On a Relative Mumford-Newstead Theorem

By

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DECLARATION

I declare that the thesis entitled **"On a relative Mumford-Newstead Theorem"** submitted by me for the degree of **Doctor of Philosophy in Mathematics** is the record of academic work carried out by me under the guidance of Professor V. Balaji and this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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CERTIFICATE

I certify that the thesis entitled **"On a Relative Mumford-Newstead Theorem"** submitted for the degree of **Doctor of Philosophy in Mathematics** by Suratno Basu is the record of research work carried out by him under my guidance and supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. I further certify that the new results presented in this thesis represent his independent work in a very substantial measure.

Chennai Mathematical Institute
Date: May, 2016.

Professor V. Balaji
Thesis Supervisor.

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Suratno Basu
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Abstract

In this thesis, we prove a relative version of the classical Mumford-Newstead theorem for a family of smooth curves degenerating to a reducible curve with a simple node. We also prove a Torelli-type theorem by showing that certain moduli spaces of torsion free sheaves on a reducible curve allows us to recover the curve from the moduli space.

Contents

Chapter 1

Introduction

Our main motivation comes from a classical theorem due to Mumford and Newstead. We will briefly explain the theorem: Let X be a smooth, projective curve of genus $g \geq 2$ over $\mathbb{C}$. We fix a line bundle L of odd degree over X. Let M_X be the moduli space of rank 2, stable vector bundles E such that $\det E \simeq L$. It is known that M_X is a smooth, projective and unirational variety. Consequently it follows, by [Ser59, Lemma 1], that the Hodge numbers $h^{0,p} = h^{p,0} = 0$.Therefore, we have the following Hodge decomposition:

$$H^3(M_X, \mathbb{C}) = H^{1,2} \oplus \overline{H^{1,2}},$$

where $\overline{\alpha}$ is the complex conjugate of $\alpha \in H^3(M_X, \mathbb{C}) = H^3(M_X, \mathbb{R}) \otimes \mathbb{C}$ and $H^{1,2} \simeq H^2(M_X, \Omega^1_{M_X})$. Let $pr_1 : H^3(M_X, \mathbb{C}) \to H^{1,2}$ be the first projection. Since $H^3(M_X, \mathbb{R}) \cap H^{1,2} = \{0\}$, we get that the image $pr_1(H^3(M_X, \mathbb{Z}))$ is a full lattice in $H^{1,2}$. We associate a complex torus corresponding to the above Hodge structure:

$$J^2(M_X) := \frac{H^{1,2}}{pr_1(H^3(M_X, \mathbb{Z}))} \tag{1.1}$$

It is known as the second intermediate Jacobian of M_X. We remark that the complex torus, defined above, varies holomorphically in an analytic family of smooth projective, unirational varieties and is a principally polarised abelian variety. It is known that the second Betti number $b_2(M_X) = 1$ ([New67]). Let ω be the unique ample, integral, Kähler class on M_X which is also a generator of $H^2(M_X, \mathbb{Z})$. Then the principal polarisation on $J^2(M_X)$ is induced by the following paring:

$$(\alpha, \beta) \mapsto \int_{M_X} \omega^{n-3} \wedge \alpha \wedge \overline{\beta}, \tag{1.2}$$

where $\alpha, \beta \in H^{1,2}$ and $n = dim_{\mathbb{C}} M_X$. We denote this polarisation on $J^2(M_X)$ by θ'. The theorem of Mumford and Newstead ([MN68, Theorem in page 1201]) asserts that there is a natural isomorphism $\phi : J(X) \to J^2(M_X)$ such that $\phi^*(\theta') = \theta$, where $J(X)$ is the Jacobian of the curve and θ is the canonical polarisation on $J(X)$. In [Bal90, Section 5, page 625]) there is a detailed proof of the fact $\phi^*(\theta') = \theta$. Hence, appealing to the classical Torelli theorem one can recover the curve X from the moduli space M_X. This approach was employed by various other authors to recover the curve X from certain other moduli spaces associated to X. For example V Balaji in [Bal90] proves a Torelli type theorem for the Seshadhri desingularisation $\tilde{N}_X$ of the rank 2 trivial determinant moduli space N_X by looking at the second intermediate Jacobian of $\tilde{N}_X$.

The main observation of Mumford and Newstead is that there is a natural isomorphism of Hodge structures between $H^1(X, \mathbb{Z})$ and $H^3(M_X, \mathbb{Z})$. In this thesis we are mainly interested in the following question: Suppose X_0 is reducible projective curve with a simple node. There is a certain moduli space M_{X_0} of rank 2, stable torsion free sheaves with fixed "odd determinant" associated to it. The question is whether there is a natural isomorphism of Mixed Hodge structures between $H^1(X_0, \mathbb{Z})$ and $H^3(M_{X_0}, \mathbb{Z})$. In this thesis we will show that this is indeed the case. In fact we will prove a stronger result in the relative setting. We now summarize the main results obtained in this thesis:

Let X_0 be a projective curve with exactly two smooth irreducible components X_1 and X_2 meeting at a simple node p. Fix two rational numbers $0<a_1, a_2<1$ such that $a_1 + a_2 = 1$ and let χ be an odd integer. Under some numerical conditions, Nagaraj and Seshadri construct in [NS97, Theorem 4.1], the moduli space $M(2, (a_1, a_2), \chi)$ of rank 2, (a_1, a_2)-semistable torsion free sheaves on X_0 with Euler charachteristic χ. Moreover, they show that $M(2, (a_1, a_2), \chi)$ is the union of two smooth, projective varieties intersecting transversally along a smooth divisor. We will observe that there exists a determinant morphism $det : M(2, (a_1, a_2), \chi) \to J^{\chi-(1-g)}(X_0)$ where $J^{\chi-(1-g)}(X_0)$ is the Jacobian parametrizing the line bundles with Euler characteristic $\chi - (1 - g)$ over X_0 (see Proposition 2.16 in Chapter 1). We further observe that the fibres of the morphism det is again the union of two smooth projective varieties intersecting transversally (see Proposition 2.17 in Appendix). Fix $\xi \in J^{\chi-(1-g)}$. We denote the fibre $det^{-1}(\xi)$ by $M_{0,\xi}$. Since $M_{0,\xi}$ is a singular variety, a priori $H^3(M_{0,\xi}, \mathbb{C})$ has an intrinsic mixed Hodge structure. Let g be the arithmetic genus of X_0. Note that $g = g_1 + g_2$, where g_i is the genus of X_i for $i = 1, 2$. Under the assumption $g_i>3$,

$i = 1, 2$, we will show that $H^3(M_{0,\xi}, \mathbb{Q}) \simeq \mathbb{Q}^{2g}$, and that it has a pure Hodge structure with Hodge numbers $h^{3,0} = h^{0,3} = 0$. Thus we have an intermediate Jacobian $J^2(M_{0,\xi})$, as defined earlier, corresponding to the Hodge structure on $H^3(M_{0,\xi}, \mathbb{C})$ which is a priori only a complex torus of dimension g.

Let $\pi : \mathcal{X} \to C$ be a proper, flat and surjective family of curves, parametrised by a smooth, irreducible curve C. Fix $0 \in C$. We assume that π is smooth outside the point 0 and $\pi^{-1}(0) = X_0$, where X_0 is as above, $g_i > 3$ for $i = 1, 2$. Let X_t be the fibre $\pi^{-1}(t)$ over $t \in C$. Fix a line bundle $\mathcal{L}$ over $\mathcal{X}$ such that the restrictions $\mathcal{L}_t$ to X_t are line bundles with Euler characteristics $\chi - (1 - g)$ for $t \neq 0$ and $\mathcal{L}_0$ is isomorphic to the line bundle ξ. In this situation, it is observed in [NS97, Lemma 7.2] that there is a family $\pi' : \mathcal{M}_{\mathcal{L}} \to C$ such that the fibre $\pi'^{-1}(t)$ over a point $t \neq 0$ is $M_{t,\mathcal{L}_t}$, the moduli space of rank 2, semistable locally free sheaves with *det* $\simeq \mathcal{L}_t$ over the smooth projective curve X_t and $\pi'^{-1}(0) = M_{0,\xi}$ (see Section 2.0.6). We should mention a related work by X Sun [Sun01]. In [Sun01] the author constructs a family of rank r fixed determinant, semistable bundles over smooth projective curves degenerating to a "fixed determinant" moduli space of rank r torsion free sheaves over X_0. We consider an analytic disc Δ around the point 0 and we denote the family $\pi' : \pi'^{-1}(\Delta) \to \Delta$ by $\{M_{t,\mathcal{L}_t}\}_{t \in \Delta}$.

With these notations we state one of the main results of this thesis :

Theorem 1.1.

1. *There is a holomorphic family* $\{J^2(M_{t,\mathcal{L}_t})\}_{t\in\Delta}$ *of intermediate Jacobians corresponding to the family* $\{M_{t,\mathcal{L}_t}\}_{t\in\Delta}$*. In other words, there is a surjective, proper, holomorphic submersion*
$$\pi_2 : J^2(\mathcal{M}_{\mathcal{L}}) \longrightarrow \Delta$$
such that $\pi_2^{-1}(t) = J^2(M_{t,\mathcal{L}_t})\ \forall\, t \in \Delta^* := \Delta \setminus \{0\}$ *and* $\pi_2^{-1}(0) = J^2(M_{0,\xi})$*. Further, there exists a relative ample class* Θ' *on* $J^2(\mathcal{M}_{\mathcal{L}})_{|\Delta^*}$ *such that* $\Theta'_{|J^2(M_{t,\mathcal{L}_t})} = \theta'_t$*, where* θ'_t *is the principal polarisation on* $J^2(M_{t,\mathcal{L}_t})$*.*

2. *There is an isomorphism*

$$\begin{array}{ccc} J^0(\mathcal{X}) & \xrightarrow[\sim]{\Phi} & J^2(\mathcal{M}_{\mathcal{L}}) \\ & {}_{\pi_1}\searrow \quad \swarrow_{\pi_2} & \\ & \Delta & \end{array} \tag{1.3}$$

such that $\Phi^*\Theta'_{|\pi_1^{-1}(t)} = \theta_t$ for all $t \in \Delta^*$, where $\pi_1 : J^0(\mathcal{X}) \to \Delta$ is the holomorphic family $\{J^0(X_t)\}_{t\in\Delta}$ of Jacobians and θ_t is the canonical polarisation on $J^0(X_t)$. In particular $J^2(\mathcal{M}_{\mathcal{L}})_0 := \pi_2^{-1}(0)$ is an abelian variety.

By the above theorem we deduce the following:

Corollary 1.2. *Let X_0 be a projective curve with exactly two smooth irreducible components X_1 and X_2 meeting at a simple node p. We further assume that g_i>3, $i = 1, 2$. Then, there is an isomorphism $J^0(X_0) \simeq J^2(M_{0,\xi})$, where $\xi \in J^{\chi}(X_0)$. In particular, $J^2(M_{0,\xi})$ is an abelian variety.*

Since $J^0(X_0)$ is isomorphic to $J^0(X_1) \times J^0(X_2)$, we observe the Jacobian $J^0(X_0)$ is independent of the nodal point in X_0. Hence, the classical Torelli theorem fails for such curves (see [MM64, Page 6]). On the other hand, it is known that under a suitable choice of the polarisation on the Jacobian $J^0(X_0)$, one can recover the normalization $\tilde{X}_0$ of X_0, but not the curve X_0. In otherwords one can recover both the components of X_0 but not the nodal point(see [Hai02, page 125]).

We see that the moduli space $M_{0,\xi}$ of rank 2 torsion free sheaves carries more information than the Jacobian $J(X_0)$. In fact, we show that we can actually recover the curve X_0 from $M_{0,\xi'}$, by following a strategy of [BBdBR01]. More precisely, we will prove the following analogue of the Torelli theorem for reducible curves:

Theorem 1.3. *Let X_0 (resp. Y_0) be the projective curve with two smooth irreducible components X_i (resp. Y_i), $i = 1, 2$ meeting at a simple node p (resp. q). We assume that $genus(X_i) = genus(Y_i)$, for $i = 1, 2$, and $X_1 \not\cong X_2$(resp. $Y_1 \not\cong Y_2$). Let $M_{0,\xi_{X_0}}$ (resp. $M_{0,\xi_{Y_0}}$) be the moduli space of rank 2, semistable torsion free sheaves E with $detE \simeq \xi_{X_0}$, $\xi_{X_0} \in J^{\chi}(X_0)$, on X_0 (resp. on Y_0). If $M_{0,\xi_{X_0}} \simeq M_{0,\xi_{Y_0}}$ then we have $X_0 \simeq Y_0$.*

1.1 Layout Of The Thesis

The general layout of this thesis is as follows: In Chapter 1 we will briefly summarize the main results of [NS97] which we crucially need for the remaining chapters. Though this Chapter mostly contains statements of the main results from [NS97] the existence of a certain determinant morphism has been explained in some details. This is quite crucial since we are mainly interested in computing the cohomologies of its fibres. I thank Prof C.S Seshadri for suggesting a way to show that the determinant morphism exists. In Chapter 3 we will compute certain cohomology groups of the moduli space $M_{0,\xi}$. The Chapter 4 contains the proof the Theorem 1.1 and in Chapter 5 we will prove the Torelli type theorem.

Chapter 2

Preliminaries

In this chapter, we briefly recall the main results in [NS97] which will be extensively used in the present work. Though this chapter is an exposition of the main results obtained in the article [NS97], the existence of certain determinant morphisms are explained with some details. Also the Theorem 2.12 is not explicitely proved in [NS97]. We believe this is a new observation. Before proceeding further we will fix the following notations. We will follow these notations in the subsequent chapters.

2.0.0.1 Notation

- Throughout we work over the field $\mathbb{C}$ of complex numbers. We assume that all the schemes are reduced, separated and of finite type over $\mathbb{C}$.
- Let $p_i : X_1 \times \cdots \times X_n \to X_i$ be the i^{th} projection, where X_i is a scheme for $i = 1, \ldots, n$. By abuse of notation, we denote $p_i^*(E_i)$ also by E_i, where E_i is a sheaf of $\mathcal{O}_{X_i}$ - modules.
- Let X be a projective scheme and E be a vector bundle over X. Then we set $h^i(E) := dim_{\mathbb{C}} H^i(X, E)$.
- By cohomology of a scheme X, we mean the singular cohomology of the space X_{ann}, the analytic space with complex analytic topology associated to X.
- Let E be coherent sheaf over X. We denote by E_p the stalk at p and by $E(p) := \frac{E_p}{m_p E_p}$ the fibre of E at $p \in X$, where m_p is the maximal ideal of the local ring at the point p.

- Let X be a smooth projective curve and E be a vector bundle over X. Then we denote $E \otimes \mathcal{O}_X(np)$ by $E(np)$, where $p \in X$ is a closed point and n is an integer.

- Let E_1, E_2 be two locally free sheaves on a projective scheme X and let q_i : $\mathbb{P}E_1 \times_X \mathbb{P}E_2 \to \mathbb{P}E_i$ for $i = 1, 2$ be two projections. Then we denote the line bundle $q_1^*\mathcal{O}(m) \otimes q_2^*\mathcal{O}(n)$ by $\mathcal{O}(m, n)$ where $\mathcal{O}(m) := \mathcal{O}(1)^{\otimes m}$.

- If Z is a closed subvariety of a smooth variety X, then we denote by $Codim(Z, X)$, the codimension of Z in X.

We recall some basic definitions from commutative algebra. All rings considered in this section are Noetherian.

Definition 2.1. Let (A, m) be a local ring and M be a finitely generated A module. We say $a_1 \cdots a_r \in m$ is a m-regular sequence if the following hold:

1. $(a_1, \cdots, a_r)M \neq M$,

2. a_i is a non-zero divisor in $\frac{M}{(a_1, \cdots, a_{i-1})M}$ for $i = 1 \cdots r$.

The depth of a A module M is defined to be the length of a maximal m- regular sequence.

Definition 2.2. An extension $A \subset B$ of reduced rings is subintegral if
(1) B is integral over A
(2) $Spec\ B \to Spec\ A$ is a bijection
(3) $\forall\ P \in Spec\ B$, $k_{A \cap P} \to k_P$ is an isomorphism, where $k_P = \frac{B_P}{PB_P}$

Definition 2.3. If $A \subset B$, both rings reduced, we say A is seminormal in B if there is no extension $A \subset C \subset B$ with $C \neq A$ and $A \subset C$ subintegral. We say A is seminormal if it is seminormal in its total ring of quotients.

Seminormal rings are characterised by the following properties:

Proposition 2.4. *([Nar93, Proposition 3.6]) A reduced ring is seminormal if* $\forall b, c \in A$ *with* $b^3 = c^2$ *there is a unique* $a \in A$ *with* $b = a^2$ *and* $c = a^3$.

A variety V is said to be seminormal if $\mathcal{O}_{V,p}$ is seminormal $\forall p \in V$.

2.0.1 Triples associated to a torsion free sheaves on a reducible nodal curve

Let X_0 be a projective curve of arithmetic genus g with exactly two smooth irreducible components X_1 and X_2 meeting at a simple node p. The arithmetic genus $g := h^1(\mathcal{O}_{X_0})$ of such a curve is $g = g_1 + g_2$, where g_i is the genus of X_i for $i = 1, 2$. Let F be coherent $\mathcal{O}_{X_0}$- module. Then we say F is a depth 1 $\mathcal{O}_{X_0}$- module if F_x is a depth 1, $\mathcal{O}_{X_0,x}$-module for all $x \in X_0$. By a torsion free sheaf over X_0 we always mean a coherent $\mathcal{O}_{X_0}$-module of depth 1. Let F be a coherent $\mathcal{O}_{X_0}$-module. Then F has depth 1 if and only if it is a pure sheaf of dimension 1 i.e. for all nonzero subsheaves $G \subset F$ we have dim $supp(G) = 1$ (see [NS97, Proposition 2.1]). We say a torsion free sheaf F is of rank (r_1, r_2) on X_0 if F restricted to X_i is of rank r_i, $i = 1, 2$. We say F is of rank r if $r_1 = r_2 = r$. Note that one of the r_i could be zero for a torsion free sheaf.

2.0.2 Hecke modification

Let V be a vector bundle over a smooth projective curve X. Fix a point $x \in X$ and $K \subset V(x)$ be a subspace of $V(x)$. There are two canonical constructions called the Hecke modifications defined as follows:

(I) $\phi : W \to V$, $Im(\phi_p) = K$, where W is a vector bundle and ϕ is a homomorphism of vector bundles, which is an isomorphism outside p.
(II) $\phi' : V \to W'$, $Ker(\phi_p) = K$, where W' is a vector bundle and ϕ is a homomorphism of vector bundles, which is an isomorphism outside p.

(See [NS97, Remark 2.4])

2.0.2.1 Equivalence between the category of torsion free sheaves and the category of triples

Let $\vec{C}$ be a category whose objects are triples (F_1, F_2, A) where F_i are vector bundles on X_i, for $i = 1, 2$ and $A : F_1(p) \to F_2(p)$ is a linear map. Let $(F_1, F_2, A), (G_1, G_2, B) \in \vec{C}$. We say $\phi : (F_1, F_2, A) \to (G_1, G_2, B)$ is a morphism if there are morphisms $\phi_i : F_i \to G_i$

of $\mathcal{O}_{X_i}$-modules for $i = 1, 2$ such that the following diagram is commutative:

$$\begin{array}{ccc} F_1(p) & \xrightarrow{\phi_1 \otimes k(p)} & G_1(p) \\ \downarrow A & & \downarrow B \\ F_2(p) & \xrightarrow{\phi_2 \otimes k(p)} & G_2(p) \end{array} \tag{2.1}$$

Let F be torsion free sheaf on X_0. Note that F is locally free outside the point p. Let $\bar{F}_1$ be the restriction of F to X_1 and $\bar{F}_2$ be the restriction of F to X_2. Let $F_1 = \bar{F}_1/(torsion)$ and $F_2' = \bar{F}_2/(torsion)$.

Proposition 2.5. *Given a torsion free sheaf F on X_0 there is a unique triple $(F_1, F_2, A) \in \vec{C}$ (up to isomorphism), where F_2 is the Hecke modification (of type (II)) of F_2', such that $F \simeq \{(f_1, f_2) \in F_1 \oplus F_2 \mid A(f_1(p)) = f_2(p)\}$. In fact the above association gives an equivalence of category between the category of torsion free sheaves and the category $\vec{C}$.*

Proof. Note that if $(F_1, F_2, A) \in \vec{C}$ be a triple then $F = \{(f_1, f_2) \in F_1 \oplus F_2 \mid A(f_1(p)) = f_2(p)\}$ is a torsion free sheaf. We will show that given a torsion free sheaf we can associate a unique triple $(F_1, F_2, A) \in \vec{C}$ such that $F = \{(f_1, f_2) \in F_1 \oplus F_2 \mid A(f_1(p) = f_2(p)\}$. From Proposition 2.2 in [NS97] we get an exact sequence:

$$0 \to F \to F_1 \oplus F_2' \to T \to 0$$

of $\mathcal{O}_{X_0}$-modules where the module T is supported at the point p with $m_{X_0,p}T = 0$ and the morphism $F \to F_1 \oplus F_2'$ is an isomorphism outside p. Let

$$N = Ker(F_1(p) \oplus F_2'(p) \to T).$$

Then, in [NS97, Remark 2.1], it is shown that the canonical projections

$$N \to F_1(p) \text{ and } N \to F_2'(p)$$

are surjective. Therefore it follows that F can be identified with the subsheaf of $F_1 \oplus F_2'$ consisting of all $f = (f_1, f_2)$ such that evaluation of f at p is in N. Let $K = Ker(N \to F_1(p))$. Then we see that $K \subset F_2'(p)$. There exists a vector bundle F_2 on X_2 and a morphism $i : F_2' \to F_2$ which is an isomorphism outside p and $ker(i_p) = K$ (see Subsection 2.0.2). Let $N^1 = Image(N)$ under the canonical homomorphism $F_1(p) \oplus F_2'(p) \xrightarrow{\theta} F_1(p) \oplus F_2(p)$. Then we see that the canonical homomorphism $N^1 \to F_1(p)$ is an isomorphism and $N = \theta^{-1}(N^1)$ (see [NS97, Remark 2.3]). Now the subspace $N^1 \subset F_1(p) \oplus F_2(p)$ induces a linear map $A : F_1(p) \to F_2(p)$ such that $N' = Graph(A)$. Therefore we conclude that

$$F \simeq \{(f_1, f_2) \in F_1 \oplus F_2 \mid (f_1(p), f_2(p)) \in Graph(A).$$

The equivalence of category is proven in the [NS97, Lemma 2.3]. □

Remark 2.6. Similarly, we define another category $\overleftarrow{C}$ whose objects are triples (F_1, F_2, A) where F_i are vector bundles over X_i for $i = 1, 2$ and $A : F_2(p) \to F_1(p)$ is a linear map. The morphism between any two such triples is defined in the same way before. In the proof of Proposition 2.5 if we set $K' = Ker(N \to F_2(p))$ then we see that

$$F \simeq \{(f_1, f_2) \in F_1' \oplus F_2' \mid B(f_2(p) = f_1(p)\},$$

where $j : F_1 \to F_1'$ is the Hecke modification such that $ker(j_p) = K'$ and $B : F_2'(p) \to F_1'(p)$ is a linear map. In fact one can again show that the category of torsion free sheaves is equivalent to the category $\overleftarrow{C}$ (see [NS97, Remark 2.9]). Moreover, we note that if the triples $(F_1, F_2, A) \in \overrightarrow{C}$ and $(F_1', F_2', B) \in \overleftarrow{C}$ correspond the same torsion free sheaf F, then they are related by the following diagram:

$$\begin{array}{ccc} F_1(p) & \xrightarrow{i_p} & F_1'(p) \\ \downarrow A & & \uparrow B \\ F_2(p) & \xleftarrow[j_p]{} & F_2'(p) \end{array} \tag{2.2}$$

where $i : F_1 \to F_1'$ (resp. $j : F_2' \to F_2$) is the Hecke modification, as explained in Subsection 2.0.2, such that $ker(i_p) = ker(A)$ (resp $Im(j_p) = Im(A)$).

