 0$.

Let $y \in \Omega$ be given. Note that $y - \Omega$ is an open subset of $\mathbb{R}^d$ containing 0. Since Ω is dense in P, it follows that $y - \Omega \cap \Omega$ is a non-empty open subset of $\mathbb{R}^d$. The fact that E has a measure zero implies that $y - \Omega \cap \Omega \cap \Omega \setminus E$ is a non-empty subset of Ω. Let $x \in y - \Omega \cap \Omega \cap \Omega \setminus E$. Then $y = z + x$ for some $z \in \Omega$. Since $\alpha_x(A) = 0$, $\alpha_y(A) = 0$. The faithfulness of α implies that $A = 0$, which is a contradiction. Hence the linear span of $\{T(a) : T \in M_*, a \in \Omega^*\}$ is norm dense in M_*.

Let $T \in M_*$ and $a \in \Omega^*$ be given. Let $x \in \Omega$ and $\{x_k\}$ be a sequence in Ω such that $x_k \to x$. Calculate as follows to observe that

$$\begin{aligned}
\beta_{x_k}(T(a)) &= \int_\Omega e^{-\langle a,y\rangle}\beta_{y+x_k}(T)d(y)\\
&= \int_{\mathbb{R}^d} e^{-\langle a,u-x_k\rangle}1_{\Omega+x_k}(u)\beta_u(T)d(u-x_k)\\
&= \int_{\mathbb{R}^d} e^{-\langle a,u-x_k\rangle}1_{\Omega+x_k}(u)\beta_u(T)d(u)
\end{aligned}$$

A simple application of dominated convergence theorem, together with Lemma 2.2.1 and Lemma I.1.5 of [10], implies that $\beta_{x_k}(T(a))$ converges weakly to $\beta_x(T(a))$. Therefore, the map $\Omega \ni x \to \langle \beta_x(T(a)), A\rangle \in \mathbb{C}$ is continuous. Hence, α is an E_0-semigroup over Ω on M □

Proposition 2.3.3. *Let $\alpha = \{\alpha_x\}_{x\in\Omega}$ be an E_0-semigroup over Ω on M. Assume that α is faithful. Then there exists a unique $\widetilde{\alpha} = \{\widetilde{\alpha}_x\}_{x\in P}$ E_0-semigroup over P on M such that $\widetilde{\alpha}_x = \alpha_x$ for all $x \in \Omega$*

Proof: Let $\{\beta_x\}_{x\in\Omega}$ be the family of linear maps on M_* such that $\langle \beta_x(T), A\rangle = \langle T, \alpha_x(A)\rangle$ for all $x \in \Omega$, $T \in M_*$ and $A \in M$ and $\beta_{x+y} = \beta_x\beta_y$ for all $x, y \in \Omega$. Let $T \in M_*$ and $A \in M$ be given. Since the map $\Omega \ni x \to \langle T, \alpha_x(A)\rangle \in \mathbb{C}$ is continuous, the map $\Omega \ni x \to \langle \beta_x(T), A\rangle \in \mathbb{C}$ is continuous.

We know that the linear span of $\{T(a) : T \in M_*, a \in \Omega^*\}$ is norm dense in M_* and the family β is norm bounded.

Let $T \in M_*$ and $a \in \Omega^*$ be given. Let $x \in P$ be given. Suppose that (x_k) is a sequence in Ω such that $x_k \to x \in P$.

Now by change of variable, we get

$$\begin{aligned}\beta_{x_k}(T(a)) &= \int_\Omega e^{-\langle a,y\rangle}\beta_{y+x_k}(T)d(y)\\ &= \int_{\mathbb{R}^d} e^{-\langle a,u-x_k\rangle}1_{\Omega+x_k}(u)\beta_u(T)d(u-x_k)\\ &= \int_{\mathbb{R}^d} e^{-\langle a,u-x_k\rangle}1_{\Omega+x_k}(u)\beta_u(T)d(u)\end{aligned}$$

By Lemma 2.2.1 and by Dominated Convergence Theorem, $\beta_{x_k}(T(a)$ converges to

$$\int_{\mathbb{R}^d} e^{-\langle a,u-x\rangle}1_{\Omega+x}(u)\beta_u(T)d(u).$$

The above integral expression is independent of the chosen sequence (x_k).

For $x \in P$, define $\widetilde{\beta}_x(T) = \lim_{k\to\infty}\beta_{x_k}(T)$ where (x_k) is any sequence in Ω such that $x_k \to x \in P$ and the limit is taken in the weak sense. Clearly, $\widetilde{\beta}_x = \beta_x$ for all $x \in \Omega$. Hence $\widetilde{\beta} = \{\widetilde{\beta}_x\}_{x\in P}$ is a family of linear maps on M_* such that for $x, y \in P\widetilde{\beta}_{x+y} = \widetilde{\beta}_x\widetilde{\beta}_y$. We claim that for $T \in M_*$ and $A \in M$, the map $P \ni x \to \langle\widetilde{\beta}_x(T), A\rangle \in \mathbb{C}$ is continuous. Observe that $\widetilde{\beta} = \{\widetilde{\beta}_x\}_{x\in P}$ is norm bounded. It is enough to check the weak continuity of $\widetilde{\beta}$ on some norm dense set in M_*. We know that the linear span of $\{T(a) : T \in M_*, a \in \Omega^*\}$ is norm dense in M_*. Let $a \in \Omega^*$ and $T \in M_*$ be given. Fix $x \in P$ and $A \in M$. Let (x_k) be a sequence in Ω such that $x_k \to x \in P$. By Lemma 2.2.1 and by Dominated

Convergence Theorem,

$$\begin{aligned}\lim_{k\to\infty}\langle\widetilde{\beta}_{x_k}(T(a)),A\rangle &= \lim_{k\to\infty}\int_{\mathbb{R}^d} e^{-\langle a,u-x_k\rangle}1_{\Omega+x_k}(u)\langle\beta_u(T),A\rangle d(u)\\ &= \int_{\mathbb{R}^d} e^{-\langle a,u-x\rangle}1_{\Omega+x}(u)\langle\beta_u(T),A\rangle d(u)\\ &= \langle\widetilde{\beta}_x(T(a)),A\rangle.\end{aligned}$$

The above calculation, to obtain the third equality, we used the hypothesis that the map $\Omega \in x \to \langle\beta_x(T),A\rangle$ is continuous. Hence our claim completes.

Fix $x \in P$ and $A \in M$. Let (x_k) be a sequence in Ω such that $x_k \to x \in P$. We claim that $\alpha_{x_k}(A)$ converges σ-weakly. Note that $\alpha_{x_k}(A) \in \{T \in M : ||T|| \leq ||A||\}$. Since $\{T \in M : ||T|| \leq ||A||\}$ is σ-weak compact, it is enough to show that any convergent subsequence of $(\alpha_{x_k}(A))$ converges to the same limit.

Claim: Suppose (x_k), (y_k) are sequences in Ω and B, $C \in M$ are such that $x_k \to x$, $y_k \to x$,$\alpha_{x_k}(A) \to B$ and $\alpha_{y_k}(A) \to C$. Then $B = C$.

Let $s \in \Omega$ be given. Note that $s+x_k \to s+x$, $s+y_k \to s+x$ and $s+x \in \Omega$. Observe that $\alpha_{s+x_k}(A) = \alpha_s(\alpha_{x_k}(A))$. Taking limits, we obtain $\alpha_{s+x}(A) = \alpha_s(B)$. Similarly, we obtain $\alpha_{s+x}(A) = \alpha_s(C)$. Thus, $\alpha_s(B - C) = 0$ for every $s \in \Omega$. The faithfulness of α implies that $B = C$. This proves our claim. Hence, any convergent subsequence of $(\alpha_{x_k}(A))$ converges to the same limit. This also implies that the limit of $(\alpha_{x_k}(A))$ is independent of the chosen sequence (x_k).

For $x \in P$ let $\widetilde{\alpha}_x(A) := \lim\limits_{k\to\infty}\alpha_{x_k}(A)$, where the limit is taken in the σ-weakly sense. Clearly, $\widetilde{\alpha}_x = \alpha_x$ for all $\in \Omega$. Since α_y is unital for all $y \in \Omega$, $\widetilde{\alpha}_x(1) = \lim\limits_{k\to\infty}\alpha_{x_k}(1) = 1$.

Let $x \in P$ be given. We claim that $\widetilde{\alpha}_x(U)$ is unitary whenever U is unitary in M. Let $U \in M$ be unitary and set $V := \widetilde{\alpha}_x(U)$. Choose a sequence (x_k) in Ω such that $x_k \to x$. Let $s \in \Omega$ be given. Note that $\alpha_{s+x_k}(U) \to \alpha_{s+x}(U)$, which is unitary. But $\alpha_{s+x_k}(U) = \alpha_s(\alpha_{x_k}(U))$ converges, by definition, to $\alpha_s(\widetilde{\alpha}_x(U) = \alpha_s(V)$. Now, $\alpha_s(V^*V) = \alpha_s(V)^*\alpha_s(V) = \alpha_{s+x}(U)^*\alpha_{s+x}(U) = 1$. Similarly $\alpha_s(VV^*) = 1$. Since α is faithful, it follows that $VV^* = V^*V = 1$. Therefore, $\widetilde{\alpha}_x(U)$ is unitary.

Claim : $\widetilde{\alpha}_x$ is a $*$-endomorphism of M. By definition, it is clear that $\widetilde{\alpha}_x$ is a $*$-linear map on M. Let U and V be given unitary.

$$\begin{aligned}\widetilde{\alpha}_x(UV) &= \lim_{k\to\infty} \alpha_{x_k}(UV)\\ &= \lim_{k\to\infty} \alpha_{x_k}(U)\alpha_{x_k}(V)\\ &= \lim_{k\to\infty} \alpha_{x_k}(U) \lim_{k\to\infty} \alpha_{x_k}(V)\\ &= \widetilde{\alpha}_x(U)\widetilde{\alpha}_x(V).\end{aligned}$$

Hence $\widetilde{\alpha}$ is a $*$-endomorphism of M.

We claim that $\widetilde{\alpha}_x$ is normal. Note that $\langle\widetilde{\beta}_x(T), A\rangle = \langle T, \widetilde{\alpha}_x(A)\rangle$ for all $T \in M_*$ and $A \in M$. It follows that $\widetilde{\alpha}_x$ is normal. Thus, $\widetilde{\alpha} = \{\widetilde{\alpha}_x : x \in P\}$ is a family of unital normal $*$-endomorphisms on M.

Cliam : $\widetilde{\alpha} = \{\widetilde{\alpha}_x\}_{x\in P}$ is an E_0 semigroup on M. Let x,$y \in P$ be given. Choose sequences (x_k) and (y_k) in Ω such that $x_k \to x$ and $y_k \to y$. Let $A \in M$ be given. Fix m and let k tends to infinity in the equation $\alpha_{x_k+y_m}(A) = \alpha_{x_k} \circ \alpha_{y_m}(A)$ to obtain the equation $\alpha_{x+y_m}(A) = \widetilde{\alpha}_x \circ \alpha_{y_m}(A)$. Now, let m tends to infinity to arrive at the equation $\widetilde{\alpha}_{x+y} = \widetilde{\alpha}_x \circ \widetilde{\alpha}_y$. This proves our claim. The faithfulness of α implies that $\widetilde{\alpha}_0$ is the identity map on M.

The σ-weak continuity of $\widetilde{\alpha} = \{\widetilde{\alpha}_x : x \in P\}$ is is immediately followed from Proposition 2.3.1. Uniqueness follows from the density of Ω. This completes the proof. □

Theorem 2.3.4. *Let $\alpha = \{\alpha_x\}_{x\in P}$ be a semigroup of unital normal $*$-endomorphisms on M such that $T \in M_*$and $A \in M$, the map $P \ni x \to T(\alpha_x(A)) \in \mathbb{C}$ is measurable. Assume that α is faithful in the sense that $\bigcap_{x\in\Omega} ker(\alpha_x) = \{0\}$. Then $\alpha = \{\alpha_x\}_{x\in P}$ is an E_0-semigroup over P on M.*

Proof : By Prop 2.3.2 and Prop 2.3.3, $\alpha = \{\alpha_x\}_{x\in P}$ is an E_0-semigroup over P on M. □

Remark 2.3.5. *The above Theorem is stated as a Theorem A in the introduction of this thesis. The proof of Theorem A first appeared in [17] (see Corollary 4.3).*

2.4 Extending Cocycle

Let $\alpha = \{\alpha_x\}_{x\in P}$ be an E_0-semigroup over P on M. A σ-weakly continuous family $\{U_x\}_{x\in P}$ of unitaries in M is called an α-cocycle if $U_{x+y} = U_x\alpha_x(U_y)$ for all x , $y \in P$.

Suppose that $\{U_x\}_{x\in P}$ is an α-cocycle. For $x \in P$, define $\beta_x = Ad(U_x)\circ\alpha_x$. Then $\beta = \{\beta_x\}_{x\in P}$ is an E_0-semigroup over P on M. We call such an E_0-semigroup a cocycle perturbation of α.

The next proposition says that every measurable cocycle is continuous.

Proposition 2.4.1. *Let $\alpha = \{\alpha_x\}_{x\in P}$ be an E_0-semigroup over P on M. Assume that α is faithful. Suppose $\{U_x : x \in \Omega\}$ is a σ-weakly measurable family of isometries in M such that $U_{x+y} = U_x\alpha_x(U_y)$ for all x , $y \in \Omega$. Then there exists a unique σ-weakly continuous family of isometries $\{\widetilde{U}_x\}_{x\in P}$ in M such that $\widetilde{U}_x = U_x$ for $x \in \Omega$. Moreover, if U_x, for $x \in \Omega$, is unitary in M , then $\widetilde{U}_x$, $x \in P$, is a unitary in M.*

Proof: The proof of σ-weak continuity of the family $\{U_x\}_{x\in\Omega}$ is same as in the Prop 2.3.1 of [7] in which where Arveson uses Connes 2×2 matrix trick.

Fix $x \in P$. Let (x_k) be any sequence in Ω such that $x_k \to x \in P$. We claim that U_{x_k} converges σ-weakly to an isometry. Since the closed unit ball of M is σ-weak compact, it is enough to show that any convergent subsequence of (U_{x_k}) converges to the same limit.

Claim : Let (x_k) and (y_k) be sequences in Ω such that $x_k \to x$, $y_k \to x$, $U_{x_k} \to U$ and $U_{y_k} \to V$ where the convergence is in σ-weak topology. We claim that $U = V$ and U is an isometry. Moreover, if U_{x_k} and U_{y_k} are unitary , then U is unitary.

Choose a point $s \in \Omega$. Letting limit k tends to infinity in the equation $U_s\alpha_s(U_{x_k}) = U_{s+x_k}$ to arrive at the equation $U_s\alpha_s(U) = U_{s+x}$. Similarly , we obtain $U_s\alpha_s(V) = U_{s+x}$. Thus, $\alpha_s(U) = \alpha_s(V)$ for $s \in \Omega$. The faithfulness of α implies that $U = V$. Calculate as follows to observe that

$$\begin{aligned}\alpha_s(U^*U) &= \alpha_s(U^*)\alpha_s(U)\\ &= \alpha_s(U^*)U_s^*U_s\alpha_s(U)\\ &= U_{s+x}^*U_{s+x}\\ &= 1\\ &= \alpha_s(1).\end{aligned}$$

Since α is faithful , $U^*U = 1$ and hence U is an isometry. Also the equation $U_s\alpha_s(U) = U_{s+x}$ tells us that if U_x, for $x \in \Omega$, is a unitary , then $\alpha_s(UU^*) = \alpha_s(1) = 1$. The faithfulness of α implies that $UU^* = 1$. Hence, U is unitary. This proves our claim. Hence, any convergent subsequence of (U_{x_k}) converges to the same limit. This also implies that the limit of (U_{x_k}) is independent of the chosen sequence (x_k).

Let (x_k) and (y_k) be sequences in Ω such that $x_k \to x$, $y_k \to y$. Fix k and m. Letting k tends to infinity in equation $U_{x_k+y_m} = U_{x_k}\alpha_{x_k}(U_{y_m})$, we get the equation $U_{x+y_m} = \widetilde{U_x}\alpha_x(U_{y_m})$. Again taking m tends to infinity in the equation $U_{x+y_m} = \widetilde{U_x}\alpha_x(U_{y_m})$, we arrive at the equation $\widetilde{U_{x+y}} = \widetilde{U_x}\alpha_x(\widetilde{U_y})$. Thus the family $\{\widetilde{U_x}\}_{x\in P}$ satisfies the α-cocycle equation. Uniqueness follows from the density of Ω. The proof is now complete. □

2.5 Examples of E_0-semigroups

Example 2.5.1. ***(CAR Flow)*** *Fix a Hilbert space $\mathcal{H}$. For $n \geq 1$ let $\mathcal{H}^{\otimes^n}$ be the n-fold tensor product of a Hilbert space $\mathcal{H}$. Let S_n be the group of all permutations of the set $\{1, 2, 3 \ldots n\}$. For $\sigma \in S_n$, let U_σ be the unitary defined on the product vectors in $\mathcal{H}^{\otimes^n}$ by the relation*

$$U_\sigma(\zeta_1 \otimes \zeta_2 \cdots \otimes \zeta_n) = \zeta_{\sigma^{-1}(1)} \otimes \zeta_{\sigma^{-1}(2)} \cdots \otimes \zeta_{\sigma^{-1}(n)}$$

where σ^{-1} is the inverse of σ. The closed subspace $\mathcal{H}^{@^n} = \{\xi \in \mathcal{H}^{\otimes^n} : U_\sigma(\xi) = \epsilon(\sigma)\xi$ for all $\sigma \in S_n\}$, where $\epsilon(\sigma) = \pm 1$ according to whether the permutation σ is even or odd and call $\mathcal{H}^{@^n}$ n-fold antisymmetric tensor product of $\mathcal{H}$ with itself n-times. For $n \geq 1$, let Q_n be the projection of $\mathcal{H}^{\otimes^n}$ onto $\mathcal{H}^{@^n}$. The image of $\zeta_1 \otimes \zeta_2 \cdots \otimes \zeta_n$ under the projection Q_n will be denoted by $\zeta_1 \wedge \zeta_2 \cdots \wedge \zeta_n$.

Let $\Gamma_a(\mathcal{H}) = \mathbb{C} + \bigoplus_{n=1}^{\infty} \mathcal{H}^{@^n}$ be an antisymmetric Fock space over $\mathcal{H}$. Note that the set $\{\wedge_{k=1}^n \zeta_k | n \in \mathbb{N}, \zeta_k \in \mathcal{H}\}$ is total in $\Gamma_a(H)$. Denote the vector $1 \oplus 0 \oplus 0 \ldots$ by Θ.