2.0.2.2 Notion of semistability

Fix an ample line bundle $\mathcal{O}_{X_0}(1)$ on X_0. Let $deg(\mathcal{O}_{X_0}(1)|_{X_i}) = c_i$, $i = 1, 2$, and $a_i = \frac{c_i}{c_1 + c_2}$. Then $0 < a_1, a_2 < 1$ and $a_1 + a_2 = 1$. We say $a = (a_1, a_2)$ a polarisation on X_0. We fix a polarization $a = (a_1, a_2)$ on X_0.

Definition 2.7. For a torsion free sheaf F of rank type (r_1, r_2), we define the rank $r := a_1 r_1 + a_2 r_2$ and the slope $\mu(F) := \frac{\chi(F)}{r}$, where $\chi(F) := h^0(F) - h^1(F)$. A torsion free sheaf F is said to be semistable(resp. stable) with respect to the polarisation $a = (a_1, a_2)$ if $\mu(G) \leq \mu(F)$(resp. $<$) for all nontrivial proper subsheaves G of F.

Let (F_1, F_2, A) be a triple. A triple (G_1, G_2, B) is said to be a subtriple of (F_1, F_2, A) if $G_i \xrightarrow{j_i} F_i$ are inclusions of $\mathcal{O}_{X_i}$-modules, $i = 1, 2$, such that the following diagram commutes-

$$\begin{array}{ccc} G_1(p) & \xrightarrow{j_1(p)} & F_1(p) \\ \downarrow A & & \downarrow B \\ G_2(p) & \xrightarrow{j_2(p)} & F_2(p) \end{array} \tag{2.3}$$

Definition 2.8. We define the Euler characteristic and the slope of a triple $(F_1, F_2, A) \in \vec{C}$ to be:

$$\chi((F_1, F_2, A)) = \chi(F_1) + \chi(F_2) - rk(F_2) \text{ and } \mu(F_1, F_2, A) = \frac{\chi((F_1, F_2, A))}{r}.$$

A triple (F_1, F_2, A) is said to be semistable(resp. stable) if $\mu(G_1, G_2, B) \leq \mu(F_1, F_2, A)$ for all nontrivial proper subtriples of (F_1, F_2, A).

Lemma 2.9. *A torsion free sheaf F on X_0 is $a = (a_1, a_2)$- semistable (resp. stable) if and only if the corresponding triple (F_1, F_2, A) is $a = (a_1, a_2)$ semistable (resp. stable).*

Proof. Let $(F_1, F_2, A) \in \vec{C}$ be the triple associated to a torsion free sheaf F on X_0. Then we have $\chi(F) = \chi(F_1, F_2, A)$ (see [NS97, Remark 2.11]). By Proposition 2.5 a torsion free subsheaf of F corresponds to a subtriple of (F_1, F_2, A). Therefore, the above statement follows. □

Let $S(r, \chi)$ be the set of all rank r, $a = (a_1, a_2)$ semistable torsion free sheaves on X_0 with Euler charachteristic χ. Note that the Hilbert polynomial $P(E, n) = (c_1 + c_2)n + \chi(E)$ for all $E \in S(r, \chi)$ (this can be easily computed from equation (2.1.3)).

Lemma 2.10. *([Ses, Septieme Partie]) Then there exists an integer m_0 such that-*

1. *$H^1(F(m)) = 0$ for all $F \in S(r, \chi)$ and $m \geq m_0$*
2. *$F(m)$ is globally generated by its sections for all $F \in S(r, \chi)$.*

2.0.3 Moduli space of rank 1 torsion free sheaves over a reducible nodal curve

2.0.3.1 Euler Characteristic bounds for rank 1 semistable sheaves

Fix three integers $\chi \neq 0$, χ_1 and $\chi_2 \neq 1$ with $\chi > \chi_i$ such that $\chi = \chi_1 + \chi_2 - 1$. Let $\mathcal{O}_{X_0}(1)$ be an ample line bundle such that $deg(\mathcal{O}_{X_0}(1)|_{X_1}) = \chi_1 - 1$ and $deg(\mathcal{O}_{X_0}(1)|_{X_2}) = \chi_2$. Since $\chi = \chi_1 + \chi_2 - 1$, the Hilbert polynomial $P(L, n) = (n+1)\chi$ for all $L \in S(1, \chi)$. Let $b_1 = \frac{\chi_1 - 1}{\chi}$ and $b_2 = \frac{\chi_2}{\chi}$. In this subsection whenever we say a semistable rank 1 torsion free sheaf we assume the semistability with respect to the polarisation $b = (b_1, b_2)$.

Lemma 2.11. *Let $L \in S(1, \chi)$ and $(L_1, L_2, \lambda) \in \vec{C}$ be the unique triple representing L. Then $\chi(L_i)$, the Euler characteristic of L_i, satisfy the following:*

$$\chi_1 \leq \chi(L_1) \leq \chi_1 + 1,\ \chi_2 - 1 \leq \chi(L_2) \leq \chi_2.$$

Moreover if L is semistable and non locally free then we have $\chi(L_1) = \chi_1$ and $\chi(L_2) = \chi_2$.

Conversely, suppose L be a rank 1 torsion free sheaf with $\chi(L_i)$ satisfy the above conditions then $L \in S(1, \chi)$.

Proof. By Lemma [NS97, Lemma 5.2] we can easily derive the following: if L is a rank 1,locally free and $(L_1, L_2, \lambda) \in \vec{C}$ be the unique triple representing L then we only have to check the semistability condition for the subtriples $(L_1(-p), 0, 0)$ and $(0, L_2, 0)$. If L is a rank 1, non locally free sheaf and $(L_1, L_2, 0)$ be the triple representing L then we only have to check the semistability for the subtriples $(L_1, 0, 0)$ and $(0, L_2, 0)$. Now by using the definition of semistability (see 2.8) we immediately get the above Lemma. □

In this subsection we prove that the moduli space of rank 1,semistable torsion free sheaves with respect to a certain choice polarisation is isomorphic to the product of the Jacobians.

Fix an integer $m \geq m_0$ such that Lemma 2.10 holds for all $F \in S(1, \chi)$ and let $P(n) = (n+1)\chi$. Let $Q(1, \chi)$ be the Quot scheme parametrising all coherent quotients

$$\mathcal{O}_{X_0}^{\oplus p(m)} \to L \to 0$$

with Hilbert polynomial $P(n)$ and $\mathcal{U}^1$ be the universal quotient sheaf of $\mathcal{O}_{X_0 \times Q(1,\chi)}^{\oplus p(m)}$ on $X_0 \times Q(1,\chi)$. Let $R(1,\chi)^{ss}$ be the open subset of $Q(1,\chi)$ such that if $q \in R(1,\chi)^{ss}$ then $\mathcal{U}_q^1 := \mathcal{U}^1|_{X_0 \times q}$ is a rank 1 semistable torsion free quotient and the natural map

$$H^0(\mathcal{O}_{X_0 \times q}) \to H^0(\mathcal{U}_q^1)$$

is an isomorphism. Note that if $L \in S(1,\chi)$ then ,by Lemma 2.10, $L(m)$ is a quotient of a trivial sheaf of rank $p(m) := h^0(L(m))$ and the natural map $H^0(\mathcal{O}_{X_0}^{\oplus p(m)}) \to H^0(L(m))$ is an isomorphism. Therefore $L(m) \simeq \mathcal{U}_q^1$ for some $q \in R(1,\chi)^{ss}$. The group $GL(p(m))$ acts on $Q(1,\chi)$ and $R(1,\chi)^{ss}$ is invariant under the action of $GL(p(m))$. Moreover, the action of $GL(p(m))$ goes down to an action of $PGL(p(m))$. By a general result in (see [Ses, Septieme partie,III, Theorem 15]) the good quotient $R(1,\chi)^{ss}//PGL(p(m))$ exists as a reduced, projective scheme. We will now study the orbit closure equivalence in $R(1,\chi)^{ss}$. Let R_0 be the open subset of $R(1,\chi)^{ss}$ consisting of only rank 1, locally free sheaves. Then $R_0 = R_1 \sqcup R_1'$ where R_1 consists of those rank 1 locally free sheaves L such that $\chi(L_1) = \chi_1$, $\chi(L_2) = \chi_2$ and R_1' consists of those rank 1 locally free sheaves L such that $\chi(L_1) = \chi_1 + 1$, $\chi(L_2) = \chi_2 - 1$. Let $J^{\chi_i}(X_i)$ be the Jacobian of isomorphism classes of line bundles over X_i with Euler characteristic χ_i, $i = 1,2$. With these notations the main theorem of this subsection is:

Theorem 2.12. *The good quotient $R(1,\chi)^{ss}//PGL(p(m))$ is isomorphic to $J^{\chi_1}(X_1) \times J^{\chi_2}(X_2)$.*

Proof. Let $q : R(1,\chi)^{ss} \to R(1,\chi)^{ss}//PGL(p(m))$ be the quotient map. Moreover, the subset $R_1 \subset R(1,\chi)^{ss}$ is smooth and irreducible. Therefore, the quotient $R_1/PGL(p(m))$ is smooth (see [HL10, 4.2.13]) and irreducible.

Claim 1: The quotient $q(R_1) = R_1/PGL(p(m))$ is isomorphic to $J^{\chi_1}(X_1) \times J^{\chi_2}(X_2)$.

To see this consider $\mathcal{U}$ over $X_0 \times R_1$. Then $\mathcal{U}^1$ is locally free and hence $\mathcal{U}_i = \mathcal{U}^1|_{X_i \times R_1}$ is locally free. Moreover, $\chi(\mathcal{U}_i|_{X_i \times q}) = \chi_i$, $i = 1,2$. Thus by the universal property of $J^{\chi_i}(X_i)$ we get a morphism $f_i : R_1 \to J^{\chi_i}$, $i = 1,2$. Therefore, we get a morphism $f = (f_1, f_2) : R_1 \to J = J^{\chi_1}(X_1) \times J^{\chi_2}(X_2)$. Clearly, this morphism is $PGL(p(m))$-invariant and the fibres of this morphism are isomorphic to the orbits of the $PGL(p(m))$ action. Therefore, we get a bijective morphism $R_1/PGL(p(m)) \to J$. Since $R_1/PGL(p(m))$ and J are integral and J is smooth, we have that $R_1/PGL(p(m))$ is isomorphic to the variety $J = J^{\chi_1}(X_1) \times J^{\chi_2}(X_2)$.

Claim 2: We have an equality:

$$R_1/PGL(p(m)) = R(1,\chi)^{ss}//PGL(p(m).$$

Note that Claims 1 and 2 together prove the theorem.

Let $[L_0] \in R(1,\chi)^{ss}//PGL(p(m)$ where [] is orbit closure equivalence class. Then we want to show there is a $L \in R_1$ such that $q(L) = [L_0]$. In other words the orbit closure $\overline{O(L)}$ intersects the orbit closure $\overline{O(L_0)}$. Suppose, L_0 is locally free with $\chi(L_0|_{X_i}) = \chi_i$, $i = 1,2$ then there is nothing to prove. So we assume that L_0 is a rank 1 torsion free but non-locally free sheaf. Let $(L_1, L_2, 0) \in \overrightarrow{C}$ be the the unique triple representing L_0. Then by Lemma 2.11 we get $\chi(L_i) = \chi_i$, $i = 1,2$. Let L be the rank 1 locally free sheaf corresponding to the triple $(L_1, L_2, \lambda) \in \overrightarrow{C}$ where $\lambda : L_1(p) \to L_2(p)$ is an isomorphism. We will show now the orbit closure $\overline{O(L)}$ intersects the orbit $O(L_0)$. For this, let $p_i : X_i \times \mathbb{A}^1 \to X_i$, $i = 1,2$, be the two projections. We again denote the pullback $p_i^* L_i$ by L_i. Since L_i, $i = 1,2$, are free $\mathcal{O}_{\mathbb{A}^1}$-module, we can choose a $\mathcal{O}_{\mathbb{A}^1}$-module homomorphism $\lambda : L_1|_{p\times\mathbb{A}^1} \to L_2|_{p\times\mathbb{A}^1}$ such that $\lambda(t) : L_1(p,t) \to L_2(p,t)$ is an isomorphism for all $t \neq 0$ and $\lambda(0) = 0$. Let G be the graph of the morphism λ in $L_1|_{p\times\mathbb{A}^1} \oplus L_1|_{p\times\mathbb{A}^1}$ and $G' := \frac{L_1|_{p\times\mathbb{A}^1} \oplus L_2|_{p\times\mathbb{A}^1}}{G}$. Let $\mathcal{L} := Ker(L_1 \oplus L_2 \to G')$ over $X_0 \times \mathbb{A}^1$. Now $L_1 \oplus L_2$ and G', being free $\mathcal{O}^1_{\mathbb{A}}$-module, are flat over $\mathbb{A}^1$. Therefore, $\mathcal{L}$ is flat over $\mathbb{A}^1$. We also see that $\mathcal{L}_t$ is the torsion free sheaf corresponding to the triple $(L_1, L_2, \lambda(t))$. Therefore, $\mathcal{L}_t \simeq L$ for all $t \neq 0$ and $\mathcal{L}_0 \simeq L_0$. Note that, as $\mathcal{L}_t \in R(1,\chi)^{ss}$ for all $t \in \mathbb{A}^1$, $H^1(\mathcal{L}_t) = 0$ and $\mathcal{L}_t$ is globally generated for all $t \in \mathbb{A}^1$. By semicontinuity theorem, we get $p_{2*}\mathcal{L}$ is locally free sheaf of rank $p(m)$ on $\mathbb{A}^1$. Since any locally free sheaf on $\mathbb{A}^1$ is free, $p_{2*}\mathcal{L} \simeq \mathcal{O}_{\mathbb{A}^1}{}^{\oplus p(m)}$. Thus we get a quotient

$$\mathcal{O}^{p(m)}_{X_0\times\mathbb{A}^1} \simeq p_2^* p_{2*}\mathcal{L} \to \mathcal{L} \to 0.$$

such that $H^0(\mathcal{O}^{p(m)}_{X_0\times t}) \to H^0(\mathcal{L}_t)$ is an isomorphism for all $t \in \mathbb{A}^1$. Hence we get a morphism $\phi : \mathbb{A}^1 \to R(1,\chi)^{ss}$ such that $\phi^*\mathcal{U}^1 \simeq \mathcal{L}$. Since $\mathcal{L}_t \simeq L$ for all $t \in \mathbb{A}^1 - 0$, $\phi(\mathbb{A}^1 - 0)$ lies in the $PGL(p(m))$ orbit of L and $\phi(0) = L_0$. Therefore, L_0 is in the orbit closure of L. Clearly, $\chi(L_i) = \chi(L_i|_{X_i}) = \chi_i$, $i = 1,2$. Thus $L \in R_1$, and we are done.

Finally suppose, L_0 is a rank 1, locally free sheaf such that $\chi(L_1) = \chi_1 + 1$ and $\chi(L_2) = \chi_1 - 1$ where (L_1, L_2, λ), $\lambda : L_2(p) \to L_1(p)$ an isomorphism, is the unique triple representing L_0. Let L be the rank 1 locally free sheaf represented by $(L_1(-p), L_2(p), \lambda) \in \overrightarrow{C}$. Then the orbit closure $\overline{O(L_0)}$ intersects the orbit closure $\overline{O(L)}$ in $R(1,\chi)^{ss}$. This easily follows from the observation: the torsion free sheaf L' represented by the triple $(L_1, L_2, 0) \in \overleftarrow{C}$ is in the orbit closure of L_0. Note, by Remark 2.6, L' is isomorphic to the torsion free sheaf represented by $(L_1(-p), L_2(p), 0) \in \overrightarrow{C}$. Thus , by Lemma 2.11, L' is semistable. By the previous argument we can show that L' is also in the orbit closure of L. Hence the orbit closure $\overline{O(L_0)}$ intersects the orbit closure $\overline{O(L)}$. Thus given any $[L_0] \in R(1,\chi)^{ss}//PGL(p(m)$ we have seen that there is a $L \in R_1$ such that $q(L) = [L_0]$. This proves Claim 2. □

2.0.4 Moduli space of rank 2 torsion free sheaves over a reducible nodal curve

2.0.4.1 Euler Characteristic bounds for rank 2 semistable sheaves

Fix an integer χ and a polarization $a = (a_1, a_2)$ on X_0 such that $a_1\chi$ is not an integer. Then we have the following Euler characteristic restrictions:

Lemma 2.13.

Let χ_1, χ_2 be the unique integers satisfying

$$a_1\chi < \chi_1 < a_1\chi + 1 \;,\; a_2\chi + 1 < \chi_2 < a_2\chi + 2 \tag{2.4}$$

and $\chi = \chi_1 + \chi_2 - 2$. If F is a rank 2, $a = (a_1, a_2)$-semistable sheaf then $\chi(F_1) = \chi_1$, $\chi(F_2) = \chi_2$ or $\chi(F_1) = \chi_1 + 1$, $\chi(F_2) = \chi_2 - 1$ and $rk(A) \geq 1$ where $(F_1, F_2, A) \in \overrightarrow{C}$ is the unique triple representing F. Moreover if F is non-locally free then $\chi(F_1) = \chi_1$ and $\chi(F_2) = \chi_2$.

Proof. See [NS97, Theorem 3.1]. □

Fix an integer χ such that $a_1\chi$ is not an integer. With these notations one of the main theorems of the article [NS97] is the following.

Theorem 2.14. *The fine moduli space $M(2, a, \chi)$ of isomorphism classes rank 2, (a_1, a_2) stable torsion free sheaves with Euler characteristic χ exists as a reduced, projective scheme. Moreover, it has exactly two smooth, irreducible components meeting transversally along a divisor whenever χ is odd.*

Remark 2.15. Since $M(2, a, \chi)$ is union of two smooth projective varieties intersecting transversally, we have $M(2, a, \chi)$ is seminormal.

2.0.5 Determinant morphism

Fix an integer χ and a polarization (a_1, a_2) on X_0 such that $a_1\chi$ is not an integer. We also fix an integer m' such that Lemma 2.10 holds for all $E \in S(2, \chi)$. Let $Q(2, \chi)$ be the Quot scheme parametrising all coherent quotients

$$\mathcal{O}_{X_0}^{\oplus p(m')} \to E \to 0$$

and $\mathcal{U}^2$ be the universal quotients sheaf of $\mathcal{O}_{X_0 \times Q(2,\chi)}^{\oplus p(m')}$ on $X_0 \times Q(2, \chi)$. Let $R(2, \chi)^{ss}$ be the open subset of $Q(2, \chi)$ such that if $q \in R(2, \chi)^{ss}$ then $\mathcal{U}_q^2 := \mathcal{U}^2|_{X_0 \times q}$ is a rank 2 semistable torsion free quotient and the natural map

$$H^0(\mathcal{O}_{X_0 \times q}) \to H^0(\mathcal{U}_q^1)$$

is an isomorphism. Let $R(2, \chi)^{ss}$ be the open subset of $Q(2, \chi)$ such that if $q \in R(2, \chi)^{ss}$ then $\mathcal{U}_q^2 := \mathcal{U}^2|_{X_0 \times q}$ is a rank 2 semistable torsion free quotient and the natural map

$$H^0(\mathcal{O}_{X_0 \times q}) \to H^0(\mathcal{U}_q^2)$$

is an isomorphism. The moduli space $M(2, a, \chi)$ is isomorphic to the good quotient $R(2, \chi)^{ss}//PGL(p(m'))$. Let $\chi_i' = \chi_i - (1 - g_i)$, $i = 1, 2$.

Proposition 2.16. *There exists a determinant morphism det* : $M(2, a, \chi) \to J^{\chi_1'}(X_1) \times J^{\chi_2'}(X_2)$ *defined by the association* $F \mapsto \frac{\wedge^2 F}{tor}$ *where tor is the maximal submodule of* $\wedge^2 F$ *with proper support on* X_0.

Proof. First note that if F is a rank 2, (a_1, a_2)-semistable torsion fre semistable torsion free sheaf with Euler charachteristic χ then $\chi(\frac{\wedge^2 F}{tor}) = \chi - (1 - g)$. Let us restrict the universal quotient sheaf $\mathcal{U}^2$ to $X_0 \times R(2, \chi)^{ss}$. Consider the sheaf $\frac{\wedge^2 \mathcal{U}^2}{T}$ where T is the maximal subsheaf of $\wedge^2 \mathcal{U}^2$ with proper support. The natural surjection map

$$\wedge^2 \mathcal{U}^2|_{X_0 \times q} \to \frac{\wedge^2 \mathcal{U}^2}{T}|_{X_0 \times q}$$

induces an isomorphism

$$\frac{\wedge^2\mathcal{U}^2|_{X_0\times q}}{tor} \simeq \frac{\wedge^2\mathcal{U}^2}{T}|_{X_0\times q},$$

for all $q \in R(2,\chi)^{ss}$ as the kernel of the surjection is supported over the nodal point p. Therefore, we get $\chi(\frac{\wedge^2\mathcal{U}^2}{T}|_{X_0\times q}) = \chi - (1-g)$ for all $q \in R(2,\chi)^{ss}$. Hence the Hilbert polynomial of $\frac{\wedge^2\mathcal{U}^2}{T}|_{X_0\times q}$ is independent of $q \in R(2,\chi)^{ss}$. Since $R(2,\chi)^{ss}$ is reduced and the Hilbert polynomial of $\frac{\wedge^2\mathcal{U}^2}{T}|_{X_0\times q}$ is independent of $q \in R(2,\chi)^{ss}$ we get that $\frac{\wedge^2\mathcal{U}^2}{T}$ is flat over $R(2,\chi)^{ss}$. Note that if $\mathcal{U}^2_q := \mathcal{U}^2|_{X_0\times q}$ is represented by the unique triple $(\mathcal{U}^2_{q_1}, \mathcal{U}^2_{q_2}, \mathcal{A}_q)$ then $\chi(\mathcal{U}^2_{q_1}) = \chi_1$, $\chi(\mathcal{U}^2_{q_2}) = \chi_2$ or $\chi(\mathcal{U}^2_{q_1}) = \chi_1 + 1$, $\chi(\mathcal{U}^2_{q_2}) = \chi_2 - 1$. Thus by Lemma 2.11 $\frac{\wedge^2\mathcal{U}^2}{T}|_{X_0\times q}$, $q \in R(2,\chi)^{ss}$ is semistable. Thus, by Lemma 2.10, there is an integer m such that

$$(i) H^1(\frac{\wedge^2\mathcal{U}^2}{T}|_{X_0\times q}(m)) = 0 \text{ and} (ii) \frac{\wedge^2\mathcal{U}^2}{T}|_{X_0\times q}(m) \text{ is globally generated.}$$

Therefore, there is an open covering $\{U_i\}$ of $R(2,\chi)^{ss}$ and morphisms $det_i : U_i \to R(1,\chi-(1-g))^{ss}$ such that, for any non-empty intersection $U_i \cap U_j$ if we denote by $det_{ij} = det_i|_{U_{ij}}$, then there exists $g \in PGL(n)(U_{ij})$ with the property $det_{ij} = g det_{ji}$ where $n = h^0(\frac{\wedge^2\mathcal{U}^2}{T}|_{X_0\times q}(m))$ (see [D9, Proposition 5.10]). Therefore, we get a well-defined morphism $det : R(2,\chi)^{ss} \to R(1,\chi-(1-g))^{ss}//PGL(n)$. Now the group $PGL(p(m'))$ acts on $R(2,\chi)^{ss}$. Clearly, if $q_1, q_2 \in R(2,\chi)^{ss}$ lie in the same orbit of $PGL(p(m'))$ then $det(q_1) = det(q_2)$. Hence we get a morphism $R(2,\chi)^{ss}//PGL(p(m')) \to R(1,\chi-(1-g))^{ss}//PGL(n)$ which we again denote by det. Therefore, we are done since we have already proven $R(1,\chi-(1-g))^{ss}//PGL(n) \simeq J^{\chi_1'}(X_1) \times J^{\chi_2'}(X_2)$ (cf. Theorem 2.12) □

For the next proposition we will assume χ to be odd. Let M_{12} and M_{21} be two components of $M(2,a,\chi)$ and D be their intersection. Then M_{12} and M_{21} are smooth.

Proposition 2.17. *The fibres of the morphism* $det : M(2,a,\chi) \to J_0 = J^{\chi_1'}(X_1) \times J^{\chi_2'}(X_2)$ *are the union of two smooth, irreducible projective varieties meeting transversally along a smooth divisor.*

Proof. Let $J^0(X_0)$ be the variety parametrising all isomorphism classes of line bundles L such that $deg(L|_{X_i}) = 0$, $i = 1, 2$. Then $J^0(X_0)$ acts on both the varieties $M(2,a,\chi)$ and J_0 by $F \to F \otimes L$. Let det_1 and det_2 be the restriction of det to M_{12} and M_{21}. Then det_i are $J^0(X_0)$ equivariant morphisms. Since M_{12}, M_{21} and J_0 are smooth we get the morphisms det_i are smooth. Since D consists of rank 2 non locally free sheaves D is invariant under the action of $J^0(X_0)$. By the same argument we get the fibres of $det_i|_D$ are smooth. Clearly, $det_i^{\xi} \cap D = det_i|_D^{-1}(\xi)$ for any $\xi \in J_0$. Therefore, $det_1^{-1}(\xi)$ and $det_2^{-1}(\xi)$ intersect transversally. □

Remark 2.18. The specific choice of polarisation in subsection 2.0.3 ensures that the "determinant" $\frac{\wedge^2 F}{tor}$ of a rank 2 semistable torsion free sheaf is again semistable. Note that if we choose a generic polarisation i.e. $a_1\chi$ is not an integer, then we can easily show that rank 1 torsion free but non locally free sheaves do not occur as semistable sheaves (see [NS97, page 113]). In this case we can easily show that the moduli space of rank 1 sheaves is isomorphic to the product of the Jacobian.

2.0.6 Relative moduli space and relative determinant morphism

Let $C = SpecR$ where R is a complete discrete valuation ring and $\mathcal{X} \to C$ be a flat family of proper, connected curves. We assume the generic fibre $\mathcal{X}_\eta$ is smooth and the closed fibre $\mathcal{X}_0$ is the curve X_0. We further assume that $\mathcal{X}$ is regular over $\mathbb{C}$. For any C scheme S we denote $\mathcal{X} \times_C S$ by $\mathcal{X}_S$. Fix an integer χ.

2.0.6.1 Relative moduli of rank 1, torsion free sheaves:

Fix a relatively ample line bundle $\mathcal{O}_{\mathcal{X}}(1)$ over $\mathcal{X}$ such that $\mathcal{O}_{\mathcal{X}}(1)|_{X_0}$ gives the polarisation of type (b_1, b_2). Let $\mathcal{Q}_1 \to C$ be the relative Quot scheme parametrising all rank 1 coherent quotients

$$\mathcal{O}_{\mathcal{X}}^{p(N)} \to \mathcal{L} \to 0.$$

which has the fixed Hilbert polynomial $p(n) := (n+1)\chi'$, $\chi' = \chi - (1-g)$, along the fibre of $\mathcal{X}$ and flat over C. Let $\mathcal{U}$ be the universal quotient sheaf of $\mathcal{O}_{\mathcal{X}_{\mathcal{Q}_1}}^{\oplus p(m)}$ on $\mathcal{X}_{\mathcal{Q}_1}$. Let $\mathcal{G} = Aut(\mathcal{O}_{\mathcal{X}}^{p(m)})$ be the reductive group scheme over C. Then $\mathcal{G}$ acts on $\mathcal{Q}_1$. Let $\mathcal{R}_1^{ss}$ be the open subvariety of $\mathcal{Q}_1$ consisting of those quotients $\mathcal{L}$ which are semistable along the fibre of $\mathcal{X}$ and the natural map $H^0(\mathcal{O}_{\mathcal{X}}^{p(m)}) \to H^0(\mathcal{L})$ is an isomorphism. We can construct a good quotient $\mathcal{J} := \mathcal{R}_1^{ss}//\mathcal{G}$, projective over C using GIT over arbitrary base. Also note that $(\mathcal{R}_1{}^{ss}//\mathcal{G})_t = \mathcal{R}_{1_t}{}^{ss}//\mathcal{G}_t$ for all $t \in C$ ([Ses77, Theorem 4]). Thus the general fibre $\mathcal{J}_\eta$ is the Jacobian $J^{\chi'}(\mathcal{X}_\eta)$ and by Theorem 2.12 the closed fibre $\mathcal{J}_0$ is isomorphic to $J^{\chi'_1}(X_1) \times J^{\chi'_2}(X_2)$.

2.0.6.2 Relative moduli of rank 2, torsion free sheaves

Fix a relatively ample line bundle $\mathcal{O}_{\mathcal{X}}(1)'$ over $\mathcal{X}$ such that $\mathcal{O}_{\mathcal{X}}(1)'|_{X_0} = \mathcal{O}_{X_0}(1)$ gives the polarisation of type (a_1, a_2) and such that $a_1\chi$ is not an integer. Let $\mathcal{Q}_2 \to C$ be the relative Quot scheme parametrising all rank 2 coherent quotients

$$\mathcal{O}_{\mathcal{X}}^{p(m')} \to \mathcal{E} \to 0.$$

which has the fixed Hilbert polynomial $p(n) := (c_1 + c_2)n + \chi$, where $c_i = deg(\mathcal{O}_{X_0}(1)|_{X_i}$, along the fibre of $\mathcal{X} \to C$ and flat over C. Let $\mathcal{G}' = Aut(\mathcal{O}_{\mathcal{X}}^{p(m')})$ be the reductive group scheme over C. Then $\mathcal{G}'$ acts on $\mathcal{Q}_2$. Let $\mathcal{R}_2^{ss}$ be the open subvariety of $\mathcal{Q}_2$ consisting of those quotients $\mathcal{E}$ which are semistable along the fibre of $\mathcal{X}$ and for whom the natural map $H^0(\mathcal{O}_{\mathcal{X}}^{p(N)}) \to H^0(\mathcal{E})$ is an isomorphism. It is shown in [NS97, Theorem 4.2] that a relative moduli space $\mathcal{M} := \mathcal{R}_2^{ss}//\mathcal{G}'$ exists and is projective over C using GIT over arbitrary base. Thus the general fibre $\mathcal{M}_\eta$ is the moduli space $M_{\mathcal{X}_\eta}(2, \chi)$ of rank 2, semistable sheaves with Euler characteristic χ and $\mathcal{M}_0$ is the moduli space $M(2, a, \chi)$. Note that if $\mathcal{X}$ is a regular surface, by [NS97, Remark 4.2], $\mathcal{R}_2^{ss}$ is smooth over $\mathbb{C}$. If we assume χ to be odd then $\mathcal{R}_2^{ss} = \mathcal{R}_2^{s}$. Therefore, $P\mathcal{G}'$ acts on $\mathcal{R}_2^s$ freely. Since $\mathcal{R}_2^{ss}$ is smooth we conclude that $\mathcal{M} = \mathcal{R}_2^s/P\mathcal{G}'$ is regular over $\mathbb{C}$.

Proposition 2.19. *There exists a morphism $Det : \mathcal{M} \to \mathcal{J}$ such that the following diagram commutes-*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{Det} & \mathcal{J} \\ & {\scriptstyle \pi'} \searrow \quad \swarrow {\scriptstyle \pi''} & \\ & C & \end{array} \tag{2.5}$$

Moreover, we have $Det|_{\mathcal{M}_0} = det$.

Proof. Let $\mathcal{U}''$ be the universal quotient sheaf on $\mathcal{X}_{\mathcal{R}_2{}^{ss}}$ and $\bar{T}$ be the maximal subsheaf of $\wedge^2\mathcal{U}''$ with proper support. Note that $\mathcal{R}_2^{ss}$ is a reduced scheme and $\chi(\frac{\wedge^2\mathcal{U}_q''}{tor}) = \chi - (1-g)$ for all $q \in \mathcal{R}_2^{ss}$. Now $\chi(\frac{\wedge^2\mathcal{U}_q''}{tor}) \simeq \frac{\wedge^2\mathcal{U}''}{\bar{T}}|_{\mathcal{X}_q}$.Thus $\chi(\frac{\wedge^2\mathcal{U}_q''}{tor}) = \chi - (1-g)$ for all $q \in \mathcal{R}_2^{ss}$. Therefore, the Hilbert polynomial of $\frac{\wedge^2\mathcal{U}''}{\bar{T}}|_{\mathcal{X}_q}$ is independent of q. Since $\mathcal{R}_2^{ss}$ is reduced the above fact implies that $\frac{\wedge^2\mathcal{U}''}{\bar{T}}$ is flat over $\mathcal{R}_2^{ss}$. Now using similar arguments as in Proposition 2.16 we get a morphism $Det' : \mathcal{R}_2^{ss} \to \mathcal{J}$. This morphism is compatible with the action of $P\mathcal{G}'$ on $\mathcal{R}_2^{ss}$. Thus we get a morphism $Det : \mathcal{M} \to \mathcal{J}$ such that $Det|_{\mathcal{M}_0} = det$. □

Remark 2.20. By similar arguments as in the proof of Proposition 2.17 we can show that Det is a smooth morphism. Fix a section $\sigma : C \to \mathcal{J}$ such that $\sigma(0) = \xi$. This corresponds to a line bundle $\mathcal{L}$ over $\mathcal{X}$ such that $\mathcal{L}|_{\mathcal{X}_0} = \xi$. Let us denote $Det^{-1}(\sigma(C))$ by $\mathcal{M}_{\mathcal{L}}$. Since both the varieties $\mathcal{M}$ and $\mathcal{J}$ are smooth we conclude that $\mathcal{M}_{\mathcal{L}}$ is smooth over $\mathbb{C}$.

2.0.7 Moduli space of triples

Fix $\xi \in J_0$ and let $det^{-1}(\xi) := M_{0,\xi}$. In this subsection we will discuss a different description of the moduli spaces $M(2, a, \chi)$ and $M_{0,\xi}$ in terms of certain moduli space of triples glued along a certain divisor. This description is given in section 5 of the article [NS97]. This description will be useful for the cohomology computations later.

The following facts are well known. For completeness we shall indicate a proof.

Fact 2.21. Let (X, x) be a smooth,projective curve together with a marked point x and $(E, 0 \subset F^2E(x) \subset E(x))$ be a parabolic vector bundle with weights $0<\beta_1<\beta_2<1$. Suppose the weights satisfy $|\beta_1 - \beta_2|<\frac{1}{2}$. Then we have-
(a) E is parabolic semistable implies E is parabolic stable.
(b) E is parabolic semistable implies E is semistable.
(c) If E is stable then any quasi parabolic structure $(E, 0 \subset F^2E(x) \subset E(x))$ is parabolic semistable with respect to the weights $0<\beta_1<\beta_2<1$.

Proof. From our assumption on weights we get that $|\frac{\beta_1+\beta_2}{2} - \beta_i|<\frac{1}{2}$ for $i = 1, 2$. Suppose E is strictly parabolic semistable. Let L be a parabolic line subbundle of E. Then we have-

$$deg(L) = \frac{deg(E)}{2} + \frac{\beta_1 + \beta_2}{2} - \beta_i.$$

Since $|\frac{\beta_1+\beta_2}{2} - \beta_i|<\frac{1}{2}$ and $deg(L)$ is an integer this is not possible. This completes the proof of (a). Let L be a line subbundle of E. The parabolic stability of E implies

$$deg(L)<\frac{deg(E)}{2} + \frac{\beta_1 + \beta_2}{2} - \beta_i.$$

Therefore, $deg(L)<\frac{deg(E)}{2} \pm \frac{1}{2}$. Since $deg(L)$ is an integer the above inequality will imply $deg(L) \leq \frac{deg(E)}{2}$. This completes the proof of (b). Let L be a subbundle of

E. If $L(x) \cap F^2E(x) \neq 0$ then we associate the weight β_1 otherwise we associate the weight β_2. Now as E is stable we have

$$deg(L) < \frac{deg(E)}{2}.$$

Since $|\frac{\beta_1+\beta_2}{2} - \beta_i| < \frac{1}{2}$ and $deg(L)$ is an integer we conclude that

$$deg(L) < \frac{deg(E)}{2} + \frac{\beta_1 + \beta_2}{2} - \beta_i.$$

This completes the proof of (c). □

The following results is proved in [NS97]

Fact 2.22. Let $(F_1, F_2, A) \in \overrightarrow{C}$ (resp. $(F_1', F_2', B) \in \overleftarrow{C}$) be a rank 2, (a_1, a_2)-semistable triple and the Euler characteristic $\chi(F_i)$, $i = 1, 2$, satisfy the inequality 2.4(resp. the inequality 2.7), then F_i(resp. F_i') are semistable over X_i for $i = 1, 2$ (see [NS97, Theorem 5.1]).

Conversely, we have the following:

Lemma 2.23. *Let F_i be rank 2 semistable bundles over X_i and the Euler characteristic $\chi(F_i)$, $i = 1, 2$, satisfies the inequalities 2.4. Let $A : F_1(p) \to F_2(p)$ be a linear map and $rk(A) = 2$, then $(F_1, F_2, A) \in \overrightarrow{C}$ is (a_1, a_2)-semistable. Moreover, if F_1 and F_2 are both stable then (F_1, F_2, A) is (a_1, a_2)-semistable if $rk(A) \geq 1$.*

Proof. Case 1: Let $rk(A) = 2$ The proof of the statement (1) follows from [Bar14, Lemma 3.1.12 page 39]. For the sake of completeness, we will give a proof here.

It is shown in [NS97, Lemma 5.2] that it is enough to check the semistability condition for the subtriples of the form (G_1, G_2, B), where $G_i = 0$ or a subbundle of F_i or of the form $F_i(-p) \subset G_i \subset F_i$, for $i = 1, 2$. Since $rk(A) = 2$, we can list out the triples that can not occur as subtriple of (F_1, F_2, A). Those subtriples are of the form:

(i) $(L_1, 0, 0)$, $\quad (ii)(F_1', L_2, 0)$, $\quad or \quad (iii)(L_1, F_2', 0)$,

where L_i are line subbundles of F_i and F_i' are the Hecke modifications of F_i for $i = 1, 2$.

If (G_1, G_2, B) is a subtriple of (F_1, F_2, A), where G_i is a subbundle of F_i, then we can easily check that:

$$\chi((G_1, G_2, B)) \leq \frac{\chi((F_1, F_2, A))}{2}.$$

Therefore, it is enough to check that the semistability condition for the subtriples is of the form:

- $(F_1(-p), 0, 0)$
- $(F_1(-p), L_2, 0)$
- $(0, F_2(-p), 0)$,
- $(L_1, F_2(-p), 0)$,
- $(0, L_2, 0)$

where L_i is a rank 1 subbundle of F_i. Now we will check the semistability condition for the subtriple of the form $(F_1(-p), 0, 0)$:

$$\begin{aligned}
\frac{\chi((F_1(-p), 0, 0))}{2a_1} - \frac{\chi}{2} &= \frac{\chi(F_1) - 2}{2a_1} - \frac{\chi}{2} \\
&= \frac{1}{2}\left[\frac{\chi(F_1) - (a_1\chi + 1) - 1}{a_1}\right] \\
&< 0 \quad (\text{since} \quad \chi(F_1) < a_1\chi + 1 \text{ by } 2.4).
\end{aligned}$$

We check the semistability condition for the triple $(F_1(-p), L_2, 0)$

$$\begin{aligned}
\mu(F_1(-p), L_2, 0) &= \frac{\chi(F_1) - 2 + \chi(L_2) - 1}{2a_1 + a_2} \\
&\leq \frac{\chi(F_1) + \frac{\chi(F_2)}{2} - 3}{2a_1 + a_2} \qquad \text{by semistability of } F_2 \\
&= \frac{2\chi(F_1) + \chi(F_2) - 6}{4a_1 + 2a_2} = \frac{2\chi(F_1) + \chi(F_2) - 6}{2(1 + a_1)} \text{ since } a_1 + a_2 = 1 \\
&\leq \frac{2(a_1\chi + 1) + (a_2\chi + 2) - 6}{2(1 + a_1)} \qquad \text{by } 2.4 \\
&= \frac{(2a_1 + a_2)\chi - 2}{2(1 + a_1)} = \frac{\chi}{2} - \frac{1}{1 + a_1} < \frac{\chi}{2} = \mu(F_1, F_2, A)
\end{aligned}$$

Similarly, for the subtriple $(0, F_2(-p), 0)$, $(L_1, F_2(-p), 0)$, $(0, L_2, 0)$ one can easily check the semistability condition. Therefore, we conclude that (F_1, F_2, A) is (a_1, a_2)-semistable.

Now suppose $rk(A) = 1$. In this case we need both F_i to be stable.

Since $rk(A) = 1$ we get a parabolic structure on F_1 given by $0 \subset ker(A) \subset F_1(p)$ and a parabolic structure on $F_2(p)$ given by $0 \subset Im(A) \subset F_2(p)$. By Fact 2.21 (c) we conclude that the above two quasi parabolic structure are parabolic stable with respect to the weights $0 < \frac{a_1}{2} < \frac{a_2}{2} < 1$. Thus by [NS97, Theorem 6.1] we get that (F_1, F_2, A) is semistable.

We will also give a direct proof: As both F_i are stable-

$$\chi(L_i) \leq \frac{\chi(F_i)}{2} - \frac{1}{2}$$

for any subbundle L_i of F_i, $i = 1, 2$. Using this and the inequality 2.4 we get -

$$\chi(L_1) < \frac{a_1 \chi}{2}.$$

Thus, for the subtriple $(L_1, 0, 0)$, we have $\mu(L_1, 0, 0) < \mu(F_1, F_2, A)$.