For each $\zeta \in \mathcal{H}$ we define a bounded linear operator $a(\zeta)$ on $\Gamma_a(\mathcal{H})$ where $a(\zeta) : \mathcal{H}_n \to \mathcal{H}_{n+1}$ by the relation

$$a(\zeta)(\wedge_{k=1}^n) = \sqrt{n+1}\zeta \wedge \zeta_1 \wedge \cdots \wedge \zeta_n \text{ for all } \zeta_k \in \mathcal{H}\ ,\ 1 \leq k \leq n\ n \in \mathbb{N}.\ a(\zeta)(\Phi) = \zeta.$$

One can find the adjoint $a^*(\zeta)$ of $a(\zeta)$ which maps $\mathcal{H}_{n+1}$ into $\mathcal{H}_n$ as follows

$$a^*(\zeta)(\wedge_{k=1}^{n+1}) = (n+1)^{-1/2} \sum_{k=1}^{n+1} (-1)^{k-1} \langle \zeta, \zeta_k \rangle \zeta \wedge \zeta_1 \cdots \wedge \hat{\zeta_k} \wedge \cdots \wedge \zeta_{n+1}$$

where the hat ˆ over ζ_k indicates its omission.

The family $\{a(\zeta)\}_{\zeta\in\mathcal{H}}$ of bounded operators satisfy the CAR relation:

For ζ,$\eta \in \mathcal{H}$ and $\lambda \in \mathbb{C}$

1 $a(\lambda\zeta + \eta) = \lambda a(\zeta) + a^\dagger(\eta)$, *2* $a(\zeta)a(\eta) + a(\eta)a(\zeta) = 0,$

3 $a(\zeta)a^*(\eta) + a^*(\eta)a(\zeta) = \langle \zeta, \eta \rangle 1.$

By Proposition 23.4 *in [19], the set* $\{a(\zeta), a^*(\eta) : \zeta, \eta \in \mathcal{H}\}$ *is irreducible in* $\mathcal{B}(\Gamma_a(\mathcal{H}))$.

Remark 2.5.2. *Suppose V is an isometry acting on a Hilbert space $\mathcal{H}$. Then there exists a unique normal $*$-endomorphism of $\mathcal{B}(\Gamma_a(\mathcal{H}))$, say α_V such that $\alpha_V(a(\zeta)) = a(V(\zeta))$ for all $\zeta \in \mathcal{H}$. For proof of this fact, the reader refers to the Proposition* 2.1.7*, page* 25 *in [7].*

Let $V = \{V_x\}_{x \in P}$ be a strongly continuous semigroup family of isometries acting on a Hilbert space $\mathcal{H}$. For every fixed $x \in P$, let $\alpha_x : \mathcal{B}(\Gamma_a(\mathcal{H})) \to \mathcal{B}(\Gamma_a(\mathcal{H}))$ be the normal $*$-endomorphism such that

$$\alpha_x(a(\zeta)) = a(V_x\zeta) \text{ for all } \zeta \in \mathcal{H}.$$

Since the the family $V = \{V_x\}_{x \in P}$ is semigroup, the family $\{\alpha_x\}_{x \in P}$ forms semigroup of unital, normal $*$-endomorphism on $\mathcal{B}(\Gamma_a(\mathcal{H}))$.

Note that the linear span of $\{a(\zeta), a^*(\eta) : \zeta, \eta \in \mathcal{H}\}$ is strongly dense in $\mathcal{B}(\Gamma_a(\mathcal{H}))$.

For every normal linear functional T on $\mathcal{B}(\Gamma_a(\mathcal{H}))$, and A belongs to the linear span of $\{a^\dagger(\zeta), a(\eta) : \zeta, \eta \in \mathcal{H}\}$, the map $P \ni x \to \langle T, \alpha_x(A) \rangle \in \mathbb{C}$ is continuous.

Suppose T is normal linear functional on $\mathcal{B}(\Gamma_a(\mathcal{H}))$. Let $A \in \mathcal{B}(\Gamma_a(\mathcal{H}))$ be given. By Kaplansky density theorem, there exists a sequence $\{A_k\}$ in the linear span of $\{a(\zeta), a^*(\eta) : \zeta, \eta \in \mathcal{H}\}$ such that $||A_k|| \leq ||A||$ and $(A_k) \to A$ in SOT. Therefore, the map $P \ni x \to \langle T, \alpha_x(A) \rangle \in \mathbb{C}$ is measurable.

Hence by Theorem 2.3.4, $\{\alpha_x\}_{x \in P}$ is an E_0-semigroup over P on $\mathcal{B}(\Gamma_a(\mathcal{H}))$.

Example 2.5.3. ***(CCR flow)*** *Fix a Hilbert space $\mathcal{H}$. For $n \geq 1$ let $\mathcal{H}^{\otimes^n}$ be the n-fold tensor product of a Hilbert space $\mathcal{H}$. Let S_n be the group of all permutations of the set $\{1, 2, 3 \dots n\}$. For $\sigma \in S_n$ let U_σ be the unitary defined on the product vectors in $\mathcal{H}^{\otimes^n}$ by the relation*

$$U_\sigma(\zeta_1 \otimes \zeta_2 \cdots \otimes \zeta_n) = \zeta_{\sigma^{-1}(1)} \otimes \zeta_{\sigma^{-1}(2)} \cdots \otimes \zeta_{\sigma^{-1}(n)}$$

where σ^{-1} is the inverse of σ. The closed subspace $\mathcal{H}^{\circledS^n} = \{\xi \in \mathcal{H}^{\otimes^n} : U_\sigma(\xi) = \xi$ for all $\sigma \in S_n\}$, and call $\mathcal{H}^{\circledS^n}$ n-fold symmetric tensor product of $\mathcal{H}$ with itself n-times.

Let $\Gamma_s(\mathcal{H}) = \mathbb{C} + \bigoplus_{n=1}^{\infty} \mathcal{H}^{\circledS^n}$ be the symmetric Fock space over $\mathcal{H}$. For $\xi \in \mathcal{H}$, define the element $e(\xi) = \oplus (n!)^{-1/2} \xi^{\otimes^n}$ in $\Gamma_s(\mathcal{H})$ and call exponential vector associated with ξ.

By Prop19.4 in[19], the set $\{e(\xi) : \xi\mathcal{H}\}$ *of all exponential vectors is linearly independent set and total in* $\Gamma_s(\mathcal{H})$*. For any* $\xi \in \mathcal{H}$*, let* $W(\xi)$ *be a unitary operator on* $\Gamma_s(\mathcal{H})$ *defined by the relation* $W(\xi)e(\eta) = e^{-1/2||\xi||^2-\langle\xi,\eta\rangle}e(\xi+\eta)$ *for all* $\eta \in \mathcal{H}$*.*

Note the map $\mathcal{H} \ni \xi \to W(\xi) \in \mathcal{B}(\Gamma_s(\mathcal{H}))$ *is strongly continuous and for* $\xi, \eta \in \mathcal{H}$ *we have* $W(\xi)W(\eta) = e^{-iIm\langle\xi,\eta\rangle}W(\xi+\eta)$*. By Prop 20.9 in [19] the family* $\{W(\xi) : \xi \in \mathcal{H}\}$ *is irreducible in* $\mathcal{B}(\Gamma_s(\mathcal{H}))$*.*

Remark 2.5.4. *Suppose* V *is an isometry acting on a Hilbert space* $\mathcal{H}$*. Then there exists a unique normal* $*$*-endomorphism of* $\mathcal{B}(\Gamma_s(\mathcal{H}))$*, say* α_V *such that* $\alpha_V(W(\xi)) = W(V(\xi))$ *for all* $\zeta \in \mathcal{H}$*. For a proof of this fact, the reader refers to the Proposition* 2.1.3*, page* 23 *in [7].*

Let $V = \{V_x\}_{x\in P}$ *be a strongly continuous semigroup family of isometries acting on a Hilbert space* $\mathcal{H}$*. For every fixed* $x \in P$*, let* $\alpha_x : \mathcal{B}(\Gamma_s(\mathcal{H})) \to \mathcal{B}(\Gamma_s(\mathcal{H}))$ *be the normal* $*$*-endomorphism such that*

$$\alpha_x(W(\xi)) = W(V_x\xi) \text{ for all } \xi \in \mathcal{H}.$$

Since the the family $V = \{V_x\}_{x\in P}$ *is semigroup, the family* $\{\alpha_x\}_{x\in P}$ *forms semigroup of unital, normal* $*$*-endomorphism on* $\mathcal{B}(\Gamma_s(\mathcal{H}))$*.*

For every normal linear functional T *on* $\mathcal{B}(\Gamma_s(\mathcal{H}))$*, and* A *belongs to the linear span of* $\{W(\xi) : \xi \in \mathcal{H}\}$*, the map* $P \ni x \to \langle T, \alpha_x(A)\rangle \in \mathbb{C}$ *is continuous. Suppose* T *is a normal linear functional on* $\mathcal{B}(\Gamma_s(\mathcal{H}))$*. Let* $A \in \mathcal{B}(\Gamma_s(\mathcal{H}))$ *be given. By Kaplansky density theorem, there exists a sequence* $\{A_k\}$ *in linear span of* $\{W(\xi) : \xi \in \mathcal{H}\}$ *such that* $||A_k|| \leq ||A||$ *and* $(A_k) \to A$ *in SOT. Therefore, the map* $P \ni x \to \langle T, \alpha_x(A)\rangle \in \mathbb{C}$ *is measurable.*

Hence by Theorem 2.3.4, $\{\alpha_x\}_{x\in P}$ *is an* E_0*-semigroup over* P *on* $\mathcal{B}(\Gamma_a(\mathcal{H}))$*.*

Example 2.5.5. ***(Tensor Product)***

Let $\alpha = \{\alpha_x\}_{x\in P}$ *and* $\beta = \{\beta_x\}_{x\in P}$ *be* E_0*-semigroups on von Neumann algebras* $B(\mathcal{H})$ *and* $\mathcal{B}(\mathcal{K})$ *respectively. The tensor product* $\alpha \otimes \beta$ *is the semigroup family* $\{(\alpha\otimes\beta)_x\}_{x\in P}$ *of unital normal* $*$*-endomorphism on* $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ *where* $(\alpha\otimes\beta)_x : \mathcal{B}(\mathcal{H}\otimes\mathcal{K}) \to \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ *by the formulla* $(\alpha\otimes\beta)_x(A\otimes B) = \alpha_x(A)\otimes\beta_x(B)$ *for all* $A \in \mathcal{B}(\mathcal{H})$ *and* $B \in \mathcal{B}(\mathcal{K})$*.*

Observe that linear span of the set $\{A \otimes B : A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{B}(\mathcal{K})\}$ *is strongly dense in* $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$*.*

Let $\xi_1, \xi_2 \in \mathcal{H}$ *and* $\eta_1, \eta_2 \in \mathcal{K}$ *be given. Suppose* $A \in \mathcal{B}(\mathcal{H})$ *and* $B \in \mathcal{B}(\mathcal{K})\}$ *are given. Then the map* $P \ni x \to \langle\alpha_x(A)(\xi_1), \xi_2\rangle\langle\beta_x(B)(\eta_1), \eta_2\rangle \in \mathbb{C}$ *is continuous. By*

Kaplansky density theorem, it follows that the map $P \ni x \to (\alpha \otimes \beta)_x \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ *is weakly measurable. Hence by Theorem 2.3.4,* $\{(\alpha \otimes \beta)_x\}_{x \in P}$ *is an* E_0*-semigroup over* P *on* $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$*.*

2.6 Applications of Theorem 2.3.4

In this section, we shall see that every strongly measurable isometric representation of P on $\mathcal{H}$ is strongly continuous.

An isometric representation of P on a separable Hilbert space $\mathcal{H}$ is a map $V : P \to \mathcal{B}(\mathcal{H})$ such that

1) For $x \in P$, V_x is an isometry;

2) For $x, y \in P$, $V_{x+y} = V_x V_y$; and

3) For $\xi \in \mathcal{H}$, the map $P \ni x \to V_x(\xi) \in \mathcal{H}$ is continuous.

Proposition 2.6.1. *Let* $V = \{V_x\}_{x \in P}$ *be a family semigroup of isometries on* $\mathcal{H}$ *such that for* $\xi \in \mathcal{H}$*, the map* $P \ni x \to V_x(\xi) \in \mathcal{H}$ *is measurable. Then for* $\xi \in \mathcal{H}$*, the map* $P \ni x \to V_x(\xi) \in \mathcal{H}$ *is continuous.*

Proof: Let $\Gamma_a(\mathcal{H}) = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\circledast^n}$ be an antisymmetric Fock space over $\mathcal{H}$. Denote the vector $1 \oplus 0 \oplus \dots$ by Θ. For every fixed $x \in P$, let $\alpha_x : \mathcal{B}(\Gamma_a(\mathcal{H})) \to \mathcal{B}(\Gamma_a(\mathcal{H}))$ be the normal $*$-endomorphism such that

$$\alpha_x(a(\zeta)) = a(V_x \zeta) \text{ for all } \zeta \in \mathcal{H}.$$

Since the family $V = \{V_x\}_{x \in P}$ is semigroup, the family $\{\alpha_x\}_{x \in P}$ forms semigroup of unital, normal $*$-endomorphism on $\mathcal{B}(\Gamma_a(\mathcal{H}))$. Note that the $\alpha = \{\alpha_x\}_{x \in P}$ forms measurable E_0-semigroups. By Theorem 2.3.4, $\alpha = \{\alpha_x\}_{x \in P}$ E_0-semigroups. Hence, for $\xi \in \mathcal{H}$, the map $P \ni x \to \alpha_x(a(\xi))\Theta = a(V_x \xi)\Theta = V_x(\xi) \in \mathcal{H}$ is continuous.

2.7 An Invariant for E_0-semigroups

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on a separable Hilbert space $\mathcal{H}$. We treat $\mathcal{B}(\mathcal{H})$ as a measurable space where the measurable structure on $\mathcal{B}(\mathcal{H})$ is given by the

σ-algebra generated by σ-weakly closed subsets of $\mathcal{B}(\mathcal{H})$. Let $p : \Omega \times \mathcal{B}(\mathcal{H}) \to \Omega$ be the projection $p(x, T) = x$.

We define the notion of a concrete product system as in [page-35 [7]] with $(0, \infty)$ replaced by Ω.

Definition 2.7.1. *Let $\mathcal{E}$ be a standrd Borel subset of $\Omega \times \mathcal{B}(\mathcal{H})$. For $x \in \Omega$, let*

$$\mathcal{E}(x) = \{T \in \mathcal{B}(\mathcal{H}) : (x, T) \in \mathcal{E}\}.$$

We say that $\mathcal{E}$ is a concrete product system over Ω if the following conditions hold:

1) *The map $p : \mathcal{E} \to \Omega$ is onto.*

2) *For $x \in \Omega$, the set of operators $\mathcal{E}(x)$ is a linear subspace of $\mathcal{B}(\mathcal{H})$. For $T, S \in \mathcal{E}(x)$, T^*S is a scalar which we denote by $\langle T, S\rangle$. With respect to the inner product, $\mathcal{E}(x)$ is a separable Hilbert space.*

3) *For $x, y \in \Omega$, the linear span of $\{TS : T \in \mathcal{E}(x)S \in \mathcal{E}(y)\}$ is dense in $\mathcal{E}(x + y)$.*

4) *There exists a sequence $\{V_n\}_{n\in\mathbb{N}}$ of measurable maps from Ω to $\mathcal{B}(\mathcal{H})$ such that for every $x \in \Omega$, the set $\{V_n(x) : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{E}(x)$*

Remark 2.7.2. *Condition (4) of Defn. 2.7.1 is equivalent to the fact that for every $x \in \Omega$, $\mathcal{E}(x)$ infinite dimensional Hilbert space and there exists a sequence $\{V_n\}_{n\in\mathbb{N}}$ of measurable maps from Ω to $\mathcal{B}(\mathcal{H})$ such that for every $x \in \Omega$, the set $\{V_n(x) : n \in \mathbb{N}\}$ is total set in $\mathcal{E}(x)$. For a proof of this equivalence, we refer the reader to Prop.7.27 of [11].*

Let $\alpha = \{\alpha_x\}_{x\in P}$ be a fixed E_0-semigroup over P. For $x \in \Omega$, the intertwining space for α_x is the set

$$\mathcal{E}(x) = \{T \in \mathcal{B}(\mathcal{H} : \alpha_x(A)T = TA\ \forall\ A \in \mathcal{B}(\mathcal{H}\}.$$

Clearly, $\mathcal{E}(x)$ is a linear subspace of $\mathcal{B}(\mathcal{H})$.By Theorem1.1.8, $\mathcal{E}(x)$ is a separable Hilbert space where the inner product is given by $\langle T, S\rangle = T^*S$.

Observe that $x \in P$, the intertwining space $\mathcal{E}(x)$ is either one dimensional or infinite dimension. Indeed for $\{\alpha_{tx}\}_{t\geq 0}$ is an E_0-semigroup and an application of Theorem 2.4.7 in [7] yields the desired conclusion.

Lemma 2.7.3. *Let $\alpha = \{\alpha_x\}_{x\in P}$ be an E_0-semigroup on $\mathcal{B}(\mathcal{H})$. Then the following are equivalent.*

1) There exists $x \in \Omega$ such that $\mathcal{E}(x)$ is one dimensional.

2) For every $x \in P$, $\mathcal{E}(x)$ is one dimensional.

3) There exists $x \in \Omega$ such that α_x is an automorphism.

4) For every $x \in P$, α_x is an automorphism.

Proof: The structure theorem for a single endomorphism 1.1.6 implies that (1) is equivalent to (3) and (2) is equivalent to (4). Thus, it is enough to prove that (3) implies (4). Let $s \in \Omega$ be such that α_s is an automorphism. For $x \in P$, let $M_x = \alpha_x(\mathcal{B}(\mathcal{H})$. Note that if $x \leq y$, then $M_x \subset M_y$. Consider a point $x \in P$. Since $s \in \Omega$ and Ω is open it follows that there exists $n \in \mathbb{N}$ such that $y =: s - \frac{x}{n} \in \Omega$. Thus $\frac{x}{n} \leq s$. As a consequence, we have $\mathcal{B}(\mathcal{H}) = M_s \subset M_{\frac{x}{n}}$. This implies that $\alpha_{\frac{x}{n}}$ is onto and hence an automorphism. Now It follows immediately that α_x is an automorphism. This completes the proof. □

Remark 2.7.4. *E_0-semigroups for which each endomorphism is an automorphism are completely classified up to cocycle conjugacy by $\mathbb{T}$-valued 2-cocycles on $\mathbb{R}^d$ (see Theorem* 3.3 *[1]). In view of this, we do not consider such degenerate cases.*

In this section, by an E_0-semigroup we mean a family $\alpha = \{\alpha_x\}_{x \in P}$ of endomorphisms on $\mathcal{B}(\mathcal{H})$ as in Defn.2.1.1 such that for every $x \in \Omega$, α_x is not onto.

Let $\mathcal{E}_\alpha = \coprod_{x \in \Omega} \mathcal{E}(x)$. Then $\mathcal{E}_\alpha$ satisfies the first three condition of Definition 2.7.1. Let $p : \mathcal{E}_\alpha \to \Omega$ be the projection $p(x, T) = x$. The fact that $\mathcal{E}_\alpha$ is a standard Borel space is proved exactly as in proposition in [7].

One more technical lemma that we need is the following. This is quite standard in the theory of measurable field of Hilbert spaces. However, we recall the proof for the reader's convenience.