For the subtriple of the form $(0, L_2, 0)$, $(F_1', L_2, 0)$ and $(L_1, F_2', 0)$ we can easily check the semistabilty condition. It only remains to check the semistabilty condition for subtriples of the form (F_1, L_2, A) and $(L_1, F_2, 0)$. Note that the above occurs as subtriples if $Ker(A) = L_1(p)$ and $Im(A) = L_2(p)$. Using the inequality 2.4 and

$$\chi(L_i) \leq \frac{\chi(F_i)}{2} - \frac{1}{2}.$$

we check the semistability condition for the subtriple (F_1, L_2, A) and $(L_1, F_2, 0)$:

$$\begin{aligned}
\mu(F_1, L_2, A) &= \frac{\chi(F_1) + \chi(L_2) - 1}{2a_1 + a_2} \\
&\leq \frac{2\chi(F_1) + \chi(F_2) - 3}{2(1 + a_1)} \\
&= \frac{\chi + \chi(F_1) - 1}{2(1 + a_1)} \\
&< \frac{\chi + a_1\chi + 1 - 1}{2(1 + a_1)} = \frac{\chi}{2}
\end{aligned}$$

and

$$\begin{aligned}\mu(L_1, F_2, 0) &= \frac{\chi(L_1) + \chi(F_2) - 2}{a_1 + 2a_2} \\ &\leq \frac{\chi(F_1) + 2\chi(F_2) - 5}{2(1 + a_2)} \\ &= \frac{\chi + \chi(F_2) - 3}{2(1 + a_2)} \\ &< \frac{\chi + a_2\chi + 2 - 1}{2(1 + a_2)} < \frac{\chi}{2} - \frac{1}{2(1 + a_1)} < \frac{\chi}{2}\end{aligned}$$

□

Remark 2.24. The same results hold true for the triples in the other direction i.e. if F_i are semistable over X_i, $i = 1, 2$ satisfying the inequality 2.6 and $rk(A) = 2$ then the triple $(F_1, F_2, A) \in \overleftarrow{C}$ is (a_1, a_2)-semistable. Moreover, if F_i are stable and $rk(A) \geq 1$ then $(F_1, F_2, A) \in \overleftarrow{C}$ is (a_1, a_2)-semistable.

(I) **Semistable triple of type (I)**: We say a rank 2, (a_1, a_2)-semistable triple $(F_1, F_2, A) \in \overrightarrow{C}$ is of type (I) if $\chi(F_i)$, $i = 1, 2$, satisfy the following inequalities:

$$a_1\chi < \chi_{X_1}(F_1) < a_1\chi + 1 \ , \ a_2\chi + 1 < \chi_{X_2}(F_2) < a_2\chi + 2 \tag{2.6}$$

and $rk(A) \geq 1$.

(II) **Semistable triple of type (II)**: We say a (a_1, a_2)-semistable triple $(F_1, F_1, B) \in \overleftarrow{C}$ is of type (II) if $\chi(F_i)$, $i = 1, 2$ satisfy the following inequalities:

$$a_1\chi + 1 < \chi_{X_1}(F_1') < a_1\chi + 2 \ , \ a_2\chi < \chi_{X_2}(F_2') < a_2\chi + 1 \tag{2.7}$$

and $rk(B) \geq 1$.

Let S be a scheme. We say $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{A})$ is a family of triples parametrised by S if $\mathcal{F}_i$'s are locally free sheaves on $X_i \times S$, $i = 1, 2$ and $\mathcal{A} : \mathcal{F}_1|_{p\times S} \to \mathcal{F}_2|_{p\times S}$ is a $\mathcal{O}_S$-module homomorphism of locally free sheaves.

Remark 2.25. Given a family of triples $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{A})$ parametrised by S we can associate a family of torsion free sheaves $\mathcal{F}$ parametrised by S i.e. a coherent sheaf $\mathcal{F}$ on $X_0 \times S$ which is flat over S such that $\mathcal{F}_s$ is torsion free for all $s \in S$. The association is the following: Let G be the locally free subsheaf of $\mathcal{F}_1|_{p\times S} \oplus \mathcal{F}_2|_{p\times S}$ generated by the graph of the homomorphism $\mathcal{A}$ and $\mathcal{L}_S := \frac{\mathcal{F}_1|_{p\times S} \oplus \mathcal{F}_2|_{p\times S}}{G}$. Consider the exact sequence-

$$0 \to \mathcal{F} \to \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{L}_S \to 0.$$

Since, $\mathcal{F}_1 \oplus \mathcal{F}_2$ and $\mathcal{L}_S$ are both flat over S. Hence $\mathcal{F}$ is flat over S.

In [NS97, Theorem 5.3] it is shown that there is a smooth, irreducible projective variety which has the coarse moduli property for families of semistable triples of type I. We denote this space by M_{12}. By the same construction one can construct another smooth, irreducible, projective variety which has the coarse moduli property of semistable triples of type (II). We denote this space by M_{21}. Let

$$D_1 := \{[(F_1, F_2, A) \in M_{12} \mid rk(A) = 1\}.$$

and

$$D_2 := \{[(F_1', F_2', B)] \in M_{21} \mid rk(B) = 1\}.$$

Then, by [NS97, Theorem 6.1] it follows that D_1 (resp. D_2) is a smooth divisor in M_{12}(resp. M_{21}). Now if $(F_1, F_2, A) \in \overrightarrow{C}$ and $rk(A) = 1$, then by Remark 2.6, we get a unique triple $(F_1', F_2', B) \in \overleftarrow{C}$ such that $rk(B) = 1$ and $\chi(F_1') = \chi(F_1) + 1$, $\chi(F_2') = \chi(F_2) - 1$. Therefore, this association defines a natural isomorphism between D_1 and D_2. Let us denote this isomorphism by Ψ and M_0 be the variety obtained by identifying the closed subschemes D_1 and D_2 via the isomorphism Ψ. Now by Remark 2.25 we get a morphism $f_1 : M_{12} \to M(2, a, \chi)$ (resp. $f_2 : M_{21} \to M(2, a, \chi)$) by associating a triple (F_1, F_2, A) to the corresponding torsion free sheaf F. Clearly f_1 and f_2 are compatible with the gluing morphism Ψ. Thus we get a morphism $M_0 \to M(2, a, \chi)$. This morphism is bijective. Also this morphism induces an isomorphism on the dense open subvariety of M_0 consisting of isomorphism classes of triples $[(F_1, F_2, A)]$ such that $rk(A) = 2$. Therefore it is a birational morphism. Thus by [Vit89, Theorem 2.4] the variety M_0 is isomorphic to the moduli space $M(2, a, \chi)$ as the latter space is projective and seminormal (see Remark 2.15) without any one dimensional component.

Let S be a finite type scheme and $\chi_i' = \chi_i - (1 - g_i)$. Given a family of type (I), (a_1, a_2) semistable triples $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{A})$ parametrised by S we get two families of line bundles $\wedge^2 \mathcal{F}_i$ over $X_i \times S$, $i = 1, 2$. Thus by the universal property of $J^{\chi_i'}(X_i)$ we get a morphism

$$det_1 : M_{12} \to J_0 := J^{\chi_1'}(X_1) \times J^{\chi_2'}(X_2).$$

such that $det_1((F_1, F_2, A)) = (\wedge^2 F_1, \wedge^2 F_2)$ for all closed points $(F_1, F_2, A) \in M_{12}$. Similarly, we get another morphism:

$$det_2 : M_{21} \to J'_0 := J^{\chi'_1+1}(X_1) \times J^{\chi'_2-1}(X_2).$$

such that $det_2((F_1, F_2, A)) = (\wedge^2 F_1, \wedge^2 F_2)$ for all closed points $(F_1, F_2, A) \in M_{21}$.

Lemma 2.26. *The fibres of det_i are smooth and the fibres of det_i intersect D_i transversally, $i = 1, 2$.*

Proof. The group $J^0(X_1) \times J^0(X_2)$ acts on M_{21} (resp. M_{21}) by $(F_1, F_2, A) \mapsto (F_1 \otimes L_1, F_2 \otimes L_2, A)$ and on J_0 (resp. J'_0) by $(M_1, M_2) \mapsto (M_1 \otimes L_1, M_2 \otimes L_2)$ where $(L_1, L_2) \in J^0(X_1) \times J^0(X_2)$. The morphism det_1 (resp. det_2) is clearly compatible with the above actions. Thus det_1 (resp. det_2) is smooth. As M_{12} (resp. M_{21}) and J_0 (resp. J'_0) are smooth, the fibres of det_1 (resp. det_2) are smooth. Clearly, the divisor D_1 (resp. D_2) is invariant under the above action. Therefore, $det_i|_{D_i}$ are smooth, $i = 1, 2$. Thus the fibres of $det_i|_{D_i}$ are also smooth. Clearly, the intersection of a fibre of det_i with D_i is the fibre of $det_i|_{D_i}$. Hence we are done. □

Fix $\xi = (\xi_1, \xi_2) \in J^{\chi'_1}(X_1) \times J^{\chi'_2}(X_2)$. Let $det_1^{-1}(\xi) := M_{12}^{\xi}$ and $det_2^{-1}(\xi') := M_{21}^{\xi'}$ where $\xi' = (\xi(p), \xi(-p))$. By Lemma 2.26 the fibre $det_1^{-1}(\xi)$ (resp. $det_2^{-1}(\xi')$) intersects D_1 (resp. D_2) transversally. Hence $D_1^{\xi} := det_1^{-1}(\xi) \cap D_1$ and $D_2^{\xi'} := det_2^{-1}(\xi') \cap D_2$. Let $M_{0,\xi}$ be the closed subvariety of M_0 obtained by gluing M_{12}^{ξ} and $M_{21}^{\xi'}$ along the closed subschemes D_1^{ξ} and $D_2^{\xi'}$ via the isomorphism Ψ.

Let *det* be the morphism defined in Proposition 2.16. We can easily show that $det^{-1}(\xi)$, $\xi \in J_0$ is isomorphic to the variety $M_{0,\xi}$. In the next chapter we will compute some of the cohomology groups of $M_{0,\xi}$.

2.0.7.1 Notation

Henceforth, we will denote by $M_{0,\xi}$, the moduli space of rank 2, (a_1, a_2)- semistable bundles with $det \simeq \xi$ and its components by M_{12} and M_{21}. We also denote the smooth divisor D_1^{ξ} in M_{12} by D_1 and the smooth divisor D_2^{ξ} in M_{21} by D_2.

We conclude this section by proving a geometric fact about the moduli space M_{12} (resp. M_{21}).

Lemma 2.27. *The moduli space M_{12} (resp, M_{21}) is a unirational variety.*

Proof. To prove M_{12} is unirational we can assume, after tensoring by line bundles, it consists of all triples (F_1, F_2, A), where F_i is semistable over X_i such that $deg(F_i)>2(2g_i - 1)$ $i = 1, 2$. Then, any such F_i can be obtained as an extension:

$$0 \to \mathcal{O}_{X_i} \to F_i \to \xi_i \to 0,$$

where $\xi_i = det(F_i)$ for $i = 1, 2$. The exact sequences of this type are classified by $V_{\xi_i} := Ext^1(\mathcal{O}_{X_i}, \xi_i) = H^1(X_i, \xi_i^*)$. Let $\mathcal{E}_i$ be the universal extension over $X_i \times V_{\xi_i}$. We denote the restriction $\mathcal{E}_{i_{|p \times V_{\xi_i}}}$ by $\mathcal{E}_{i_p}$. Clearly, $Hom(\mathcal{E}_1, \mathcal{E}_2)$ parametrises a family of triples in the sense we have defined family of triples and if (F_1, F_2, A) is a triple corresponding to the closed point $A \in Hom(\mathcal{E}_1, \mathcal{E}_2)$ then F_i's are the extensions of the type described before. Now as the F_i's are semistable if we choose an isomorphism $A : F_1(p) \to F_2(p)$ then by Lemma 2.22, (F_1, F_2, A) is semistable. Thus we conclude that the set of points W where the corresponding triple is semistable is a nonempty Zariski open set of $Hom(\mathcal{E}_1, \mathcal{E}_2)$. Therefore, by the coarse moduli property of M_{12}, we get a morphism from W to M_{12}. Clearly the morphism $W \to M_{12}$ is surjective. Hence, M_{12} is a unirational variety. The same argument shows the moduli space M_{21} is also a unirational variety. □

Chapter 3

Topology of $M_{0,\xi}$

3.1 Introduction

In this chapter, our main aim is to outline a strategy to compute the cohomology groups of $M_{0,\xi}$ and compute explicitly the third cohomology group. We make the following convention: Let X be a topological space. By $H^k(X)$ we mean the singular cohomology groups of X with the coefficients in $\mathbb{Q}$, $k \geq 0$. Whenever we obtain any results for other coefficients, e.g $\mathbb{Z}$, we will specifically mention it. Suppose X and Y are varities over $\mathbb{C}$. Whenever we say $X \to Y$ is a topological fibre bundle, we assume the underlying topology of X and Y to be the complex analytic topology.

3.2 Some Topological Facts About Fixed Determinant Moduli Space over Smooth Projective Curve

Let Y be a smooth,projective curve of genus $g_Y \geq 2$ and M_Y be the moduli space of rank 2 semistable bundles with fixed determinant. The cohomology groups of M_Y are quite well studied in the literature. When the determinant is odd M_Y is a smooth projective variety of dimension $3g_Y - 3$ and the cohomology groups with integral coefficients are completely known. When the determinant is even M_Y need not be smooth. In fact it is known that the singular locus of M_Y is precisely the complement $M_Y \setminus M_Y^s$ if $g_Y \geq 3$ where M_Y^s is the open subset consisting of stable bundles (see [NR69, Theorem 1]). In this case also the Betti numbers are determined in the work of [BS07]. We will summarize some of the results concerning the cohomology groups of M_Y in both the cases i.e. odd determinant and even determinant:

Lemma 3.1. 1. *Let* M_Y *be the moduli space of rank* 2 *semistable bundles with odd determinant. Then* M_Y *is a smooth, projective rational variety ([New75]) and hence it is simply connected and* $H^3(M_Y, \mathbb{Z})_{tor} = 0$*. Furthermore,* $b_1(M_Y) = 0$*,* $b_2(M_Y) = 1$*,* $b_3(M_Y) = 2g_Y$*, where* b_i *are the Betti numbers ([New67]).*

2. *Let* M_Y *be the moduli space of rank* 2 *semistable bundles with even determinant. Then* M_Y^s *is a simply connected variety ([BBGN07, Proposition 1.2]). Furthermore, we have* $b_1(M_Y) = 0$*,* $b_2(M_Y) = 1$ *and* $b_3(M_Y^s) = 2g_Y$*, where* b_i *are the Betti numbers ([Nit89], [BS07, Section 3]).*

3.3 Cohomology Computaion Of $M_{0,\xi}$

In the last chapter we have seen that the moduli space $M_{0,\xi}$ is the union of two smooth projective varieties M_{12} and M_{21}. In this section our first aim is to compute some of the cohomology groups of M_{12} and M_{21} and then using Mayer-Vietris long exact sequence we will compute the cohomology groups of $M_{0,\xi}$. In the final section we will mainly compute the third cohomology group of $M_{0,\xi}$.

Let M_1 (resp. M_1') be the moduli space of rank 2, semistable bundles over X_1 with $det \simeq \xi_1$ (resp. with $det \simeq \xi_1(p)$) and M_2 (resp. M_2') be the moduli space of rank 2, semistable bundes over X_2 with $det \simeq \xi_2$ (resp. $\det \simeq \xi_2(-p)$) where ξ_i's are line bundles of degree $d_i = \chi_i - 2(1 - g_i)$ for $i = 1, 2$ and the integers χ_1, χ_2 satisfy the inequality 2.4. Since χ is odd, one of the integer in the pair (d_1, d_2) is odd and the other is even. We assume that d_1 is odd and d_2 is even. Therefore, M_1 and M_2' are smooth projective varieties. Let M_2^s be the open subvariety of M_2 consisting of all the isomorphism classes of stable bundles over X_2 and $M_1'^s$ be the open subvariety of M_1' consisting of all the isomorphism classes of stable bundles over X_1. Note that $M_2 \setminus M_2^s$ is precisely the singular locus of M_2 if $g_2 \geq 3$ and $M_1' \setminus M_1'^s$ is precisely the singular locus of M_1' if $g_1 \geq 3$.

Let us denote the open subvariety $M_1 \times M_2^s$ of $M_1 \times M_2$ by B. We will show the following,

Proposition 3.2. *There is a surjective morphism* $p : M_{12} \to M_1 \times M_2$*. Moreover,* $p : P \to B$ *is a topological* $\mathbb{P}^3$*-bundle where* $P := p^{-1}(B)$*.*

Proof. Let S be a finite type scheme and $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{A})$ be a family of triples parametrised by S such that $(\mathcal{F}_{1_s}, \mathcal{F}_{2_s}, \mathcal{A}_s)$ is (a_1, a_2)-semistable of type (I) for all $s \in S$ where $\mathcal{F}_{i_s} := \mathcal{F}_i|_{X_0 \times s}$. We also assume $\wedge^2 \mathcal{F}_{i_s} \simeq \xi_i$, $i = 1, 2$. Then by Fact 2.22 $\mathcal{F}_{i_s}$, $i = 1, 2$, are semistable for all $s \in S$. Thus we get a morphism $p : M_{12} \to M_1 \times M_2$. Let $([F_1], [F_2]) \in M_1 \times M_2$. Choose any isomorphism $A : F_1(p) \to F_2(p)$. Then, by Fact 2.23 (F_1, F_2, A) is (a_1, a_2)-semistable. Therefore, p is surjective.

Now we show that $p : P \to B$ is a topological $\mathbb{P}^3$-bundle. Let $b = ([F_1], [F_2]) \in B$. Our first claim is the fibre $p^{-1}(b)$ is homeomorphic to $\mathbb{P}Hom(F_1(p), F_2(p)) \simeq \mathbb{P}^3$. Let $A \in Hom(F_1(p), F_2(p))$ and $A \neq 0$. Since both F_i's are stable, by Lemma 2.23, (F_1, F_2, A) is (a_1, a_2)-stable. Thus we get a morphism $i_b : Hom(F_1(p), F_2(p)) \setminus 0 \to M_{12}$. Clearly, $i_b(Hom(F_1(p), F_2(p))) \setminus 0) = p^{-1}(b)$. Note that (F_1, F_2, A) and $(F_1, F_2, \lambda A)$ are isomorphic for all $\lambda \in \mathbb{C}^*$. Thus i_b descends to a morphism i_b : $\mathbb{P}Hom(F_1(p), F_2(p)) \to p^{-1}(b)$. Now we show that i_b is injective. Then the claim will follow. Let $A, B \in \mathbb{P}Hom(F_1(p), F_2(p))$ are distinct points. Then the triples (F_1, F_2, A), $F_1, F_2, B)$ are non isomorphic. Suppose, (F_1, F_2, A) and (F_1, F_2, B) are isomorphic as triples. Then there are isomorphisms $\phi_i : F_i \to F_i$, $i = 1, 2$ such that we have the following commutative diagram:

$$\begin{array}{ccc} F_1(p) & \xrightarrow{\phi_1(p)} & F_1(p) \\ \downarrow A & & \downarrow B \\ F_2(p) & \xrightarrow{\phi_2(p)} & F_2(p) \end{array} \tag{3.1}$$

Since F_i are stable the only isomorphisms of F_i are λI for some scalor λ. Thus we have $\phi_i(p) = \lambda_i I$, $i = 1, 2$. From the commutativity of the above diagram we get $B\lambda_1 = \lambda_2 A$. Thus $B = \lambda_1^{-1}\lambda_2 A$. Hence a contradiction as A and B are district in $\mathbb{P}Hom(F_1(p), F_2(p))$. Therefore, the morphism is injective. Since the fibres of $p : P \to B$ are compact, $p : P \to B$ is a proper, analytic map.

Our next claim is that the induced map $dp : T_F \to T_{p(F)}$ at the level of Zariski tangent space is surjective for all $F = (F_1, F_2, A) \in P$. Let $(F_1, F_2) \in B$. Since F_i are both stable, $i = 1, 2$, the Zariski tangent space $T_{F_i} \simeq H^1(End(F_i))_0$ where $H^1(End(F_i))_0 = Ker(tr^1 : H^1(End(F_i)) \to H^1(\mathcal{O}_{X_i}))$ and tr^1, the trace homomorphism (see [HL10, Theorem 4.5.4]). Thus the tangent space $T_{(F_1,F_2)}B \simeq H^1(End(F_1))_0 \times H^1(End(F_2))_0$. Now a cocycle in $H^1(End(F_i))$ corresponds to a locally free sheaf $\mathcal{F}_i$ over $X_i \times D$ such that $\mathcal{F}_{it_0} \simeq F_i$ where $D = Spec\frac{\mathbb{C}[\varepsilon]}{\varepsilon^2}$ and $t_0 = (\varepsilon)$. Choose an isomorphism $\mathcal{A} : F_1(p) \to F_2(p)$. Then clearly, A lifts to a $\mathcal{O}_D$-module homomorphism

$\mathcal{A} : F_1|_{p\times D} \to F_2|_{p\times D}$. Thus we get a triple $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{A})$ parametrised by D such that $(\mathcal{F}_{1t_0}, \mathcal{F}_{2t_0}, \mathcal{A}_{t_0}) = (F_1, F_2, A)$. By the coarse moduli property of M_{12} we get a morphism $x : D \to M_{12}$ such that $x(t_0) = (F_1, F_2, A)$. In otherwords we get a point of the Zariski tangent space at (F_1, F_2, A). Thus dp is surjective. Therefore, p is a proper, surjective, holomorphic submersion. Hence, $p : P \to B$ is a topological $\mathbb{P}^3$-bundle. □

Remark 3.3. By the same arguments as before we get a morphism $p' : M_{21} \to M_1' \times M_2'$, where M_1' is the moduli space of rank 2 semistable bundles over X_1 with $det \simeq \xi_1'$ and M_2' be the moduli space of rank 2 semistable bundles over X_2 with $det \simeq \xi_2'$, where $\xi_1' := \xi_1(p)$ and $\xi_2' := \xi_2(-p)$. Moreover, $p' : \overline{P} \to B'$ is a $\mathbb{P}^3$-fibration where $B' = M_1^s \times M_2$ and $\overline{P} = p'^{-1}(B')$.

3.3.1 Codimension computations

In the following proposition we compute the codimension of the complement of the open subvariety P in M_{12} (resp. the complement of $\overline{P}$ in M_{21}).

Let K' denote the complement of P in M_{12} and K_2' denote the complement of $\overline{P}$ in M_{21}. Then we have

Proposition 3.4.

(a) *Codim*$(K', M_{12}) = g_2 - 1$. *where* g_2 *is the genus of* X_2
(b) *Codim*$(K_2', M_{21}) = g_1 - 1$ *where* g_1 *is the genus of* X_1.

Proof. We will only show (a). The proof of (b) is similar. Note that if $(F_1, F_2, A) \in K'$ then F_2 is a strictly semistable bundle on X_2. Therefore, $K' = p^{-1}(M_1 \times K)$, where $K = M_2 \setminus M_2^s$ and p is the morphism as in Proposition 3.2. . Now $F_2 \in K$ if and only if there is a short exact sequence

$$0 \to L_2 \to F_2 \to L_1 \to 0,$$

for some line bundles L_1, L_2 with $deg(L_i) = \frac{d_2}{2}$, $i = 1, 2$. Clearly, $L_1 \otimes L_2 \simeq det(F_2) \simeq \xi_2$. Thus K consists of all S-equivalance classes $[L_1 \oplus L_2]$ of semistable bundles on X_2 where $L_1, L_2 \in J^{d_2'}(X_2)$, $d_2' = \frac{d_2}{2}$ such that $L_1 \otimes L_2 \simeq \xi_2$. Let K^0 be the subset of K consisting of all S- equivalence classes $[L_1 \oplus L_2]$ such that $L_1 \not\cong L_2$. Then by [NR69, Lemma 4.3] K^0 is an open and dense subset of K. Let $K'' = p^{-1}(M_1 \times K^0)$. Then K''

is open and dense in K'. Therefore, we get $dim(K') = dim(K'')$. Now we will find a parameter variety of isomorphism classes of all (a_1, a_2)-semistable triples (F_1, F_2, A) where $F_1 \in M_1$ and $F_2 \in \mathbb{P}Ext^1(L_1, L_2)$ for some $L_i \in J^{d_2'}(X_2)$, $i = 1, 2$ with $L_1 \not\cong L_2$ and show that this parameter variety has same dimension as K''.

Let $J^{\xi_2} = \{(L_1, L_2) \in J^{d_2'}(X_2) \times J^{d_2'}(X_2) \mid L_1 \otimes L_2 \simeq \xi_2\}$. Note that $J^{d_2'}(X_2)$ is isomorphic to J^{ξ_2} by $L \mapsto (L, \xi_2 \otimes L^{-1})$. Therefore, J^{ξ_2} is a closed subvariety of $J^{d_2'}(X_2) \times J^{d_2'}(X_2)$ of dimension g_2. Let $J' = \{(L_1, L_2) \in J^{d_2}(X_2) \times J^{d_2'}(X_2) \mid L_1 \not\cong L_2\}$. Then, clearly J' is an open and dense subvariety of $J^{d_2'}(X_2) \times J^{d_2'}(X_2)$. Let $J'^{\xi_2} := J' \cap J^{\xi_2}$.

We will construct a projective bundle $\mathbb{P}$ over J' such that the fibre over a point $(L_1, L_2) \in J'$ is isomorphic to $\mathbb{P}Ext^1(L_1, L_2)$: Let $\mathcal{L}$ be the Poincare line bundle over $X_2 \times J^{d_2'}(X_2)$ and $\mathcal{L}_i := (id \times p_i)^*\mathcal{L}$ where $p_i : J^{d_2'}(X_2) \times J^{d_2'}(X_2) \to J^{d_2'}(X_2)$ is the ith projection for $i = 1, 2$. Then $V := R^1 p_{J'_*} Hom(\mathcal{L}_1, \mathcal{L}_2)$ is a locally free sheaf of rank $g_2 - 1$ over J' where $p_{J'} : X_2 \times J' \to J'$ is the projection. Let $\mathbb{P}$ over J' be the projective bundle associated to V. Then the fibre over a point $(L_1, L_2) \in J'$ is isomorphic to $\mathbb{P}Ext^1(L_1, L_2)$. Let $P' = \mathbb{P}|_{J'^{\xi_2}}$. Let $\mathcal{G}$ be the universal extension over $X_2 \times P'$ (see [NR69, Proposition 3.1]) and F be a universal bundle over $X_2 \times M_1$ (note that F exists as the degree and rank of the vector bundles in M_1 are coprime).

Let $\mathcal{G}_p := \mathcal{G}|_{p \times P'}$ and $F_p := F|_{p \times M_1}$. Clearly, $Hom(F_p, G_p)$ parametrises a family of triples of type (I) such that every closed point in $Hom(F_p, G_p)$ corresponds to a triple (F_1, F_2, A) where $F_1 \in M_1$ and $F_2 \in P'$. Note that if $E \in \mathbb{P}Ext^1(L_1, L_2)$ then $Aut(E) \simeq \mathbb{C}^*$ whenever $L_1 \not\cong L_2$ (see [NR69, Lemma 4.1]). Let $A, B \in \mathbb{P}Hom(F_p, \mathcal{G}_p)$ be two distinct closed points and (F_1, F_2, A), (G_1, G_2, B) be the corresponding triples. Then (F_1, F_2, A) and (G_1, G_2, B) are non isomorphic. This follows from the two facts: if $E_1 \in M_1$ and $E_2 \in P'$ then $Aut(E_i) \simeq \mathbb{C}^*$. If $E_1, E_2 \in \mathbb{P}Ext^1(L_1, L_2)$ are distinct then E_1 and E_2 are non isomorphic ([NR69, Lemma 3.3]). Let K_1 be the subset of $\mathbb{P}Hom(F_p, \mathcal{G}_p)$ whose closed points correspond to the triples $(F_1, \mathcal{G}_2, A)$ such that $rk(A) = 2$. Then K_1 is an open subset in $\mathbb{P}Hom(F_p, \mathcal{G}_p)$. Note that by Lemma 2.23, any closed point of K_1 is semistable. Therefore, by the coarse moduli property of M_{12}, we get a morphism $i_K : K_1 \to M_{12}$. By the above discussions i_K is injective. Clearly, the image $i_K(K_1)$ is dense in K'' since if $(F_1, F_2, A) \in K'' \setminus i_K(K_1)$ then $F_2 \simeq L_1 \oplus L_2$ for some $L_1, L_2 \in J^{d_2'}(X_2)$. Therefore, $dim(K_1) = dim(K'')$

We have $\dim(M_1) = 3g_1 - 3$ and $\dim(P') = 2g_2 - 2$. Therefore, $dim(\mathbb{P}Hom(F_p, \mathcal{G}_p)) = dim(K_1) = 3g_1 - 3 + 2g_2 - 2 + 3 = 3g_1 + 2g_2 - 2$. Note that P is a dense open subvariety of M_{12}, therefore $dim(P) = dim(M_{12})$. Now, from the proof of Proposition 3.2, $p : P \to B$ is flat with fibres isomorphic to $\mathbb{P}^3$ as algebraic varieties. Therefore, $dim(P) = dim(B) + 3 = 3(g_1 + g_2) - 3$. Hence, we have $\dim M_{12} = 3g_1 + 3g_2 - 3$. Since $Codim(K', M_{12}) = dim(M_{12}) - dim(K')$ and $dim(K_1) = dim(K'')$, we see

$$\begin{aligned} Codim(K', M_{12}) &= 3g_1 + 3g_2 - 3 - 3g_1 - 2g_2 + 2 \\ &= g_2 - 1. \end{aligned}$$ □

Now we recall a well-known fact (see [BS07, Lemma 12]).

Lemma 3.5. *Let X be a smooth projective variety and $k := Codim(X/U)$, where U is an open subset of X . Then we have $H^i(X, \mathbb{Z}) \simeq H^i(U, \mathbb{Z})$ for all $i<2k-1$.*

Using the above Lemma and Proposition 3.4 we immediately get the following

Proposition 3.6. *With the above notations,*

(i) $H^i(M_{12}, \mathbb{Z}) \simeq H^i(P, \mathbb{Z})$ *for* $i<2k-1$ *where* $k = g_2 - 1$.

(ii) $H^i(M_{21}, \mathbb{Z}) \simeq H^i(\overline{P}, \mathbb{Z})$ *for* $i<2k'-1$ *where* $k' = g_1 - 1$.

3.3.2 Computation of cohomology groups of M_{12} (resp. M_{21})

In this subsection we will outline the strategy to compute the Betti numbers of the component M_{12}(resp. M_{21}) and compute the third cohomology of $M_{0,\xi}$ in full details. First we compute the Betti numbers of P (resp. $\overline{P}$) using the Leray-Hirsh Theorem:

Theorem 3.7. *(Leray-Hirsh) Let $f : X \to Y$ be a topological fibre bundle with fibres isomorhic to F. Suppose, $e_1, \cdots, e_n \in H^*(X)$ such that $H^*(X_y)$ is freely generated by $i_y^* e_1, \cdots, i_y^* e_n$ for all $y \in Y$ where $X_y = f^{-1}(y)$ and $i_y : X_y \to X$ are the inclusions. Then $H^*(X)$ is freely generated as a $H^*(Y)$-module by $e_1, \cdots, e_n$.*

Proposition 3.8. *The k-th Betti number $b_k(P) = \sum_{l+m=k} b_l(B) b_m(\mathbb{P}^3)$ (resp. $b_k(\overline{P}) = \sum_{l+m=k} b_l(\overline{B}) b_m(\mathbb{P}^3)$.*

Proof. Since P and B are both smooth varieties and $p : P \to B$ is a submersion we get p is smooth. Therefore, the fibres of $p|_P$ are smooth. From the proof of Proposition 3.2, it follows that the fibres of p are isomorphic to $\mathbb{P}^3$ as algebraic varieties. Choose a relatively ample line bundle L over P. Now p_*L is locally free by Zariski Main theorem. Therefore, we get that the dimension of $H^0(p^{-1}(b), L|_{p^{-1}(b)})$ is constant for all $b \in B$. Hence, $L|_{p^{-1}(b)} = \mathcal{O}(k)$ for some k>0 for all $b \in B$. Consider the cohomolgy classes $c_1(L)$, $c_1(L)^2$, $c_1(L)^3$. We denote by $j_b : p^{-1}(b) \to P$ the inclusion. Then $H^*(p^{-1}(b))$ is freely generated by $j_b^* c_1(L)$, $j_b^* c_1(L)^2$ and $j_b^* c_1(L)^3$ for all $b \in B$. Thus using the Leray-Hirsch theorem we get:

$$b_k(P) = \sum_{l+m=k} b_l(B) b_m(\mathbb{P}^3).$$

where $b_k(X)$ denotes the kth Betti number of a space X. □

As a corollary of the above Proposition we immediately get:

Corollary 3.9. *(i)* $b_1(P) = 0$ *(resp.*$b_1(\overline{P}) = 0$*) , (ii)* $b_2(P) = 3$ *(resp.* $b_2(\overline{P}) = 3$*) and (iii)* $b_3(P) = 2g$ *(resp.* $b_3(\overline{P}) = 2g$*) where* g *is the arithmetic genus of* X_0 *and* b_i*'s are the Betti numbers,* $i = 1, 2, 3$.

Proof. By Lemma 3.1 and the Kunneth formula it follows that $b_1(B) = b_1(M_1) + b_1(M_2^s) = 0$, $b_2(B) = b_2(M_1) + b_2(M_2^s) = 1 + 1 = 2$ and $b_3(B) = b_3(M_1) + b_3(M_2^s) = 2g_1 + 2g_2 = 2g$. Thus by Proposition 3.8 we get $b_1(P) = 0$, $b_2(P) = 3$ and $b_3(P) = 2g$. □

Remark 3.10. By the above proposition all the Betti numbers of P can be computed using the above argument as the Betti numbers of the varieties M_1 and M_2^s are well known (see [BS07, page 113]).

Let g_1, g_2>3. Then as a consequence of Proposition 3.6 and Corollary 3.9 we immediately get:

Theorem 3.11. *With the notations above,*

1. $H^1(M_{12}) = 0$ *(resp.* $H^1(M_{21}) = 0$*).*
2. $H^2(M_{12}) \simeq \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$ *(resp.* $H^2(M_{21}) \simeq \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$*).*
3. $H^3(M_{12}) \simeq \mathbb{Q}^{2g}$ *(resp.* $H^2(M_{21}) \simeq \mathbb{Q}^{2g}$*).*

Remark 3.12. M_{12} is a smooth, projective unirational variety by Lemma 2.0.7.1. Therefore, by a result of Serre ([Ser59]), M_{12} is a simply connected variety.

3.3.3 Continuation of the cohomology computation

We have $M_{0,\xi} = M_{12} \cup M_{21}$. Let $D = M_{12} \cap M_{21}$. We will compute the third cohomology group of $M_{0,\xi}$ using the Mayer-Vietoris sequence. Before we compute the cohomology group we will make few more observations.

Let P_1 be the moduli space of rank 2, parabolic semistable bundles $(F_1, 0 \subset F^2F_1 \subset F_1(p))$ on X_1 with parabolic weights $0<\frac{a_1}{2}<\frac{a_2}{2}<1$ and $detF_1 \simeq \xi_1$. Let P_2 be the moduli space of rank 2, parabolic semistable bundles $(F_2, 0 \subset F^2F_2(p) \subset F_2(p))$ on X_2 with parabolic weights $0<\frac{a_1}{2}<\frac{a_2}{2}<1$ and $detF_2 \simeq \xi_2$. By Fact 2.21 (a) any parabolic semistable bundle in P_1 (resp. in P_2) is parabolic stable. Therefore, one can show that P_i's are smooth, $i = 1, 2$. Since $E \in P_i$ is semistable by Fact 2.21 (b), we get morphisms $q_i : P_i \to M_i$, $i = 1, 2$. Let $P_2^s = q_2^{-1}(M_2^s)$. Thus we get a morphism $q := (q_1, q_2) : P_1 \times P_2 \to M_1 \times M_2$ such that $q^{-1}(B) = P_1 \times P_2^s$ where $B := M_1 \times M_2^s$. Now by using the same argument given in [NS97, Theorem 6.1] we can show that there is an embedding $i : P_1 \times P_2 \to M_{12}$ such that the image is isomorphic to D. Thus we have a commutative diagram of morphisms:

$$\begin{array}{ccc} P_1 \times P_2 & \xrightarrow{\quad i \quad} & M_{12} \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & M_1 \times M_2 & \end{array} \tag{3.2}$$

where p is the morphism in Proposition 3.2.

Let $q_1' : P_1 \to M_1'$ be the morphism defined by $E_1 \to E_1'$ and $q_2' : P_2 \to M_2'$ defined by $E_2 \to E_2'$ where E_i' are the Hecke modifications of E_i. Then we get another commutative diagram of morphisms:

$$\begin{array}{ccc} P_1 \times P_2 & \xrightarrow{\quad j \quad} & M_{21} \\ & {\scriptstyle q'}\searrow \quad \swarrow{\scriptstyle p} & \\ & M_1' \times M_2' & \end{array} \tag{3.3}$$

where the morphism $q' : P_1 \times P_2 \to M_1' \times M_2'$ is given by the association $(E_1, E_2) \mapsto (E_1', E_2')$ and p' is the morphism in Remark 3.3.

In the following lemma we summarize some topological facts about the moduli spaces P_i, $i = 1, 2$.

Lemma 3.13. *(i)* P_i*,* $i = 1, 2$*, are smooth, projective and rational variety being* $\mathbb{P}^1$*-bundles associated to algebraic vector bundles over coprime moduli spaces (see Remark 5.4). In particular,* P_i *are simply connected and* $Pic(P_i) \simeq H^2(P_i, \mathbb{Z})$ *for* $i = 1, 2$*.*
(ii) $H^1(P_i, \mathbb{Z}) = 0$*,* $H^2(P_i, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ *and* $H^3(P_i, \mathbb{Z}) \simeq \mathbb{Z}^{2g_i}$ *for* $i = 1, 2$ *(follows from the previous statement).*

Lemma 3.14. (1) $q^* : H^3(P_1 \times P_2^s) \simeq H^3(M_1 \times M_2^s)$.
(2) $p^* : H^3(P) \simeq H^3(M_1 \times M_2^s)$.

Proof. (1) From the fact 2.21 (c) it follows that the topological fibre of $q : P_1 \times P_2^s \to M_1 \times M_2^s$ is $\mathbb{P}^1 \times \mathbb{P}^1$. Using the similar arguments given in 3.2 and 3.9 we can show that q is a topological $\mathbb{P}^1 \times \mathbb{P}^1$- bundle satisfying the hypothesis of the Leray-Hirsch theorem. Thus by Leray-Hirsch theorem $q^* : H^3(P_1 \times P_2^s) \simeq H^3(M_1 \times M_2^s)$.

(2) The proof is already given in 3.9. □

Lemma 3.15. $i^* : H^3(P_1 \times P_2) \simeq H^3(M_{12})$ *where* $i : P_1 \times P_2 \to M_{12}$ *is the inclusion.*

Proof. First note that $i(P_1 \times P_2^s) \subset P$. Therefore, we have-

$$\begin{array}{ccc} P_1 \times P_2^s & \xrightarrow{\quad i \quad} & P \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & M_1 \times M_2^s & \end{array} \tag{3.4}$$

By the commutativity of the above diagram we get $i^* p^* = q^*$. Since, by Lemma 3.14, p^* and q^* are isomorphisms, we get $i^* : H^3(P) \to H^3(P_1 \times P_2^s)$ is an isomorphism. By an argument given in [Bal88, Proposition 7] we can show that $Codim(K, P_1 \times P_2) = g_2 - 1$ where $K = P_1 \times P_2 \setminus P_1 \times P_2^s$. Thus, by Lemma 3.5, we get $i_1^* : H^3(P_1 \times P_2) \to H^3(P_1 \times P_2^s)$ is an isomorphism where $i_1 : P_1 \times P_2^s \to P_1 \times P_2$ is the inclusion. Also, we have shown $i_2^* : H^3(M_{12}) \to H^3(P)$ is an isomorphism where $i_2 : P \to M_{21}$ is the inclusion. Thus $i^* : H^3(M_{12}) \to H^3(P_1 \times P_2)$ is an isomorphism. □

It is known that the Picard groups $Pic(M_i)$ (resp. $Pic(M_i')$), $i = 1, 2$, are isomorphic to $\mathbb{Z}$ ([DN89, Theorem B]). Let θ_i (resp. θ_i'), $i = 1, 2$, be the unique ample generators of $Pic(M_i)$ (resp. $Pic(M_i')$). Let us denote the projections $M_1 \times M_2 \to M_i$

by s_i; the projections $M'_1 \times M'_2 \to M'_i$ by s'_i and the projections $P_1 \times P_2 \to P_i$ by r_i, $i = 1, 2$. Then we immediately get the following relations:

$$r_1^* q_1^* \theta_1 = q^* s_1^* \theta_1, \quad r_1^* q_1'^* \theta_1' = q'^* s_1'^* \theta_1',$$

$$r_2^* q_2^* \theta_2 = q^* s_2^* \theta_2, \quad r_2^* q_2'^* \theta_2' = q'^* s_2'^* \theta_2'.$$

Let $\Theta_1 := q^* s_1^* \theta_1$, $\Theta_2 := q'^* s_1'^* \theta_1'$, $\Theta_3 := q^* s_2^* \theta_2$ and $\Theta_4 := q'^* s_2'^* \theta_2'$.

Lemma 3.16. *The line bundles* Θ_i, $i = 1, \cdots 4$ *are linearly independent on* $P_1 \times P_2$

Proof. Note that $q_1^* \theta_1'$ and $q_1'^* \theta_1'$ are linearly independent line bundles over P_1. This follows from the observation: The line bundles θ_1 (resp. θ_1') is ample over M_1 (resp. M'_1) and M_1, M'_1 are normal varieties. Moreover, by Fact 2.21 the fibre $q_1^{-1}(F)$ is isomorphic to $\mathbb{P}^1$ for all $F \in M_1$ and the fibre $q_1'^{-1}(F)$ is isomorphic to $\mathbb{P}^1$ for all $F \in M_1'^s$. The image of the morphism, inside some projective space, defined by the linear system corresponding to the sufficiently large power of $q_1^* \theta_1$ (resp. $q_1'^* \theta_1'$) will be isomorphic to M_1 (resp. M'_1) (this follows by Lemma 5.6). Since M_1 and M'_1 are not isomorphic we have $q_1^* \theta_1'$ and $q_1'^* \theta_1'$ are linearly independent. Now $r_i : P_1 \times P_2 \to P_i$ are the projection maps, $i = 1, 2$. Therefore, $\Theta_1 = r_1^* q_1^* \theta_1$ and $\Theta_2 = r_1^* q_1'^* \theta_1'$ are linearly independent. Similarly Θ_3 and Θ_4 are linearly independent. Next we will show that the relation $\Theta_1^{a_1} \otimes \Theta_2^{a_2} = \Theta_3^{a_3} \otimes \Theta_4^{a_4}$, with all a_i non zero will never occur. The above relation would imply $r_1^* L = r_2^* M$ where L is a non trivial line bundle on P_1 and M is a nontrivial line bunlde on P_2. But this is impossible as $r_1^* L|_{q_1^{-1}(x)}$ is trivial but $r_2^* M|_{q_1^{-1}(x) \simeq P_2} \simeq M$ is non trivial for $x \in P_1$. From the above observation we conclude that Θ_i are linealy independent, $i = 1, 2, 3, 4$. □

In chapter 5 we will observe that the P_i's are rational varieties and $Pic(P_i) \simeq \mathbb{Z} \oplus \mathbb{Z}$, $i = 1, 2$. Thus $Pic(P_1 \times P_2) \simeq Pic(P_1) \times Pic(P_2) \simeq \mathbb{Z}^4$. Therefore, the line bundles Θ_i's generate the Picard group $Pic(P_1 \times P_2)$, $i = 1, 2, 3, 4$. Now we will prove the following:

Lemma 3.17. *The morphism* $i^* - j^* : H^2(M_{12}) \oplus H^2(M_{21}) \to H^2(P_1 \times P_2) \simeq H^2(D)$, *is surjective where* i *and* j *are the inclusions in the diagrams 3.2, 3.3.*

Proof. Since the P_i are rational varieties, we have that $c_1 : Pic(P_1 \times P_2) \to H^2(P_1 \times P_2, \mathbb{Z})$ is an isomorphism where c_1 is the first chern class homomorphism. By Lemma 3.13 we get $Pic(P_1 \times P_2) \simeq H^2(P_1 \times P_2, \mathbb{Z}) \simeq \mathbb{Z}^4$ Therefore, by Lemma 3.16

$H^2(P_1 \times P_2)$ is generated by $c_1(\Theta_1)$, $c_1(\Theta_2)$, $c_1(\Theta_3)$ and $c_1(\Theta_4)$. From commutative diagrams 3.2 and 3.3 we get

$$\Theta_1 = i^*(p^* s_1^* \theta_1),\ \Theta_2 = j^*(p'^* s_1'^* \theta_1'),\ \Theta_3 = i^*(p^* s_2^* \theta_2) \text{ and } \Theta_4 = j^*(p'^* s_2'^* \theta_2').$$

Therefore, $H^2(M_{12}) \oplus H^2(M_{21}) \to H^2(P_1 \times P_2^s)$ is surjective. □

We will now prove the main theorem of this section.