Lemma 2.7.5. *Let $(X, \mathcal{B})$ be a measurable space and $\{p(x) : x \in X\}$ be a σ-weakly measurable family of infinite projections in $\mathcal{B}(\mathcal{H})$.*

1) Then there exists a σ-weakly measurable family of isometries $\{w(x) : x \in X\}$ in $B(H)$ such that $w(x)w(x)^* = p(x)$.

2) Suppose q is an infinite projection in $B(H)$. Then there exists a σ-weakly measurable family of partial isometries $\{w(x) : x \in X\}$ in $B(H)$ such that $w(x)^*w(x) = q$ and $w(x)w(x)^* = p(x)$.

Proof For $x \in X$, let $\mathcal{H}_x := \{\xi \in \mathcal{H} : p(x)\xi = \xi\}$. Then $\mathcal{H}_x$ is an infinite dimensional Hilbert space for every $x \in X$. Let $\Gamma := \{f : X \to \mathcal{H} : f$ is weakly measurable such that $p(x)f(x) = f(x)\}$. Let $\{e_1, e_2, ,\}$ be an orthonormal basis for $\mathcal{H}$. Observe that $\{p(.)e_i : i = 1, 2, \} \subset \Gamma$ and for every $x \in X$, $\{p(x)e_i : x \in X\}$ is total in $\mathcal{H}_x$. By Theorem 1.3.2 (see [11]), it follows that there exists $u_1, u_2, \in \Gamma$ such that for every $x \in X$, $\{u_1(x), u_2(x), \}$ is an orthonormal basis for $\mathcal{H}_x$. Let $x \in X$ be given. Let $w(x)$ be the isometry sending $e_k \to u_k(x)$. It is now clear that $\{w(x) : x \in X\}$ is a σ-weakly-measurable family of isometries such that for $x \in X$, $w(x)w(x)^* = p(x)$. This proves (1). Suppose q is an infinite projection in $\mathcal{B}(\mathcal{H})$. Let $\{w(x) : x \in X\}$ be a family of isometries as in (1). Choose an isometry $v \in \mathcal{B}(\mathcal{H})$ such that $vv^* = q$. It is immediate that $\{w(x)v^* : x \in X\}$ is the desired family of partial isometries. This proves (2) and the proof is now complete. □

We need the following lemma which is really lemma 2.4.8 in [7] adapted to our situation.

Lemma 2.7.6. *Let $\alpha = \{\alpha_x\}_{x\in P}$ be an E_0-semigroup over P. Fix $x_0 \in \Omega$, there exists weakly measurable family $\{U_x\}_{x\in\Omega}$ of unitaries in $\mathcal{B}(\mathcal{H})$ such that $U_x\alpha_{x_0}(A) = \alpha_x(A)U_x$ for all $A \in \mathcal{B}(\mathcal{H})$ and $x \in \Omega$.*

Proof: Let Q be a non-zero minimal projection in $\mathcal{B}(\mathcal{H})$. Then there exists a unit vector $\zeta_0 \in \mathcal{H}$ such that $Q = \theta_{\zeta_0,\zeta_0}$. Then for $x \in \Omega$, $\alpha_x(Q)$ is a projection which has the dimension n. Note that the family $\{\alpha_x(Q)\}_{x\in\Omega}$ of projections is weakly measurable. First, we claim that there exist weakly measurable family $V = \{V_x\}_{x\in\Omega}$ of partial isometries in $\mathcal{H}$ such that $V_x^*V_x = \alpha_{x_0}(Q)$ and $V_xV_x^* = \alpha_x(Q)$ for all $x \in \Omega$.

Put $\mathcal{H}_x$ = range of $\alpha_x(Q)$. Let $\{e_m\}$ be an orthonormal basis for $\mathcal{H}$. For $m \geq 1$, let the map $p_m : \Omega \to \mathcal{H}$ be defined by the relation $p_m(x) = \alpha_x(Q)e_m$ for all $x \in \Omega$. First we observe that for $x \in \Omega$, the set $\{p_m(x) : m \in \mathbb{N}\}$ forms a total set in $\mathcal{H}_x$. Note that for each m, p_m is continuous. It follows that the family $\{\mathcal{H}_x\}_{x\in\Omega}$ forms a measurable field of Hilbert space.

By Theorem 1.3.2, there exists sequence $\{u_m\}$ of measurable sections defined on Ω such that $\{u_m(x)\}_{i=1}^{\infty}$ forms an orthonormal basis for $\mathcal{H}_x$ for all $x \in \Omega$. For $x \in \Omega$ let V_x be a partial isometry defined by $V_x(u_m(x_0)) = u_m(x)$ on range of $\alpha_{x_0}(Q)$ and zero on orthogonal complement of range of $\alpha_{x_0}(Q)$. It is now an easy verification that $V = \{V_x\}_{x\in\Omega}$ is weakly measurable family of partial isometries in $\mathcal{H}$ such that $V_x^*V_x = \alpha_{x_0}(Q)$ and $V_xV_x^* = \alpha_x(Q)$ for all $x \in \Omega_n$. As in the proof of Arveson in [7], for $x \in \Omega$ there exist a unitary U_x such that $U_x(\alpha_{x_0}(A)V_{x_0}\xi) = \alpha_x(A)V_x\xi$ for all $A \in \mathcal{B}(\mathcal{H})$. Then $\{U_x\}_{x\in\Omega}$

forms a weakly measurable because the family $V = \{V_x\}_{x\in\Omega}$ is weakly measurable. By definition, we have $U_x\alpha_{x_0}(A) = \alpha_x(A)U_x$ for all $A \in \mathcal{B}(\mathcal{H})$ and $x \in \Omega$. □

Theorem 2.7.7. *Let $\alpha = \{\alpha_x\}_{x\in P}$ be an E_0-semigroup over P. Then $\mathcal{E}_\alpha = \coprod_{x\in\Omega} \mathcal{E}(x)$ is concrete product system over Ω.*

Proof: Note that $\mathcal{E}_\alpha$ is the standard Borel space which satisfies the first three condition of Definition 2.7.1. Let $\{V_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{E}(x_0)$. By lemma 2.7.6, let $U = \{U_x\}_{x\in\Omega}$ be a weakly measurable family of unitaries in $\mathcal{B}(\mathcal{H})$ such that $U_x\alpha_{x_0}(A) = \alpha_x(A)U_x$ for all $A \in \mathcal{B}(\mathcal{H})$ and $x \in \Omega$. Note that for $x \in \Omega$, $\{U_xV_i\}$ forms an orthonormal basis for $\mathcal{E}(x)$. Then for each i, $\{U_xV_i\}_{x\in\Omega}$ is a family of measurable section defined on Ω. This proves our claim. □

We define the notion of cocycle conjugacy among E_0-semigroups as in Arveson's [7]. For $A \in \mathcal{B}(\mathcal{H})$ and a unitary $U \in \mathcal{B}(\mathcal{H})$, let $Ad(U)(A) = UAU^*$. Let us begin with definition.

Definition 2.7.8. *Let $\alpha = \{\alpha_x\}_{x\in P}$ be an E_0-semigroup over P acting on $\mathcal{B}(\mathcal{H})$. A σ-weakly continuous family $\{U_x\}_{x\in P}$ of unitaries in M is called an α-cocycle if*

$$U_{x+y} = U_x\alpha_x(U_y) \text{ for all } x\ ,\ y \in P.$$

Given an α-cocycle $\{U_x\}_{x\in P}$, it is routine to verify that the family $\{Ad(U_x)\circ\alpha_x\}_{x\in P}$ forms an E_0-semigroup over P. Let $\beta = \{\beta_x\}_{x\in P}$ be an E_0-semigroup over P acting on $\mathcal{B}(\mathcal{H})$. We say β is a cocycle perturbation of α if there exist an α-cocycle $\{U_x\}_{x\in P}$ such that for $x \in P$, $\beta_x = Ad(U_x)\circ\alpha_x$.

Definition 2.7.9. *Let $\alpha = \{\alpha_x\}_{x\in P}$ and $\beta = \{\beta_x\}_{x\in P}$ be E_0 semigroups on $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ respectively. We say that*

- *α is conjugate to β if there exists a unitary $U : \mathcal{H} \to \mathcal{K}$ such that for every $x \in P$, $\beta_x = Ad(U)\circ\alpha_x\circ Ad(U^*)$.*

- *α is cocycle conjugate to β if there exists a unitary $U : \mathcal{H} \to \mathcal{K}$ such that the E_0-semigroup $\{Ad(U)\circ\alpha_x\circ Ad(U^*)\}_{x\in P}$ is a cocycle perturbation of β.*

We observe that cocycle conjugacy is an equivalence relation.

Definition 2.7.10. *Let $\mathcal{E} \subseteq \Omega\times\mathcal{B}(\mathcal{H})$ and $\mathcal{F} \subseteq \Omega\times\mathcal{B}(\mathcal{H})$ be concrete product systems. We say that $\mathcal{E}$ is isomorphic to $\mathcal{F}$ if for every $x \in \Omega$, there exist a unitary operator $U_x : \mathcal{E}(x) \to \mathcal{F}(x)$ such that the maps*

$$\mathcal{E} \ni (x,T) \to (x,U_xT) \in \mathcal{F} \text{ and } \mathcal{F} \ni (x,T) \to (x,U_x^*T) \in \mathcal{E}$$

are measurable and $U_{x+y}(ST) = U_x(S)U_y(T)$ *for all* $x \in \Omega$, $S \in \mathcal{E}(x)$ *and* $T \in \mathcal{F}$.

Our aim is to show that two E_0-semigroups are cocycle conjugate if and only if their product systems are isomorphic.

Lemma 2.7.11. *Let* $\mathcal{E} = \coprod_{x\in\Omega} \mathcal{E}(x)$ *be a concrete product system such that* $[\mathcal{E}(x)\mathcal{H}] = \mathcal{H}$ *for every* $x \in \Omega$. *Then there exist a unique* E_0*-semigroup* $\alpha = \{\alpha_x\}_{x\in P}$ *on* $\mathcal{B}(\mathcal{H})$ *such that for every* $x \in \Omega$, $\alpha_x(A)T = TA$ *for all* $A \in \mathcal{B}(\mathcal{H})$ *and* $T \in \mathcal{E}(x)$.

Moreover, $\mathcal{E} = \mathcal{E}_\alpha$ *is the concrete product system associated with* α.

Proof: Fix $x \in \Omega$. Note that $\mathcal{E}(x)$ is a separable, norm closed subspace of $B(\mathcal{H})$ such that T^*S is a scalar for every $S, T \in \mathcal{E}(x)$. Hence by Theorem 1.1.8, it follows that there exists a unique normal $*$-endomorphism α_x of $B(\mathcal{H})$ such that

$$\mathcal{E}(x) = \{T \in B(\mathcal{H}) : \alpha_x(A)T = TA \text{ for } A \in B(\mathcal{H})\}.$$

Recall from Proposition 1.1.8, it follows that α_x is unital for every $x \in \Omega$.

Let $x, y \in \Omega$ be given. Since $\mathcal{E}(x+y)$ is the closure of the linear span of the set $\{uv : u \in \mathcal{E}(x), v \in \mathcal{E}(y)\}$ and by Prop.1.1.9, it follows that $\alpha_{x+y} = \alpha_x \circ \alpha_y$. Consequently $\{\alpha_x\}_{x\in\Omega}$ is a semigroup of normal $*$-endomorphisms of $B(\mathcal{H})$.

Since $\mathcal{E}$ is a measurable field of Hilbert spaces, it follows that there exists measurable sections $V_1, V_2, \cdots$ such that for every $x \in \Omega$, $\{V_i(x)\}_{i=1}^\infty$ forms an orthonormal basis for $E(x)$. Hence by Theorem 1.1.8, for $x \in \Omega$, α_x is given by the equation

$$\alpha_x(A) = \sum_{n=1}^{\infty} V_n(x)AV_n(x)^* \tag{2.7.1}$$

where the sum in Equation 2.7.1 is a strongly convergent sum. The measurability of ϕ and Equation 3.1.1 implies that for $A \in B(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$, the map $\Omega \ni x \to \langle \alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$ is measurable. By Proposition 2.3.2, it follows that for $A \in B(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$, the map $\Omega \ni x \to \langle \alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$ is continuous. Again by Proposition 2.3.3, it follows that $\{\alpha_x\}_{x\in\Omega}$ extends to a unique E_0-semigroup which we still denote by $\alpha := \{\alpha_x\}_{x\in P}$. Let $\mathcal{E}_\alpha = \coprod_{x\in\Omega} \mathcal{E}_\alpha(x)$ be a concrete product system associated with α. Note that the intertwining space $\mathcal{E}_\alpha(x)$ for α equals to $\mathcal{E}(x)$. Therefore, $\mathcal{E} = \mathcal{E}_\alpha$. Hence the proof completes. □

Theorem 2.7.12. *Let $\alpha = \{\alpha_x\}_{x\in P}$ and $\beta = \{\beta_x\}_{x\in P}$ be E_0-semigroups on $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ respectively. Then α and β are cocycle conjugate if and only if their product systems are isomorphic.*

Proof: We assume that α and β act on same $\mathcal{B}(\mathcal{H})$, because replacing β with a conjugate version of itself.

Assume that β is a cocycle perturbation of α. Let $\{U_x\}_{x\in P}$ be an α-cocycle such that $\beta_x = U_x\alpha_x(A)U_x^*$ for all $A \in \mathcal{B}(\mathcal{H})$ and $x \in P$. We claim that for every $x \in \Omega$, $U_x\mathcal{E}_\alpha(x) = \mathcal{E}_\beta(x)$.

Indeed for $T \in \mathcal{E}_\alpha(x)$ and $A \in \mathcal{B}(\mathcal{H})$ we have

$$\begin{aligned}\beta_x(A)U_xT &= U_x\alpha_x(A)U_x^*U_xT\\ &= U_x\alpha_x(A)T\\ &= U_xTA.\end{aligned}$$

The above equality shows that $U_x\mathcal{E}_\alpha(x) \subseteq \mathcal{E}_\beta(x)$. Similar argument for the opposite inclusion.

Let $\theta : \mathcal{E}_\alpha \to \mathcal{E}_\beta$ be the map defined by the following relation $\theta((x,T)) = (x, U_xT)$ for all $(x,T) \in \mathcal{E}_\alpha$. It is routine to verify that θ is a product system isomorphism. Hence $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are isomorphic.

Suppose α and β are two E_0-semigroups acting on $\mathcal{B}(\mathcal{H})$. Assume that $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are isomorphic. Let $\theta : \mathcal{E}_\alpha \to \mathcal{E}_\beta$ be the isomorphism. Suppose there exists sequence of mesurable functions $V_n : \Omega \to \mathcal{E}_\alpha$ such that for $x \in \Omega$ $\{V_n(x)\}_{n=1}^\infty$ is orthonormal basis for $\mathcal{E}_\alpha(x)$.

Put $W_n = \theta \circ V_n$ for $n \in \mathbb{N}$. We get a sequence of measurable functions $W_n : \Omega \to \mathcal{E}_\beta$ such that for $x \in \Omega$ $\{W_n(x)\}_{n=1}^\infty$ is an orthonormal basis for $\mathcal{E}_\beta(x)$. For $x \in \Omega$, put $U_x = \sum_{n=1}^\infty W_n(x)V_n(x)^*$. Since $\sum_{n=1}^\infty W_n(x)W_n(x)^* = 1 = \sum_{n=1}^\infty V_n(x)V_n(x)^*$, U_x is a unitary operator.

We claim that for $x \in \Omega$, $U_xT = \theta(T)$ for all $T \in \mathcal{E}_\alpha(x)$. Indeed, for

$$\begin{aligned} U_xT &= (\sum_{n=1}^{\infty} W_n(x)V_n(x)^*)T \\ &= \sum_{n=1}^{\infty} \theta(V_n(x))\langle T, V_n(x)\rangle \\ &= \theta(\sum_{n=1}^{\infty} \langle T, V_n(x)\rangle V_n(x)) \\ &= \theta(T). \end{aligned}$$

Hence U_x is the unique unitary operator such that $U_xT = \theta(T)$ for all $T \in \mathcal{E}_\alpha(x)$, because the set $\{T\xi : T \in \mathcal{E}_\alpha(x), \xi \in \mathcal{H}\}$ is total in $\mathcal{H}$.

Note that the map $\Omega \ni x \to U_x \in \mathcal{B}(\mathcal{H})$ is σ-weakly measurable. The measurable family $\{U_x\}_{x\in\Omega}$ satisfies the cocycle equation left to the reader. Prop 2.4.1 implies that there exists a unique σ-weakly continuous family of unitaries $\{\widetilde{U}_x\}_{x\in P}$ in M such that $\widetilde{U}_x = U_x$ for $x \in \Omega$. Note that $\beta_x = U_x\alpha_x(A)U_x^*$ for all $A \in \mathcal{B}(\mathcal{H})$ and $x \in \Omega$. Prop 2.3.1 implies that β is a cocycle perturbation of α. □

Remark 2.7.13. *The above Theorem is stated as a Theorem B in the introduction of this thesis. The proof of Theorem B first appeared in [17] (see Theorem 2.9).*

Chapter 3

Essential Representation of Discrete Product Systems

Before discussing the product systems over the closed convex cone in $\mathbb{R}^d$, first, we explore the product systems over finitely generated subsemigroups of $\mathbb{Z}^d$.

3.1 Discrete Product Systems

In this section, we recall the notion of a product systems of Hilbert spaces over a semigroup. What follows in this section is well known and is based on [12], the monograph [7] and [8]. We have included this section to make this chapter easily readable and self contained.

Let S be a countable cancellative semigroup containing the identity element e. The following definition is equivalent to the one given in [12].

Definition 3.1.1. *By a product system of Hilbert spaces over S, we mean a disjoint union of non-zero separable Hilbert spaces over S, say $E := \coprod_{t \in S} E(t)$, such that*

(1) the Hilbert space $E(e) = \mathbb{C}$,

(2) for $s, t \in S$, there exists a unitary $u_{s,t} : E(s) \otimes E(t) \to E(st)$ such that for $r, s, t \in S$, $u_{rs,t}(u_{r,s} \otimes id) = u_{r,st}(id \otimes u_{s,t})$, and

(3) for $s \in S$, $u_{s,e}(x \otimes \lambda) = \lambda x$ and $u_{e,s}(\lambda \otimes x) = \lambda x$ for $\lambda \in E(e)$ and $x \in E(s)$.

Let $s, t \in S$ and $(x, y) \in E(s) \times E(t)$ be given. We write $u_{s,t}(x \otimes y)$ as xy.

Let $E := \coprod_{t\in S} E(t)$ and $F := \coprod_{t\in S} F(t)$ be product systems over S. We say E is isomorphic to F if for every $t \in S$, there exists a unitary operator $\theta_t : E(t) \to F(t)$ such that $\theta_{st}(xy) = \theta_s(x)\theta_t(y)$ for $s, t \in S$ and $(x, y) \in E(s) \times E(t)$.

Definition 3.1.2. *Let $E := \coprod_{t\in S} E(t)$ be a product system over S. Let $\mathcal{H}$ be a separable Hilbert space. By a representation of E on $\mathcal{H}$, we mean a map $\phi : E \to B(\mathcal{H})$ such that*

(1) for $x, y \in E$, $\phi(xy) = \phi(x)\phi(y)$, and

(2) for $s \in S$ and $x, y \in E(s)$, $\phi(y)^*\phi(x) = \langle x|y\rangle$.