Theorem 3.18. *(i)* $H^2(M_{0,\xi}) \simeq \mathbb{Q}^2$ *and (ii)* $H^3(M_{0,\xi}) \simeq \mathbb{Q}^{2g}$*.*

Proof. By Lemma 3.17, 3.15 and using the Mayer-Vietoris sequence we get-

$$0 \to H^2(M_{0,\xi}) \to H^2(M_{12}) \oplus H^2(M_{21}) \to H^2(D) \to 0.$$

and

$$0 \to H^3(M_{0,\xi}) \to H^3(M_{12}) \oplus H^3(M_{21}) \to H^3(D) \to 0.$$

Now $b_2(M_{12}) = b_2(M_{21}) = 2$ by Theorem 3.11 and $b_2(D) = b_2(P_1 \times P_2) = 4$ by Lemma 3.13. Therefore, $b_2(M_{0,\xi}) = 2$.

Also we have that $b_3(M_{12}) = b_3(M_{21}) = 2g$ by Theorem 3.11 and $b_3(D) = b_3(P_1 \times P_2) = 2g$ by Lemma 3.13. Therefore, $b_3(M_{0,\xi}) = b_3(M_{12}) + b_3(M_{21}) - b_3(D) = 2g$. □

No we will discuss the Hodge structure on $H^3(M_{0,\xi}, \mathbb{Z})$ (see the next chapter for related definitions).

3.3.4 Hodge structure on $H^3(M_0, \mathbb{Z})$

Let us first recall the definitions of the Hodge structure of weight k and the mixed Hodge structure.

Definition 3.19. Let $V_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module of finite rank. We say that the module $V_{\mathbb{Z}}$ admits a Hodge structure of weight k if

$$V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}.$$

with $V^{p,q} = \overline{V^{q,p}}$, where $\bar{\alpha}$ denotes the conjugate of α.

Definition 3.20. Let $V_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module of finite rank. A mixed Hodge structure on $V_{\mathbb{Z}}$ consists of the following data: a decreasing filtration (weight filtration)

$$0 \subseteq \cdots \subseteq W_k \subseteq \cdots \subseteq W_m = V_{\mathbb{Q}}.$$

, $k \leq m$ defined over $\mathbb{Q}$ and an increasing filtration (Hodge filtration),

$$V_{\mathbb{C}} = F^0 \supseteq \cdots \supseteq F^p \supseteq \cdots \supseteq F^k = 0.$$

, defined over $\mathbb{C}$ such that on the each quotients $\frac{W_i}{W_{i-1}}$ the Hodge filtration $\{F^p\}$ induces a pure Hodge structure of weight i.

We say a mixed Hodge structure pure if there is no weight filtration.

Theorem 3.21. *The Hodge structure on* $H^3(M_0, \mathbb{Z})$ *is pure of weight* 3 *with* $h^{3,0} = h^{0,3} = 0$.

Proof. We have the following short exact sequence:

$$0 \to H^3(M_{0,\xi}, \mathbb{C}) \xrightarrow{r^*} H^3(M_{12}, \mathbb{C}) \oplus H^3(M_{21}, \mathbb{C}) \xrightarrow{i^*-j^*} H^3(D, \mathbb{C}) \to 0.$$

where all the morphisms are the morphism of Hodge structures. Thus $Ker(i^* - j^*)$ is a pure sub Hodge structure of $H^3(M_{12}, \mathbb{C}) \oplus H^3(M_{21}, \mathbb{C})$ of weight 3. This induces a Hodge structure of weight 3 on $H^3(M_{0,\xi}, \mathbb{C})$ as $H^3(M_{0,\xi}, \mathbb{C})$ is isomorphic to $Ker(i^* - j^*)$. Since M_{12} and M_{21} are smooth unirational varieties (see Lemma 2.0.7.1) and their intersection $D = M_{12} \cap M_{21}$ is also a smooth unirational variety, we have $h^{3,0}(M_{12}) = h^{3,0}(M_{21}) = 0$ and $h^{3,0}(D) = 0$. Hence we conclude $h^{3,0}(Ker(i^* - j^*)) = h^{3,0}(H^3(M_{0,\xi}, \mathbb{Z})) = 0$. This completes the proof. □

Remark 3.22. Thus we can define the intermediate Jacobian as in 1.1 corresponding to the Hodge structure on $H^3(M_{0,\xi}, \mathbb{Z})$. We will denote this intermediate Jacobian by $J^2(M_0)$.

Chapter 4

Degeneration of the intermediate Jacobian of the moduli space

In this chapter we will review the basic Hodge theory, variation of Hodge structures. We will also state the existence limit Mixed Hodge structure in the case of family of a smooth projecting varieties degenerating to a normal crossing variety. Finally we will use these results to prove the main theorem of this chapter. We follow the notations of the subsection 2.0.6 in Chapter 1.

4.1 Review of Hodge Theory

Let Y be a smooth projective variety over $\mathbb{C}$. Then the cohomology group $H^k(Y, \mathbb{C})$ admits the following functorial Hodge decomposition:

$$H^k(Y, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$$

with $H^{q,p} \simeq \overline{H^{p,q}}$ where $-$ denotes the complex conjugation on $H^k(Y, \mathbb{C})$ induced by the complex conjugation on $\mathbb{C}$. Fix a very ample line bundle L on Y and a Hermitian metric h on L. Then the class of the Chern form $\omega_{L,h}$ (see [Voi02, page 75]) is equal to the first Chern class $c_1(L) \in H^2(Y, \mathbb{Z}) \subset H^2(Y, \mathbb{R})$ ([Voi02, Theorem 7.10]). Since L is very ample $c_1(L)$ is positive definite. Such a closed 2 form is known as Kähler form. The Lefschetz operator $L : H^k(Y, \mathbb{Z}) \to H^{k+2}(Y, \mathbb{Z})$ corresponding to the

Kähler form $c_1(L)$ is defined to be:

$$L(\alpha) := c_1(L) \wedge \alpha.$$

. For $r \leq n$ the primitive cohomology is defined to be

$$H^r(Y)_{prim} := Ker(L^{n-r+1})$$

where $n := dim_{\mathbb{C}}(Y)$. We have the following Lefschetz decomposition:

Theorem 4.1. *([Voi02, Theorem 6.4]) For $k \leq n$ the operator*

$$L^{n-k} : H^k(Y, \mathbb{C}) \to H^{2n-k}(Y, \mathbb{C})$$

is an isomorphism and we have the decomposition

$$H^k(Y, \mathbb{C}) \simeq \bigoplus_{2r \leq k} L^r H^{k-2r}(Y, \mathbb{C})_{prim}.$$

Note that the above decomposition is compatible with the Hodge decomposition. The operator L induces an bilinear form Q on $H^k(Y, \mathbb{R})$ given by:

$$Q(\alpha, \beta) := \int_Y c_1(L)^{n-k} \wedge \alpha \wedge \beta. \tag{4.1}$$

Then Q is alternating if k is odd, symmetric otherwise. The induced Hermitian form:

$$H(\alpha, \beta) = i^k Q(\alpha, \bar{\beta})$$

satisfies the following properties (see [Voi02, page 160]):
(i) The Hodge decomposition and the Lefschetz decomposition is orthogonal for H.
(ii) $i^{p-q-k}(-1)^{\frac{k(k-1)}{2}} H(\alpha, \alpha) > 0$ for α nonzero of type (p, q) and $\alpha \in H^k(Y, \mathbb{C})_{prim}$.
We make the following definition:

Definition 4.2. Let $V_{\mathbb{Z}}$ be a free $\mathbb{Z}$ module. We say $V_{\mathbb{Z}}$ has an integral Hodge structure of weight k if

$$V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q} \text{ and } H^{q,p} = \overline{H^{p,q}}$$

. The integral Hodge structure $(V_{\mathbb{Z}}, H^{p,q})$ of weight k is polarised if there is an intersection form Q defined on $V_{\mathbb{C}}$ which is alternating if k is odd, symmetric otherwise and satisfies the conditions (i), (ii) above.

By above we see that the primitive cohomology group $H^k(Y, \mathbb{Z})_{prim} \subset H^k(Y, \mathbb{R})_{prim}$ has a polarised integral Hodge structure.

4.1.0.1 Variation of Hodge structure

Let $\mathcal{X}$, B be complex manifolds and $\phi : \mathcal{X} \to B$ be a holomorphic map. Let $X_t := \phi^{-1}(t)$ denote the the fibre of ϕ above the point t.

Definition 4.3. We say that $\phi : \mathcal{X} \to B$ is an analytic family of complex manifolds if ϕ is a proper, holomorphic submersion.

Consider the sheaves $H^k_A := R^k\phi_* A$ where A is a ring of coefficients (usually $\mathbb{Z}$,$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), considered as the constant sheaf of stalk A, and $R^k\phi_*$ is the k-th derived functor of the functor ϕ_* from the category of sheaves over $\mathcal{X}$ to the category of sheaves over B. Then we can show that, as ϕ is a proper, holomorphic submersion, H^k_A is a local system with stalks isomorphic to $H^k(X, A)$. Let V^k be the holomorphic vector bundle over B whose sheaf of section is $\mathcal{H}^k := H^k_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_B$. Then V^k is equipped with a flat connection

$$\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega^1_B$$

as follows: For $\sigma \in \mathcal{H}^k$, $\sigma = \sum_i \alpha_i \sigma_i$ in a basis σ_i of a local trivialisation, we set

$$\nabla(\sigma) := \sum_i \sigma_i d\alpha_i \in \mathcal{H}^k \otimes \Omega^1_B.$$

This is well defined since if we choose another basis σ'_i they are related by constant transition matrix. Clearly, $\nabla^2 = 0$. Therefore, the connection ∇ is flat. This flat connection is known as Gauss-Mannin connection.

Consider now a family $\phi : \mathcal{X} \to B$ of smooth, projective manifolds. Let V^k be the holomorphic bundle corresponding to this family defined above. Then we have the following theorem:

Theorem 4.4. *([Voi02, Theorem 10.3]) The Hodge filtration $F^p(t) \subset H^k(X_t, \mathbb{C})$, $t \in B$ gives a filtration of V^k by holomorphic subbundles F^pV^k, called the Hodge subbundles. These* *bundles satisfies the transversality property*

$$\nabla F^p V^k \subset F^{p-1} V^k \otimes \Omega^1_B$$

4.1.0.2 Polarized Variation Of Hodge structure

We say the datum $(V^k, F^p V^k, V^k_{\mathbb{Z}})$ a Variation of Hodge structure. We say $(V^k, F^p V^k, V^k_{\mathbb{Z}})$ is polarised if there exists a non- degenerate $(-1)^k$- symmetric bilinear form Q on V^k, defined over $\mathbb{Z}$, such that for all $t \in B$ the Hodge structure on V_t is polarized in a sense of Definition 4.2.

Remark 4.5. The variation of Hodge structure arising from the primitive cohomology groups $\{H^k(\mathcal{Y}_t, \mathbb{Z})_{prim}\}_{t \in B}$ is an example of polarized variation of Hodge structure.

4.1.0.3 Intermediate Jacobian

Let $k = 2m - 1$ be positive odd integer. The Weil operator $C_W : H^k(Y, \mathbb{C}) \to H^k(Y, \mathbb{C})$ is defined to be:

$$C_W(\alpha) = i^{p-q} \sum_{p+q=k} \alpha^{p,q}, \ \alpha = \sum_{p+q} \alpha^{p,q} \in H^k(Y, \mathbb{C}).$$

Note that C_W is $\mathbb{C}$ linear isomorphism, $C_W(H^k(Y, \mathbb{R}) = H^k(Y, \mathbb{R})$ and $C_W^2 = -Id$. Therefore, C_W defines a complex structure on $H^k(Y, \mathbb{R})$. We define

$$J^m_W(Y) := \frac{H^k(Y, \mathbb{R})}{H^k(Y, \mathbb{Z})}.$$

Then $J^m_W(Y)$ is a complex torus where the complex structure on $H^k(Y, \mathbb{R})$ is defined by C_W. The above complex torus is known as the Weil intermediate Jacobian.

Proposition 4.6. *The complex torus $J^m_W(Y)$ is projective. In otherwords it is an abelian variety.*

Proof. To show that $J^m_W(Y)$ is projective we need to show that there exists an integral Kähler form ω on $J^m_W(Y)$. Note that $H^2(J^m_W(Y), \mathbb{Z}) = \wedge^2 H^1(J^m_W(Y), \mathbb{Z})$. Now $H^1(J^m_W(Y), \mathbb{Z}) \simeq H^{2m-1}(Y, \mathbb{Z})^*$. Therefore, the bilinear form Q, as it is alternating, induces an element $\omega \in H^2(J^m_W, \mathbb{Z})$. We first check that ω is of type $(1, 1)$ at each point of $J_W^m(Y)$. It is enough to check this for the complexified tangent space $T_e J \otimes \mathbb{C}$ at identity. This means that the $\mathbb{C}$ bilinear extension of ω on a 2- form on $T_e J \otimes \mathbb{C}$ vanishes on $\wedge^2 T_e^{1,0}$. But this bilinear extension is the alternating form Q on $H^{2m-1}(Y, \mathbb{C})$ and $T_e^{1,0} J$ is the complex subspace $H^{2m-1}(Y, \mathbb{R})$ where the complex structure is given by the operator C_W. Therefore, the above fact follows from the following observation: note that $H^{2m-1}(Y, \mathbb{R})$ with the complex structure C_W is isomorphic to a direct summand W of $\bigoplus_{p+q=2m-1} H^{p,q}$ such that if $H^{p,q}$ belongs to W then $H^{q,p}$ does not belong to W. Let $\alpha, \beta \in W$. Let $\alpha^{p,q}$ be a component of α and $\beta^{p',q'}$ be a component of β . By the the above $p' \neq q$ and $q' \neq p$. Thus $L^{n-k} \wedge \alpha^{p,q} \wedge \beta^{p',q'} = 0$ since it is of type $(n-k+p+p', n-k+q+q')$ and $H^{2n}(Y, \mathbb{C})$ only has class of (n,n)-type. Therefore, $Q(\alpha, \beta) = 0$. Next we will show that the Hermitian form,induced by ω, on the holomorphic tangent space is positive definite at every point of $J_W^m(Y)$. It is enough to check for the tangent space at the identity. The tangent space at the identity T_e can be identified with $H^{2m-1}(Y, \mathbb{R})$ where the complex structure is given by the operator C_W. We will show that $Q(\alpha, C_W \alpha)$>0 for all $\alpha \in H^{2m-1}(Y, \mathbb{R})$ and α nonzero. Suppose α be a primitive class of odd degree k'. Then we have

$$
\begin{aligned}
Q(\alpha, C_W(\alpha)) &= Q\Big(\sum_{p<q, p+q=k'} \alpha^{p,q} + \sum_{p<q, p+q=k'} \overline{\alpha^{p,q}}, \sum_{p<q, p+q=k'} i^{p-q} \alpha^{p,q} + \sum_{p<q, p+q=k'} i^{q-p} \overline{\alpha^{p,q}}\Big) \\
&= Q\Big(\sum_{p+q=k', p<q} i^{q-p} Q(\alpha^{p,q}, \overline{\alpha^{p,q}}) + \sum_{p+q=k', p<q} i^{p-q} Q(\overline{\alpha^{p,q}}, \alpha^{p,q}\Big) \\
&= 2 \sum_{p+q=k', p<q} i^{p-q-k} H(\alpha^{p,q}, \alpha^{p,q}) \\
&>0.
\end{aligned}
$$

Now any class $\alpha \in H^k(Y, \mathbb{R})$ has a unique decomposition

$$\alpha = \sum_r L^r \alpha_r,$$

where the α_r are of degree $k-2r \leq inf(n, 2n-k)$ and are primitive in the sense that $L^{n-k+2r+2}\alpha_r = 0$ in $H^{2n-k+2r+2}(Y, \mathbb{R})$. We have

$$Q(\sum_r L^r\alpha_r, C_W \sum_r L^r\alpha_r) = \sum_r Q(L^r\alpha_r, L^r C_W\alpha_r) \text{ since } if\ r \neq s\ Q(L^r\alpha_r, C_W L^s\alpha_s) = 0$$
$$= \sum_r Q(\alpha_r, C_W\alpha_r)$$
$$>0 \text{ since } Q(\alpha_r, C_W\alpha_r)>0 \text{ for all } r.$$

□

There is another complex operator $C_H : H^k(Y, \mathbb{C}) \to H^k(Y, \mathbb{C})$ defined to be:

$$C_H(\alpha) = \sum\nolimits_{p+q=k} i^{\frac{p-q}{|p-q|}}\alpha^{p,q},\ \alpha = \sum_{p+q=k} \alpha^{p,q} \in H^k(Y, \mathbb{C}).$$

Again we note that $C_H(H^k(Y, \mathbb{R}) = H^k(Y, \mathbb{R})$ and $C_H^2 = -Id$. We define a complex torus by

$$J_H^m(Y) := \frac{H^k(Y, \mathbb{R})}{H^k(Y, \mathbb{Z})}.$$

where the complex structure on $H^k(Y, \mathbb{R})$ is defined by C_H. It is known as the Griffiths intermediate Jacobian. We remark that $J_H^m(Y)$ need not be an abelian variety.

Remark 4.7. $J_W^m(Y)$ and $J_H^m(Y)$ coincide if $|p-q| = 1$.

Set $F^pH^(Y, \mathbb{C}) := \bigoplus_{r \geq p} H^{r,k-r}$. This gives an increasing filtration on

$$H^k(Y, \mathbb{C}) = F^0H^k(Y, \mathbb{C}) \supset .. \supset F^pH^k(Y, \mathbb{C}) \supset .. \supset F^kH^k(Y, \mathbb{C})$$

and $H^k(Y, \mathbb{C}) = F^p \oplus \overline{F^{k-p+1}}$. It is easy to see that

$$J_H^m(Y) = \frac{H^{2m-1}(Y, \mathbb{C})}{F^mH^{2m-1}(Y, \mathbb{C}) \oplus H^{2m-1}(Y, \mathbb{Z})}.$$

Proposition 4.8. *Let $\phi : \mathcal{Y} \to B$ be a family of smooth projective manifolds. Then there is a family $\phi' : \mathcal{J}_H \to B$ of Griffiths intermediate Jacobians over B i.e. $\mathcal{J}_{Ht} = J_H^k(X_t)$, $t \in B$ and $k = 2m-1$.*

Proof. Let $L \to B$ be the fibre bundle over B with fibres isomorphic to $H^k(X_t, \mathbb{Z})$ corresponding to the local system $H^k_{\mathbb{Z}}$. We set $W^k := \frac{V^k}{F^kV^k}$. Since $H^k(X_t, \mathbb{Z}) \cap$

$F^k H^k(X_t, \mathbb{C}) = (e)$, $L_t \subset W_t^k$ are lattices of maximal rank for all $t \in B$. We have already observed that $\frac{W_t^k}{L_t}$ is the Griffiths intermediate Jacobian $J_H^k(X_t)$. Thus we get an inclusion $j : L \to W^k$ such that $j_t : L_t \to W_t^k$ is cocompact for all t. Therefore, there is a unique complex manifold structure on $\frac{W^k}{L}$ such that $W^k \to \frac{W^k}{L}$ is local analytic isomorphism and this makes $\frac{W^k}{L} \to B$ a holomorphic submersion, where $\frac{W^k}{L}$ is the quotient space defined by the fibrewise equivalence i.e. by the equivalence relation $v, w \in W^k$ are related if $v, w \in W_t^k$ for some t and $v - w \in L_t$. □

4.1.1 Degeneration of Hodge structures

Let $\mathcal{Y}$ be a smooth projective variety and C be an irreducible, smooth projective curve. Let $\phi : \mathcal{Y} \to C$ be a morphism and let Y_t denotes the scheme theoretic fibre of ϕ above the point $t \in C$.

Definition 4.9. We say $\phi : \mathcal{Y} \to C$ is an algebraic family of projective varieties if ϕ is a proper, flat and surjective morphism.

Note that, as C is smooth, there is a dense open subset U of C such that $\phi : \phi^{-1}(U) \to U$ is smooth. Therefore, since C is a curve, only over the finite number of points the fibre of ϕ is not smooth. We assume that the fibres of ϕ are reduced. Consider such an algebraic family $\phi : \mathcal{Y} \to C$. Let Y_p, the fibre over a point $p \in C$, be the union of smooth, irreducible projective varieties intersecting transversally. Let us consider the complex analytic topology on C. As, C is smooth every point of C has a complex analytic neighborhood (analytically) isomorphic to unit disk. We choose an analytic neighborhood of p isomorphic to the unit disk. Let us again denote this neighborhood by Δ. Let $\mathcal{Y}_\Delta := \phi^{-1}(\Delta)$. Then ϕ_Δ is a proper, holomorphic map such that $\phi|\Delta^* : \mathcal{Y}_\Delta^* \to \Delta^*$ is a submersion where $\Delta^* = \Delta \setminus p$ and $\mathcal{Y}_\Delta^* = \mathcal{Y}_\Delta \setminus p$.

Theorem 4.10. *(Deligne canonical extension, [Del70, page 91-92]) Let (V, ∇) be a holomorphic vector bundle over Δ^* together with a flat connection ∇ which has unipotent monodromy T. Then there exists a unique extension $(\bar{V}, \bar{\nabla})$ of (V, ∇) over Δ such that the extended connection $\bar{\nabla}$ is logarithmic with residue $log(T)$.*

From now on we will simply denote the family $\phi_\Delta : \mathcal{Y}_\Delta \to \Delta$ by $\phi : \mathcal{Y} \to \Delta$.

Fix a positive integer k. We have a flat vector bundle (V^k, ∇) over Δ^* corresponding to the local system $R^k \pi_* \mathbb{C}$. Since the fibre Y_0 is union smooth projective varieties intersecting transversally one can show that ∇ has unipotent monodromy T with unipotent index atmost $k+1$ [Lan73]. Therefore, there is a canonical extension (see Theorem 4.10) $(\bar{V}^k, \bar{\nabla})$.

Theorem 4.11. *([Sch73]) The Hodge subbundles $\mathcal{F}^p$ of V^k extend to a holomorphic subbundle $\bar{\mathcal{F}}^p$ of $\bar{V}^k$ such that*

$$\bar{\nabla}(\bar{\mathcal{F}}^p) \subset \bar{\mathcal{F}^{p-1}} \otimes \Omega^1_{\Delta}(log(0)).$$

Let $\bar{V}^k(0)$ and $\bar{\mathcal{F}}^p(0)$ be the fibres of $\bar{V}^k$ and $\bar{\mathcal{F}}^p$ at $0 \in \Delta$. Let $\overline{V_{\mathbb{Z}}}^k := j_* V^k_{\mathbb{Z}}$ where $j : \Delta^* \to \Delta$ is the inclusion and $\overline{V_{\mathbb{Z}}}^k(0)$ be the fibre at 0. Then $\{\bar{\mathcal{F}}^p(0)\}$ gives a filtration on $\bar{V}^k(0)$. The nilpotent operator N acts on the fibre $\bar{V}^k(0)$ inducing a decresing filtration

$$0 \subset W_0 \subset \cdots \subset W_{2k} = \bar{V}^k(0).$$

This is called the monodromy weight filtration (see [Mor84, page 106]. W Schmid [Sch73] showed that for each $t \in \mathbb{C}^*$, the data $(t^N \overline{V}^k_{\mathbb{Z}}(0), \overline{\mathcal{F}^p}(0), W_r)$ defines a mixed Hodge structure (see [Hai02, Theorem 10] for the statement). We call this mixed Hodge structure limiting mixed Hodge structure. As the fibre $Y_0 = Y_1 \cup Y_2 \cup \cdots \cup Y_n$ is the union of smooth projective varieties such that every m-fold intersection $Y_{i_1} \cap .. \cap Y_{i_2}$ is smooth, using Mayer- Vietoris type spectral sequence one can induce a functorial mixed Hodge structure on $H^k(Y_0, \mathbb{Q})$ (see [Mor84, page 105]). Clemens and Schmid showed that there is a natural morphism $i^* : H^k(Y_0, \mathbb{Q}) \to \bar{V}^k_0$ of mixed Hodge structure of $(0,0)$-type (see [Mor84, Clemens-Schmid I,page 108]).

We end this section by recalling few results from [Mor84]. Let $Y_0 = Y_1 \cup Y_2 \cup \cdots \cup Y_n$. Define the dual graph Γ of Y_0 to be the simplicial complex with one vertex p_i for each component Y_i of Y_0 such that the simplex $<p_{i_0}, \cdots, p_{i_k}>$ belongs to Γ if and only if $Y_{i_0} \cap \cdots \cap Y_{i_k} \neq \phi$.

Lemma 4.12. *Let Y_i, $i = 1, \cdots, n$, be smooth projective curves. Then $N : \bar{V}^1_0 \to \bar{V}^1_0$ is trivial if and only if $H^1(|\Gamma|) = 0$ where N is the nilpotent operator acting $\bar{V}^1_0$.*

Proof. The proof follows from [Mor84, Corollary 2, page 109]. □

Though we have stated the above results in a very general set up, we are going to apply these results in a very particular situation in the next section.

4.2 The Main Theorem

Let $\pi : \mathcal{X} \to C$ be a proper, flat and surjective family of curves, parametrised by a smooth, irreducible curve C. We assume that $\mathcal{X}$ is a smooth variety over $\mathbb{C}$. Fix a point $0 \in C$. We assume that π is smooth outside the point 0 and $\pi^{-1}(0) = X_0$ where X_0 is a reducible curve with two smooth, irreducible components meeting at a node. Fix a line bundle $\mathcal{L}$ over $\mathcal{X}$ such that the restriction $\mathcal{L}_t$ to X_t is a line bundle with Euler characteristic $\chi - (1-g)$ for all $t \in C$ where g is the genus of X_t. We denote the restriction $\mathcal{L}|_{X_0}$ by ξ. In chapter 1 we have shown that there is a proper, flat, surjective family $\pi' : \mathcal{M}_{\mathcal{L}} \to C$ such that $\pi'^{-1}(0) = M_{0,\xi}$, the moduli space of rank 2, stable torsion free sheaves with determinant ξ and for $t \neq 0$, $\pi'^{-1}(t) = M_{t,\mathcal{L}_t}$, the moduli space of rank 2 stable bundles on X_t with determinant L_t (see Proposition 2.19 and Remark 2.20). Moreover, $\mathcal{M}_{\mathcal{L}}$ is smooth over $\mathbb{C}$. Choose a neighbourhood of the point 0 which is analytically isomorphic to the open unit disk Δ such that both the morphisms $\pi'|_{\Delta^*}$ and $\pi|_{\Delta^*}$ are smooth, $\Delta^* := \Delta - 0$. Denote the family $\pi'|_{\Delta^*} : \pi'^{-1}(\Delta^*) \to \Delta^*$ by $\{M_t\}_{t\in\Delta^*}$ and the family $\pi|_{\Delta^*} : \pi^{-1}(\Delta^*) \to \Delta^*$ by $\{X_t\}_{t\in\Delta^*}$.

4.2.0.1 Variation of Hodge structure corresponding to the family $\{M_t\}_{t\in\Delta^*}$ and $\{X_t\}_{t\in\Delta^*}$

Since $\pi'|_{\Delta^*}$ (resp.$\pi|_{\Delta^*}$) is smooth we get a local system $R^i\pi'_*\mathbb{Z}$ (resp. $R^i\pi_*\mathbb{Z}$) for $i \geq 0$, of free abelian groups. The fibre over a point $t \in \Delta^*$ of the local system $R^i\pi'_*\mathbb{Z}$ (resp. $R^i\pi_*\mathbb{Z}$) is isomorphic to $H^i(M_t, \mathbb{Z})$ (resp. $H^i(X_t, \mathbb{Z})$). Let $H_{\mathbb{Z}}(\mathcal{M}) := R^3\pi'_*\mathbb{Z}$ and $H_{\mathbb{Z}}(\mathcal{X}) := R^1\pi_*\mathbb{Z}$. Let $H_{\mathbb{C}}(\mathcal{M})$ (resp. $H_{\mathbb{C}}(\mathcal{X})$) be the holomorphic bundle over Δ^* whose sheaf of section is $H_{\mathbb{Z}}(\mathcal{M}) \otimes_{\mathbb{Z}} \mathcal{O}_{\Delta^*}$ (resp. $H_{\mathbb{Z}}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathcal{O}_{\Delta^*}$). Then $H_{\mathbb{C}}(\mathcal{M})$(resp. $H_{\mathbb{C}}(\mathcal{X})$) admits a flat connection $\nabla_{\mathcal{M}}$ (resp. $\nabla_{\mathcal{X}}$). Let T (resp. T') be the monodromy operator defined by the flat connection $\nabla_{\mathcal{M}}$ (resp. $\nabla_{\mathcal{X}}$) corresponding to the positive generator of $\pi_1(\Delta^*, t_0)$. Since the fibre $\pi'^{-1}(0)$ (resp. $\pi^{-1}(0)$) is a union of two smooth projective varities intersecting transversally, T (resp. T') is unipotent. The unipotency index of T is atmost 4 and T' is atmost 2. Let $(H_{\mathbb{C}}(\mathcal{M}), F^p, H_{\mathbb{Z}}(\mathcal{M}))$ (resp. $(H_{\mathbb{C}}(\mathcal{M}), G^q, H_{\mathbb{Z}}(\mathcal{M})))$ be the variation of Hodge structure corresponding to the local system $H_{\mathbb{Z}}(\mathcal{M})$ (resp.$H_{\mathbb{Z}}(\mathcal{X})$). Then by Proposition 4.8 we get a family $\pi' : J^2(\mathcal{M}^*_{\mathcal{L}}) \to \Delta^*$ (resp. $\pi : J^0(\mathcal{X}^*) \to \Delta^*$) such that $J^2(\mathcal{M}^*_{\mathcal{L}})_t = J^2(M_{t,\mathcal{L}_t})$ (resp. $J^0(\mathcal{X}^*_t) = J^0(X_t)$ for all $t \in \Delta^*$). Note that as $M_{t,\mathcal{L}_t}$ are rational varieties,$t \in M_{t,\mathcal{L}_t}$ we have $h^{0,3}(M_{t,\mathcal{L}_t}) = h^{3,0}(M_{t,\mathcal{L}_t}) = 0$. Therefore, by Remark 4.7 we see that $J^2(M_{t,\mathcal{L}_t})$

coincides with the Weil intermediate Jacobian. In otherwords $J^2(M_{t,\mathcal{L}_t})$ is an abelian variety for all $t \in \Delta^*$.

Let ω_t be the unique ample, integral, Kähler class which generates $H^2(M_{t,\mathcal{L}_t}, \mathbb{Z})$ for all $t \in \Delta^*$. Then there are polarisations Θ_t on $J^2(M_{t,\mathcal{L}_t})$ induced by the intersection forms -

$$(\alpha, \beta) \mapsto \int_{M_{t,\mathcal{L}_t}} \omega^{n-3} \wedge \alpha \wedge \beta, \tag{4.2}$$

where $\alpha, \beta \in H^1(M_{t,\mathcal{L}_t}, \mathbb{R})$ and $n = dim_{\mathbb{C}} M_{M_{t,\mathcal{L}_t}}$, $t \in \Delta^*$. This polarisations $\{\Theta'_t\}_{t \in \Delta^*}$ fit together to give a relative polarization Θ' on $J^2(\mathcal{M}^*)$. Let θ_t be the polarization on $J^1(X_t)$ for all $t \in \Delta^*$ induced by the intersection form:

$$(\alpha, \beta) \mapsto \int_{X_t} \alpha \wedge \beta, \tag{4.3}$$

where $\alpha, \beta \in H^1(X_t, \mathbb{R})$. Then $\{\theta_t\}_{t \in \Delta^*}$ fit together to give a relative polarization Θ on $J^1(\mathcal{X}^*)$.

We denote canonical extension of $H_{\mathbb{C}}(\mathcal{M})$ (resp. $H_{\mathbb{C}}(\mathcal{X})$) over Δ by $\overline{H}_{\mathbb{C}}(\mathcal{M})$ (resp. $\overline{H}_{\mathbb{C}}(\mathcal{X})$). Let $\overline{H}_{\mathbb{Z}}(\mathcal{M}) := j_*(H_{\mathbb{Z}}(\mathcal{M}))$ (resp. $\overline{H}_{\mathbb{Z}}(\mathcal{X}) := j_*(H_{\mathbb{Z}}(\mathcal{X}))$ where $j : \Delta^* \to \Delta$ is an inclusion.

4.2.0.2 Limiting mixed Hodge structure on the fibre $\overline{H}(\mathcal{M})(0)$ and $\overline{H}(\mathcal{X})(0)$

Lemma 4.13. *The limiting Hodge structure* $(t^{N'}\overline{H}_{\mathbb{Z}}(\mathcal{X})(0), \overline{G^q}(0), W'_r)$ *is pure and is isomorphic to the Hodge structure on* $H^1(X_0) \simeq H^1(X_1) \oplus H^1(X_2)$*.*

Proof. Since the singular fiber X_0 is the union of two smooth curves meeting at a node we have $H^1(|\Gamma|) = 0$ where Γ is the dual graph associated to X_0. Therefore, by Lemma 4.12, we get $N' = 0$. Thus there is no weight filtration and hence the limiting Hodge structure is pure. Now we have a morphism of MHS, $i^* : H^1(X_0, \mathbb{C}) \to \overline{H}(\mathcal{X})(0)$ of $(0,0)$ type (see [Mor84, Clemens-Schmid I,page 108]). By Local Invariance Cycle Theorem [Mor84, page 108]), it is known that:

$$Ker(N') = Im(i^*).$$

Since $Ker(N') = \overline{H}(0)$, i^* is surjecive. Now $rk(H^1(X_0, \mathbb{C})) = 2g = rk(\overline{H}(\mathcal{X})(0)))$. Therefore, $i^* : H^1(X_0, \mathbb{C}) \simeq H^1(X_1, \mathbb{C}) \oplus H^1(X_2, \mathbb{C}) \to \overline{H}(\mathcal{X})(0)$ is an isomorphism of Hodge structure. (see [Mor84, page 111]). □

As a consequence of the above Lemma we get the following:

Corollary 4.14. *There is a holomorphic family $\pi_2 : J^0 \to \Delta$ extending the family $\pi_2 : J^{0*} \to \Delta^*$ such that $\pi_2^{-1}(0) = J^0(X_0)$.*

Proof. Since $N' = 0$, we get that $\overline{G^1}(0) \cap \overline{H}_{\mathbb{Z}}(\mathcal{X})(0) = (0)$. As a consequence $\overline{H}_{\mathbb{Z}}(\mathcal{X})(0)$ is a full lattice inside $\overline{H}_{\mathbb{C}}(\mathcal{X})(0)/\bar{G}^1(0)$. Thus there is a holomorphic family $\pi_2 : J^0(\mathcal{X}) \to \Delta$ extending the family $\pi_1 : J^{0*} \to \Delta^*$ such that $\pi_2^{-1}(0) = V/\overline{H}_{\mathbb{Z}}(0)$ where $V := \overline{H}_{\mathbb{C}}(0)/\overline{F^1}(0)$. By Lemma 4.13, it follows that $\pi_2^{-1}(0) \simeq J^0(X_0)$. □

Next we shall show,

Lemma 4.15. *There is an isomorphism $\overline{\phi} : \overline{H_{\mathbb{C}}}(\mathcal{X}) \to \overline{H_{\mathbb{C}}}(\mathcal{M})$ of flat vector bundles such that $\overline{\phi}(\overline{G^q}) = \overline{F^{q+1}}$ and $\overline{\phi}(\overline{H_{\mathbb{Z}}}(\mathcal{X})) = \overline{H_{\mathbb{Z}}}(\mathcal{M})$, $q = 0, 1$.*

Proof. Let $\mathcal{U}$ be the relative universal bundle over $\mathcal{X}^* \times_{\Delta^*} \mathcal{M}^*$ i.e. $\mathcal{U}|_{X_t \times M_t}$ is the corresponding universal bundle. Now if we consider $(1,3)$ Kunneth-component $[c_2(\mathcal{U})|_{X_t \times M_t}]_{1,3} \in H^1(X_t, \mathbb{Z}) \otimes H^3(M_t, \mathbb{Z})$ of $c_2(\mathcal{U}|_{X_t \times M_t})$, then we get a morphism $\phi_t : H^1(X_t, \mathbb{Z}) \to H^3(M_t, \mathbb{Z})$, $t \in \Delta^*$ such that $\phi_t(G^q(t)) \subseteq F^{q+1}(t)$ for $q = 0, 1$ (see [MN68]). Thus we get a morphism $\phi : H_{\mathbb{C}}(\mathcal{X}) \to H_{\mathbb{C}}(\mathcal{M})$ of flat vector bundles, such that $\phi(G^q) \subseteq F^{q+1}$, $q = 0, 1$. By the Mumford-Newstead theorem [MN68, Proposition 1, page 1204] we conclude that $\overline{\phi}$ is an isomorphism. Further, we have $\phi^*(\Theta') = \Theta$ (see [Bal90, Section 5,page 625]). Since the Deligne canonical extension is unique the morphism ϕ extends to an isomorphism $\bar{\phi} : \overline{H}(\mathcal{X}) \to \overline{H}(\mathcal{M})$. Moreover, as the filtration $\{\overline{G^q}(0)\}$ (resp. $\{\overline{F^p}(0)\}$) is canonically determined by the filtrations $\{G^q(t)\}$ (resp. $\{F^q(t)\}$)(see [Mor84, Theorem(Schmid),page 116]), we get $\phi_0(\overline{G^q}(0)) = \overline{F^{q+1}}(0)$. □

Now we will state and prove the main theorem of this chapter:

Theorem 4.16.

1. *There is a holomorphic family* $\{J^2(M_{t,\mathcal{L}_t})\}_{t\in\Delta}$ *of intermediate Jacobians corresponding to the family* $\{M_{t,\mathcal{L}_t}\}_{t\in\Delta}$. *In other words, there is a surjective, proper, holomorphic submersion*
$$\pi_2 : J^2(\mathcal{M}_{\mathcal{L}}) \longrightarrow \Delta$$
such that $\pi_2^{-1}(t) = J^2(M_{t,\mathcal{L}_t})\ \forall\ t \in \Delta^* := \Delta \setminus \{0\}$ *and* $\pi_2^{-1}(0) = J^2(M_{0,\xi})$. *Further, we show that there exists a relative ample class* Θ' *on* $J^2(\mathcal{M}_{\mathcal{L}})_{|\Delta^*}$ *such that* $\Theta'_{|J^2(M_{t,\mathcal{L}_t})} = \theta'_t$, *where* Θ'_t *is the principal polarisation on* $J^2(M_{t,\mathcal{L}_t})$.

2. *There is an isomorphism*
$$\begin{array}{ccc} J^0(\mathcal{X}) & \xrightarrow[\sim]{\Phi} & J^2(\mathcal{M}_{\mathcal{L}}) \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & \Delta & \end{array} \tag{4.4}$$
such that $\Phi^*\Theta'_{|\pi_1^{-1}(t)} = \Theta_t$ *for all* $t \in \Delta^*$, *where* $\pi_1 : J^0(\mathcal{X}) \to \Delta$ *is the holomorphic family* $\{J^0(X_t)\}_{t\in\Delta}$ *of Jacobians and* Θ_t *is the canonical polarisation on* $J^0(X_t)$. *In particular,* $J^2(\mathcal{M}_{\mathcal{L}})_0 := \pi_2^{-1}(0)$ *is an abelian variety.*

Proof. Proof of (1): By Lemma 4.15 we get that the local system $H_{\mathbb{Z}}(\mathcal{M})$ is isomorphic to the local system $H_{\mathbb{Z}}(\mathcal{X})$ over Δ^*. Since by Lemma 4.13 the local system $H_{\mathbb{Z}}(\mathcal{X})$ has trivial monodromy, therefore the local system $H_{\mathbb{Z}}(\mathcal{M})$ also has trivial monodromy. Hence $N = 0$. Thus we have $\overline{F^2}(0) \cap \overline{H}_{\mathbb{Z}}(\mathcal{M})(0) = (0)$. As a consequence we have a holomorphic family $\pi_1 : J^2(\mathcal{M}_{\mathcal{L}}) \to \Delta$ extending the family $\pi_1 : J^{2*} \to \Delta^*$ such that $\pi^{-1}(0) = V'/\overline{H}_{\mathbb{Z}}(\mathcal{M})(0)$ where $V' = \overline{H}_{\mathbb{C}}(\mathcal{M})(0)/\overline{F}^2(0)$. Now we claim: $\pi_1^{-1}(0) \simeq J^2(M_{0,\xi})$. By Theorem 3.21, we see that the Hodge structure on $H^3(M_{0,\xi}, \mathbb{Z})$ is pure and it has rank $2g$. Now there is a morphism $i^* : H^3(M_{0,\xi}, \mathbb{Z}) \to \overline{H}_{\mathcal{M}}(0)$ of MHS of $(0,0)$ type and $Ker(N) = Im(i^*)$. Since $Im(N) = 0$ and both the Hodge structures have the same rank $2g$, $\overline{H}_{\mathbb{Z}}(\mathcal{M})(0)$ and $H^3(M_{0,\xi}, \mathbb{Z})$ are isomorphic as Hodge structures. This completes the proof of (1).

Proof of (2): This immediately follows from Lemma 4.15. □

As a corollary of the theorem we get the following result:

Corollary 4.17. *Let* X_0 *be a projective curve with exactly two smooth irreducible components* X_1 *and* X_2 *meeting at a simple node* p. *We further assume that* $g_i > 3$, $i = 1, 2$. *Then, there is an isomorphism* $J^0(X_0) \simeq J^2(M_{0,\xi})$, *where* $\xi \in J^{\chi}(X_0)$. *In particular,* $J^2(M_{0,\xi})$ *is an abelian variety.*

Proof. By our genus assumption: g_i>3 for i=1,2, we get X_i's have finite number of automorphism $i = 1, 2$. Therefore, the curve X_0 is stable i.e., it has finite number of automorphisms. As the moduli space of stable curves is complete, we get an algebraic family $r : \mathcal{X} \to \mathbb{P}^1$ such that $r_1^{-1}(t)$ is smooth if $t \neq t_0$ and $r_1^{-1}(t_0) = X_0$. Moreover, we can choose $\mathcal{X}$ to be regular over $\mathbb{C}$. Therefore, by Theorem 4.16, we get $J^2(M_{0,\xi}) \simeq J(X_0)$. Hence, $J^2(M_{0,\xi})$ is an abelian variety. □

Chapter 5

Torelli type Theorem for the moduli space of rank 2 deg 1 fixed determinant torsion free sheaves over a reducible curve

In this chapter our goal is to investigate the moduli space $M_{0,\xi}$ more carefully, and show that we can actually recover the curve X_0 i.e. both the components as well as the node, from the moduli space $M_{0,\xi}$ following a strategy given in [BBdBR01].

5.1 Pointed Torelli Theorem for Parabolic moduli space

Let X be a smooth projective curve and E be a rank n algebraic vector bundle on X. Fix a finite number of points $p_1, \cdots, p_m$ on X.