The representation ϕ is called essential if for every $t \in S$, $\overline{\phi(E(t))\mathcal{H}} = \mathcal{H}$.

Let $E := \coprod_{t\in S} E(t)$ be a product system over S and let $\phi : E \to B(\mathcal{H})$ be an essential representation. Then for every $t \in P$, $\phi|_{E(t)}$ is linear. For a proof of this, adapt the argument in Section 3.2 of [7] to our setup. Also the map $\phi : E(t) \to \phi(E(t)) \subset B(\mathcal{H})$ is an unitary.

Fix $t \in S$. Note that $\phi(E(t))$ is a separable, norm closed subspace of $B(\mathcal{H})$ such that T^*S is a scalar for every $S, T \in \phi(E(t))$. Hence by Theorem 1.1.8, it follows that there exists a unique normal $*$-endomorphism α_t of $B(\mathcal{H})$ such that

$$\phi(E(t)) = \{T \in B(\mathcal{H}) : \alpha_t(A)T = TA \text{ for } A \in B(\mathcal{H})\}.$$

Since ϕ is an essential and Prop. 1.1.8, $\alpha_t(1) = 1$ for $t \in S$.

Let $s, t \in S$ be given. Since $E(st)$ is the closure of the linear span of the set $\{uv : u \in E(s), v \in E(t)\}$, it follows from (2) of Definition 3.1.2 that $\phi(E(st))$ is the closed linear span of $\{ST : S \in \phi(E(s)), T \in \phi(E(t))\}$. Hence by Prop.1.1.9, it follows that $\alpha_{st} = \alpha_s \circ \alpha_t$. Consequently $\{\alpha_t\}_{t\in S}$ is an E_0-semigroup of unital normal $*$-endomorphisms of $B(\mathcal{H})$.

Since E is a measurable field of Hilbert space, it follows that there exists sections $v_1, v_2, \cdots$ such that for every $s \in S$, $\{v_i(t)\}_{i=1}^d$ forms an orthonormal basis for $E(t)$. Consequently, for $t \in S$, $\{\phi(v_i(t))\}$ is an orthonormal basis for $\phi(E(t))$. Hence by Theorem 1.1.8, for $x \in \Omega$, α_t is given by the equation

$$\alpha_t(A) = \sum_{n=1}^{d} \phi(v_n(t))A\phi(v_n(t))^* \tag{3.1.1}$$

where the sum in Equation 3.1.1 is strongly convergent sum if d is infinite. Let $F := \coprod_{t\in S} \phi(E(t))$ be the product system associated to α. For $t \in S$, put $\theta_t = \phi|_{E(t)}$. Note that for $t \in S$, θ_t is a unitary between $E(t)$ and $\phi(E(t)$. Since ϕ is multiplicative, $\theta_{st}(uw) = \theta_s(u)\theta_t(w)$ for $(u, w) \in E(s) \times E(t)$ and $s, t \in S$. Hence E is isomorphic to F.

Remark 3.1.3. *It is known that a product system over S is isomorphic to a product system associated to an E_0-semigroup over S. This is due to the fact that any product system has a representation on a separable Hilbert space. (See Lemma 1.10 of [12]).*

For $t, s \in S$, we write $t \geq s$ if there exists $a \in S$ such that $t = sa$. Since P is cancellative, it follows that for $t, s \in S$, if $t \geq s$ then there exists a unique element in S, denoted $s^{-1}t$, such that $t = s(s^{-1}t)$.

Let $E := \coprod_{t\in S} E(t)$ be a product system over S. Let $t, s \in S$ be such that $t \geq s$. Choose $a \in S$ such that $t = sa$. For $v \in E(s)$ and $w \in E(t)$, there exists a unique element denoted $v^*w \in E(a)$ such that $\langle x|v^*w\rangle = \langle vx|w\rangle$ for every $x \in E(a)$. Note that

$$||v^*w|| \leq ||v||||w|| \tag{3.1.2}$$

for $v \in E(s)$ and $w \in E(t)$.

Lemma 3.1.4. *Let $E := \coprod_{t\in S} E(t)$ be a product system over S. Let $t, s, r \in S$ be such that $t \geq s$. Then for $v \in E(s)$, $w_1 \in E(t)$ and $w_2 \in E(r)$, $v^*(w_1w_2) = (v^*w_1)w_2$.*

Proof. Let $a \in S$ be such that $t = sa$. Consider elements $v \in E(s)$, $w_1 \in E(t)$ and $w_2 \in E(r)$. To show $v^*(w_1w_2) = (v^*w_1)w_2$, it is enough to show that

$$\langle v^*(w_1w_2)|u\rangle = \langle (v^*w_1)w_2|u\rangle$$

for every $u \in E_{ar}$. Since $\{xy : x \in E(a), y \in E(r)\}$ is total in E_{ar}, it suffices to show that for $x \in E(a), y \in E(r)$,

$$\langle v^*(w_1w_2)|xy\rangle = \langle (v^*w_1)w_2|xy\rangle.$$

To that end, let $x \in E(a)$ and $y \in E(r)$ be given. Calculate as follows to find that

$$\begin{aligned}\langle v^*(w_1w_2)|xy\rangle &= \langle w_1w_2|v(xy)\rangle \\ &= \langle w_1w_2|(vx)y\rangle \\ &= \langle w_1|vx\rangle\langle w_2|y\rangle \\ &= \langle v^*w_1|x\rangle\langle w_2|y\rangle \\ &= \langle (v^*w_1)w_2|xy\rangle.\end{aligned}$$

This completes the proof. □

We also need the following Lemma whose proof is obtained by merely translating the proof of Lemma 2.4 of [8] to our setup. Thus we omit the proof.

Lemma 3.1.5. *Let $E := \coprod_{t\in S} E(t)$ be a product system over S. Let $t, s \in S$ be such that $t \geq s$. Suppose that $\{v_i\}_{i=1}^d$ is an orthonormal basis for $E(s)$. Here d is the dimension of $E(s)$. Then $\sum_{i=1}^{d} ||v_i^*\xi||^2 = ||\xi||^2$ for every $\xi \in E(t)$.*

3.2 Examples of semigroups of $\mathbb{Z}^d$

Let us give a few examples of subsemigroups of $\mathbb{Z}^d$ before we proceed further.

(1) The semigroup $\mathbb{N}^d$ is a finitely generated subsemigroup of $\mathbb{Z}^d$. For $i = 1, 2, \cdots, d$, let $e_i \in \mathbb{Z}^d$ be the vector whose co-ordinate at the i^{th}-place is 1 and 0 elsewhere. Then $\{e_i : i = 1, 2, \cdots, d\}$ generates $\mathbb{N}^d$.

(2) Let $r, s \geq 0$ be such that $r + s = d$. Then $\mathbb{N}^r \times \mathbb{Z}^s$ is finitely generated. The set $\{e_i : i = 1, 2, \cdots, r\} \cup \{\pm\, e_i : i = r + 1, \cdots, d\}$ generates $\mathbb{N}^r \times \mathbb{Z}^s$. Moreover the group of invertibles of $\mathbb{N}^r \times \mathbb{Z}^s$ is $\{0\} \times \mathbb{Z}^s$. Suppose $r_1, s_1, r_2, s_2 \in \mathbb{N}$ be such that $r_1 + s_1 = r_2 + s_2 = d$. Then $\mathbb{N}^{r_1} \times \mathbb{Z}^{s_1}$ is isomorphic to $\mathbb{N}^{r_2} \times \mathbb{Z}^{s_2}$ if and only if $r_1 = r_2$ and $s_1 = s_2$.

(3) Let $d = 1$ and $m \geq 1$ be given. Let $S_m := \mathbb{N}\backslash\{1, 2, \cdots, m\}$ i.e.

$$S_m = \{0\} \cup \{m + 1, m + 2, m + 3, \cdots\}.$$

We claim that $\{m + 1, m + 2, \cdots, 2m + 1\}$ generates S_m. Let $\overline{S_m}$ be the subsemigroup of $\mathbb{N}$ generated by $\{m+1, m+2, \cdots, 2m+1\}$. We claim that for every

$x \geq 2m+2$, the set $\{y : m+1 \leq y \leq x\} \subset \overline{S_m}$. We prove this by induction on x. Since $2m+2 = 2(m+1)$, it is clear that $\{y : m+1 \leq y \leq 2m+2\} \subset \overline{S_m}$. Now suppose $x \geq 2m+2$ be such that $\{y : m+1 \leq y \leq x\} \subset \overline{S_m}$. Then $x - m \geq m+1$ and $x - m < x$. Hence $x - m \in \overline{S_m}$. Now $x+1 = (x-m)+m+1 \in \overline{S_m}$. This shows that $\{y : m+1 \leq y \leq x+1\} \subset \overline{S_m}$. Thus by induction, it follows that $\overline{S_m} = S_m$.

We claim that S_m is isomorphic to S_n if and only if $m = n$. Call an element $x \in S_m$ reducible ([9], Page 32) if there exists $y, z \in S_m$ such that $x = y + z$ with $y, z \neq 0$. Call an element $x \in S_m$ irreducible if x is not reducible. An easy induction argument implies that the set of irreducible elements of S_m are $\{m+1, m+2, \cdots, 2m+1\}$. Thus the cardinality of the set of irreducible elements of S_m is $m+1$. This shows that S_m is isomorphic to S_n if and only if $m = n$.

(4) Let $m \geq 1$ and let S_m be the semigroup considered in (3). We claim that S_m is not isomorphic to $\mathbb{N}^k$ for every k. Let $\{e_i : i = 1, 2, \cdots, k\}$ be the generators considered in (1). It is clear that the irreducible elements of $\mathbb{N}^k$ are $\{e_1, e_2, \cdots, e_k\}$. Given distinct irreducible elements $x, y \in S_m$ there exists $r, s \geq 1$ such that $rx = sy$. But given distinct irreducible elements $x, y \in \mathbb{N}^k$, $rx = sy$ if and only if $r = s = 0$. This shows that S_m is not isomorphic to $\mathbb{N}^k$ for any k.

3.3 Construction of an essential representation

Fix $d \geq 1$. Let $S \subset \mathbb{Z}^d$ be a non-zero finitely generated subsemigroup. Then $S - S$ is a subgroup of $\mathbb{Z}^d$ and hence isomorphic to $\mathbb{Z}^m$ for some m. Thus with no loss of generality, we can assume that $S - S = \mathbb{Z}^d$. The semigroup S is fixed for the rest of this section.

Proposition 3.3.1. *Let $E := \coprod_{s \in S} E(s)$ be a product system over S. Suppose that $E(s)$ is 1-dimensional for every $s \in S$. Then there exists an E_0-semigroup over S $\alpha := \{\alpha_s\}_{s \in S}$ on $B(\ell^2(\mathbb{Z}^d))$ such that the product system associated to α is isomorphic to E.*

Proof. For $s \in S$, choose a unit vector $e_s \in E(s)$. Then for every $r, s \in S$, there exists a unique scalar, denoted $\omega(r, s) \in \mathbb{T}$, such that $e_r e_s = \omega(r, s) e_{r+s}$. The associativity of the multiplication of the product system E implies that ω is a multiplier on S i.e. for $r, s, t \in S$,

$$\omega(r, s)\omega(r+s, t) = \omega(r, s+t)\omega(s, t).$$

By Theorem 2.2 of [15], it follows that ω extends to a multiplier on $\mathbb{Z}^d$. We denote the extension also by ω.

Let $\{\delta_z : z \in \mathbb{Z}^d\}$ be the standard orthonormal basis for $\ell^2(\mathbb{Z}^d)$. For $x \in \mathbb{Z}^d$, let U_x be the unitary on $\ell^2(\mathbb{Z}^d)$ defined by the equation

$$U_x(\delta_z) = \omega(x,z)\delta_{x+z}.$$

Note that $U_xU_y = \omega(x,y)U_{x+y}$ for all $x, y \in \mathbb{Z}^d$.

For $s \in S$, let α_s be the automorphism of $B(\ell^2(\mathbb{Z}^d))$ defined by the formula

$$\alpha_s(A) = U_sAU_s^*.$$

Then $\alpha := \{\alpha_s\}_{s\in S}$ is an E_0-semigroup over S on $B(\ell^2(\mathbb{Z}^d))$. Let $F := \coprod_{s\in S} F(s)$ be the product system associated to the E_0-semigroup over$S$, α. It is clear that for $s \in S$, $F(s)$ is spanned by U_s. For $s \in S$, let $\theta_s : E(s) \to F(s)$ be the unitary such that $\theta_s(e_s) = U_s$. Now it is immediate that $\theta := \coprod_{s\in S} \theta_s : E \to F$ is an isomorphism of product systems. This completes the proof. □

Remark 3.3.2. *Suppose $S = \mathbb{Z}^d$ and $E := \coprod_{s\in\mathbb{Z}^d} E(s)$ is a product system over S. Then for every $s \in \mathbb{Z}^d$, $E(s) \otimes E(-s) \cong \mathbb{C}$. This implies that $E(s)$ is 1-dimensional for every $s \in S$. Thus by Proposition 3.3.1, it follows that E is isomorphic to a product system associated to an E_0-semigroup over S.*

Hereafter we assume that $S \neq \mathbb{Z}^d$. Let us make a few preliminary observations regarding the semigroup S. Let $\{e_1, e_2, \cdots, e_r\}$ be a set of generators for the semigroup S i.e. $S = \{\sum_{i=1}^r m_ie_i : m_i \in \mathbb{N}\}$. Let $a := \sum_{i=1}^r e_i$. For $x, y \in \mathbb{Z}^d$, we write $x \geq y$ if $x - y \in S$. We use the above notations for the rest of this chapter. We have the following archimedean principle.

Lemma 3.3.3. *Let $x \in \mathbb{Z}^d$ be given. Then there exists $n \geq 1$ such that $na \geq x$. As a consequence, we have $-ka \notin S$ for every $k \geq 1$.*

Proof: Since $S - S = \mathbb{Z}^d$, there exists integers $m_1, m_2, \cdots, m_r$ such that $x = \sum_{i=1}^r m_ie_i$. Let $n \geq 1$ be such that $n \geq m_i$ for each i. Then $na - x = \sum_{i=1}^r (n - m_i)e_i \in S$.

Suppose $-ka \in S$ for some $k \geq 1$. Then $-a = -ka + (k-1)a \in S$. This implies that $-na \in S$ for every $n \geq 1$. Let $x \in \mathbb{Z}^d$ be given. Then there exists $n \geq 1$ such that

$na \geq -x$ or in other words, $na + x \in S$. Hence $x = (na + x) + (-na) \in S$. This forces that $S = \mathbb{Z}^d$ which is a contradiction since we have assumed that $S \neq \mathbb{Z}^d$.

Lemma 3.3.4. *The intersection* $\bigcap_{n=0}^{\infty}(S+na) = \emptyset$. *Moreover the sequence* $\{S+na\}_{n\geq 0}$ *is a decreasing sequence of subsets of* S.

Proof. Suppose $y \in \bigcap_{n=0}^{\infty}(S+na)$. Then $y-na \in S$ for every $n \geq 0$. By Lemma 3.3.3, there exists $n_0 \geq 1$ such that $n_0a-y \in S$. Note that $-a = (y-(n_0+1)a)+(n_0a-y) \in S$ which is a contradiction to Lemma 3.3.3. It is clear that $\{S + na\}_{n\geq 0}$ is a decreasing sequence of subsets of S. This completes the proof. □

Notations: For $k \geq 0$, let $L_k := (S+ka)\backslash(S+(k+1)a)$. Then Lemma 3.3.4 implies that $\{L_k : k \geq 0\}$ is a disjoint family of subsets of S whose union is S. Observe that for $k \geq 0$, $ka \in L_k$. Also note that for $k \geq 0$, $S + ka = \coprod_{m\geq k} L_m$. Since $S = \coprod_{k\geq 0} L_k$, for $s \in S$, there exists a unique non-negative integer denoted $n(s)$ such that $s \in L_{n(s)}$. Note that for $s \in S$, $n(s + a) = n(s) + 1$. Also observe that for $s \in S$, $s - n(s)a \in L_0$ and if $s \in L_0$ then $s + ka \in L_k$ for $k \geq 0$. For $z \in \mathbb{Z}^d$, let $L_z = (L_0 + z) \cap S$. We use the above notations throughout this chapter.

Let $E := \coprod_{s\in S} E(s)$ be a product system over S which is fixed for the rest of this section. We assume that there exists $s \in S$ such that $E(s)$ is not 1-dimensional. Our goal in this section is to construct an essential representation of E on an infinite dimensional separable Hilbert space. Let $e \in E(a)$ be a unit vector which is fixed for the rest of this section.

Let $\mathcal{V}$ denote the vector subspace of sections of E which are square integrable over L_z for every $z \in \mathbb{Z}^d$. More precisely, let $f : S \to E$ be a section. Then $f \in \mathcal{V}$ if and only if for every $z \in \mathbb{Z}^d$,

$$\sum_{s\in L_z} ||f(s)||^2 < \infty.$$

As is customary, an empty sum equals zero. Let $f \in \mathcal{V}$ and $k \geq 0$ be given. We say that f is k-stable if $f(s + a) = f(s)e$ for $s \geq ka$. Note that if f is k-stable then f is k_1-stable for $k_1 \geq k$. Let $f \in \mathcal{V}$ be given. We say that f is stable if f is k-stable for some $k \geq 0$. Denote the set of stable sections in $\mathcal{V}$ by $\mathcal{S}$. Note that $\mathcal{S}$ is a vector subspace of $\mathcal{V}$.

Let $f \in \mathcal{V}$. We say that f is eventually zero if there exists $k \geq 0$ such that $f(s) = 0$ for $s \geq ka$. Denote the set of eventually zero sections in $\mathcal{V}$ by $\mathcal{N}$. Note that $\mathcal{N} \subset \mathcal{S}$ and $\mathcal{N}$ is a vector subspace of $\mathcal{S}$.

Let $f, g \in \mathcal{S}$ be given. Since f and g are square integrable over L_z for every $z \in \mathbb{Z}^d$, it follows that for every $k \geq 0$, the sum $\sum_{s \in L_k} \langle f(s)|g(s)\rangle$ exists.