Definition 5.1. A parabolic structure on E over the points p_j, $j = 1, \cdots, m$ consists of the following data:
(i) A flag of subspaces of

$$E(p) := F_1 E(p_j) \supset F_2 E(p_j) \supset .. \supset F_{r_j} E(p_j), \; j = 1, \cdots, m$$

(ii) weights $\alpha_1, \cdots, \alpha_{r_j} \in \mathbb{R}$ attached to the subspaces $F_1(p_j), \cdots, F_{r_j}(p_j)$ such that $0 \leq \alpha_1 < \alpha_2 ... < \alpha_{r_j} < 1$.

For a parabolic vector bundle $(E, \{F_j E(p_j)\}, \{\alpha_i^j\})$ we define the parabolic degree to be:

$$Pardeg(E) = deg(E) + \sum_j \sum_i \alpha_i^j k_i^j.$$

We say a parabolic bundle E is semistable (resp. stable) if for all proper parabolic subbundle $F \subset E$ we have

$$\frac{pardeg(F)}{rk(F)} \leq \frac{pardeg(E)}{rk(E)} \ resp\ (<)$$

We have the following Theorem:

Theorem 5.2. *([MS80, Theorem 4.1]) The coarse moduli space of S-equivalence classes of rank r parabolic semistable bundles with fixed parabolic degree exists as a normal projective variety of dimension $r^2(g-1) + 1 + dim\mathcal{F}_j$ where $\mathcal{F}_j$ are the flag varieties determined by the quasi parabolic structures at the points p_j, $j = 1, \cdots, m$.*

Now we consider parabolic bundles of rank 2 with fixed determinant η_X of degree d and parabolic structure over a point p on X. We choose parabolic weights $0 \leq \alpha_1 < \alpha_2 < 1$ so that parabolic semistable (= parabolic stable). Then the moduli space of parabolic bundles is a smooth, projective variety. We denote this moduli space by M_X^{par}. We state the main theorem of [BBdBR01]:

Theorem 5.3. *([BBdBR01, Theorem 2.1] Suppose, X and Y be two smooth projective curves. Let M_X^{par} (resp. M_Y^{par}) be the moduli spaces of isomorphism classes of rank 2 parabolic stable vector bundles with fixed determinant, defined in the above paragraph. If $M_X^{par} \simeq M_Y^{par}$ then there exists an isomorphism $\sigma : X \to Y$ such that $\sigma(p) = q$.*

5.2 A Torelli type theorem for the singular curve X_0.

In Chapter 2, we have described the moduli space $M_{0,\xi}$ of rank 2, $a = (a_1, a_2)$ stable torsion free sheaves with det $E \simeq \xi$ over X_0.

Let $\pi : \tilde{X}_0 \to X_0$ be the normalization map and $\pi^{-1}(p) = \{x_1, x_2\}$, where $p \in X_1 \cap X_2$. Note that $\tilde{X}_0 = X_1 \sqcup X_2$, the disjoint union of X_1 and X_2. Fix a line bundle ξ on X_0 and let $\xi_i = \xi|_{X_i}$, $i = 1, 2$. Recall that the moduli space $M_{0,\xi}$ of rank 2 stable torsion free sheaves with determinant ξ over X_0 is the union of two irreducible, smooth, projective varieties intersecting transversally along a divisor D. We have also observed that D is isomorphic to the product $P_1 \times P_2$, where P_1 is the moduli space of rank 2 parabolic semistable bundles $(F_1, 0 \subset F^2F_1(x_1) \subset F_1(x_1))$ over X_1 with $det \simeq \xi_1$ and weights $(\frac{a_1}{2}, \frac{a_2}{2})$, and P_2 is the moduli space of rank 2 parabolic semistable bundles $(F_2, 0 \subset F^2F_2(x_2) \subset F_2(x_2))$ over X_2 with $det \simeq \xi_2$ and weights $(\frac{a_1}{2}, \frac{a_2}{2})$, where $a = (a_1, a_2)$ is the polarisation on X_0. Without loss of generality, we can assume that $deg(\xi_1) = 1$ and $deg(\xi_2) = 0$.

Let M_1 (resp. M'_1) be the moduli space of rank 2, deg 1, semistable bundles over X_1 with det $E \simeq \xi_1$ (resp. moduli space of rank 2, deg 0 semistable bundles over X_2 with $detE \simeq \xi_1(-x_1)$).

Note that $Pic(M_1) \simeq \mathbb{Z}$ (resp. $Pic(M'_1) \simeq \mathbb{Z}$). Let θ_1 (resp. θ'_1) be the unique ample generator of $Pic(M_1)$ (resp. of $Pic(M'_1)$). It is known that there exists a unique rank 2 bundle $\mathcal{E}$ over $X_1 \times M_1$ such that $\wedge^2\mathcal{E}_{x_1} \simeq \theta_1$, where $\mathcal{E}_{x_1} := \mathcal{E}_{|x_1 \times M_1}$ (see [Ram73, Definition 2.10]). Since the weights $0<\frac{a_1}{2}, \frac{a_2}{2}<1$ are very small, we can show that: $P_1 \simeq \mathbb{P}(\mathcal{E}_{x_1})$ (see [Bal88, Proposition 6] and [DB02, Theorem 3.7, (1)]). Therefore, it follows that $Pic(P_1) \simeq Pic(M_1) \oplus Pic(\mathbb{P}^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

We define a morphism $\pi'_1 : \mathbb{P}(\mathcal{E}_{x_1}) \to M'_1$ as follows: any closed point of $\mathbb{P}(\mathcal{E}_{x_1})$ over $E \in M_1$ looks like $\{E, V(x_1)\}$, where $V(x_1)$ is a one dimensional subspace of the fibre $E(x_1)$. Consider the vector bundle V which fits into the following exact sequence

$$0 \to V \to E \to (i_{x_1})_*(E(x_1)/V(x_1)) \to 0. \tag{5.1}$$

As $E(x_1)/V(x_1)$ is a 1-dimensional vector space supported over the point x_1, it follows that $det(V) \simeq \xi_1(-x_1)$. We can easily check that V is semistable (see [Bal88, page 11]).

Thus we get a Hecke correspondence:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}_{x_1}) & \xrightarrow{\pi'_1} & M'_1 \\ \downarrow{\scriptstyle \pi_1} & & \\ M_1 & & \end{array} \tag{5.2}$$

Similarly, let M_2 (resp. M'_2) be the moduli space of rank 2, deg 1 semistable bundles over X_2 with $detE \simeq \xi_2(x_2)$ (resp. the moduli space of rank 2, deg 0 semistable bundles over X_2 with $detE \simeq \xi_2$).

Let θ_2 (resp. θ'_2) be the unique ample generator of $Pic(M_2)$ (resp. $Pic(M'_2)$). Then there is a unique universal bundle $\mathcal{E}'$ over $X_2 \times M_2$ such that $\wedge^2 \mathcal{E}'_{x_2} \simeq \theta_2$ where $\mathcal{E}'_{x_2} := \mathcal{E}'_{|x_2 \times M_2}$.

Again, for the choice of weights $0<\frac{a_1}{2}, \frac{a_2}{2}<1$, we have $P_2 \simeq \mathbb{P}(\mathcal{E}'_{x_2})$ (see [DB02, Theorem 3.7 (2)]) and we have a Hecke correspondence as in the previous case:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}'_{x_2}) & \xrightarrow{\pi'_2} & M'_2 \\ \downarrow{\scriptstyle \pi_2} & & \\ M_2 & & \end{array} \tag{5.3}$$

So, we have the following:

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{E}_{x_1}) \times \mathbb{P}(\mathcal{E}'_{x_2}) & \xrightarrow{p_1} & \mathbb{P}(\mathcal{E}_{x_1}) & \xrightarrow{\pi'_1} & M'_1 \\ & & \downarrow{\scriptstyle \pi_1} & & \\ & & M_1 & & \end{array} \tag{5.4}$$

and

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{E}_{x_1}) \times \mathbb{P}(\mathcal{E}'_{x_2}) & \xrightarrow{p_2} & \mathbb{P}(\mathcal{E}'_{x_2}) & \xrightarrow{\pi'_2} & M'_2 \\ & & \downarrow{\scriptstyle \pi_2} & & \\ & & M_2 & & \end{array} \tag{5.5}$$

Remark 5.4. Note that M_1 and M_2 are smooth ,projective, rational varieties. Now P_1 (resp. P_2) is isomorphic to the projective bundle $\mathbb{P}(\mathcal{E}_{x_1})$ (resp. $\mathbb{P}(\mathcal{E}'_{x_2})$). Therefore, the P_i's are rational varieties, $i = 1, 2$.

For the rest of the section we will fix the following notations:

$$\vartheta_1 := (\pi_1 \circ p_1)^* \theta_1, \quad \vartheta_2 := (\pi'_1 \circ p_1)^* \theta'_1,$$

$$\vartheta_3 := (\pi_2 \circ p_2)^* \theta_2, \quad \vartheta_4 := (\pi'_2 \circ p_2)^* \theta'_2.$$

Proposition 5.5. *The numerically effective cone of $P_1 \times P_2$ is generated by the line bundles ϑ_i, $i = 1, 2, 3, 4$.*

Proof. Clearly, ϑ_i, $i = 1, \cdots, 4$ are numerically effective (nef) line bundles as they are the pull backs of the ample line bundles. First we show that ϑ_i, $i = 1, \cdots, 4$, are linearly independent. Note that $\pi_1^*\theta_1$ and $\pi_1'^*\theta_1'$ are linearly independent over $\mathbb{Z}$ (see the proof of [BBdBR01, page 4, Theorem 2.1] for an argument). Therefore, $\vartheta_1 = p_1^*\pi_1^*\theta_1$ and $\vartheta_2 := p_1^*\pi'^*\theta_1'$ are linearly independent. By similar reason ϑ_3 and ϑ_4 are linearly independent. Now we show that the relation $\vartheta_1^{a_1} \otimes \vartheta_2^{a_2} = \vartheta_3^{a_3} \otimes \vartheta_4^{a_4}$ for some $a_i \neq 0, i = 1, \cdots 4$ will not occur. Suppose, $\vartheta_1^{a_1} \otimes \vartheta_2^{a_2} = \vartheta_3^{a_3} \otimes \vartheta_4^{a_4}$. Then this would imply $p_1^*(\pi_1^*\theta_1{}^{a_1} \otimes \pi_1'^*\theta_1'{}^{a_2}) = p_2^*(\pi_2^*\theta_2{}^{a_3} \otimes \pi_2'^*\theta_2'{}^{a_4})$. But this is impossible for the following reason: The line bundle $p_1^*(\pi_1^*\theta_1 \otimes \pi_1'^*\theta_1')$ is trivial on the fibres of p_1. But as the fibres of p_1 are P_2 and $\pi_2^*\theta_2 \otimes \pi_2'^*\theta_2'$ is a non trivial line bundle on P_2 we get $p_2^*(\pi_2^*\theta_2 \otimes \pi_2'^*\theta_2')$ is non trivial on the fibres of p_1. From the above observation, it follows that ϑ_i, $i = 1, \cdots 4$ are linearly independent. Since P_1 and P_2 are both rational varieties we get $Pic(P_1 \times P_2) \simeq Pic(P_1) \times Pic(P_2) \simeq \mathbb{Z}^4$. Therefore, any nef line bundle on $P_1 \times P_2$ is a non negative linear combination of $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$.

Next we show that $\bigotimes_{i=1}^{4} \vartheta_i^{a_i}$ is ample if $a_i > 0$ for all $i = 1, \ldots, 4$. It is enough to show that $\bigotimes_{i=1}^{4} \vartheta_i$ is ample. We observe that $\pi_1^*\theta_1 \otimes \pi_1'^*\theta_1'$ (resp. $\pi_2^*\theta_2 \otimes \pi_2'^*\theta_2'$) is ample on P_1(resp. on P_2)(see the proof of [BBdBR01, Theorem 2.1, page 4, 3rd paragraph]). Therefore , $\bigotimes_{i=1}^{4} \vartheta_i$ is ample on $P_1 \times P_2$.

Finally, we have to show $\bigotimes_{i=1}^{4} \vartheta_i^{a_i}$ is not ample if $a_i = 0$ for some i. Now fix $j \in \{1, \ldots, 4\}$ such that $a_j = 0$. Then $\bigotimes_{i=1, i\neq j}^{4} \vartheta_i$ is not ample as it is the pull back of an ample line bundle from $P_k \times M_l$ or $P_k \times M_k'$ for $l, k \in \{1, 2\}$, $k \neq l$. Next we observe that if $i \in \{1, 2\}$ and $j \in \{3, 4\}$, then $\vartheta_i \otimes \vartheta_j$ is not ample. Since in this case it is pull back of an ample line bundle from $M_k \times M_l$ or $M_k' \times M_l'$ for $k, l \in \{1, 2\}$, $k \neq l$. We have already observed that $\vartheta_1 \otimes \vartheta_2$ and $\vartheta_3 \otimes \vartheta_4$ is not ample.

So, from the above observations, we conclude the proposition. □

Lemma 5.6. *Let $f : X \to Y$ be a projective morphism with Y, a normal variety. Suppose, each fibre of f is a rational variety. Let L be a line bundle on Y then $H^0(X, f^*L) \simeq H^0(Y, L)$.*

Proof. Since the fibres of f are connected and Y is normal we have $\mathcal{O}_Y \simeq f_*\mathcal{O}_X$. Thus $L \simeq f_*f^*L$. Since all the fibres of f are rational we get $H^i(X_y, L_y) = H^i(X_y, \mathcal{O}_{X_y}) = 0$ for all $i > 0$. Hence $H^0(X, f^*L) \simeq H^0(Y, f_*f^*L) = H^0(Y, L)$ □

Remark 5.7. Note that $\pi_1^*\theta_1 \otimes \pi_1'^*\theta_1'$ (resp. $\pi_2^*\theta_2 \otimes \pi_2'^*\theta_2'$) is ample on P_1 (resp. on P_2) (see the proof of 5.5). Now $\vartheta_1 \otimes \vartheta_2 = p_1^*(\pi_1^*\theta_1 \otimes \pi_1'^*\theta_1')$ and $\vartheta_3 \otimes \vartheta_4 = p_1^*(\pi_1^*\theta_3 \otimes \pi_1'^*\theta_4')$. Since P_1 and P_2 are both rational varaities, by Lemma 5.6, the image of the morphism $|(\vartheta_1 \otimes \vartheta_2)^n| : P_1 \times P_2 \to \mathbb{P}^N$ is isomorphic to P_1 for some $n \gg 0$. Similarly, the image of the morphism $(\vartheta_3 \otimes \vartheta_4)^m : P_1 \times P_2 \to \mathbb{P}^M$ is isomorphic to P_2 for some $m \gg 0$.

Lemma 5.8. *Let θ be a nef but not ample line bundle on $P_1 \times P_2$ (i.e., θ lies in the boundary of the nef cone of $P_1 \times P_2$) and $\theta \neq \vartheta_1^a \otimes \vartheta_2^b$ or $\vartheta_3^c \otimes \vartheta_4^d$, where a, b, c and d are some positive integers. Let Z be the image of the morphism $P_1 \times P_2 \to \mathbb{P}^{N'}$ induced by the linear system $|\theta^n|$ for some large n. Then we have $dim(Z) \neq dim(P_i)$ for $i = 1, 2$.*

Proof. Assume that $\theta \neq \vartheta_1 \otimes \vartheta_2$ *or* $\vartheta_3 \otimes \vartheta_4$. Then θ is either of the form $\bigotimes_{i \neq j} \vartheta_i$ for $i, j \in \{1, \dots, 4\}$ or $\vartheta_i \otimes \vartheta_j$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$ (see the proof of 5.5).

Fix $j \in \{1, 2, 3, 4\}$. If θ is of the form $\bigotimes_{\substack{i=1 \\ i \neq j}}^{4} \vartheta_i$, then the image Z of the morphism $|\theta^n|$, for sufficiently large n, is either isomorphic to $P_k \times M_l$ or $P_k \times M_l'$ for $k, l \in \{1, 2\}$, $k \neq l$.

If θ is of the form $\vartheta_i \otimes \vartheta_j$ then the image Z of the morphism $|\theta^n|$ is either isomorphic to $M_k \times M_l$ or $M_k \times M_l'$, for $k, l \in \{1, 2\}$, $k \neq l$. In both the cases we see $dim(Z) \neq dim(P_i)$ and hence we are done. □

Now we prove the main theorem of this section:

Let X_0 (resp. Y_0) be a reducible curve with two components X_1, X_2 (resp. Y_1, Y_2) meeting transversally at a point p (resp. q). Let $\pi_1 : \tilde{X}_0 \to X_0$ (resp. $\pi_2 : \tilde{Y}_0 \to Y_0$) be the normalisation map and $\pi_1^{-1}(p) = \{x_1, x_2\}$, $\pi_2^{-1}(q) = \{y_1, y_2\}$. We will make the following assumption on the components of X_0 and Y_0.

- $g(X_i) = g(Y_i) \geq 2$ for $i = 1, 2$.
- $X_1 \ncong X_2$ (resp. $Y_1 \ncong Y_2$).

Fix $\xi_{X_0} \in J^{\chi}(X_0)$ (resp. $\xi_{Y_0} \in J^{\chi}(Y_0)$). Let $M_{0,\xi_{X_0}}$ (resp. $M_{0,\xi_{Y_0}}$) be the moduli space of rank 2, $a = (a_1, a_2)$-stable torsion free sheaves with $detE \simeq \xi_{X_0}$ (resp.$detE \simeq \xi_{Y_0}$) on X_0 (resp. on Y_0). Let $D \subset M_{0,\xi_{X_0}}$ (resp. $D' \subset M_{0,\xi_{Y_0}}$) be the singular locus of $M_{0,\xi_{X_0}}$

(resp. $M_{0,\xi_{Y_0}}$) and P_i (resp. P'_i) be the parabolic moduli spaces, described before, with parabolic structure over x_i (resp. y_i). Then $D \simeq P_1 \times P_2$ and $D' \simeq P'_1 \times P'_2$. Now we have the following Torelli type theorem.

Theorem 5.9. *If $M_{0,\xi_{X_0}} \simeq M_{0,\xi_{Y_0}}$ then we have $X_0 \simeq Y_0$.*

Proof. Let $\Psi : M_{0,\xi_{X_0}} \simeq M_{0,\xi_{Y_0}}$ be an isomorphism. Then $\Psi(D) = D'$ as D is the singular locus of $M_{0,\xi}$. Therefore, Ψ induces an isomorphism $\Psi : P_1 \times P_2 \simeq P'_1 \times P'_2$. Now if we can show that the above statement will imply $P_i \simeq P'_{\sigma(i)}$ for $i \in \{1,2\}$ and σ is a permutation on $\{1,2\}$. Then by [BBdBR01, Theorem 2.1], we get an isomorphism $f_i : X_i \to Y_{\sigma(i)}$ such that $f_i(x_i) = y_{\sigma(i)}$. Hence, we get $X_0 \simeq Y_0$. We will show that if $\Psi : P_1 \times P_2 \simeq P'_1 \times P'_2$, then $P_i \simeq P'_{\sigma(i)}$.

Let $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4$ be the generators of the nef cone of $P'_1 \times P'_2$ as in Proposition 5.5. Let $N := \varsigma_1 \otimes \varsigma_2$ and $N' := \varsigma_3 \otimes \varsigma_4$. Then Ψ^*N, Ψ^*N' lie in the boundary of the nef cone of $P_1 \times P_2$. Note that, for sufficiently large n, m, the image of the morphism $|\Psi^*N^n|$ is isomorphic to P'_1 and the image of the morphism $|\Psi^*N'^m|$ is isomorphic to P'_2.

Now we claim that $\Psi^*(N) = \vartheta_1^a \otimes \vartheta_2^b$ or $\vartheta_3^c \otimes \vartheta_4^d$ for some $a, b, c, d > 0$. Otherwise, by Lemma 5.8, the dimension of the image of $|\Psi^*(N)^n|$ will be different from the dimension of P'_1. Suppose that $\Psi^*(N) = \vartheta_1^a \otimes \vartheta_2^b$ for some $a, b > 0$. Then, by our assumption $Y_1 \not\cong Y_2$, we have $\Psi^*(N') = \vartheta_3^c \otimes \vartheta_4^d$ for some $c, d > 0$. Therefore, by Remark 5.7, for sufficiently large $n, m \gg 0$, the images of the morphisms defined by the linear systems $|\Psi^*N^n|$ and $|\Psi^*N'^m|$ will be isomorphic to P_1 and P_2. Hence, we have isomorphisms $\Phi_1 : P_1 \to P'_1$ and $\Phi_2 : P_2 \to P'_2$ such that the following diagrams commute:

$$\begin{array}{ccc} P_1 \times P_2 & \xrightarrow{\Psi} & P'_1 \times P'_2 \\ \downarrow{\scriptstyle |\Psi^*(N)^n|} & & \downarrow{\scriptstyle |N^n|} \\ P_1 & \xrightarrow{\Phi_1} & P'_1 \end{array} \tag{5.6}$$

$$\begin{array}{ccc} P_1 \times P_2 & \xrightarrow{\Psi} & P'_1 \times P'_2 \\ \downarrow{\scriptstyle |\Psi^*(N')^m|} & & \downarrow{\scriptstyle |N'^m|} \\ P_2 & \xrightarrow{\Phi_2} & P'_2 \end{array} \tag{5.7}$$

Therefore, by Theorem 5.3, there is an isomorphism $f_1 : X_1 \to Y_1$ such that $f_1(x_1) = y_1$ and an isomorphism $f_2 : X_2 \to Y_2$ such that $f_2(x_2) = y_2$.

Suppose that $\Psi^*(N) = \vartheta_2^c \otimes \vartheta_4^d$, c, d>0. Then, by similar arguments as above, we can show that $P_2 \simeq P_1'$ and $P_1 \simeq P_2'$. Therefore, there is an isomorphism $f_1' : X_2 \to Y_1$ such that $f_1'(x_2) = y_1$ and an isomorphism $f_2'(x_1) = y_2$ Hence, we conclude $X_0 \simeq Y_0$. This completes the proof. □

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