Proposition 3.3.5. *Let $f, g \in \mathcal{S}$. Then the sequence $\Big(\sum_{s \in L_k} \langle f(s)|g(s)\rangle\Big)_{k=1}^{\infty}$ converges.*

Proof. Without loss of generality, we can assume that f and g are k_0-stable for some $k_0 \geq 0$. Let $k \geq k_0$ be given. Note that the map $L_{k_0} \ni s \to s + (k - k_0)a \in L_k$ is a bijection. Now calculate as follows to observe that

$$\begin{aligned}
\sum_{s \in L_k} \langle f(s)|g(s)\rangle &= \sum_{s \in L_{k_0}} \langle f(s + (k-k_0)a)|g(s + (k-k_0)a\rangle \\
&= \sum_{s \in L_{k_0}} \langle f(s)e^{k-k_0}|g(s)e^{k-k_0}\rangle \quad \text{(Since } f \text{ and } g \text{ are } k_0\text{-stable)} \\
&= \sum_{s \in L_{k_0}} \langle f(s)|g(s)\rangle.
\end{aligned}$$

This shows that the sequence $\Big(\sum_{s \in L_k} \langle f(s)|g(s)\rangle\Big)_{k=1}^{\infty}$ is eventually constant and hence converges. This completes the proof. ☐

For $f, g \in \mathcal{S}$, let

$$\langle f|g\rangle := \lim_{k \to \infty} \Big(\sum_{s \in L_k} \langle f(s)|g(s)\rangle\Big).$$

Then $\langle\ |\ \rangle$ defines a semi-definite inner product on $\mathcal{S}$. Let $f \in \mathcal{S}$ be given. Note that $\langle f|f\rangle = 0$ if and only if $f \in \mathcal{N}$. It is straightforward to see that if $f \in \mathcal{N}$ then $\langle f|f\rangle = 0$. Now, let $f \in \mathcal{S}$ be such that $\langle f|f\rangle = 0$. Assume that f is k_0-stable for some $k_0 \geq 0$. Then the proof of Proposition 3.3.5 implies that $\sum_{s \in L_k} ||f(s)||^2 = 0$ for every $k \geq k_0$. This implies that f vanishes on L_k for $k \geq k_0$. Hence f vanishes on $\coprod_{k \geq k_0} L_k = S + k_0 a$. Consequently, we have $f \in \mathcal{N}$. Thus $\langle\ |\ \rangle$ descends to a positive definite inner product on $\mathcal{S}/\mathcal{N}$ which we still denote by $\langle\ |\ \rangle$. Let $\mathcal{H}$ be the completion of the pre-Hilbert space $\mathcal{S}/\mathcal{N}$.

Remark 3.3.6. *Let $f, g \in \mathcal{S}$. Assume that f and g are k_0-stable for some $k_0 \geq 0$. Then the proof of Proposition 3.3.5 shows that*

$$\langle f|g\rangle = \sum_{s \in L_{k_0}} \langle f(s)|g(s)\rangle.$$

Proposition 3.3.7. *The Hilbert space $\mathcal{H}$ is separable and is non-zero.*

Proof. For $k \geq 0$, let $\mathcal{H}_k := \bigoplus_{s \in L_k} E(s)$. Clearly $\mathcal{H}_k$ is separable for each $k \geq 0$. Fix $k \geq 0$. Let $\xi \in \mathcal{H}_k$ be given. Define a section $\widetilde{\xi} : S \to E$ by the following formula:

$$\widetilde{\xi}(s) := \begin{cases} \xi(s - n(s)a + ka)e^{n(s)-k} & \text{if } s \geq ka, \\ \\ 0 & \text{elsewhere.} \end{cases}$$

Note that the above definition makes sense since for $s \geq ka$, $n(s) \geq k$. We claim the following.

(1) The section $\widetilde{\xi} \in \mathcal{V}$ and is k-stable.

(2) For $s \in L_k$, $\widetilde{\xi}(s) = \xi(s)$.

Let $z \in \mathbb{Z}^d$ be given. Note that $\sum_{s \in L_z} ||\widetilde{\xi}(s)||^2 = \sum_{s \in A} ||\widetilde{\xi}(s)||^2$ where $A := \{s \in L_z : s \geq ka\}$. If A is empty, there is nothing to prove. Suppose that A is non-empty. We claim that the map $A \ni s \to s - n(s)a + ka \in L_k$ is injective. Let $s_1, s_2 \in A$ be such that $s_1 - n(s_1)a + ka = s_2 - n(s_2)a + ka$. To show $s_1 = s_2$, it is enough to prove $n(s_1) = n(s_2)$. Suppose not. Without loss of generality, we can assume that $n(s_2) > n(s_1)$. Note that $s_2 - z = (s_1 - z) + (n(s_2) - n(s_1))a \in S + a$ which contradicts the fact that $s_2 \in L_z = ((S + z) \backslash (S + z + a)) \cap S$. Let B be the image of the map $A \ni s \to s - n(s)a + ka \in L_k$. Now calculate as follows to observe that

$$\begin{aligned} \sum_{s \in L_z} ||\widetilde{\xi}(s)||^2 &= \sum_{s \in A} ||\widetilde{\xi}(s)||^2 \\ &= \sum_{s \in A} ||\xi(s - n(s)a + ka)||^2 \\ &= \sum_{s \in B} ||\xi(s)||^2 \\ &\leq \sum_{s \in L_k} ||\xi(s)||^2 \\ &< \infty. \end{aligned}$$

This shows that $\widetilde{\xi} \in \mathcal{V}$. Let $s \geq ka$ be given. Calculate as follows to observe that

$$\begin{aligned}\widetilde{\xi}(s+a) &= \xi(s+a-n(s+a)a+ka)e^{n(s+a)-k} \\ &= \xi(s+a-(n(s)+1)a+ka)e^{n(s)+1-k} \quad (\text{Since } n(s+a)=n(s)+1) \\ &= \xi(s-n(s)a+ka)e^{n(s)-k}e \\ &= \widetilde{\xi}(s)e.\end{aligned}$$

This proves that $\widetilde{\xi}$ is k-stable. This proves (1). Note that for $s \in L_k$, $n(s) = k$. Now (2) follows from the definition. Remark 3.3.6 together with (1) and (2) implies that the map $\mathcal{H}_k \ni \xi \to \widetilde{\xi} + \mathcal{N} \in \mathcal{H}$ is an isometry which we denote by V_k.

Let $f \in \mathcal{S}$ be given. Assume that f is k-stable for some $k \geq 0$. Let $\xi \in \mathcal{H}_k$ be defined by $\xi(s) = f(s)$. Suppose $s \geq ka$. Note that $s = t+(n(s)-k)a$ where $t = (s-n(s)a)+ka$. Observe that $t \in L_k$ and in particular $t \geq ka$. Since f is k-stable it follows that

$$\begin{aligned}f(s) &= f(t+(n(s)-k)a) \\ &= f(t)e^{n(s)-k} \\ &= f(s-n(s)a+ka)e^{n(s)-k} \\ &= \xi(s-n(s)a+ka)e^{n(s)-k} \\ &= \widetilde{\xi}(s).\end{aligned}$$

Thus we have shown that $\widetilde{\xi} - f$ is eventually zero. Consequently $\widetilde{\xi} + \mathcal{N} = f + \mathcal{N}$. Hence $\{f + \mathcal{N} : f \in \mathcal{S}\} = \bigcup_{k=0}^{\infty} V_k\mathcal{H}_k$. This implies that $\bigcup_{k=0}^{\infty} V_k\mathcal{H}_k$ is dense in $\mathcal{H}$. As each $\mathcal{H}_k$ is separable, it follows that $\mathcal{H}$ is separable. Since each $\mathcal{H}_k$ is non-zero, it is clear that $\mathcal{H}$ is non-zero. This completes the proof. □.

We need the following two important lemmas before defining a representation of E on $\mathcal{H}$.

Lemma 3.3.8. *Let $k \geq 0$ and $b \geq ka$ be given. For every $x \in L_k$, the intersection $\{x + ma : m \geq 0\} \cap L_b$ is singleton. For $x \in L_k$, let $\chi(x) \in L_b$ be such that*

$$\{\chi(x)\} = \{x + ma : m \geq 0\} \cap L_b.$$

Then the map $L_k \ni x \to \chi(x) \in L_b$ is a bijection.

Proof. Recall that $L_b = (S+b)\backslash(S+b+a)$ and $L_k = (S+ka)\backslash(S+(k+1)a)$. Let $x \in L_k$ be given. By Lemma 3.3.3, there exists $m \geq 0$ such that $ma-(b-x) = x+ma-b \in S$. Let

$m(x)$ be the least non-negative integer such that $x + m(x)a \in S + b$. Suppose $m(x) = 0$. Since $b \geq ka$ and $x \notin S+(k+1)a$, it follows that $x = x+m(x)a \notin S+b+a$. In this case, $x + m(x)a \in L_b$. Now suppose $m(x) \geq 1$. Then by definition $x + (m(x) - 1)a \notin S + b$. Hence $x + m(x)a \notin S + b + a$. In this case too, $x + m(x)a \in L_b$. This proves that the intersection $\{x + ma : m \geq 0\} \cap L_b$ is non-empty.

Suppose $x + ma \in (S + b)\backslash(S + b + a)$. By the definition of $m(x)$, it follows that $m \geq m(x)$. Suppose $m > m(x)$. Write $m = n + m(x)$ with $n \geq 1$. Observe that $x + ma = (x + m(x)a) + na \in S + b + na \subset S + b + a$. Hence $x + ma \in S + b + a$ which contradicts the fact that $x+ma \in (S+b)\backslash(S+b+a)$. As a consequence, we have $m = m(x)$. This implies that the intersection $\{x + ma : m \geq 0\} \cap L_b$ is singleton.

Let χ be the map described in the statement of the Lemma. We claim that χ is 1-1. Let $x_1, x_2 \in L_k$ be such that $\chi(x_1) = \chi(x_2)$. Then $x_1+m(x_1)a = x_2+m(x_2)a$. It is enough to prove that $m(x_1) = m(x_2)$. Suppose not. Without loss of generality, we can assume that $m(x_1) < m(x_2)$. Then $x_1 = x_2 + (m(x_2) - m(x_1))a \in S + ka + (m(x_2) - m(x_1))a$. Since $S+ka+(m(x_2)-m(x_1))a$ is a subset of $S+(k+1)a$, it follows that $x_1 \in S+(k+1)a$. This contradicts the fact that $x_1 \in (S + ka)\backslash(S + (k + 1)a)$. Hence χ is 1-1.

We claim that χ is onto. Let $y \in (S + b)\backslash(S + b + a)$ be given. Since $y \in S + b$ and $b \geq ka$, it follows that $y = y - 0.a \in S + ka$. Hence the set $\{m \in \mathbb{N} : y - ma \in S + ka\}$ is non-empty. We claim that $\{m \in \mathbb{N} : y - ma \in S + ka\}$ is bounded. Suppose not. Then there exists a sequence (m_ℓ) such that $m_\ell \to \infty$ and $y - m_\ell a \in S + ka$. By Lemma 3.3.3, it follows that there exists $m_0 \geq 0$ such that $m_0 a - y + ka \in S$. Choose ℓ such that $m_\ell > m_0$. Then

$$(m_0 - m_\ell)a = (m_0 a - y + ka) + (y - m_\ell a - ka) \in S$$

which is a contradiction to Lemma 3.3.3 since $m_0 - m_\ell < 0$. This proves that the set $\{m \in \mathbb{N} : y - ma \in S + ka\}$ is bounded. Let m_0 be the largest non-negative integer such that $y - m_0 a \in S + ka$. Then $y - (m_0 + 1)a \notin S + ka$ or in other words $y - m_0 a \notin S + (k + 1)a$. Hence $y - m_0 a \in (S + ka)\backslash(S + (k + 1)a)$. Set $x = y - m_0 a$. Then $y = x+m_0 a \in \{x+ma : m \geq 0\}\cap L_b$. Since the intersection $\{x+ma : m \geq 0\}\cap L_b$ is singleton, it follows that $\chi(x) = y$. This proves that χ is onto. This completes the proof. ◻

Lemma 3.3.9. *Let $f, g \in \mathcal{S}$ be given. Assume that f and g are k-stable for some $k \geq 0$. Let $b \in S$ be such that $b \geq ka$. Then*

$$\langle f|g\rangle = \sum_{s \in L_b} \langle f(s)|g(s)\rangle.$$

Proof. Let $\chi : L_k \to L_b$ be the bijection described in Lemma 3.3.8. For $x \in L_k$, let $m(x) \geq 0$ be the unique non-negative integer such that $\chi(x) = x + m(x)a$. Now calculate as follows to observe that

$$\begin{aligned}
\sum_{s \in L_b} \langle f(s)|g(s)\rangle &= \sum_{x \in L_k} \langle f(\chi(x))|g(\chi(x))\rangle \\
&= \sum_{x \in L_k} \langle f(x + m(x)a)|g(x + m(x)a)\rangle \\
&= \sum_{x \in L_k} \langle f(x)e^{m(x)}|g(x)e^{m(x)}\rangle \quad \text{(Since } f \text{ and } g \text{ are } k\text{-stable)} \\
&= \sum_{x \in L_k} \langle f(x)|g(x)\rangle \\
&= \langle f|g\rangle \quad \text{(by Remark 3.3.6)}.
\end{aligned}$$

This completes the proof. □

Let $b \in S$ and $v \in E(b)$ be given. For $f \in \mathcal{S}$, let $\phi_0(v)f : S \to E$ be the section defined by

$$(\phi_0(v)f)(s) := \begin{cases} vf(s-b) & \text{if } s \geq b \\ 0 & \text{elsewhere.} \end{cases}$$

Let $f \in \mathcal{S}$ be given. We leave it to the reader to verify that $\phi_0(v)f \in \mathcal{V}$. Suppose that f is k-stable. Choose $k_0 \geq 0$ such that $k_0 a \geq b$. Set $k_1 = k_0 + k$. Let $s \in S$ be such that $s \geq k_1 a$. Then calculate as follows to observe that

$$\begin{aligned}
(\phi_0(v)f)(s+a) &= vf(s+a-b) \\
&= vf(s-b)e \quad \text{(Since } s - b \geq k_1 a - b = ka + (k_0 a - b) \geq ka) \\
&= (\phi_0(v)f)(s)e.
\end{aligned}$$

This proves that $\phi_0(v)f$ is k_1-stable.

Proposition 3.3.10. *Let $b \in S$ and $u, v \in E(b)$ be given. Then for $f \in \mathcal{S}$,*

$$\langle \phi_0(u)f|\phi_0(v)f\rangle = \langle u|v\rangle\langle f|f\rangle.$$

Proof. Let $f \in \mathcal{S}$ be given. Assume that f is k-stable for some $k \geq 0$. Choose $k_0 \geq 0$ such that $k_0a \geq b$ and set $k_1 = k_0 + k$. Then $\phi_0(u)f$ and $\phi_0(v)f$ are k_1-stable. Now calculate as follows to observe that

$$\begin{aligned}
\langle \phi_0(u)f|\phi_0(v)f\rangle &= \sum_{s \in L_{k_1}} \langle \phi_0(u)f(s)|\phi_0(v)f(s)\rangle \quad \text{(by Remark 3.3.6)} \\
&= \sum_{s \in L_{k_1}} \langle uf(s-b)|vf(s-b)\rangle \\
&= \langle u|v\rangle \sum_{s \in L_{k_1}} \langle f(s-b)|f(s-b)\rangle \\
&= \langle u|v\rangle \sum_{s \in L_{k_1a-b}} \langle f(s)|f(s)\rangle \\
&= \langle u|v\rangle\langle f|f\rangle. \quad \text{(Since } k_1a - b \geq ka \text{ and by Lemma 3.3.9)}
\end{aligned}$$

In the above calculation, to obtain the fourth equality, we have used the fact that the map $L_{k_1} \ni s \to s - b \in L_{k_1a-b}$ is a bijection. This completes the proof. ☐

Let $b \in S$ and $v \in E(b)$ be given. Prop. 3.3.10 implies that for $f \in \mathcal{S}$,

$$\langle \phi_0(v)f|\phi_0(v)f\rangle = ||v||^2\langle f|f\rangle.$$

As a consequence, it follows that there exists a unique bounded linear operator, denoted $\phi(v)$, on $\mathcal{H}$ such that $\phi(v)(f + \mathcal{N}) = \phi_0(v)f + \mathcal{N}$ for every $f \in \mathcal{S}$. Prop. 3.3.10 implies that for $u, v \in E(b)$, $\phi(v)^*\phi(u) = \langle u|v\rangle$. It is clear that $\phi : E \to B(\mathcal{H})$ is multiplicative. Thus ϕ is a representation of E on $\mathcal{H}$. Our goal is to show that ϕ is essential.

Remark 3.3.11. *The Hilbert space $\mathcal{H}$ is infinite dimensional. To see this, observe that we have assumed that there exists $b \in S$ such that $E(b)$ is not 1-dimensional. Let $\{v_i\}_{i=1}^d$ be an orthonormal basis for $E(b)$ where d is the dimension of $E(b)$. Since ϕ is a representation $\{\phi(v_i)\}_{i=1}^d$ is a family of isometries with orthogonal range projections. But $d \geq 2$. This implies that $\mathcal{H}$ is infinite dimensional.*

Let $v \in E(a)$ and $f \in \mathcal{S}$ be given. Define a section $f_v : S \to E$ by the following formula

$$f_v(s) = v^* f(s+a).$$

We leave it to the reader to verify that $f_v \in \mathcal{S}$. We merely indicate that to show $f_v \in \mathcal{V}$, one needs to use Eq. 3.1.2 and to show that f_v is stable one needs to use Lemma 3.1.4. Note that if f is k-stable then f_v is k-stable.

Lemma 3.3.12. *Let $v \in E(a)$ be given. For $f \in \mathcal{S}$, $\phi(v)^*(f+\mathcal{N}) = f_v + \mathcal{N}$.*

Proof. Let $f \in \mathcal{S}$ be given. To show that $\phi(v)^*(f+\mathcal{N}) = f_v + \mathcal{N}$, it suffices to show that for $g \in \mathcal{S}$, $\langle \phi(v)^*(f+\mathcal{N}) | g + \mathcal{N}\rangle = \langle f_v + \mathcal{N} | g + \mathcal{N}\rangle$. Let $g \in \mathcal{S}$ be given. Without loss of generality, we can assume that f and g are k-stable for some $k \geq 0$. Then $\phi_0(v)g$ is $k+1$-stable. Now calculate as follows to observe that

$$\begin{aligned}
\langle \phi(v)^*(f+\mathcal{N}) | g+\mathcal{N}\rangle &= \langle f+\mathcal{N} | \phi(v)(g+\mathcal{N})\rangle \\
&= \sum_{s \in L_{k+1}} \langle f(s) | \phi_0(v) g(s)\rangle \quad \text{(by Remark 3.3.6)} \\
&= \sum_{s \in L_{k+1}} \langle f(s) | v g(s-a)\rangle \\
&= \sum_{s \in L_{k+1}} \langle v^* f(s) | g(s-a)\rangle \\
&= \sum_{s \in L_k} \langle v^* f(s+a) | g(s)\rangle \\
&= \langle f_v + \mathcal{N} | g + \mathcal{N}\rangle. \quad \text{(by Remark 3.3.6)}
\end{aligned}$$

In the above calculation, to obtain the fifth equality, we have used the fact that the map $L_k \ni s \to s+a \in L_{k+1}$ is a bijection. This completes the proof. □

Recall that $\{e_i : i = 1, 2, \cdots, r\}$ are the chosen generators of S and $a = \sum_{i=1}^{r} e_i$.

Theorem 3.3.13. *The representation ϕ is essential.*

Proof. Let $\alpha := \{\alpha_s\}_{s \in S}$ be the E-semigroup over S associated to ϕ. To show that α_s is unital for every s, it suffices to show that α_a is unital. To see this, note that $\alpha_s(1) \leq \alpha_t(1)$ if $s \geq t$. Hence if α_a is unital, it follows that α_{e_i} is unital for every $i = 1, 2, \cdots, r$. But S is generated by $\{e_i : i = 1, 2, \cdots, r\}$. This forces that α_s is unital for every $s \in S$ provided α_a is unital.

Let $\{v_i\}_{i=1}^d$ be an orthonormal basis for $E(a)$ where d denotes the dimension of $E(a)$. We claim that $\sum_{i=1}^{d} \phi(v_i)\phi(v_i)^* = 1$. Here the sum is interpreted in the strong sense if d is infinite. Since $\{\phi(v_i)\phi(v_i)^*\}_{i=1}^d$ forms a mutually orthogonal family of projections, it is enough to show that

$$\sum_{i=1}^{d} \langle \phi(v_i)\phi(v_i)^*(f+\mathcal{N})|f+\mathcal{N}\rangle = ||f+\mathcal{N}||^2$$

for every $f \in \mathcal{S}$.

Let $f \in \mathcal{S}$ be given. Assume that f is k-stable. Then f_{v_i} is k-stable for every i. Now calculate as follows to observe that

$$\begin{aligned}
\sum_{i=1}^{d} ||\phi(v_i)^*(f+\mathcal{N})||^2 &= \sum_{i=1}^{d} ||f_{v_i}+\mathcal{N}||^2 \\
&= \sum_{i=1}^{d} \sum_{s \in L_k} ||f_{v_i}(s)||^2 \\
&= \sum_{i=1}^{d} \sum_{s \in L_k} ||v_i^* f(s+a)||^2 \\
&= \sum_{s \in L_k} \sum_{i=1}^{d} ||v_i^* f(s+a)||^2 \\
&= \sum_{s \in L_k} ||f(s+a)||^2 \quad (\text{ by Lemma 3.1.5}) \\
&= \sum_{s \in L_k} ||f(s)||^2 \quad (\text{since } f \text{ is } k\text{-stable}) \\
&= ||f+\mathcal{N}||^2 \quad (\text{by Remark 3.3.6}).
\end{aligned}$$

In the fourth equality of the above calculation, we have interchanged the order of summation which is permissible since the terms involved are non-negative. This completes the proof. □

Summarizing the above Theorem, we state as the following Theorem.

Theorem 3.3.14. *Let S be a finitely generated sub-semigroup of $\mathbb{Z}^d$. Let $E := \coprod_{t \in S} E(t)$ be a discrete product system over S. Then the discrete product system E admits an essential* *representation. Consequently, there exists an E_0-semigroup over S whose product system is isomorphic to E.*

Remark 3.3.15. *The above Theorem is stated as a Theorem C in the introduction of this thesis. The proof of Theorem C first appeared in [16] (see Theorem 3.13).*

Chapter 4

Existence of E_0-semigroups over closed convex cone in $\mathbb{R}^d$

In Chapter 2, we studied the concrete product system associated with an E_0-semigroup acting on $\mathcal{B}(\mathcal{H})$. We shall define the notion of an abstract product system like in the one parameter case in [2]. We will show that every abstract product system is nothing but concrete product associated with an E_0-semigroup acting on $\mathcal{B}(\mathcal{H})$.

Let P be closed convex cone in $\mathbb{R}^d$ which is spanning i.e. $P - P = \mathbb{R}^d$ and pointed i.e. $P \cap -P = \{0\}$. Denote the interior of P by Ω. Note that Ω is a subsemigroup of $\mathbb{R}^d$ and $P + \Omega \subseteq \Omega$. Observe that $\Omega - \Omega = \mathbb{R}^d$.

The case when $P = [0, \infty)$ was first settled by Arveson using the machinery of the spectral C^* -algebra of a product system. The proof is technical and long. Later Skeide in [26] found a simpler proof. Consequently, Arveson himself found a simpler proof in [8]. We should also mention that Skeide proved in [26] that the E_0 -semigroups arising out of his construction and Arveson's construction are in fact conjugate and not just cocycle conjugate.

4.1 Abstract Product system And Representation

We begin by defining the notion of an abstract product system over Ω. We imitate Arveson's definition in [7] (Page 68, Definition 3.1.1.).

Definition 4.1.1. *By an abstract product system over Ω, we mean a standard Borel space E together with a measurable surjection $p : E \to \Omega$ such that the following holds.*

(1) For $x \in \Omega$, $E(x) := p^{-1}(x)$ is a non-zero separable Hilbert space.

(2) *There exists an associative multiplication* $E \times E \ni (u,v) \to uv \in E$ *such that* $p(uv) = p(u) + p(v)$ *for* $u, v \in E$. *Also the multiplication* $E \times E \ni (u,v) \to uv \in E$ *is measurable.*

(3) *Let* $x, y \in \Omega$ *be given. Then there exists a unitary* $u_{x,y} : E(x) \otimes E(y) \to E(x+y)$ *such that* $u_{x,y}(u \otimes v) = uv$ *for* $(u,v) \in E(x) \times E(y)$.

(4) *Let* $\Delta := \{(u,v) \in E \times E : p(u) = p(v)\}$. *The maps* $\Delta \ni (u,v) \to u + v \in E$ *and* $\Delta \ni (u,v) \to \langle u|v\rangle \in \mathbb{C}$ *are measurable.*

(5) *The map* $\mathbb{C} \times E \ni (\lambda, u) \to \lambda u \in E$ *is measurable.*

(6) *As a measurable field of Hilbert spaces,* E *is trivial by which we mean the following: There exists a separable Hilbert space* $\mathcal{H}_0$ *and a Borel isomorphism* $\theta : E \to \Omega \times \mathcal{H}_0$ *such that* $\pi_1 \circ \theta = p$ *and for every* $x \in \Omega$, *the map* $\pi_2 \circ \theta : E(x) \to \mathcal{H}_0$ *is a unitary. Here by* π_1 *and* π_2, *we mean the first and second projections from* $\Omega \times \mathcal{H}_0$ *onto* Ω *and* $\mathcal{H}_0$ *respectively. We shall abbreviate the above by saying that as a measurable field of Hilbert spaces,* E *is isomorphic to* $\Omega \times \mathcal{H}_0$ *and write* $E \simeq \Omega \times \mathcal{H}_0$.

We usually suppress the surjection p and simply write E as $E = \coprod_{x \in \Omega} E(x)$.

Remark 4.1.2. *Let* $\mathcal{H}$ *be an infinite dimensional separable Hilbert space and* $\alpha := \{\alpha_x\}_{x \in P}$ *be an* E_0*-semigroup over* P. *Then the product system associated to* α *is an abstract product system.*

Definition 4.1.3. *Let* $E := \coprod_{x \in \Omega} E(x)$ *and* $F := \coprod_{x \in \Omega} F(x)$ *be abstract product systems over* Ω. *We say that* E *is isomorphic to* F *if for every* $x \in \Omega$, *there exists a unitary map* $\theta_x : E(x) \to F(x)$ *such that*

(1) *for* $x, y \in \Omega$ *and* $(u,v) \in E(x) \times E(y)$, $\theta_{x+y}(uv) = \theta_x(u)\theta_y(v)$, *and*

(2) *the map* $\theta := \coprod_{x \in \Omega} \theta_x : E \to F$ *is a Borel isomorphism.*

Since E *and* F *are standard Borel spaces, it is enough to require that the map* θ *in* (2) *is* 1-1, *onto and measurable (See Page 70, Theorem 3.3.2. of [2]).*

Let us give a few more examples of closed convex cone before we proceed further.

(1) Let $\mathbb{R}^d_+ := \{(x_1, x_2, \cdots, x_d) : \text{ for every } i,\ x_i \geq 0\}$. Clearly $\mathbb{R}^d_+$ is a closed convex cone which is both spanning and pointed.

(2) Let V be a finite dimensional real inner product space. Denote the space of symmetric linear operators on V by $\mathcal{S}(V)$ and the cone of positive operators on V by $\mathcal{P}(V)$. Then it is clear that $\mathcal{P}(V)$ is a closed convex cone in $\mathcal{S}(V)$ which is both spanning and pointed.

(3) Let $d \geq 3$ and $Q := \{(x_1, x_2, \cdots, x_d) : x_d \geq 0, x_d^2 \geq \sum_{k=1}^{d-1} x_k^2\}$. Then Q is a closed convex cone in $\mathbb{R}^d$ which is both pointed and spanning. The cone Q is often called the forward light cone in $\mathbb{R}^d$.

Remark 4.1.4. *The cones described in* (1), (2) *and* (3) *are examples of what are called symmetric cones. There is a well known bijective correspondence between symmetric cones and Euclidean Jordan algebras from which it follows that the cones described in* (1), (2) *and* (3) *are mutually non-isomorphic. We will not enter into a long digression trying to distinguish the cones in* (1), (2) *and* (3) *by elementary means as it is not relevant to the rest of this article. To know more about the exact relationship between Jordan algebras and symmetric cones, we refer the reader to the first five chapters of the monograph [10]. Of course, there are other possibilities like polyhedral cones whose classification we are not aware of if one such exists. We satisfy ourselves with the reasoning that demonstrates that the cone of positive operators on a finite dimensional inner product space is not isomorphic to* $\mathbb{R}_+^k$ *for any* k *when the dimension of the underlying vector space is atleast 2.*

Let $P \subset \mathbb{R}^d$ be a closed convex cone and $F \subset P$ be a closed subsemigroup of P (i.e. $x+y \in F$ if $x, y \in F$) containing the origin 0. We say that F is a face of P if F is closed and if $y + z \in F$ with $y, z \in P$ then $y, z \in F$. It is routine to see that a face of P is always a closed convex cone. Let $F \subset P$ be a face. Then F is said to be one dimensional if the linear span of F i.e. $< F >= F - F$ is one dimensional. Note the following.

(a) Let V be a finite dimensional real inner product space of dimension atleast 2. Denote the linear space of symmetric linear operators on V by $\mathcal{S}(V)$ and the cone of positive operators on V by $\mathcal{P}(V)$. Denote the set of one dimensional faces of $\mathcal{P}(V)$ by $\mathcal{F}$ and the set of rank one projections on V by $\mathcal{Q}$. It follows immediately from the spectral theorem that the map $\mathcal{Q} \ni Q \to \{\lambda Q : \lambda \geq 0\} \in \mathcal{F}$ is a bijection. This implies in particular that the set of one dimensional faces is uncountable.

(b) On the other hand, the set of one dimensional faces of $\mathbb{R}_+^k$ has cardinality k.

Hence, it follows that the cone of positive operators on a finite dimensional inner product space is not isomorphic to $\mathbb{R}_+^k$ for any $k \geq 1$ when the dimension of the underlying vector space is atleast 2.

Let us make a few preliminary observations regarding the dimension of the fibres of an abstract product system. We will also drop the adjective "abstract" and simply call an abstract product system over Ω a product system over Ω.

Lemma 4.1.5. *Let $E := \coprod_{x\in\Omega} E(x)$ be a product system over Ω. For $x \in \Omega$, let $d(x)$ be the dimension of $E(x)$. Then either $d(x) = 1$ for all $x \in \Omega$ or $d(x) = \infty$ for all $x \in \Omega$.*

Proof. Note that $d(x) \neq 0$ for every $x \in \Omega$. Condition (6) of Definition 4.1.1 implies that $d(x) = d(y)$ for every $x, y \in \Omega$. Thus it is enough to show that $d(a) = 1$ or $d(a) = \infty$ for some $a \in \Omega$. Let $a \in \Omega$ be given. Then $E(a) \otimes E(a) \cong E(2a)$. Hence $d(a) = d(2a) = d(a)^2$. This implies $d(a) = 1$ or $d(a) = \infty$. This completes the proof. □

Proposition 4.1.6. *Let $E := \coprod_{x\in\Omega} E(x)$ be a product system over Ω such that the fibre $E(x)$ is one-dimensional for every $x \in \Omega$. Then there exists an E_0-semigroup $\alpha := \{\alpha_x\}_{x\in P}$ on $B(L^2(\mathbb{R}^d))$ such that for $x \in P$, α_x is an automorphism and the product system associated to α is isomorphic to E.*

Proof. Since $E \simeq \Omega \times \mathbb{C}$, it follows that there exists a measurable section $e : \Omega \to E$ such that for $x \in \Omega$, $||e(x)|| = 1$ and $E(x)$ is spanned by $e(x)$. Let $x, y \in \Omega$ be given. Then there exists a unique scalar denoted $\omega(x, y) \in \mathbb{T}$ such that $e(x)e(y) = \omega(x, y)e(x + y)$. Observe that for $x, y \in \Omega$, $\omega(x, y) = \langle e(x)e(y)|e(x + y)\rangle$. This implies that the function $\Omega \times \Omega \ni (x, y) \to \omega(x, y) \in \mathbb{T}$ is measurable. The associativity of the multiplication of the product system implies that ω is a multiplier on Ω i.e. for $x, y, z \in \Omega$,

$$\omega(x, y)\omega(x + y, z) = \omega(x, y + z)\omega(y, z).$$

By Theorem 3.3. of [15], it follows that ω extends to a multiplier on $\mathbb{R}^d$. We denote the extension again by ω. For $x \in \mathbb{R}^d$, let U_x be the unitary on $L^2(\mathbb{R}^d)$ defined by the following formula

$$U_x f(y) = \omega(x, y - x)f(y - x)$$

for $f \in L^2(\mathbb{R}^d)$. Note that $U_xU_y = \omega(x, y)U_{x+y}$ for $x, y \in \mathbb{R}^d$. Also observe that for $f, g \in L^2(\mathbb{R}^d)$, the map $\mathbb{R}^d \ni x \to \langle U_x f|g\rangle$ is measurable. For a proof of this fact, we refer the reader to the paragraph preceding Theorem 3.3. of [15].

For $x \in P$, let α_x be the automorphism of $B(L^2(\mathbb{R}^d))$ defined by the formula

$$\alpha_x(A) = U_x A U_x^*.$$

It is clear that $\alpha := \{\alpha_x\}_{x \in P}$ is a semigroup of unital normal $*$-endomorphisms of $B(L^2(\mathbb{R}^d))$. The weak measurability of $\{U_x\}_{x \in P}$ implies that for every $A \in B(L^2(\mathbb{R}^d))$ and $f, g \in L^2(\mathbb{R}^d)$, the map $P \ni x \to \langle \alpha_x(A) f | g \rangle$ is measurable. Now 2.3.4 implies that α is an E_0-semigroup.

Let $F := \coprod_{x \in \Omega} F(x)$ be the product system associated to the E_0-semigroup α. Then it is clear that for every $x \in \Omega$, $F(x)$ is spanned by U_x. For $x \in \Omega$, let $\theta_x : E(x) \to F(x)$ be the unitary such that $\theta_x(e(x)) = U_x$. Then the map $\theta := \coprod_{x \in \Omega} \theta_x : E \to F$ is 1-1, onto and preserves the multiplication. To see that θ is measurable, let $\mu : \Omega \times \mathbb{C} \to E$ be defined by $\mu(x, \lambda) = \lambda e(x)$ and let $\nu : \Omega \times \mathbb{C} \to F$ be defined by $\nu(x, \lambda) = (x, \lambda U_x)$. Then μ and ν are measurable. Moreover, μ is 1-1 and onto. Since the spaces involved are standard, it follows that μ^{-1} is measurable. Note that $\theta = \nu \circ \mu^{-1}$. Hence θ is measurable. This completes the proof. □

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. We endow $B(\mathcal{H})$ with the measurable structure induced by the weak operator topology.

Definition 4.1.7. *Let $E := \coprod_{x \in \Omega} E(x)$ be a product system over Ω. By a representation of E on $\mathcal{H}$, we mean a map $\phi : E \to B(\mathcal{H})$ such that*

(1) the map ϕ is measurable,

(2) for $u, v \in E$, $\phi(uv) = \phi(u)\phi(v)$, and

(3) for $x \in \Omega$ and $u, v \in E(x)$, $\phi(v)^*\phi(u) = \langle u | v \rangle$.

The representation ϕ is called essential if $\overline{\phi(E(x))\mathcal{H}} = \mathcal{H}$ for every $x \in \Omega$.

Let $\phi : E \to B(\mathcal{H})$ be a representation. Then ϕ restricted to each fibre is linear. The proof is exactly the same as in the 1-dimensional case and hence we omit the proof. For the proof in the 1-dimensional case, we refer the reader to Page 71 of [7]. Moreover Condition (3) implies that ϕ restricted to each fibre is isometric.

Fix $x \in \Omega$. Note that $\phi(E(x))$ is a separable, norm closed subspace of $B(\mathcal{H})$ such that T^*S is a scalar for every $S, T \in \phi(E(x))$. Hence by Theorem 1.1.8, it follows that there exists a unique normal $*$-endomorphism α_x of $B(\mathcal{H})$ such that

$$\phi(E(x)) = \{T \in B(\mathcal{H}) : \alpha_x(A)T = TA \text{ for } A \in B(\mathcal{H})\}.$$

Recall from Prop. 1.1.7 that $\alpha_x(1)$ is the projection onto the closed subspace $\overline{\phi(E(x))\mathcal{H}}$.

Let $x, y \in \Omega$ be given. Since $E(x+y)$ is the closure of the linear span of the set $\{uv : u \in E(x), v \in E(y)\}$, it follows from (2) of Definition 4.1.7 that $\phi(E(x+y))$ is the closed linear span of $\{ST : S \in \phi(E(x)), T \in \phi(E(y))\}$. Hence by Prop.1.1.9, it follows that $\alpha_{x+y} = \alpha_x \circ \alpha_y$. Consequently $\{\alpha_x\}_{x\in\Omega}$ is a semigroup of normal $*$-endomorphisms of $B(\mathcal{H})$. Now suppose that ϕ is essential. Then by Prop 1.1.8, it follows that α_x is unital for every $x \in \Omega$.

Since E is a measurable field of Hilbert space, it follows that there exists measurable sections $e_1, e_2, \cdots$ such that for every $x \in \Omega$, $\{e_i(x)\}_{i=1}^{\infty}$ forms an orthonormal basis for $E(x)$. Consequently, for $x \in \Omega$, $\{\phi(e_i(x))\}$ is an orthonormal basis for $\phi(E(x))$. Hence by Theorem 1.1.8, for $x \in \Omega$, α_x is given by the equation

$$\alpha_x(A) = \sum_{n=1}^{\infty} \phi(e_n(x))A\phi(e_n(x))^* \tag{4.1.1}$$

where the sum in Equation 4.1.1 is a strongly convergent sum. The measurability of ϕ and Equation 4.1.1 implies that for $A \in B(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$, the map $\Omega \ni x \to \langle \alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$ is measurable. By Proposition 2.3.2, it follows that for $A \in B(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$, the map $\Omega \ni x \to \langle \alpha_x(A)\xi|\eta\rangle \in \mathbb{C}$ is continuous. Again by Proposition 2.3.3, it follows that $\{\alpha_x\}_{x\in\Omega}$ extends to a unique E_0-semigroup which we still denote by $\alpha := \{\alpha_x\}_{x\in P}$. The constructed E_0-semigroup α is called the E_0-semigroup associated to the essential representation ϕ.

Proposition 4.1.8. *Let $\phi : E \to B(\mathcal{H})$ be an essential representation and let $\alpha := \{\alpha_x\}_{x\in P}$ be the E_0-semigroup associated to ϕ. Then E is isomorphic to the product system associated to α.*

Proof. Let $F := \coprod_{x\in\Omega} F(x)$ be the product system associated to α. For $x \in \Omega$, by the definition of α_x, $F(x) = \phi(E(x))$. Now the map $E \ni u \to (p(u), \phi(u)) \in F$ is an isomorphism of product systems. Here $p : E \to \Omega$ is the canonical surjection that comes equipped with the product system E. This completes the proof. □

4.2 Construction of an essential representation

Assume that P is proper subset of $\mathbb{R}^d$. We fix an element $a \in \Omega$ for the rest of this section. For $x, y \in \mathbb{R}^d$, we write $y <_\Omega x$ if $x - y \in \Omega$. We have the following archimedean principle.

Lemma 4.2.1. *Let $x \in \mathbb{R}^d$. Then there exists a positive integer n_0 such that $x <_\Omega n_0 a$.*

Proof: Note that the sequence $a - \frac{x}{n} \to a \in \Omega$. But Ω is an open subset of $\mathbb{R}^d$. Hence there exists a positive integer n_0 such that $n \geq n_0$ implies $a - \frac{x}{n} \in \Omega$. This implies that $na - x \in \Omega$ for $n \geq n_0$. In particular, $n_0 a - x \in \Omega$. This completes the proof.

Lemma 4.2.2. *The interior Ω of P does not contains $-a$.*

Proof: Let $x \in \mathbb{R}^d$ be given. Then there exists positive integer n such that $x + na \in \Omega$. If $-a \in \Omega$, then $x \in \Omega$ which is a contradiction to $P \neq \mathbb{R}^d$.

Lemma 4.2.3. *The intersection $\displaystyle\bigcap_{n=0}^{\infty}(\Omega + na) = \emptyset$. Also $\{\Omega + na\}_{n=0}^{\infty}$ is a decreasing sequence of subsets of Ω.*

Proof. Suppose $y \in \displaystyle\bigcap_{n=0}^{\infty}(\Omega + na)$. Then $y - na \in \Omega$ for every $n \geq 0$. By Lemma 4.2.1, it follows that there exists a positive integer n_0 such that $n_0 a - y \in \Omega$. Now observe that $-a = (y - (n_0 + 1)a) + (n_0 a - y) \in \Omega$ which is a contradiction since $-a \in \Omega$. It is clear that $\{\Omega + na\}_{n=0}^{\infty}$ is a decreasing sequence of subsets of Ω. □

Let us fix a few notations which will be used throughout this chapter. For $k \in \mathbb{N}$, let $L_k := (\Omega + ka) \backslash (\Omega + (k+1)a)$. Then Lemma 4.2.3 implies that $\{L_k : k \in \mathbb{N}\}$ is a disjoint family of measurable subsets of Ω. Note that for $k \in \mathbb{N}$, $\Omega + ka = \displaystyle\coprod_{m \geq k} L_m$. Since $\Omega = \displaystyle\coprod_{k \in \mathbb{N}} L_k$, it follows that given $x \in \Omega$, there exists a unique non-negative integer $n(x)$ such that $x \in L_{n(x)}$. Since $n(x) = k$ for $x \in L_k$, it is clear that the map $\Omega \ni x \to n(x) \in \mathbb{N}$ is measurable. Note that $n(x + a) = n(x) + 1$ for $x \in \Omega$. Also observe that for $x \in \Omega$, $x - n(x)a \in L_0$ and for $x \in L_0$ and $k \in \mathbb{N}$, $x + ka \in L_k$.

We need the fact that L_k has non-zero Lebesgue measure for every $k \in \mathbb{N}$. In what follows, we will not use any other measure except the Lebesgue measure on $\mathbb{R}^d$. Also we denote the Lebesgue measure on $\mathbb{R}^d$ by λ. Since $L_k = L_0 + ka$, it suffices to show that $L_0 = \Omega \backslash (\Omega + a)$ has positive measure. Observe that $\Omega \backslash (\Omega + a)$ contains the open set

$\Omega\backslash(P+a)$ which is non-empty since $\frac{a}{2} \in \Omega\backslash(P+a)$. Now it follows immediately that $\Omega\backslash(\Omega+a)$ has positive measure. For $z \in \mathbb{R}^d$, let $L_z := (L_0+z)\cap\Omega$.

Let $p : E \to \Omega$ be a product system which is fixed for the rest of this section. We assume that the fibres of E are infinite dimensional. Let $e \in E(a)$ be a unit vector which is fixed for the rest of this section. Our goal is to exhibit an essential representation of E on an infinite dimensional separable Hilbert space.

Let $\mathcal{V}$ denote the vector space of measurable sections of E which are square integrable over L_z for every $z \in \mathbb{R}^d$. More precisely, let $f : \Omega \to E$ be a measurable section. Then $f \in \mathcal{V}$ if and only if for every $z \in \mathbb{R}^d$, $\int_{L_z} ||f(x)||^2 dx < \infty$.

Let $f \in \mathcal{V}$ and $k \in \mathbb{N}$ be given. We say that f is k-stable if $f(x+a) = f(x)e$ for almost all $ka <_\Omega x$ i.e. the measurable set $\{x \in \Omega + ka : f(x+a) \neq f(x)e\}$ has measure zero. Note that if f is k-stable and $k_1 \geq k$ then f is k_1-stable. We say a section in $\mathcal{V}$ is stable if it is k-stable for some $k \in \mathbb{N}$. Denote the set of stable sections in $\mathcal{V}$ by $\mathcal{S}$. Then it is clear that $\mathcal{S}$ is a vector subspace of $\mathcal{V}$.

Let $f \in \mathcal{V}$. We say that f is null if there exists $k \in \mathbb{N}$ such that $f(x) = 0$ for almost all $ka <_\Omega x$ i.e. there exists $k \in \mathbb{N}$ such that $\{ka <_\Omega x : f(x) \neq 0\}$ has measure zero. Denote the set of null sections in $\mathcal{V}$ by $\mathcal{N}$. Then it is clear that $\mathcal{N}$ is a vector subspace of $\mathcal{V}$. We leave it to the reader to verify that $\mathcal{N} \subset \mathcal{S}$.

Lemma 4.2.4. *Let $f \in \mathcal{S}$ be given. Assume that f is k-stable for some $k \in \mathbb{N}$. Then for every $m \geq 1$, $f(x+ma) = f(x)e^m$ for almost all $ka <_\Omega x$.*

Proof. We prove this by induction on m. Let $A_m := \{ka <_\Omega x : f(x+ma) \neq f(x)e^m\}$. The fact that f is k-stable implies that A_1 has measure zero. Now assume that A_m has measure zero. Let $ka <_\Omega x$ be given. Suppose $x \notin A_m$ and $x + ma \notin A_1$. Then calculate as follows to observe that

$$\begin{aligned}
f(x+(m+1)a) &= f(x+ma+a) \\
&= f(x+ma)e \quad (\text{Since } x+ma \notin A_1) \\
&= f(x)e^m e \quad (\text{Since } x \notin A_m) \\
&= f(x)e^{m+1}.
\end{aligned}$$

This implies that $A_{m+1} \subset A_m \cup ((\Omega+ka)\cap(A_1 - ma)) \subset A_m \cup (A_1 - ma)$. Since A_m and A_1 have measure zero, it follows that A_{m+1} has measure zero. This completes the proof. □

Let $f, g \in \mathcal{S}$ be given. Since f and g are square integrable over L_z for every $z \in \mathbb{R}^d$, it follows that the integral $\int_{L_k} \langle f(x)|g(x)\rangle dx$ exists for every $k \in \mathbb{N}$.

Lemma 4.2.5. *Let $f, g \in \mathcal{S}$ be given. Assume that f and g are k_0-stable for some $k_0 \in \mathbb{N}$. Then for $k \geq k_0$,*

$$\int_{L_k} \langle f(x)|g(x)\rangle dx\rangle = \int_{L_{k_0}} \langle f(x)|g(x)\rangle dx.$$

Proof. Let $k > k_0$ be given. Note that the map $L_{k_0} \ni x \to x + (k - k_0)a \in L_k$ is a measurable bijection which is measure preserving. Calculate as follows to observe that

$$\begin{aligned}
\int_{L_k} \langle f(x)|g(x)\rangle dx &= \int_{L_{k_0}} \langle f(x + (k - k_0)a)|g(x + (k - k_0)a)\rangle dx \\
&= \int_{L_{k_0}} \langle f(x)e^{k-k_0}|g(x)e^{k-k_0}\rangle dx \quad \text{(by Lemma 4.2.4)} \\
&= \int_{L_{k_0}} \langle f(x)|g(x)\rangle.
\end{aligned}$$

This completes the proof. □

Let $f, g \in \mathcal{S}$ be given. Thanks to Lemma 4.2.5, $\lim_{k\to\infty} \int_{L_k} \langle f(x)|g(x)\rangle dx$ exists. Define

$$\langle f|g\rangle = \lim_{k\to\infty} \int_{L_k} \langle f(x)|g(x)\rangle dx.$$

Observe that $\langle\ |\ \rangle$ defines a semi-definite inner product on $\mathcal{S}$. Let $f \in \mathcal{S}$. We claim that $\langle f|f\rangle = 0$ if and only if $f \in \mathcal{N}$. It is clear that if $f \in \mathcal{N}$, then $\langle f|f\rangle = 0$. Now let $f \in \mathcal{S}$ be such that $\langle f|f\rangle = 0$. Assume that f is k_0-stable for some $k_0 \in \mathbb{N}$. Lemma 4.2.5 implies that $\int_{L_k} ||f(x)||^2 dx = 0$ for every $k \geq k_0$. Consequently for every $k \geq k_0$, $f(x) = 0$ for almost all $x \in L_k$. Since $\Omega + k_0 a = \coprod_{k \geq k_0} L_k$, it follows that $f(x) = 0$ for almost all $k_0 a <_\Omega x$. This proves that $f \in \mathcal{N}$.

Thus $\langle\ |\ \rangle$ descends to a positive definite inner product on $\mathcal{S}/\mathcal{N}$ which we still denote by $\langle\ |\ \rangle$. Let $\mathcal{H}$ be the completion of the pre-Hilbert space $\mathcal{S}/\mathcal{N}$. Next we show that the Hilbert space $\mathcal{H}$ is an infinite dimensional separable Hilbert space.

Proposition 4.2.6. *The Hilbert space $\mathcal{H}$ is separable and is infinite dimensional.*

Proof. Let $k \in \mathbb{N}$ be fixed. Consider a measurable section $\xi : L_k \to E$ which is square integrable. Define a section $\widetilde{\xi} : \Omega \to E$ by the following formula

$$\widetilde{\xi}(x) := \begin{cases} \xi(x - n(x)a + ka)e^{n(x)-k} & \text{if } ka <_\Omega x, \\ \\ 0 & \text{elsewhere.} \end{cases}$$

Note that for $x \in L_k$, $\widetilde{\xi}(x) = \xi(x)$. For $m > k$ and $x \in L_m$, $\widetilde{\xi}(x) = \xi(x - ma + ka)e^{m-k}$. This implies that $\widetilde{\xi}$ is measurable on each L_m for $m \geq k$. Hence $\widetilde{\xi}$ is measurable on $\coprod_{m \geq k} L_m = \Omega + ka$. It is clear that $\widetilde{\xi}$ is measurable on the complement of $\Omega + ka$. Consequently it follows that $\widetilde{\xi}$ is a measurable section.

We claim that $\widetilde{\xi} \in \mathcal{V}$. Let $z \in \mathbb{R}^d$ be given. Set $A := L_z \cap (\Omega + ka)$. Since $\widetilde{\xi}$ vanishes on $L_z \backslash A$, it follows that $\int_{L_z} ||\widetilde{\xi}(x)||^2 dx = \int_A ||\widetilde{\xi}(x)||^2 dx$. If A is empty, then the last equality implies that $\int_{L_z} ||\widetilde{\xi}(x)||^2 dx = 0 < \infty$ and hence $\widetilde{\xi} \in \mathcal{V}$. Suppose that A is non-empty. Let $\chi : A \to L_k$ be the map defined by

$$\chi(x) = x - n(x)a + ka.$$

The measurability of the map $\Omega \ni x \to n(x) \in \mathbb{N}$ implies that χ is measurable. We claim that χ is 1-1. Let $x_1, x_2 \in A$ be such that $\chi(x_1) = \chi(x_2)$. To prove $x_1 = x_2$, it suffices to show that $n(x_1) = n(x_2)$. Suppose not. Without loss of generality, we can assume that $n(x_1) < n(x_2)$. Note that $x_1 - z \in \Omega$ and $n(x_2) - n(x_1)$ is a natural number greater than or equal to 1. Since $\{\Omega + na : n \geq 1\}$ is a decreasing sequence of subsets, it follows that $x_2 - z = (x_1 - z) + (n(x_2) - n(x_1))a \in \Omega + a$ which is a contradiction to the fact that $x_2 - z \in L_0 = \Omega \backslash (\Omega + a)$. This contradiction implies that $n(x_1) = n(x_2)$ and consequently $x_1 = x_2$. This proves that χ is 1-1. Since A and L_k are $\mathbb{R}^d_\delta$ subsets of $\mathbb{R}^d$, it follows that A and L_k are Polish spaces. Let B be the image of χ. Then by Theorem 3.3.2. of [2], it follows that B is a Borel subset of L_k and $\chi : A \to B$ is a Borel isomorphism.

We claim that χ is measure preserving. Let $C \subset A$ be a Borel subset. For $m \geq k$, let $C_m := \{x \in C : n(x) = m\}$. Then $C = \coprod_{m \geq k} C_m$. As a consequence, we have

$\chi(C) = \coprod_{m \geq k} \chi(C_m) = \coprod_{m \geq k} (C_m - ma + ka)$. Now calculate as follows to observe that

$$\begin{aligned}\lambda(\chi(C)) &= \sum_{m \geq k} \lambda(C_m - ma + ka) \\ &= \sum_{m \geq k} \lambda(C_m) \\ &= \lambda(\coprod_{m \geq k} C_m) \\ &= \lambda(C).\end{aligned}$$

This shows that χ is measure preserving. Calculate as follows to observe that

$$\begin{aligned}\int_{L_z} ||\widetilde{\xi}(x)||^2 dx &= \int_A ||\widetilde{\xi}(x)||^2 dx \\ &= \int_A ||\xi(x - n(x)a + ka)||^2 dx \\ &= \int_A ||\xi(\chi(x))||^2 dx \\ &= \int_B ||\xi(x)||^2 dx \\ &\leq \int_{L_k} ||\xi(x)||^2 dx \\ &< \infty.\end{aligned}$$

This shows that $\widetilde{\xi} \in \mathcal{V}$. Next we claim $\widetilde{\xi}$ is k-stable. Let $ka <_\Omega x$ be given. Calculate as follows to observe that

$$\begin{aligned}\widetilde{\xi}(x + a) &= \xi(x + a - n(x + a)a + ka)e^{n(x+a)-k} \\ &= \xi(x + a - (n(x) + 1)a + ka)e^{n(x)+1-k} \quad (\text{ Since } n(x + a) = n(x) + 1) \\ &= \xi(x - n(x)a + ka)e^{n(x)-k}e \\ &= \widetilde{\xi}(x)e.\end{aligned}$$

This proves that $\widetilde{\xi}$ is k-stable. Note that $\widetilde{\xi}(x) = \xi(x)$ for $x \in L_k$. Let $\mathcal{H}_k = L^2(L_k, E)$. By Lemma 4.2.5, it follows that the map $\mathcal{H}_k \ni \xi \to \widetilde{\xi} + \mathcal{N}$ is well-defined and is an isometry which we denote by V_k.

Let $f \in \mathcal{V}$ be given. Assume that f is k-stable for some $k \in \mathbb{N}$. Let $\xi : L_k \to E$ be defined by $\xi(x) = f(x)$ for $x \in L_k$. For $m \geq 1$, let $A_m := \{x > ka : f(x+ma) \neq f(x)e^m\}$. By Lemma 4.2.4, it follows that A_m has measure zero for every $m \geq 1$. Let $m > k$ and $x \in L_m$ be given. Suppose $x \notin A_{m-k} + (m-k)a$. Then calculate as follows to observe that

$$\begin{aligned} f(x) &= f(x - ma + ka + (m-k)a) \\ &= f(x - ma + ka)e^{m-k} \quad (\text{Since } x - ma + ka \notin A_{m-k} \text{ and } ka <_\Omega x - ma + ka.) \\ &= \widetilde{\xi}(x) \end{aligned}$$

Therefore $\{x \in L_m : f(x) \neq \widetilde{\xi}(x)\} \subset A_{m-k} + (m-k)a$. Since $A_{m-k} + (m-k)a$ has measure zero, it follows that $\{x \in L_m : f(x) \neq \widetilde{\xi}(x)\}$ has measure zero. Thus for every $m > k$, $f(x) = \widetilde{\xi}(x)$ for almost all $x \in L_m$. By definition, f agrees with $\widetilde{\xi}$ on L_k. Hence $f(x) = \widetilde{\xi}(x)$ for almost all $x \in \coprod_{m \geq k} L_m = \Omega + ka$. Consequently $f - \widetilde{\xi} \in \mathcal{N}$. This proves that $f + \mathcal{N} = \widetilde{\xi} + \mathcal{N}$. Thus we have shown that $\{f + \mathcal{N} : f \in \mathcal{S}\} = \bigcup_{k=0}^{\infty} V_k\mathcal{H}_k$. Since each $\mathcal{H}_k$ is separable, it follows that $\mathcal{H}$ is separable. As each $\mathcal{H}_k$ is infinite dimensional, it follows that $\mathcal{H}$ is infinite dimensional. This completes the proof. $\square$

We need the following two important lemmas before defining a representation of E on $\mathcal{H}$. Fix $k \in \mathbb{N}$. Let $b \in \Omega$ be such that $ka <_\Omega b$. Recall that $L_k = (\Omega + ka)\backslash(\Omega + (k+1)a)$ and $L_b = (\Omega + b)\backslash(\Omega + b + a)$. Let $x \in L_k$ be given. By Lemma 4.2.1, there exists $m_0 \in \mathbb{N}$ such that $m_0 a - (b - x) = x + m_0 a - b \in \Omega$ or in other words $x + m_0 a \in \Omega + b$. Let $m(x)$ be the least non-negative integer such that $x + m(x)a \in \Omega + b$.

Lemma 4.2.7. *With the foregoing notations, we have the following.*

(1) For every $x \in L_k$, $x + m(x)a \in L_b$.

(2) For every $x \in L_k$, the intersection $\{x + ma : m \in \mathbb{N}\} \cap L_b$ is singleton.

(3) The map $\chi : L_k \to L_b$ defined by $\chi(x) = x + m(x)a$ is a measurable bijection.

(4) The map $\chi : L_k \to L_b$ is measure preserving.

Proof. Let $x \in L_k$ be given. Suppose $m(x) = 0$. Then since $x \not\geq_\Omega (k+1)a$ and $ka <_\Omega b$, it follows that $x \not\geq_\Omega b + a$. Hence $x = x + m(x)a \in L_b$. Suppose $m(x) \geq 1$. Then by definition $x + (m(x) - 1)a \notin \Omega + b$ or in other words, $x + m(x)a \notin \Omega + b + a$. In this case too, $x + m(x)a \in L_b$. This proves (1).

Let $x \in L_k$ be given. Suppose $x + ma \in L_b$ for some $m \in \mathbb{N}$. Then $x + ma \in \Omega + b$. Hence by the definition of $m(x)$, it follows that $m \geq m(x)$. To prove (2), it suffices to show that $m = m(x)$. Suppose not. Then $m > m(x)$. Write $m = m(x) + n$ with $n \geq 1$. Now $x + ma = (x + m(x)a) + na \in (\Omega + b) + na \subset \Omega + b + a$ which is a contradiction since $x + ma \notin \Omega + b + a$. This contradiction proves that $m = m(x)$. This proves (2).

Let $x_1, x_2 \in L_k$ be such that $\chi(x_1) = \chi(x_2)$. Then $x_1 + m(x_1)a = x_2 + m(x_2)a$. To show $x_1 = x_2$, it suffices to show that $m(x_1) = m(x_2)$. Suppose not. Without loss of generality, we can assume that $m(x_1) < m(x_2)$. Then $x_1 = x_2 + (m(x_2) - m(x_1))a \in \Omega + ka + (m(x_2) - m(x_1))a \subset \Omega + (k+1)a$ which is a contradiction since $x_1 \not>_\Omega (k+1)a$. This contradiction implies that $m(x_1) = m(x_2)$ and consequently $x_1 = x_2$. This proves that χ is 1-1.

Let $y \in L_b$ be given. Then $ka <_\Omega b <_\Omega y$. Hence the set $\{n \in \mathbb{N} : y - na \in \Omega + ka\}$ is non-empty, for it contains 0. We claim that the set $\{n \in \mathbb{N} : y - na \in \Omega + ka\}$ is bounded. Suppose not. Then there exists a sequence (n_ℓ) such that $n_\ell \to \infty$ and $y - n_\ell a \in \Omega + ka \subset P$. Hence $y \in \bigcap_{n=0}^{\infty}(\Omega + na)$ which is contradiction to 4.2.3. Let n_0 be the largest non-negative integer such that $y - n_0 a \in \Omega + ka$. Then $y - (n_0+1)a \notin \Omega + ka$ or in other words $y - n_0 a \notin \Omega + (k+1)a$. Let $x := y - n_0 a$. Then $x \in L_k$ and $y = x + n_0 a \in L_b$. Since the intersection $\{x + ma : m \in \mathbb{N}\} \cap L_b$ is singleton, it follows that $y = \chi(x)$. This proves that χ is onto.

To show χ is measurable, it is enough to show that $L_k \ni x \to m(x) \in \mathbb{N}$ is measurable. Let $r \in \mathbb{R}$ be given. We claim that $\{x \in L_k : m(x) \geq r\}$ is a closed subset of L_k. Let (x_n) be a sequence in L_k such that $m(x_n) \geq r$ and $x_n \to x \in L_k$. Then the sequence $x_n + m(x)a \to x + m(x)a \in \Omega + b$. But $\Omega + b$ is an open subset of $\mathbb{R}^d$ containing $x + m(x)a$. Hence $x_n + m(x)a \in \Omega + b$ eventually. By the definition of the function m, it follows that $m(x_n) \leq m(x)$ eventually. Thus $m(x) \geq r$. This proves that $\{x \in L_k : m(x) \geq r\}$ is a closed subset of L_k. As a consequence, we obtain that the function m is measurable and consequently χ is measurable. This proves (3).

Since L_k and L_b are $\mathbb{R}^d_\delta$-subsets of $\mathbb{R}^d$, it follows that L_k and L_b are Polish spaces. Hence by Theorem 3.3.2. of [2], it follows that χ is a Borel isomorphism. Let $A \subset L_k$ be a measurable subset. For $n \in \mathbb{N}$, let $A_n := \{x \in A : m(x) = n\}$. Then $A = \coprod_{n=0}^{\infty} A_n$ and

$\chi(A) = \coprod_{n=0}^{\infty} \chi(A_n) = \coprod_{n=0}^{\infty}(A_n + na)$. Now calculate as follows to observe that

$$\begin{aligned}\lambda(\chi(A)) &= \sum_{n=0}^{\infty} \lambda(A_n + na) \\ &= \sum_{n=0}^{\infty} \lambda(A_n) \\ &= \lambda(\coprod_{n=0}^{\infty} A_n) \\ &= \lambda(A).\end{aligned}$$

This proves (4). This completes the proof. □

Lemma 4.2.8. *Let $f, g \in \mathcal{S}$ be given. Assume that f and g are k-stable for some $k \in \mathbb{N}$. Let $b \in \Omega$ be such that $ka <_\Omega b$. Then*

$$\langle f|g\rangle = \int_{L_b} \langle f(x)|g(x)\rangle dx.$$

Proof. Let $L_k \ni x \to m(x) \in \mathbb{N}$ and $\chi : L_k \to L_b$ be the functions considered in Lemma 4.2.7. For $n \in \mathbb{N}$, let $A_n := \{x \in L_k : m(x) = n\}$. Then $L_k = \coprod_{n=0}^{\infty} A_n$. Now calculate as follows to observe that

$$\begin{aligned}\int_{L_b} \langle f(x)|g(x)\rangle dx &= \int_{L_k} \langle f(\chi(x))|g(\chi(x))\rangle dx \quad \text{(Since } \chi \text{ is measure preserving)} \\ &= \sum_{n=0}^{\infty} \int_{A_n} \langle f(\chi(x))|g(\chi(x))\rangle dx \\ &= \sum_{n=0}^{\infty} \int_{A_n} \langle f(x+na)|g(x+na)\rangle dx \\ &= \sum_{n=0}^{\infty} \int_{A_n} \langle f(x)e^n|g(x)e^n\rangle dx \quad \text{(by Lemma 4.2.4)} \\ &= \sum_{n=0}^{\infty} \int_{A_n} \langle f(x)|g(x)\rangle dx \\ &= \int_{L_k} \langle f(x)|g(x)\rangle dx = \langle f|g\rangle \quad \text{(by Lemma 4.2.5)}.\end{aligned}$$

This completes the proof. □

Let $b \in \Omega$ and $v \in E(b)$ be given. For $f \in \mathcal{S}$, let $\phi_0(v)f : \Omega \to E$ be the measurable section defined by

$$(\phi_0(v)f)(x) := \begin{cases} vf(x-b) & \text{if } b <_\Omega x \\ \\ 0 & \text{elsewhere.} \end{cases}$$

Let $f \in \mathcal{S}$ be given. We leave it to the reader to verify that $\phi_0(v)f \in \mathcal{V}$. Assume that f is k-stable for some $k \geq 1$. Set $A := \{x > ka : f(x+a) \neq f(x)e\}$. Then A has measure zero. Choose $k_0 \in \mathbb{N}$ such that $b <_\Omega k_0 a$ and set $k_1 = k_0 + k$. We claim that $\phi_0(v)f$ is k_1-stable. Let $k_1 a <_\Omega x$ and $x \notin A + b$ be given. Calculate as follows to observe that

$$\begin{aligned} (\phi_0(v)f)(x+a) &= vf(x+a-b) \quad (\text{Since } b <_\Omega k_0 a <_\Omega k_1 a <_\Omega x+a) \\ &= vf(x-b)e \quad (\text{Since } ka <_\Omega k_1 a - b <_\Omega x - b \text{ and } x - b \notin A) \\ &= (\phi_0(v)f)(x)e. \end{aligned}$$

Hence the set $\{k_1 a <_\Omega x : (\phi_0(v)f)(x+a) \neq (\phi_0(v)f)(x)e\}$ is contained in $A + b$ which has measure zero. This proves that $\phi_0(v)f$ is k_1-stable.

Proposition 4.2.9. *Let $b \in \Omega$ and $u, v \in E(b)$ be given. Then for $f \in \mathcal{S}$,*

$$\langle \phi_0(u)f | \phi_0(v)f \rangle = \langle u|v\rangle \langle f|f\rangle.$$

Proof. Assume that f is k-stable for some $k \geq 1$. Choose $k_0 \geq 1$ such that $b <_\Omega k_0 a$ and set $k_1 = k_0 + k$. Now calculate as follows to observe that

$$\begin{aligned} \langle \phi_0(u)f | \phi_0(v)f \rangle &= \int_{L_{k_1}} \langle (\phi_0(u)f)(x) | (\phi_0(v)f)(x) \rangle dx \\ &= \int_{L_{k_1}} \langle uf(x-b) | vf(x-b) \rangle dx \\ &= \langle u|v\rangle \int_{L_{k_1}} \langle f(x-b) | f(x-b) \rangle dx \\ &= \langle u|v\rangle \int_{L_{k_1 a - b}} \langle f(x) | f(x) \rangle dx \\ &= \langle u|v\rangle \langle f|f\rangle \quad (\text{Since } ka <_\Omega k_1 a - b \text{ and by Lemma 4.2.8}). \end{aligned}$$

This completes the proof. □

Let $b \in \Omega$ and $v \in E(b)$ be given. Proposition 4.2.9 implies that for $f \in \mathcal{S}$,

$$\langle \phi_0(v)f|\phi_0(v)f\rangle = ||v||^2\langle f|f\rangle.$$

As a consequence, it follows that there exists a unique bounded linear operator, denoted $\phi(v)$, on $\mathcal{H}$ such that $\phi(v)(f+\mathcal{N}) = \phi_0(v)f+\mathcal{N}$ for every $f \in \mathcal{S}$. Proposition 4.2.9 implies that for $u, v \in E(b)$, $\phi(v)^*\phi(u) = \langle u|v\rangle$. It is clear that $\phi : E \to B(\mathcal{H})$ is multiplicative. Next we verify that ϕ is measurable. We need the following remark.

Remark 4.2.10. *Let $(X, \mathcal{B}_X)$ be a measurable space and $(Y, \mathcal{B}_Y, \mu)$ be a σ-finite measure space. Let $\mathcal{B}_X \otimes \mathcal{B}_Y$ be the product σ-algebra on $X \times Y$. Consider a measurable function $f : X \times Y \to \mathbb{C}$. Suppose that for every $x \in X$, the map $Y \ni y \to f(x,y) \in \mathbb{C}$ is integrable. We claim that the function $X \ni x \to \int f(x,y)d\mu(y) \in \mathbb{C}$ is measurable. By considering the real and imaginary parts of f separately, we can assume that f is real valued.*

Write $f = f_1 - f_2$ where f_1 is the positive part of f and f_2 is the negative part of f. Fix a point $x_0 \in X$. Let δ_{x_0} be the Dirac-delta measure on $(X, \mathcal{B}_X)$. It follows from Tonelli's theorem applied to the product measure $\delta_{x_0} \times \mu$ on $(X \times Y, \mathcal{B}_X \otimes \mathcal{B}_Y)$ that the functions $X \ni x \to \int f_1(x,y)d\mu(y)$ and $X \ni x \to \int f_2(x,y)d\mu(y)$ are measurable. Since for $x \in X$,

$$\int f(x,y)d\mu(y) = \int f_1(x,y)d\mu(y) - \int f_2(x,y)d\mu(y)$$

it follows that the function $X \ni x \to \int f(x,y)d\mu(y) \in \mathbb{C}$ is measurable.

Proposition 4.2.11. *The map $\phi : E \to B(\mathcal{H})$ is measurable. Hence ϕ is a representation of E on $\mathcal{H}$.*

Proof. For $k \geq 1$, let $E_k := \{v \in E : 0 <_\Omega p(v) <_\Omega ka\}$. Then E_k is a measurable subset of E for every k and $\bigcup_{k=1}^\infty E_k = E$. Thus it suffices to show that ϕ restricted to E_k is measurable for every k. Fix $k \geq 1$. It suffices to show that for $f \in \mathcal{S}$, the map $E_k \ni v \to \langle \phi_0(v)f|f\rangle \in \mathbb{C}$ is measurable. Let $f \in \mathcal{S}$ be given. Assume that f is k_0-stable for some $k_0 \geq 1$. Then for $v \in E_k$, $\phi_0(v)f$ is k_0+k-stable. Hence by Lemma 4.2.5, it follows that

$$\langle \phi_0(v)f|f\rangle = \int_{L_{k_0+k}} \langle vf(x-p(v))|f(x)\rangle dx.$$

The above integral representation together with Remark 4.2.10 implies that the function $E_k \ni v \to \langle \phi_0(v)(f)|f\rangle \in \mathbb{C}$ is measurable. This completes the proof. □

Our goal is to show that the representation ϕ is essential.

Remark 4.2.12. *We need the following before we proceed further.*

(1) *Let $x, y \in \Omega$ be such that $x < y$. For $v \in E(x)$ and $w \in E(y)$, there exists a unique element denoted $v^*w \in E(y - x)$ such that $\langle v^*w|u\rangle = \langle w|vu\rangle$ for $u \in E(y - x)$. Note that for $v \in E(x)$ and $w \in E(y)$,*

$$||v^*w|| \leq ||v||||w||. \tag{4.2.2}$$

Let $x, y, z \in \Omega$ be such that $x < y$. Let $v \in E(x)$, $w_1 \in E(y)$ and $w_2 \in E(z)$ be given. Then

$$v^*(w_1w_2) = (v^*w_1)w_2. \tag{4.2.3}$$

The proof of this fact is exactly similar to the proof of Lemma 2.4 of [8]. Hence we omit the proof.

(2) *Let $x, y \in \Omega$ be such that $x < y$. Let $\{v_1, v_2, \cdots ,\}$ be an orthonormal basis for $E(x)$. Then for $\xi \in E(y)$,*

$$\sum_{i=1}^{\infty} ||v_i^*\xi||^2 = ||\xi||^2. \tag{4.2.4}$$

The proof of this fact is exactly similar to the proof of Lemma 2.4 of [8]. Hence we omit the proof.

Let $v \in E(a)$ be given. For $f \in \mathcal{S}$, let $f_v : \Omega \to E$ be defined by

$$f_v(x) = v^*f(x + a).$$

Let $f \in \mathcal{S}$ be given. Note that f_v is a section. To see that f_v is measurable, let $s : \Omega \to E$ be a measurable section. Then the map $\Omega \ni x \to \langle f_v(x)|s(x)\rangle = \langle f(x + a)|vs(x)\rangle \in \mathbb{C}$ is measurable. This implies that f_v is measurable. We leave it to the reader to verify that $f_v \in \mathcal{S}$. We only indicate that to prove $f_v \in \mathcal{V}$, one needs to use Inequality 4.2.2 and to prove f_v is stable one needs to appeal to Equation 4.2.3. Note that if f is k-stable then f_v is k-stable.

Lemma 4.2.13. *Let $v \in E(a)$ and $f \in \mathcal{S}$ be given. Then $\phi(v)^*(f + \mathcal{N}) = f_v + \mathcal{N}$.*

Proof. It suffices to prove that for every $g \in \mathcal{S}$, $\langle \phi(v)^*(f+\mathcal{N})|g+\mathcal{N}\rangle = \langle f_v+\mathcal{N}|g+\mathcal{N}\rangle$. Let $g \in \mathcal{S}$ be given. Without loss of generality, we can assume that f and g are k-stable for some $k \geq 1$. Note that $\phi_0(v)g$ is $k+2$-stable. Now calculate as follows to observe that

$$\begin{aligned}
\langle \phi(v)^*(f+\mathcal{N})|g+\mathcal{N}\rangle &= \langle f|\phi_0(v)g\rangle \\
&= \int_{L_{k+2}} \langle f(x)|vg(x-a)\rangle dx \quad \text{(by Lemma 4.2.5)} \\
&= \int_{L_{k+1}} \langle f(x+a)|vg(x)\rangle dx \\
&= \int_{L_{k+1}} \langle v^* f(x+a)|g(x)\rangle dx \\
&= \int_{L_{k+1}} \langle f_v(x)|g(x)\rangle dx \\
&= \langle f_v+\mathcal{N}|g+\mathcal{N}\rangle \quad \text{(by Lemma 4.2.5)}.
\end{aligned}$$

This completes the proof. □

Theorem 4.2.14. *The representation ϕ is essential.*

Proof. Let $\alpha := \{\alpha_x\}_{x\in\Omega}$ be the semigroup of normal $*$-endomorphisms associated to ϕ. In order to show that α_x is unital for each $x \in \Omega$, it suffices to prove that α_a is unital. Indeed, suppose that α_a is unital. Then $\alpha_{na} = \alpha_a^n$ is unital for every $n \geq 1$. Let $x \in \Omega$ be given. Choose $n \geq 1$ such that $x <_\Omega na$. Write $na = x + y$ with $y \in \Omega$. Then $1 = \alpha_{na}(1) = \alpha_x(\alpha_y(1)) \leq \alpha_x(1)$. Hence α_x is unital. Thus it suffices to show that α_a is unital.

Let $\{v_1, v_2, \cdots\}$ be an orthonormal basis for $E(a)$. We claim that

$$\sum_{i=1}^{\infty} \phi(v_i)\phi(v_i)^* = 1$$

where the sum is a strongly convergent sum. Since $\{\phi(v_i)\phi(v_i)^*\}_{i=1}^{\infty}$ is a sequence of pairwise orthogonal projections, it suffices to show that

$$\sum_{i=1}^{\infty} \langle \phi(v_i)\phi(v_i)^*(f+\mathcal{N})|f+\mathcal{N}\rangle = ||f+\mathcal{N}||^2$$

for every $f \in \mathcal{S}$.

Let $f \in \mathcal{S}$ be given. Assume that f is k-stable for some $k \geq 1$. Then f_{v_i} is k-stable for every i. Now calculate as follows to observe that

$$\begin{aligned}
\sum_{i=1}^{\infty} ||\phi(v_i)^*(f+\mathcal{N})||^2 &= \sum_{i=1}^{\infty} ||f_{v_i}+\mathcal{N}||^2 \\
&= \sum_{i=1}^{\infty} \int_{L_k} ||v_i^* f(x+a)||^2 dx \quad \text{(by Lemma 4.2.5)} \\
&= \int_{L_k} \Big(\sum_{i=1}^{\infty} ||v_i^* f(x+a)||^2 \Big) dx \\
&= \int_{L_k} ||f(x+a)||^2 dx \quad \text{(by Equality 4.2.4)} \\
&= \int_{L_k} ||f(x)||^2 dx \quad \text{(Since } f \text{ is } k\text{-stable)} \\
&= ||f+\mathcal{N}||^2 \quad \text{(by Lemma 4.2.5).}
\end{aligned}$$

In the third equality of the above calculation, we have interchanged the summation and the integral which is permissible since the terms involved are non-negative. This completes the proof. □

We end this section by stating what we have done so far as a Theorem whose proof is a direct consequence of Theorem 4.2.14 and Proposition 4.1.8.

Theorem 4.2.15. *Let $E := \coprod_{x \in \Omega} E(x)$ be a product system over Ω such that $E(x)$ is infinite dimensional for every $x \in \Omega$. Denote the representation of E constructed in Proposition 4.2.11 by ϕ. Let $\alpha := \{\alpha_x\}_{x \in P}$ be the E_0-semigroup associated to ϕ. Then E is isomorphic to the product system associated to α.*

Remark 4.2.16. *The above Theorem is stated as a Theorem D in the introduction. The proof of Theorem D first appeared in [18] (see Theorem 3.14).*

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